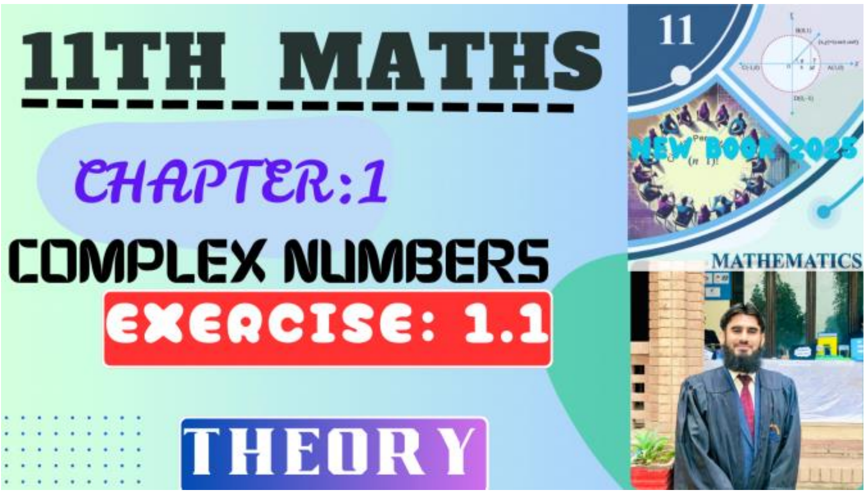
N11EX 1.1

Monday, May 12, 2025 12:49 PM



Complex numbers

The number of the form $z = a + bi$ where $a, b \in R$ and $i = \sqrt{-1}$ are called complex numbers.

$(\pm 2)^2 = 4$

$\sqrt{4} = 2$

$\sqrt{-4} = 2\sqrt{-1}$ $\boxed{= 2i}$

Set of complex numbers is denoted by C.

Mathematically

$C = \{z \mid z = a + bi \ \wedge a, b \in R \ \& \ i = \sqrt{-1}\}$

such that

$z = a + bi$

Real part

imaginary Part

- a is called **real part** and b is called **imaginary part** of complex number.
- Real part is denoted by $\boldsymbol{Rez}$ and imaginary part by $\boldsymbol{Im\ z}$ of complex numbers
- ★ Every real number is a complex number with 0 as its imaginary part

$z = a + 0i = a + 0 = a$

$z = a + i \notin \mathbb{R}$

Conjugate of complex numbers:

Let $z = a + bi$ be a complex number, then $a - bi$ is called the complex conjugate of $a + bi$

$a + bi$.

It is denoted by $\bar{z}$.

For example $4 - 5i$ is a complex conjugate of $5 + 4i$.

$z = 2 + 3i \qquad \bar{z} = 2 - 3i$

❖ A real number is self-conjugate.

$z = a + 0i = a + 0 = a$

$\bar{z} = a$

Operations on complex numbers

$z_1 = a + bi$

$z_2 = c + di$

The symbols $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{k}$ represent real numbers.

(i) Addition $(a + bi) + (c + di) = (a + c) + i(b + d)$

(ii) $k(a + bi) = ka + ikb$

(iii) Subtraction $(a + bi) - (c + di)$
$= a + bi - c - di$
$= (a - c) + i(b - d)$

(iv) Multiplication $(a + bi)(c + di)$
$= ac + adi + bci + bdi^2$
$= ac + adi + bci - bd$
$= (ac - bd) + (ad + bc)i$

$\because i^2 = -1$

Complex number as ordered pairs of real numbers

The complex number $a + bi$ can be written as an ordered pair (a, b)

real part   img part

In ordered pair form some properties are as follows:

i. Equality: $(a, b) = (c, d) \Leftrightarrow a = c \wedge b = d$

$z_1 = a + bi = (a, b)$

$z_2 = c + di = (c, d)$

ii. Addition: $(a, b) + (c, d) = (a + c, b + d)$

iii. Multiplication: $(a, b)(c, d) = (ac - bd, ad + bc)$

iv. Subtraction: $(a, b) - (c, d) = (a - c, b - d)$

v. Product of real number with complex number:
If k is any real number, then $k(a, b) = (ka, kb)$

