VISCOUS FLUID-I

INSTRUCTOR :- Dr MUHAMMAD QASIM

Physics

- Mechanics
  - Relatavistic Mechanics
  - Rigid Body Mechanics
    - Statics
    - Dynamics
      - Kinematics
      - Kinetics
  - Quantum Mechanics
  - Deformable Body Mechanics
  - Fluid Mechanics
    - Fluid Statics
    - Fluid Dynamics
      - Aerodynamics
      - Hydrodynamics
- Thermo-dynamics
- Electricit

Physics :- Physics is the natural science that involves the study of Matter and its motion and behaviour through space and time, along with related concepts such as energy and force.
One of the most fundamental scientific disciplines, the main goal of physics is to understand how the universe behaves.

* Types of Physics *

1) **Mechanics :-** Mechanics is the branch of physical sciences which deals with the state of rest or motion of bodies that are subjected to the action of forces.
Mechanics is the study of forces that act on bodies and the resultant motion that those bodies experience.
With roots in physics and mechanics, Engineering mechanics is the basis of all the mechanical sciences.

2) **Thermodynamics :-** Thermodynamics is a branch of physics

concerned with heat and temprature and their relation to other forms of energy and work.

Thermodynamics applied to a wide variety of topics in science and engineering, especially physical chemistry, chemical engineering and mechanical engineering.

3) **Electricity:-** Electricity is the set of physical phenomena associated with the presence and motion of electric charge. Although initially considered a phenomenon separate from magnetism, since the development of Maxwell's equations, both are recognized as part of a single phenemonon: electromagnetism. Various common phenomena are related to electricity including lightining, electric discharges and many others.

The presence of an electric charge, which can be either positive or negative, produces an electric field. The movement of electric charges is an electric current and produces a magnetic field.

* Types of Mechanics *

i) **Relatavistic Mechanics:-** In physics Relatavistic Mechanics reffers to mechanics compatible with special relativity (SR) and general relativity (GR). It provides a non-quantum mechanical description of a system of particles, or a fluid, in cases where the velocities of moving objects are comparable to the speed of light c.

ii) **Quantum Mechanics:-** The branch of mechanics that deals with the mathematical description of the motion and ~~description~~ interaction of subatomic particles, incorporating the concepts of quantization of energy, wave-particle duality, the uncertainty principle and the correspondance principle.

Quantum mechanics (also known as quantum physics) including quantum field theory, is a fundamental theory in physics which describes nature at the smallest scale of energy levels of atoms and subatomic

iii) **Rigid Body Mechanics:-** In physics a rigid Body is an idealization of a solid body in which deformation is neglected. In other words, the distance between any two given points of a rigid body remains constant in time regardless of external force exerted on it.

iv) **Deformable Body Mechanics:-** The basic assumption in rigid body mechanics is that the bodies involves do not deform under applied load.

* In mechanics, any body that changes its shape and/or volume while being acted upon by any kind of external force is known as deformable body and study of that kind is known as deformable ~~by~~ body mechanics.

v) **Fluid Mechanics:-** Fluid mechanics is a branch of physics concerned with the mechanics of fluid (liquid, gases and plasmas) and the forces on them. Fluid mech- has a wide range of applications including mechanical engineering, civil engineering, chemical engineering, biomedical engineering, geophysics, astrophysics and biology.

Types of Rigid Body Mechanics

1) **Statics:-** The branch of mechanics concerned with bodies at rest and forces in equilibrium.

② Statics is the branch of mechanics that is concerned with the analysis of loads (forces and torque) acting on physical systems that do not experience an acceleration ($a=0$), but rather, are in static equilibrium with their environment.

2) **Dynamics:-** The branch of mechanics concerned with the motion of bodies under the action of forces.

Types of Fluid Mechanics

1) **Fluid Statics:-** Fluid statics or Hydrostatics is the branch of fluid Mechanics that studies fluid at rest. It encompasses the study of conditions under which fluids or at rest in stable equilibrium as opposed to fluid dynamics.

2) Fluid Dynamics:- Fluid dynamics is a branch of fluid mechanics that describes the flow of fluids (liquids and gases). It has several sub-disciplines including aerodynamics (the study of air and other gases in motion) and Hydrodynamics (the study of liquids in motion)

* Types of Dynamics *

i) Kinematics :- Kinematics is a branch of classical mechanics that describes the motion of points, bodies (objects), and system of bodies (group of objects) without considering the mass of each or the forces that caused the motion.

ii) Kinetics :- The branch of mechanics concerned with the effects of forces on the motion of a body or system of bodies, especially of forces that do not originate within the system itself. Also called dynamics.

Physics

★ **Fluid:-** In fluid is a substance that continuously deforms (flows) under the applied shear stress. Fluids are a subset of the phases of matter and include liquids, plasmas, and to some extent, plastic solids.

② A substance that has no fixed shape and yields easily to external pressure.

③ Is a substance able to flow easily.

★ **Fluid Flow:-** Fluid such as gases and liquids in motion is called as fluid flow. Motion of a fluid subjected to unbalanced forces. This motion continues as long as unbalanced forces are applied.

★ Types of Fluid

1:- **Ideal Fluid:-** A fluid which can not be compressed and have no viscosity falls in the category of ideal fluid. Ideal fluid can not found in actual practice but it is an imaginary fluid because all the fluids that ~~can~~ exist in the environment have some viscosity. There is no ideal fluid in reality.

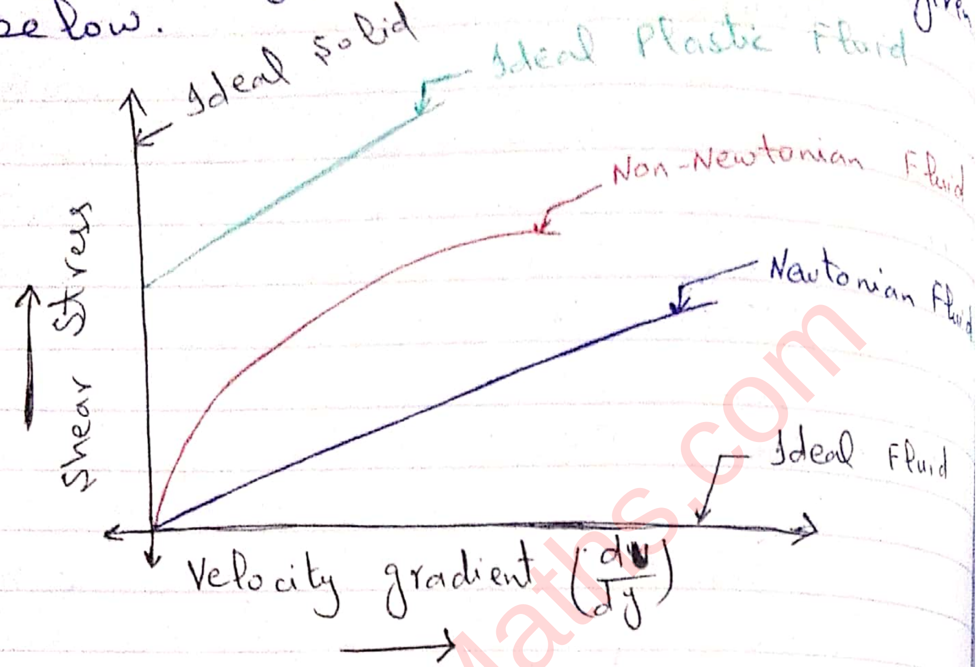
2. **Real Fluid :-** A fluid which has some viscosity is called real fluid. Actually all the fluids existing or present in the environment are called real fluids for example water and air etc.

3. **Newtonian Fluid:-** If a real fluid obeys the Newton's law of viscosity (i.e the shear stress is directly proportional to the shear strain (velocity gradient)) then it is known as Newtonian fluid.

4. **Non-Newtonian Fluid:-** If a real fluid does not obey the Newton's law of viscosity (i.e. shear stress is not directly proportional the rate of shear strain or velocity gradient is known as non Newtonian fluid. e.g. Plaster, slurries etc.

5. **Ideal Plastic Fluid:-** A fluid having the value of shear stress more than the yield value and shear stress is proportional to the shear strain (velocity gradient) is known as ideal plastic fluid.

The graph between the shear stress and velocity gradient for the different types of fluids are given below.



6.- **Incompressible Fluid** :- A fluid, in which the density of fluid does not change with change in external force or pressure, is known as incompressible fluid. All liquids are considered in this category.

7. **Compressible Fluid** :- A fluid, in which the desity of the fluid change while change in external force or pressure, is known as compressible fluid. All gases are considered in this category.

Tabular Representation of Fluid Types

Types of Fluid	Density	Viscosity
Ideal Fluid	Constant	Zero
Real Fluid	Variable	Non zero
Newtonian Fluid	Constant/Var	$T = u(du/dy)$
Non-Newtonian	—"—	$T \neq u(du/dy)$
Incomppressible	Constant	Non zero/Zero
Compressible	Variable	—"—

* Types of Flow

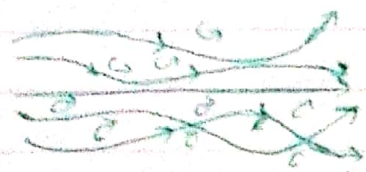
Rotational or Irrotational Flow:-

To classify any flow as rotational or irrotational the angular motion of the fluid elements is analyzed. If the angle between the angle between the two intersecting lines of the boundary of the fluid element changes while moving in the flow, then the flow is a rotational flow. But if the fluid element rotates as a whole and there is no change in angles between the boundary lines then the flow can not be Rotational flow, so it is irrotational flow.

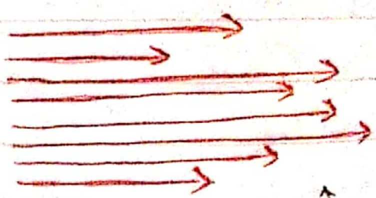
This means there should be some deformation in the fluid element in a rotational flow.

2. Laminar Flow:- The flow of a fluid moving with a moderate speed and fluid flows in parallel layers, with no disruption between the layers. At low velocities, the fluid tends to flow without lateral mixing, and adjacent layers slide past one another like playing cards is known as laminar flow or streamline flow.

3. Turbulant Flow:- Turbulant flow is a type of fluid flow, in which the fluid undergoes irregular fluctuations, or mixing, in contrast to laminar flow. In turbulant flow the speed of the fluid at a point is continuously undergoing changes in both magnitude and direction.



Turbulant Flow



Laminar Flow

4. Steady Flow:- A flow that is not a function of time is called steady flow. Steady state flow refers to the condition where the fluid properties at a point in the system do not change over time.

i.e $\frac{d\eta}{dt} = 0$; where η is fluid prop[erty] it may be velocity or dens[ity]

5. **Unsteady Flow:-** Time dependent flow is known as unsteady (so called transient) flow. i.e

$$\frac{d\eta}{dt} \neq 0$$

6. **Uniform Flow:-** Flow of fluid in which each particle move along its line of flow with constant speed and in which the cross section of each stream tube remain unchanged.

7. **Non-Uniform Flow:-** Flow is said to be non-uniform, where there is a change in velocity of the flow at different points in a flowing fluid, for a given time. For example, the flow of liquids under pressure through long pipelines of varying diameter is reffered to as non-uniform flow

Difference Between Uniform and Steady Flow:- In uniform flow, flow velocity is the same magnitude and direction at every point in the fluid. A steady flow is one in which the conditions (velocity, pressure and cross-section) may differ from point to point but do not change with time.

Normal Stress:- Normal stress σ is the force per unit area exerted perpendicularly to the surface over which it acts.

Shear Stress:- Shear (or tangential) stress τ is the force per unit area exerted tangentially to the surface over which it acts.

★ The SI unit of stress is pascal
$$Pa = N/m^2$$

★ Dimension of $[\sigma]$ or $[\tau]$ is $ML^{-1}T^{-2}$

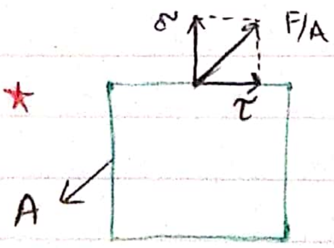
★ 

Fig. 1

In fig 1. normal stress σ and shear stress τ due to force F acting on surface A

★ Typical examples of normal and

shear stress are respectively, pressure and friction.

* The general term to classify these quantities is tensor. The order of tensor refers to the number of directions associated with them

→ Scaler :- Tensor of order zero
→ Vector :- Tensor of order one
→ Stress :- Tensor of order two.

* Scaler quantities like pressure, temperature have magnitude but zero direction property * Vector quantities like velocity have magnitude and one direction associated with them. * Quantities like stress have magnitude and two directions associated with them. The direction of force and direction (orientation) of area on which it acts.

Fluid:- Fluid is a substance that continuously deforms under the action of shear stress.

OR "A fluid is a substance that at rest can not with stand (resist) shear stresses)

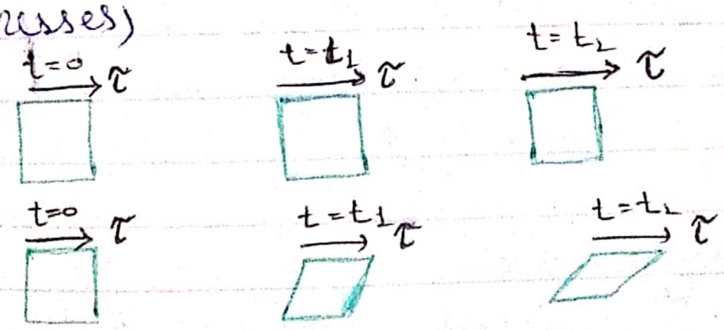
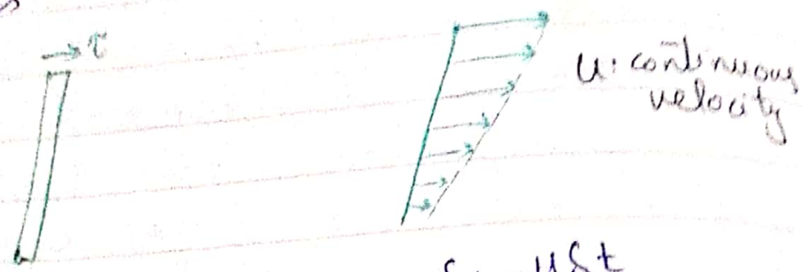


Fig 2.

In figure 2. Behaviour of a small rectangular piece of solid and fluid under the action of shear stress



δx = finite
(Solid)

$\delta x = u \delta t$
(Fluid)

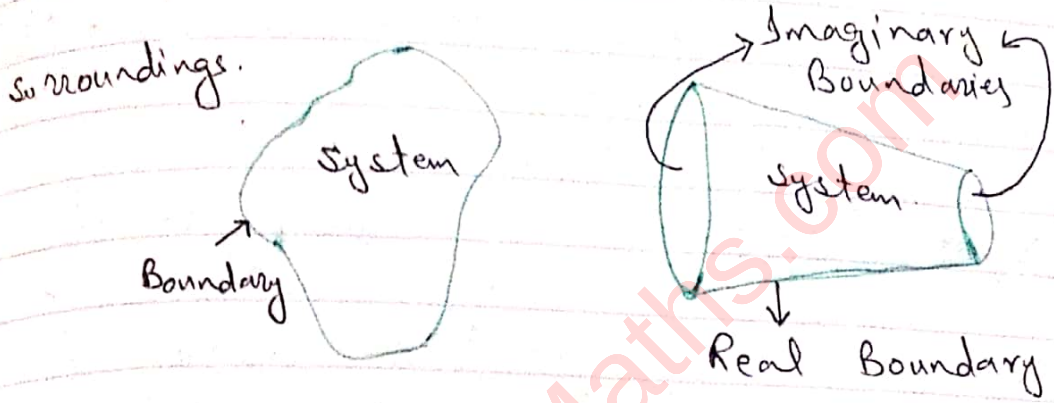
* Solid objects to shear stress deforms (below the elastic limit of deformation) until it reaches the equilibrium deformation, maintaining its shape thereafter.
* A fluid does not recover its original shape when the stress is removed.

System:- A system refers to a fixed, identifiable quantity of mass which is seperated from its surroundings by its boundaries. In fluid mechanics an infinitesimal lump of fluid is considered as a system and is referred as a fluid element or a particle.

Surroundings:- The mass or region outside the system is called the surrounding.

