Calculus and Analytical Geometry

1 Core Topics

The course covers:
Functions, Domain and Range. Introduction to limits: Limits and Continuity, Techniques of funding limits, Indeterminate forms of limits, Introduction to functions: Continuous and discontinuous functions and their applications, Differential calculus: Concept and idea of differentiation, Geometrical and Physical meaning of derivatives, Rules of differentiation, Techniques of differentiation, Rates of change, Tangents and Normal lines, Chain rule, implicit differentiation, linear approximation, Applications of differentiation: Extreme value functions, Mean value theorems, Maxima and Minima of a function for single-variable, Concavity. Integral calculus: Concept and idea of Integration, Indefinite Integrals, Techniques of integration, Riemann sums and Definite Integrals, Applications of definite integrals, Improper integral, Applications of Integration; Area under the curve. **Book recommended for this course is Calculus by Thomas 13^{th} edition.**

1.1 Definition of a Function

A function f from a set D to a set Y is a rule that assigns a unique value $f(x)$ in Y to each x in D.

- **Input:** x is the independent variable.
- **Output:** $y = f(x)$ is the dependent variable.

1.2 Domain and Range

- **Domain:** The set D of all possible input values.
- **Range:** The set of all $f(x)$ values as x varies throughout D.
- In Calculus, domain and range are often sets of real numbers.

1.3 Examples of Domain and Range

1. $y = x^2$: Domain $(-\infty, \infty)$, Range $[0, \infty)$.
2. $y = \frac{1}{x}$: Domain $(-\infty, 0) \cup (0, \infty)$, Range $(-\infty, 0) \cup (0, \infty)$.
3. $y = \sqrt{x}$: Domain $[0, \infty)$, Range $[0, \infty)$.
4. $y = \sqrt{4 - x}$: Domain $(-\infty, 4]$, Range $[0, \infty)$.
5. $y = \frac{x^2 - 4}{x - 2} = x + 2$: Domain $(-\infty, \infty)$.
6. $y = \sqrt{1 - x^2}$: Domain $[-1, 1]$, Range $[0, 1]$.

1.4 Vertical Line Test

A function f can have only one value $f(x)$ for each x in its domain. Therefore, no vertical line can intersect the graph of a function more than once.

- A circle is not the graph of a function.
- $y = \sqrt{1 - x^2}$ (upper semicircle) and $y = -\sqrt{1 - x^2}$ (lower semicircle) are functions.

1.5 Even and Odd Functions

- **Even Function:** $f(-x) = f(x)$. Symmetric about the y-axis. Example: $y = x^2$, $y = x^4$.
- **Odd Function:** $f(-x) = -f(x)$. Symmetric about the origin. Example: $y = x$, $y = x^3$

1.6 Step Functions

- **Least Integer Function (Ceiling):** Denoted by $\lceil x \rceil$. Smallest integer greater than or equal to x.
- **Greatest Integer Function (Floor):** Denoted by $\lfloor x \rfloor$. Greatest integer less than or equal to x. Examples: $\lfloor 2.4 \rfloor = 2$, $\lfloor 1.9 \rfloor = 1$.

1.7 Increasing and Decreasing Functions

Let f be defined on an interval I with $x_1, x_2 \in I$:

1. **Increasing:** If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$.
2. **Decreasing:** If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$.

1.8 Everyday Life Uses of Calculus

- **Physics:** Integration is used to calculate the center of mass and center of gravity.
- **Computer Graphics:** Uses Linear Algebra and Analytic Geometry.
- **Scientific Computing:** Computer Algebra Systems compute integrals and derivatives symbolically or numerically.

2 Common Functions

2.1 Power Functions

A function $f(x) = x^a$, where a is a constant, is called a power function.

2.1.1 (a) $a = n$ (A Positive Integer)

For $n = 1, 2, 3, 4, 5$, the functions are $y = x, y = x^2, y = x^3, y = x^4, y = x^5$.

- As the power n gets larger, the curves tend to flatten towards the x-axis on the interval $(-1, 1)$ and rise more steeply for $|x| > 1$.
- **Even-powered functions:** Decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.
- **Odd-powered functions:** Increasing over the entire real line $(-\infty, \infty)$.

2.1.2 (b) $a = -1$ and $a = -2$

- $f(x) = x^{-1} = \frac{1}{x}$: Domain $x \neq 0$, Range $y \neq 0$. Symmetric about the origin; decreasing on $(-\infty, 0)$ and $(0, \infty)$.
- $g(x) = x^{-2} = \frac{1}{x^2}$: Domain $x \neq 0$, Range $y > 0$. Symmetric about the y-axis; increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

2.1.3 (c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}$

- $y = \sqrt{x} = x^{1/2}$: Domain $[0, \infty)$, Range $[0, \infty)$.
- $y = \sqrt[3]{x} = x^{1/3}$: Defined over all real x. Domain $(-\infty, \infty)$, Range $(-\infty, \infty)$.
- $y = x^{2/3} = (x^{1/3})^2$: Domain $(-\infty, \infty)$, Range $[0, \infty)$.

2.2 Polynomials

A function p is a polynomial if:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are real constants (coefficients).

- **Domain:** All polynomials have the domain $(-\infty, \infty)$.
- **Degree:** If $a_n \neq 0$, then n is the degree of the polynomial.
- **Quadratic:** Degree 2, $p(x) = ax^2 + bx + c$.
- **Cubic:** Degree 3, $p(x) = ax^3 + bx^2 + cx + d$.

2.3 Linear Functions

A function of the form $f(x) = mx+b$ is a linear function.

- If $b = 0$, the line passes through the origin.
- If $m = 0$, it is a constant function.
- If $m = 1$ and $b = 0$, $f(x) = x$ is the identity function.

2.4 Rational and Algebraic Functions

- **Rational Function:** A ratio of two polynomials, $f(x) = \frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$.
- **Algebraic Function:** Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots).

2.5 Transcendental Functions

These functions are not algebraic and include:

- **Trigonometric Functions:** e.g., $f(x) = \sin x, f(x) = \cos x$.
- **Exponential Functions:** $f(x) = a^x$ (where $a > 0, a \neq 1$). Range is always $(0, \infty)$.
- **Logarithmic Functions:** $f(x) = \log_a x$, which are the inverse of exponential functions.
- **Example:** The Catenary is a transcendental function.

3 Sum, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided to produce new functions (except where the denominator is zero).
If f and g are functions, then for every x that belongs to the domain of both f and g (i.e., $x \in D(f) \cap D(g)$), we define the following:

- $(f + g)(x) = f(x) + g(x)$
- $(f - g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$

At any point in $D(f) \cap D(g)$ where $g(x) \neq 0$, we can define the quotient:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Functions can also be multiplied by constants. If c is a real number, then:

$$(cf)(x) = cf(x)$$

Example: Given $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$:

- $D(f) = [0, \infty)$
- $D(g) = (-\infty, 1]$
- Common domain: $D(f) \cap D(g) = [0, \infty) \cap (-\infty, 1] = [0, 1]$
- $(f + g)(x) = \sqrt{x} + \sqrt{1-x}$, Domain: $[0, 1]$
- $(f - g)(x) = \sqrt{x} - \sqrt{1-x}$, Domain: $[0, 1]$
- $(fg)(x) = \sqrt{x(1-x)}$, Domain: $[0, 1]$
- $\left(\frac{f}{g}\right)(x) = \sqrt{\frac{x}{1-x}}$, Domain: $[0, 1)$ (exclude $x = 1$ to avoid division by zero)

4 Composite Functions

Composition is another way of combining functions. **Definition:** If f and g are functions, the composite function $f \circ g$ (f composed with g) is defined by:

$$(f \circ g)(x) = f(g(x))$$

Example: If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find:

1. $(f \circ g)(x) = f(g(x)) = \sqrt{x+1}$, Domain: $[-1, \infty)$
2. $(g \circ f)(x) = g(f(x)) = \sqrt{x} + 1$, Domain: $[0, \infty)$
3. $(f \circ f)(x) = f(f(x)) = \sqrt{\sqrt{x}} = x^{1/4}$, Domain: $[0, \infty)$
4. $(g \circ g)(x) = g(g(x)) = (x+1)+1 = x+2$, Domain: $(-\infty, \infty)$

5 Shifting a Graph of a Function

5.1 Vertical Shifts

- $y = f(x) + k$ shifts the graph up k units ($k > 0$).
- $y = f(x) - k$ shifts the graph down k units ($k > 0$).

5.2 Horizontal Shifts

- $y = f(x + h)$ shifts the graph left h units ($h > 0$).
- $y = f(x - h)$ shifts the graph right h units ($h > 0$).

6 Scaling and Reflecting Formulas

6.1 Scaling (for $c > 1$)

- $y = cf(x)$: Stretches the graph vertically by a factor c.
- $y = \frac{1}{c}f(x)$: Compresses the graph vertically by factor c.
- $y = f(cx)$: Compresses the graph horizontally by factor c.
- $y = f(x/c)$: Stretches the graph horizontally by factor c.

6.2 Reflections

- $y = -f(x)$: Reflects the graph across the x-axis.
- $y = f(-x)$: Reflects the graph across the y-axis.

7 Limit of a Function

7.1 Definition

Let f be a function. If as x approaches a from both left and right sides of a, $f(x)$ approaches to a special number L, then L is called the limit as x approaches a:

$$\lim_{x \to a} f(x) = L$$

7.2 Example

Find the limit of $f(x) = \frac{x^2-4}{x-2}$. As x approaches 2:

- $f(1.999) = 3.99$
- $f(2.001) = 4.001$

The limit appears to be 4.

8 Informal Description of the Limit of a Function

For the function $f(x) = \frac{x^2-1}{x-1}$, the function is defined for all real numbers x except $x = 1$. For $x \neq 1$, we can simplify the formula by factoring:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \text{ for } x \neq 1$$

Even though f is not defined at $x = 1$, the function has a limit of 2 as x approaches 1.

9 Limits of Identity and Constant Functions

1. If f is the identity function $f(x) = x$, then for any value of c:

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c$$

2. If f is the constant function $f(x) = k$, then for any value c:

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k$$

10 Examples of Functions with No Limit

A function may not have a limit at a particular point for several reasons:

- **Jump Discontinuity:** The unit step function $U(x)$ has no limit at $x = 0$ because it jumps from 0 (for $x < 0$) to 1 (for $x > 0$).
- **Infinite Growth:** The function $g(x) = \frac{1}{x}$ has no limit as $x \to 0$ because its values grow arbitrarily large in absolute value.
- **Oscillation:** The function $f(x) = \sin(1/x)$ has no limit as $x \to 0$ because its values oscillate between $+1$ and -1 infinitely often as it approaches zero.

11 The Limit Laws

If L, M, c and k are real numbers and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then:

1. **Sum Rule:** $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. **Difference Rule:** $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. **Constant Multiple Rule:** $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. **Product Rule:** $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. **Quotient Rule:** $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$
6. **Power Rule:** $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. **Root Rule:** $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$, n a positive integer

12 Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then:

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0$$

13 Exercises and Infinite Limits

13.1 Exercise 2.4

1. True or False Statements based on a graph (implied):

(a) $\lim_{x\to -1^+} f(x) = 1$ (True)

(b) $\lim_{x\to 0^-} f(x) = 1$ (False)

(c) $\lim_{x\to 0^-} f(x) = 0$ (True)

(d) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$ (True)

3. Piecewise Function: Let $f(x)$ be defined as:

$$f(x) = \begin{cases} 3 - x & x < 2 \\ \frac{x}{2} + 1 & x > 2 \end{cases}$$

(a) Find limits as $x \to 2$:

- Right-hand limit: $\lim_{x\to 2^+} f(x) = \frac{2}{2} + 1 = 2$
- Left-hand limit: $\lim_{x\to 2^-} f(x) = 3 - 2 = 1$

(b) Does $\lim_{x\to 2} f(x)$ exist? No, the limit does not exist because $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$.
(c) Limit at $x = 4$: $\lim_{x\to 4^-} f(x) = \frac{4}{2} + 1 = 3$.