Properties of the fundamental operations on complex numbers:

- **Additive identity** in C is $(0, 0)$ $= I$

$z = a + bi = (a, b)$

- Every complex number has **additive inverse** $(-a, -b)$
- **Multiplicative identity** is $(1, 0)$ $= 1 + 0i = 1$

$Z + I = (a, b) + (0, 0)$
$= (a, b) = Z$ ✓

- Every non-zero complex number has **multiplicative inverse**. The multiplicative inverse of (a, b) *is* $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

$Z = a + bi$

$Z^{-1} = \frac{1}{Z} = \frac{1}{a+bi}$

$= \frac{1}{a+bi} \times \frac{a-bi}{a-bi}$

$= \frac{a-bi}{a^2-(bi)^2}$

$Z^{-1} = \frac{a-bi}{a^2+b^2}$

- The set C of complex numbers **does not satisfy the order axioms**. In fact there is no sense in saying that one complex number is greater or less than the other.

Argand diagram:

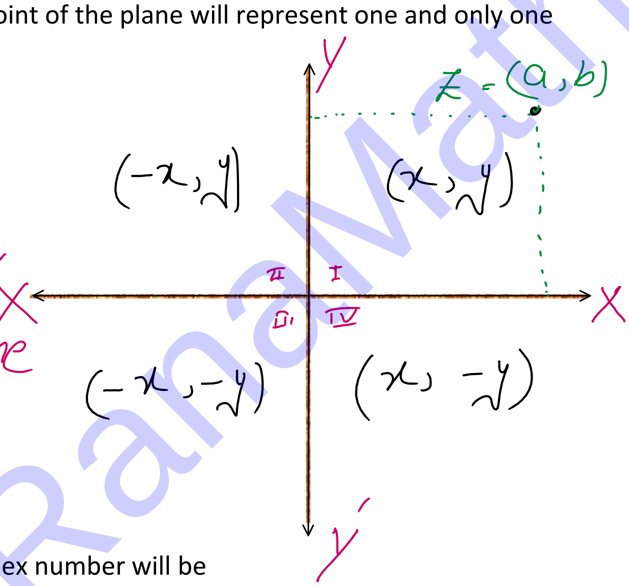
The figure representing one or more complex numbers on the complex plane is called **an argand diagram.**

Every complex number will be represented by one and only one point of the coordinate plane and every point of the plane will represent one and only one complex number.

$\mathbb{R}$ $\mathbb{C}$

Z-plane

complex plane



★ The components of the complex number will be the coordinates of the point representing it.

★ In this representation the $x - axis$ is called the real axis and $y - axis$ is called the imaginary axis.

★ The coordinate plane itself is called the complex plane or $z - plane$.

★ The argand diagram is a way of representing one or more complex numbers on the complex plane.

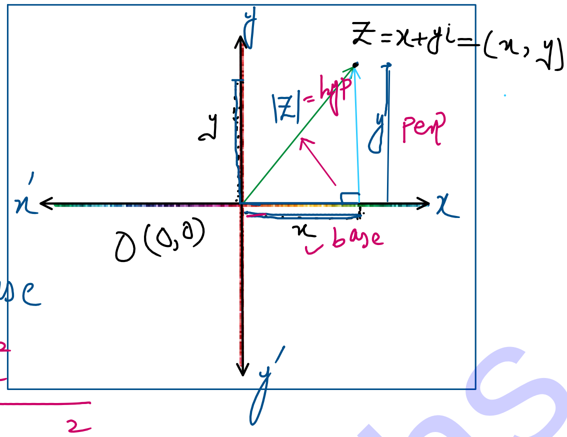
★ Points on the $x - axis$ represent real numbers whereas the points on the $y - axis$

represent imaginary numbers.