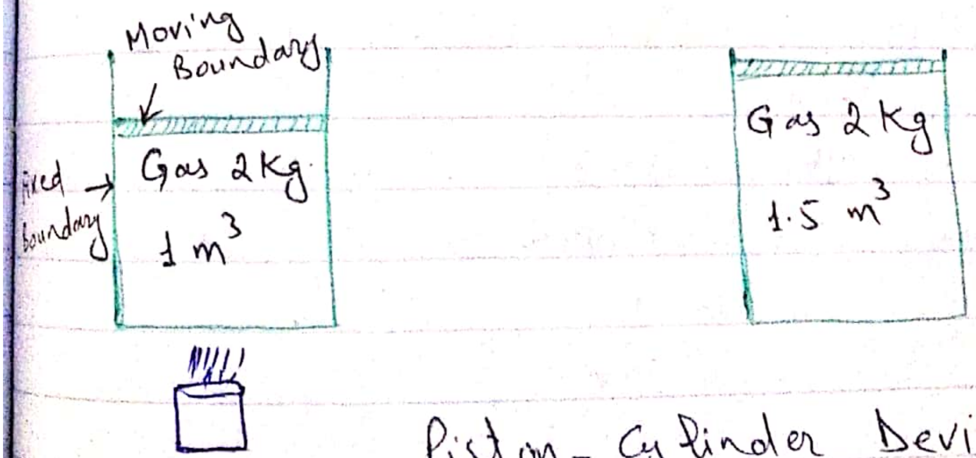
Boundary- The real or imaginary surface that separates the system from the surroundings is called boundary. (also control surface)

* The Boundary of the system can be fixed or movable.



Closed System:- A closed system also known as control mass consists of a fixed amount of mass and no mass can cross its boundary. But energy in the form of heat can cross the boundary.

* Volume of closed system does not have to be fixed.

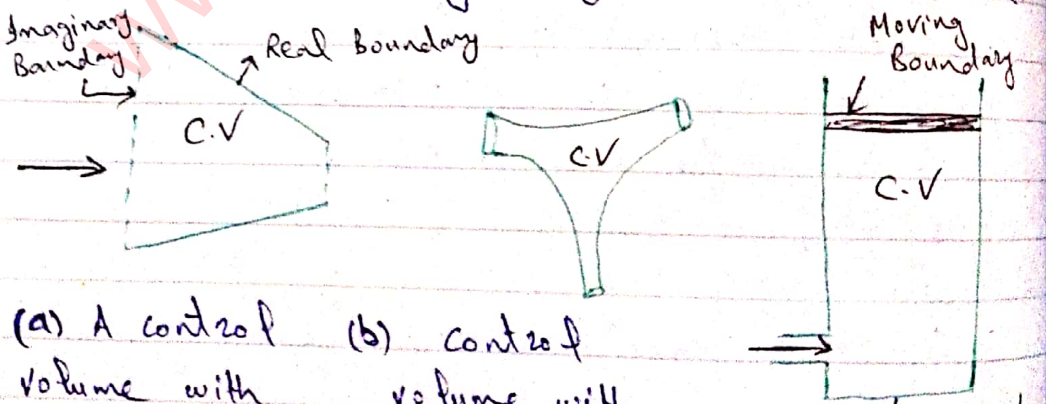


Piston- Cylinder Device.

In piston cylinder device, we are focusing on the gas, here it is our system. The inner ~~system of~~ surface of the piston and the cylinder form the boundary, and since no mass is crossing the boundary, it is a closed system. Note that the energy in the form of heat may cross the boundary, and part of the boundary (inner surface of piston) may move.

Open System:- An open system or control volume (CV) is properly selected region in space. It usually encloses a device that involves mass flow such as compressor, turbine, nozzle, car radiator, water heater.

* Both mass and energy can cross the boundary of control volume.



(a) A control volume with real and imaginary boundaries

(b) control volume with moving boundary.

Property:- Any characteristic of a system is called a property. Some familiar properties are pressure 'P', temperature 'T', volume 'V', mass 'm' and viscosity μ.

* Properties are considered to be either intesive or extensive.

Intensive Properties:- Intensive Properties are those properties that are independant of system size or the amount of material in the system such as density, boiling point, pressure, temperature, thermal conductivity etc.

→ Boiling point of water is 100°C at a pressure of one atmosphere which remains true regardless of quantity.

→ Each material has a certain constant density regardless of the amount present. Hence density is an intensive property. Whether 2kg or 2 gram of a substance is present, if mass is divided by volume present, the result is the same value.

Extensive Properties:- Extensive properties are those properties whose values depend on size or amount of the matter in a system. For example mass, length, volume, heat capacity. The amount of heat required to melt a small ice cube at constant temperature and pressure is an extensive property as the amount of heat required to melt small ice cube would be much less than the amount of heat required to melt a big block of ice, so it is dependent on the quantity.

★ Ratio of two extensive properties of same object or system is an extensive property.

→ Ratio of an object's mass and volume, which are two extensive properties, is density which is an intensive property.

→ specific heat capacity or specific heat is heat capacity per unit mass of material, ratio of two extensive properties is an extensive property.

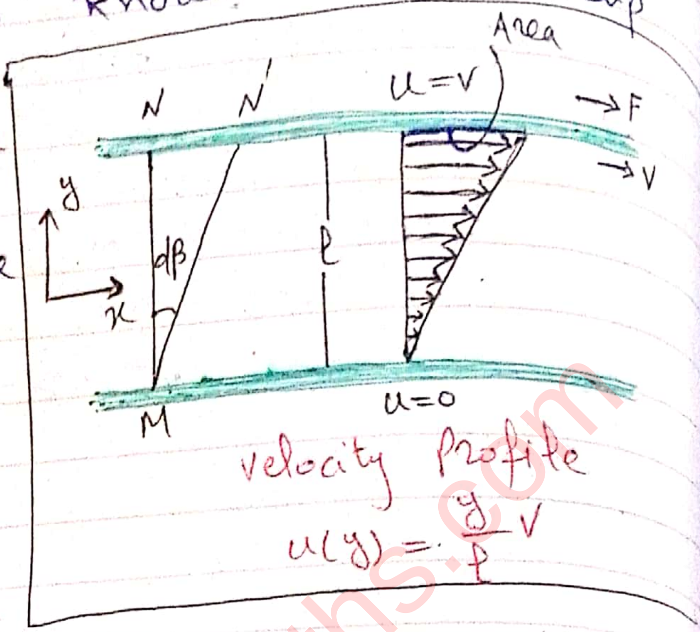
★ **Viscosity**:- The viscosity of a fluid is a measure of its "resistance to deformation"

★ Viscosity is due to internal frictional force that develops between different layers of fluid as they forced to move relative to each other.

* It corresponds to the informal concept of "thickness". For example honey has a much higher viscosity than water.
* Different fluids with different viscosties flow at different speed when same force is applied to them.
* Force required to induce movement is larger for move viscous fluids as compared to less viscous fluids

To obtain a relation for viscosity, consider a fluid layer between two very large parallel plates ~~immerced in a large body of~~ (or equivalently, two parallel plates immersed in a large body of a fluid) seperated by a distance 'l'. Now a constant force $\vec{F}$ is applied to the upper plate while the lower plate is held fixed. The upper plate moves continusly under the influence of this force at a constant speed 'v'.

The fluid is direct contact with the upper plate sticks to the plate surface and move with it at same speed (i.e there is no slip at the boundary and this condition is known as no slip condition)



The shear stress acting on this profile (fluid) layer is

$$\tau = \frac{F}{A} \longrightarrow ①$$

Where 'A' is the contact area between the plate and fluid, and the fluid layer deforms continuously under the influence of shear stress. The fluid in contact with the layer plate assume the velocity of that plate which is zero (because of no-slip condition) For steady and laminar flow, the fluid velocity between plates varies linearly between '0' and 'V'. Thus velocity profile and velocity gradients are

$$u(y) = \frac{y}{l} V \longrightarrow ②$$

$$\text{and} \quad \frac{du}{dy} = \frac{V}{l} \longrightarrow ③$$

Where 'y' is vertical distance from lower plate. During a differential time 'dt', the sides of fluid particles along a vertical line 'MN' rotate through a differential angle '$d\beta$'. The angular displacement or deformation.

Can be expressed as

$$d\beta \approx \tan d\beta \longrightarrow (4)$$

The upper plate moves a differential distance $da = v\,dt \longrightarrow (5)$

From (4) $d\beta \approx \tan d\beta = \frac{da}{l} = \frac{v\,dt}{l}$

$$= \frac{v}{l}\,dt = \frac{du}{dy}\,dt \quad \text{using (5)}$$

Hence the deformation under the influence of shear stress 'τ' becomes

$$\frac{d\beta}{dt} = \frac{du}{dy} \longrightarrow (6)$$

(6) shows that the rate of deformation of a fluid element is equivalent to the velocity gradient $\frac{du}{dy}$. Also, as the rate of deformation is directly proportional to the shear stress τ so

$$\tau \propto \frac{d\beta}{dt} \quad \text{i.e} \quad \tau \propto \frac{du}{dy} \longrightarrow (7)$$

or $$\tau = \mu \frac{du}{dy} \longrightarrow (8)$$

(8) is known as Newton's law of viscosity. Fluids which obey Newton's law of viscosity (i.e for which rate of deformation is linearly proportional to shear stress or velocity gradient) Most common fluids such as water, air and oils are Newtonian fluids

as $[\tau] = \left[\frac{F}{A}\right] = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$

$$[\mu] = [\tau] / \left[\frac{du}{dy}\right] = \frac{ML^{-1}T^{-2}}{LT^{-1}/L} = ML^{-1}T^{-1}$$

Kinematic Viscosity:- In fluid mechanics the ratio of dynamic viscosity to density appears frequently i.e

$$\nu = \frac{\mu}{\rho}$$

This ratio is known as kinematic viscosity

$$[\nu] = \frac{ML^{-1}T^{-1}}{ML^{-3}} = L^2 T^{-1}$$

units of kinematic viscosity are m^2/s and its dimension is $L^2 T^{-1}$.

MUHAMMAD TAHIR WATTOO
SP18-PMT-005 (0344-8563284)

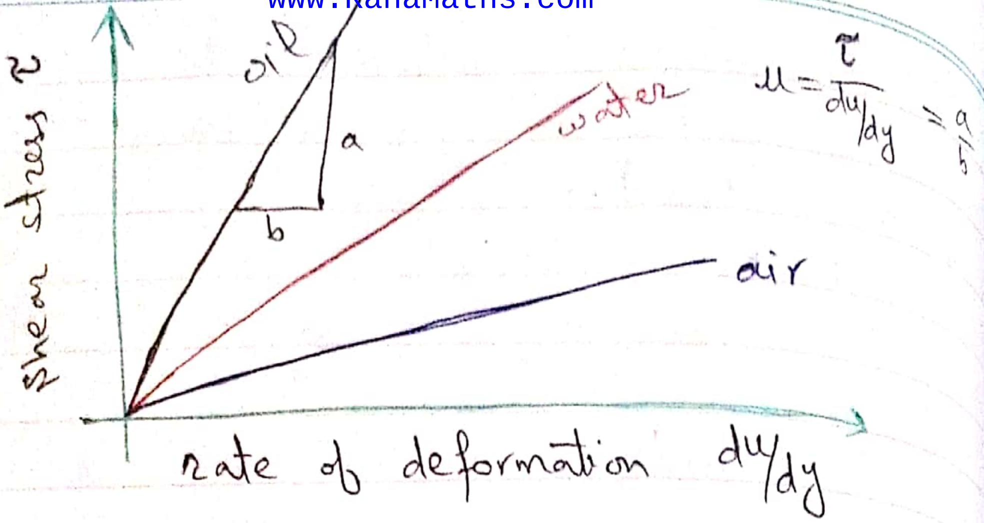
Classification of Fluids

* **Real Fluids:-** (Viscous Fluid) All fluids for which $\mu \neq 0$ are called real or viscous fluids.

* **Ideal Fluids:-** (Inviscid Fluid) All fluids for which $\mu = 0$ are ideal inviscid fluids.

Note that all natural fluids including the synthetic fluids are real fluids and exibit viscosity effects. An inviscid fluid is a frictionless fluid does not exist in nature. However, many fluids under certain engineering conditions (applications) show negligible viscosity effects and can be treated as inviscid fluids.

* **Newtonian Fluids:-** A fluid which obeys the Newton's law of ~~motion~~ viscosity is called a Newtonian fluid. The graphical representation of of shear stress versus shear rate / rate of deformation / velocity gradient is a straight line passes through origin. the slope of the straight line gives the viscosity for the given fluid.



★ Non-Newtonian Fluid:- A non-Newtonian fluid is a fluid whose flow properties differ in any way from those of Newtonian fluids. For example ; Tooth paste, butter, Ketchup, plastic polymers, gels, paints, greases etc are the non-Newtonian fluids.

★ A non-Newtonian fluid does not obey the Newtonian law of viscosity. In this case, shear stress is non-linearly proportional to the velocity gradient. Mathematically for such fluids, power law model holds i.e.

$$\tau \propto \left(\frac{du}{dy}\right)^n \qquad n \neq 1$$

$$\tau = K\left(\frac{du}{dy}\right)^n$$

Where 'n' is called the flow behaviour index is a measure of how the fluid deviates from the Newtonian fluid, (to compare

(to compare different) and
K is called the consistency index
is a measure of the consistency
of fluid.

For $K = \mu$ and $n = 1$ above equ reduces to Newtonian's law of viscosity

$$\tau = K\left(\frac{du}{dy}\right)^{n-1}\left(\frac{du}{dy}\right)$$

$$= \eta \frac{du}{dy}$$

where $\eta = K\left(\frac{du}{dy}\right)^{n-1}$ is called apparent viscosity.

Thus the viscosity of Non-Newtonian fluids is dependent on shear rate/deformation rate.

* Instrument used to measure the viscosity of Newtonian fluids is called viscometer. For the fluids with viscosties which vary with flow conditions an instrument called a rheometer is used.

* **Psuedoplastic/Shearthinning Fluids) $n<1$**

Shearthinning fluids are those in which apparent viscosity decreases with increasing deformation rate. Polymer solution, nail polish, paints, Ketchup, tomato sauce are examples of shearthinning fluids. Another name of shearthinning fluids is psuedoplastic fluids.

When shaken or squeezed out of a bottle, Ketchup will thin and flow but will retain its shape on a burger or plate.

Dilatant (or shear thickening) Fluid:-

If apparent viscosity increase with increasing deformation rate is $n > 1$

Example:- Suspension of starch

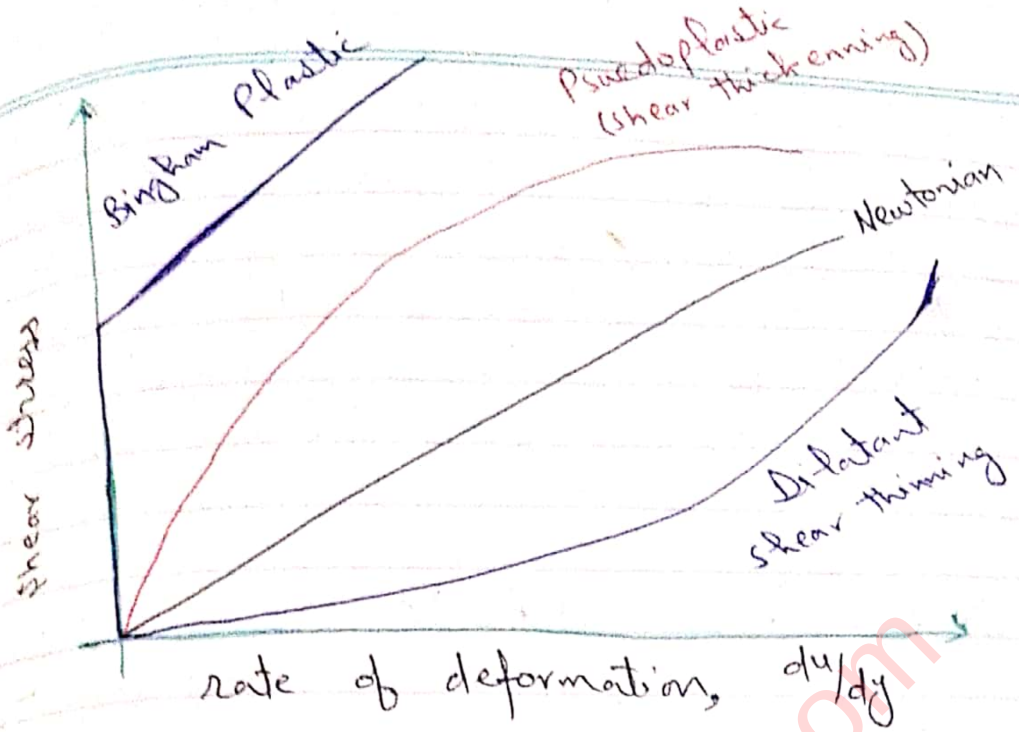
Ideal or Bingham Plastic:-

A Bingham plastic is a material that behaves as a solid at low stresses, but flow as a viscous fluid at a higher stress and exhibit a linear relationship between stress and rate of deformation.

that is, some finite shear stress must be applied before the material flow. This minimum stress is known as yield stress

$$\tau = \tau_y + \mu_p \frac{du}{dy}$$

Example:- mayoinaise, musturd, toothpaste, jellies



Time Dependent Non-Newtonian Fluids

(a) **Thixotropic Fluids** :- Thixotrophy is the property of certain fluids that are thick (viscous) under normal conditions, but flow (become thin, less viscous) over time when stressed. They then take a fixed time to return to a more viscous state. In more technical language, some Non-Newtonian fluids show decrease in apparent viscosity with time under constant applied stress. Some clays are thixotropic, some paints. synovial fluid found in joints of some bones.

b) **Rheopectic Fluid** :- Rheopecty or rheopexy is the rare property of fluids to show a time dependent change in viscosity. The longer the fluid undergoes shearing force, the higher its viscosity.