13.2 Sandwich Theorem (Squeeze Theorem)

Theorem: Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c (except possibly at $x = c$) and that:

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

Then, $\lim_{x\to c} f(x) = L$.
Example: Given $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$ for all $x \neq 0$. Find $\lim_{x\to 0} u(x)$.

- Limit of lower bound: $\lim_{x\to 0}(1 - \frac{x^2}{4}) = 1$
- Limit of upper bound: $\lim_{x\to 0}(1 + \frac{x^2}{2}) = 1$

By the Sandwich Theorem, $\lim_{x\to 0} u(x) = 1$.

13.3 Limits Involving Infinity

Finite Limits as $x \to \pm\infty$: The symbol ∞ does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds.
Rules:

1. $\lim_{x\to\pm\infty} k = k$ (where k is constant)
2. $\lim_{x\to\pm\infty} \frac{1}{x} = 0$
3. $\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$ (for positive integer n)

Example: Evaluate $\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2}$. Divide numerator and denominator by the highest power of x (which is x^2):

$$\lim_{x\to\infty} \frac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{2}{x^2}} = \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3}$$

Example: Evaluate $\lim_{x\to\infty} \frac{11x+2}{2x^3-1}$. Divide by x^3:

$$\lim_{x\to\infty} \frac{\frac{11}{x^2} + \frac{2}{x^3}}{2 - \frac{1}{x^3}} = \frac{0+0}{2-0} = 0$$

13.4 Infinite Limits

Example: Find $\lim_{x\to\infty}(\sqrt{x^6 + 5x^3} - x^3)$. Multiply by the conjugate:

$$= \lim_{x\to\infty}(\sqrt{x^6 + 5x^3} - x^3) \cdot \frac{\sqrt{x^6 + 5x^3} + x^3}{\sqrt{x^6 + 5x^3} + x^3}$$
$$= \lim_{x\to\infty} \frac{(x^6 + 5x^3) - x^6}{\sqrt{x^6 + 5x^3} + x^3}$$
$$= \lim_{x\to\infty} \frac{5x^3}{\sqrt{x^6(1 + \frac{5}{x^3})} + x^3}$$
$$= \lim_{x\to\infty} \frac{5x^3}{x^3\sqrt{1 + \frac{5}{x^3}} + x^3}$$
$$= \lim_{x\to\infty} \frac{5}{\sqrt{1 + \frac{5}{x^3}} + 1} = \frac{5}{\sqrt{1+0} + 1} = \frac{5}{2}$$

13.5 Asymptotes

Horizontal Asymptote: A line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either:

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

Vertical Asymptote: A line $x = a$ is a vertical asymptote if $\lim_{x\to a^\pm} f(x) = \pm\infty$.

14 Limits at Infinity

We consider the behavior of functions as x approaches positive or negative infinity.

14.1 Basic Limits

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$
$$\lim_{x\to+\infty} \frac{1}{x} = 0$$

In general, for any positive integer n:

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$$

14.2 Laws of Limits at Infinity

Behaviors involving infinity:

- $\infty + \infty = \infty$
- As $x \to \infty$, $x^2 \to \infty$.
- As $x \to \infty$, $-x^2 \to -\infty$.

15 Examples: Limits of Rational Functions

15.1 Example 1: Equal Degrees

Evaluate $\lim_{x\to\infty} \frac{x^2+1}{x^2+2x+1}$.

Divide numerator and denominator by the highest power of x in the denominator (x^2):

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2+1}{x^2+2x+1} &= \lim_{x\to\infty}\frac{\frac{x^2}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{2x}{x^2}+\frac{1}{x^2}}\\ &= \lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{1+\frac{2}{x}+\frac{1}{x^2}}\\ &= \frac{1+0}{1+0+0} = 1\end{aligned}$$

15.2 Example 2: Higher Degree in Denominator

Evaluate $\lim_{x\to\infty}\frac{x^2+1}{x^3+3x+1}$.
Divide by x^3:

$$\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{1}{x^3}}{1+\frac{3}{x^2}+\frac{1}{x^3}} = \frac{0+0}{1+0+0} = 0$$

Since the limit is 0, $y = 0$ is a horizontal asymptote.

15.3 Example 3: Higher Degree in Numerator

Evaluate $\lim_{x\to\infty}\frac{x^3+1}{x^2+1}$.
Divide by x^2:

$$\begin{aligned}\lim_{x\to\infty}\frac{\frac{x^3}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}} &= \lim_{x\to\infty}\frac{x+\frac{1}{x^2}}{1+\frac{1}{x^2}}\\ &= \frac{\infty+0}{1+0} = \infty\end{aligned}$$

16 Limits Involving Radicals

Evaluate $\lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{3x-6}$.
Recall that $\sqrt{x^2} = |x|$. Since $x\to-\infty$, we have $|x| = -x$ (or $x < 0$). We divide the numerator by $\sqrt{x^2}$ and the denominator by $-|x|$ (which is effectively dividing by the same magnitude, respecting signs):

$$\begin{aligned}\lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{3x-6} &= \lim_{x\to-\infty}\frac{\frac{\sqrt{x^2+2}}{\sqrt{x^2}}}{\frac{3x}{-\sqrt{x^2}}-\frac{6}{-\sqrt{x^2}}}\\ &\text{(Using } x = -\sqrt{x^2} \text{ for } x<0)\\ &= \lim_{x\to-\infty}\frac{\sqrt{1+\frac{2}{x^2}}}{\frac{3x}{-(-x)}-\frac{6}{-x}}\\ &= \lim_{x\to-\infty}\frac{\sqrt{1+\frac{2}{x^2}}}{3-\frac{6}{x}}\end{aligned}$$

(Simplification step based on sign)
Alternatively, dividing by x (where $x = -|x|$):

$$\begin{aligned}&= \lim_{x\to-\infty}\frac{\frac{\sqrt{x^2+2}}{-\sqrt{x^2}}}{\frac{3x}{x}-\frac{6}{x}}\\ &= \frac{-\sqrt{1+0}}{3-0} = -\frac{1}{3}\end{aligned}$$

Thus, $y = -1/3$ is a horizontal asymptote.

17 Asymptotes

An asymptote of a function is a curve or a line such that the distance between the function and the line tends to zero as they tend to infinity.

17.1 Types of Asymptotes

1. Horizontal Asymptotes
2. Vertical Asymptotes
3. Oblique (Slant) or Curvilinear Asymptotes

17.2 Horizontal Asymptote Example

For $y = \frac{1}{x}$:

$$\lim_{x\to\infty}\frac{1}{x} = 0, \quad \lim_{x\to-\infty}\frac{1}{x} = 0$$

Therefore, $y = 0$ is a horizontal asymptote.

17.3 Vertical Asymptote Example

Consider $f(x) = \frac{x-1}{x^2-1}$. Simplifying the expression (for $x \neq 1$):

$$f(x) = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}$$

The denominator is zero at $x = -1$. Thus, $x = -1$ is a vertical asymptote.

17.4 Domain and Range Example

For a function like $y = \frac{1}{x-3} + 7$:

- Vertical Asymptote: $x = 3$
- Domain: $D = (-\infty, 3) \cup (3, \infty)$
- Range: $R = (-\infty, 7) \cup (7, \infty)$ (assuming horizontal asymptote shift)

18 Oblique and Curvilinear Asymptotes

18.1 Oblique Asymptotes

Consider the function $f(x) = \frac{x^2+1}{x+1}$. By performing long division:

$$f(x) = \frac{x^2+1}{x+1} = x - 1 + \frac{2}{x+1}$$

As $x\to\infty$, the term $\frac{2}{x+1}\to 0$. Therefore, the line $y = x - 1$ is an oblique asymptote.

18.2 Curvilinear Asymptotes

Consider the function $f(x) = \frac{x^3+1}{x+2}$. By performing long division ($x^3 + 1$ divided by $x + 2$):

$$f(x) = x^2 - 2x + 4 - \frac{7}{x+2}$$

As $x\to\infty$, the fraction approaches 0. Therefore, the parabola $y = x^2 - 2x + 4$ is a curvilinear asymptote.

18.3 Limit at Infinity Example

Evaluate $\lim_{x\to\infty} \frac{x^3+x^2+1}{5x^4+2x+1}$. Divide numerator and denominator by the highest power of x in the denominator (x^4):

$$\lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^4}}{5+\frac{2}{x^3}+\frac{1}{x^4}} = \frac{0}{5} = 0$$

19 Continuity

19.1 Continuity at a Point

Definition: A function f is continuous at a point $x = a$ if the following three conditions are met:

1. $f(a)$ is defined.
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x) = f(a)$.

19.2 Example: Discontinuity

Check the continuity of the function at $x = 3$:

$$g(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 5, & x = 3 \end{cases}$$

Solution:

1. Value of function: $g(3) = 5$.
2. Limit as $x \to 3$:
$$\begin{aligned} \lim_{x\to 3} g(x) &= \lim_{x\to 3} \frac{x^2-9}{x-3} \\ &= \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} \\ &= \lim_{x\to 3}(x+3) = 6 \end{aligned}$$
3. Conclusion: Since $\lim_{x\to 3} g(x) \neq g(3)$ $(6 \neq 5)$, the function $g(x)$ is discontinuous at $x = 3$.

20 Continuity of a Polynomial

20.1 Example: Absolute Value

Check the continuity of $f(x) = |x|$.

$$|x| = \begin{cases} -x, & x < 0 \\ 0, & x = 0 \\ x, & x > 0 \end{cases}$$

The function is continuous on $(-\infty, 0)$ and $(0, \infty)$ because it behaves like a polynomial. At $x = 0$:

- $f(0) = 0$
- $\lim_{x\to 0} f(x) = 0$

Since the limit equals the function value, f is continuous at $x = 0$. Thus, it is continuous everywhere.

21 Continuity on an Interval

A function f is continuous on a closed interval $[a, b]$ if it is continuous at every point in (a, b) and:

- $\lim_{x\to a^+} f(x) = f(a)$ (Right continuous at a)
- $\lim_{x\to b^-} f(x) = f(b)$ (Left continuous at b)

21.1 Example

Check the continuity of $f(x) = \sqrt{9-x^2}$ on $[-3, 3]$.

1. Let c be an arbitrary point in $(-3, 3)$.
$$\lim_{x\to c} f(x) = \lim_{x\to c} \sqrt{9-x^2} = \sqrt{9-c^2} = f(c)$$
So, f is continuous on the open interval $(-3, 3)$.
2. At endpoints:
   - At $x = -3$: $\lim_{x\to -3^+} \sqrt{9-x^2} = 0 = f(-3)$.
   - At $x = 3$: $\lim_{x\to 3^-} \sqrt{9-x^2} = 0 = f(3)$.

Therefore, $f(x)$ is continuous on $[-3, 3]$.