★ In an Argand diagram, the complex number $x + yi$ is uniquely represented by the order pair (x, y)

Modulus of complex number:

The real number $\sqrt{x^2 + y^2}$ is called the modulus of the complex number $\mathrm{x} + yi$ and it is denoted by $|x + yi|$



$$hyp^2 = perp^2 + base^2$$

$$|Z|^2 = y^2 + x^2$$

$$|Z| = \sqrt{y^2 + x^2}$$

$$\boxed{|Z| = \sqrt{x^2 + y^2}}$$

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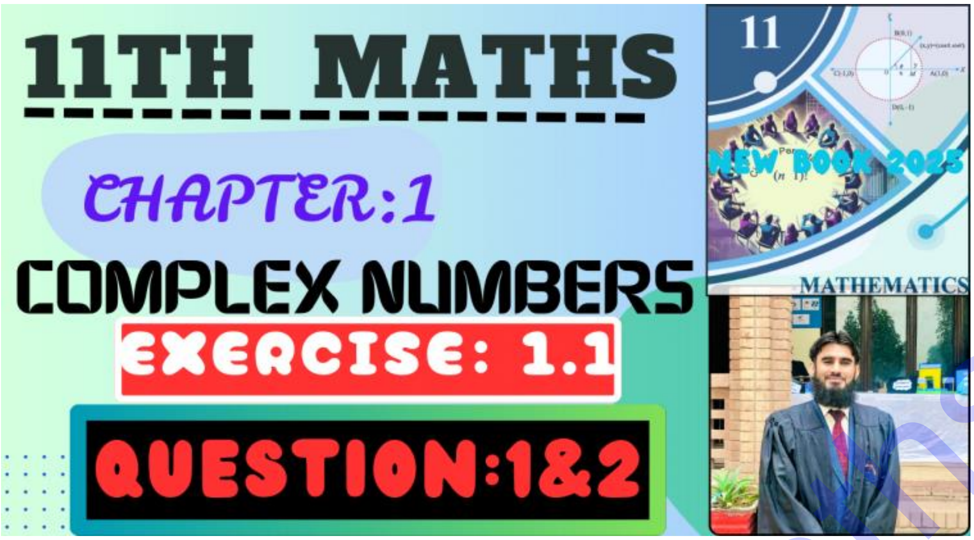