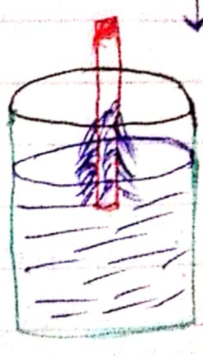
Example:- lubricants, thicken or solidify when shaken. Printer ink, gypsum paste.

(C) ViscoElastic Fluids:- some fluids after deformation partially return to their original shape, when applied stress is released.

Example:- Biopolymers, metals at high temperature, semi crystalline polymers.

Weissenberg Effect / Rod Climbing Effect:-

When a rod is rotated in a beaker containing an Non-Newtonian solution, the solution moves in the opposite direction of motion of rod and climbs up the rod. this effect is known as weissenberg/rod climbing effect.



Die Swell Effect:-

As a polymer exit a die, the diameter of liquid stream increases by us to by upto an order of magnitude.

* Tensors are infact any physical quantity that can be represented by a scaler, vector or matrix

* The order or rank of a tensor refers to the number of directions associated with them.

Index Notation:- The conventions used to represent the components of a vector are typically (u, v, ω) and (u_1, u_2, u_3). The second one i.e (u_1, u_2, u_3) is referred as index notation or indicial notation

$$u_i = u_1\hat{e}_1 + u_2\hat{e}_2 + u_3\hat{e}_3 = (u_1, u_2, u_3) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

→ Vector is a first order tensor

→ Order or rank of a tensor is the number of subscripts it contains

$p \rightarrow$ zeroth order tensor p

$u_i \rightarrow$ implies a vector $u_i = (u_1, u_2, u_3)$

$\sigma_{ij} \rightarrow$ implies a 2nd order tensor

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Summation Index:- If the same

index is repeated in a term, it implies a summation

$$\sum_{i=1}^{3} \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} \rightarrow \text{scaler}$$

$$\frac{\partial u_i}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \rightarrow \text{scaler}$$

$$\sigma_{ij} n_j = \sigma_{i1} n_1 + \sigma_{i2} n_2 + \sigma_{i3} n_3 \rightarrow \text{vector}$$

* The index showing up twice is called the "repeated" or dummy index.

* Indices showing up once are called "free" indices because you can substitute any coordinate dimension to get an equation in that dimension.
In an equation, the free index must be the same in every term

$A_{ij} x_j = b_i$ $\quad$ $i \rightarrow$ free index

$A_{ij} x_j = b_j$ (wrong)

Some Special Tensors:-

1) Kronecker delta / substitution tensor / Identity tensor. It is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{identity matrix}$$

$$\delta_{ij} u_i = \delta_{1j} u_1 + \delta_{2j} u_2 + \delta_{3j} u_3$$

$$= u_j$$

if $j=1$ $\quad \delta_{11} \rightarrow \delta_{ij} u_i = u_1$

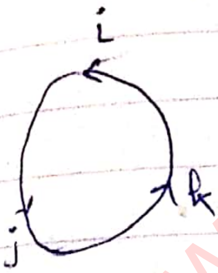
$j=2$ $\quad \delta_{22} \rightarrow \quad = u_2$

$j=3$ $\quad \delta_3 \rightarrow \quad = u_3$

2) Permutation / Alternating Tensor:-

ϵ_{ijk} (3rd order tensor; has 27 components)

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, 312 \\ 0 & \text{if any two indices repeated} \\ -1 & \text{if } ijk = 321, 132, 213 \end{cases}$$



used to form vector product / cross product

$$c_i = \epsilon_{ijk} a_j b_k \quad , \quad \bar{c} = \bar{a} \times \bar{b}$$

Addition:- Vector and tensor addition in tensor notation is written as

$$c_i = a_i + b_i$$

and $$c_{ij} = a_{ij} + b_{ij}$$

Product:- Dot product of two vectors $\bar{a}$ and $\bar{b}$ is written as $a_i b_i$

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

* Symmetric & Antisymmetric Tensors:-

Transpose of tensor obtained by interchanging two indices; the transpose of $A_{ij} = A_{ji}$

→ A tensor is said to be symmetric if it is equal to its transpose

i.e $A_{ij} = A_{ji}$

→ A tensor is said to be antisymmetric if it is equal to negative of its transpose i.e $A_{ij} = -A_{ji}$

→ $$A_{ij} = \frac{1}{2}A_{ij} + \frac{1}{2}A_{ji} + \frac{1}{2}A_{ij} - \frac{1}{2}A_{ji}$$

$$= \underbrace{\frac{1}{2}(A_{ij}+A_{ji})}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(A_{ij}-A_{ji})}_{\text{Antisymmetric}}$$

Let

$$C_{ij} = \frac{1}{2}(A_{ij}+A_{ji}) = \frac{1}{2}(A_{ji}+A_{ij}) = C_{ji}$$

$$B_{ij} = \frac{1}{2}(A_{ij}-A_{ji}) = \frac{-1}{2}(A_{ji}-A_{ij}) = -B_{ji}$$

Hence any tensor can be represented by a linear combination of symmetric and anti-symmetric tensors.

Differentiation of a scalar Function

$f(x)$ with respect to j is

$$\frac{\partial f}{\partial x_j} = f_{,j}$$

* Differential of a vector $\bar{V}$ is

$$\frac{\partial \bar{V}}{\partial x_j} = \left(\frac{\partial V_1}{\partial x_j}, \frac{\partial V_2}{\partial x_j}, \frac{\partial V_3}{\partial x_j}\right) = V_{i,j}$$

Divergence:- Divergence of a vector is a scaler.

As $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x_i}$

$$\nabla \cdot \bar{V} = \left(\frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}\right) = \frac{\partial V_i}{\partial x_i}$$

$$= V_{i,i}$$

Curl of a Vector:-

$$\nabla \times \bar{F} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left[\left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right), \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right), \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)\right]$$

$$(\nabla \times \bar{F})_i = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j}$$

$$= \epsilon_{ijk} F_{k,j}$$

$$\epsilon_{2jk} \frac{\partial F_k}{\partial x_j} = \epsilon_{213} \frac{\partial F_3}{\partial x_2} + \epsilon_{231} \frac{\partial F_1}{\partial x_3}$$

Gradient of a Scaler Function:- Gradient of a scaler function

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \frac{\partial F}{\partial x_i} = F_{,i}$$

Gradient of a vector function

$$\nabla \bar{F} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} \end{bmatrix}$$

$$= \frac{\partial F_i}{\partial x_j} = F_{i,j}$$

Note that gradient always raises the rank of tensor by 1.

Note:- $(\bar{V}\cdot\bar{\nabla})\bar{V} = \left(V_i \frac{\partial}{\partial x_i}\right) V_j$

$$= \left[\left(V_1 \frac{\partial V_1}{\partial x_1} + V_2 \frac{\partial V_1}{\partial x_2} + V_3 \frac{\partial V_1}{\partial x_3}\right), \left(V_1 \frac{\partial V_2}{\partial x_1} + V_2 \frac{\partial V_2}{\partial x_2} + V_3 \frac{\partial V_2}{\partial x_3}\right), \left(V_1 \frac{\partial V_3}{\partial x_1} + V_2 \frac{\partial V_3}{\partial x_2} + V_3 \frac{\partial V_3}{\partial x_3}\right)\right]$$

Field:- The term field refers to a scaler, vector or tensor quantity described by continuous function of time and space coordinates.

Description of a fluid field:- There are two ways to describe the motion of a fluid

1) Lagrangian Description (Info for fixed fluid particle)
2) Eulerian Description (Info for (at) fix point)

Lagrangian View Point:- The Lagrangian view point of fluid mechanics is a natural extension of a particle mechanics. We focus attention on material particles as they move through flow. Each particle in flow is labeled or identified by its original position x_i°. The indipendent variables in the Lagrangian viewpoint are the initial position x_i° and the time $\hat{t}$.

The position x_i of fluid particle is given by $x_i = x_i(x_i^\circ, \hat{t})$

This function given the path of particles with time $\hat{t}$ as a parameter.

The velocity and acceleration of a particle is defined by

$$v_i = \frac{dx_i}{dt}(x_i^\circ, t) = \frac{dx_i}{d\hat{t}}(x_0, y_0, z_0, \hat{t})$$

$$a_i = \frac{d^2 x_i}{d\hat{t}^2}$$

The temperature in Lagrangian variables is given by $T = T_L(x_i^\circ, \hat{t})$

Eulerian Description:- The Eulerian view point has us with a fixed point in space x_i as time proceeds. i.e fluid domain is not followed as it deforms, but rather flows is on fixed spatial domain through which fluid flows.

$$u_i(x_i,t) = \begin{bmatrix} u(x_i,t) \\ v(x_i,t) \\ w(x_i,t) \end{bmatrix}$$

$$\rho = \rho_E(x_i,t)$$

$$T = T_E(x_i,t)$$

Substantial/material derivative:-

Let F be any fluid property of interest (pressure, density, velocity, ...)

$F_L(x_i^\circ, \hat{t}) \longrightarrow$ Lagrangian

$F_E(x_i,t) \longrightarrow$ Eulerian

$$F = F_L(x_i^\circ, t) = F_E(x_i,t)$$

Equating these functions make sense only if

$$F = F_L(x_i^\circ, \hat{t}) = F_E(x_i = x_i(x_i^\circ, \hat{t}), t = \hat{t})$$

$$\frac{\partial F_L}{\partial \hat{t}} = \frac{\partial F_E}{\partial x_i}\frac{\partial x_i}{\partial t} + \frac{\partial F_E}{\partial t}\frac{\partial t}{\partial \hat{t}}$$

$$= \frac{\partial F_E}{\partial t} + u_i \frac{\partial F_E}{\partial x_i}$$

$$\underbrace{\frac{\partial}{\partial \hat{t}} = \frac{D(\ )}{Dt}}_{\text{Lagrangian Derivative}} \equiv \underbrace{\frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}}_{\text{Eulerian frame of reference}}$$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i}$$

$D\rho/Dt \rightarrow$ time rate of change of density of the given fluid element as it moves through space.

Here our eyes are locked on fluid element as it is moving and we are watching the density element change as it moves through a point say P_i. This is different from $\left(\frac{\partial \rho}{\partial t}\right)_{P_i}$ which is physically the time rate of change of density at a fixed point P_i i.e. For $(\partial \rho/\partial t)_{P_i}$ we fix our eyes on stationary point and watch the density due to transient fluctuation in flow field

$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + (V.\nabla)\rho$$

$$(V.\nabla)\rho = u_i \frac{\partial \rho}{\partial x_i}$$ convective derivative

which is physically time rate of change due to movement of fluid element from one location to an other in the flow field where flow properties are spatially different.

$$\frac{D\bar{V}}{Dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i}$$

Note:- $\bar{V}.\bar{\nabla}$ is not strictly a correct vector operator, as it does not obey the commutative rule

$\bar{V}.\bar{\nabla} \neq \bar{\nabla}.\bar{V}$ (psuedo- vector)

MUHAMMAD TAHIR WATTOO

PhD MATHEMATICS

COMSATS UNIVERSITY
ISLAMABAD

SP18-PMT-005
03448563284

Material/Substantial/Total Derivative

The substantial derivative is the total time change of a property experienced/measured by an observer moving with particle.

→ Material derivative is a Lagrangian concept.

→ The material derivative is a link between Eulerian and Lagrangian description.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\bar{V}\cdot\bar{\nabla})$$

$\frac{D}{Dt}$ → Lagrangian rate of change

$\frac{\partial}{\partial t}$ → Eulerian rate of change

$(\bar{V}\cdot\bar{\nabla})$ → Convective rate of change

(i) $$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + (\bar{V}\cdot\bar{\nabla})\rho$$

$$= \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z}$$

$$\Rightarrow \frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + u_i\frac{\partial\rho}{\partial x_i}$$

Here $\frac{D\rho}{Dt}$ ⇒ Instantenceous rate of change of density. It is time rate of change of density of the given fluid element as it is moves through space.

→ $D\rho/Dt$ is different from $\partial\rho/\partial t$ which is physically the time rate of change of density at a fixed point which is due to transient fluctuation in the flow field.

→ $(\bar{V}\cdot\bar{\nabla})\rho$ is called convective change of density. is the time rate of change due to movement of fluid element from one location to another in flow field where flow properties are spatially different.

Acceleration:-

$$\bar{a} = \frac{D\bar{V}}{Dt} = \frac{\partial \bar{V}}{\partial t} + (\bar{V}\cdot\bar{\nabla})\bar{V}$$

convective acceleration

Local acceleration

In cartesian coordinates

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

In tensor notation

$$a_i = \frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}$$

effect of fluid particle moving to a new location in flow, where velocity is different change of density at a fixed point which is due to transient fluctuation in the flow field.

→ $(\bar{V}\cdot\bar{\nabla})\rho$ is called convective change of density. is the time rate of change due to movement of fluid element from one location to another in flow field where flow properties are spatially different.

Acceleration:-

$$\bar{a} = \frac{D\bar{V}}{Dt} = \underbrace{\frac{\partial \bar{V}}{\partial t}}_{\text{Local acceleration}} + \underbrace{(\bar{V}\cdot\bar{\nabla})\bar{V}}_{\text{convective acceleration}}$$

In cartesian coordinates

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

In tensor notation

$$a_i = \frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}$$

effect of fluid particle moving to a new location in flow, where velocity is different

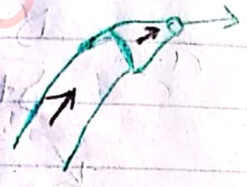
concept of steady flow:- $\left(\frac{\partial}{\partial t} \equiv 0\right)$

A steady flow is strictly an Eulerian concept i.e flow is steady when in the Eulerian description none of variables depends on time

$$\frac{\partial \bar{v}}{\partial t} = 0 \;, \quad \frac{\partial \rho}{\partial t} = 0 \;, \quad \frac{\partial T}{\partial t} = 0$$

Examples

Consider the steady flow of water through a garden hoze nozzle. We define the steady in Eulerian frame of refrence to be when the properties in the flow field do not change with respect to time. Since the velocity at exist of the nozzle is larger than that at the nozzle entrance, fluid particle clearly accelerates even flow is steady.



→ The acceleration is non zero because of convective acceleration.

→ $\frac{D}{Dt} \equiv 0$ does not mean steady since the flow could speed up at some points and slow down at other. i.e while the flow is steady from the point of view of Eulerian refrence frame it is not steady from Lagrangian refrence frame moving with a fluid particle that enters the nozzle and accelerates as it passes through the nozzle.

Concept of Incompressible Flow:-

$\frac{D\rho}{Dt} = 0$ (Density of fluid particle is constant)

→ Incompressible flow is strictly a Lagrangian concept

→ For steady flow $\frac{\partial \rho}{\partial t} = 0$

i.e density at a particular point in the flow field is constant, and would allow particle to change density as they move from point to point i.e $(\bar{V}\cdot\nabla)\rho \neq 0$

→ All substances are compressible through a certain extent i.e of variable density. However, in many particular solutions the density variations are so small that they can be neglected and density can be considered constant. Flows that are modeled assuming constant density are called incompressible.

$$\frac{\partial \rho}{\partial t} = 0 \quad , \quad \frac{\partial \rho}{\partial x_i} = 0$$

Uniform Flow:- The flow is defined as uniform when in the flow field the velocity and other properties do not change from point to point at any instant of time.

→ For example, flow from a long straight pipe of constant diameter is uniform flow, on other hand flow through a pipe of reducing section or through curved pipe or through a converging / diverging channel.