22 Exercise 2.6 Solutions

22.1 Q.9: Sandwich Theorem

Evaluate $\lim_{x\to\infty} \frac{\sin 2x}{x}$. We know that for all real numbers:

$$-1 \leq \sin 2x \leq 1$$

Divide inequality by x (for $x > 0$):

$$-\frac{1}{x} \leq \frac{\sin 2x}{x} \leq \frac{1}{x}$$

Since $\lim_{x\to\infty}(-\frac{1}{x}) = 0$ and $\lim_{x\to\infty}(\frac{1}{x}) = 0$, by the Sandwich Theorem:

$$\lim_{x\to\infty} \frac{\sin 2x}{x} = 0$$

22.2 Q.13

Evaluate $\lim_{x\to\infty} \frac{2x+3}{5x+7}$.

$$\lim_{x\to\infty} \frac{x(2+\frac{3}{x})}{x(5+\frac{7}{x})} = \frac{2+0}{5+0} = \frac{2}{5}$$

22.3 Q.21

Evaluate $\lim_{x\to\infty} \frac{3x^4+5x^2-1}{6x^3-7x+3}$. Divide by x^3:

$$\lim_{x\to\infty} \frac{3x+\frac{5}{x}-\frac{1}{x^3}}{6-\frac{7}{x^2}+\frac{3}{x^3}} = \frac{\infty}{6} = \infty$$

22.4 Q.23

Evaluate $\lim_{x\to\infty} \sqrt{\frac{8x^2-3}{2x^2+x}}$.

$$\sqrt{\lim_{x\to\infty} \frac{x^2(8-\frac{3}{x^2})}{x^2(2+\frac{1}{x})}} = \sqrt{\frac{8}{2}} = \sqrt{4} = 2$$

23 Slope of a Line

The slope of a line is defined as the rise over the run:

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2-y_1}{x_2-x_1}$$

24 Secant Line

A secant line is a line joining any two points on a curve. Consider a curve $y = f(x)$ and two points $P(x, f(x))$ and $Q(x+h, f(x+h))$. The slope of the secant line (m_{sec}) represents the average rate of change:

$$m_{sec} = \frac{f(x+h) - f(x)}{h}$$

25 Tangent Line

As the point Q approaches P (i.e., $h \to 0$), the secant line becomes the tangent line. The slope of the tangent line (m_{tan}) is the limit of the secant slope:

$$m_{tan} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

This limit is also known as the derivative of f at x, denoted by $\frac{dy}{dx}$ or $f'(x)$.

26 Examples: Definition of Derivative

26.1 Example 1: $f(x) = x^2$

Find the derivative with respect to x using the definition.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h\to 0} (2x+h) = 2x \end{aligned}$$

26.2 Example 2: $f(x) = \sqrt{x}$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

26.3 Example 3: Instantaneous Velocity

Consider a particle whose position is given by $S(t) = 1 + 5t - 2t^2$. Find the instantaneous velocity $v(t)$.

$$\begin{aligned} v(t) &= \lim_{h\to 0} \frac{S(t+h) - S(t)}{h} \\ &= \lim_{h\to 0} \frac{[1 + 5(t+h) - 2(t+h)^2] - [1 + 5t - 2t^2]}{h} \\ &= \lim_{h\to 0} \frac{1 + 5t + 5h - 2(t^2 + h^2 + 2th) - 1 - 5t + 2t^2}{h} \\ &= \lim_{h\to 0} \frac{5h - 2h^2 - 4th}{h} \\ &= \lim_{h\to 0} (5 - 2h - 4t) = 5 - 4t \end{aligned}$$

27 Finding the equation* of the Tangent Line

To find the tangent line to $y = f(x)$ at a point $x = x_1$:

1. Evaluate $y_1 = f(x_1)$ to get the point of tangency (x_1, y_1).
2. Find $f'(x)$ and evaluate $m = f'(x_1)$ to get the slope.
3. Substitute m, x_1, and y_1 into the point-slope form:
$$y - y_1 = m(x - x_1)$$

27.1 Example 4

Find the tangent line of $f(x) = 2x^2 - 3$ at $(2, 5)$.

1. Point is given: $(2, 5)$. Check: $f(2) = 2(4) - 3 = 5$.
2. Find $f'(x)$:
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{[2(x+h)^2 - 3] - [2x^2 - 3]}{h} \\ &= \lim_{h\to 0} \frac{2(x^2 + 2xh + h^2) - 3 - 2x^2 + 3}{h} \\ &= \lim_{h\to 0} \frac{4xh + 2h^2}{h} = 4x \end{aligned}$$
Slope at $x = 2$: $m = f'(2) = 4(2) = 8$.
3. equation*:
$$y - 5 = 8(x - 2) \implies y = 8x - 11$$

27.2 Example 5

Find the tangent line for $y = \frac{x+3}{1-x}$ at $x = -2$.

1. Find y_1: $y(-2) = \frac{-2+3}{1-(-2)} = \frac{1}{3}$. Point: $(-2, 1/3)$.
2. Find derivative (using definition):
$$\begin{aligned} y'(x) &= \lim_{h\to 0} \frac{\frac{x+h+3}{1-(x+h)} - \frac{x+3}{1-x}}{h} \\ &= \lim_{h\to 0} \frac{(x+h+3)(1-x) - (x+3)(1-x-h)}{h(1-x-h)(1-x)} \\ &= \ldots \text{ (simplifying numerator) } \ldots \\ &= \lim_{h\to 0} \frac{4h}{h(1-x-h)(1-x)} = \frac{4}{(1-x)^2} \end{aligned}$$
Slope at $x = -2$: $m = \frac{4}{(1-(-2))^2} = \frac{4}{3^2} = \frac{4}{9}$.
3. equation*:
$$y - \frac{1}{3} = \frac{4}{9}(x + 2)$$

28 Differentiation Rules

28.1 Derivative of a Constant

If f is a constant function $f(x) = c$, then:

$$\frac{d}{dx}(c) = 0$$

28.2 Power Rule

If n is a positive integer (holds for real numbers where defined):

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Examples:

- $y = x^{\pi} \implies \frac{dy}{dx} = \pi x^{\pi-1}$
- $y = x^{1/3} \implies \frac{dy}{dx} = \frac{1}{3}x^{1/3-1} = \frac{1}{3}x^{-2/3}$

28.3 Constant Multiple Rule

If u is a differentiable function of x and c is a constant:

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

Example: $y = 4x^8 \implies \frac{dy}{dx} = 4(8x^7) = 32x^7$.

28.4 Sum Rule

If u and v are differentiable functions:

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

Example: $y = 2x^6 + x^{-9}$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^6) + \frac{d}{dx}(x^{-9}) = 12x^5 - 9x^{-10}$$

29 Derivative of Logarithmic Functions

29.1 Basic Formulas

1. Limit definition of e:

$$\lim_{v\to 0}(1+v)^{\frac{1}{v}} = e$$

2. Natural Logarithm:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

3. General Logarithmic Function ($\log_b x$):

$$\frac{d}{dx}(\log_b x) = \frac{d}{dx}\left[\frac{\ln x}{\ln b}\right] = \frac{1}{\ln b}\cdot\frac{1}{x} = \frac{1}{x\ln b}$$

4. Chain Rule for Natural Logarithm:

$$\frac{d}{dx}[\ln u] = \frac{1}{u}\cdot\frac{du}{dx}$$

5. Chain Rule for General Logarithm:

$$\frac{d}{dx}[\log_b u] = \frac{1}{u\ln b}\cdot\frac{du}{dx}$$

29.2 Examples

1. Find $\frac{d}{dx}[\ln(x^2+1)]$:

$$\begin{aligned}\frac{d}{dx}[\ln(x^2+1)] &= \frac{1}{x^2+1}\cdot\frac{d}{dx}(x^2+1)\\ &= \frac{1}{x^2+1}\cdot(2x)\\ &= \frac{2x}{x^2+1}\end{aligned}$$

2. Find $\frac{d}{dx}\left[\ln\left(\frac{x^2\sin x}{\sqrt{1+x}}\right)\right]$: First, simplify using logarithmic properties:

$$\begin{aligned}y &= \ln\left(\frac{x^2\sin x}{(1+x)^{1/2}}\right)\\ &= \ln(x^2) + \ln(\sin x) - \ln((1+x)^{1/2})\\ &= 2\ln x + \ln(\sin x) - \frac{1}{2}\ln(1+x)\end{aligned}$$

Now, differentiate:

$$\begin{aligned}\frac{dy}{dx} &= 2\left(\frac{1}{x}\right) + \frac{1}{\sin x}(\cos x) - \frac{1}{2(1+x)}(1)\\ &= \frac{2}{x} + \cot x - \frac{1}{2(1+x)}\end{aligned}$$

30 Derivative of $\ln|x|$

To find the derivative of $f(x) = \ln|x|$ where $x \neq 0$:

Case 1: $x > 0$

$$|x| = x \implies \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Case 2: $x < 0$

$$|x| = -x \implies \frac{d}{dx}(\ln(-x)) = \frac{1}{-x}\cdot\frac{d}{dx}(-x) = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$$

Conclusion:

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x}, \quad x \neq 0$$

31 Logarithmic Differentiation Example

Find the derivative of $y = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}$.

Take the natural log of both sides:

$$\begin{aligned}\ln y &= \ln\left(\frac{x^2(7x-14)^{1/3}}{(1+x^2)^4}\right)\\ \ln y &= \ln(x^2) + \ln((7x-14)^{1/3}) - \ln((1+x^2)^4)\\ \ln y &= 2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2)\end{aligned}$$

Differentiate both sides with respect to x:

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{2}{x} + \frac{1}{3(7x-14)}(7) - \frac{4}{1+x^2}(2x)\\ \frac{1}{y}\frac{dy}{dx} &= \frac{2}{x} + \frac{7}{21(x-2)} - \frac{8x}{1+x^2}\\ \frac{1}{y}\frac{dy}{dx} &= \frac{2}{x} + \frac{1}{3(x-2)} - \frac{8x}{1+x^2}\end{aligned}$$

Multiply by y to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \left[\frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\right]\left(\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}\right)$$

32 Derivatives of Exponential Functions

32.1 Formulas

1. Base b: $\frac{d}{dx}[b^x] = b^x \ln b$
2. Base e: $\frac{d}{dx}[e^x] = e^x$
3. Chain Rule (Base b): $\frac{d}{dx}[b^u] = b^u \ln b \cdot \frac{du}{dx}$
4. Chain Rule (Base e): $\frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}$

32.2 Examples

- $\frac{d}{dx}[2^x] = 2^x \ln 2$
- $\frac{d}{dx}[e^{\cos x}] = e^{\cos x} \cdot \frac{d}{dx}(\cos x) = e^{\cos x}(-\sin x) = -\sin x\, e^{\cos x}$

32.3 Variable Base and Exponent

Find $\frac{dy}{dx}$ for $y = (x^2+1)^{\sin x}$.
Take ln of both sides:

$$\ln y = \ln((x^2+1)^{\sin x})$$
$$\ln y = \sin x \cdot \ln(x^2+1)$$

Differentiate implicitly:

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[\sin x \cdot \ln(x^2+1)]$$
$$\frac{1}{y}\frac{dy}{dx} = \sin x \cdot \frac{1}{x^2+1}(2x) + \ln(x^2+1)\cdot(\cos x)$$
$$\frac{dy}{dx} = y\left[\frac{2x\sin x}{x^2+1} + \cos x \ln(x^2+1)\right]$$
$$\frac{dy}{dx} = (x^2+1)^{\sin x}\left[\frac{2x\sin x}{x^2+1} + \cos x \ln(x^2+1)\right]$$

33 Derivatives of Inverse Trigonometric Functions

33.1 Formulas

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$
$$\frac{d}{dx}[\cos^{-1} u] = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$
$$\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2}\frac{du}{dx}$$
$$\frac{d}{dx}[\cot^{-1} u] = -\frac{1}{1+u^2}\frac{du}{dx}$$
$$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$
$$\frac{d}{dx}[\csc^{-1} u] = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

33.2 Example

Find $\frac{dy}{dx}$ for $y = \sec^{-1}(e^x)$.

$$\frac{dy}{dx} = \frac{1}{|e^x|\sqrt{(e^x)^2-1}} \cdot \frac{d}{dx}(e^x)$$
$$= \frac{1}{e^x\sqrt{e^{2x}-1}} \cdot e^x$$
$$= \frac{1}{\sqrt{e^{2x}-1}}$$

34 Lecture 15: Linearization and Differentials

34.1 Linearization

If f is differentiable at $x = a$, then the approximating function:

$$L(x) = f(a) + f'(a)(x-a)$$

is the linearization of f at a. The approximation $f(x) \approx L(x)$ is the standard linear approximation of f at a. The point $x = a$ is the center of the approximation.