EXERCISE 1.1

1. Simplify

(i) i^9

$= i^{8+1}$

$\because a^{m+n} = a^m \cdot a^n$

$= i^8 \cdot i^1 = (i^2)^4 \cdot i$

$\because i^2 = -1$

$= (-1)^4 \cdot i = (1) \cdot i = i$ Ans

(ii) i^{17}

$= i^{16+1}$

$\because a^{m+n} = a^m \cdot a^n$

$= i^{16} \cdot i = (i^2)^8 \cdot i$

$\because i^2 = -1$

$= (-1)^8 \cdot i$

$= (1) \cdot i = i$ Ans

(iii) $(-1)^{19}$

$= (-i)^{19}$

(iii) $(-\sqrt{-1})^{19}$

$= (-i)^{19}$

$= -i^{19}$

$= -i^{18+1}$

$= -(i^2)^9 \cdot i^1$

$\because i^2 = -1$

$= -(-1)^9 \cdot i$

$= -(-1) \cdot i$

$= 1 \cdot i = i$ Ans

(iv) $(-1)^{\frac{-21}{2}}$

$\because i^2 = -1$

$= (i^{\cancel{2}})^{\frac{-21}{\cancel{2}}}$

$= i^{-21} = \frac{1}{i^{21}}$

$= \frac{1}{i^{20+1}} = \frac{1}{i^{20} \cdot i^1}$

$= \frac{1}{(i^2)^{10} \cdot i} = \frac{1}{(-1)^{10} \cdot i}$

$= \frac{1}{1 \cdot i} = \frac{1}{i}$

$= \frac{1}{i} \times \frac{i}{i}$

$= \frac{i}{i^2} = \frac{i}{-1}$

$= -i$ Ans

2. Prove that $\bar{z} = z$ iff z is real.

Let $z = a + bi \in \mathbb{C}$ then

$\bar{z} = a - bi$

Suppose that

$\bar{z} = z$

$a - bi = a + bi$

$\cancel{a} + bi - \cancel{a} + bi = 0$

$2bi = 0$

$2i \neq 0$ then $b = 0$

$z = a + bi = a + 0i = a$

$\Rightarrow z$ is real number.

Conversely

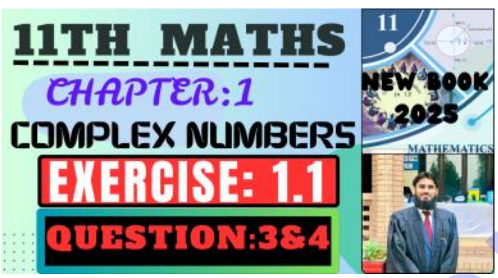
Suppose that z is real

$z = a$

$\bar{z} = a$

$\Rightarrow \bar{z} = z$

....



EXERCISE 1.1

3. For $z \in C$, show that:

i. $\dfrac{z+\bar{z}}{2} = Re(z)$

Let $z = a+bi \in C$ then

$\bar{z} = a-bi$

$$L.H.S = \frac{z+\bar{z}}{2}$$

$$= \frac{a+\cancel{bi}+a-\cancel{bi}}{2}$$

$$= \frac{2a}{2}$$

$$= a$$

$$= Re(z) = R.H.S$$

ii. $\frac{z-\bar{z}}{2i} = Im(z)$

Let $z = a+bi \in C$ then

$\bar{z} = a-bi$

$L.H.S = \frac{z-\bar{z}}{2i}$

$= \frac{a+bi-(a-bi)}{2i}$

$= \frac{\cancel{a}+bi-\cancel{a}+bi}{2i}$

$= \frac{\cancel{2}b\cancel{i}}{\cancel{2}\cancel{i}}$

$= b = Im(z)$

iii. $|z|^2 = z.\bar{z}$

Let $z = a+bi \in C$ then

$\bar{z} = a-bi$

$R.H.S = z\cdot\bar{z}$

$= (a+bi)(a-bi)$

$\because a^2-b^2 = (a+b)(a-b)$

$= a^2-(bi)^2$

$= a^2-b^2i^2$ $\quad \because i^2=-1$

$= a^2+b^2$ —— ①

$z = a+bi$

$= a^2 + b^2$ —— (i)

$z = a + bi$

$|z| = \sqrt{a^2 + b^2}$

sq. b.s

$|z|^2 = a^2 + b^2$ —— (ii)

from (i) & (ii)

$|z|^2 = z \cdot \bar{z}$

iv. $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

Let $z = a + bi \in C$ then

$\bar{z} = a - bi$

$|z| = \sqrt{a^2 + b^2}$

sq. b.s

$|z|^2 = a^2 + b^2$

R.H.S $= \frac{\bar{z}}{|z|^2} = \frac{a - bi}{a^2 + b^2}$

$= \frac{a}{a^2 + b^2} - \frac{bi}{a^2 + b^2}$ —— (i)

L.H.S $= \frac{1}{z} = \frac{1}{a + bi}$

$= \frac{1}{a + bi} \times \frac{a - bi}{a - bi}$

$= \frac{a - bi}{a^2 - (bi)^2}$

$= \frac{a - bi}{a^2 - b^2 i^2}$ $\because i^2 = -1$

$= \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{bi}{a^2 + b^2}$ —— (ii)