★ Steady uniform flow

$$\Rightarrow \frac{\partial}{\partial t} \equiv 0 \,,\; u_i \frac{\partial}{\partial x_i} \equiv 0$$

★ Steady non-uniform flow

$$\Rightarrow \frac{\partial}{\partial t} \equiv 0 \,,\; u_i \frac{\partial}{\partial x_i} \not\equiv 0$$

★ Unsteady uniform flow

$$\Rightarrow \frac{\partial}{\partial t} \not\equiv 0 \,,\; u_i \frac{\partial}{\partial x_i} \equiv 0$$

★ Unsteady non-uniform flow

$$\Rightarrow \frac{\partial}{\partial t} \not\equiv 0 \,,\; u_i \frac{\partial}{\partial x_i} \not\equiv 0$$

Problems.-1:- Consider steady, incompressible two dimensional flow through a converging duct. A simple approximation velocity field is given by

$$\vec{V} = (u, v) = (u_0 + bx)\,i - by\,j$$

Calculate the material acceleration of the fluid particles passing through this duct.

Solution

$$u = u_0 + b_x \quad ; \quad v = -b_y$$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= 0 + (u_0 + b_x)b + (-by)0 + 0$$

$$= b(u_0 + bx)$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$= 0 + (u_0 + bx)0 + (-by)(-b) + 0$$

$$= b^2 y$$

In vector form

$$\bar{a} = b(u_0 + bx)\,i + b^2 y\,j$$

For positive x and b fluid particles accelerate in +ve x direction. Even this flow is steady there is still a non-zero acceleration field (Non-uniform flow)

Problem 2:- The pressure field of problem (1) is given by

$$P = P_0 - \frac{\rho}{2}\left[2u_0 b x + b^2(x^2 + y^2)\right]$$

where P_0 is the pressure at $x=0$. Generate an expression for the rate of change of pressure following a fluid particle.

Solution

Flow is steady, incompressible, two dimensional.

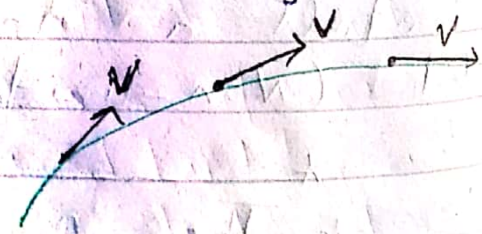
By definition, material derivative when applied to pressure produces the rate of change of pressure following a fluid particle

$$\frac{DP}{Dt} = \underbrace{\cancel{\frac{\partial P}{\partial t}}}_{\text{steady}} + u\frac{\partial P}{\partial x} + v\frac{\partial P}{\partial y} + \underbrace{\cancel{w\frac{\partial P}{\partial z}}}_{\text{Two dimensional}}$$

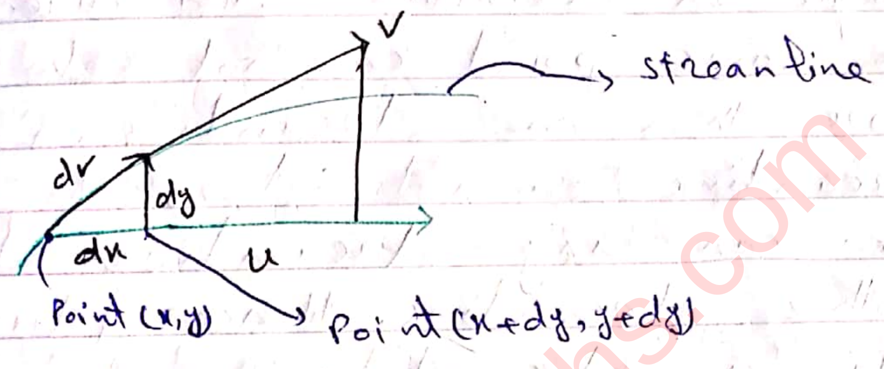
$$= (u_0 + bx)\left[-\rho u_0 b - \rho b^2 x\right] + (-by)\left[-\rho b^2 y\right]$$

$$\frac{DP}{Dt} = \rho\left[-u_0^2 b - 2u_0 b^2 x + b^3(y^2 - x^2)\right]$$

Streamlines:- The streamline is the line tangent at every point to the velocity vector i.e. stream line is a curve drawn in fluid such that tangent to it at every point is in the direction of velocity vector $\bar{V}$. So, it indicates the velocity direction.



Equation of Stream Line:- Consider an infinitesimal arc length $d\bar{r} = dx\,i + dy\,j$ along a streamline. By definition of streamline $d\bar{r}$ must be parallel to local velocity vector $\bar{V} = u\,i + v\,j$



Therefore

$$d\bar{r} \times \bar{V} = \bar{0}$$

$$\begin{vmatrix} i & j & k \\ dx & dy & 0 \\ u & v & 0 \end{vmatrix} = \bar{0}$$

$$-u\,dy + v\,dx = 0$$

$$\frac{dx}{u} = \frac{dy}{v} \longrightarrow (1)$$

Cylindrical coordinates

$$\begin{vmatrix} e_r & e_\theta & 0 \\ dr & r\,d\theta & 0 \\ V_r & V_\theta & 0 \end{vmatrix}$$

$$\frac{dr}{dv} = \frac{v\,d\theta}{V_\theta}$$

equation (1) is called equation of stream line for two dimensional flow field.

A streamtube consists of a bundle of streamlines, much lik a communication cable consists of bundle of fibre optic cables.

→ By definition of streamline it is clear that components of velocity normal to streamline is zero, so no flow cross the streamline.

→ The instanteneous velocity at a point in the fluid flow must be unique in magnitude and direction, the same point can not belong to more than one streamline. So, a streamline cannot intersect itself nor can any streamline intersect another streamline. Hence fluid within a streamtube must remain there and can not cross the boundary. Thus all the fluid entering at one end of the streamtube must eventually leave from the other end. Hence a streamtube behave like a solid tube.

Path line :- A path line is the actual path travelled by an individual fluid particle over some time period. That is, if we fix our attention on a fluid particle, the curve which particle describe through the motion called a path-line. Since it is the time history of the position of a fluid particle it is best described using the Lagrangian description.

Note:- When the motion is steady, so the fluid pattern does not vary with time, the pathlines coincides with streamlines. However, when the fluid motion is unsteady, flow pattern varies with time and the path line in general doesnot coincides with streamlines.

* Since the particle velocity is known at each spatial point, the trajectory coordinates $x(t)$, $y(t)$, $z(t)$ can be obtained by integrating the equation of motion

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w$$

As boundary condition will need the position of particle at a given time. Then, the variable time "t" can be eliminated to reach the equation of trajectory in explicit or implicit form

Problem 3:- Find the flow stream lines of velocity field

$$\vec{V} = (u, v) = (U_0 + bx)\hat{i} - by\hat{j}$$

Solution

Equation of streamline

$$\frac{dx}{u} = \frac{dy}{v}$$

$$\frac{dx}{U_0 + bx} = \frac{dy}{-by}$$

$$\frac{1}{b}\ln(u_0+bx)+\frac{1}{b}\ln(by)=\frac{1}{b}\ln c_1$$

$$\ln\left[(u_0+bx)\,y\right]=\frac{1}{b}\ln c_1$$

$$\Rightarrow \quad y(u_0+bx)=c_1^{1/b}$$

$$\Rightarrow \quad y=\frac{c}{u_0+bx}$$

Each value of 'c' yields a unique streamline.

Problem 4:- Converging duct flow is modeled by steady two dimensional velocity field as of problem ③ $\bar{V}=(u_0+bx)\hat{i}-by\hat{j}$. A fluid particle is located on the x-axis at $x=x_A$ at time $t=0$. At some later time t, fluid particles has moved downstream to new location $x=x_{A'}$. Generate an analytical expression for x-location of fluid particle at some $t=0$ arbitrary time times in terms of initial location x_A and constants u_0 and b.

Solution

x-component of velocity of fluid particle

$$\frac{dx_{\text{particle}}}{dt}=u=u_0+bx_{\text{particle}}$$

$$\frac{dx}{dt} = u = u_0 + bx$$

$$\frac{dx}{u_0 + bx} = dt$$

Integrating $\frac{1}{b}\ln(u_0 + bx) = t - \frac{1}{b}\ln C_1$

$$\ln\left[C_1(u_0 + bx)\right] = t$$

$$u_0 + bx = C_2 e^{bt} \longrightarrow \text{(i)}$$

at $t = 0$, $x = x_A$

$$u_0 + bx_A = C_2 e^{b(0)}$$

$$C_2 = u_0 + bx_A$$

(i) $\Rightarrow$ $u_0 + bx = [u_0 + bx_A]e^{bt}$

$$x = \frac{1}{b}\left[(u_0 + bx_A)e^{bt} - u_0\right]$$

$$x = x_A' = \frac{1}{b}\left[(u_0 + bx_A)e^{bt} - u_0\right]$$

varify at $t = 0$, $x = x_A$

$\Rightarrow$ $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho$

$$= \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$$

We observe that the equation $\frac{D\rho}{Dt} = 0$

would imply that fluid is incompressible. On the other hand, the equation $\frac{\partial \rho}{\partial t} = 0$ would imply that the flow is steady i.e density is forever the same at a particular point in space. Thus in this case, the density is constant w.r.t time but it may vary at different points due to convective effect alone.

Thus we have established the equivalence of operators

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{V}.\nabla \longrightarrow ①$$

or in cartesian form

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \longrightarrow ②$$

applicable to both scaler or vector functions of position and time, provided that these functions are associated with properties of the moving fluid.

Material Derivative in Cylinderical Coordinates:- We know that in cartesian coordinate

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \rightarrow ①$$

where u, v and w are the velocity components in x, y and z direction respectively. Also

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

therefore $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$, $z = z$

Let V_r, V_θ, V_z be the velocity components in the r, θ and z directions respectively.

Then $$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x}$$

$$= \frac{x}{r}\frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial \theta} + 0$$

$$= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

Now $u = V_r\cos\theta - V_\theta\sin\theta$, therefore

$$u\frac{\partial}{\partial x} = (V_r\cos\theta - V_\theta\sin\theta)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$= \cos^2\theta V_r\frac{\partial}{\partial r} - \frac{\sin\theta\cos\theta}{r}V_r\frac{\partial}{\partial \theta} - \sin\theta\cos\theta V_\theta\frac{\partial}{\partial r}$$

$$+ \frac{\sin^2\theta}{r}V_\theta\frac{\partial}{\partial \theta} \longrightarrow (2)$$

similarly $$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial z}\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial y} = \frac{y}{r}\frac{\partial}{\partial r} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial z}\frac{\partial z}{\partial y}$$

$$= \frac{y}{r}\frac{\partial}{\partial r} + \frac{x}{r^2}\frac{\partial}{\partial \theta} + 0 = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

and $v = V_r\sin\theta + V_\theta\cos\theta$, therefore

$$v\frac{\partial}{\partial y} = (V_r\sin\theta + V_\theta\cos\theta)\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$= \sin^2\theta V_r\frac{\partial}{\partial r} + \frac{\sin\theta\cos\theta}{r}V_r\frac{\partial}{\partial \theta} +$$

$$\sin\theta\cos\theta V_\theta\frac{\partial}{\partial r} + \frac{\cos^2\theta}{r}V_\theta\frac{\partial}{\partial \theta} \longrightarrow (3)$$

Finally $$\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial}{\partial z}\frac{\partial z}{\partial z}$$

$$= 0 + 0 + \frac{\partial}{\partial z}(1) = \frac{\partial}{\partial z}$$

But $\omega = V_z$ therefore, $\omega \frac{\partial}{\partial z} = V_z \frac{\partial}{\partial z}$ → (4)

Substituting from (2), (3), and (4) in equ (1) we get

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\sin^2\theta + \cos^2\theta) V_r \frac{\partial}{\partial r} + (\sin^2\theta + \cos^2\theta) \frac{V_\theta}{r} \frac{\partial}{\partial \theta} + V_z \frac{\partial}{\partial z}$$

or $$\frac{D}{Dt} = \frac{\partial}{\partial t} + V_r \frac{\partial}{\partial r} + \frac{V_\theta}{r} \frac{\partial}{\partial \theta} + V_z \frac{\partial}{\partial z}$$

Material Derivative in Spherical Coordinates:- We know that

$$x = r\sin\theta\cos\phi \;,\; y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

and $$r = \sqrt{x^2+y^2+z^2} \;;\; \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$$
$$\phi = \tan^{-1} y/x$$

Let V_r, V_θ and V_ϕ be the velocity components in the r, θ and ϕ directions respectively, then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial x}$$
$$= \frac{x}{r}\frac{\partial}{\partial r} + \frac{xz}{r^2\sqrt{x^2+y^2}}\frac{\partial}{\partial \theta} - \frac{y}{x^2+y^2}\frac{\partial}{\partial \phi}$$
$$= \sin\theta\cos\phi \frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \longrightarrow (6)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial y}$$

$$= \frac{y}{r}\frac{\partial}{\partial r} + \frac{yz}{z^2\sqrt{x^2+y^2}}\frac{\partial}{\partial \theta} + \frac{x}{x^2+y^2}\frac{\partial}{\partial \phi}$$

$$= \sin\theta \sin\phi \frac{\partial}{\partial r} + \frac{\cos\theta \sin\phi}{r}\frac{\partial}{\partial \theta} +$$

$$\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \longrightarrow (7)$$

& $$\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial z}$$

$$= \frac{z}{r}\frac{\partial}{\partial r} - \frac{\sqrt{x^2+y^2}}{r^2}\frac{\partial}{\partial \theta} + 0$$

$$= \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \longrightarrow (8)$$

Now

$$\left.\begin{aligned} u &= V_r \sin\theta \cos\phi + V_\theta \cos\theta \cos\phi - V_\phi \sin\phi \\ v &= V_r \sin\theta \sin\phi + V_\theta \cos\theta \sin\phi + V_\phi \cos\phi \\ w &= V_r \cos\theta - V_\theta \sin\theta \end{aligned}\right\} \rightarrow (9)$$

Substituting equ (6) to (9) in equ (1) we get

$$D/Dt = \frac{\partial}{\partial t} + (V_r \sin\theta \cos\phi + V_\theta \cos\theta \cos\phi - V_\phi \sin\phi)\left(\sin\theta \cos\phi \frac{\partial}{\partial z} + \frac{\cos\theta \cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{z\sin\theta}\frac{\partial}{\partial \phi}\right) + (V_z \sin\theta \sin\phi + V_\theta \cos\theta \sin\phi + V_\phi \cos\phi)\left(\sin\theta \sin\phi \frac{\partial}{\partial r} + \frac{\cos\theta \cos\phi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\right)$$

$$+ (V_z \cos\theta - V_\theta \sin\theta)\left(\cos\theta \frac{\partial}{\partial z} - \frac{\sin\theta}{z}\frac{\partial}{\partial \theta}\right)$$

After simplification, we get.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta) V_r \frac{\partial}{\partial r}$$
$$+ (\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta)\frac{V_\theta}{r}\frac{\partial}{\partial\theta} +$$
$$\left(\frac{\sin^2\phi}{r\sin\theta} + \frac{\cos^2\phi}{r\sin\theta}\right) V_\phi \frac{\partial}{\partial\phi}$$

or

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial\theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \longrightarrow (10)$$

is the expression for the material derivative in spherical polar coordinate.

MUHAMMAD TAHIR WATTOO

PhD MATHEMATICS

COMSATS UNIVERSITY
ISLAMABAD

SP18-PMT-005
0344-8563284

Body Forces:- Forces acting upon the whole material volume which are developed without physical contact are called body forces.

Forces due to gravity, electric field, magnetic field are body forces.

→ This force can be defined in two different ways.