34.1.1 Examples

1. Local Linearization of Square Root

(a) Find the local linearization of $f(x) = \sqrt{x}$ at $a = 1$.

(b) Use the linearization to approximate $\sqrt{1.1}$.

Solution:

(a) Find the derivative and value at $a = 1$:

$$f(x) = \sqrt{x} \implies f(1) = 1$$
$$f'(x) = \frac{1}{2\sqrt{x}} \implies f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

Construct the linearization function:

$$L(x) = f(1) + f'(1)(x-1) = 1 + \frac{1}{2}(x-1)$$

(b) Approximate $\sqrt{1.1}$:

$$\sqrt{1.1} \approx L(1.1) = 1 + \frac{1}{2}(1.1-1) = 1 + \frac{1}{2}(0.1) = 1.05$$

(Actual value ≈ 1.0488)

2. Linearization of Sine Find the linearization of $f(x) = \sin x$ at $a = 0$ and approximate $\sin(2°)$.
Solution:

$$f(x) = \sin x \implies f(0) = 0$$
$$f'(x) = \cos x \implies f'(0) = 1$$

Linearization:

$$L(x) = 0 + 1(x-0) \implies L(x) = x$$

To approximate $\sin(2°)$, first convert degrees to radians:

$$2° = 2 \times \frac{\pi}{180} \approx 0.0349 \text{ radians}$$

Approximation:

$$\sin(2°) \approx L(0.0349) = 0.0349$$

34.2 Differentials

Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable. The differential dy is defined as:

$$dy = f'(x)dx$$

34.2.1 Examples

1. Relationship between dy and dx For $y = x^2$, find dy at $x = 1$.

$$dy = 2xdx \implies \text{at } x = 1, \quad dy = 2dx$$

2. Comparing Δy and dy Given $y = \sqrt{x}$:

(a) Find Δy and dy.

(b) Evaluate Δy and dy at $x = 4$ with $\Delta x = dx = 3$.

Solution:

(a) Formulas:

$$\Delta y = f(x + \Delta x) - f(x) = \sqrt{x + \Delta x} - \sqrt{x}$$
$$dy = f'(x)dx = \frac{1}{2\sqrt{x}}dx$$

(b) Evaluate at $x = 4, \Delta x = 3$:

$$\Delta y = \sqrt{4+3} - \sqrt{4} = \sqrt{7} - 2 \approx 2.65 - 2 = 0.65$$
$$dy = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4} = 0.75$$

34.3 Error in Differential Approximation

- True change: $\Delta f = f(a + \Delta x) - f(a)$
- Differential estimate: $df = f'(a)\Delta x$
- Approximation error: $|\Delta f - df|$

As $dx \to 0$, the error $\epsilon \to 0$.

35 Related Rates

The problem of finding a rate of change from other known rates of change is called a related rates problem.

35.1 Examples

1. Spherical Balloon Volume $V = \frac{4}{3}\pi r^3$. Differentiate with respect to time t:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

2. Area of a Circle Area $A = \pi r^2$. equation* relating rates:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

3. Linear Relation Given $y = 5x$ and $\frac{dx}{dt} = 2$. Find $\frac{dy}{dt}$.

$$\frac{dy}{dt} = 5\frac{dx}{dt} = 5(2) = 10$$

4. Multi-variable equation* Given $r + s^2 + v^3 = 12$, $\frac{dr}{dt} = 4$, and $\frac{ds}{dt} = -3$. Find $\frac{dv}{dt}$ when $r = 3$, $s = 1$, and $v = 2$.

Solution: Differentiate the equation* with respect to t:

$$\frac{dr}{dt} + 2s\frac{ds}{dt} + 3v^2\frac{dv}{dt} = 0$$

Substitute known values:

$$4 + 2(1)(-3) + 3(2)^2\frac{dv}{dt} = 0$$
$$4 - 6 + 12\frac{dv}{dt} = 0$$
$$-2 + 12\frac{dv}{dt} = 0$$
$$12\frac{dv}{dt} = 2 \implies \frac{dv}{dt} = \frac{1}{6}$$

5. Volume of a Cylinder Volume $V = \pi r^2 h$. How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ and $\frac{dr}{dt}$?

(a) r **is constant:**

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

(b) h **is constant:**

$$\frac{dV}{dt} = 2\pi rh\frac{dr}{dt}$$

(c) **Neither is constant (Product Rule):**

$$\frac{dV}{dt} = \pi\left(r^2\frac{dh}{dt} + h \cdot 2r\frac{dr}{dt}\right) = \pi r^2\frac{dh}{dt} + 2\pi rh\frac{dr}{dt}$$

36 Related Rates

36.1 Question 6

Given $x = y^3 - y$ and $\frac{dy}{dt} = 5$. What is $\frac{dx}{dt}$ when $y = 2$?
Solution: Differentiating x with respect to t:

$$\frac{dx}{dt} = (3y^2 - 1)\frac{dy}{dt}$$

Substitute $y = 2$ and $\frac{dy}{dt} = 5$:

$$\frac{dx}{dt} = [3(2)^2 - 1](5)$$
$$= (12 - 1)(5)$$
$$= 11(5) = 55$$

Note: The lecture note calculation shows $3(2)^2(5) - 5 = 60 - 5 = 55$.

36.2 Question 11

The edge length x of a cube decreases at the rate of 5 m/min. When $x = 3$m, at what rate does the cube's (a) surface area and (b) volume change?
Given: $\frac{dx}{dt} = -5$ m/min.
(a) Surface Area Change The surface area of a cube is $S = 6x^2$.

$$\frac{dS}{dt} = 12x\frac{dx}{dt}$$
$$\text{At } x = 3: \quad \frac{dS}{dt} = 12(3)(-5)$$
$$= -180 \text{ m}^2/\text{min}$$

(b) Volume Change The volume of a cube is $V = x^3$.

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

$$\text{At } x = 3: \quad \frac{dV}{dt} = 3(3)^2(-5) = 3(9)(-5) = -135 \text{ m}^3/\text{min}$$

36.3 Question 29: Hemispherical Bowl

Water is draining from a hemispherical bowl of radius 13 m at the rate of 6 m^3/min. The volume of water in a spherical bowl is given by $V = \frac{\pi}{3}y^2(3R - y)$, where y is the depth of the water and R is the radius of the bowl. Given:

- $R = 13$ m
- $\frac{dV}{dt} = -6 \text{ m}^3/\text{min}$

(a) Rate of change of water level Find $\frac{dy}{dt}$ when $y = 8$ m.
Differentiate the volume formula with respect to t:

$$V = \frac{\pi}{3}(3Ry^2 - y^3)$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[6Ry\frac{dy}{dt} - 3y^2\frac{dy}{dt}\right]$$

$$\frac{dV}{dt} = \pi(2Ry - y^2)\frac{dy}{dt}$$

Solve for $\frac{dy}{dt}$:

$$\frac{dy}{dt} = \frac{1}{\pi(2Ry - y^2)}\frac{dV}{dt}$$

Substitute $R = 13$, $y = 8$, and $\frac{dV}{dt} = -6$:

$$\frac{dy}{dt} = \frac{-6}{\pi[2(13)(8) - (8)^2]} = \frac{-6}{\pi[208 - 64]} = \frac{-6}{144\pi} = -\frac{1}{24\pi} \text{ m/min}$$

(b) Rate of change of the radius of the water surface Let r be the radius of the water surface. From the geometry of the sphere (Pythagorean theorem on the cross-section):

$$R^2 = r^2 + (R - y)^2$$

Substitute $R = 13$:

$$169 = r^2 + (13 - y)^2$$

$$r^2 = 169 - (169 - 26y + y^2)$$

$$r^2 = 26y - y^2$$

$$r = \sqrt{26y - y^2}$$

(c) Find $\frac{dr}{dt}$ when $y = 8$ Differentiate r with respect to t:

$$\frac{dr}{dt} = \frac{d}{dt}(26y - y^2)^{1/2} = \frac{1}{2}(26y - y^2)^{-1/2}(26 - 2y)\frac{dy}{dt} = \frac{13 - y}{\sqrt{26y - y^2}}\frac{dy}{dt}$$

Substitute $y = 8$ and $\frac{dy}{dt} = -\frac{1}{24\pi}$:

$$\frac{dr}{dt} = \frac{13 - 8}{\sqrt{26(8) - 64}}\left(-\frac{1}{24\pi}\right) = \frac{5}{\sqrt{208 - 64}}\left(-\frac{1}{24\pi}\right) = \frac{5}{\sqrt{144}}\left(-\frac{1}{24\pi}\right) = \frac{5}{12}\left(-\frac{1}{24\pi}\right) = -\frac{5}{288\pi} \text{ m/min}$$

37 Points of Non-differentiability

A function fails to be differentiable at a point for the following reasons:

1. Discontinuity
2. Corner
3. Points of Vertical Tangency
4. Cusp

37.1 Corner

Example: $f(x) = |x|$. Using the definition of the derivative at $x = 0$:

$$f'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h}$$

Since $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$:
Left-hand derivative:

$$f'(0^-) = \lim_{h\to 0^-}\frac{|0+h| - |0|}{h} = \lim_{h\to 0^-}\frac{-h}{h} = -1$$

Right-hand derivative:

$$f'(0^+) = \lim_{h\to 0^+}\frac{h}{h} = 1$$

Since the left and right limits are not equal, $f(x)$ is not differentiable at $x = 0$.

37.2 Points of Vertical Tangency

We say that a continuous curve $y = f(x)$ has a vertical tangent at x if:

$$\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \infty \quad \text{or} \quad -\infty$$

This means the slope of the tangent line becomes undefined (vertical).
Example: Consider $y = x^{1/3}$. Find the derivative at $x = 0$:

$$f'(0) = \lim_{h\to 0}\frac{(0+h)^{1/3} - 0^{1/3}}{h} = \lim_{h\to 0}\frac{h^{1/3}}{h} = \lim_{h\to 0}\frac{1}{h^{2/3}} = \infty$$

37.3 Cusp

A cusp occurs when the derivative approaches ∞ from one side and $-\infty$ from the other.
Example: Consider $y = x^{2/3}$. Find the derivative at $x = 0$:

$$f'(0) = \lim_{h\to 0} \frac{(0+h)^{2/3} - 0^{2/3}}{h}$$
$$= \lim_{h\to 0} \frac{h^{2/3}}{h} = \lim_{h\to 0} \frac{1}{h^{1/3}}$$

Analyzing the limits from both sides:

- Right-hand limit: $\lim_{h\to 0^+} \frac{1}{h^{1/3}} = +\infty$
- Left-hand limit: $\lim_{h\to 0^-} \frac{1}{h^{1/3}} = -\infty$

Thus, there is a cusp at $x = 0$.

38 Extreme Values of Functions

38.1 Definitions

Let f be a function with domain D. Then f has an absolute maximum value on D at a point c if:

$$f(x) \leq f(c) \quad \text{for all } x \text{ in } D$$

And an absolute minimum value at c if:

$$f(x) \geq f(c) \quad \text{for all } x \text{ in } D$$

Minimum and maximum values are called extreme values. Maxima and minima are also referred to as global maxima or minima.

38.2 Examples

1. $y = x^2$ on $(-\infty, \infty)$:

- No absolute maximum.
- Absolute minimum of 0 at $x = 0$.

2. $y = x^2$ on $[0, 2]$:

- Absolute maximum of 4 at $x = 2$.
- Absolute minimum of 0 at $x = 0$.

3. $y = x^2$ on $(0, 2]$:

- Absolute maximum of 4 at $x = 2$.
- No absolute minimum (since $x = 0$ is not included).

4. $y = x^2$ on $(0, 2)$:

- No absolute maximum.
- No absolute minimum.