L.H.S = R.H.S

$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ Ans

4. Find the multiplicative inverse of each of the following numbers:

i. $(-4, 7)$

Let $z = (-4, 7)$

$z = -4 + 7i$

multiplicative inverse $= z^{-1} = \frac{1}{z}$

$= \frac{1}{-4+7i}$

$= \frac{1}{-4+7i} \times \frac{-4-7i}{-4-7i}$

$= \frac{-4-7i}{(-4)^2 - (7i)^2}$ $\quad \because a^2 - b^2 = (a-b)(a+b)$

$= \frac{-4-7i}{16 - 49i^2}$ $\quad \because i^2 = -1$

$= \frac{-4-7i}{16+49} = \frac{-4-7i}{65}$

$\frac{1}{z} = \frac{-4}{65} - \frac{7}{65}i$ Ans.

$\frac{1}{z} = \left(\frac{-4}{65}, -\frac{7}{65}\right)$

ii. $\left(\sqrt{2}, -\sqrt{5}\right)$

$z = (\sqrt{2}, -\sqrt{5}) = (a, b)$

$a = \sqrt{2}, \quad b = -\sqrt{5}$

Multiplicative inverse

$z^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

$z^{-1} = \left(\frac{\sqrt{2}}{(\sqrt{2})^2 + (-\sqrt{5})^2}, \frac{+\sqrt{5}}{(\sqrt{2})^2 + (-\sqrt{5})^2}\right)$

$\left(\sqrt{2}\right)+\left(-\sqrt{5}\right) \qquad \left(\sqrt{2}\right)+\left(-\sqrt{5}\right)$

$$= \left(\frac{\sqrt{2}}{2+5}, \quad \frac{\sqrt{5}}{2+5}\right)$$

$$z^{-1} = \left(\frac{\sqrt{2}}{7}, \quad \frac{\sqrt{5}}{7}\right) \qquad \text{Ans.}$$

iii. $(1, 0)$

Let $z = (1, 0)$

$z = 1 + 0i$

$z = 1$

Multiplicative inverse

$$z^{-1} = \frac{1}{z} = \frac{1}{1} = 1$$

$$z^{-1} = 1 + 0i = (1, 0)$$

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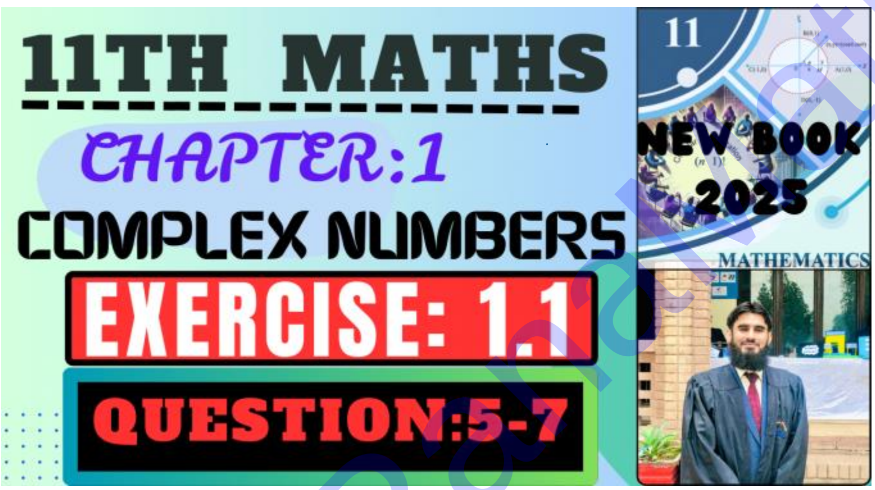