The first way is a volume force, which is defined as the exerted force per unit volume $\bar{f}_V = \frac{\text{force}}{\text{volume}}$

Dimension of $\bar{f}_V = \frac{MLT^{-2}}{L^3} = ML^{-2}T^{-2}$

units $= N/m^3$

It can also be defined as massic force, the force exerted over the body per unit mass $\bar{f}_m = \frac{\text{force}}{\text{mass}}$

Dimension of $\bar{f}_m = \frac{MLT^{-2}}{M} = LT^{-2}$

Unit $= N/Kg$

The differential force $d\bar{f}_V$ acting on a differential element of volume dV and mass $dm = \rho\, dV$ can be calculated as

$$d\bar{F}_V = f_V\, dV \longrightarrow (i)$$

$$d\bar{F}_V = f_m\, dm = f_m \rho\, dV \longrightarrow (ii)$$

The total force over a given domain V can be calculated by integration

$$\bar{F}_v = \int_V f_v \, dv \longrightarrow (iii)$$

$$\bar{F}_v = \int_V \rho \bar{f}_m \, dv \longrightarrow (iv)$$

Note:- Note that the both type of forces (per unit mass or volume) can be used to compute the total force F_v exerted over a body. As a consequence

$$\bar{f}_v = \rho \bar{f}_m \longrightarrow (v) \qquad ML^{-2}T^{-2} = (ML^{-3})(LT^{-2}) = ML^{-2}T^{-2}$$

Example:- (i) The gravity $\bar{f}_v = \rho \bar{g}$ where ρ is the fluid density (with dimension ML^{-3}) and $\bar{g}$, the acceleration of the gravitational field (with dimensions LT^{-2}) and $|g| = 9.81\ m/s^2$

$$[\rho g] = [ML^{-3}][LT^{-2}] = ML^{-2}T^{-2}$$

ii) The electromagnetic force

$$\bar{f}_v = \rho_e \bar{E} + \bar{J} \times \bar{B}$$

$\bar{f}_v$: force dencity (force per unit volume)

ρ_e: charge density (charge per unit volume)

$\bar{E}$:- electric field

$\bar{B}$:- Magnetic field

$\bar{J}$:- current density

Surface Forces:- These forces act upon the surface of fluid element by its surroundings through direct contact. For example pressure.

→ Viscous force (i.e. frictional) force in a moving fluid are also surface forces.

→ These forces are computed from stresses

$$\bar{f}_s = \frac{\text{force}}{\text{surface}}$$

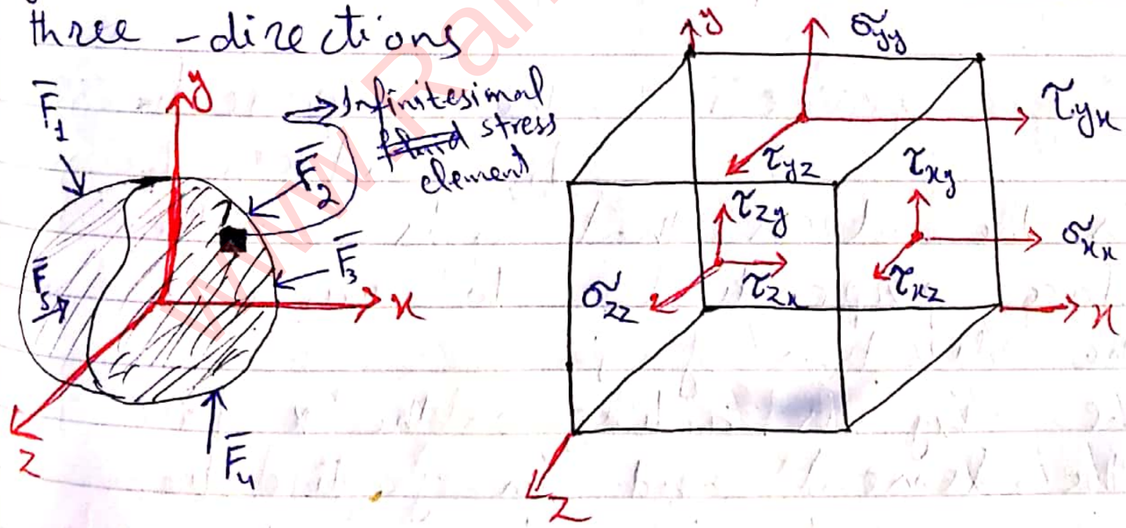
The force acting on a differential element of surface ds is

$$d\bar{F}_s = \bar{f}_s\, ds \longrightarrow (i)$$

whose integration over the given surface s yields the total force

$$\bar{F}_s = \int_s \bar{f}_s\, ds \longrightarrow (ii)$$

Stress Tensor:- consider an arbitrary material subject to different forces. Due to these forces stress get induced in the material in all three-directions



Let us consider an infinitesimal stress element which shows the state of stress at a point in given stress ~~at a point~~ material / body.

→ stress element consists of six faces

in three dimensions. The face perpendicular to positive direction is positive face and to negative direction is called negative face.

→ The stress element is subject to three different normal stresses (σ_{xx}, σ_{yy}, σ_{zz}) and six different shear stresses (τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, τ_{zy}).

→ The complete stress matrix (a multi-dimensional array) is given below

$$\tau = \tau_{ij} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Two indices

$3^2 = 9$

→ Tensor of order/rank 2.

Sign Convection:- The component τ_{ij} of a stress tensor is the stress that acts on the plane face perpendicular to axis i and in the direction of j

i.e τ_{xy} is acting on the x-face of the element and in y-direction.

σ_{xx} is acting on the x-face (face ⊥ to x axis) and in x-direction

→ shear stress is +ve when it acts on positive face (or for ground face where the normal vector is aligned to coordinate axis) and +ve direction of an axis. It is also +ve when it acts on negative face (or background face where the normal vector is in opposite direction to the coordinate axis) and negative direction.

→ Shear stress is negative when it acts in positive face and negative direction. It is also negative when it acts in negative face and +ve direction of axis.

→ An important property of τ is its symmetry i.e $\tau_{ij} = \tau_{ji}$

Stress Vector on any surface:-

$f_{si} = \tau_{ij} n_j \rightarrow$ inner product of stress tensor $\bar{\bar{\tau}}$ and normal vector $\bar{n}$ $(\bar{\bar{\tau}} \cdot \bar{n})$

i=1 $f_{s1} = \tau_{1j} n_j$

$= \sigma_{11} n_1 + \tau_{12} n_2 + \tau_{13} n_3$

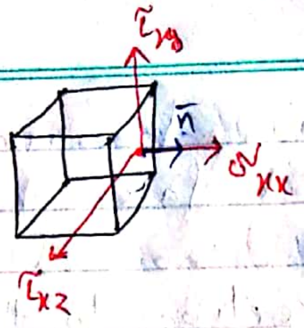
i=2 $f_{s2} = \tau_{2j} n_j$

$= \tau_{21} n_1 + \sigma_{22} n_2 + \tau_{23} n_3$

i=3 $f_{s3} = \tau_{3j} n_j$

$= \tau_{31} n_1 + \tau_{32} n_2 + \sigma_{33} n_3$

i.e
$$\begin{bmatrix} f_{s1} \\ f_{s2} \\ f_{s3} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$



$$\sigma_{xx} = \begin{bmatrix} \sigma_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \sigma_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$

$$\bar{n} = [1, 0, 0]$$

$$f_{s_1} = \sigma_{11} = \sigma_{xx}$$
$$f_{s_2} = \tau_{12} = \tau_{xy}$$
$$f_{s_3} = \tau_{13} = \tau_{xz}$$

surface force

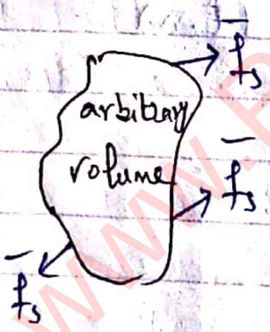
$$\vec{f_s} = [\sigma_{xx}, \tau_{xy}, \tau_{xz}]$$

forces stress in x-direction.

$$\bar{f}_s = \tau_{ij} n_j = \bar{\bar{\tau}} \cdot \bar{n}$$

→ If $\bar{\bar{\tau}}$ is known, the surface force acting on any direction can be calculat[ed]

→ f_{s_1}, f_{s_2} and f_{s_3} are called stress vectors



force per unit area acting locally at point

Total force due to these surface forces, we have to integrate

$$\bar{F}_s = \int_s \bar{f}_s \, ds = \int_s \bar{\bar{\tau}} \cdot \bar{n} \, ds$$

$\int \bar{\nabla} \cdot \bar{\bar{\tau}} \, dV$

divergence theo[rem]

$$= \int_s \tau_{ij} n_j \, ds$$

Note:- Divergence of a tensor is a vector.

$$\bar{\nabla}\cdot\bar{\bar{\tau}} = \frac{\partial}{\partial x_i}\tau_{ij}$$

$$(\bar{\nabla}\cdot\bar{\bar{\tau}})_1 = \frac{\partial}{\partial x_i}\tau_{i1}$$

$$= \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial z}$$

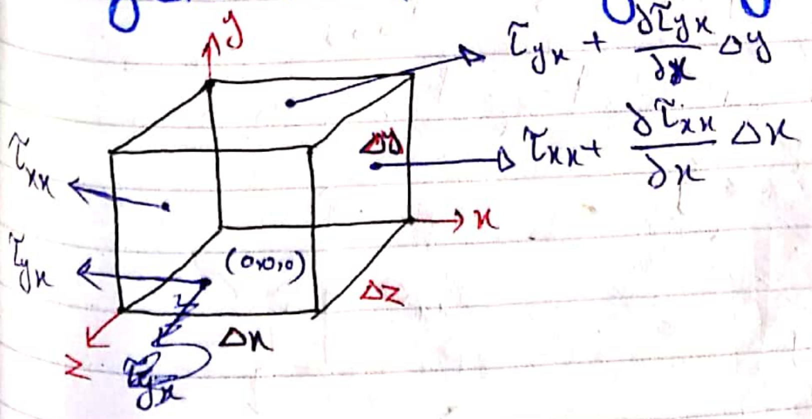
$$\Rightarrow (\bar{\nabla}\cdot\bar{\bar{\tau}})_x = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$(\bar{\nabla}\cdot\bar{\bar{\tau}})_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$(\bar{\nabla}\cdot\bar{\bar{\tau}})_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}$$

$$(\bar{\nabla}\cdot\bar{\bar{\tau}}) = \left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}, \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}, \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right]$$

Physical Meaning of $\nabla\cdot\tau$



X-Forces:- $\left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}\Delta x - \tau_{xx}\right)\Delta y \Delta z$

$$+\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y - \tau_{yx}\right)\Delta x \Delta z$$

$$+\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\Delta z - \tau_{zx}\right)\Delta x \Delta y$$

$$F_x = \underbrace{\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)}_{\text{same as first component of stress tensor}} \underbrace{\Delta x \Delta y \Delta z}_{\text{volume}}$$

$\nabla \cdot \bar{\bar{\tau}}$ = Net force per unit volume.

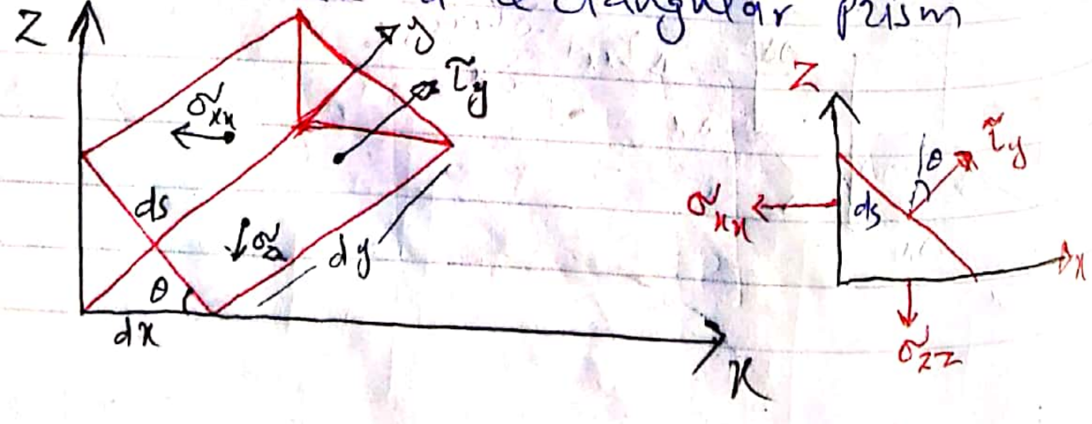
Concept of Pressure:- Let us take fluid at rest. According to definition fluid at rest can not with stand any shear stress/tangential stress i.e.

$$\tau_{12} = \tau_{13} = \tau_{23} = 0$$

$$\tau_{xy} = \tau_{xz} = \tau_{yz} = 0$$

$$\bar{\bar{\tau}} = \tau_{ij} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \longrightarrow (i)$$

Let us take a rectangular prism



Equilibrium of forces in x-direction

$$-\sigma_{xx}\,dz\,dy + \tau_y\,ds\,dy\sin\theta = 0 \longrightarrow (ii)$$

Also $\dfrac{dz}{ds} = \sin\theta \longrightarrow (iii)$

$$dz = ds\sin\theta$$

from (ii) and (iii)

$$-\sigma_{xx}\,ds\,dy\sin\theta + \tau_y\,ds\,dy\sin\theta = 0$$

or

$$-\sigma_{xx} + \tau_y = 0$$

i.e $\sigma_{xx} = \tau_y$ (for any z)

Pressure is independent of direction at any point as long as there are no shearing stress pressure

Pascal's Law

Similarly

$$\sigma_{xx} = \sigma_{zz}$$

Hence, in a fluid at rest, the normal stresses are identical, in any direction. This normal stress is called pressure $P>0$, and because it is negative, so that

$$\tau_y = \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -P$$

Therefore the stress tensor in a fluid at rest is

$$\bar{\bar{\tau}} = \tau_{ij} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} = -PI$$

or, $\tau_{ij} = -P\delta_{ij}$

$i=1, j=1$ $\quad \sigma_{11} = -P\delta_{11} = -P$ $\quad$ $i=1, j=2$

$i=j=2$ $\quad \sigma_{22} = -P\delta_{22} = -P$ $\quad$ $-P\delta_{12} = 0$

* Note that pressure is a scaler and it acts equally in any spacial direction.
* The direction of the force that the pressure causes is determined by the normal to the surface.
* Minus sign indicates that pressure is a negative normal stress, also called compression, that acts in opposite direction to exterior normal.
* This type of tensor, proportional to identity tensor, is called an isotropic tensor.

Fundamental Equation of Fluid Statics:-

Let us take an arbitrary fluid volume V, with surface S. For a fluid at rest a force balance yield

$$\sum \bar{F} = 0$$

Taking into considration body and surface forces acting over the volume V.

$$\bar{F}_V + \bar{F}_s = \bar{0} \longrightarrow (i)$$

i.e

$$\int_V \rho f_m \, dV + \int_S \tau_{ij} n_j \, ds = 0 \longrightarrow (ii)$$

as

$$\tau_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} = -pI = -p\delta_{ij}$$

$$\int_V \rho f_m \, dV + \int_S (-p)\,\delta_{ij} n_j \, ds = 0$$

$$\int_V \rho f_m \, dV - \int_S p n_j \, ds = 0$$

Gauss or divergence

$\int \bar{G}\cdot\hat{n}\, ds = \int_V \bar{\nabla}\cdot\bar{G}\, dV$

Applying divergence / Gauss theorem to transform 2nd integral to volume integral

$$\int_V (\rho f_m - \nabla p)\, dV = 0$$

since above equation holds for any volume, taking $V \to 0$

$$\rho f_m - \nabla p = 0$$

$$\nabla p = \rho f_m$$

For fluid in static equilibrium, the gradient of pressure is force per unit volume.

* The net force caused by pressure depends on the pressure gradient, i.e. on spatial variations of pressure. If the pressure is uniform, then it causes not force over fluid particle.

Fundamental Equation of Fluid Statics:-

Let us take an arbitrary fluid volume V, with surface S. For a fluid at rest a force balance yields

$$\sum F = 0$$

considering body and surface forces acting over the volume V,

$$\bar{F}_V + \bar{F}_S = 0 \longrightarrow (1)$$

i.e

$$\int_V \rho f_m dV + \int_S \bar{\bar{\tau}} \cdot \bar{n} \, dS$$

or

$$\int_V \rho f_m dV + \int_S \tau_{ij} n_j dS = 0 \longrightarrow (2)$$

As fluid at rest can only withstand normal stress and the stress tensor takes the form

$$\tau_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} = -p\delta_{ij} \longrightarrow (3)$$

substituting (3) in (2) we get

$$\int_V \rho f_m dV + \int_S (-p\delta_{ij} n_j) dS = 0$$

or

$$\int_V \rho f_m dV + \int_S -p n_i dS = 0 \longrightarrow (4)$$

By applying Gauss/Divergence theorem

to transform 2nd integral in ④ into volume integral

$$\int_V \rho f_m \, dV + \int_V -\nabla P \, dV = 0$$

or $$\int_V (\rho f_m - \nabla P) \, d\bar{V} = 0$$

$$\rho f_m = \bar{\nabla} P \longrightarrow ⑤$$

This is the fundamental equation of fluid static.

In cartesian components

$$\begin{Bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{Bmatrix} = \rho \begin{Bmatrix} f_{mx} \\ f_{my} \\ f_{mz} \end{Bmatrix} \longrightarrow ⑥$$

* The presence of body force leads to a non-zero gradient of pressure parallel to body force even in a stationary fluid.