39 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$ and $f(x_2) = M$ such that:

$$m \leq f(x) \leq M \quad \text{for every other } x \text{ in } [a, b]$$

40 Local Extreme Values

Definition: A function f has a local maximum value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c. Similarly, f has a local minimum value at c if $f(x) \geq f(c)$ for all x in an open interval containing c.

40.1 The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then:

$$f'(c) = 0$$

41 Critical Point

An interior point of the domain of a function f where f' is zero or undefined is a critical point of f.

41.1 Examples

1. Find all critical points of $f(x) = x^3 - 3x + 1$.

$$f'(x) = 3x^2 - 3 = 0$$
$$3(x^2 - 1) = 0$$
$$3(x+1)(x-1) = 0$$
$$x = 1, \quad x = -1$$

2. Find all critical points of $f(x) = 3x^{5/3} - 15x^{2/3}$.

$$f'(x) = 3\left(\frac{5}{3}\right)x^{2/3} - 15\left(\frac{2}{3}\right)x^{-1/3}$$
$$= 5x^{2/3} - 10x^{-1/3}$$
$$= 5x^{-1/3}(x-2) = \frac{5(x-2)}{x^{1/3}}$$

- $f'(x) = 0$ if $x = 2$.
- $f'(x)$ is undefined if $x = 0$.

Thus, $x = 0$ and $x = 2$ are critical points.

42 Rolle's Theorem

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

42.1 Example

$f(x) = x^2 - 5x + 6$ on $[2, 3]$.

- $f(2) = 4 - 10 + 6 = 0$
- $f(3) = 9 - 15 + 6 = 0$
- $f(a) = f(b)$, so Rolle's Theorem applies.

Find c:

$$f'(x) = 2x - 5$$
$$f'(c) = 0 \implies 2c - 5 = 0 \implies c = \frac{5}{2} = 2.5$$

Since $2.5 \in (2, 3)$, the theorem is verified.

43 Mean Value Theorem

Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval (a, b). Then there is at least one point c in (a, b) at which:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

43.1 Example

Find c for $f(x)$ on $[0, 2]$ such that the tangent is parallel to the secant line passing through $(0, 0)$ and $(2, 4)$. (implied $f(x) = x^2$)

$$\text{Slope } m = \frac{4 - 0}{2 - 0} = 2$$
$$f'(x) = 2x \implies f'(c) = 2c$$
$$2c = 2 \implies c = 1$$

44 Monotonic Functions and First Derivative Test

Corollary: Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

- If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
- If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

44.1 Examples

1. Find intervals of increase/decrease for $f(x) = x^2 - 4x + 3$.

$$f'(x) = 2x - 4 = 2(x - 2)$$

- $f'(x) \geq 0$ if $x > 2$ (Increasing on $[2, \infty)$)
- $f'(x) < 0$ if $x < 2$ (Decreasing on $(-\infty, 2]$)

2. Find intervals for $f(x) = x^3$. $f'(x) = 3x^2$. Since $3x^2 \geq 0$ for all x, f is increasing on $(-\infty, \infty)$.
3. Find intervals for $f(x) = 3x^4 + 4x^3 - 12x^2 + 21$.

$$\begin{aligned} f'(x) &= 12x^3 + 12x^2 - 24x \\ &= 12x(x^2 + x - 2) \\ &= 12x(x + 2)(x - 1) \end{aligned}$$

Critical points: $x = 0, x = -2, x = 1$. Testing intervals:

- $x < -2$: $(-)(-)(-) \implies$ Decreasing on $(-\infty, -2)$
- $-2 < x < 0$: $(-)(+)(-) \implies$ Increasing on $(-2, 0)$
- $0 < x < 1$: $(+)(+)(-) \implies$ Decreasing on $(0, 1)$
- $x > 1$: $(+)(+)(+) \implies$ Increasing on $(1, \infty)$

45 Curve Sketching

45.1 Procedure

To sketch the graph of a function $y = f(x)$, follow these steps:

1. Identify the domain of the function.
2. Find the derivatives y' and y''.
3. Find the critical points of f (where $f'(x) = 0$ or is undefined) and identify the behavior at each.
4. Find intervals where the curve is increasing or decreasing.
5. Find points of inflection and determine the concavity ($f'' > 0$ concave up, $f'' < 0$ concave down).
6. Identify asymptotes that may exist (vertical, horizontal, or oblique).
7. Plot key points, such as intercepts and the points found in steps 3-5, and sketch the curve together with any asymptotes.

45.2 Example 1: Polynomial Function

Sketch the graph of $f(x) = x^4 - 4x^3 + 10$.
1. Domain: $(-\infty, \infty)$
2. Derivatives:

$$f'(x) = 4x^3 - 12x^2$$
$$f''(x) = 12x^2 - 24x$$

3. Critical Points: Set $f'(x) = 0$:

$$4x^3 - 12x^2 = 0$$
$$4x^2(x - 3) = 0$$

Critical points are $x = 0$ and $x = 3$.
4. Intervals of Increase/Decrease:

- $(-\infty, 0)$: Test $x = -1 \implies f'(-1) = -4 - 12 = -16 < 0$ (Decreasing)
- $(0, 3)$: Test $x = 1 \implies f'(1) = 4 - 12 = -8 < 0$ (Decreasing)
- $(3, \infty)$: Test $x = 4 \implies f'(4) = 4(64) - 12(16) > 0$ (Increasing)

Conclusion:

- No relative extremum at $x = 0$ (sign does not change).
- Relative minimum at $x = 3$. Value $f(3) = 81 - 108 + 10 = -17$.

5. Concavity and Inflection Points: Set $f''(x) = 0$:

$$12x^2 - 24x = 0$$
$$12x(x - 2) = 0$$

Possible inflection points at $x = 0$ and $x = 2$.

- $(-\infty, 0)$: Test $x = -1 \implies f''(-1) = 12 + 24 > 0$ (Concave Up)

- $(0,2)$: Test $x = 1 \implies f''(1) = 12 - 24 < 0$ (Concave Down)
- $(2,\infty)$: Test $x = 3 \implies f''(3) > 0$ (Concave Up)

Inflection points are at $x = 0$ ($y = 10$) and $x = 2$ ($y = 16 - 32 + 10 = -6$).

45.3 Example 2: General Shape from Derivative

Sketch the general shape knowing $y' = 2 + x - x^2$.
Critical Points:

$$2 + x - x^2 = 0$$
$$(2 - x)(1 + x) = 0 \implies x = 2, x = -1$$

Intervals:

- $(-\infty, -1)$: $y' < 0$ (Decreasing)
- $(-1, 2)$: $y' > 0$ (Increasing)
- $(2, \infty)$: $y' < 0$ (Decreasing)

Local minimum at $x = -1$, Local maximum at $x = 2$.
Concavity: $y'' = 1 - 2x$. Inflection point at $x = 1/2$.

- $x < 1/2$: $y'' > 0$ (Concave Up)
- $x > 1/2$: $y'' < 0$ (Concave Down)

46 Plotting Rational Functions

46.1 Example 3

Sketch the graph of $f(x) = \frac{x^2+4}{2x}$.
1. Domain: $(-\infty, 0) \cup (0, \infty)$
2. Derivatives: Rewrite as $f(x) = \frac{x}{2} + \frac{2}{x} = \frac{1}{2}x + 2x^{-1}$.

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2}$$
$$f''(x) = \frac{4}{x^3}$$

3. Critical Points: $f'(x) = 0 \implies x^2 - 4 = 0 \implies x = \pm 2$.

- At $x = -2$: $f'(-3) > 0$ (Inc), $f'(-1) < 0$ (Dec) $\implies$ Relative Max.
- At $x = 2$: $f'(1) < 0$ (Dec), $f'(3) > 0$ (Inc) $\implies$ Relative Min.

4. Concavity:

- $x < 0$: $f''(x) < 0$ (Concave Down)
- $x > 0$: $f''(x) > 0$ (Concave Up)

5. Asymptotes:

- Vertical Asymptote: $x = 0$.
- Oblique Asymptote: $y = \frac{x}{2}$ (since $f(x) = \frac{x}{2} + \frac{2}{x}$ and $\frac{2}{x} \to 0$ as $x \to \infty$).

46.2 Example 4

Sketch $y = -\frac{x^2-2}{x^2-1} = \frac{2-x^2}{x^2-1}$.
1. Domain: $\mathbb{R} \setminus \{-1, 1\}$
2. Derivatives:

$$y' = \frac{-2x}{(x^2 - 1)^2}$$
$$y'' = \frac{2(3x^2 + 1)}{(x^2 - 1)^3}$$

3. Analysis:

- **Critical Point:** $x = 0$. $y'(0) = 0$.
- **Test:** For $x < 0$, $y' > 0$ (Increasing). For $x > 0$, $y' < 0$ (Decreasing).
- **Extremum:** Local Maximum at $(0, -2)$.

4. Asymptotes:

- Vertical Asymptotes: $x = 1, x = -1$.
- Horizontal Asymptote: $y = -1$ (limit as $x \to \infty$).

47 Second Derivative Test for Local Extrema

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails.

47.1 Example

Find the local extrema of $f(x) = 3x^5 - 5x^3$.
Solution: Find derivatives:

$$f'(x) = 15x^4 - 15x^2$$
$$f''(x) = 60x^3 - 30x$$

Find critical points ($f'(x) = 0$):

$$15x^4 - 15x^2 = 0$$
$$15x^2(x^2 - 1) = 0$$
$$15x^2(x - 1)(x + 1) = 0$$

Critical points are $x = 0, x = 1, x = -1$.
Testing Critical Points:

- At $x = -1$:

  $$f''(-1) = 60(-1)^3 - 30(-1) = -60 + 30 = -30 < 0$$

  Local Maximum.
- At $x = 1$:

  $$f''(1) = 60(1)^3 - 30(1) = 60 - 30 = 30 > 0$$

  Local Minimum.
- At $x = 0$:

  $$f''(0) = 0$$

  Test Inconclusive. (First derivative test would show f decreases on both sides, so no extremum).

48 Indeterminate Forms and L'Hopital's Rule

48.1 Introduction

John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero. The rule is known as L'Hopital's Rule.
Theorem: Suppose $f(a) = 0$ and $g(a) = 0$, and that f and g are differentiable on an open interval I containing a. Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

48.2 Examples (0/0 Form)

1. Evaluate $\lim_{x\to 0} \frac{3x-\sin x}{x}$**.** Substitution gives 0/0. Apply L'Hopital's Rule:

$$\lim_{x\to 0} \frac{3-\cos x}{1} = 3 - 1 = 2$$

2. Evaluate $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$**.** Substitution gives 0/0. Apply Rule:

$$\lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$$

3. Evaluate $\lim_{x\to 0} \frac{x-\sin x}{x^3}$**.** Substitution gives 0/0.

$$\lim_{x\to 0} \frac{1-\cos x}{3x^2} = \left(\frac{0}{0}\right)$$
$$\lim_{x\to 0} \frac{\sin x}{6x} = \left(\frac{0}{0}\right)$$
$$\lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$$

48.3 Indeterminate Forms $(\infty/\infty, \infty \cdot 0, \infty - \infty)$

L'Hopital's Rule also applies if $\lim f(x) = \pm\infty$ and $\lim g(x) = \pm\infty$.
Examples: 1. Evaluate $\lim_{x\to\infty} \frac{x}{e^x}$**.** Form ∞/∞.

$$\lim_{x\to\infty} \frac{1}{e^x} = 0$$

2. Evaluate $\lim_{x\to 0^+} x \ln x$**.** Form $0 \cdot (-\infty)$. Rewrite as $\frac{\ln x}{1/x}$ (Form $-\infty/\infty$):

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

3. Evaluate $\lim_{x\to 0}(\frac{1}{x} - \frac{1}{\sin x})$**.** Form $\infty - \infty$. Combine fractions:

$$\lim_{x\to 0} \frac{\sin x - x}{x \sin x} \quad \left(\frac{0}{0}\right)$$

Apply Rule:

$$\lim_{x\to 0} \frac{\cos x - 1}{x\cos x + \sin x} \quad \left(\frac{0}{0}\right)$$

Apply Rule again:

$$\lim_{x\to 0} \frac{-\sin x}{-x\sin x + \cos x + \cos x} = \frac{0}{0+1+1} = 0$$

49 Applied Optimization

49.1 Example 1: Sum and Product

Find two numbers whose sum is 60 and whose product is a maximum.
Solution: Let x and y be the two numbers.

$$\text{Sum: } S = x + y = 60 \implies y = 60 - x$$
$$\text{Product: } P = xy$$

Substitute y into the product equation*:

$$P(x) = x(60 - x)$$
$$P(x) = 60x - x^2$$

To find the maximum, find the derivative and set it to zero:

$$P'(x) = 60 - 2x = 0$$
$$2x = 60 \implies x = 30$$

Find y:

$$y = 60 - 30 = 30$$

Maximum Product: $P = (30)(30) = 900$.

