Chp 15
Multiple Integerals:

Definate Integral

Area under the curve is called definate Integral.

$$\int_a^b f(x)\,dx = F(a) - F(b)$$

Double Integral as Volume

$$\text{Volume} = \lim_{n\to\infty} S_n = \iint_R f(x,y)\,dA$$

where $\Delta A_k \to 0$ as $n \to \infty$

Fubini's Theorem. (1st form)

If $f(x,y)$ is continuous throughout the rectangular region R: $a \le x \le b$ & $c \le y \le d$ then,

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Fubini's Theorem (Stronger form)

Let $f(x,y)$ be continous on region R:

① If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 & g_2 are continous on $[a,b]$ the

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

② If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$

where h_1 & h_2 are continuous on $[c,d]$ then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

$\longleftrightarrow$

Ex 15.1:

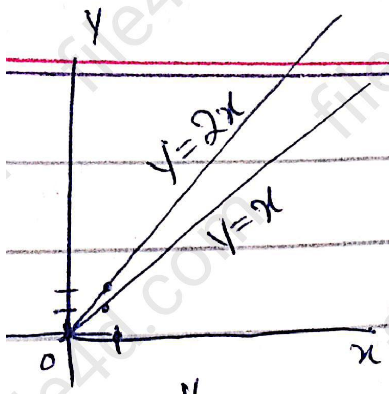
⇒ Finding Limits of Integral:

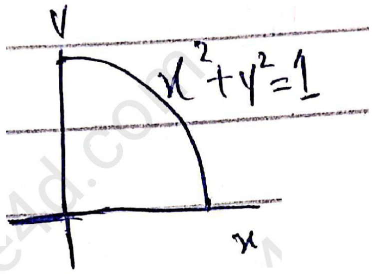
⇒ When y depends on x then we write limit of y function in terms of x and limits of x are constant.

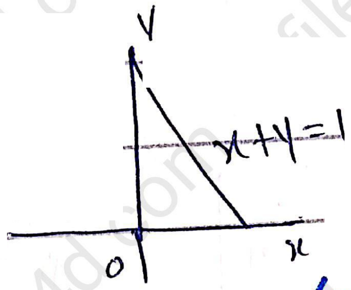
⇒ When x depends on y, then write limits of y in constant terms and x in terms of y.

⇒ When we find limits of x, draw a horizontal line, the line enters the region having smaller limit and the line leaves the region having large limit.

lines moves from $\begin{cases}\text{left to right} \\ \text{down to up.}\end{cases}$

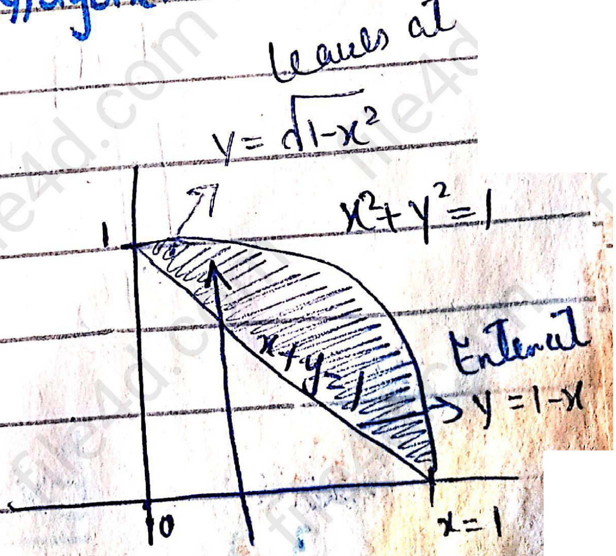






Sketch regions

$$\int_0^1\int_{1-x}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx.$$



$y=\sqrt{1-x^2}$, $y=1-x$

$x=0$, $x=1$

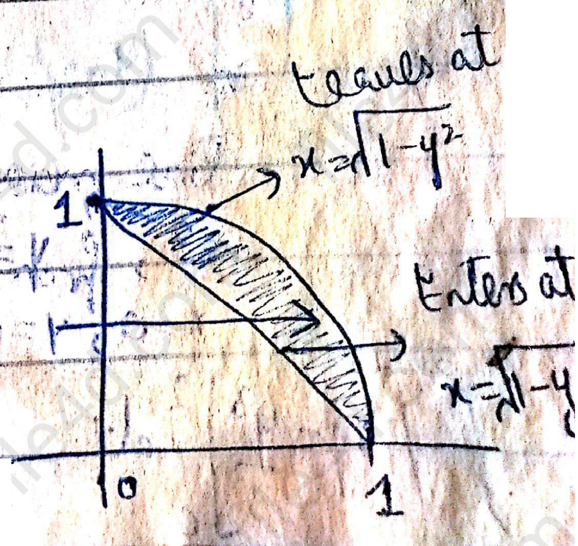
$y^2+x^2=1$, $y+x=1$

ellipse , line.

Reverse order:

$\sqrt{1-y^2} = x$, $x=1-y$

$$\int_0^1\int_{1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$$



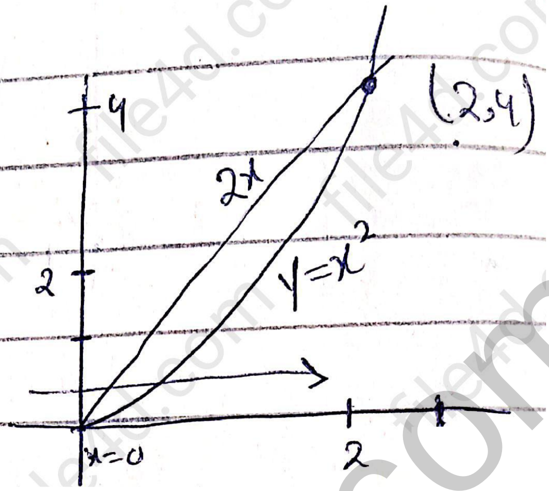
Ex 4: $\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx.$

Sketch region.

$x=0,\ x=2$

$y=x^2$, $y=2x$

parabola, line



Reverse order.

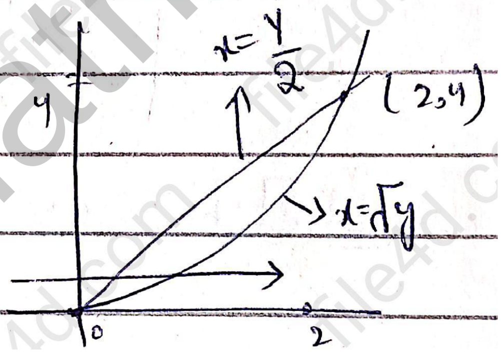
at $x=0,\ y=0$

at $x=2,\ y=4.$

So, $0 \le y \le 4.$

$x=\sqrt{y}$, $x=\frac{y}{2}$

(leaves) (enters)



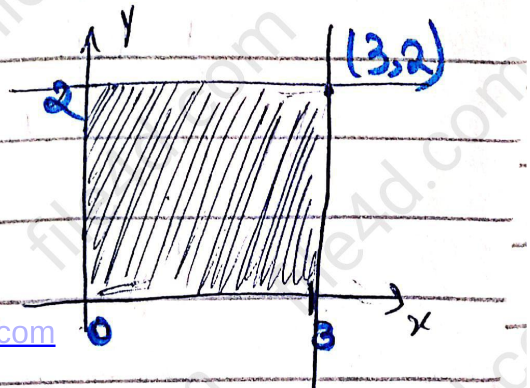
$$\int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy$$

Ex 15.1: Sketch & Evaluate.

① $\int_0^3 \int_0^2 (4-y^2)\,dy\,dx.$

$x=3$, $y=2.$ line, line.

$x=0$, $y=0$



$$\int_0^3 \left[4y - \frac{y^3}{3}\right]_0^2 dx$$

$$= \int_0^3 \left(4(2) - \frac{2^3}{3}\right) dx$$

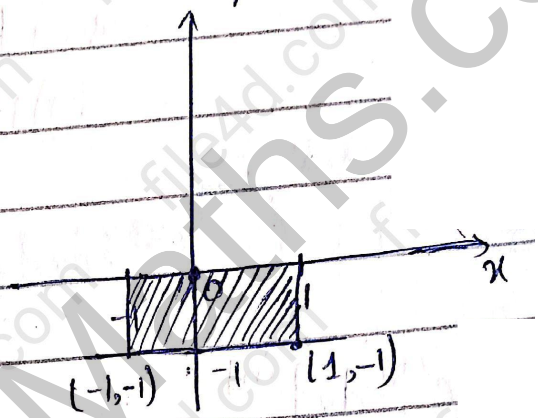
$$= \int_0^3 \left(\frac{24-8}{3}\right) dx \Rightarrow \frac{16}{3}\left[x\right]_0^3$$

$$= \frac{16}{3}\left[3-0\right] = 16$$

③ $\int_{-1}^{0}\int_{-1}^{1} (x+y+1)\,dx\,dy$

$x=1,\ x=-1$

$y=0,\ y=-1$



$$\int_{-1}^{0}\left[\frac{x^2}{2} + xy + x\right]_{-1}^{1} dy$$

$$= \int_{-1}^{0}\left[\left(\frac{1^2}{2}+y+1\right) - \left(\frac{(-1)^2}{2} + (-1)y + (-1)\right)\right] dy$$

$$= \int_{-1}^{0}\left(\cancel{\tfrac{1}{2}} + y + 1 - \cancel{\tfrac{1}{2}} + y + 1\right) dy$$

$$= \int_{-1}^{0}(2y+2)\,dy$$

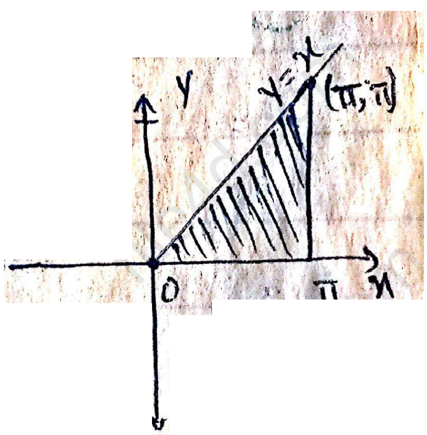
$$= 2\left[\frac{y^2}{2} + y\right]_{-1}^{0}$$

$$= 2\left[0 - \left(\frac{(-1)^2}{2} + (-1)\right)\right]$$

$$= 2\left[0 - \frac{1}{2} + 1\right]$$

$$= \cancel{2}\left(\frac{1}{\cancel{2}}\right) = 1$$

⑤ : $\int_0^{\pi}\int_0^{x} x\sin y\,dy\,dx.$



$$= \int_0^{\pi} -x\cos y\Big|_0^{x} dx$$

$$= \int_0^{\pi} \left(-x\cos x + x\right) dx$$

$$= \underbrace{\int_0^{\pi} -x\cos x\,dx}_{\mathrm{I}\ \ \mathrm{II}} + \int_0^{\pi} x\,dx$$

$$= -x\sin x\Big|_0^{\pi} - \int_0^{\pi}(-1)\sin x\,dx + \frac{x^2}{2}\Big]_0^{\pi}$$

$$= -x\sin x\Big|_0^{\pi} + (-\cos x)\Big]_0^{\pi} + \frac{\pi^2}{2} - 0$$

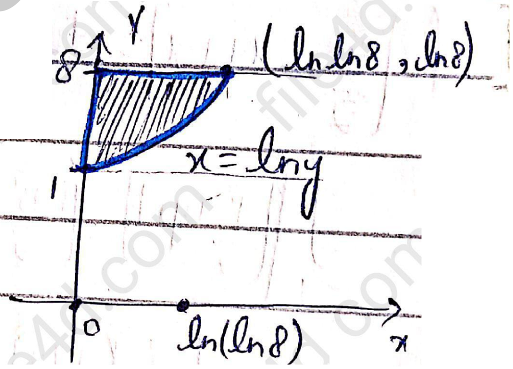
$$= -\pi\sin\pi - 0\sin 0 - \cos\pi + \cos 0 + \pi^2/2$$

$$= -\cos\pi + 1 + \pi^2/2$$

$$= -(-1)+1+\frac{\pi^2}{2} = 1+1+\frac{\pi^2}{2} = 2+\frac{\pi^2}{2}$$

④ $\displaystyle\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy$

$$= \int_1^{\ln 8}\left[\frac{e^{x+y}}{1}\,dx\right]_0^{\ln y}$$



$$= \int_1^{\ln 8}\left(e^{\ln y+y} - e^{0+y}\right)dy$$

$$= \int_1^{\ln 8}\left(e^{\ln y}\cdot e^{y} - e^{y}\right)dy$$

$$= \int_1^{\ln 8}\left(\underset{\mathrm{I}\ \mathrm{II}}{ye^{y}} - e^{y}\right)dy$$

Integeration by parts

$$= ye^{y}\Big|_1^{\ln 8} - \int_1^{\ln 8}(1)e^{y} - e^{y}\Big|_1^{\ln 8}$$

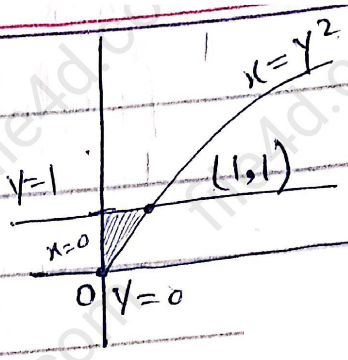
$$= \ln 8\, e^{\ln 8} - 1e^{1} - e^{\ln 8} + e^{1} - e^{\ln 8} + e^{1}$$

$$= 8\ln 8 - 8 - 8 + e^{1}$$

$$= 8\ln 8 - 16 + e^{1}$$

(9) $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy$

$y=1,\ x=y^2$



$$= \int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy$$

$$= \int_0^1 \left[3y^3 \frac{e^{xy}}{y(1)} \right]_0^{y^2} dy$$

$$= \int_0^1 \left[3y^2 e^{xy} \right]_0^{y^2} dy \Rightarrow \int_0^1 \left(3y^2 e^{y^3} - 3y^2 e^0\right) dy$$

$$= \int_0^1 \left(3y^2 e^{y^3} - 3y^2(1)\right) dy$$

$$= e^{y^3} - \frac{3y^3}{3} \Big]_0^1$$

$$= e^{y^3} - y^3 \Big|_0^1$$

$$\cancel{= e^1 - 1 - e^0 + 0^3}$$

$$\cancel{= e^1 - 1 - e^1 + 1}$$

$$= (e^1 - 1^3) - (e^0 - 0)$$

$$= e^1 - 1 - 1$$

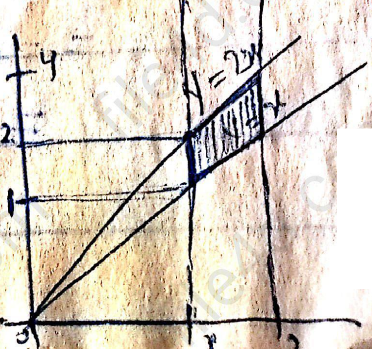
$$= e - 2$$

$\underline{e^{xy}}$

e^{xy}

$3y^2 \frac{e^y}{y(1)}$

(11) Quadrilateral: $f(x,y) = x/y$ over region bounded by lines $x=1$, $x=2$, $y=x$, $y=2x$



$$= \int_1^2 \int_x^{2x} \frac{x}{y}\, dy\, dx$$

$$= \int_1^2 \left[x \ln y \right]_x^{2x} dx$$

$$= \int_1^2 x(\ln 2x - \ln x)\,dx$$

$$= \int_1^2 x \ln\left(\frac{2x}{x}\right) dx$$

$$= \ln 2 \int_1^2 x \cdot dx$$

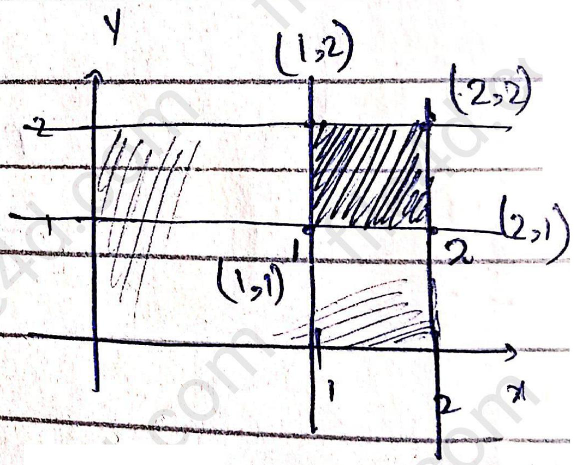
$$= \ln 2 \cdot \frac{x^2}{2}\Big]_1^2$$

$$= \ln 2\left(\frac{(2)^2}{2} - \frac{1^2}{2}\right)$$

$$= \ln 2\left(2 - \frac{1}{2}\right)$$

$$= \ln 2\left(\frac{3}{2}\right) \Rightarrow \frac{3}{2}\ln 2$$

⑫. Square. $f(x,y) = \frac{1}{xy}$ over $1 \le x \le 2$, $1 \le y \le 2$



$$= \int_1^2 \int_1^2 \frac{1}{xy}\,dx\,dy$$

$$= \int_1^2 \left[\frac{1}{y}\ln x\right]_1^2 dy$$

$$= \int_1^2 \frac{1}{y}(\ln 2 - \ln 1)\,dy$$

$$= \ln 2 \int_1^2 \frac{1}{y}\,dy$$

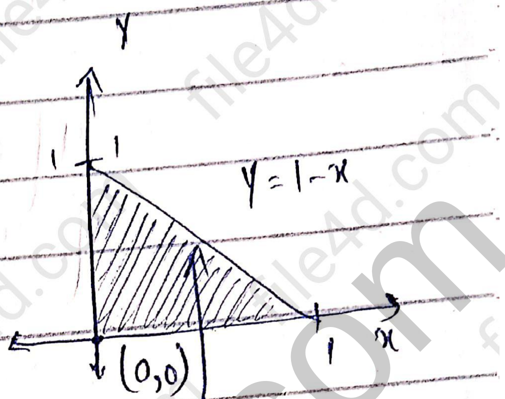
$$= \ln 2\,[\ln y]_1^2$$

$$= \ln 2 \cdot [\ln 2 - \ln 1]$$

$$= (\ln 2)^2$$

(13) $f(x,y) = x^2+y^2$ over the region having vertices: $(0,0), (1,0), (0,1)$

$$= \int_0^1 \int_0^{1-x} (x^2+y^2)\,dy\,dx$$



Enters $y=0$

leaves $y = 1-x$

$$= \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_0^{1-x} dx.$$

$$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx.$$

$$= \int_0^1 \left(x^2 - x^3 + \frac{(1-x)^3}{3}\right) dx.$$

$$= \int_0^1 \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{(1-x)^4(-1)}{3\times 4}\right]_0^1$$

$$= \left[\frac{1}{3} - \frac{1}{4} - \frac{(1-1)^4}{12}\right] - \left[\frac{0}{3} - \frac{0}{4} - \frac{(1-0)^4}{12}\right]$$

$$= \frac{1}{3} - \frac{1}{4}\ 0 - 0 + \frac{1}{12}$$

$$= \frac{4-3+1}{12}$$

$$= \frac{2}{12} = \frac{1}{6}$$

(17) $$\int_{-2}^{0} \int_{v}^{-v} 2\,dp\,dv$$

$$= \int_{-2}^{0} \left[2p\right]_v^{-v} dv$$

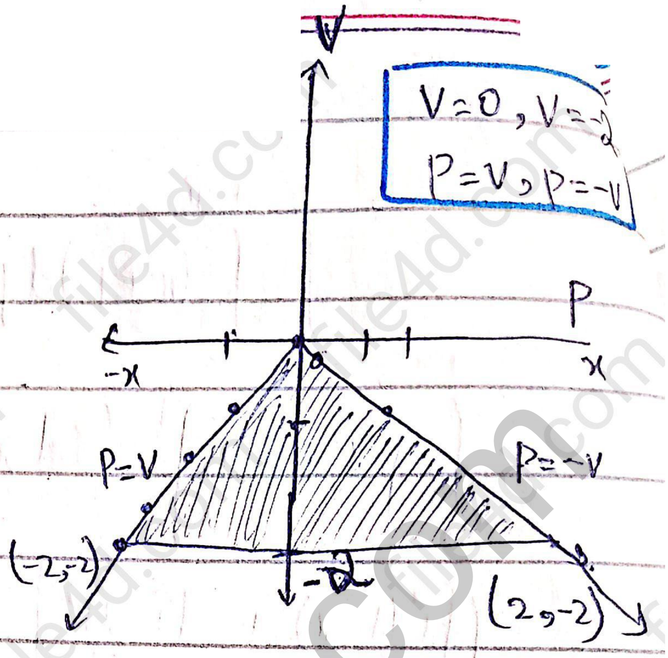
$$= \int_{-2}^{0} \left(2(-v) - 2(v)\right) dv$$

$$= \int_{-2} -4v\,dv$$

$$= -4\frac{v^2}{2}\Big]_{-2}^{0}$$

$$= -4\left[\frac{0}{2} - \left(\frac{(-2)^2}{2}\right)\right]$$

$$= -4\left[-\frac{\cancel{4}\,2}{\cancel{2}}\right] = 8$$



(21) $\int_0^1 \int_2^{4-2x} dy\,dx.$

$x=1,\ x=0$

$y=4-2x,\ y=2$

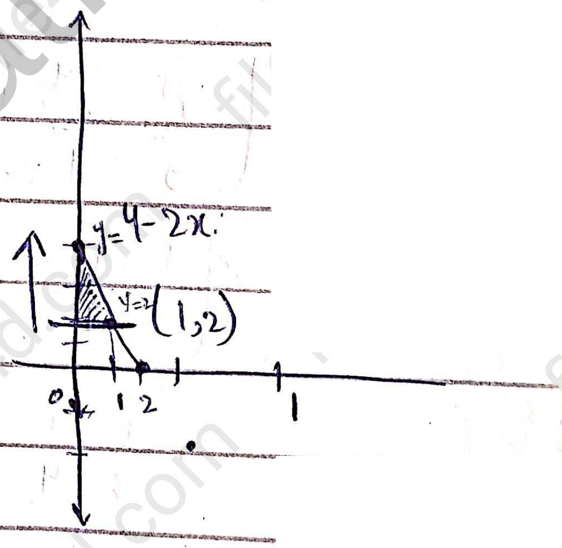
$$= \int_0^1 \int_2^{4-2x} dy\,dx$$

= at $x=1$

$\boxed{y=2}$

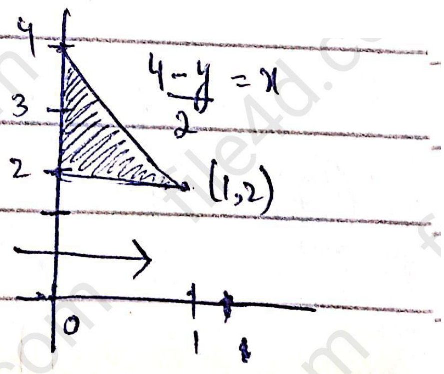
at $x=0$, $\boxed{y=4}$

So $2 \le y \le 4$.