5. Separate into real and imaginary parts (write as a simple complex number):

(i) $\dfrac{2-7i}{4+5i}$

$a+bi$

$$= \frac{2-7i}{4+5i} \times \frac{4-5i}{4-5i}$$

$a-bi$

$$= \frac{2(4-5i) - 7i(4-5i)}{4^2-(5i)^2}$$

$\because a^2-b^2$
$=(a+b)(a-b)$

$$= \frac{8-10i-28i+35i^2}{16-25i^2}$$

$\because i^2=-1$

$$= \frac{8-38i-35}{16+25}$$

$$= \frac{8 - 38i - 35}{16 + 25}$$

$$= \frac{8 - 35 - 38i}{41}$$

$$= \frac{-27 - 38i}{41}$$

$$= \frac{-27}{41} - \frac{38}{41}i \quad \text{Ans.}$$

(ii) $\frac{(-2+3i)^2}{1+i}$

$$= \frac{(3i-2)^2}{1+i}$$

$\because (a-b)^2 = a^2+b^2-2ab$

$$= \frac{(3i)^2 + 2^2 - 2(3i)(2)}{1+i}$$

$$= \frac{9i^2 + 4 - 12i}{1+i} \qquad \because i^2 = -1$$

$$= \frac{9(-1) + 4 - 12i}{1+i}$$

$$= \frac{-9+4-12i}{1+i}$$

$$= \frac{-5-12i}{1+i}$$

$$= \frac{-5-12i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{-5(1-i) - 12i(1-i)}{1^2 - i^2}$$

$$= \frac{-5+5i-12i+12i^2}{1-(-1)}$$

$$= \frac{-5-7i+12(-1)}{1+1}$$

$$= \frac{-5-7i-12}{2} = \frac{-17-7i}{2}$$

$$= \frac{-17}{2} - \frac{7}{2}i$$

$$= \frac{-17}{9} - \frac{7}{9}i$$

Ans.

(iii) $\frac{i}{1+i}$

$$= \frac{i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{i(1-i)}{1^2 - i^2}$$

$\because a^2 - b^2 = (a-b)(a+b)$

$$= \frac{i - i^2}{1 - (-1)}$$

$$= \frac{i - (-1)}{1+1}$$

$$= \frac{i+1}{2} = \frac{1+i}{2}$$

$$= \frac{1}{2} + \frac{i}{2}$$ Ans.

(iv) $\frac{(4+3i)^2}{4-3i}$

$\because (a+b)^2 = a^2+b^2+2ab$

$= \frac{4^2+(3i)^2+2(4)(3i)}{4-3i}$

$= \frac{16+9i^2+24i}{4-3i}$

$= \frac{16+9(-1)+24i}{4-3i}$

$= \frac{16-9+24i}{4-3i}$

$= \frac{7+24i}{4-3i}$

$= \frac{7+24i}{4-3i} \times \frac{4+3i}{4+3i}$

$= \frac{7(4+3i)+24i(4+3i)}{(4)^2-(3i)^2}$

$= \frac{28+21i+96i+72i^2}{16-9i^2}$

$= \frac{28+117i-72}{16+9}$ $\qquad \because i^2=-1$

$= \frac{-44+117i}{25}$

$= -\frac{44}{25} + \frac{117}{25}i$ Ans.

$$= \frac{-44}{25} + \frac{117}{25} i \quad \text{Ans.}$$

6. If $z_1 = 2 + i, z_2 = 3 - 2i, z_3 = 1 + 3i$ *then express* $\frac{\overline{z_1}\,\overline{z_3}}{z_2}$ in the form of a $+ bi$.