49.2 Example 2: Difference and Product

Find two numbers whose difference is 40 and whose product is a minimum.
Solution: Let x and y be the numbers.

$$\text{Difference: } d = y - x = 40 \implies y = 40 + x$$
$$\text{Product: } P = xy$$

Substitute y:

$$P(x) = x(40 + x) = 40x + x^2$$

Find critical point:

$$P'(x) = 40 + 2x = 0$$
$$2x = -40 \implies x = -20$$

Find y:

$$y = 40 + (-20) = 20$$

Minimum Product: $P = (-20)(20) = -400$.

49.3 Exercise Q:10 - Rectangular Tank

A 1125 ft^3 open-top rectangular tank with a square base x ft on a side and y ft deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also the cost of excavation.
Given Information (inferred from standard optimization problems of this type):

- Volume $V = x^2y = 1125$.
- Minimize Cost (Material + Excavation).

(Note: The full cost function details are cut off in the provided text, but the setup usually involves minimizing surface area or a weighted cost function subject to the volume constraint).
General Setup: Constraint: $y = \frac{1125}{x^2}$. Surface Area (Open Top): $A = x^2 + 4xy$. Substitute y:

$$A(x) = x^2 + 4x\left(\frac{1125}{x^2}\right) = x^2 + \frac{4500}{x}$$

To minimize, differentiate:

$$A'(x) = 2x - \frac{4500}{x^2} = 0 \implies 2x^3 = 4500 \implies x^3 = 2250$$

(Solution continues based on specific cost parameters if they were distinct from area).

50 Newton's Method

Newton's method is a technique to approximate the roots of a real-valued function $f(x) = 0$.

50.1 Derivation

Given a function $y = f(x)$ and an initial guess x_n. The equation* of the tangent line at $(x_n, f(x_n))$ is:

$$y - f(x_n) = f'(x_n)(x - x_n)$$

To find the next approximation x_{n+1}, we find where this tangent line crosses the x-axis (set $y = 0$ and solve for x):

$$0 - f(x_n) = f'(x_n)(x - x_n)$$
$$-\frac{f(x_n)}{f'(x_n)} = x - x_n$$
$$x = x_n - \frac{f(x_n)}{f'(x_n)}$$

Thus, the iterative formula is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{provided } f'(x_n) \neq 0$$

50.2 Example 1

Find the positive root of the equation* $f(x) = x^2 - 2 = 0$.

$$f(x) = x^2 - 2$$
$$f'(x) = 2x$$

Formula: $x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}$.
Let initial guess $x_0 = 1$.

- **Iteration 1:**

$$x_1 = 1 - \frac{1^2 - 2}{2(1)} = 1 - \frac{-1}{2} = 1.5$$

- **Iteration 2:**

$$x_2 = 1.5 - \frac{(1.5)^2 - 2}{2(1.5)} = 1.5 - \frac{2.25 - 2}{3} = 1.5 - \frac{0.25}{3} \approx 1.4167$$

50.3 Example 2

Find the x-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$. equation* to solve: $x^3 - x = 1 \implies x^3 - x - 1 = 0$.

$$f(x) = x^3 - x - 1$$
$$f'(x) = 3x^2 - 1$$

Formula: $x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$.
Let guess $x_0 = 1.5$.

- **Iteration 1:**

$$x_1 = 1.5 - \frac{(1.5)^3 - 1.5 - 1}{3(1.5)^2 - 1}$$
$$= 1.5 - \frac{3.375 - 2.5}{6.75 - 1} = 1.5 - \frac{0.875}{5.75} \approx 1.3478$$

51 Antiderivatives

51.1 Definition

A function $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$ for all x in the domain. The general antiderivative is denoted by:

$$\int f(x)\,dx = F(x) + C$$

where C is an arbitrary constant.

51.2 Power Rule for Integration

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

51.3 Examples

1. Find the general antiderivative of $f(x) = x^2$.

$$\int x^2\,dx = \frac{x^3}{3} + C$$

2. Find the antiderivative of $f(x) = \frac{1}{x^3} = x^{-3}$.

$$\int x^{-3}\,dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$$

3. Find the general antiderivative of $f(x) = \sin x$.

$$\int \sin x\,dx = -\cos x + C$$

52 Initial Value Problems

A differential equation* $\frac{dy}{dx} = f(x)$ with an initial condition $y(x_0) = y_0$ is called an initial value problem.

52.1 Example

Find the curve $y = f(x)$ whose derivative is $\frac{dy}{dx} = 3x^2 - 1$ and which passes through the point $(1, 4)$.

1. Integrate to find the general solution:

$$y = \int (3x^2 - 1)\,dx = x^3 - x + C$$

2. Apply initial condition $y(1) = 4$:

$$4 = (1)^3 - (1) + C \implies 4 = 0 + C \implies C = 4$$

3. Particular Solution:

$$y = x^3 - x + 4$$

52.2 Physics Application

Derive the equation* of position s for a body moving with constant acceleration a, initial velocity v_0, and initial position s_0.

$$\frac{d^2s}{dt^2} = a$$
$$\frac{ds}{dt} = \int a\,dt = at + C_1$$

At $t = 0$, velocity is v_0, so $C_1 = v_0$. Thus, $v(t) = at + v_0$.

$$s(t) = \int (at + v_0)\,dt = \frac{1}{2}at^2 + v_0t + C_2$$

At $t = 0$, position is s_0, so $C_2 = s_0$.

$$s(t) = \frac{1}{2}at^2 + v_0t + s_0$$

53 Absolute Extrema on Closed Intervals

53.1 Example 1

Find the absolute values of the function $f(x) = 4x^3 - 39x^2 + 90x + 2$ on $[1, 6]$.
Solution: Find the derivative and critical points:

$$f'(x) = 12x^2 - 78x + 90 = 0$$
$$6(2x^2 - 13x + 15) = 0$$
$$2x^2 - 13x + 15 = 0$$
$$2x^2 - 10x - 3x + 15 = 0$$
$$2x(x - 5) - 3(x - 5) = 0$$
$$(2x - 3)(x - 5) = 0$$

Critical points: $x = \frac{3}{2} = 1.5$ and $x = 5$.
Evaluate $f(x)$ at critical points and endpoints $\{1, 1.5, 5, 6\}$:

- $f(1) = 4(1)^3 - 39(1)^2 + 90(1) + 2 = 4 - 39 + 90 + 2 = 57$
- $f(1.5) = 62.75$
- $f(5) = -23$
- $f(6) = 2$ (Wait, checking calculation: $4(216) - 39(36) + 90(6) + 2 = 864 - 1404 + 540 + 2 = 2$)

Conclusion:

- Absolute Maximum: 62.75 at $x = 1.5$
- Absolute Minimum: -23 at $x = 5$

53.2 Example 2

Find the absolute extrema of $g(t) = 8t - t^4$ on $[-2, 1]$.
Solution:

$$g'(t) = 8 - 4t^3$$

Set $g'(t) = 0$:

$$4t^3 = 8$$
$$t^3 = 2 \implies t = \sqrt[3]{2} \approx 1.26$$

The critical point $t = \sqrt[3]{2}$ is not in the interval $[-2, 1]$. Evaluate at endpoints:

- $g(-2) = 8(-2) - (-2)^4 = -16 - 16 = -32$ (Minimum)
- $g(1) = 8(1) - (1)^4 = 7$ (Maximum)

54 Local Extrema

54.1 Example 1

Identify local maximum and minimum values of $f(x) = x^2 - 4x$.
Solution:

$$f'(x) = 2x - 4 = 2(x - 2)$$

Critical point at $x = 2$.

- For $x < 2$, $f'(x) < 0$ (Decreasing)
- For $x > 2$, $f'(x) > 0$ (Increasing)

Local minimum at $x = 2$. Value: $f(2) = (2)^2 - 4(2) = 4 - 8 = -4$.

54.2 Example 2

Find local extrema for $f(x) = 2x^3 + 3x^2 - 12x$.
Solution:

$$f'(x) = 6x^2 + 6x - 12$$
$$= 6(x^2 + x - 2)$$
$$= 6(x + 2)(x - 1)$$

Critical points: $x = -2$ and $x = 1$.
Test intervals:

- At $x = -2$: Local Maximum.
- At $x = 1$: Local Minimum.

Evaluate:

$$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) = -16 + 12 + 24 = 20$$

54.3 Example 3 (Trigonometric)

Find extrema for $f(x) = \sin 2x$ on $[0, \pi]$.
Solution:

$$f'(x) = 2\cos 2x$$

Set $f'(x) = 0$:

$$\cos 2x = 0 \implies 2x = \frac{\pi}{2}, \frac{3\pi}{2} \implies x = \frac{\pi}{4}, \frac{3\pi}{4}$$

- At $x = \pi/4$: Local Maximum.
- At $x = 3\pi/4$: Local Minimum.

55 Concavity and Inflection Points

55.1 Definition

- **Concave Up:** f' is increasing ($f'' > 0$) on an open interval I.
- **Concave Down:** f' is decreasing ($f'' < 0$) on an open interval I.
- **Inflection Point:** A point where the concavity changes.

55.2 Example 4

Analyze concavity for $f(x) = x^3 - 9x^2 + 7x$.
Solution:

$$f'(x) = 3x^2 - 18x + 7$$
$$f''(x) = 6x - 18$$

Set $f''(x) = 0 \implies 6(x-3) = 0 \implies x = 3$.

- $x < 3$: $f'' < 0$ (Concave Down on $(-\infty, 3)$).
- $x > 3$: $f'' > 0$ (Concave Up on $(3, \infty)$).

Inflection point at $x = 3$:

$$f(3) = 3^3 - 9(3)^2 + 7(3) = 27 - 81 + 21 = -33$$

Point: $(3, -33)$.

55.3 Example 5

Analyze concavity for $f(x) = x^4 + 4x^3 + 1$.
Solution:

$$f'(x) = 4x^3 + 12x^2$$
$$f''(x) = 12x^2 + 24x$$

Set $f''(x) = 0$:

$$12x(x+2) = 0 \implies x = 0, x = -2$$

Sign Chart for f'':

- $(-\infty, -2)$: $f'' > 0$ (Concave Up).
- $(-2, 0)$: $f'' < 0$ (Concave Down).
- $(0, \infty)$: $f'' > 0$ (Concave Up).

Inflection points at:

- $x = 0 \implies f(0) = 1$. Point $(0, 1)$.
- $x = -2 \implies f(-2) = 16 - 32 + 1 = -15$. Point $(-2, -15)$.

56 Second Derivative Test for Local Extrema

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

56.1 Example 6

Find local extrema for $f(x) = 2x^3 - 12x^2$.
Solution:

$$f'(x) = 6x^2 - 24x = 6x(x-4)$$
$$f''(x) = 12x - 24$$

Critical points: $x = 0, x = 4$.