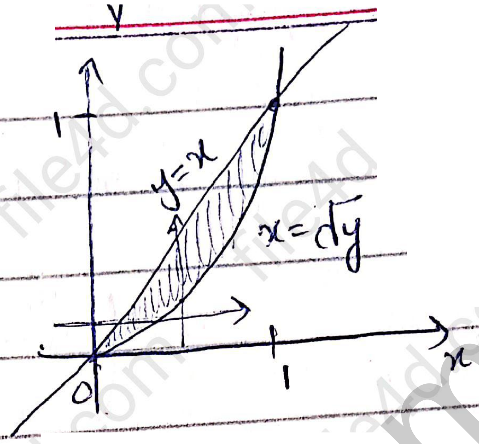


$$= \int_2^4 \int_0^{4-y/2} dx\,dy.$$

Inverse order.



(3) $\int_0^1 \int_y^{\sqrt{y}} dx\,dy$



$y=0, \quad y=1$

$x=\sqrt{y}, \quad x=y.$

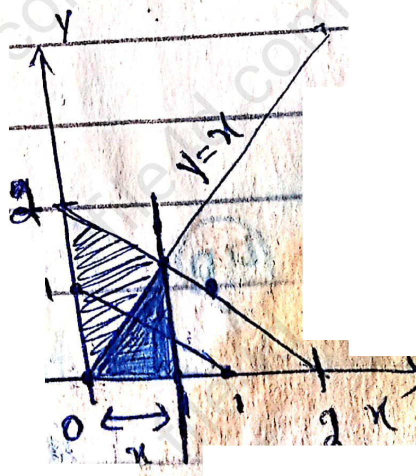
Inverse order:

$\boxed{x^2=y}, \quad \boxed{y=x}$

$$= \int_0^1 \int_{x^2}^{x} f(x,y)\,dy\,dx$$

So, reversed Integral is $\int_0^1 \int_{x^2}^{x} dy\,dx$

(4) : Find Volume of region $z=x^2+y^2$ bounded by $y=x, x=0, x+y=2$



y depend on x.

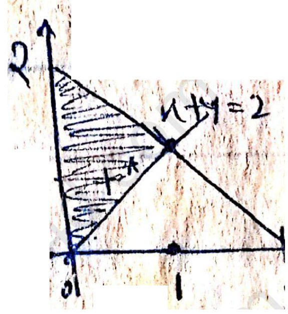
$x \le y \le 2-x.$

$x=0$,

Putting $y=x$ in $x+y=2$

$x+x=2$

$\boxed{x=1}$



So, $0 \le x \le 1$

$$= \int_0^1 \int_x^{2-x} (x^2+y^2)\,dy\,dx$$

$$= \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_x^{2-x} dx \Rightarrow \int_0^1 \left[x^2(2-x) + \frac{(2-x)^3}{3} \right.$$

$$= \int_0^1 \left[2x^2 - x^3 + \frac{(2-x)^3}{3} - x^3 - \frac{x^3}{3}\right] dx$$

$$= \int_0^1 \left(2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}\right) dx.$$

$$= \left[\frac{2x^3}{3} - \frac{7x^4}{3\times4} + \frac{(2-x)^4(-1)}{3\times4}\right]_0^1$$

$$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(0 - 0 - \frac{(2-0)^4}{12}\right)$$

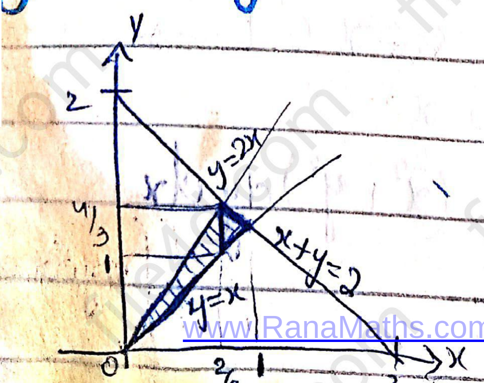
$$= \left(\frac{8-7-1}{12} + \frac{16}{12}\right)$$

$$= \frac{8-8}{12} + \frac{16}{12}$$

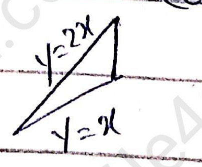
$$= 0 + 4/3$$

$$= 4/3$$

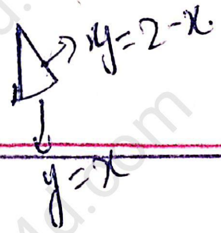
(40) Triangular region: $\iint_R xy\,dA$ where R is the region bounded by lines $y = x$, $y = 2x$ & $x+y=2$.



(1)



$0 \le x \le \frac{2}{3}$



$\frac{2}{3} \le x \le 1$

$$\int_0^{2/3}\int_x^{2x} xy\,dy\,dx + \int_{2/3}^{1}\int_x^{2-x} xy\,dy\,dx$$

$$= \int_0^{2/3} \left.\frac{xy^2}{2}\right]_x^{2x} dx + \int_{2/3}^1 \left.\frac{xy^2}{2}\right]_x^{2-x} dx$$

$$= \int_0^{2/3}\left(\frac{4x^3}{2} - \frac{x^3}{2}\right)dx + \int_{2/3}^1 \left(\frac{x(2-x)^2}{2} - \frac{x(x^2)}{2}\right)$$

$$= \int_0^{2/3} \frac{3x^3}{2}dx + \int_{2/3}^1 \frac{x(-4x+x^2+4)}{2} - \frac{x^3}{2}$$

$$= \left[\frac{3x^4}{2\times4}\right]_0^{2/3} + \int_{2/3}^1\left(-2x^2 + \frac{x^3}{\cancel{2}} + \frac{4x}{\cancel{2}} - \frac{x^3}{\cancel{2}}\right)dx$$

$$= \frac{3}{8}\left(\frac{2}{3}\right)^4 - 0 + \left[\frac{-2x^3}{3} + \frac{\cancel{2}x^2}{\cancel{2}}\right]_{2/3}^1$$

$$= \frac{3}{8}\times\frac{16}{81}\,2 + \left[\left(\frac{-2(1)}{3}+1\right) - \left(-\frac{2}{3}\left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^2\right)\right]$$

$$= \frac{6}{81} + \frac{1}{3} + \frac{16}{81} - \frac{4}{9}$$

$$= \frac{13}{81} \quad \text{Ans.}$$

Reverse order:

$$\iint xy\,dx\,dy + \iint xy\,dx\,dy$$

Ex 15.2

Area of bounded region (closed)

⇒ The function must be constant

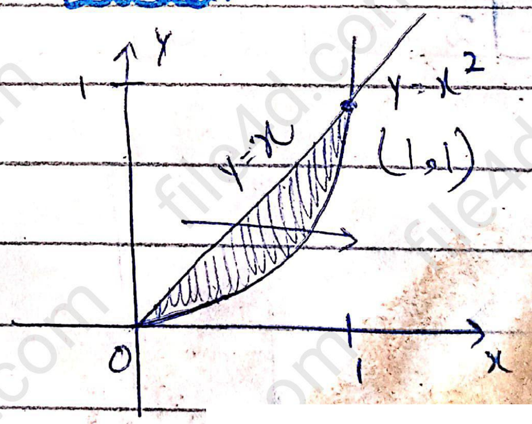
$$\text{Area} = \iint_R dA$$

$f(x,y)$ - is constant.

⇒ for finding Area of triangles, rectangles, square, we can use this defination in which function will be constant.

Examples:

① Find area of Region bounded by $y=x$, $y=x^2$ in 1st Quad.



$x=0, \quad x=1$

$y=0, \quad y=1$

$$\text{Area} = \int_0^1 \int_y^{\sqrt{y}} dx\,dy$$

$$= \int_0^1 \left[x \right]_y^{\sqrt{y}} dy$$

$$= \int_0^1 (\sqrt{y} - y)\, dy$$

$$= \left[\frac{y^{1/2+1}}{1/2+1} - \frac{y^2}{2} \right]_0^1$$

$$= \left(\frac{y^{3/2}}{3/2} - \frac{y^2}{2} \right)_0^1$$

$$= \left(\frac{2}{3} y^{3/2} - \frac{y^2}{2} \right)_0^1$$

$$= \left[\frac{2(1)}{3} - \frac{1}{2} - 0 \right]$$

$$= \frac{4-3}{6} = \frac{1}{6}.$$

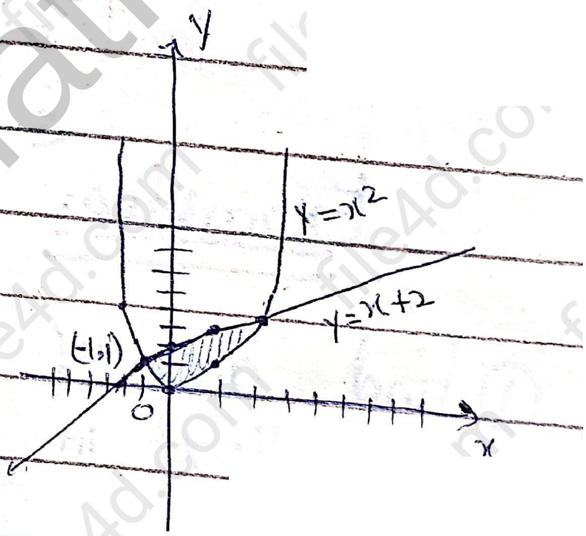
Example 2:

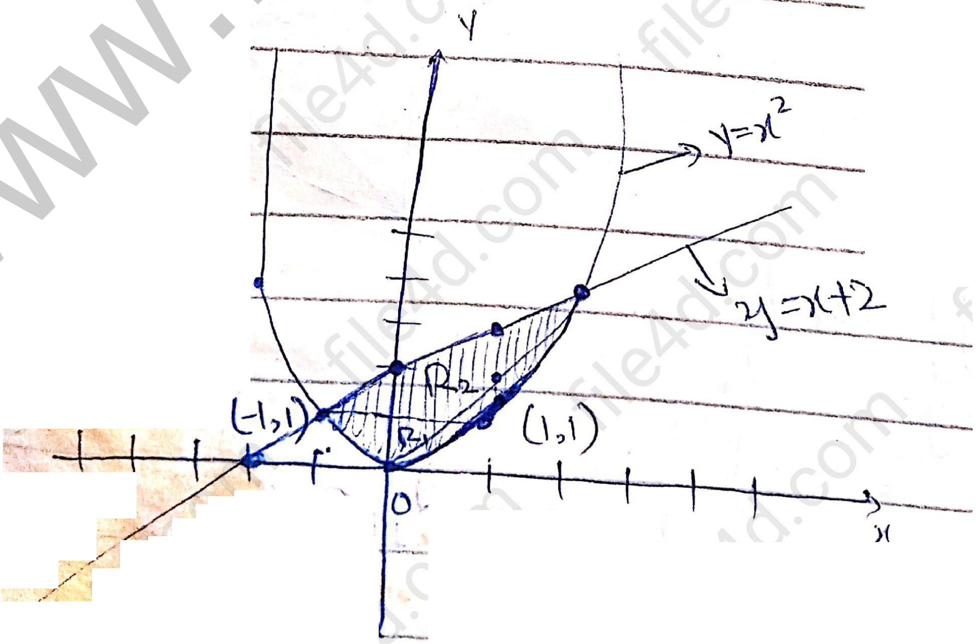
Find Area :- $y = x^2, y = x+2$

$y = x^2$

$x = \pm\sqrt{y}$

x	0	1	2	-1	-2
$y=x^2$	0	1	4	1	4
$y=x+2$	2	3	4	1	0





Area $=$ Area of R_1 + Area of R_2

$$= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\,dy$$

Reverse order:

$$\int_{-1}^{1} \int_{x^2}^{x+2} dy\,dx + \int_{-1}^{2} \int_{x^2}^{x+2} dy\,dx$$

$$\boxed{\text{Area} = \int_{-1}^{2} \int_{x^2}^{x+2} dy\,dx}$$

$$\text{Area} = \int_{-1}^{2} \Big[y\Big]_{x^2}^{x+2} dx$$

$$\text{Area} = \int_{-1}^{2} (x+2-x^2)\,dx$$

$$\text{Area} = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$\text{Area} = \frac{2^2}{2} + 2(2) - \frac{2^3}{3} - \left(\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right)$$

$$= \frac{4}{2} + \frac{4}{1} - \frac{8}{3} - \frac{1}{2} + \frac{2}{1} - \frac{1}{3}$$

$$= \frac{4+8-1+4}{2} - \frac{9}{3}$$

$$\text{Area} = \frac{15}{2} - \cancel{\frac{9}{3}}3 = \frac{15-6}{2} = \frac{9}{2}$$

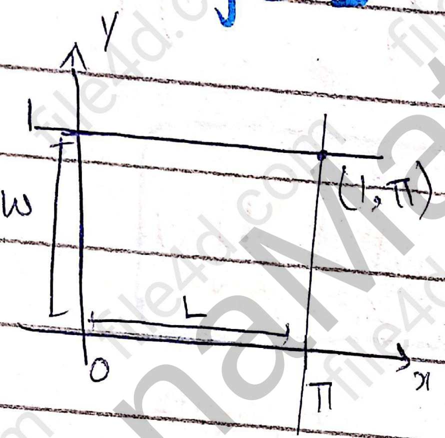
⇒ Average value of f over R

$$= \frac{1}{\text{area of } R} \iint_R f\,dA$$

Ex:3: Find Avg value of $f = x\cos xy$ over rectangle $0 \le x \le \pi$, $0 \le y \le 1$

⇒ Length of Rectangle $= \pi$

⇒ Width $= 1$



Area of Rectangle $= 1 \times \pi = \pi$

OR: Area of region $= \iint_R f(x,y)\,dA$

$$= \int_0^1\int_0^\pi (1)\,dx\,dy$$

as f must be constant for area of bounded region.

$$\int_0^1 [x]_0^\pi\,dy$$

$$= \int_0^1 (\pi - 0)\,dy$$

$$= \pi y\Big]_0^1$$

$$\pi(1-0) = \pi$$

So, Area of Rectangle is π

$$\text{Average value} = \frac{1}{\pi}\int_0^1\int_0^\pi x\cos xy \, dx\, dy$$

Or, (easy way).

$$= \frac{1}{\pi}\int_0^\pi\int_0^1 x\cos xy \, dy\, dx$$

$$= \frac{1}{\pi}\int_0^\pi \left[\frac{x\sin xy}{1\cdot x}\right]_0^1 dx$$

$$= \frac{1}{\pi}\int_0^\pi (\sin x - 0)\, dx$$

$$= \frac{1}{\pi}\left(-\cos x\right)_0^\pi$$

$$= -\frac{1}{\pi}\left[\cos\pi - \cos 0\right]$$

$$= -\frac{1}{\pi}\left[-1-1\right]$$

$$= -\frac{1}{\pi}\left[-2\right] = \frac{2}{\pi}$$

$$\boxed{\text{Average value} = \frac{2}{\pi}}$$

$\longleftrightarrow$

Formulas:

Mass: $M = \iint_R \delta(x,y)\, dA$

First moment:

$\Rightarrow$ **Along x-axis:** $M_x = yM$ or $\iint_R y\delta(x,y)\, dA$

$\Rightarrow$ **Along y-axis:** $M_y = xM$ or $\iint_R x\,\delta(x,y)\, dA$

Centre of mass:

$$\bar{x} = \frac{M_y}{M}$$

$$\bar{y} = \frac{M_x}{M}$$

2nd Moment / Moment of Inertia

⇒ About x-axis:

$$I_x = \iint_R y^2 \delta(x,y)\, dA$$

⇒ About y-axis

$$I_y = \iint_R x^2 \delta(x,y)\, dA$$

⇒ About line: $I_L = \iint_R r^2(x,y)\,\delta(x,y)\, dA$

⇒ About origin: $I_o = I_x + I_y$

$$I_o = \iint_R (x^2+y^2)\,\delta(x,y)\, dA$$

Radii of gyration:

$I_x = MR_x^2$

1. About x-axis $\quad R_x = \sqrt{I_x/M}$
2. About y-axis $\quad R_y = \sqrt{I_y/M}$
3. About ~~o-axis~~ origin $\quad R_o = \sqrt{I_o/M}$

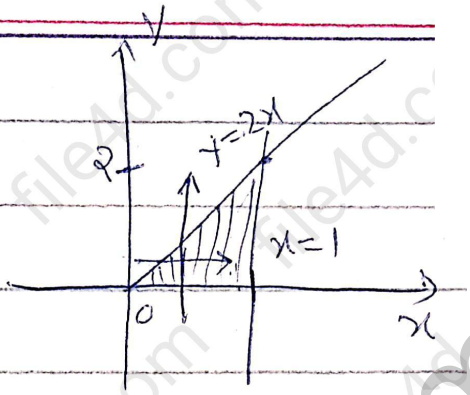
Example 18

$x=1$, $y=2x$ in Quadrant 1:

$\delta(x,y) = 6x+6y+6$

Find M, M_x, M_y, Centre of Mass?