Hydrostatic Equation:- For the body force due to the acceleration of gravity can be written as

$$\bar{f}_V = \rho f_m = \rho \bar{g} = \begin{Bmatrix} 0 \\ 0 \\ -\rho g \end{Bmatrix} \longrightarrow ⑦$$

substituting ⑦ into ⑥ we get

$$\partial P / \partial x = 0 \longrightarrow ⓐ$$

$$\left.\begin{array}{l} \dfrac{\partial P}{\partial y} = 0 \longrightarrow (b) \\[2ex] \partial P/\partial z = -\rho g \longrightarrow (c) \end{array}\right| \longrightarrow (8)$$

The first two equations imply $P \neq P(x, y)$ i.e $P = P(z)$ only. For incompressible fluid density being constant. (8c) can be written as

$$\frac{dP}{dz} = -\rho g \longrightarrow (9)$$

integrating between two points, we get

$$\int_{P_1}^{P_2} \frac{dP}{dz} = -\rho g \int_{z_1}^{z_2} dz \longrightarrow (10)$$

$$P_2 - P_1 = -\rho g(z_2 - z_1) \longrightarrow (11)$$

* The pressure difference between two points inside a liquid depends only on height difference between points.
* The pressure increases linearly with depth.

⇒ Recall the velocity gradient/Deformation tensor:- $D_{ij} = \nabla \bar{V} = \text{grad}\, \bar{V} = \dfrac{\partial u_i}{\partial x_j} \longrightarrow (12)$

$$D_{ij} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\[2ex] \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} \\[2ex] \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\[2ex] \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{bmatrix}$$

Any 2nd order tensor can be decomposed into symmetric and anti symmetric part

$$D_{ij} = \frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \longrightarrow (13)$$

$$\Rightarrow D_{ij} = \epsilon_{ij} + \omega_{ij} \longrightarrow (14)$$

where $\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \longrightarrow (15)$ is symmetric part i.e. $\epsilon_{ij} = \epsilon_{ji}$ and

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \longrightarrow (16)$$ is anti-sym Part i.e $\omega_{ij} = -\omega_{ji}$

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

$$\epsilon_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{\partial u_1}{\partial x_1}$$

i.e $\epsilon_{xx} = \partial u / \partial x$

Similarly $\epsilon_{yy} = \frac{\partial v}{\partial y}$ & $\epsilon_{zz} = \frac{\partial w}{\partial z}$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \epsilon_{21}$$

i.e. $\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \epsilon_{yx}$

Similarly

$$\epsilon_{xz} = \epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\epsilon_{zy} = \epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Trace:- $\text{Trace}(\epsilon_{ij}) = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$

$$= \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$$

$$= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$\Rightarrow \text{trace}(\epsilon_{ij}) = \frac{\partial u_i}{\partial x_i} \rightarrow \text{divergence of } \bar{V}$$

$$\longrightarrow (17)$$

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\Rightarrow \omega_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_1}{\partial x_1}\right) = 0 = \omega_{22} = \omega_{33}$$

$$\omega_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = -\omega_{21}$$

i.e $\omega_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = -\omega_{yx}$

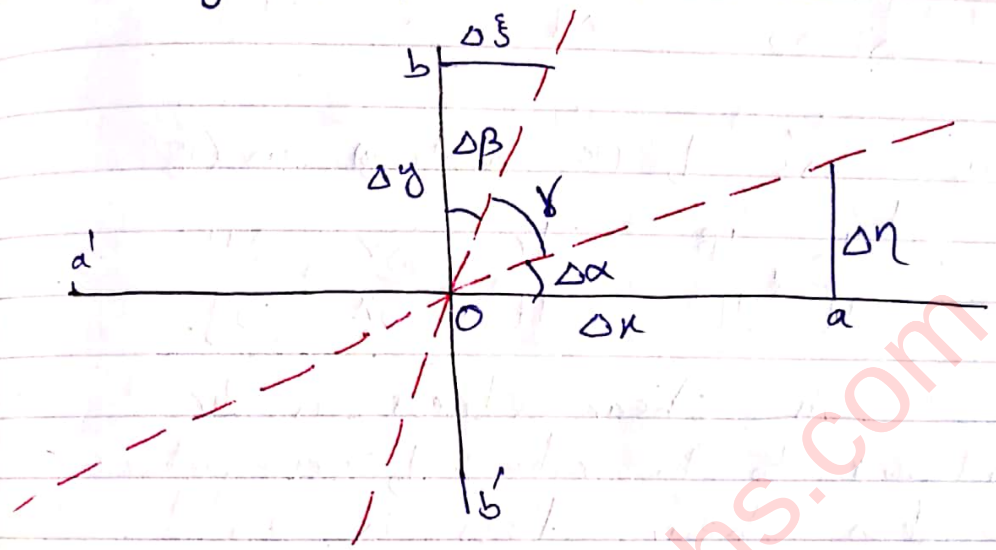
$$\omega_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = -\omega_{zx}$$

$$\omega_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) = -\omega_{zy}$$

$$\omega_{ij} = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} = \begin{bmatrix} \omega_{xx} & \omega_{xy} & \omega_{xz} \\ \omega_{yx} & \omega_{yy} & \omega_{yz} \\ \omega_{zx} & \omega_{zy} & \omega_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{bmatrix}$$

Angular Deformation:- Angular deformation of fluid element involves the changes in the angle between two mutually perpendicular lines



Now consider two lines oa and ob. The rate of angular deformation of the fluid element is the rate decrease of the angle between the two lines oa and ob. Thus the rate of angular deformation

$$-\frac{d\gamma}{dt} = \frac{d\alpha}{dt} + \frac{d\beta}{dt} \longrightarrow (18)$$

Now $\frac{d\alpha}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\alpha}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\eta/\Delta x}{\Delta t}$ | $S = r\theta$, $\theta = \frac{S}{r}$

as $\frac{\Delta\eta}{\Delta t} = \Delta v = \frac{\partial v}{\partial x}\Delta x$ | $S = vt$, $\Delta v = v - v_0$

$$\frac{\Delta\eta}{\Delta x} = \frac{\partial v}{\partial x}\Delta t$$

$$\frac{d\alpha}{dt} = \lim_{\Delta t \to 0} \frac{(\partial v/\partial x)\Delta x \Delta t/\Delta x}{\Delta t}$$

$$= \frac{\partial v}{\partial x} \longrightarrow (19)$$

$$\frac{d\beta}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\beta}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\xi/\Delta y}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(\partial u/\partial y)\Delta y \Delta t/\Delta y}{\Delta t} = \frac{\partial u}{\partial y} \longrightarrow (20)$$

Making use of (19) and (20) in (18) we get

$$\frac{d\alpha}{dt} + \frac{d\beta}{dt} = -\frac{d\gamma}{dt} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

As shear stress is directly proportional to the rate of angular deformation shear stress is related to velocity gradients through the fluid viscosity. Hence

$$\tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

Flux:- The flux is a quantity used to measure the amount of a property transported across a surface per unit time.

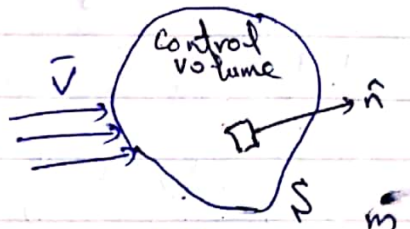
Volumetric Flow Rate:- The volumetric flow rate Q is the volume of fluid that crosses the surface per unit time

$$Q = \int_S \bar{v}\cdot\bar{n}\,ds = \int_S u_i n_i\,ds$$

normal comp of velocity × Area of surface

$[Q] = L^3T^{-1}$, units m^3/s

$\bar{v}\cdot\bar{n} = u_i n_i$ is the normal component of velocity to the surface. Therefore, a positive value of $u_i n_i$ means that the flow locally out of volume and negative sign means that it is into volume



Mass Flow Rate:-

The mass flux $\dot{m}$ or G is the mass per unit time that flows across a surface

$$\dot{m} = G = \int_S \rho \bar{V}.\bar{n}\, ds = \int_S \rho u_i n_i\, ds$$

$$[\dot{m}] = MT^{-1} \; ; \text{ unit } kg/s$$

$$[\rho u_i] = ML^{-3}LT^{-1} = \frac{M}{L^2T}$$

Law of Conservation of Mass:-

(Continuity Equation)

The mass of fluid inside the control volume is $\int_V \rho\, dV$

By law of conservation of mass

$$\begin{pmatrix}\text{Time rate of decrease} \\ \text{of mass inside} \\ \text{control volume}\end{pmatrix} = \begin{pmatrix}\text{Net mass flow out of} \\ \text{control volume through} \\ \text{surface } S\end{pmatrix} \longrightarrow (1)$$

Time rate of decrease of mass inside the control volume is given by

$$-\frac{\partial}{\partial t}\int_V \rho\, dV \longrightarrow (2)$$

Net mass flow out of the entire

control volume through the control surface is given by $\int_S \rho u_i n_i \, ds \longrightarrow (3)$

substituting (2) and (3) in (1) we get

$$\frac{\partial}{\partial t}\int_V \rho \, dV + \int_S \rho u_i n_i \, ds = 0$$

or $$\int_V \frac{\partial \rho}{\partial t} dV + \int_S \rho u_i n_i \, ds = 0 \longrightarrow (4)$$

Applying the divergence theorem, the surface integral in (4) can be expressed as volume integral. i.e.

$$\int_S \rho u_i n_i \, ds = \int_V \frac{\partial}{\partial x_i}(\rho u_i) \, dV \longrightarrow (5)$$

substituting (5) in (4) we get

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_V \frac{\partial}{\partial x_i}(\rho u_i) \, dV = 0$$

or $$\int_V \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i)\right] dV = 0$$

i.e $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \longrightarrow (6)$$

$$\frac{\partial}{\partial x_i}(\rho u_i) = \rho \frac{\partial u_i}{\partial x_i} + u_i \frac{\partial \rho}{\partial x_i} \longrightarrow (7)$$

$$\boxed{\begin{array}{l}\nabla \cdot (\rho \bar{V}) \\ = \rho \nabla \cdot \bar{V} + \bar{V} \cdot \nabla \rho\end{array}}$$

In view of (7) eqn (6) becomes

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_i}{\partial x_i} + u_i \frac{\partial \rho}{\partial x_i} = 0 \longrightarrow (8)$$

In cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial z}\right) + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}$$

$$+ w\frac{\partial \rho}{\partial z} = 0 \longrightarrow (9)$$

For incompressible fluids ρ is constant

$$(9) \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\text{i.e} \quad \frac{\partial u_i}{\partial x_i} = 0 \Rightarrow (\bar{\nabla}\cdot\bar{V}) = 0$$

From (8)

$$\underbrace{\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i}}_{\text{material derivative of } \rho} + \rho \frac{\partial u_i}{\partial x_i} = 0 \longrightarrow (10)$$

$$\text{i.e} \quad \frac{1}{\rho}\frac{d\rho}{dt} + \frac{\partial u_i}{\partial x_i} = 0 \longrightarrow (11)$$

As we follow a fluid element through the flow field, its density changes as $\frac{\partial u_i}{\partial x_i} = \nabla\cdot\bar{V}$ changes

Incompressibility continuity equation in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial (u_z)}{\partial z} = 0$$

Rotation:- Rotation is a vector quantity

A particle moving in a general 3-dimensional flow field may rotate about all three coordinates axis

$$\Omega = \Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k}$$

$$\Omega = \Omega_1 \hat{i} + \Omega_2 \hat{j} + \Omega_3 \hat{k}$$

$\Omega_x \rightarrow$ Rotation about x-axis

$\Omega_y \rightarrow$ Rotation about y-axis

$\Omega_z \rightarrow$ Rotation about z-axis

$$\bar{\nabla} \times \bar{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}$$

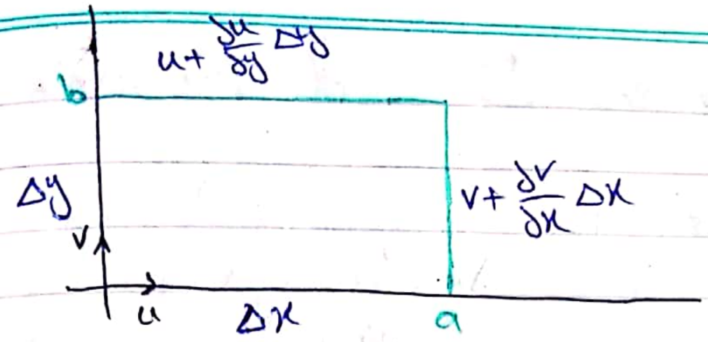
$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

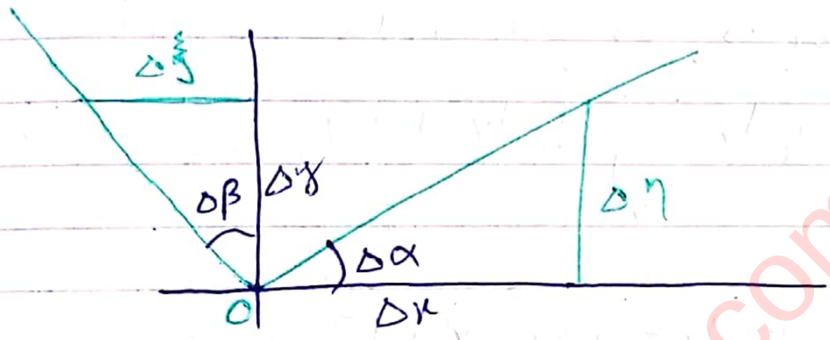
★

$$\omega_{ij} = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix}$$

$$\omega_{12} = \frac{1}{2}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right] \qquad \because \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\omega_{13} = \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right] \quad \& \quad \omega_{32} = \frac{1}{2}\left[\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right]$$





Rotation of line oa

$$\omega_{oa} = \lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta \eta / \Delta x}{\Delta t} \longrightarrow ①$$

Now $\dfrac{\Delta \eta}{\Delta t} = \Delta v = \dfrac{\partial v}{\partial x} \Delta x$

$$\Delta \eta = \frac{\partial v}{\partial x} \Delta x \Delta t$$

$$\frac{\Delta \eta}{\Delta x} = \frac{\partial v}{\partial x} \Delta t$$

$$\frac{1}{\Delta t} \frac{\Delta \eta}{\Delta x} = \frac{\partial v}{\partial x} \Rightarrow \omega_{oa} = \frac{\partial v}{\partial x}$$

And $\omega_{ob} = -\dfrac{\partial u}{\partial y}$

Total rotation $\omega_{oa} + \omega_{ob} = \left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} \right)$

A ———— A

Law of Conservation of Momentum:

Time rate of change of linear momentum of a material system is equal to the sum of applied surface and body forces as we follow the fluid.

$$\frac{d}{dt}\int_V \rho u_i \, dv = \int_V \rho f_i \, dv + \int_S T_{ij} n_j \, ds \longrightarrow ①$$

$$\int_V \frac{d}{dt}(\rho u_i)\, dv = \int_V \rho f_i \, dv + \int_S T_{ij} n_j \, ds \longrightarrow ②$$

Applying Gauss theorem

$$\int_V \frac{d}{dt}(\rho u_i)\, dv = \int_V \rho f_i \, dv + \int_V \frac{\partial}{\partial x_j}(T_{ij})\, dv \longrightarrow ③$$

For incomppressible fluids

$$\int_V \left[\rho \frac{du_i}{dt} - \rho f_i - \frac{\partial}{\partial x_j} T_{ij}\right] dv = 0$$

$$\Rightarrow \rho \frac{du_i}{dt} - \frac{\partial}{\partial x_j} T_{ij} - \rho f_i = 0 \qquad i = 1, 2, 3$$

or $$\rho \frac{du_i}{dt} = \frac{\partial}{\partial x_j} T_{ij} + \rho f_i = 0 \longrightarrow ④$$

$$\rho \left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right] = \frac{\partial}{\partial x_j} T_{ij} + \rho f_i = 0 \longrightarrow ⑤$$

Navier Stokes Equation:-

The Navier-Stokes equations, named after Claude-Louis Navier (1785–1836) and George Gabriel Stokes (1819–1903) describes the equation of motion of Newtonian viscous fluids

For incompressible, Newtonian viscous fluids, the total stress tensor is given by

$$T_{ij} = -p\delta_{ij} + \mu \tau_{ij} \longrightarrow (6)$$

$-p\delta_{ij}$: Inviscid stress

$\mu\tau_{ij}$: Viscous stress tensor/shear stress tensor and is zero when fluid is at rest.

$$\Rightarrow T_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \longrightarrow (7)$$

$$\frac{\partial T_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(-p\delta_{ij}\right) + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right)$$

$$= \frac{-\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \mu \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)$$

$$\boxed{\text{as } \frac{\partial u_j}{\partial x_j} = 0 \quad \bar{\nabla}\cdot\bar{V} = 0}$$

$$= \frac{-\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \mu \frac{\partial}{\partial x_i}(0)$$

$$\frac{\partial T_{ij}}{\partial x_j} = \frac{-\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} \longrightarrow (8)$$

substituting in (5) we get

$$\rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right] = \frac{-\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \rho f_i = 0$$

or $$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \cdot \frac{\partial^2 u_i}{\partial x_j^2} + f_i = 0 \rightarrow (9)$$

Equ (8.9) is known as Navier-Stokes equations. where $\nu = \mu/\rho$ is the kinematic viscosity. $[\nu] = L^2T^{-1}$

In cartesian coordinate and in component form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + f_x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + f_y$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + f_z \rightarrow (10)$$

** ———————— **

MUHAMMAD TAHIR WATTOO

0344-8563284

SP18-PMT-005

Non-Dimensional Variable:- A non-dimensional variable is a veriable with null dimensions i.e. without dimensions.