- At $x = 0$: $f''(0) = -24 < 0 \implies$ Local Max.
- At $x = 4$: $f''(4) = 48 - 24 = 24 > 0 \implies$ Local Min.

56.2 Example 7

Find local extrema for $f(x) = 4x^3 - 6x^2 - 24x + 1$.
Solution:

$$f'(x) = 12x^2 - 12x - 24 = 12(x^2 - x - 2) = 12(x-2)(x+1)$$
$$f''(x) = 24x - 12$$

Critical points: $x = 2, x = -1$.

- At $x = -1$: $f''(-1) = -24 - 12 = -36 < 0 \implies$ Local Max.
- At $x = 2$: $f''(2) = 48 - 12 = 36 > 0 \implies$ Local Min.

57 5.3 The Definite Integral

57.1 Definitions

The symbol $\int$ is the integral sign. The function $f(x)$ is the integrand, x is the variable of integration, a is the lower limit of integration, and b is the upper limit of integration.

58 Properties of Definite Integrals

58.1 Rules Satisfied by Definite Integrals

1. **Order of Integration:**
$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$
2. **Zero Width Interval:**
$$\int_a^a f(x)\,dx = 0$$
3. **Constant Multiple:**
$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$
4. **Sum and Difference:**
$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

5. **Additivity:** If f is integrable on the three intervals determined by a, b, and c:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

6. **Max-Min Inequality:** If f has a maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then:

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$$

7. **Domination:**
   - If $f(x) \ge g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.
   - If $f(x) \ge 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \ge 0$.

59 Examples

59.1 Example: Additivity

Given $\int_{-1}^1 f(x)\,dx = -2$ and $\int_1^4 f(x)\,dx = 5$. Evaluate $\int_{-1}^4 f(x)\,dx$.

$$\int_{-1}^4 f(x)\,dx = \int_{-1}^1 f(x)\,dx + \int_1^4 f(x)\,dx$$
$$= -2 + 5 = 3$$

59.2 Example: Max-Min Inequality

Show that $\int_0^1 \sqrt{1+\cos x}\,dx \le \sqrt{2}$.
Solution: We know that $\int_a^b f(x)\,dx \le \max f \cdot (b-a)$. The maximum value of $\sqrt{1+\cos x}$ on $[0, 1]$ occurs at $x = 0$ (since cosine decreases on $[0, 1]$).

$$\max f = \sqrt{1+\cos 0} = \sqrt{1+1} = \sqrt{2}$$

Therefore:

$$\int_0^1 \sqrt{1+\cos x}\,dx \le \sqrt{2} \cdot (1-0) = \sqrt{2}$$

59.3 Average Value

The average value of a continuous function on $[a, b]$ is defined as:

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

60 Exercises 5.3

60.1 Question 10

Given: $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, $\int_7^9 h(x)\,dx = 4$.
(a) Find $\int_1^9 -2f(x)\,dx$.

$$-2\int_1^9 f(x)\,dx = -2(-1) = 2$$

(b) Find $\int_7^9 [f(x) + h(x)]\,dx$.

$$\int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx = 5 + 4 = 9$$

(c) Find $\int_1^7 f(x)\,dx$.

$$\int_1^9 f(x)\,dx = \int_1^7 f(x)\,dx + \int_7^9 f(x)\,dx$$

$$-1 = \int_1^7 f(x)\,dx + 5 \implies \int_1^7 f(x)\,dx = -6$$

60.2 Question 41

Evaluate $\int_3^1 7\,dx$.

$$\int_3^1 7\,dx = [7x]_3^1 = 7(1) - 7(3) = 7 - 21 = -14$$

60.3 Question 55

Find the average value of $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$.

$$\begin{aligned} av(f) &= \frac{1}{\sqrt{3}-0}\int_0^{\sqrt{3}} (x^2-1)\,dx \\ &= \frac{1}{\sqrt{3}}\left[\frac{x^3}{3} - x\right]_0^{\sqrt{3}} \\ &= \frac{1}{\sqrt{3}}\left[\left(\frac{(\sqrt{3})^3}{3} - \sqrt{3}\right) - 0\right] \\ &= \frac{1}{\sqrt{3}}\left[\frac{3\sqrt{3}}{3} - \sqrt{3}\right] \\ &= \frac{1}{\sqrt{3}}[\sqrt{3} - \sqrt{3}] = 0 \end{aligned}$$

60.4 Question 72

Maximize: For what values of a and b is the integral $\int_a^b (x - x^2)\,dx$ maximized? To maximize the integral, we integrate over the interval where the integrand is non-negative.

$$x - x^2 \ge 0 \implies x(1-x) \ge 0$$

This holds for $0 \le x \le 1$. Thus, $a = 0$ and $b = 1$.
Minimize: For what values of a and b is the integral $\int_a^b (x^4 - 2x^2)\,dx$ minimized? We integrate over the interval where the integrand is negative.

$$x^4 - 2x^2 \le 0$$
$$x^2(x^2-2) \le 0$$

Since $x^2 \ge 0$, we need $x^2 - 2 \le 0 \implies x^2 \le 2 \implies -\sqrt{2} \le x \le \sqrt{2}$. Thus, $a = -\sqrt{2}$ and $b = \sqrt{2}$.

61 5.4 The Fundamental Theorem of Calculus

Theorem (Mean Value Theorem for Definite Integrals): If f is continuous on $[a, b]$, then at some point c in $[a, b]$:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Theorem (Part 1): If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b).

62 5.5 Indefinite Integrals and Substitution Method

62.1 Indefinite Integrals

The indefinite integral is defined as:

$$\int f(x)\,dx = F(x) + C$$

62.2 Properties of Indefinite Integrals

Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$ respectively, and that k is a constant. Then:

1. **Constant Multiple Rule:** A constant factor can be moved through an integral sign.

$$\int kf(x)\,dx = k\int f(x)\,dx = kF(x) + C$$

2. **Sum Rule:** An antiderivative of a sum is the sum of the antiderivatives.

$$\begin{aligned}\int [f(x)+g(x)]\,dx &= \int f(x)\,dx + \int g(x)\,dx \\ &= F(x) + G(x) + C\end{aligned}$$

3. **Difference Rule:**

$$\begin{aligned}\int [f(x)-g(x)]\,dx &= \int f(x)\,dx - \int g(x)\,dx \\ &= F(x) - G(x) + C\end{aligned}$$

62.3 Examples

1. Evaluate $\int 4\cos x\,dx$:

$$4\int \cos x\,dx = 4\sin x + C$$

2. Evaluate $\int (x+x^2)\,dx$:

$$\int x\,dx + \int x^2\,dx = \frac{x^2}{2} + \frac{x^3}{3} + C$$

3. Evaluate $\int \frac{\cos x}{\sin^2 x}\,dx$: Rewrite using trigonometric identities:

$$\begin{aligned}\int \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}\,dx &= \int \csc x\cot x\,dx \\ &= -\csc x + C\end{aligned}$$

62.4 Substitution Method

If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

62.4.1 Examples of Substitution

1. Evaluate $\int (x^2+1)^2\cdot 2x\,dx$: Let $u = x^2+1$, then $du = 2x\,dx$.

$$\int u^2\,du = \frac{u^3}{3} + C = \frac{(x^2+1)^3}{3} + C$$

2. Evaluate $\int \cos(7x+5)\,dx$: Let $u = 7x+5$, then $du = 7\,dx \implies dx = \frac{1}{7}\,du$.

$$\int \cos u\cdot\frac{1}{7}\,du = \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7x+5) + C$$

3. Evaluate $\int x^2\sin(x^3)\,dx$: Let $u = x^3$, then $du = 3x^2\,dx \implies x^2\,dx = \frac{1}{3}\,du$.

$$\int \sin u\cdot\frac{1}{3}\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos(x^3) + C$$

4. Evaluate $\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz$: Let $u = z^2+1$, then $du = 2z\,dz$.

$$\begin{aligned}\int \frac{1}{u^{1/3}}\,du &= \int u^{-1/3}\,du \\ &= \frac{u^{2/3}}{2/3} + C = \frac{3}{2}(z^2+1)^{2/3} + C\end{aligned}$$

63 5.6 Definite Integral Substitutions

63.1 Formula

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

63.2 Examples

1. Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$: Let $u = x^3+1$.

- $du = 3x^2\,dx$
- Lower limit: $x = -1 \implies u = (-1)^3 + 1 = 0$
- Upper limit: $x = 1 \implies u = (1)^3 + 1 = 2$

$$\begin{aligned}\int_0^2 \sqrt{u}\,du &= \int_0^2 u^{1/2}\,du \\ &= \left[\frac{2}{3}u^{3/2}\right]_0^2 \\ &= \frac{2}{3}(2^{3/2} - 0) = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}\end{aligned}$$

2. Evaluate $\int_0^2 x(x^2+1)^3\,dx$: Let $u = x^2+1 \implies du = 2x\,dx \implies x\,dx = \frac{1}{2}\,du$.

- $x = 0 \implies u = 1$
- $x = 2 \implies u = 5$

$$\begin{aligned}\int_1^5 u^3\cdot\frac{1}{2}\,du &= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^5 \\ &= \frac{1}{8}(5^4 - 1^4) = \frac{1}{8}(625-1) = \frac{624}{8} = 78\end{aligned}$$

64 Improper Integrals: (1st Kind)

Integrals with infinite limits of integration are improper integrals of Type I.

1. If $f(x)$ is continuous on $[a, \infty)$, then:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then:

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then:

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

In each case, if the limit is finite, we say that the improper integral **converges** and the limit is the value of the improper integral. If the limit fails to exist, the improper integral **diverges**.

65 Examples

65.1 Example 1

Is the area under the curve $y = \frac{\ln x}{x^2}$ from $x = 1$ to $x = \infty$ finite? If so, what is the value?
Solution:

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx$$

Using integration by parts: Let $u = \ln x \implies du = \frac{1}{x}dx$. Let $dv = x^{-2}dx \implies v = -\frac{1}{x}$.

$$\begin{aligned}\int \frac{\ln x}{x^2}\,dx &= -\frac{\ln x}{x} - \int -\frac{1}{x}\cdot\frac{1}{x}\,dx \\ &= -\frac{\ln x}{x} + \int x^{-2}\,dx \\ &= -\frac{\ln x}{x} - \frac{1}{x}\end{aligned}$$

Evaluating limits:

$$\begin{aligned}\lim_{b\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^b &= \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b}\right) - \left(-\frac{\ln 1}{1} - \frac{1}{1}\right) \\ &= (0-0) - (0-1) = 1\end{aligned}$$

(Note: $\lim_{b\to\infty}\frac{\ln b}{b} = 0$ by L'Hopital's Rule). Thus, the integral converges to 1.

65.2 Example 2

Evaluate $\int_{-\infty}^\infty \frac{dx}{1+x^2}$.
Solution:

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2}$$

Evaluating the first part:

$$\begin{aligned}\int_{-\infty}^0 \frac{dx}{1+x^2} &= \lim_{a\to-\infty}[\tan^{-1} x]_a^0 \\ &= \tan^{-1}(0) - \lim_{a\to-\infty}\tan^{-1}(a) \\ &= 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}\end{aligned}$$

Evaluating the second part:

$$\begin{aligned}\int_0^\infty \frac{dx}{1+x^2} &= \lim_{b\to\infty}[\tan^{-1} x]_0^b \\ &= \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$

Total Integral $= \frac{\pi}{2} + \frac{\pi}{2} = \pi$.