$$M = \iint_R \delta(x,y)\, dA$$



$$M = \int_0^2 \int_{y/2}^{1} (6x+6y+6)\, dx\, dy$$

OR

$$M = \int_0^1 \int_0^{2x} (6x+6y+6)\, dy\, dx$$

$$M = \int_0^1 \left[\left(6xy + \frac{6y^2}{2} + 6y\right)\right]_0^{2x} dx$$

$$M = \int_0^1 \left[6x(2x) + \frac{\cancel{6}^{3}(2x)^2}{\cancel{2}} + 6(2x)\right] - 0\, dx$$

$$M = \int_0^1 \left(12x^2 + 12x^2 + 12x\right) dx$$

$$M = \int_0^1 \left(24x^2 + 12x\right) dx$$

$$M = \left.\frac{24x^3}{3} + \frac{12x^2}{2}\right]_0^1$$

$$M = \frac{\cancel{24}^{8}}{\cancel{3}} + \frac{\cancel{12}^{6}}{\cancel{2}} = 14$$

$$\boxed{M = 14}$$

First Moments

$$M_x = \int_0^1 \int_0^{2x} y(6x+6y+6)\, dy\, dx$$

$$= \int_0^1 \int_0^{2x} (6xy + 6y^2 + 6y)\, dy\, dx$$

$$= \int_0^1 \left[\frac{3\cancel{6}xy^2}{\cancel{2}} + \frac{6y^3}{3} + \frac{6y^2}{2}\right]_0 dx$$

$$= \int_0^1 \left[3xy^2 + 2y^3 + 3y^2\right]_0^{2x} dx$$

$$= \int_0^1 3x(2x)^2 + 2(2x)^3 + 3(2x)^2) \, dx$$

$$= \int_0^1 (12x^3 + 16x^3 + 12x^2) \, dx$$

$$= \left[\frac{12x^4}{4} + \frac{16x^4}{4} + \frac{12x^3}{3}\right]_0^1$$

$$= \left[3x^4 + 4x^4 + 4x^3\right]_0^1$$

$$= 3+4+4 = 11.$$

$$\boxed{M_x = 11}$$

$$M_y = \int_0^1 \int_0^{2x} x(6x+6y+6) \, dy\,dx$$

$$= \int_0^1 \int_0^{2x} (6x^2+6xy+6x) \, dy\,dx$$

$$= \int_0^1 \left[6x^2y + \frac{6xy^2}{2} + 6xy\right]_0^{2x} dx$$

$$= \int_0^1 \left[6x^2(2x) + 3x(2x)^2 + 6x(2x)\right] dx$$

$$= \int_0^1 (12x^3 + 12x^3 + 12x^2) \, dx$$

$$= \left[\frac{24x^4}{4} + \frac{12x^3}{3}\right]_0^1$$

$$= \frac{24}{4} + \frac{12}{3}$$

$$= 6+4 = 10$$

$$\boxed{M_y = 10}$$

Centre of Mass:

$$\bar{y} = \frac{M_x}{M} = \frac{11}{14}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\cancel{10}}{\cancel{14}} = \frac{5}{7}$$

Find moment of Inertia & radii of gyration?

$$M = 14$$

$$I_y = \int_0^1\int_0^{2x} x^2(6x+6y+6)\,dy\,dx$$

$$= \int_0^1\int_0^{2x} (6x^3+6x^2y+6x^2)\,dy\,dx$$

$$= \int_0^1 \left[6x^3y + \frac{\overset{3}{\cancel{6}}x^2y^2}{\cancel{2}} + 6x^2y\right]_0^{2x} dx$$

$$= \int_0^1 \left[6x^3(2x) + 3x^2(2x)^2 + 6x^2(2x)\right] dx$$

$$= \int_0^1 (12x^4 + 12x^4 + 12x^3)\,dx$$

$$= \left(\frac{24x^5}{5} + \frac{12x^4}{4}\right)_0^1$$

$$= \frac{24}{5} + 3 = \frac{39}{5}$$

$$\boxed{I_y = 39/5}$$

$$I_x = \int_0^1\int_0^{2x} y^2(6x+6y+6)\,dy\,dx$$

$$I_x = \int_0^1\int_0^{2x} (6xy^2+6y^3+6y^2)\,dy\,dx$$

$$I_x = \int_0^1 \left[\frac{6xy^3}{3}+\frac{6y^4}{4}+\frac{6y^3}{3}\right]_0^{2x} dx$$

$$I_x = \int_0^1 \left(2x(2x)^3+\frac{3}{2}(2x)^4+2(2x)^3\right) dx$$

$$I_x = \int_0^1 (16x^4+24x^4+16x^3)\,dx$$

$$I_x = \int_0^1 (40x^4+16x^3)\,dx$$

$$I_x = \left[\frac{40x^5}{5}+\frac{16x^4}{4}\right]_0^1$$

$$I_x = 8+4 = 12$$

$$\boxed{I_x = 12}$$

$$I_0 = I_x + I_y = 12 + \frac{39}{5} = \frac{99}{5}$$

$$\boxed{I_0 = \frac{99}{5}}$$

Radii:

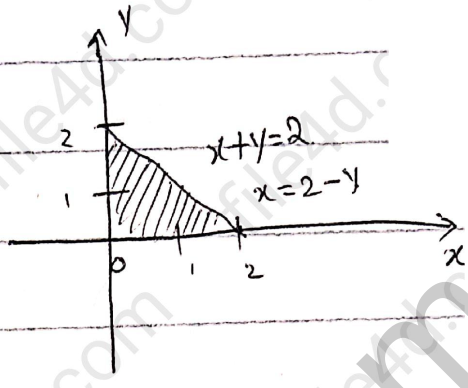
$$R_x = \sqrt{I_x/M} = \sqrt{\frac{12}{14}} \approx 0.93$$

$$R_y = \sqrt{I_y/M} = \sqrt{\frac{39/5}{14}} \approx 0.75$$

$$R_0 = \sqrt{I_0/M} = \sqrt{\frac{99/5}{14}} \approx 1.19$$

Ex 15.2

① The coordinate axes and line $x+y=2$



$$Area = \int_0^2\int_0^{2-y} dx\,dy$$

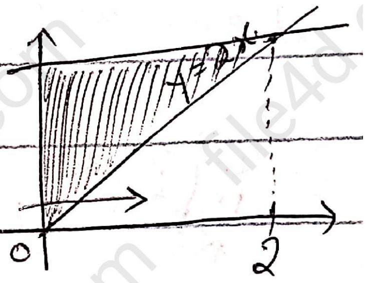
$$Area = \int_0^2 \left[x\right]_0^{2-y} dy$$

$$Area = \int_0^2 (2-y)\,dy = 2y - \frac{y^2}{2}\Big]_0^2 = 2(2) - \frac{2^2}{2} = 4 - \frac{4}{2}$$

$$= 4-2$$

$$\boxed{Area = 2}$$

② The line $x=0$, $y=2x$ & $y=4$



$$Area = \int_0^4\int_0^{y/2} dx\,dy$$

$$Area = \int_0^4 \frac{y}{2}\,dy$$

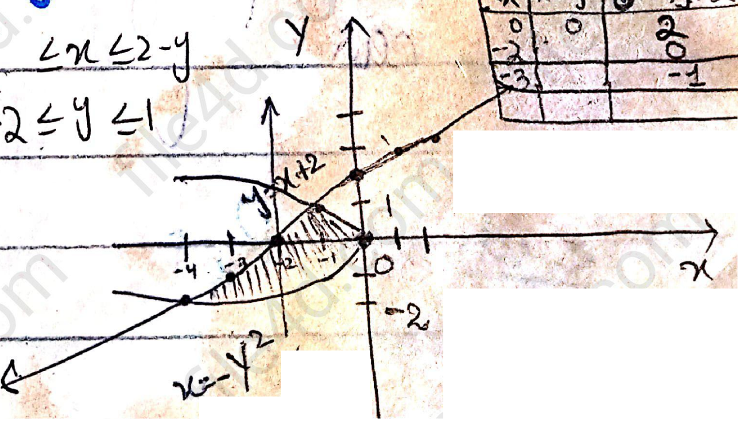
$$Area = \frac{y^2}{4}\Big]_0^4 = \frac{4^2}{4} = 4 \text{ Ans}$$

③ parabola $x=-y^2$ & line $y=x+2$

x	$x=-y^2$	$x=y+2$
0	0	2
-2		0
-3		-1

$$A = \int_{-2}^{1}\int_{-y^2}^{2-y} dx\,dy \qquad y-2 \qquad -y^2 \le x \le 2-y$$

$$-2 \le y \le 1$$



$$A = \int_{-2}^{1} \left[2-y-(-y^2)\right] dy$$

$$A = \int_{-2}^{1} (2 - y + y^2)\,dy$$

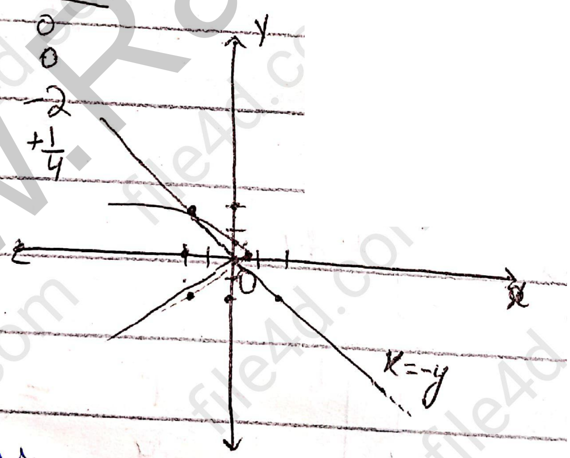
$$A = \left. 2y - \frac{y^2}{2} + \frac{y^3}{3} \right]_{-2}^{1}$$

$$A = \left(2 - \frac{1}{2} + 2^2\right) - \left(2(-2) - \frac{(-2)^2}{2} + \frac{(-2)^3}{3}\right)$$

$$A = +4 + \frac{4}{2} + \frac{8}{3}$$

(4) Parabola $x = y - y^2 \rightarrow (1)$ & line $y = -x \rightarrow (2)$

y	$x=-y$	$x = y - y^2$
0	0	0
1	-1	0
2	-2	-2
1/2	-1/2	+1/4

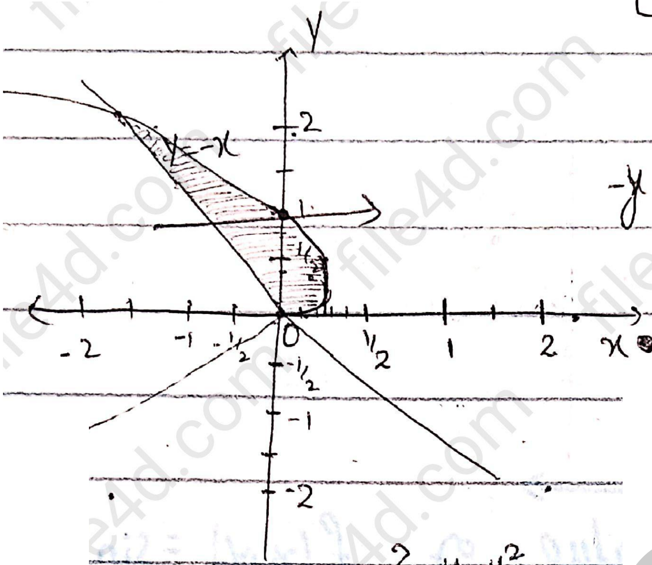


Notes

By Comparing two equations we can find point of intersection b/w two curves or line.

4) $x = y - y^2 \rightarrow (1)$ $\quad y = -x \rightarrow (2)$

y	$-y = +x$	$x = y - y^2$
1	-1	0
$1/2$	$-1/2$	$0.25 = 1/4$
$2/5$	$-2/5$	$0.24, 6/25$



$-y \le x \le y - y^2$

$0 \le y \le 2$

Putting (2) in (1)

$-y = y - y^2$

$-y - y = -y^2$

$-2y = -y^2$

$y^2 - 2y = 0$

$y(y-2) = 0$

$y = 0, y = 2$

at $y = 0, x = 0$

at $y = 2, x = -2$

$0 = y - y^2$

$y(1-y) = 0$

$y = 1, y = 0$

$(0,0), (0,1)$

$$\text{Area} = \int_0^2 \int_{-y}^{y-y^2} dx\,dy$$

$$= \int_0^2 \left[y - y^2 + y \right] dy$$

$$= \int_0^2 (2y - y^2)\,dy$$

$$= 2\frac{y^2}{2} - \frac{y^3}{3} \Big]_0^2$$

$$= \frac{2(2)^2}{2} - \frac{2^3}{3} = 4 - \frac{8}{3} = \frac{4}{3}$$

9) $$\int_0^6 \int_{y^2/3}^{2y} dx\,dy$$

$$= \int_0^6 \left(2y - \frac{y^2}{3}\right) dy = \frac{2y^2}{2} - \frac{y^3}{9}\Big]_0^6 = 6^2 - \frac{6^3}{9} = 36 - \frac{216}{9}$$

$$\boxed{A = 12}$$

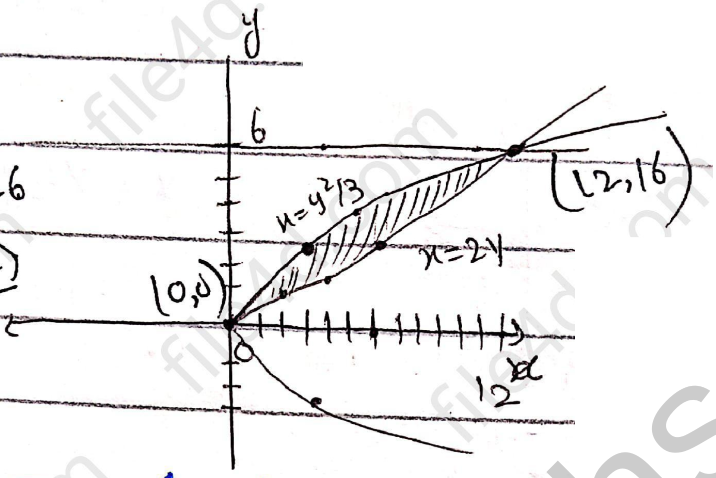
$x = 2y \qquad y = 0$

$x = y^2/3 \qquad y = 6$

at $y=0$, $x=0$; at $y=0$, $\boxed{x=0}$

at $y=6$, $x=12$; at $y=6$, $\boxed{x=12}$

$(12,6)$, $(0,0)$.



⑮ Average value of $f(x,y) = \sin(x+y)$.

ⓐ Rectangle $0 \le x \le \pi$ $0 \le y \le \pi$

$$\text{Area} = \int_0^{\pi}\int_0^{\pi} \sin(x+y)\,dx\,dy$$

$$= \int_0^{\pi} \left[-\frac{\cos(x+y)}{(1+0)} \right]_0^{\pi} dy$$

$$= \int_0^{\pi} \left[-\cos(\pi+y) - (-\cos(0+y)) \right]_0^{\pi} dy$$

$$= \int_0^{\pi} \cancel{-\cos\pi}\left(-\cos(\pi+y) + \cos y\right) dy$$

$$= -\sin(\pi+y) + \sin y \Big]_0^{\pi}$$

$$= -\sin(2\pi) + \sin\pi - 0$$

$$= 0$$

Area integral is 0

For Average value, finding Area of

Rectangle: $L \times W$

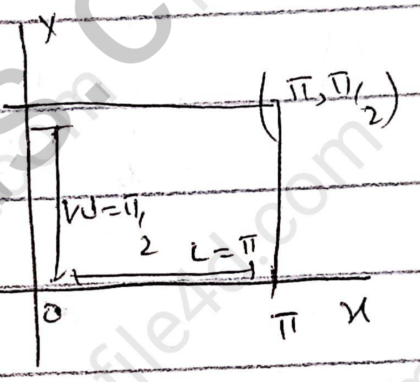
$\pi \times \pi = \pi^2$

$$\text{Average Value} = \frac{1}{\text{Area of } R} \iint_R f(x,y)$$

$$= 0$$

(b) Rectangle $0 \le x \le \pi$, $0 \le y \le \pi/2$

$$= \int_0^{\pi/2}\int_0^{\pi} f(x,y)\,dx\,dy$$



$$= \int_0^{\pi/2}\int_0^{\pi} \sin(x+y)\,dx\,dy$$

$$\int_0^{\pi/2} -\cos(x+y)\Big]_0^{\pi} dy$$

$$= \int_0^{\pi/2} \left(-\cos(\pi+y) + \cos(0+y)\right) dy$$

$$= -\sin(\pi+y) + \sin y\Big]_0^{\pi/2}$$

$$= -\sin\left(\pi+\pi/2\right) + \sin\pi/2 - 0$$

$$= -\sin\left(\frac{3\pi}{2}\right) + \sin\pi/2$$

$$= -(-1) + 1 = 2$$

$$\text{Area of Rectanle} = \pi \times \pi/2 = \pi^2/2$$

$$\text{Average value} = \frac{1}{\pi^2/2}(2)$$

$$= 4/\pi^2$$

(19): Finding centre of mass,

$\delta = 3$, bounded by lines $x=0$, $y=x$ & parabola $y = 2-x^2$ in 1st Quadrant ①

$$\int_0^2\int_y^{\sqrt{2-y}} 3\,dx\,dy \quad \text{in 1st Quad}$$

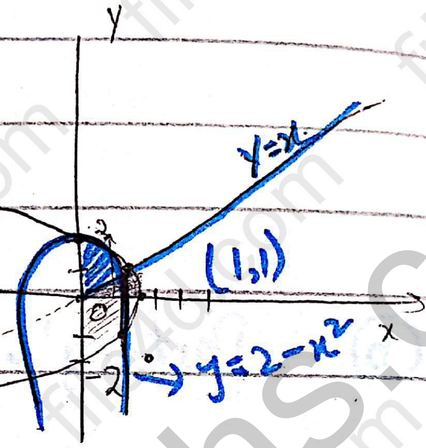
put $y=-x$ in ①

$\Rightarrow -x+x^2=2$

$\Rightarrow x(x+1)=2$

$x=2$, $\boxed{x=1}$

not Possible



OR,

$$M = \int_0^1\int_x^{2-x^2} 3\,dy\,dx \quad \left(= 3\int_0^1 \left[\frac{x^2}{2}\right]_x^{2-x^2} dy\right.$$

Limit of x

$$= 3\int_0^1 \left(\frac{(2-x^2)^2}{2} - \frac{x}{2}\right) dy$$

$$= 3\int_0^1 \left[\frac{4+x^4-4x^2}{2} - \frac{x}{2}\right] dy$$

$$= 3\int_0^1 \left[\frac{4x}{2} + \frac{x^5}{10} - \frac{4x^3}{6} - \frac{x^2}{4}\right]_0^1 dy$$

$$= 3 \left.\right)$$

$x = 2-x^2$

$x+x^2-2=0$

$x^2+2x-x-2=0$

$x(x+2)-1(x+2)$

$x=-2, x=1$

neglect

$$M = 3\int_0^1 (2-x^2-x)\,dx$$

$$M = 3\left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_0^1$$

$$M = 3\left[2 - \frac{1}{3} - \frac{1}{2}\right]$$

$$M = 3\left[\frac{12-2-3}{6}\right]$$

$$\boxed{M = \frac{7}{2}}$$

$$M_x = 3\int_0^1\int_x^{2-x^2} y\,dy\,dx$$

$$= 3\int_0^1 \left[\frac{y^2}{2}\right]_x^{2-x^2} dx$$

$$= \frac{3}{\cancel{2}}\int_0^1 \left[\frac{(2-x^2)^2}{2} - \frac{x}{2}\right] dx$$

$$= \frac{3}{\cancel{2}}\int_0^1 \left[\frac{4+x^4-4x^2}{2} - \frac{x^2}{2}\right] dx$$

$$= \frac{3}{\cancel{2}}\left[\frac{4x}{2} + \frac{x^5}{10} - \frac{4x^3}{6} - \frac{x^3}{3\times 2}\right]_0^1$$

$$= \frac{3}{\cancel{2}}\left[\frac{4}{2} + \frac{1}{10} - \frac{4}{36} - \frac{1}{6}\right]$$

$$= \frac{3}{\cancel{2}}\left[2 + \frac{1}{10} - \frac{2}{3} - \frac{1}{4}\right]$$

$$\boxed{M_x = 19/5}$$

$$M_y = 3\int_0^1\int_x^{2-x^2} x\,dy\,dx$$

$$= 3\int_0^1 \left[xy\right]_x^{2-x^2} dx$$

$$= 3\int_0^1 \left[x(2-x^2) - x(x)\right] dx$$

$$= 3\int_0^1 \left[2x - x^3 - x^2\right] dx$$

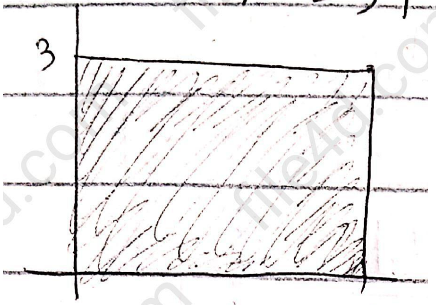
$$= 3\left[\frac{2x^2}{\cancel{2}} - \frac{x^4}{4} - \frac{x^3}{3}\right]_0^1$$

$$= 3\left[1 - \frac{1}{4} - \frac{1}{3}\right] = \cancel{3}\left[\frac{12-3-4}{4\times\cancel{12}}\right] = \frac{5}{4}$$