$z_1 = 2 + i \qquad \overline{z_1} = 2 - i$

$z_3 = 1 + 3i \qquad \overline{z_3} = 1 - 3i$

$$\frac{\overline{z_1}\,\overline{z_3}}{z_2} = \frac{(2-i)(1-3i)}{3-2i}$$

$$= \frac{2(1-3i) - i(1-3i)}{3-2i}$$

$$= \frac{2 - 6i - i + 3i^2}{3-2i}$$

$$= \frac{2 - 7i - 3}{3-2i}$$

$$= \frac{-1 - 7i}{3-2i}$$

$$= \frac{-1-7i}{3-2i} \times \frac{3+2i}{3+2i}$$

$$= \frac{-1(3+2i) - 7i(3+2i)}{(3)^2 - (2i)^2}$$

$$= \frac{-3-2i-21i-14i^2}{9-4i^2}$$

$\because i^2 = -1$

$$= \frac{-3-23i+14}{9+4}$$

$$= \frac{11-23i}{13}$$

$$= \frac{11}{13} - \frac{23i}{13} \quad \text{Ans}$$

7. If $z_1 = 2 + 7i$ and $z_2 = -5 + 3i$, then evaluate the following:

i. $|2z_1 - 4z_2|$

$z = a + bi$, $|z| = \sqrt{a^2 + b^2}$

$$2z_1 - 4z_2 = 2(2+7i) - 4(-5+3i)$$
$$= 4 + 14i + 20 - 12i$$
$$= 24 + 2i$$

$$|2z_1 - 4z_2| = \sqrt{(24)^2 + 2^2}$$
$$= \sqrt{576 + 4}$$
$$= \sqrt{580}$$
$$= \sqrt{2^2 \times 145}$$
$$= \sqrt{2^2}\sqrt{145}$$

$$\begin{array}{r|l} 2 & 580 \\ \hline 2 & 290 \\ \hline 5 & 145 \\ \hline & 29 \end{array}$$

$= \sqrt{4}\sqrt{145}$

$= 2\sqrt{145}$ Ans

ii. $|3z_1 + 2\bar{z}_1|$

If $z_1 = 2 + 7i$ and $z_2 = -5 + 3i$

$z_1 = 2+7i$ $\quad \bar{z}_1 = 2-7i$

$3z_1 + 2\bar{z}_1 = 3(2+7i)+2(2-7i)$

$$3Z_1 + 2\bar{Z}_1 = 3(2+7i) + 2(2-7i)$$

$$= 6 + 21i + 4 - 14i$$

$$= 10 + 7i$$

$$|3Z_1 + 2\bar{Z}_1| = \sqrt{10^2 + 7^2}$$

$$= \sqrt{100 + 49}$$

$$= \sqrt{149} \quad \text{Ans}$$

iii. $\left|-7z_2 + 2\bar{z}_2\right|$

If $z_1 = 2 + 7i$ *and* $z_2 = -5 + 3i$

$z_2 = -5+3i \qquad \bar{z}_2 = -5-3i$

$-7z_2 + 2\bar{z}_2 = -7(-5+3i)+2(-5-3i)$

$= 35-21i-10-6i$

$= 25-27i$

$|-7z_2+2\bar{z}_2| = \sqrt{25^2+(-27)^2}$

$= \sqrt{625+729}$

$= \sqrt{1354}$ Ans

iv. $\left|(z_1 + z_2)^3\right|$

If $z_1 = 2 + 7i$ and $z_2 = -5 + 3i$

$$z_1 + z_2 = 2 + 7i + (-5 + 3i)$$
$$= 2 + 7i - 5 + 3i$$
$$= 10i - 3$$
$$(z_1 + z_2)^3 = (10i - 3)^3$$
$$\because (a-b)^3 = a^3 - b^3 - 3ab(a-b)$$
$$(z_1 + z_2)^3 = (10i)^3 - 3^3 - 3(10i)(3)(10i - 3)$$
$$= 1000i^3 - 27 - 90i(10i - 3)$$
$$= 1000i^2 \cdot i - 27 - 900i^2 + 270i$$
$$= 1000(-1) \cdot i - 27 - 900(-1) + 270i$$
$$= -1000i - 27 + 900 + 270i$$

$= 1000(-1) \cdot i - 27 - 900(-1) + 270i$

$= -1000i - 27 + 900 + 270i$

$= 873 - 730i$

$$|(z_1+z_2)^3| = \sqrt{(873)^2 + (-730)^2}$$

$$= \sqrt{762129 + 532900}$$

$$= \sqrt{1295029}$$

$$= 109\sqrt{109} \quad \text{Ans}$$

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