66 Improper Integrals: (2nd Kind)

Integrals of functions with vertical asymptotes (infinite discontinuity) within the limits of integration.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a:
$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$$
2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b:
$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$$
3. If f is discontinuous at c where $a < c < b$:
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

66.1 Examples

1. Evaluate $\int_0^3 \frac{dx}{(x-1)^{2/3}}$: Vertical asymptote at $x = 1$.

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

Evaluate first part:

$$\begin{aligned}\lim_{b\to 1^-}\int_0^b (x-1)^{-2/3}\,dx &= \lim_{b\to1^-}[3(x-1)^{1/3}]_0^b \\ &= 3(0) - 3(-1)^{1/3} = 3\end{aligned}$$

Evaluate second part:

$$\begin{aligned}\lim_{c\to 1^+}\int_c^3 (x-1)^{-2/3}\,dx &= \lim_{c\to1^+}[3(x-1)^{1/3}]_c^3 \\ &= 3(2)^{1/3} - 3(0) = 3\sqrt[3]{2}\end{aligned}$$

Total $= 3 + 3\sqrt[3]{2}$.
2. Evaluate $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$: Discontinuity at $x = 1$.

$$\begin{aligned}\lim_{b\to1^-}\int_0^b \frac{1}{\sqrt{1-x^2}}\,dx &= \lim_{b\to1^-}[\sin^{-1} x]_0^b \\ &= \sin^{-1}(1) - \sin^{-1}(0) \\ &= \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$

67 Convergence Tests

67.1 Limit Comparison Test (LCT)

If $f(x)$ and $g(x)$ are positive continuous functions on $[a, \infty)$, and if:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge or both diverge.

67.2 Example

Show that $\int_1^\infty \frac{dx}{1+x^2}$ converges by comparison with $\int_1^\infty \frac{1}{x^2}\,dx$.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{1+x^2} = 1$$

Since $L = 1$ (finite and positive) and $\int_1^\infty \frac{1}{x^2}\,dx$ converges ($p = 2 > 1$), the integral $\int_1^\infty \frac{dx}{1+x^2}$ also converges.

67.3 Example

Investigate convergence of $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ using $g(x) = \frac{1}{x}$.

$$\lim_{x\to\infty} \frac{1-e^{-x}}{x} \cdot \frac{x}{1} = \lim_{x\to\infty} (1 - e^{-x}) = 1 - 0 = 1$$

Since $\int_1^\infty \frac{1}{x}\,dx$ diverges (harmonic), the given integral diverges.

68 Integral Formulas

1. $\int k\,dx = kx + C$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C, \quad a > 0, a \neq 1$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = \ln|\csc x - \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$
22. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$

69 Integration by Parts

Formula:

$$\int u\,dv = uv - \int v\,du$$

69.1 Examples

1. Evaluate $\int x\cos x\,dx$: Let $u = x \implies du = dx$. Let $dv = \cos x\,dx \implies v = \sin x$.

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x - (-\cos x) + C = x\sin x + \cos x + C$$

2. Evaluate $\int \ln x\,dx$: Let $u = \ln x \implies du = \frac{1}{x}\,dx$. Let $dv = dx \implies v = x$.

$$\int \ln x\,dx = x\ln x - \int x\left(\frac{1}{x}\right)dx = x\ln x - \int dx = x\ln x - x + C$$

3. Evaluate $\int x^2 e^x\,dx$: Let $u = x^2 \implies du = 2x\,dx$. Let $dv = e^x\,dx \implies v = e^x$.

$$\int x^2 e^x\,dx = x^2 e^x - \int 2xe^x\,dx = x^2e^x - 2\int xe^x\,dx$$

Apply integration by parts again for $\int xe^x\,dx$: Let $u = x \implies du = dx$. Let $dv = e^x\,dx \implies v = e^x$.

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x$$

Substitute back:

$$\int x^2e^x\,dx = x^2e^x - 2(xe^x - e^x) + C = x^2e^x - 2xe^x + 2e^x + C$$

4. Evaluate $\int e^x\cos x\,dx$: Let $I = \int e^x \cos x\,dx$. Let $u = \cos x \implies du = -\sin x\,dx$. Let $dv = e^x\,dx \implies v = e^x$.

$$I = e^x\cos x - \int e^x(-\sin x)\,dx = e^x\cos x + \int e^x \sin x\,dx$$

Apply parts again for $\int e^x \sin x\,dx$: Let $u = \sin x \implies du = \cos x\,dx$. Let $dv = e^x\,dx \implies v = e^x$.

$$\int e^x\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx = e^x\sin x - I$$

Substitute back into original equation*:

$$I = e^x\cos x + (e^x\sin x - I)$$
$$2I = e^x(\cos x + \sin x)$$
$$I = \frac{e^x}{2}(\cos x + \sin x) + C$$

69.2 Tabular Method

Used for integrals like $\int x^n e^{ax}\,dx$ or $\int x^n \sin(ax)\,dx$.
Example: $\int x^2e^x\,dx$

Derivatives of u	Integrals of dv
x^2	e^x
$2x$	e^x
2	e^x
0	e^x

Multiply diagonals with alternating signs (+, -, +, ...):

$$\int x^2e^x\,dx = +(x^2)(e^x) - (2x)(e^x) + (2)(e^x) + C = e^x(x^2 - 2x + 2) + C$$

70 Integrating Products of Sines and Cosines

70.1 Procedure

To evaluate integrals of the form $\int \sin^m x \cos^n x \, dx$:

1. **If n is odd:**
   - Split a factor of $\cos x$.
   - Use the identity $\cos^2 x = 1 - \sin^2 x$.
   - Substitute $u = \sin x$.
2. **If m is odd:**
   - Split a factor of $\sin x$.
   - Use the identity $\sin^2 x = 1 - \cos^2 x$.
   - Substitute $u = \cos x$.
3. **If m and n are both even:**
   - Use the relevant identities to reduce the powers:

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

70.2 Examples

1. Evaluate $\int \sin^4 x \cos^5 x \, dx$: Since $n = 5$ is odd:

$$\begin{aligned}\int \sin^4 x \cos^5 x \, dx &= \int \sin^4 x (\cos^2 x)^2 \cos x \, dx \\ &= \int \sin^4 x (1 - \sin^2 x)^2 \cos x \, dx\end{aligned}$$

Let $u = \sin x$, $du = \cos x \, dx$:

$$\begin{aligned}&= \int u^4 (1 - u^2)^2 \, du \\ &= \int u^4 (1 - 2u^2 + u^4) \, du \\ &= \int (u^4 - 2u^6 + u^8) \, du \\ &= \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C \\ &= \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C\end{aligned}$$

2. Evaluate $\int \sin^3 x \cos^2 x \, dx$: Since $m = 3$ is odd:

$$\begin{aligned}\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\ &= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx\end{aligned}$$

Let $u = \cos x$, $du = -\sin x \, dx$:

$$\begin{aligned}&= \int (1 - u^2) u^2 (-du) \\ &= \int (u^4 - u^2) \, du \\ &= \frac{u^5}{5} - \frac{u^3}{3} + C \\ &= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C\end{aligned}$$

3. Evaluate $\int \sin^2 x \cos^4 x \, dx$: Since both powers are even:

$$\begin{aligned}&\int \sin^2 x \cos^4 x \, dx \\ &= \int \sin^2 x (\cos^2 x)^2 \, dx \\ &= \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx \\ &= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) \, dx\end{aligned}$$

(Note: Expansion and further integration would follow using standard methods for even powers).

71 Integrating Products of Tangents and Secants

71.1 Procedure

To evaluate integrals of the form $\int \tan^m x \sec^n x \, dx$:

1. **If n is even:**
   - Split a factor of $\sec^2 x$.
   - Use identity $\sec^2 x = \tan^2 x + 1$.
   - Substitute $u = \tan x$.
2. **If m is odd:**
   - Split a factor of $\sec x \tan x$.
   - Use identity $\tan^2 x = \sec^2 x - 1$.
   - Substitute $u = \sec x$.
3. **If n is odd and m is even:**
   - Use integration by parts or reduction formulas.

71.2 Examples

1. Evaluate $\int \tan^2 x \sec^4 x \, dx$: Since $n = 4$ is even:

$$\int \tan^2 x \sec^2 x \sec^2 x \, dx = \int \tan^2 x (1 + \tan^2 x) \sec^2 x \, dx$$

Let $u = \tan x$, $du = \sec^2 x \, dx$:

$$\begin{aligned}&= \int u^2 (1 + u^2) \, du \\ &= \int (u^2 + u^4) \, du \\ &= \frac{u^3}{3} + \frac{u^5}{5} + C \\ &= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C\end{aligned}$$

2. Evaluate $\int \tan^2 x \sec x \, dx$:

$$\int (\sec^2 x - 1) \sec x \, dx = \int (\sec^3 x - \sec x) \, dx$$

Using reduction formula for $\int \sec^3 x \, dx$:

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x|$$

Therefore:

$$\int \sec^3 x\,dx - \int \sec x\,dx$$
$$= \left(\frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x|\right)$$
$$- \ln|\sec x + \tan x|$$
$$= \frac{1}{2}\sec x\tan x - \frac{1}{2}\ln|\sec x + \tan x| + C$$

72 Trigonometric Substitution

72.1 Substitution Method

Expression	Substitution	Simplification
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

72.2 Examples

1. Evaluate $\int \frac{dx}{x^2\sqrt{4-x^2}}$: Let $x = 2\sin\theta \implies dx = 2\cos\theta\,d\theta$.

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\,d\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}$$
$$= \int \frac{2\cos\theta\,d\theta}{4\sin^2\theta\cdot 2\cos\theta}$$
$$= \frac{1}{4}\int \frac{1}{\sin^2\theta}\,d\theta$$
$$= \frac{1}{4}\int \csc^2\theta\,d\theta$$
$$= -\frac{1}{4}\cot\theta + C$$

Substitute back: $\sin\theta = \frac{x}{2}$. From the right triangle, adjacent side is $\sqrt{4-x^2}$. So $\cot\theta = \frac{\sqrt{4-x^2}}{x}$.

$$= -\frac{\sqrt{4-x^2}}{4x} + C$$

2. Find the arc length of the curve $y = \frac{x^2}{2}$ from $x = 0$ to $x = 1$: Formula: $L = \int_a^b \sqrt{1+(y')^2}\,dx$.

$$y' = x \implies (y')^2 = x^2$$
$$L = \int_0^1 \sqrt{1+x^2}\,dx$$

Let $x = \tan\theta \implies dx = \sec^2\theta\,d\theta$. Limits:

- $x = 0 \implies \theta = 0$
- $x = 1 \implies \theta = \frac{\pi}{4}$

$$\int_0^{\pi/4} \sqrt{1+\tan^2\theta}\sec^2\theta\,d\theta = \int_0^{\pi/4} \sec\theta\cdot\sec^2\theta\,d\theta$$
$$= \int_0^{\pi/4} \sec^3\theta\,d\theta$$

Using reduction formula or integration by parts:

$$= \left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^{\pi/4}$$
$$= \frac{1}{2}[\sqrt{2}(1) + \ln(\sqrt{2}+1)] - \frac{1}{2}[1(0) + \ln(1)]$$
$$= \frac{1}{2}[\sqrt{2} + \ln(\sqrt{2}+1)] \approx 1.148$$

3. Evaluate $\int \frac{dx}{x\sqrt{x^2-25}}$ for $x \geq 5$: Let $x = 5\sec\theta \implies dx = 5\sec\theta\tan\theta\,d\theta$.

$$\int \frac{5\sec\theta\tan\theta\,d\theta}{5\sec\theta\sqrt{25\sec^2\theta - 25}} = \int \frac{5\sec\theta\tan\theta\,d\theta}{5\sec\theta\cdot 5\tan\theta}$$
$$= \frac{1}{5}\int d\theta$$
$$= \frac{1}{5}\theta + C$$

Since $x = 5\sec\theta \implies \sec\theta = \frac{x}{5} \implies \theta = \sec^{-1}\left(\frac{x}{5}\right)$.

$$= \frac{1}{5}\sec^{-1}\left(\frac{x}{5}\right) + C$$