$\boxed{M_y = 5/4}$

20: finding moments & radii of gyration

$x=3,\ y=3$



$$M = \int_0^3\int_0^3 3\,dx\,dy$$

$$M = \int_0^3 \Big[3x\Big]_0^3 dy$$

$$M = \int_0^3 (3)(3)\,dy$$

$$M = 9y\Big]_0^3 = 9(3) = 27$$

$$\boxed{M=27}$$

(a) I_x :

$$= \int_0^3\int_0^3 y^2\,\delta(x,y)\,dx\,dy$$

$$= 3\int_0^3\int_0^3 y^2\,dx\,dy$$

$$= 3\int_0^3 \Big[y^2 x\Big]_0^3 dy$$

$$= 3\int_0^3 y^2(3)\,dy$$

$$= \Big[9\cdot\frac{y^3}{3}\Big]_0^3 = 9\frac{(3)^3}{3} = 9(3)^2 = 81$$

$$\boxed{I_x = 81}$$

(b) I_y: $= \int_0^3\int_0^3 3x^2\,dx\,dy$

$$= \int_0^3 \left[\frac{\cancel{3}x^3}{\cancel{3}}\right]_0^3 dy$$

$$= \int_0^3 [x^3]_0^3\,dy = \int_0^3 (3^3 - 0)\,dy = \int_0^3 27\,dy$$

$$= 27y\Big]_0^3$$

$$= 27\times 3 = 81$$

$$\boxed{I_y = 81}$$

$$I_o = I_x + I_y = 81 + 81 = 162$$

Radii of guyration:

$\Rightarrow$ $R_x = \sqrt{I_x/M} = \sqrt{81/27} = \sqrt{3}$

$\Rightarrow$ $R_y = \sqrt{I_y/M} = \sqrt{\frac{81}{27}} = \sqrt{3}$

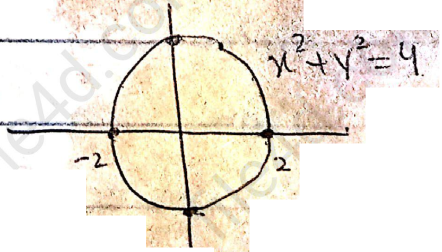
(27): Finding moment of Interia about x-axis if $\delta = 1$ bounded by circle $x^2+y^2=4$, Also find I_y.

$I_x = ?$

$$I_x = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y^2\,dy\,dx$$

$y^2 = 4-x^2$

$y = \pm\sqrt{4-x^2}$



$$I_x = \int_{-2}^{2}\left[\frac{y^3}{3}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left(\frac{\left((4-x^2)^{1/2}\right)^3}{3} + \frac{\left((4-x^2)^{1/2}\right)^{3/2}}{3} \right) dx$$

$$= \int_{-2}^{2} \left(\frac{(4-x^2)^{3/2}}{3} + \frac{(4-x^2)^{3/2}}{3} \right) dx$$

$$= \frac{2}{3} \int_{-2}^{2} (4-x^2)^{3/2} dx$$

$$= \frac{2}{3} \int_{-2}^{2} (4-x^2)^{3/2} dx$$

let $x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$

if $x \to 2, \quad \theta = +\pi/2$

$x \to -2, \quad \theta = -\pi/2$

$$= \frac{2}{3} \int_{-\pi/2}^{\pi/2} (4 - 4\sin^2\theta)^{3/2} \, 2\cos\theta \, d\theta$$

$$= \frac{2}{3} \int_{-\pi/2}^{\pi/2} 4^{3/2} (1-\sin^2\theta)^{3/2} \, 2\cos\theta \, d\theta$$

Using Identity

$\cos 2\theta = 2\cos^2\theta - 1$

$\frac{\cos 2\theta + 1}{2} = \cos^2\theta$

$$= \frac{2 \times 8 (2^2)^{3/2}}{3} \int_{-\pi/2}^{\pi/2} (\cos^2\theta)^{3/2} \, 2\cos\theta \, d\theta$$

$$= \frac{16 \times 2}{3} \int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta$$

$$= \frac{16 \times 2}{3} \int_{-\pi/2}^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right)^2 d\theta$$

$\cos^2 2\theta = \frac{1+\cos 4\theta}{2}$

$$= \frac{16\times 2}{3}\int_{-\pi/2}^{\pi/2}\left(1+\frac{\cos^2 2\theta + 2\cos 2\theta}{4}\right)d\theta$$

$$= \frac{8}{3}\int_{-\pi/2}^{\pi/2}\left[1+\frac{1+\cos 4\theta}{2}+2\cos 2\theta\right]d\theta$$

$$= \frac{8}{3}\left[\theta+\frac{\theta}{2}+\frac{1}{2}\left(\frac{\sin 4\theta}{2}\right)+\frac{2\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}d\theta$$

$$= \frac{8}{3}\left(\left[\frac{\pi}{2}+\frac{\pi/2}{2}+0+0\right]-\left[\frac{-\pi}{2}+\frac{(-\pi/2)}{2}+0\right]\right)$$

$$= \frac{8}{3}\left[\frac{\pi}{2}+\frac{\pi}{4}+\frac{\pi}{2}+\frac{\pi}{4}\right] = \frac{8}{3}\left(\frac{2\pi+\pi+\pi+2\pi}{4}\right)$$

$$I_x = \frac{8}{3}\left[\frac{6\pi}{4}\right] \qquad \boxed{I_x = 4\pi}$$

By Symmetric: $I_x = I_y$:

$$\boxed{I_y = 4\pi}$$

$$I_0 = I_x + I_y$$

$$I_0 = 4\pi + 4\pi$$

$$\boxed{I_0 = 8\pi}$$

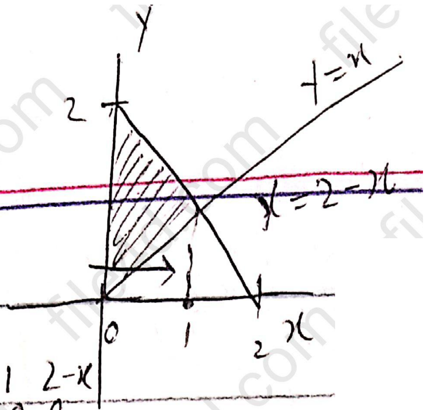
(33) Triangular region bounded by y-axis & line $y=x$, $y=2-x$. Find centre of mass if $\delta(x,y) = 6x+3y+3$

$y = 2-x$, $y = x$

$x = 2-x$

$2x = 2$

$\boxed{x=1}$



$$M = \int_0^1 \int_x^{2-x} \delta(x,y)\,dy\,dx$$

$$M = \int_0^1 \int_x^{2-x} (6x+3y+3)\,dy\,dx$$

$$M = \int_0^1 \left[6xy + \frac{3y^2}{2} + 3y\right]_x^{2-x} dx$$

$$M = \int_0^1 \left[6x(2-x) + \frac{3(2-x)^2}{2} + 3(2-x) - \left(6x^2 + \frac{3x^2}{2} \right.\right.$$

$$M = \int_0^1 \left[12x - 6x^2 + \frac{3}{2}(4+x^2-4x) + 6 - 3x - 6x^2 - \frac{3}{2}x^2 - 3x\right] dx$$

$$M = \left[\frac{12x^2}{2} - \frac{6x^3}{3} + \frac{3}{2}\left[4x + \frac{x^3}{3} - \frac{4x^2}{2}\right] + 6x - \frac{3x^2}{2} - \frac{6x^3}{3} - \frac{3x^3}{2\cdot 3} - \frac{3x^2}{2}\right]_0^1$$

$$M = \left[\frac{12}{2} - \frac{6}{3} + \frac{3}{2}\left[4 + \frac{1}{3} - \frac{4}{2}\right] + 6 - \frac{3}{2} - 2 - \frac{1}{2} - \frac{3}{2}\right]$$

$$M = 6 - 2 + 6 + \frac{1}{2} - \frac{3}{2} + 6 - \frac{3}{2} - 2 - \frac{1}{2} - \frac{3}{2}$$

$$M = 6 + 6 - 2 - 2 - 3 - 3 + 6$$

$$M = 6 + 6 - 4$$

$$M = 12 - 4$$

$$\boxed{M = 8}$$

$$\boxed{M = 8}$$

$$M_y = \int_0^1\int_x^{2-x} x\,(6x+3y+3)\,dy\,dx$$

$$M_y = \int_0^1\int_x^{2-x} (6x^2+3xy+3x)\,dy\,dx$$

$$M_y = \int_0^1 \left[6x^2y + 3x\frac{y^2}{2} + 3xy\right]_x^{2-x} dx$$

$$M_y = \int \left\{6x^2(2-x) + \frac{3}{2}x(2-x)^2 + 3(x)(2-x)\right\} - \left\{6x^2(x) + \frac{3x(x)^2}{2} + 3x(x)\right\} dx$$

$$M_y = \int_0^1 \left(12x^2 - 6x^3 + \frac{3x}{2}(2-x)^2 + 6x - 3x^2 - 6x^3 - \frac{3}{2}x^3 - 3x^2\right) dx$$

$$M_y = \int_0^1 \left(12x^2 - 6x^3 + \frac{3x}{2}(4-4x+x^2) + 6x - 3x^2 - 6x^3 - \frac{3}{2}x^3 - 3x^2\right) dx$$

$$M_y = \int_0^1 \left(12x^2 - 6x^3 + 6x - 6x^2 + \cancel{\tfrac{3}{2}x^3} + 6x - 3x^2 - 6x^3 - \cancel{\tfrac{3}{2}x^3} - 3x^2\right) dx$$

$$M_y = \int_0^1 \left[\left(12x^2 + \cancel{\tfrac{6}{2}x^2} - \cancel{6x^2}\right) - 12x^3 + 6x + 6x + 8(2)\right] dx$$

$$M_y = \int_0^1 (-12x^3 + 12x)\,dx$$

$$M_y = -\frac{12x^4}{4} + \frac{12x^2}{2}\Big]_0^1$$

$$M_y = -3x^4 + 6x^2\Big]_0^1$$

$$M_y = -3+6$$

$$\boxed{M_y = 3}$$

$$M_x = \int_0^1\int_x^{2-x} y\,(6x+3y+3)\,dy\,dx$$

$$M_x = \int_0^1 \int_x^{2-x} (6xy + 3y^2 + 3y)\,dy\,dx$$

$$M_x = \int_0^1 \left[\frac{6xy^2}{2} + \frac{3y^3}{3} + \frac{3y^2}{2} \right]_x^{2-x} dx$$

$$M_x = \int_0^1 \left[3xy^2 + y^3 + \frac{3}{2}y^2 \right]_x^{2-x} dx$$

$$M_x = \int_0^1 \left(3x(2-x)^2 + (2-x)^3 + \frac{3}{2}(2-x)^2 \right) - \left(3x^3 + x^3 + \frac{3}{2}x^2 \right) dx$$

$$M_x = \int_0^1 \left(3x(4x + x^3 - 4x^2) + (2-x)^3 + \frac{3}{2}(4 + x^2 - 4x) - 4x^3 - \frac{3}{2}x^2 \right) dx$$

$$M_x = \int_0^1 \left(12x + 3x^3 - 12x^2 + (2-x)^3 + 6 + \cancel{\frac{3}{2}x^2} - 6x - 4x^3 - \cancel{\frac{3}{2}x^2} \right) dx$$

$(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$

$$M_x = \int_0^1 \left(\cancel{12x} + \cancel{3x^3} - 12x^2 + 8 + 6x^2 - \cancel{12x} - \cancel{x^3} + 6 - 6x - 4x^3 \right) dx$$

$$M_x = \int_0^1 \left(14 - 6x - 6x^2 - 2x^3 \right) dx$$

$$M_x = 14x - \frac{6x^2}{2} - \frac{6x^3}{3} - \frac{2x^4}{4} \Big|_0^1$$

formula

$\boxed{(a-b)^3 = a^3 - b^3 - 3ab(a-b)}$

$$M_x = 14 - 3 - 2 - \frac{1}{2}$$

$$M_x = 9 - \frac{1}{2}$$

$$M_x = \frac{18-1}{2} = \frac{17}{2}$$

$$\boxed{M_x = \frac{17}{2}}$$

Centre of mass.

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M.}$$

$$\bar{x} = \frac{3}{8}, \quad \bar{y} = \frac{17}{16}$$

(36) Centre of mass.

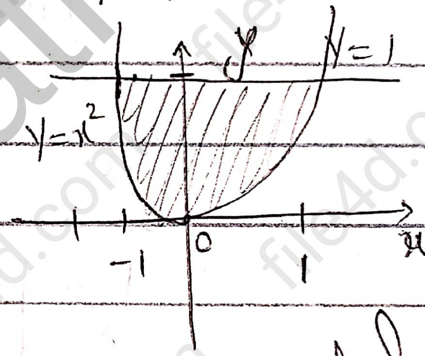
$y = 1$

$y = x^2$ (upper parabola)

$\delta(x, y) = y + 1.$

$\sqrt{x^2} = \sqrt{1}$

$\boxed{x = \pm 1}$



$$M = \int_{-1}^{1} \int_{x^2}^{1} (y+1)\, dy\, dx$$

$$M = \int_{-1}^{1} \left[\frac{y^2}{2} + y \right]_{x^2}^{1} dx$$

Be careful while putting limits

$$M = \int_{-1}^{1} \left(\frac{1}{2} + 1 \right) - \left[\frac{(x^2)^2}{2} + x^2 \right] dx$$

$$M = \int_{-1}^{1} \left(\frac{3}{2} - \frac{x^4}{2} - x^2 \right) dx$$

$$M = \left[\frac{3x}{2} - \frac{x^5}{10} - \frac{x^3}{3} \right]_{-1}^{1}$$

$$M = \left[\frac{3}{2} - \frac{1}{10} - \frac{1}{3} - \left(\frac{3}{2}(-1) - \frac{(-1)^5}{10} - \frac{(-1)^3}{3} \right) \right]$$

$$M = \frac{3}{2} - \frac{1}{10} - \frac{1}{3} + \frac{3}{2} - \frac{1}{10} - \frac{1}{3} = \frac{32}{15}$$

$$\boxed{M = \frac{32}{15}}$$

$$M_x = \int_{-1}^{1}\int_{x^2}^{1} y(y+1)\,dy\,dx$$

$$M_x = \int_{-1}^{1}\left[\frac{y^3}{3}+\frac{y^2}{2}\right]_{x^2}^{1} dx$$

$$M_x = \int_{-1}^{1}\left[\left(\frac{1}{3}+\frac{1}{2}\right)-\left(\frac{(x^2)^3}{3}+\frac{(x^2)^2}{2}\right)\right]dx$$

$$M_x = \int_{-1}^{1}\left(\frac{5}{6}-\frac{x^6}{3}-\frac{x^4}{2}\right)dx$$

$$M_x = \left[\frac{5}{6}x-\frac{x^7}{21}-\frac{x^5}{10}\right]_{-1}^{1}$$

$$M_x = \left[\frac{5}{6}-\frac{1}{21}-\frac{1}{10}\right]-\left[\frac{5}{6}(-1)-\frac{(-1)^7}{21}-\frac{(-1)^5}{10}\right]$$

$$M_x = \frac{5}{6}-\frac{1}{21}-\frac{1}{10}+\frac{5}{6}-\frac{1}{21}-\frac{1}{10}$$

$$\boxed{M_x = \frac{48}{35}}$$

$$M_y = \int_{-1}^{1}\int_{x^2}^{1} x(y+1)\,dy\,dx$$

$$= \int_{-1}^{1}\left[\frac{xy^2}{2}+xy\right]_{x^2}^{1} dx$$

$$= \int_{-1}^{1} \left[\frac{x(x^2)^2}{2} + x(x^2)\right] dx$$

$$= \int_{-1}^{1} \left[\frac{x^5}{2} + x^3\right] dx$$

$$= \left.\frac{x^6}{12} + \frac{x^4}{4}\right]_{-1}^{1}$$

$$= \frac{1}{12} + \frac{1}{4} - \left(\frac{(-1)^6}{12} + \frac{(-1)^4}{4}\right)$$

$$= \frac{1}{12} + \frac{1}{4} - \frac{1}{12} - \frac{1}{4}$$

$$\boxed{M_y = 0}$$

Centre of mass

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$

$$\bar{x} = \frac{0}{M}, \quad \bar{y} = \frac{\cancel{48}^{24}/\cancel{35}_{7}}{\cancel{32}^{15}/\cancel{15}_{3}}$$

$$\boxed{\bar{x} = 0}, \quad \boxed{\bar{y} = \frac{9}{14}}$$

Moment of Inertia: (2nd Moment)

about y-axis.

$$I_y = \iint_{x^2}^{1} x^2 (y+1)\, dy\, dx$$

$$I_y = \int_{-1}^{1} \left[\frac{x^2y^2}{2} + x^2y\right]_{x^2}^{1} dx$$

$$I_y = \int_{-1}^{1} \left(\frac{x^2}{2} + x^2 - \frac{x^6}{2} - x^4 \right) dx$$

$$I_y = \int_{-1}^{1} \left(\frac{3x^2}{2} - \frac{x^6}{2} - x^4 \right) dx$$

$$I_y = \left[\frac{x^3}{6} + \frac{x^3}{3} - \frac{x^7}{14} - \frac{x^5}{5} \right]_{-1}^{1}$$

$$I_y = \frac{1}{6} + \frac{1}{3} - \frac{1}{14} - \frac{1}{5} - \left(\frac{-1}{6} - \frac{1}{3} + \frac{1}{14} + \frac{1}{5} \right)$$

$$I_y = \frac{1}{6} + \frac{1}{3} - \frac{1}{14} - \frac{1}{5} + \frac{1}{6} + \frac{1}{3} - \frac{1}{14} - \frac{1}{5}$$

$$I_y = 2\left(\frac{1}{6} + \frac{1}{3} - \frac{1}{14} - \frac{1}{5} \right)$$

$$\boxed{I_y = \frac{16}{35}}$$

Radii of gyration:

$$R_y = \sqrt{\frac{I_y}{M}}$$

$$R_y = \sqrt{\frac{16/35}{32/15}}$$

$$\boxed{R_y = \sqrt{\frac{3}{14}}}$$

Ex 15.3 :

$$\iint_R f(x,y)\,dx\,dy$$

$$\iint_R f(r,\theta)\, r\,dr\,d\theta$$

But there is only one way to find limit of Integration (No reverse order)

constant limit of θ $\leftarrow \iint_R f(r,\theta)\, r\,dr\,d\theta$

limit of r depend on θ

⇒ Change cartisean Integral into an equivalent Integral. then evaluate polar integrals.