Non-Dimensionalization:- Conversion of a system of dimensional equations to a system that contains only non-dimensional quantities.

Note:- ★ A combination of non-dimensional variables is dimensionless and change of units does not effect the value of a dimensionless number.

★ Dimensional analysis is useful in scaling-up the lab experiments, in producing dimensionless numbers that governing equations.

★ The problem has the fewest variables and the simplest mathematical structure when expressed in non dimensional variables.

★ A flow depends on relevant dimensionless numbers, rather than seperately on individuals geometric dimensions (such as density, viscosity)

Example:- Consider the equation of motion of object launched vertically from the surface of earth

$$\frac{d^2y}{dt^2} = \frac{-gR^2}{(y+R)^2} \longrightarrow (1)$$

$$y(0) = 0 \quad , \quad y'(0) = V_0$$

y = height ; v_0 = initial velocity , R = radius of earth , g = gravitational acceleration

The main concept of non-dimensionlization is to seperates variables and parameters of the equation, and express the variables in terms of parameters.

Variable	Parameter
$y = [L]$	$g = [L/T^2]$
$t = [T]$	$R = [L]$
	$V_0 = [L/T]$

Now express the variables in terms of parameters.

First, we try to find dimensionless time $\bar{t} = t/a$, a is unknown untill we find the value.

Looking into the list of parameters we can combine R and V_0 to get dimension of $[T]$ i.e let

$$a = \frac{R}{V_0} = \left[\frac{L}{LT^{-1}}\right] = [T]$$

$$\bar{t} = \frac{t}{a} = \frac{t}{R/V_0} = \frac{V_0 t}{R}$$

Similarly, $\bar{y} = y/b$ as R has the dimension of $[L]$ so setting $b = R$ i.e

$$\bar{y} = y/R$$

Now from ①

$$\left(\frac{V_0^2}{R^2}\right)(R)\frac{d^2\bar{y}}{d\bar{t}^2} = \frac{-gR^2}{R^2(\bar{y}+1)^2}$$

$$\frac{V_0^2}{R}\frac{d^2\bar{y}}{d\bar{t}^2} = \frac{-g}{(\bar{y}+1)^2}$$

V_0, R, g
3 Parameters to one parameter α

$$\frac{d^2\bar{y}}{d\bar{t}^2} = -\frac{\alpha}{(\bar{y}+1)^2} \longrightarrow (2)$$

dimensionless equation

with $\bar{t} = \frac{V_0 t}{R}$, $\bar{y} = \frac{y}{R}$, $\alpha = \frac{gR}{V_0^2}$

non-dimensional quantities.

Another Proof Let

$$\bar{t} = t/t_c, \quad \bar{y} = y/y_c$$

i.e $t = \bar{t}\,t_c$, $y = \bar{y}\,y_c$

$$\frac{d^2y}{dt^2} = \frac{-gR^2}{(y+R)^2}$$

$$\frac{y_c}{t_c^2}\frac{d^2\bar{y}}{d\bar{t}^2} = \frac{-gR^2}{(y_c\bar{y}+R)^2}$$

or

$$\frac{y_c}{g t_c^2}\frac{d^2\bar{y}}{d\bar{t}^2} = \frac{-1}{\left(1+\frac{y_c}{R}\bar{y}\right)^2}; \quad \bar{y}(0)=0, \quad \bar{y}'(0) = \frac{t_c}{y_c}V_0$$

For dimensionless equation

$$\frac{y_c}{g t_c^2} = 1, \quad \frac{t_c}{y_c}V_0 = 1, \quad \frac{y_c}{R} = 1$$

From $y_c/R = 1 \Rightarrow$ $y_c = R$, so $\boxed{\bar{y} = \frac{y}{R}}$

From $\frac{y_c}{g t_c^2} = 1 \Rightarrow t_c^2 = \frac{y_c}{g} = \frac{R}{g}$

or $t_c = \sqrt{R/g}$ i.e. $\bar{t} = \sqrt{\frac{g}{R}}\,t$

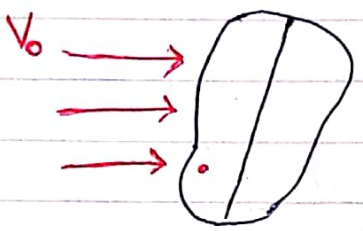
Also from $\frac{t_c}{y_c} V_0 = 1 \Rightarrow t_c = \frac{y_c}{V_0} = \frac{R}{V_0}$

i.e. $\boxed{\bar{t} = \frac{V_0 t}{R}}$

we may have more than one choice to non-dimensionalize the problem.

Consider the NSE for steady flow

$$\rho\left[u_j \frac{\partial u_i}{\partial x_j}\right] = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$



V_0 = characteristic velocity
L = characteristic length

Imagine we have the flow with some characteristic velocity V_0 and some object which is involved in that flow (such as flow over an object, for example diameter of pipe in case of flow inside a pipe). Let us define the dimensions

$$\bar{x}_i = \frac{x_i}{L}, \quad \bar{u}_i = \frac{u_i}{V_0}, \quad \bar{p} = \frac{p}{\mu V_0/L} \rightarrow \text{shear stress (force per unit area)}$$

i.e

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad \bar{u} = \frac{u}{V_0}, \quad \bar{v} = \frac{v}{V_0}$$

$$\rho\left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \rightarrow (3)$$

$$\rho\left[(\bar{u}V_0) \frac{\partial(\bar{u}V_0)}{\partial(\bar{x}L)} + (\bar{v}V_0)\frac{\partial(\bar{u}V_0)}{\partial(\bar{y}L)}\right] = -\frac{\partial(\mu V_0/L)}{\partial(\bar{x}L)}$$

$$+\mu\left[\frac{\partial^2(\bar{u}V_0)}{\partial(\bar{x}L)^2} + \frac{\partial^2(\bar{u}V_0)}{\partial(\bar{y}L)^2}\right]$$

$$\frac{\rho V_0^2}{L}\left[\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right]=-\frac{\mu V_0}{L^2}\frac{\partial \bar{p}}{\partial \bar{x}}+\frac{\mu V_0}{L^2}\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}+\partial^2\bar{u}/\partial\bar{y}^2\right)$$

$$\frac{\rho V_0^2/L}{\mu V_0/L^2}\left[\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right]=\frac{-\partial \bar{p}}{\partial \bar{x}}+\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right]$$

The number stems from dividing the convective term by the viscous diffusion term representing the ratio of inertial to viscous forces is known as Reynolds number (Osborne Reynolds, 1842–1912, Irish scientist)

$$Re=\frac{\text{convection}}{\text{diffusion}}\approx\frac{\text{inertial forces}}{\text{viscous forces}}$$

$$=\frac{\rho V_0^2/L}{\mu V_0/L^2}=\left(\frac{\rho V_0^2}{L}\right)\left(\frac{L^2}{\mu V_0}\right)=\frac{V_0 L}{\nu}$$

$$\Rightarrow R_e=\frac{V_0 L}{\nu} \qquad [R_e]=\left[\frac{LT^{-1}L}{L^2T^{-1}}\right]=[-]$$

Note that we assume that pressure term is of same order of magnitude because we choose viscous shear stress to normalize the magnitude of pressure.

Re tells us that what forces dominates the flow field.

$Re \ll 1$ (L.H.S $\to 0$), The viscous forces are dominant. The convective term can be neglected compare to the viscous term. The flows in which viscous forces dominates the inertial forces are termed as creeping flows [Every viscous flows, or small length scales]

$Re >> 1$, convective (or inertial) forces are dominant. Note that, In this case the viscous forces can not be neglected everywhere in flow field, only away from the interfaces. In particular, in areas close to interfaces and boundary layers, the viscous forces are of same order of magnitude as inertial forces, because in the vicinity of interfaces, very strong gradient may exist.

Suppose pressure scale $[P]$ is to be determined i.e.

$$\bar{x} = \frac{x}{L}, \quad \cancel{\bar{u} = \frac{V_0}{L}, \bar{v} =} \quad \bar{y} = y/L$$

$$\bar{p} = \frac{p}{[P]} \quad \bar{u} = \frac{u}{V_0} \quad \bar{v} = \frac{v}{V_0}$$

From ③ we have

$$\frac{\rho V_0^2}{L}\left[\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right] = -\frac{[P]}{L}\frac{\partial \bar{P}}{\partial \bar{x}} + \frac{\mu V_0}{L^2}\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right) \longrightarrow (4)$$

It is clear from ④ we have two choices for characteristic pressure

$$\frac{[P]}{L} = \frac{\mu V_0}{L^2} \quad \text{i.e.} \quad [P] = \frac{\mu V_0}{L}$$

and $\frac{[P]}{L} = \frac{\rho V_0^2}{L}$ i.e $[P] = \rho V_0^2$

In confined viscous flow, $\frac{\mu V_0}{L}$ is the obvious choice for pressure. Infact, in these flows, the motion is mostly due to competing pressure and shear stress gradients, according to

$$\frac{\partial p}{\partial x} \approx \frac{\partial \tau_{yx}}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2}$$

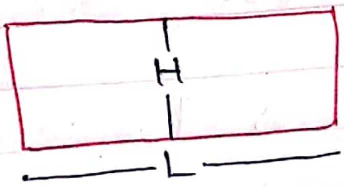
Therefore pressure can be seen as equivalent to a viscous stress which is measured in the units of $\mu V_0/L$.

In inviscid flows, the effects of viscosity are minimized and thus viscous pressure unit $\mu V_0/L$ is not appropriate as these flows are driven by pressure gradients and/or inertia according to Euler's equation

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}$$

Thus in this case pressure can be viewed as equivalent to kinetic energy which is the units of ρV_0^2

* If L and H are of same order of magnitude. One of them can be used for scaling both x and y i.e



$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L} \quad \text{or} \quad \bar{x} = \frac{x}{H}, \quad \bar{y} = y/H$$

In case H and L are not of the

same order of magnitude, then two spatial coordinates are scaled as

$$\bar{x} = \frac{x}{L} \quad , \quad \bar{y} = \frac{y}{H}$$

Drag Coefficient:- Another important dimensionless number arises when the friction at a solid interface τ_0 is calculated

$$\bar{\tau}_0 = \frac{\tau_0}{\mu V_0/L} = \frac{\tau_0}{\frac{1}{2}\rho V_0^2} \frac{\frac{1}{2}\rho V_0 L}{\mu}$$

$$= \frac{1}{2} C_D Re$$

where $C_D = \dfrac{\tau_0}{\frac{1}{2}\rho V_0^2}$

Example $\bar{x} = \dfrac{x}{L}$, $\bar{y} = \dfrac{y}{L}$, $\bar{t} = \dfrac{t}{L/V_0}$,

$\bar{P} = P/\mu V_0/L$

Then NSE

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \rho g$$

is dimensionless form yields

$$\frac{\rho V_0 L}{\mu}\left(\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) = -\frac{\partial \bar{p}}{\partial \bar{x}} + \left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right) + \frac{\rho g L^2}{\mu V_0}$$

$$Re = \frac{\rho V_0 L}{\mu} = \frac{V_0 L}{\nu}$$

$$\text{Stokes number} = S_t = \frac{\rho g L^2}{\mu V_0} = \frac{g L^2}{\nu V_0}$$

is another dimensionless number which is the ratio of body forces to the viscous forces.

Solutions of Navier Stokes Equations

Incompressible Newtonian ~~fluid~~ flow is governed by a system of four partial differential equations (PDEs); these are the continuity equation and three components of Navier Stoke equations. The pressure and three velocity components are the primary unknowns, which are in general functions of time and of spatial coordinates.

The main difficulty in the solution of these system of PDEs (i.e. NSEs) arise from their non-linearities (that are convective acceleration terms). Therefore, theoratically study of fluid mechanics is almost based on some type of approximate solutions of the governing equations. There are two basic approaches to obtaining such solutions. One is to introduce a discretized approximations, such as finite difference, finite element or spectral methods to convert non linear PDE to a set of non-linear algebraic equations and then attempt to solve the equations numerically.

The 2nd class of approximate solution techniques that has been very effective in the context of fluid mechanics are analytical methods i.e. asymptotic methods / perturbation + or non-perturbative methods (such as Homotopy analysis method, adomian decomposition method) Analytical solution calculations may sometimes be easier or more convenient than numerical methods (which may require very large compotational facilities). Moreover, Analytical approach yields greater and immediate physical insight. Analytical approximations can solve serve as a critical bench mark for numerical methods.

There are few exceptional classes of problems for which an exact analytical solution is possible. We will try to focus on them. This will give us a chance to review analytical solution techniques of PDEs and important concepts about scaling and non-dimensionalization.

Simplification of NSEs for Unidirectional Flow:- Recall the NSEs

$$\left.\begin{aligned} \rho\left[\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right] &= -\frac{\partial P}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j^2} \\ \text{or } \rho\left[\frac{\partial \bar{V}}{\partial t}+\bar{V}\cdot\nabla\bar{V}\right] &= -\bar{\nabla}P+\mu\bar{\nabla}^2\bar{V} \end{aligned}\right\} \longrightarrow (1)$$

One should expect the exact solutions for the problem for which non linear terms in (1) are identically equal to zero i.e.

$$u_j\frac{\partial u_i}{\partial x_j}=0 \quad (\bar{V}\cdot\bar{\nabla}V=0) \longrightarrow (2)$$

i.e. the gradient of velocity u_i be orthogonal to u_i itself. There are several flows that satisfy this condition. The most important is the class of so called unidirectional flows. Suppose we choose our coordinate system such that x-axis is aligned with flow and velocity field has the form

$$u_i = u(x,y,z,t)e_x = [u(x,y,z,t),0,0] \longrightarrow (3)$$

From continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

we see that equ (2) is only possible if

$$u_i = [u(y,z,t),0,0]$$
$$= u(y,z,t)\,\hat{e_x} \longrightarrow (4)$$

where e_x is the unit vector and y and z are spacial coordinates in the plane orthogonal to $\hat{e}_x$.

Hence, the non linear term in NSE ① is identically zero for any flow that satisfy ②

$$u_j \frac{\partial u_i}{\partial x_j} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial z}$$

$$= u\frac{\partial u}{\partial x} = 0 \longrightarrow ⑤$$

u must be independent of position in flow direction

Therefore because the z and y components of velocity u_i are zero it follows from NSE

$$\frac{\partial P}{\partial z} = 0 = \frac{\partial P}{\partial y} \longrightarrow ⑥$$

Thus $P = P(x,t)$ only

The only non-zero component of NSE is the x-component. i.e. $u = u(y,z,t)$ so,

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \longrightarrow ⑦$$

From ⑦ we may note that the term on L.H.S and last two terms on R.H.S are at most functions of y, z and t. Thus for any unidirectional flow $\partial P/\partial x$ can not depend on x but only on t so let

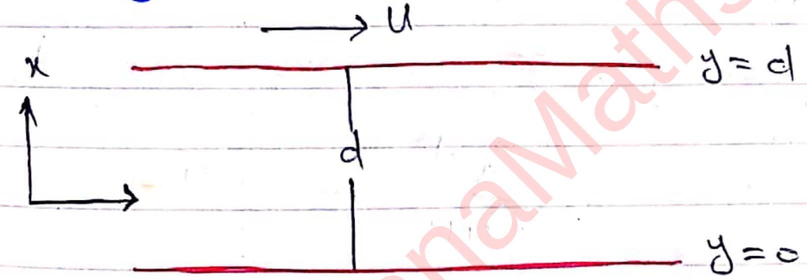
$$-\frac{\partial P}{\partial x} = G(t)$$

So that

$$\rho \frac{\partial u}{\partial t} = G(t) + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial t^2}\right) \longrightarrow (8)$$

Hence, to solve unidirectional flow problem, we have to solve (8) subject to initial and boundary conditions. It is clear that (8) has non-trivial solution only if $G(t)$ is non-zero or the value of u is non-zero on one (or more) of the boundaries of the flow domain.