① $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\,dx$

$x=1$, $y=\sqrt{1-x^2} \Rightarrow x^2+y^2=1$

$x=-1$, $y=0$

$x^2+y^2=r^2$

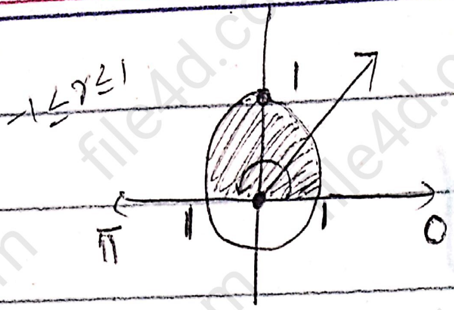
$1=r^2 \Rightarrow r=\pm 1 \Rightarrow \boxed{r=1}$

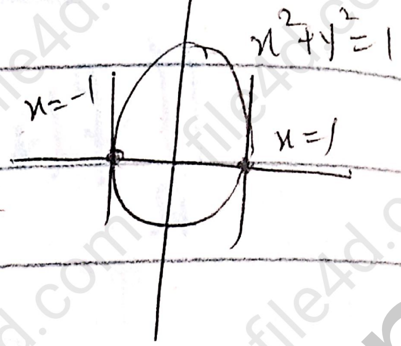
$\theta = \tan^{-1}\left(\frac{0}{x}\right) = \tan^{-1}(0) = 0,\pi$

$\boxed{\theta = 0,\pi}$

In Polar form

In Cartisian form





$$\text{Area} = \int_0^{\pi}\int_0^{1} r\,dr\,d\theta$$

$$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^1 d\theta$$

$$= \frac{1}{2}\left[\theta\right]_0^{\pi} = \frac{\pi}{2}$$

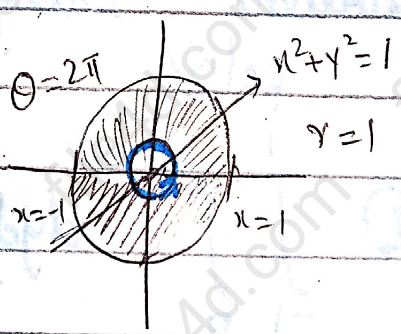
② $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx$

$y = \sqrt{1-x^2} \Rightarrow y^2+x^2=1 \Rightarrow r^2=1$

$x=1,$ $\quad y=-\sqrt{1-x^2} \Rightarrow y^2+x^2=1.$

$x=-1$

$$\text{Area} = \int_0^{2\pi}\int_0^{1} r\,dr\,d\theta$$



$$= \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^1 d\theta$$

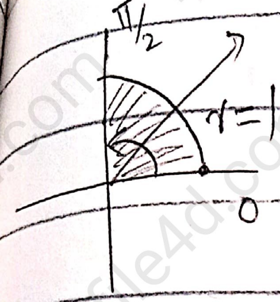
$$= \frac{1}{2}\left[\theta\right]_0^{2\pi} = \frac{2\pi}{2} = \pi$$

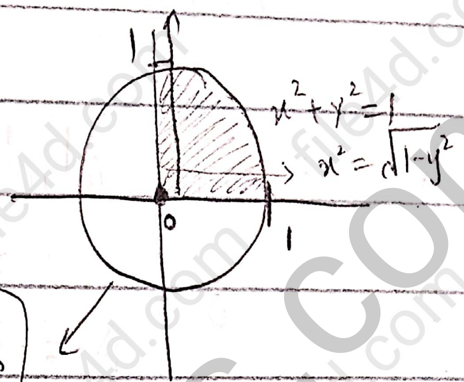
③ $\int_0^1\int_0^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy$

$$\Rightarrow \quad x^2+y^2=r^2$$

$y=1$, $\quad x=\sqrt{1-y^2} \Rightarrow x^2+y^2=1 \Rightarrow r^2=1$

$y=0$, $\quad x=0$





In polar form

In Cartesian form.

$$\text{Area} = \int_0^{\pi/2}\int_0^{1} r^2 (r\,dr\,d\theta)$$

$$A = \int_0^{\pi/2}\int_0^{1} r^3\,dr\,d\theta$$

$$A = \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$A = \frac{1}{4}\left[\theta\right]_0^{\pi/2}$$

$$A = \frac{1}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{8}$$

$$\boxed{A = \frac{\pi}{8}}$$

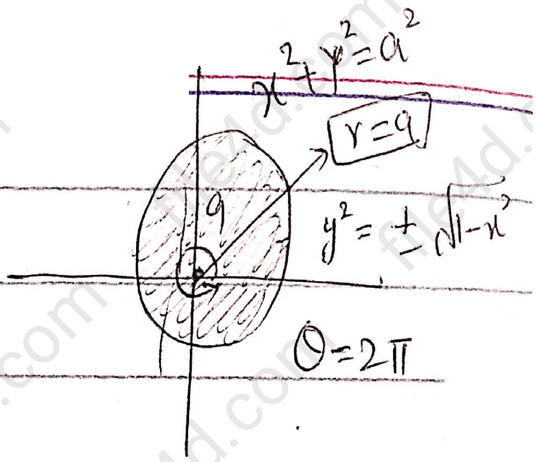
(5) $$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$$

$x=a$, $\quad y=\sqrt{a^2-x^2} \Rightarrow y^2+x^2=a^2 \Rightarrow r^2=a^2$

$x=-a$, $\quad \Rightarrow \boxed{r=a}$

$0 \le \theta \le 2\pi$

$r = a$



$$A = \int_0^{2\pi}\int_0^{a} r\,dr\,d\theta$$

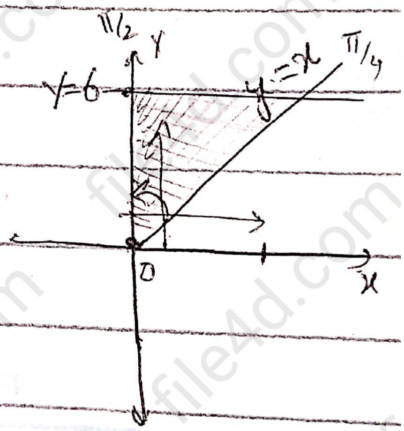
$$A = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^{a} d\theta$$

$$A = \int_0^{2\pi}\frac{a^2}{2}\,d\theta$$

$$A = \frac{a^2}{2}\left[\theta\right]_0^{2\pi} \Rightarrow \frac{a^2}{\cancel{2}}(\cancel{2}\pi)$$

$$\boxed{A = \pi a^2}$$

⑦. $\displaystyle\int_0^6\int_0^y x\,dx\,dy$



$$\left[\begin{matrix} x=y, & y=6, \\ x=0, & y=0, \end{matrix}\right]$$

But In Polar form,

$x = r\cos\theta$

$$\text{Area} = \int_{\pi/4}^{\pi/2}\int r\cos\theta\, r\,dr\,d\theta$$

$y=6$

$r\sin\theta=6$

$r=\text{cosec}\,6$

$$A = \int_{\pi/4}^{\pi/2}\int_0^{6\,\text{cosec}\,\theta} r^2\cos\theta\,dr\,d\theta$$

$$A = \int_{\pi/4}^{\pi/2}\left[\cos\theta\,\frac{r^3}{3}\right]_0^{6\,\text{cosec}\,\theta} d\theta$$

$$A = \int_{\pi/4}^{\pi/2} \left[\cos\theta \frac{(6\,\mathrm{cosec}\,\theta)^3}{3}\right] d\theta$$

$$A = 72\int_{\pi/4}^{\pi/2} (\cos\theta \cdot \mathrm{cosec}^3\theta)\, d\theta$$

$$A = 72\int_{\pi/4}^{\pi/2} \left(\frac{\cos\theta}{\sin\theta}\cdot \mathrm{cosec}^2\theta\right) d\theta$$

$$A = 72\int_{\pi/4}^{\pi/2} (\cot\theta\cdot\mathrm{cosec}^2\theta)\, d\theta$$

$\frac{d}{dx}(\cot\theta) = \mathrm{cosec}^2\theta$

$$A = 72 \; \cancel{x} \left[-\frac{(\cot\theta)^2}{2}\right]_{\pi/4}^{\pi/2}$$

$$A = -\frac{\cancel{72}^{\,36}}{\cancel{2}}\left[\left(\cot\frac{\pi}{2}\right)^2 + \left(\cot\left(\frac{\pi}{4}\right)\right)^2\right]$$

$$A = -36\left[0 - (1)\right)$$

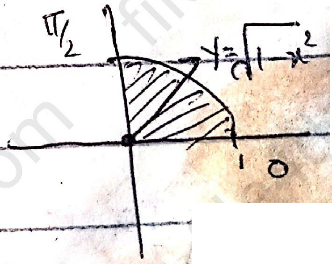
$$\boxed{A = 36}$$

(12) : $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\,dy\,dx$

$\Rightarrow x^2+y^2=r^2$

$x=0, \quad y=\sqrt{1-x^2} \Rightarrow y^2+x^2=1 \Rightarrow r^2=1 \Rightarrow \boxed{r=1}$

$x=1, \quad y=0$



$$A = \int_0^{\pi/2}\int_0^1 \underbrace{e^{-r^2}}_{II}\,\underbrace{r\,dr\,d\theta}_{I}$$

$$\cancel{A = \int_0^{\pi/2}\left(\frac{re^{-r^2}}{-2r} - \int(1)\frac{e^{-r^2}}{-2r}\,dr\right)_0^1 d\theta}$$

$$A = \frac{1}{-2}\int_0^{\pi/2}\int_0^1 \left(e^{-r^2}(-2r)\right)\cancel{r}\,dr\,d\theta$$

$\because$ Becoz derivative exists

$$A = -\frac{1}{2}\int_0^{\pi/2}\left[(e^{-r^2} r) d\theta\right]_0^1$$

$$A = -\frac{1}{2}\int_0^{\pi/2}\left(e^{-1^2}_{\cdot 1} - e^{-0^2}_{\cdot 0}\right) d\theta$$

$$A = -\frac{1}{2}\int_0^{\pi/2}\left(e^{-1} - 1\right) d\theta$$

$$A = -\frac{1}{2}\int_0^{\pi/2}\left(\frac{1}{e} - 1\right) d\theta$$

$$A = -\frac{1}{2}\left[+\frac{1}{e}(\theta) - \theta\right]_0^{\pi/2}$$

$$A = \left[\frac{-1\theta}{2e} + \frac{\theta}{2}\right]_0^{\pi/2}$$

$$A = \frac{-\pi/2}{2e} + \frac{\pi/2}{2}$$

$$A = \frac{\pi}{2}\left(\frac{-1+e}{2e}\right)$$

$$A = \pi\frac{(+e+1)}{4e}$$

$$\boxed{A = \frac{\pi(e-1)}{4e}}$$

$\theta\left(\frac{-1}{2e} + \frac{1}{2}\right)$

$\theta\left(\frac{-1+e}{2e}\right)$

$\theta\left(\frac{-1+e}{2e}\right)$

$\frac{\pi}{2}\frac{(e-1)}{2e}$

$\frac{\pi(e-1)}{4e}$

(18). Area of region that inside the Cardiode $r = 1+\cos\theta$, outside the circle $r=1$

$$\text{Area} = \int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta} r\,dr\,d\theta$$

$r = 1+\cos\theta$

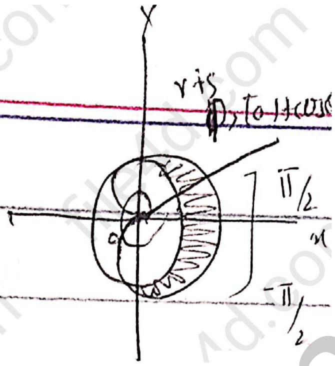
$1 = 1+\cos\theta$

$\theta = \cos^{-1}(0)$

$-\pi/2, \pi/2$

Due to Symmetry.

$$A = 2\int_0^{\pi/2}\int_0^{1+\cos\theta} r\,dr\,d\theta$$



$$A = 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{1+\cos\theta} d\theta$$

$$A = \int_0^{\pi/2}(1+\cos\theta)^2\,d\theta$$

$$A = \int_0^{\pi/2}\left(1+\cos^2\theta + 2\cos\theta\right)d\theta$$

$$A = \int_0^{\pi/2}\left(1 + \frac{1+\cos 2\theta}{2} + 2\cos\theta\right)d\theta$$

$$A = \left[\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{2} + 2\cos\theta\right]_0^{\pi/2}$$

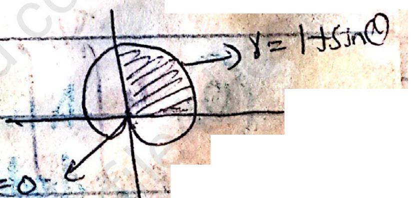
$$A = \frac{\pi}{2} + \frac{\pi}{4} + 0 + 0 + 2\sin\left(\frac{\pi}{2}\right)$$

$$A = \frac{\pi}{4} + 2(1)$$

$$\boxed{A = \frac{\pi+8}{4}}$$

(21). Find area of region cut from 1st Quad by $r = 1+\sin\theta$.

$$\text{Area} = \int_0^{\pi/2}\int_0^{1+\sin\theta} r\,dr\,d\theta$$



$$A = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{1+\sin\theta} d\theta$$

$$A = \frac{1}{2}\int_0^{\pi/2}(1+\sin\theta)^2\,d\theta$$

$$A = \frac{1}{2}\int_0^{\pi/2} (1+\sin^2\theta + 2\sin\theta)\,d\theta$$

$\cos 2\theta = 1-2\sin^2\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$A = \frac{1}{2}\int_0^{\pi/2} \left(1+ \frac{1-\cos 2\theta}{2} + 2\sin\theta\right) d\theta$$

$$A = \frac{1}{2}\left[\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{2} - 2\cos\theta\right]_0^{\pi/2}$$

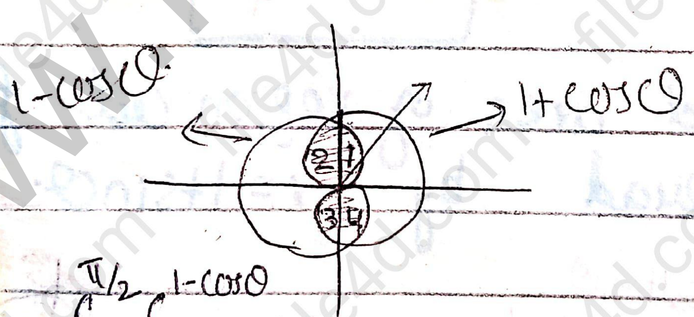
$$A = \frac{1}{2}\left(\left[\frac{\pi}{2} + \frac{\pi/2}{2} - 0 - 0\right] - \left[0 - 2\cos 0\right]\right)$$

$$A = \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{4} + 2\right)$$

$$A = \frac{\pi}{4} + \frac{\pi}{8} + \frac{\cancel{2}}{\cancel{2}}$$

$$\boxed{A = \frac{3\pi}{8} + 1}$$

(22) Area come to Interior of two Cardiode $r = 1+\cos\theta,\ r = 1-\cos\theta$



$$A = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta$$

$$A = 4\int_0^{\pi/2} \left[\frac{(1-\cos\theta)^2}{2}\right] d\theta$$

$$A = \frac{\cancel{4}^{2}}{\cancel{2}}\int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$A = 2\int_0^{\pi}\left[1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right] d\theta$$

$$\cancel{A = 2\left[\theta - 2(+\sin\theta) + \frac{\theta}{2} + \frac{(-\sin 2\theta)}{2\times 2}\right]_0^{\pi/2}}$$

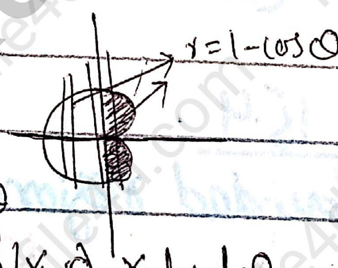
$$A = 2\left(\frac{\pi}{2} \mp 2\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{4} - \frac{\sin 2(\pi/2)}{4}\right)$$

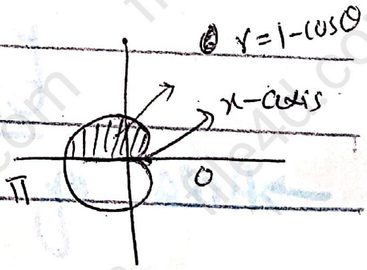
$$A = 2\left[+\frac{\pi}{2} \mp 2 + \frac{\pi}{4}\right]$$

$$A = 2\left[\frac{\pi + 3\pi}{4} - 2\right]$$

$$\boxed{A = \frac{3\pi}{2} - 4}$$

(23): Find 1st moment about x-axis
$M_x = ?$ $\delta(x,y) = 3$,
bounded below by x-axis & cardiode
$r = 1-\cos\theta$ (above)





$$M_x = \int_0^{\pi}\int_0^{1-\cos\theta} y\,\delta(x,y)\, r\,dr\,d\theta$$

And we know $y = r\sin\theta$

$$M_x = \int_0^{\pi}\int_0^{1-\cos\theta} (r\sin\theta)\, 3r\,dr\,d\theta$$

$$M_x = 3\int_0^{\pi}\int_0^{1-\cos\theta} r^2\sin\theta\,dr\,d\theta$$

$$M_x = 3\int_0^{\pi}\left[\sin\theta\,\frac{r^3}{3}\right]_0^{1-\cos\theta} d\theta$$

$$M_x = \int_0^{\pi} \left(\sin\theta\,(1-\cos\theta)^3\right) d\theta$$

$$M_x = \int_0^{\pi} \sin\theta \left(1-\cos\theta\right)^3 d\theta$$

$$M_x = \left.\left(\frac{(1+\cos\theta)^4}{4}\right)\right]_0^{\pi}$$

$$M_x = \left[\left(\frac{(1+\cos\pi)^4}{4}\right)\right] - \left[\left(\frac{(1-\cos 0)^4}{4}\right)\right]$$

$$M_x = \frac{(1+1)^4}{4} - 0$$

$$M_x = \frac{2^4}{4} = \frac{\cancel{16}\,4}{\cancel{4}}$$

$$\boxed{M_x = 4}$$

Ex 15.4:

→ Area of bounded region.

$$A = \iint_R c\, dA$$

→ Volume of Region in plane

$$V = \iint_R f(x,y)\,dx\,dy$$

→ In Space,

$$V = \iiint_R f(x,y,z)\,dx\,dy\,dz$$

It must be constant

or

$$V = \iiint_R dV$$

Average value of F over D $= \frac{1}{\text{Volume of } D} \iiint_D F\,dV$

ex: Find Average value.

$$\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz$$

Bounded by $x=2, y=2, z=2$ in 1st Quad of space. $(0,0,0)$

Volume of cube $= x^3 = 2^3 = 8$

$$V = \int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz$$

$$V = \int_0^2\int_0^2 \left[yz\left(\frac{x^2}{2}\right)\right]_0^2 dy\,dz$$

$$V = \int_0^2\int_0^2 \left[yz\left(\frac{2^2}{2}\right)\right] dy\,dz$$

$$V = 2\int_0^2 \left.\frac{zy^2}{2}\right|_0^2 dz$$

$$V = \int_0^2 z(2)^2\,dz$$

$$V = 4\int_0^2 z\,dz \Rightarrow 4\frac{z^2}{2}\Big]_0^2$$

$$V = 4\frac{(2)^2}{2} = 8$$

Average value $= \frac{1}{8}(8) = 1$

Ex 15.4.

② $x=1, y=2, z=3$ in 1st octant.

Six Iterated triple Integrals

ⓐ $\int_0^3\int_0^2\int_0^1 dx\,dy\,dz$ ⓑ $\int_0^2\int_0^3\int_0^1 dx\,dz\,dy$

ⓒ $\int_0^3\int_0^1\int_0^2 dy\,dx\,dz$ ⓓ $\int_0^1\int_0^3\int_0^2 dy\,dz\,dx$

ⓔ $\int_0^1\int_0^2\int_0^3 dz\,dy\,dx$ ⓕ $\int_0^2\int_0^1\int_0^3 dz\,dx\,dy$

Evaluating:

$$V = \int_0^3\int_0^2\int_0^1 dx\,dy\,dz$$

$$V = \int_0^3\int_0^2 [x]_0^1\,dy\,dz$$

$$V = \int_0^3\int_0^2 1\,dy\,dz$$

$$V = \int_0^3 [y]_0^2\,dz$$

$$V = \int_0^3 2\,dz$$

$$V = 2z\Big|_0^3 = 2(3) = 6$$

$$\boxed{V=6}$$

⑤ $z = 8 - x^2 - y^2 \rightarrow ①$

$z = x^2 + y^2 \rightarrow ②$

ⓐ Limits of x in terms of y, z.

$x^2 = 8 - z - y^2 \implies x = \pm\sqrt{8-z-y^2}$

Also, from ② $x = \pm\sqrt{z-y^2}$ or $x = \pm\sqrt{4-y^2}$

ⓑ Limit of y in terms of x, z

from ①, $y = \pm\sqrt{8-z-x^2}$

from ②, $y = \pm\sqrt{z-x^2} \longrightarrow \pm\sqrt{4-x^2}$

ⓒ limit of z in terms of x, y

$$z = 8-x^2-y^2$$

$$z = x^2+y^2.$$

Comparing ① & ②.

$$x^2+y^2 = 8-x^2-y^2 \implies \frac{2y^2}{2}+\frac{2x^2}{2}-\frac{8}{2}=0$$

$$y^2+x^2-4=0$$

$$x^2+y^2=4. \implies \boxed{z=4}$$

from ① put $(x,y)=(0,0)$ so,

$\boxed{z=8}$

ⓓ Limit of x in term of z,

$\boxed{\text{Put } y=0}$ $\pm\sqrt{z} = x$ and $x^2 = 8-z$

$x = \pm\sqrt{8-z}$

ⓔ Limit of y in Terms of z.

$\boxed{\text{Put } x=0}$ $y = \pm\sqrt{z}$, $y = \pm\sqrt{8-z}$

ⓖ $$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx$$

for x put $y=0$

$x^2+0^2=4$

$\boxed{x=\pm 2}$

Using ②

y in Terms of x.

$x^2+y^2=4$

$y = \pm\sqrt{4-x^2}$

(b) $$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dx\,dy$$

$x^2+y^2=4$

$\Rightarrow x = \pm\sqrt{4-y^2}$

Evaluating

$$V = 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} (8-x^2-y^2)-(x^2+y^2)\,dx\,dy$$

$$V = 4\int_{0}^{2}\int_{0}^{\sqrt{4-y^2}} (8-2x^2-2y^2)\,dx\,dy$$

$$V = 4\int_{0}^{2}\left[8x-\frac{2x^3}{3}-2y^2x\right]_0^{\sqrt{4-y^2}} dy$$

$$V = 4\int_{0}^{2}\left(8\sqrt{4-y^2}-\frac{2}{3}(\sqrt{4-y^2})^3-2y^2\sqrt{4-y^2}\right)dy$$

Too, difficult, So we change into polar form.

$$V = 4\int_{0}^{2}\int_{0}^{\sqrt{4-y^2}} 8-2(x^2+y^2)\,dx\,dy$$

$$V = 4\int_{0}^{\pi/2}\int_{0}^{2}(8-2r^2)\,r\,dr\,d\theta$$

$$V = 8\int_{0}^{\pi/2}\int_{0}^{2}(4-r^2)\,r\,dr\,d\theta$$

only evaluating

$$V = \frac{\cancel{4}\cancel{8}}{-2}\int_{0}^{\pi/2}\int_{0}^{2}(4-r^2)(-2r)\,dr\,d\theta$$

$$V = -4\int_{0}^{\pi/2}\left[\frac{(4-r^2)^2}{2}\right]_0^2 d\theta$$

$$V = -4\int_{0}^{\pi/2}\left[0-\left(\frac{(4-0)^2}{2}\right)\right]d\theta$$

$$V = 4\int_{0}^{\pi/2} 8\,d\theta$$

$$V = 32\int_0^{\pi/2} d\theta$$

$$V = 32\left(\frac{\pi}{2}\right) = 16\pi$$

$$\boxed{V = 16\pi}$$

(c) $\displaystyle\int_4^8 \int_{-\sqrt{8-z}}^{+\sqrt{8-z}} \int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\,dy\,dz \longrightarrow R_1$

$4 \le z \le 8$

$\displaystyle\int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dy\,dz \longrightarrow R_2$

$0 \le z \le 4$

So,

$$\int_4^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\,dy\,dz + \int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dy\,dz$$

(d) $\displaystyle\int_{-2}^{2} \int_{y^2}^{4} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dz\,dy + \int_{-2}^{2} \int_{4}^{8-y^2} \int_{-\sqrt{8-y^2-z}}^{\sqrt{8-z-y^2}} dx\,dz\,dy$

(e) $\displaystyle\int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-x^2}}^{+\sqrt{z-x^2}} dy\,dx\,dz + \int_4^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-x^2-z}}^{\sqrt{8-x^2-z}} dy\,dx\,dz$

(b): $z = x^2+y^2$, $z = 2y$ $\Rightarrow$ $2y > x^2+y^2$

(a) $$\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_{x^2+y^2}^{2y} dz\,dx\,dy$$

(b) $$\int_{-1}^{1} \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} \int_{x^2+y^2}^{2y} dz\,dy\,dx$$

$z = x^2+y^2 \longrightarrow (1)$

$z = 2y \longrightarrow (2)$

Comp (1) & (2) $\quad x^2+y^2 = 2y$

$x^2+y^2-2y = 0$

for limit of x in Terms of y $\Rightarrow$ $x^2+(y-1)^2 = 1$

$x^2 = 1-(y-1)^2$

$x^2 = \cancel{1}+y^2+2y\cancel{-1}$

$\boxed{x = \pm\sqrt{2y-y^2}}$

when $y=0$

$\boxed{x = 0}$

put $x=0$

$0^2+(y-1)^2 = 1$

$\sqrt{(y-1)^2} = \sqrt{1}$

$y-1 = \pm 1$

$\boxed{y = 0, 2}$

at $y=0$, $x=0$

at $y=2$, $x=0$

at $y=1$, $\boxed{x = \pm 1}$

for limit of y in terms of x

$x^2+(y-1)^2 = 1$

$\sqrt{(y-1)^2} = \sqrt{1-x^2}$

$y = \pm\sqrt{1-x^2}+1$

or

$y = 1 \pm \sqrt{1-x^2}$

⑦ : $\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx$

$= \int_0^1\int_0^1 \left[x^2z + y^2z + \frac{z^3}{3}\right]_0^1 dy\,dx$

$= \int_0^1\int_0^1 \left[x^2+y^2+\frac{1}{3}\right] dy\,dx$

$= \int_0^1 \left[x^2y + \frac{y^3}{3} + \frac{1}{3}y\right]_0^1 dx.$

$= \int_0^1 \left(x^2+\frac{1}{3}+\frac{1}{3}\right) dx$

$= \int_0^1 \left(x^2+\frac{2}{3}\right) dx \Rightarrow \left[\frac{x^3}{3}+\frac{2x}{3}\right]_0^1$

$= \frac{1}{3}+\frac{2}{3} = \frac{3}{3} = 1$

⑧ : $\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy$

$= \int_0^{\sqrt{2}}\int_0^{3y} \left[z\right]_{x^2+y^2}^{8-x^2-y^2} dx\,dy$

$= \int_0^{\sqrt{2}}\int_0^{3y} (8-x^2-y^2-x^2-3y^2)\,dx\,dy$

$= \int_0^{\sqrt{2}}\int_0^{3y} (8-2x^2-4y^2)\,dx\,dy$

$= \int_0^{\sqrt{2}} \left[8x - \frac{2x^3}{3} - 4y^2x\right]_0^{3y} dy$

$= \int_0^{\sqrt{2}} \left[8(3y) - \frac{2}{3}(3y)^3 - 4y^2(3y)\right] dy$

$= \int_0^{\sqrt{2}} (24y - 18y^3 - 12y^3)\,dy$

$$= \int_0^{\sqrt{2}} (24y - 30y^3)\, dy$$

$$= \left[\frac{24y^2}{2} - \frac{30y^4}{4}\right]_0^{\sqrt{2}}$$

$$= 12(\sqrt{2})^2 - \frac{30}{4}(\sqrt{2})^4$$

$$= 24 - \frac{30 \times 4}{4} =$$

$$= 24 - 30 = -6$$

10: $\int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz\,dy\,dx.$

$$= \int_0^1 \int_0^{3-3x} (3 - 3x - y)\, dy\, dx$$

$$= \int_0^1 \left[3y - 3xy - \frac{y^2}{2}\right]_0^{3-3x} dx$$

$$= \int_0^1 \left[3(3-3x) - 3x(3-3x) - \frac{(3-3x)^2}{2}\right] dx$$

$$= \int_0^1 \left[(3-3x)^2 - \frac{(3-3x)^2}{2}\right] dx$$

$$= \frac{1}{2}\int_0^1 (3-3x)^2\, dx.$$

$$= \frac{1}{(-3)2}\int_0^1 (3-3x)^2(-3)\, dx$$

$$= \frac{1}{-6}\left[\frac{(3-3x)^3}{3}\right]_0^1$$

$$= -\frac{1}{6}\left[\frac{(3-3(1))^3}{3} - \frac{(3-0)^3}{3}\right] = +\frac{1}{2\cancel{6}}(\cancel{9}^3) = \frac{3}{2}$$

⑬ $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz\,dy\,dx$

$= \int_0^3 \int_0^{\sqrt{9-x^2}} \underbrace{\sqrt{9-x^2}}_{\text{constant}} \, dy\,dx.$ Be carefull

$= \int_0^3 \left[\sqrt{9-x^2}\, y\right]_0^{\sqrt{9-x^2}} dx$

$= \int_0^3 \left(\sqrt{9-x^2}\right)^2 dx.$

$= \int_0^3 (9-x^2)\,dx$

$= \left[9x - \frac{x^3}{3}\right]_0^3 = 9(3) - \frac{3^3}{3} = 27 - 9 = 18$

㉔. $x+z=1, \quad y+2z=2$

$V = \int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\,dz\,dx$

$V = \int_0^1 \int_0^{1-x} (2-2z)\,dz\,dx.$

$V = \int_0^1 \left[2z - \frac{2z^2}{2}\right]_0^{1-x} dx.$

$V = \int_0^1 \left[2(1-x) - (1-x)^2\right] dx.$

$V = \int_0^1 (2 - 2x - 1 + 2x - x^2)\,dx$

$V = \int_0^1 (1 - x^2)\,dx$

$$= \int_0^1 (1-x^2)\,dx = \left[x-\frac{x^3}{3}\right]_0^1 = 1-\frac{1}{3} = \frac{2}{3}$$

(31) $x+y=4.$

$y^2+4z^2=16$

$$\int_0^4 \int_0^{\sqrt{\frac{16-y^2}{2}}} \int_0^{4-y} dx\,dz\,dy.$$

$$V = \int_0^4 \int_0^{\sqrt{\frac{16-y^2}{2}}} (4-y)\,dz\,dy$$

$$V = \int_0^4 \left[4z - yz\right]_0^{\sqrt{\frac{16-y^2}{2}}} dy$$

$$V = \int_0^4 (4-y)\frac{\sqrt{16-y^2}}{2}\,dy$$

$$\therefore V = \int_0^4 \frac{\cancel{4}\,2}{\cancel{2}}\sqrt{16-y^2}\,dy + \frac{1}{-2}\int_0^4 -2y\,\frac{\sqrt{16-y^2}}{2}\,dy$$

Let $y = 4\sin\theta \Rightarrow dy = 4\cos\theta\,d\theta$

$$V = 2\int_0^{\pi/2} \sqrt{16(1-\sin^2\theta)}\;4\cos\theta\,d\theta + \frac{1}{-2\times 2}\left[\frac{(16-y^2)^{3/2}}{3/2}\right]_0^4$$

$$= 8\int_0^{\pi/2} \cos\theta\cdot 4\cos\theta\,d\theta$$

$$= 32\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta + \frac{1}{-4}\times\frac{2}{3}\left((16-4^2)^{3/2} - (16-0)^{3/2}\right)$$

$$(-64)\quad 32$$

$$= 32\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} + \frac{2}{3\times 2}$$

$$= 32\frac{\left(\frac{\pi}{2}\right)}{2} - \frac{32}{3}$$

$$= \cancel{8}\cancel{32}\left(\frac{\pi}{\cancel{4}}\right) - \frac{32}{3}$$

$$\boxed{V = 8\pi - \frac{32}{3}}$$

Moments in 3 Dimention:

$$I_x = \iiint (y^2 + z^2)\,\delta\,dV$$

$$I_y = \iiint (x^2 + z^2)\,\delta\,dV$$

$$I_z = I_0 = \iiint (x^2 + y^2)\,\delta\,dV$$

Centre of mass.

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}.$$

$$\iiint x\,\delta\,dv, \quad \iiint y\,\delta\,dv, \quad \iiint z\,\delta\,dv.$$

⇒ For two paraboloids $\theta = 0$ to $\pi/2$.

⇒ for ~~two~~ circle and paraboloid

$\theta = 0$ to 2π

find Centre of mass

R: $x^2+y^2 \le 4$. in plane $z=0$

& paraboloid $z=4-x^2-y^2$.

$$M_{xy} = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} z\delta \, dz\, dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[\frac{z^2}{2}\right]_0^{4-x^2-y^2} \delta \, dy\, dx$$

$$= \frac{\delta}{2}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-x^2-y^2)^2 \, dy\, dx$$

In Polar, (0 to 2π

$$= \frac{\delta}{2}\int_{0}^{2\pi}\int_{0}^{2} (4-r^2)^2 \, r\, dr\, d\theta$$

$$= \frac{1}{-2}\times\frac{\delta}{2}\int_{0}^{2\pi}\int_{0}^{2} (4-r^2)^2 (-2r)\, dr\, d\theta$$

$$= \frac{\delta}{-4}\int_{0}^{2\pi}\left[\frac{(4-r^2)^3}{3}\right]_0^2 d\theta$$

$$= \frac{\delta}{-12}\int_{0}^{2\pi}\left((4-2^2)^3-(4-0)^3\right) d\theta$$

$$= \frac{+\delta}{12}\int_{0}^{2\pi} 64\, d\theta$$

$$= \frac{\delta \times \cancel{64}^{16}}{3\,\cancel{12}}\left[\theta\right]_0^{2\pi}$$

$$= \frac{\delta\times 16}{3}(2\pi) = \frac{32\pi\delta}{3}$$

Note:- As circle is in xy plane so, M_{yz} and $M_{xz} = 0$

$\Rightarrow \bar{x} = \bar{y} = 0$

$$\boxed{M_{xy} = \frac{32\pi\delta}{3}}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\frac{32\pi\delta}{3}}{8\pi\delta} = \frac{4}{3}$$

$$M = \int_0^{2\pi}\int_0^2 \delta(4-r^2)r\,dr\,d\theta = \frac{-1}{2}\int_0^{2\pi}\int_0^2 \delta(4-r^2)(-2r)\,dr\,d\theta$$

$$M = -\frac{\delta}{2}\int_0^{2\pi}\left[\frac{(4-r^2)^2}{2}\right]_0^2 d\theta \Rightarrow M = +\frac{\delta}{2}\int_0^{2\pi}\left(\frac{16}{2}\right) = 4\delta(2\pi) = 8\pi\delta$$

Q6: $x^2+4y^2=4$, plane $z=0$

$z = 2-x$

$$M = \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{+\sqrt{\frac{4-x^2}{2}}}\int_0^{2-x} dz\,dy\,dx$$

Due to symmetry,

$$= 4\int_0^2\int_0^{+\sqrt{\frac{4-x^2}{2}}}(2-x)\,dy\,dx$$

$$= 4\int_0^2\left[2y-xy\right]_0^{\sqrt{\frac{4-x^2}{2}}}dx$$

$$= 4\int_0^2\left[2\sqrt{\frac{4-x^2}{2}} - x\left(\sqrt{\frac{4-x^2}{2}}\right)\right]dx$$

$$= 2\cancel{4}\int_0^2 (2-x)\left(\sqrt{\frac{4-x^2}{\cancel{2}}}\right)dx$$

$$= 2\int_0^2 (2-x)\sqrt{4-x^2}\,dx$$

Let $x = 2\sin\theta$ $x \to 0$ then $\theta \to 0$

$dx = 2\cos\theta$ $x \to 2$ then $\theta = \pi/2$

$$= 2\int_0 (2-2\sin\theta)\sqrt{4-4\sin^2\theta}\,(2\cos\theta)\,d\theta$$

$$= 8\int_0^{\pi/2} (2-2\sin\theta)\cos^2\theta\, d\theta$$

$$= 16\int_0^{\pi/2} (1-\sin\theta)(\cos^2\theta)\, d\theta$$

$$= 16\int_0^{\pi/2} (1-\sin\theta)\left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{\cancel{16}^{\,8}}{\cancel{2}}\int_0^{\pi/2} 1 + \cos 2\theta - \sin\theta - \sin\theta\cos 2\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$$= 8\left[\theta + \frac{\sin 2\theta}{2} - \cos\theta - \frac{(\cos^3\theta)}{3}\right]_0^{\pi/2}$$

$$= 8\left[\pi/2 + 0 - 0 - 0\right]$$

$$\boxed{V = 4\pi}$$

OR

$$= 16\int_0^{\pi/2} (\cos^2\theta - \sin\theta\cos^2\theta)\, d\theta$$

$$= 16\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2} + \frac{\sin\theta\cos^3\theta}{3}\right) d\theta$$

$$= 16\left[\frac{\theta}{2} + \frac{\sin 2\theta}{2} + \frac{\cos^3\theta}{2}\right]_0^{\pi/2}$$

$$= 16\left(\frac{\pi/2}{2}\right) = \cancel{16}^{\,4}\times\frac{\pi}{\cancel{4}}$$

$$\boxed{M = 4\pi}$$

$$M_{xy} = 2\int_{-2}^{2}\int_{0}^{\frac{\sqrt{4-x^2}}{2}}\int_{0}^{2-x} z\,dz\,dy\,dx$$

$$M_{xy} = 2\int_{-2}^{2}\int_{0}^{\frac{\sqrt{4-x^2}}{2}}\left[\frac{z^2}{2}\right]_0^{2-x} dy\,dx$$

$$= \cancel{2}\int_{-2}^{2}\int_{0}^{\frac{\sqrt{4-x^2}}{2}} \frac{(2-x)^2}{\cancel{2}}\,dy\,dx$$

$$= \int_{-2}^{2}\int_{0}^{\frac{\sqrt{4-x^2}}{2}} (2-x)^2\,dy\,dx$$

$$= \int_{-2}^{2} (2-x)^2\,y\,dx$$

$$= \int_{-2}^{2} (2-x)^2\left(\frac{\sqrt{4-x^2}}{2}\right)dx$$

$$= \int_{-2}^{2} \frac{(2-x)^2\sqrt{4-x^2}}{2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{2} (2-x)^2\sqrt{4-x^2}\,dx$$

Let $x = 2\sin\theta$
$dx = 2\cos\theta$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (2-2\sin\theta)^2\sqrt{4-4\sin^2\theta}\;2\cos\theta\,d\theta$$

$$= \frac{4\times}{\cancel{2}}\int_{0}^{\pi/2} 2^2(1-\sin\theta)^2 \times 2\sqrt{1-\sin^2\theta} \times 2\cos\theta\,d\theta$$

$$= 16\int_{0}^{\pi/2} \cos^2\theta\,(1-2\sin\theta+\sin^2\theta)\,d\theta$$

$$= 16\int_{0}^{\pi/2} (\cos^2\theta - 2\sin\theta\cos^2\theta + \cos^2\theta\sin^2\theta)\,d\theta$$

$$= 16\int_0^{\pi/2} \frac{1+\cos 2\theta}{2} + 2\cos^2\theta(-\sin\theta) + \cos^2\theta(1-\cos^2\theta)\,d\theta$$

$$= 16\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2} + 2\cos^2\theta(-\sin\theta) + \frac{1+\cos 2\theta}{2} - \cos^4\theta\right)d\theta$$

$$= 16\int_0^{\pi/2} \cancel{2}\left(\frac{1+\cos 2\theta}{\cancel{2}}\right) + 2\cos^2\theta(-\sin\theta) - \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= 16\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2} + 16\left[\frac{2\cos^3\theta}{3}\right]_0^{\pi/2} - \frac{16}{4}\left[\int_0^{\pi/2}\left(1+2\cos 2\theta+\cos^2 2\theta\right)d\theta\right]$$

$$= 16\left[\frac{\pi}{2}\right] + 0 - 4\left[\int_0^{\pi/2}\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta\right]$$

$$= 8\pi - 4\left[\theta + \frac{2\sin 2\theta}{2} + \frac{\theta}{2} + \frac{\sin 4\theta}{8}\right]_0^{\pi/2}$$

$$= 8\pi - 4\left(\frac{\pi}{2} + 0 + \frac{\pi/2}{2} + 0\right)$$

$$= 8\pi - 4\left(\frac{\pi}{2}+\frac{\pi}{4}\right)$$

$$= 8\pi - \cancel{4}\left(\frac{3\pi}{\cancel{4}}\right)$$

$$= 5\pi$$

$$\boxed{M_{xy} = 5\pi}$$

~~My~~ $M_{xz} = 2\int_{-2}^{2}\int_{-\frac{\sqrt{4-x^2}}{2}}^{\frac{\sqrt{4-x^2}}{2}}\int_{0}^{2-x} y\,dz\,dy\,dx$

$$= 2\int_{-2}^{2}\int_{-\frac{\sqrt{4-x^2}}{2}}^{\frac{\sqrt{4-x^2}}{2}} y(2-x)\,dy\,dx$$

$$= 2\int_{-2}^{2}\left(\frac{2y^2}{2} - \frac{xy^2}{2}\right)_0^{\frac{\sqrt{4-x^2}}{2}} dx$$

$$= \int_{-2}^{2} \left[(2-x) \left[\frac{y^2}{2} \right]_{-\frac{\sqrt{4-x^2}}{2}}^{+\frac{\sqrt{4-x^2}}{2}} \right] dx.$$

$$= \int_{-2}^{2} (2-x) \left[\frac{\left(\frac{\sqrt{4-x^2}}{2}\right)^2}{2} - \frac{\left(-\frac{\sqrt{4-x^2}}{2}\right)^2}{2} \right] dx.$$

$$= \frac{1}{2} \int_{-2}^{2} (2-x) \left[\left(\frac{\sqrt{4-x^2}}{2} \right)^2 - \left(\frac{-\sqrt{4-x^2}}{2} \right)^2 \right] dx$$

$$= \frac{1}{2} \int_{-2}^{2} (2-x) \left(\frac{4-x^2 - 4-x^2}{2} \right) dx$$

$$= 0.$$

$$M_{yz} = \int_{-2}^{2} \int_{-\frac{\sqrt{4-x^2}}{2}}^{\frac{\sqrt{4-x^2}}{2}} \int_{0}^{2-x} x \, dz \, dy \, dx$$

$$= \int_{-2}^{2} \int_{-\frac{\sqrt{4-x^2}}{2}}^{+\frac{\sqrt{4-x^2}}{2}} x(2-x) \, dy \, dx.$$

$$= \int_{-2}^{2} (2x - x^2) \left[y \right]_{-\frac{\sqrt{4-x^2}}{2}}^{\frac{\sqrt{4-x^2}}{2}} dx$$

$$= \int_{-2}^{2} (2x - x^2) \left[\frac{\sqrt{4-x^2}}{2} + \frac{\sqrt{4-x^2}}{2} \right] dx.$$

$$= \int_{-2}^{2} (2x - x^2) \, 2 \left(\frac{\sqrt{4-x^2}}{2} \right) dx$$

$$= \int_{-2}^{2} \underbrace{(2x - x^2)}_{\to x(2-x)} \sqrt{4-x^2}\, dx.$$

let $x = 2\sin\theta$ $\quad x \to -2,\ \theta \to -\pi/2$

$dx = 2\cos\theta$ $\quad x \to 2,\ \theta \to \pi/2$

$$= \int_{-\pi/2}^{\pi/2} \cancel{2x \to} (2 - 2\sin\theta)(2\sin\theta)\sqrt{4 - 4\sin^2\theta}\,(2\cos\theta)\,d\theta$$

$$= 2\int_0^{\pi/2} 2(1-\sin\theta)(\sin\theta)\,2(\cos\theta)\,2(\cos\theta)\,d\theta.$$

$$= 16\int_0^{\pi/2} (1-\sin\theta)(\sin\theta)(\cos^2\theta)\,d\theta$$

$$= 16\int_0^{\pi/2} (\sin\theta\cos^2\theta - \sin^2\theta\cos^2\theta)\,d\theta$$

$$= 16\left[\int_0^{\pi/2} -\cos^2\theta(-\sin\theta)\,d\theta\right]_0^{\pi/2} - \int_0^{\pi/2} \cos^2\theta(1-\cos^2\theta)\,d\theta$$

$$= 16\left[\frac{-\cos^3\theta}{3}\right]_0^{\pi/2} - 16\int_0^{\pi/2} (\cos^2\theta - \cos^4\theta)\,d\theta.$$

$$= 0 - 16\int_0^{\pi/2} \frac{1+\cos 2\theta}{2} - \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= -16\int_0^{\pi/2} \frac{1+\cos 2\theta}{2} - \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$

Due To Symmetry

$$= -16\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} - \frac{\theta}{4} - \frac{2\sin 2\theta}{8} - \frac{\theta}{8} - \frac{\sin 4\theta}{32}\right]_0^{\pi/2} d\theta$$

$$= -16\left[\frac{\pi/2}{2} + 0 - \frac{\pi/2}{8} - \frac{\pi/2}{4}\right]$$

$$= 2\times 16\left[\frac{\pi}{4} - \frac{\pi}{16} - \frac{\pi}{16}\right] - \left[\frac{\pi}{4} - \right.$$

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$$= -32\left(\frac{4\pi - 2\pi - \pi}{16}\right)$$

$$= -32\left(\frac{4\pi - 3\pi}{16}\right)$$

$$= -32\left(\frac{\pi}{16}\right)$$

$$= -2\pi$$

Ex 15.6

Triple Integrals with cylindrical coordinate. (r, θ, z)

Rectangular coordinates (x, y, z)

Cylindricular coordinates (r, θ, z)

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x$$

Example ①:

Find limits of Integration bounded below by plane $z=0$ by cylinder

$$x^2 + (y-1)^2 = 1$$

and above by paraboloid = $z = x^2 + y^2$

$\longrightarrow$ limit of z $\boxed{z=0, \ z=r^2}$

$$x^2 + (y-1)^2 = 1$$

$$x^2 + y^2 - 2y + 1 = 1$$

$$r^2 - 2r\sin\theta = 0$$

Put $r^2 = x^2 + y^2$ & $y = r\sin\theta$

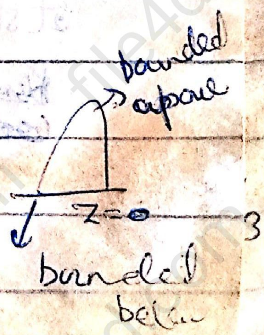
$$r(r - 2\sin\theta) = 0$$

$$r = 0, \ r = 2\sin\theta$$

Put $r=0 \Rightarrow 0 = 2\sin\theta$

$$\theta = \sin^{-1}(0)$$

$$\theta = 0, \pi, 2\pi \ldots$$



Put $x = 0$,

$$\sqrt{(y-1)^2} = \sqrt{1}$$

$$y - 1 = \pm 1$$

$y = 0, 2$, so, $\theta = 0$ to π

$$\int_{f_1(c)}^{f_2(c)} \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} dz\,r\,dr\,d\theta$$

$$= \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} f(r,\theta,z)\,dz\,r\,dr\,d\theta$$

Example 2:

Find centroid? if $x^2+y^2=4$, above by Paraboloid $z=x^2+y^2$ and banded below by ~~x-y plane~~ $\to z=0$

$\boxed{z=0}$ in xy plane

$z=x^2+y^2 \Rightarrow \boxed{z=r^2}$

$0 \le z \le r^2$

$r^2=4$

$r=\pm 2$ so r is 0 to 2

$0 \le r \le 2$

It shows that ~~Remember~~ ~~Concept~~ θ is 0 to 2π

$0 \le \theta \le 2\pi$

$$M = \int_0^{2\pi} \int_0^{2} \int_0^{r^2} dz\,r\,dr\,d\theta$$

$$M = \int_0^{2\pi} \int_0^{2} r^2 \cdot r\,dr\,d\theta$$

$$M = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^2 d\theta$$

$$M = \int_0^{2\pi} \frac{2^4}{4} d\theta = \int_0^{2\pi} \frac{16}{4} d\theta = \int_0^{2\pi} 4 d\theta$$

$$M = 4(\theta)_0^{2\pi}$$

$$M = 4(2\pi)$$

$$\boxed{M = 8\pi}$$

$$M_{xy} = \int_0^{2\pi}\int_0^{2}\int_0^{r^2} z\,dz\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \left[\frac{z^2}{2}\right]_0^{r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \frac{r^4}{2} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \frac{r^5}{2} dr\,d\theta \Rightarrow = \int_0^{2\pi} \left[\frac{r^6}{2\times 6}\right]_0^2 d\theta$$

$$= \int_0^{2\pi} \left(\frac{2^6}{2\times 6}\right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{\cancel{32}^{16}}{3\cancel{6}}\right) d\theta$$

$$= \frac{16}{3}(\theta)_0^{2\pi} = \frac{16}{3}(2\pi) = \frac{32\pi}{3}$$

$$M_{yz} = \int_0^{2\pi}\int_0^{2}\int_0^{r^2} x\,dx\,r\,dr\,d\theta = 0$$

$$\bar{x} = 0, \quad \bar{y} = 0$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\cancel{16}\pi/3}{\cancel{8}\times\frac{\pi}{3}} = \frac{\cancel{32}^{4}\pi/3}{8\pi} = 4/3$$

so centroid is $(0, 0, 4/3)$

Ex 15.6

① $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\, r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^1 \left(\sqrt{2-r^2} - r\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{-2}\left(\sqrt{2-r^2}\cdot r(-2) - r^2\right) dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{-(2-r^2)^{3/2}}{2\times 3/2} - \frac{r^3}{3}\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left[-\frac{(2-r^2)^{3/2}}{3} - \frac{r^3}{3}\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(-\frac{1}{3} - \frac{1}{3} + \frac{(2)^{3/2}}{3}\right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{-2+2^{3/2}}{3}\right) d\theta$$

$$= \frac{-2+\sqrt{8}}{3}\left(\theta\right)_0^{2\pi}$$

$$= \frac{-2+\sqrt{8}}{3}\left(2\pi\right)$$

$$= \frac{\left(-2+2\sqrt{2}\right)2\pi}{3}$$

$$= \frac{2\left(\sqrt{2}-1\right)2\pi}{3}$$

$$V = 4\pi\left(\frac{\sqrt{2}-1}{3}\right)$$

③ : $\int_0^{2\pi}\int_0^{\theta/2\pi}\int_0^{3+24r^2} dz\, r\, dr\, d\theta$

$$\int_0^{2\pi}\int_0^{\theta/2\pi} (3+24r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\theta/2\pi} (3r+24r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{3r^2}{2} + \frac{24r^4}{4}\right]_0^{\theta/2\pi} d\theta$$

$$= \int_0^{2\pi} \left[\frac{3}{2}r^2 + 6r^4\right]_0^{\theta/2\pi} d\theta$$

$$= \int_0^{2\pi} \left[\frac{3}{2}\left(\frac{\theta}{2\pi}\right)^2 + 6\left(\frac{\theta}{2\pi}\right)^4\right] d\theta$$

$$= \int_0^{2\pi} \left(\frac{3\theta^2}{8\pi^2} + \frac{6\theta^4}{16\pi^4}\right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{3\theta^2}{8\pi^2} + \frac{3}{8}\frac{\theta^4}{\pi^4}\right) d\theta$$

$$= \frac{3}{8}\left[\frac{\theta^3}{3\pi^2} + \frac{\theta^5}{5\pi^4}\right]_0^{2\pi}$$

$$= \frac{3}{8}\left[\frac{(2\pi)^3}{3\pi^2} + \frac{(2\pi)^5}{5\pi^4}\right]$$

$$= \frac{3}{8}\left[\frac{8\pi^3}{3\pi^2} + \frac{32\pi^5}{5\pi^4}\right]$$

$$= \frac{3}{8}\left[\frac{8}{3}\pi + \frac{32}{5}\pi\right]$$

$$= \pi + \frac{12\pi}{5}$$

$$= \frac{17\pi}{5}$$

(5) $\int_0^{2\pi}\int_0^1\int_r^{1/\sqrt{2-r^2}} 3\,dz\,r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^1 3\left(\frac{1}{\sqrt{2-r^2}} - r\right) r\,dr\,d\theta$$

$$= \frac{3}{-}\int_0^{2\pi}\int_0^1 \left(\frac{(-2r)}{(2-r^2)^{1/2}} \times \frac{1}{-2} - r^2\right) dr\,d\theta$$

$$= 3\int_0^{2\pi}\left[\frac{-1}{2}\frac{(2-r^2)^{-1/2+1}}{-1/2+1} - \frac{r^3}{3}\right]_0^1 d\theta$$

$$= 3\int_0^{2\pi}\left[\frac{-1}{\cancel{2}}\frac{(2-1^2)^{+1/2}}{1/\cancel{2}} - \frac{1}{3} - \left(-\frac{1}{2}(2)^{+1/2} - \cancel{0}\right)\right] d\theta$$

$$= 3\int_0^{2\pi}\left(\frac{-1}{\cancel{2}} - \frac{1}{3} + \frac{\sqrt{2}}{2}\right) d\theta$$

$$= 3\int_0^{2\pi}\left(\frac{-4}{3} + \sqrt{2}\right) d\theta$$

$$= 3\int_0^{2\pi}\left(\sqrt{2} - \frac{4}{3}\right) d\theta$$

$$= 3\left(\sqrt{2} - \frac{4}{3}\right)(2\pi)$$

$$= \pi(6\sqrt{2} - 8)$$

7. $$\int_0^{2\pi}\int_0^{3}\int_0^{z/3} r^3\,dr\,dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3}\left[\frac{r^4}{4}\right]_0^{z/3} dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3}\left(\frac{z^4/3^4}{4}\right) dz\,d\theta$$

$$= \frac{1}{324}\int_0^{2\pi}\int_0^{3} z^4\,dz\,d\theta$$

$$= \frac{1}{324}\int_0^{2\pi}\left[\frac{z^5}{5}\right]_0^3 d\theta$$

$$= \frac{1}{324}\int_0^{2\pi}\frac{3^5}{5}\,d\theta$$

$$= \frac{3^5}{3^4\times 4}\times\frac{1}{5}\left[\theta\right]_0^{2\pi}$$

$$= \frac{3(2\pi)}{20} =$$

$$= \frac{3\pi}{10}$$

11. D is the region bounded by plane $z=0$, above by sphere $x^2+y^2+z^2=4$ & on the sides of cylinder $x^2+y^2=1$. find Volume of D

(a) $dz\,dr\,d\theta$

(c) $d\theta\,dz\,dr$

(b) $dr\,dz\,d\theta$

(a) $\int_0^{2\pi}\int_0^{1}\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$

(b) $\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{1} r\,dr\,dz\,d\theta + \int_0^{2\pi}\int_{\sqrt{3}}^{2}\int_0^{\sqrt{4-z^2}} r\,dr\,dz\,d\theta$

(c) $\int_0^{1}\int_0^{\sqrt{4-r^2}}\int_0^{2\pi} r\,d\theta\,dz\,dr$

$x^2+y^2+z^2=4$ $\quad z=0$

$x^2+y^2=1$

$r^2+z^2=4$ $\quad\quad r^2=1$

$\sqrt{z^2}=\pm\sqrt{4-r^2}$ $\quad\quad r=\pm 1$

$z=\sqrt{4-r^2}$ $\quad\quad$ and 0 to 1

(neglect -ve value) $\quad\quad 0\le r\le 1$

limits

$z=0$

$z=2$

$z=\sqrt{3}$

Comparing (1) & (2)

$x^2+y^2+z^2=4$

$x^2+y^2=1$

Put (1) in (2)

$1+z^2=4$

$z^2=4-1$

$z^2=3$

$\boxed{z=\pm\sqrt{3}}$ (neglect -ve)

for z constant put $x, y = 0$

$z^2=4$

$z=2$

$0\le z\le\sqrt{3}$, break on 2,

$0\le z\le\sqrt{3}$ & $\sqrt{3}\le z\le 2$

(14) $$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}(x^2+y^2)\,dz\,dx\,dy.$$

Convert into Equivalent Integral in cylindrical coordinates: & then evaluate.

As $-1 \le y \le 1$

So, $0 \le \theta \le \pi$

or $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

0 to π $y=-1$

$y=1$

$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{0}^{r\cos\theta} r^2\, r\,dz\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{0}^{r\cos\theta} r^3\,dz\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\left(r^3 z\right)_{0}^{r\cos\theta} dr\,d\theta$$

$x = \sqrt{1-y^2}$

$x^2+y^2=1$

$r^2=1$

$\boxed{r=1}$

$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{1} r^3(r\cos\theta)\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\cos\theta\left[\frac{r^{4+1}}{4+1}\right]_0^1 d\theta$$

$$= \frac{1}{5}\int_{-\pi/2}^{\pi/2}(1^5)\cos\theta\,d\theta$$

$$= \frac{2}{5}\int_{0}^{\pi/2}\cos\theta\,d\theta$$

$$= \frac{2}{5}\left(+\sin\theta\right)_0^{\pi/2}$$

$$= \frac{2}{5}(1) = \frac{2}{5}$$

⇒ Spherical coordinate :-

$-(\rho, \phi, \theta)$

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

(21). $\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$= \int_0^{\pi}\int_0^{\pi}\left[\frac{\rho^3}{3}\right]_0^{2\sin\phi} \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{3}\int_0^{\pi}\int_0^{\pi} (2\sin\phi)^3 \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{3}\int_0^{\pi}\int_0^{\pi} 8 \cdot \sin^4\phi \, d\phi \, d\theta$$

Using Identity

$\cos 2\theta = 1 - 2\sin^2\theta$

$\Rightarrow 2\sin^2\theta = 1 - \cos 2\theta$

$$= \frac{8}{3}\int_0^{\pi}\int_0^{\pi} (\sin^2\phi)^2 \, d\phi \, d\theta$$

$$= \frac{8}{3}\int_0^{\pi}\int_0^{\pi} \left(\frac{1-\cos 2\theta}{2}\right)^2 d\phi \, d\theta$$

$$= \frac{\cancel{8}^{2}}{3} \times \frac{1}{\cancel{4}} \int_0^{\pi}\int_0^{\pi} (1 - 2\cos 2\phi + \cos^2 2\phi) \, d\phi \, d\theta$$

$$\int^{\pi}\int^{\pi} \left(1 - 2\cos 2\phi + \frac{1 + \cos 4\phi}{2}\right) d\phi \, d\theta$$

$$\left[- \frac{2\sin 2\phi}{2} + \frac{\phi}{2} + \frac{\sin 4\phi}{4}\right]_0^{\pi} d\theta$$

$$+ \frac{\pi}{2} + 0) \, d\theta \Rightarrow \frac{2}{\cancel{3}} \times \frac{\cancel{3}\pi}{2}\int_0^{\pi} 1 \cdot d\theta$$

$= \pi \times \pi = \pi^2$

(23) $$\int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{(1-\cos\phi)}{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\left[\frac{\rho^3}{3}\sin\phi\right]_0^{\frac{(1-\cos\phi)}{2}} d\rho \, d\phi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{(1-\cos\phi)}{2}} \left(\frac{1-\cos\phi}{2}\right)^3 \sin\phi \, \cancel{d\rho} \, d\phi \, d\theta$$

$$= \frac{1}{24}\int_0^{2\pi}\left[\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi} d\theta$$

$$= \frac{1}{24}\int_0^{2\pi}\left[\frac{(1-\cos\pi)^4}{4} - \frac{(1-\cos 0)^4}{4}\right] d\theta$$

$$= \frac{1}{96}\int_0^{2\pi}\left((2)^4 - (1-1)^4\right) d\theta$$

$$= \frac{1}{96}\int_0^{2\pi}(16-0)\, d\theta$$

$$= \frac{16}{96}[\theta]_0^{2\pi} = \frac{16}{96}(2\pi)$$

(27) $$\int_0^{2}\int_{-\pi}^{0}\int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho$$

$$= \int_0^2 \int_{-\pi}^{0} \rho^3 \left. \frac{(-\cos 2\phi)}{2} \right|_{+\pi/4}^{\pi/2} d\theta\, d\rho$$

$$= \int_0^2 \int_{-\pi}^{0} \left[\frac{\rho^3}{2} \left(-\frac{\cos 2(\frac{\pi}{2})}{2} + \frac{\cos 2(\pi/4)}{2} \right) \right] d\theta\, d\rho$$

$$= \int_0^2 \int_{-\pi}^{0} \frac{\rho^3}{2} \left(-(-1) + 0 \right) d\theta\, d\rho$$

$$= \frac{1}{2} \int_0^2 \int_{-\pi} \rho^3\, d\theta\, d\rho$$

$$= \frac{1}{2} \int_0^2 \rho^3 \left[\theta\right]_{-\pi}^{0} d\rho$$

$$= \frac{1}{2} \int_0^2 \rho^3 (0 - (-\pi))\, d\rho$$

$$= \frac{\pi}{2} \int_0^2 \rho^3\, d\rho$$

$$= \frac{\pi}{2} \times \left[\frac{\rho^4}{4} \right]_0^2$$

$$= \frac{\pi}{2} \left[\frac{2^4}{4} - \frac{0}{4} \right]$$

$$= \frac{\pi}{\cancel{2}} \left[\frac{\cancel{16}^{\,2}\,\cancel{4}}{\cancel{4}} \right] = 2\pi$$

32 (a) : $z = \sqrt{x^2 + y^2}$.

$z = 1,$

find : $d\rho\, d\phi\, d\theta = ?$

$z = 1 \rightarrow ①$, $z = \sqrt{x^2+y^2}$

$z = \sqrt{r^2}$

$z = r \rightarrow ②$

we need limits of ϕ & ρ.

Using ① , Using ②

$z = r$

$\rho\cos\phi = 1$ $\qquad$ $\rho\cos\phi = \rho\sin\phi$

$\rho = \dfrac{1}{\sin\phi}$ $\qquad$ $\cos\phi = \sin\phi$

$1 = \dfrac{\sin\phi}{\cos\phi}$

$\boxed{\rho = \operatorname{cosec}\phi}$

$1 = \tan\phi$

$\phi = \tan^{-1}(1) = \dfrac{\pi}{4}$

$\boxed{\phi = \dfrac{\pi}{4}}$

So, limits of Integration are.

$$dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\operatorname{cosec}\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Ex 15.7.

Substitutions in Multiple Integrals

Suppose, G is the region in uv-plane.
R is the region in xy-plane.
the we transfermed G in to R be using equations:

$$x = g(u,v) \quad , \quad y = h(u,v).$$

If f, g, h have continuous partial derivative and J(u,v) is zero only at Isolated point if

$$\iint_R f(x,y)\,dx\,dy = \iint f(g(u,v), h(u,v))\,du\,dv\,|J(u,v)|$$

Jacobian:

$$x = g(u,v), \quad y = h(u,v)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$$

Transformation in To Polar coordinates

Verification

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

As we know, $x = r\cos\theta$, $y = r\sin\theta$

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(1)$$

$$|J(r,\theta)| = |r|.$$

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\,|J(r,\theta)|\,dr\,d\theta$$

$$= \iint_R f(r\cos\theta, r\sin\theta)\,|r|\,dr\,d\theta$$

⇒ **Transformation into Cylindricular coordinate**

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

→ Cylindricular coordinates are $\cancel{(\rho, \phi, \theta}$ (r, θ, z).

$$J(r,\theta,z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$|J| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\theta\,(r\cos\theta) + r\sin\theta\,(\sin\theta) - 0$$

$$= r\,[\cos^2\theta + \sin^2\theta] = r\,[1] = r$$

$$|J(r,\theta,z)| = |r|.$$

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_R f(r\cos\theta, r\sin\theta, z)\,|J(r,\theta,z)|\,dr\,d\theta\,dz$$

$$= \iiint_R f(r\cos\theta, r\sin\theta, z)\,|r|\,dr\,d\theta\,dz$$

Verification for spherical coordinate

$r = \rho\sin\phi$, $z = \rho\cos\phi$

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$

$$J(\rho,\phi,\theta) = \begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\theta\cos\phi & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\theta\cos\phi & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \left[\sin\phi\cos\theta\left(0 + \rho^2\sin^2\phi\cos\theta\right) - \rho\cos\theta\cos\phi\left(0 - \rho\sin\phi\cos\phi\cos\theta\right) - \rho\sin\phi\sin\theta\left(-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta\right)\right]$$

$$= \rho^2\sin^3\phi\left(\cos^2\theta + \sin^2\theta\right) + \rho^2\left(\rho\cos^2\phi\sin\phi\right)\left(\cos^2\theta + \sin^2\theta\right)$$

$$= \rho^2\sin^3\phi + \rho^2\cos^2\phi\sin\phi$$

$$= \rho^2 \sin\phi \, (\sin^2\phi + \cos^2\phi)$$

$$= \rho^2 \sin\phi$$

$$J(\rho, \phi, \theta) = \rho^2 \sin\phi$$

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_R f(\rho,\phi,\theta)\,|J(\rho,\phi,\theta)| \times d\rho\,d\phi\,d\theta$$

$$= \iiint_R f(\rho,\phi,\theta)\,(\rho^2 \sin\phi)\,d\rho\,d\phi\,d\theta$$

Ex 1. Evaluate, $\displaystyle\int_0^4 \int_{x=y/2}^{x=y/2+1} \frac{2x-y}{2}\,dx\,dy.$

By transformation eq,

$$u = \frac{2x-y}{2}, \quad v = \frac{y}{2}$$

Limits of u

$x = \frac{2u+y}{2}$ $\quad$ $\boxed{y = 2v}$

$x = \frac{2u+2v}{2}$

$\boxed{x = u+v}$

at $x = \frac{y}{2}$, at $x = \frac{y}{2}+1$

$\frac{y}{2} = u+v$, $\frac{y}{2}+1 = u+v$

$2v = u+v$, $y+2 = 2u+2v$

$\boxed{u=0}$, $2v+2 = 2u+2v$

$\boxed{u=1}$

$$= \int_0^2 \int_0^1 u\,|J(u,v)|\,du\,dv$$

$$= \int_0^1 \int_0^1 u \cdot |J(u,v)|\,du\,dv$$

$$|J(u,v)| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$$

$$= \iint_R u \, |J(u,v)| \, du \, dv$$

$$= \int_0^2 \int_0^1 u \cdot (2) \, du \, dv$$

limits of v

$y = 2v$

at $y = 0$, $v = 0$

at $y = 4$, $v = 2$

Ex 2: $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 \, dy \, dx$

$\boxed{x = 0}, \boxed{x = 1}$

$\boxed{y = 1-x}, \boxed{y = 0}$

Applying transformation eqs. $x = \frac{u}{3} - \frac{v}{3}$

$y = \frac{2u}{3} + \frac{v}{3}$

① $x = 0$	$0 = \frac{u}{3} - \frac{v}{3} = \boxed{u = v}$
② $y = 0$	$2u = -v$ or $\boxed{v = -2u}$
$\sqrt{x+y}$ ③ $y = 1-x$	$\sqrt{\frac{u}{3} - \frac{v}{3} + \frac{2u}{3} + \frac{v}{3}} = \frac{3u}{3} = \sqrt{u}$ $\frac{2u}{3} + \frac{v}{3} = 1 - \frac{u}{3} + \frac{v}{3}$ $= \frac{3u}{3} = 1$ $\boxed{u = 1}$
④ for u, $v = 0$, $y = 0$	$\boxed{u = 0}$

$$= \iint_R f(u,v) \, |J(u,v)| \, dv \, du$$

$$= \int_0^1 \int_{-2u}^{u} \sqrt{u} \left(\frac{2u}{3} + \frac{v}{3} - 2\left(\frac{u}{3} - \frac{v}{3}\right) \right)^2 |J(u,v)| \, dv \, du$$

$$= \int^1 \int^u \sqrt{u} \left(\frac{2u}{3} + \frac{v}{3} - \frac{2u}{3} + \frac{2v}{3} \right)^2$$

$$= \int_0^1 \int_{-2u}^{u} \sqrt{u}\, v^2 \, |J(u,v)| \, dv\, du \quad —(1)$$

$$|J(u,v)| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial u}\left(\frac{u}{3} - \frac{v}{3}\right) & \frac{\partial}{\partial v}\left(\frac{u}{3} - \frac{v}{3}\right) \\ \frac{\partial}{\partial u}\left(\frac{2u}{3} + \frac{v}{3}\right) & \frac{\partial}{\partial v}\left(\frac{2u}{3} + \frac{v}{3}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{1}{9} + \frac{2}{9}$$

$$= \frac{\cancel{3}^{1}}{\cancel{9}_{3}} = \frac{1}{3}$$

$$|J(u,v)| = \left|\frac{1}{3}\right|$$

Put in (1)

$$= \frac{1}{3} \int_0^1 \int_{-2u}^{u} \sqrt{u}\, v^2 \, dv\, du$$

$$= \frac{1}{3} \int_0^1 \left[\frac{\sqrt{u}\, v^3}{3}\right]_{-2u}^{u} du$$

$$= \frac{1}{9} \int_0^1 \sqrt{u}\left(u^3 + (2u)^3\right) du$$

$u^{3+1/2} = u^{7/2}$

$$= \frac{1}{9} \int_0^1 \left(u^{7/2} + 8u^{7/2}\right) du$$

$$= \frac{1}{9} \left[\frac{u^{7/2+1}}{7/2+1} + \frac{8u^{7/2+1}}{7/2+1}\right]_0^1$$

$$= \frac{1}{9} \left[\frac{2}{9} u^{9/2} + \frac{2 \times 8}{9} u^{9/2}\right]_0^1$$

$$= \frac{1}{9}\left[\frac{2}{9}+\frac{16}{9}\right] = \frac{1}{9}\left[\frac{18}{9}\right]^{2} = \frac{2}{9}$$

Ex 3:

$$\int_0^3\int_0^4\int_{x=y/2}^{x=y/2+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$$

$$u=\frac{2x-y}{2},\quad v=\frac{y}{2},\quad w=\frac{z}{3}$$

$\Rightarrow 2u+2v=2x \Rightarrow \boxed{x=u+v},\ \boxed{y=2v},\ \boxed{3w=z}$

$z=0,\quad z=3,$

$y=0,\quad y=4,$

$x=\frac{y}{2},\quad x=\frac{y}{2}+1,$

Putting limits in Transformation eq.

$z=0$	$\boxed{w=0}$	$z=3,$	$\boxed{w=1}$
$y=0$	$\boxed{v=0}$	$y=4$	$\boxed{v=2}$
$x=\frac{y}{2}$	$u=\frac{2\left(\frac{y}{\cancel{2}}\right)-y}{2}$	$x=\frac{y}{2}+1$	$u=\frac{2\left(\frac{y}{2}+1\right)-y}{2}$
	$u=\frac{y-y}{2}$		$u=\frac{\cancel{2}\left(\frac{y+2}{\cancel{2}}\right)-y}{2}$
	$\boxed{u=0}$		$u=\frac{\cancel{y}+2-\cancel{y}}{2}$
			$\boxed{u=1}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial u}(u+v) & \frac{\partial}{\partial v}(u+v) & \frac{\partial}{\partial w}(u+v) \\ \frac{\partial}{\partial u}(2v) & \frac{\partial}{\partial v}(2v) & \frac{\partial}{\partial w}(2v) \\ \frac{\partial}{\partial u}(3w) & \frac{\partial}{\partial v}(3w) & \frac{\partial}{\partial w}(3w) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$$= 1(6) - 1(0-0) + 0 = 6.$$

$$= \int_0^1 \int_0^2 \int_0^1 (u+w) J(u,v,w)\, du\, dv\, dw.$$

$$= \int_0^1 \int_0^2 \int_0^1 (u+w) . 6 \, du\, dv\, dw$$

$$= 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + uw\right]_0^1 dv\, dw.$$

$$= 6 \int_0^1 \int_0^2 \left[\frac{1}{2} + w\right] dv\, dw.$$

$$= 6 \int_0^1 \left[\frac{1}{2}v + wv\right]_0^2 dw$$

$$= 6 \int_0^1 \left(\frac{1}{\cancel{2}}(\cancel{2}) + w(2)\right) dw.$$

$$= 6 \int_0^1 (1 + 2w)\, dw.$$

$$= 6\left[w + \frac{2w^2}{2}\right]_0^1$$

$$= 6\left(1 + \frac{\cancel{2}}{\cancel{2}}\right) = 6(1+1) = 6(2) = 12$$

① $u = x - y$, $v = 2x + y$. for x, y in terms of u.

find: $\dfrac{\partial(x,y)}{\partial(u,v)}$

$u = x - y$, $v = 2x + y$ Put ①

$u + y = x$, $v = 2(u+y) + y$

Put ② $u + \left(\dfrac{v-2u}{3}\right) = x$, $v = 2u + 2y + y$

$v = 2u + 3y$

$\dfrac{3u + v - 2u}{3} = x$, $\boxed{\dfrac{v-2u}{3} = y}$ —— ②

$\boxed{\dfrac{u+v}{3} = x} \to$ ③

$$\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{9} + \frac{2}{9}$$

$$= \frac{3}{9} = \frac{1}{3}$$

②: Solve, $u = x + 2y$, $v = x - y$.

for x, y in Terms of u & v, find

$\dfrac{\partial(x,y)}{\partial(u,v)}$

$u = x + 2y$, $v = x - y$

$u - 2y = x$ —— ① , Put ① $v = u - 2y - y$

$v = u - 3y$

Put ②. $u - 2\left(\dfrac{u-v}{3}\right) = x$, $\Rightarrow \boxed{y = \dfrac{u-v}{3}} \to$ ②

$$\boxed{\frac{u+2v}{3} = x}$$

So,

$$\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial u}\left(\frac{u}{3}+\frac{2v}{3}\right) & \frac{\partial}{\partial v}\left(\frac{u}{3}+\frac{2v}{3}\right) \\ \frac{\partial}{\partial u}\left(\frac{u}{3}-\frac{v}{3}\right) & \frac{\partial}{\partial v}\left(\frac{u}{3}-\frac{v}{3}\right) \end{vmatrix}$$

$$= \begin{vmatrix} 1/3 & 2/3 \\ \frac{1}{3} & -1/3 \end{vmatrix} = \frac{-1}{9} - \frac{2}{9} = \frac{-3}{9}$$

$$\boxed{J(u,v) = \frac{-1}{3}}$$

③ Solve the system,

$u = 3x+2y$, $v = x+4y$.

Find $\frac{\partial(x,y)}{\partial(u,v)} = ?$

$u = 3x+2y$, $v = x+4y$.

$\frac{u-3x}{2} = y$ — ①, put ①,

$v = x + \left(\frac{2}{4}\left(\frac{u-3x}{2}\right)\right)$,

$v = x + 2u - 6x$

$v = -5x + 2u$

$\boxed{\frac{2u-v}{5} = x} \rightarrow$ ②

put ② in ①

$\frac{1}{2}\left(u - 3\left(\frac{2u-v}{5}\right)\right) = y$

$\frac{1}{2}\left[u - \frac{6u}{5} + \frac{3v}{5}\right] = y$

$$\left(\frac{1}{2}-\frac{6}{10}\right)u+\frac{3}{10}v=y$$

$$\cancel{\left(\frac{5-12}{10}\right)u+\frac{3v}{5}=y}$$

$$\left(\frac{5-6}{10}\right)u+\frac{3}{10}v=y$$

$$-\frac{u}{10}+\frac{3v}{10}=y$$

$$\boxed{\frac{1}{10}(3v-u)=y}$$

$$J(u,v)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial}{\partial u}\left(\frac{2u}{5}-\frac{v}{5}\right) & \frac{\partial}{\partial v}\left(\frac{2u}{5}-\frac{v}{5}\right)\\ \frac{\partial}{\partial u}\left(\frac{1}{10}(3v-u)\right) & \frac{\partial}{\partial v}\left(\frac{1}{10}(3v-u)\right)\end{vmatrix}$$

$$=\begin{vmatrix}\frac{2}{5} & -\frac{1}{5}\\ -\frac{1}{10} & 3/10\end{vmatrix}$$

$$=\left(\frac{2}{5}\right)\left(\frac{3}{10}\right)-\left(\frac{1}{5}\right)\left(\frac{1}{10}\right)=\frac{3}{25}-\frac{1}{50}$$

$$J(u,v)=\frac{6-1}{50}=\frac{5}{50}=\frac{1}{10}$$

$$\boxed{J(u,v)=\frac{1}{10}}$$

⑤ Evaluate $\int_0^4 \int_{x=y/2}^{x=y/2+1} \frac{2x-y}{2}\,dx\,dy$

$$= \int_0^4 \left[\frac{1}{2}\left(\cancel{2}\frac{x^2}{\cancel{2}} - \frac{xy}{\cancel{1}}\right)\right]_{y/2}^{y/2+1} dy$$

$$= \frac{1}{2}\int_0^4 \left[x^2 - \frac{xy}{1}\right]_{y/2}^{y/2+1} dy$$

$$= \frac{1}{2}\int_0^4 \left[\left(\frac{y}{2}+1\right)^2 - \left(\frac{y}{2}+1\right)\left(\frac{y}{1}\right)\right] - \left[\left(\frac{y^2}{12}\right) - \left(\frac{y}{2}\right)\left(\frac{y}{2}\right)\right] dy$$

$$= \frac{1}{2}\int_0^4 \left[\cancel{\frac{y^2}{4}} + y + 1 - \cancel{\frac{y^2}{4}} - \frac{y}{1} - \cancel{\frac{y^2}{4}} + \cancel{\frac{y^2}{4}}\right] dy$$

$$= \frac{1}{2}\int_0^4 \left[y\left(1-\frac{1}{1}\right)+1\right] dy$$

$$= \frac{1}{2}\int_0^4 \left(y\left(\frac{0}{1}\right)+1\right) dy$$

$$= \frac{1}{2}\left[\left(\frac{y^2}{2}\right)\frac{1}{2} + y\right]_0^4$$

$$= \frac{1}{2}\left[\frac{y^2}{4}+y\right]_0^4 = \frac{1}{2}\left[\frac{4^2}{4}+4\right]$$

$$= \frac{1}{2}\left[\frac{16}{4}+4\right] =$$

$$= \frac{1}{2}\int_0^4 1\,dy = \frac{1}{2}(y)_0^4 = \frac{1}{2}(4-0)$$

$$= 2$$

15. (a) $\dfrac{\partial(x,y)}{\partial(u,v)}$ for $x = u\cos v$, $y = u\sin v$

$|J(u,v)| = ?$

$x = u\cos v$, $y = u\sin v$.

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial u}(u\cos v) & \frac{\partial}{\partial v}(u\cos v) \\ \frac{\partial}{\partial u}(u\sin v) & \frac{\partial}{\partial v}(u\sin v) \end{vmatrix}$$

$$= \begin{vmatrix} \cos v(1) & u(-\sin v) \\ \sin v(1) & u(\cos v) \end{vmatrix}$$

u, v $\quad \frac{\partial y}{\partial v}$

$$= u\cos^2 v - (-u\sin^2 v)$$

$$= u(\cos^2 v + \sin^2 v)$$

$x = r\cos\theta$

$\therefore$ by using identity $\cos^2\theta + \sin^2\theta = 1$ $\quad = u(1)$

$\frac{\partial x}{\partial \theta}$

$= u.$

$\frac{\partial r}{\partial r}$

(b) $x = u\sin v$, $y = u\cos v$.

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial u}(u\sin v) & \frac{\partial}{\partial v}(u\sin v) \\ \frac{\partial}{\partial u}(u\cos v) & \frac{\partial}{\partial u}(u\cos v) \end{vmatrix}$$

$$= \begin{vmatrix} \sin v(1) & u(\cos v) \\ \cos v(1) & u(-\sin v) \end{vmatrix}$$

$$= -\sin^2 v(u) - u(\cos^2 v) \quad (1) = -u(\sin^2 v + \cos^2 v) = -u$$