Steady Unidirectional Flow:-



Consider the steady two dimensional flow between two infinite parallel plate boundaries. Let x-axis being parallel to flow direction. We also assume that the pressure gradient G is non-zero constant and that the upper boundary moves in same direction as the pressure gradient with a constant velocity U

$$u = u(x, y)$$

$$u_i = [u(x,y), 0, 0]$$

as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0 \quad , u \neq u(x)$$

$$u = u(y) \text{ only}$$

From ⑧ we get

$$0 = G + \mu\left(\frac{d^2u}{dy^2}\right)$$

$$\frac{d^2u}{dy^2} = \frac{-G}{\mu} \longrightarrow (9)$$

corresponding boundary conditions are

$$\left.\begin{array}{lll} u = 0 & \text{at} & y = 0 \\ u = U & \text{at} & y = d \end{array}\right\} \longrightarrow (10)$$

To non-dimensionalize, we define the following dimensionless quantities

$$\bar{u} = \frac{u}{U}, \quad \bar{y} = \frac{y}{d} \longrightarrow (11)$$

From (11) using in (9) and (10) we get

$$\frac{d^2\bar{u}}{d\bar{y}^2} = \frac{-Gd^2}{\mu U} \longrightarrow (12)$$

$$\left.\begin{array}{lll} \bar{u} = 0 & \text{at} & \bar{y} = 0 \\ \bar{u} = 1 & \text{at} & \bar{y} = 1 \end{array}\right\} \longrightarrow (13)$$

We see that the dimensionless problem and its solution depends on a single dimensionless parameter.

Let

$$K = \frac{Gd^2}{\mu U} \longrightarrow (14)$$

as $[G] = \left[\frac{dP}{dx}\right] = \frac{ML^{-1}T^{-2}}{L} = ML^{-2}T^{-2}$

$$[d] = L, \quad [\mu] = ML^{-1}T^{-1}, \quad [U] = LT^{-1}$$

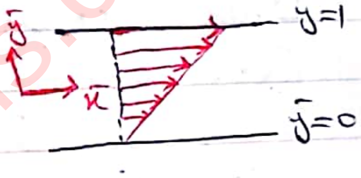
$$[K] = \frac{(ML^{-2}T^{-2})L^2}{(ML^{-1}T^{-1})(LT^{-1})} = [-]$$

Note that this parameter is just the ratio of two possible velocity scales Gd^2/μ and U, one characterized by the magnitude of the boundary velocity. The solution of (12,13) is

$$\bar{u}(\bar{y}) = -\left(\frac{Gd^2}{\mu U}\right)\left(\frac{\bar{y}^2}{2} - \frac{\bar{y}}{2}\right) + \bar{y}$$

$$\bar{u}(\bar{y}) = -K\left(\frac{\bar{y}^2}{2} - \frac{\bar{y}}{2}\right) + \bar{y} \longrightarrow (15)$$

(15) is called velocity profile or velocity distribution.



(a) If $K = 0$ i.e. $\dfrac{Gd^2}{\mu U} = 0$

$$\bar{U} = \bar{y}$$

when $K \ll 1$ i.e $\dfrac{Gd^2}{\mu U} \ll 1$, we see that from (15)

$$\bar{U} \sim \bar{y} \longrightarrow (16) \quad \text{(Plane connective flow)}$$

In this case the fluid motion is dominated by the motion of the boundary and the velocity profile reduces to a linear (simple) shear flow

Converting back to ~~shear~~ dimensional variable

$$\bar{u} = \frac{u}{U}, \quad \bar{y} = y/d$$

(16) becomes $\dfrac{u}{U} \sim \dfrac{y}{d}$ or

$$u(y) = \left(\frac{U}{d}\right) y$$

If $\frac{Gd^2}{\mu U} >> 1$ the quadratic contribution to $\bar{u}$ becomes arbitrary large as $Gd^2/\mu U$ increases. This increase of $\bar{u}$ does not mean that actual velocities in the system are blowing up. we can make $Gd^2/\mu U$ arbitrary large by simply taking U arbitrary closed to zero while holding Gd^2/μ constant.

Unidirectional & One-Dimensional Flow and Heat Transfer Problems:- ?

Second Choice:-

$$\bar{u} = \frac{u}{u_c} \quad ; \quad u_c = \frac{Gd^2}{\mu}$$

$$\bar{y} = \frac{y}{d} \longrightarrow (11.1)$$

In dimensionless form equ (10....)

$$\frac{d^2\bar{u}}{d\bar{y}^2} = -1 \longrightarrow (11.2)$$

$$\bar{u} = 0 \quad \text{at} \quad \bar{y} = 0 \longrightarrow (11.3)$$

$$\bar{u} = \frac{U\mu}{Gd^2} \quad \text{at} \quad \bar{y} = 1 \longrightarrow (11.4)$$

$$\bar{u}(\bar{y}) = \frac{-1}{2}(\bar{y}^2 - \bar{y}) + \frac{U\mu}{Gd^2}\,\bar{y} \longrightarrow (11.5)$$

Recall from (10.15)

$$\bar{u}(\bar{y}) = -\frac{Gd^2}{\mu U}\left(\frac{\bar{y}^2}{2} - \frac{\bar{y}}{2}\right) + \bar{y} \longrightarrow (10.15)$$

$$\frac{Gd^2}{\mu U} \to 0. \qquad \bar{u}(\bar{y}) \sim \bar{y}$$

$$u_c = \frac{Gd^2}{\mu} \Rightarrow \frac{u}{Gd^2/\mu} = -\frac{1}{2}\left(\frac{y^2}{d^2} - \frac{y}{d}\right) + \frac{U\mu}{Gd} \rightarrow (11.6)$$

$$\{u_c = U\} \Rightarrow \frac{u}{U} = \frac{-Gd^2}{\mu U}\left(\frac{y^2}{2d^2} - \frac{y}{2d}\right) + \frac{y}{d} \rightarrow (11.7)$$

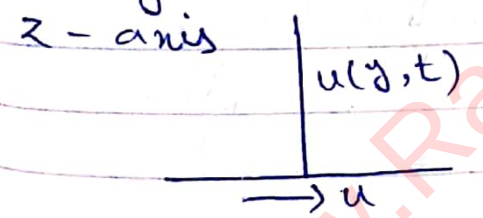
$$(11.5) \Rightarrow \frac{U\mu}{Gd^2} \rightarrow 0, \quad \bar{u}(\bar{y}) = -\frac{1}{2}(\bar{y}^2 - \bar{y})$$

$$u(y) = -\frac{Gd^2}{\mu}\left(\frac{y^2}{d^2} - \frac{y}{d}\right)$$

$$u_c = U \quad \text{if} \quad \frac{Gd^2}{\mu} << 0$$

$$u_c = \frac{Gd^2}{\mu} \quad \text{if} \quad \frac{Gd^2}{\mu} >> 0$$

Axis Symmetric Flow:- Axis symmetric flows of contriently studies in cylindrical coordinates (r, θ, z) with



$u(y,0) = 0 \quad t < 0$

$u(0,t) = 0 \quad t > 0$

$u(y,t) \rightarrow 0; \; y \rightarrow \infty$

coinciding the axis of symmetry of flow

→ Axis symmetry means that there is no variation of velocity w.r.t θ i.e.

$$\frac{\partial \bar{v}}{\partial \theta} = 0 \qquad \bar{v} = [u_r(r,\theta,z), u_\theta(r,\theta,z), u_z(r,\theta,z)$$

$$\text{i.e} \quad \frac{\partial u_r}{\partial \theta} = 0, \quad \frac{\partial u_\theta}{\partial \theta} = 0, \quad \frac{\partial u_z}{\partial \theta} = 0$$

Continuity equation for incompressible fluids (viscous flows.

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}=0 \longrightarrow (i)$$

as for unidirectional axis symmetric flow

$$\bar{V}=[0,0,u_z(r,\theta,z)] \longrightarrow (ii)$$

From (i) and (ii)

$$\frac{\partial u_z}{\partial z}=0 \Rightarrow u_z \neq u_z(z)$$

Also for axis symmetric flow

$$\frac{\partial u_z}{\partial \theta}=0 \Rightarrow u_z \neq u_z(\theta)$$

So $u_z = u_z(r)$

$\bar{V}=[0, v(x,t), 0] \rightarrow$ unidirectional vertical flow

$$\bar{V}=[0,0,u_z(r)]$$

From NSE

r-component

$$\rho\left[\frac{\partial u_r}{\partial z}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{\partial^2\theta}{r}+u_z\frac{\partial u_z}{\partial z}\right]=-\frac{\partial P}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right)\Big] \longrightarrow (3)$$

θ-component

$$0=-\partial P/\partial r \longrightarrow (4)$$

z-component

$$\rho\left[\frac{\partial u_z}{\partial z}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right]=-\frac{\partial P}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right] \longrightarrow (5)$$

From ③ and ④

$$\frac{\partial P}{\partial r} = 0 = \frac{\partial P}{\partial \theta} \Rightarrow P \neq P(r,\theta)$$

$$0 = -\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right]$$

$$0 = \underbrace{\left[-\frac{\partial P}{\partial z}\right.}_{G} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right]$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right) = \frac{-G}{\mu} \longrightarrow (i)$$

$$u_z = 0 \quad \text{at} \quad r = R \longrightarrow (ii)$$

$$\bar{r} = \frac{r}{R}, \quad \bar{u}_z = \frac{u_z}{GR^2/\mu}$$

$$\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\left(\bar{r}\frac{d\bar{u}_z}{d\bar{r}}\right) = -1$$

$$\bar{u}_z = 0 \qquad \text{at} \quad \bar{r} = 1$$

$$\frac{d}{d\bar{r}}\left(\bar{r}\frac{d\bar{u}_z}{d\bar{r}}\right) = -\bar{r}$$

$$\bar{r}\frac{d\bar{u}_z}{d\bar{r}} = -\frac{\bar{r}^2}{2} + C_1$$

$$\frac{d\bar{u}_z}{d\bar{r}} = -\frac{\bar{r}}{2} + \frac{C_1}{\bar{r}}$$

$$\bar{u}_z(\bar{r}) = -\frac{\bar{r}^2}{4} + C_1 \ln\bar{r} + C_2$$

$$\bar{r} = 1, \quad \bar{u}_z = 0 \longrightarrow (*)$$

$$C_2 = 1/4$$

$$\bar{u}_z(r) = -\frac{\bar{r}^2}{4} + \overset{0}{C_1 \ln\bar{r}} + \frac{1}{4}$$

$$\bar{u}_z(\bar{r}) = -\frac{\bar{r}^2}{4} + \frac{1}{4}$$

$$\bar{u}_z(\bar{r}) = \frac{1}{4}(1 - \bar{r}^2)$$

$$u_z(r) = \frac{GR^2}{\mu}\left(1 - \frac{r^2}{R^2}\right)$$

$$T_{rr} = 2\mu\left(\frac{\partial u_z}{\partial r}\right)$$

$$T_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)$$

$$T_{zz} =$$

$$T_{r\theta} = T_{\theta r}$$

$$T_{rz} = T_{zr} = \mu\left[\frac{\partial u_z}{\partial r} + \cancel{\frac{\partial u_r}{\partial z}}^{\,0}\right]$$

volume flow rate

$$Q = \int_0^{2\pi} \theta \, d\theta \int_0^R u_z \, r \, dr$$

$$\Rightarrow Q = 2\pi \int_0^R \left[\frac{GR^2}{\mu}\left(1 - \frac{z^2}{R^2}\right) r \, dr\right.$$

→ CRC (Papanat) all of these thing are given in this book (Please consult)

From lecture 10 we have

$$\frac{d^2u}{dy^2} = \frac{-G}{\mu}$$

$u=0$ at $y=0$

$u=U$ at $y=d$

Now, we define the dimensionless velocity

$$\left.\begin{array}{ll} \bar{u} = \dfrac{u}{u_c}, & u_c = \dfrac{Gd^2}{\mu} \\ \bar{y} = \dfrac{y}{y_c}, & y_c = d \end{array}\right\} \longrightarrow (11.1)$$

In terms of these dimensionless variables we get

$$\frac{d^2\bar{u}}{d\bar{y}^2} = -1 \longrightarrow (11.2)$$

$$\left.\begin{array}{ll} \text{and } \bar{u} = 0 & \text{at } \bar{y} = 0 \\ \bar{u} = \dfrac{U\mu}{Gd^2} & \text{at } \bar{y} = 1 \end{array}\right\} \longrightarrow (11.3)$$

Solving (11.2) and (11.3) we get

$$\bar{u}(\bar{y}) = -\frac{1}{2}(\bar{y}^2 - \bar{y}) + \frac{U\mu}{Gd^2}\bar{y} \longrightarrow (11.4)$$

Hence as

$\dfrac{U\mu}{Gd^2} \to 0$, (11.4) reduces to

$$\bar{u}(\bar{y}) = -\frac{1}{2}(\bar{y}^2 - \bar{y}) \longrightarrow (11.5)$$

This (11.5) parabolic velocity profile is characteristic of pressure-gradient-driven flow in complete absence of boundary motion and is called

Poiseuille flow.

(11.4) in dimensionless form is

$$u(y) = -\frac{Gd^2}{2\mu}\left(\frac{y^2}{d^2} - \frac{y}{d}\right) + U\frac{y}{d} \longrightarrow (11.6)$$

Again we can see that the flow is linear combination of plane poiseuille flow and linear shear flow with relative magnitude by ratio of $\frac{Gd^2}{\mu}$ to U.

In conclusion, the appropriate choices for U_c are

$$\left.\begin{aligned} U_c &= U \quad \text{if} \quad Gd^2/\mu \ll U \\ U_c &= \frac{Gd^2}{\mu} \quad \text{if} \quad Gd^2/\mu \gg U \end{aligned}\right\} \longrightarrow (11.7)$$

Axisymmetric flows:- Axis symmetric flows are conveniently studied in a cylindrical coordinate system (r, θ, z) with z-axis coinciding the axis of symmetry of flow.

Axis symmetry means there is no variation of velocity with angle θ i.e. $\partial V/\partial\theta = 0$

Axisymmetric Rectlinear Flow:-

In unidirectional ~~velocity~~ flow in which only the axial velocity component u_z is non-zero. Typical flows are cylindrical tubes and annuli, Generally

$$\bar{V} = [u_2(r,\theta,z), u_\theta(r,\theta,z), u_z(r,\theta,z)]$$

In axis symmetric rectlinear flow

$$u_r = u_\theta = 0$$

$$\bar{V} = [0, 0, u_z(r,\theta,z)] \longrightarrow (11.8)$$

As in cylindrical coordinates the continuity equation for incompressible flow is given by

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \longrightarrow (11.9)$$

becomes $\frac{\partial u_z}{\partial z} = 0 \Rightarrow u_z \neq u_z(z)$

also using axisymmetric condition $\partial u/\partial \theta = 0$ i.e $u_z \neq u_z(\theta)$. So we get

$$\boxed{u_z = u_z(r,t)} \longrightarrow (11.10)$$

from NSE in cylindrical coordinates

r-comp

$$\rho\left[\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_z}{\partial z}\right]$$

$$= -\frac{\partial P}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2}\right] \longrightarrow (11.11)$$

θ-comp

$$\rho\left[\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z}\right]$$

$$= -\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}\right.$$

$$\left. + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2}\right] \longrightarrow (11.12)$$

z-compont

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right)$$

$$= -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta}\right.$$

$$\left. + \frac{\partial^2 u_z}{\partial u_z}\right] \longrightarrow (11.13)$$

using (11.10) we get from (11.11) → (11.13)

$$\frac{\partial p}{\partial r} = 0 = \frac{\partial p}{\partial \theta} \longrightarrow (11.14)$$

$$-\frac{\partial p}{\partial z} + \mu\frac{1}{r}\frac{d}{dr}\left(r\frac{\partial u_z}{\partial r}\right) = 0 \longrightarrow (11.15)$$

(11.14) $\Rightarrow$ $p \neq p(r,\theta)$, $p = p(z_i)$

Let $-\frac{\partial p}{\partial z} = G$

So $\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right) = -G$

or $\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right) = \frac{-G}{\mu} \longrightarrow (11.16)$

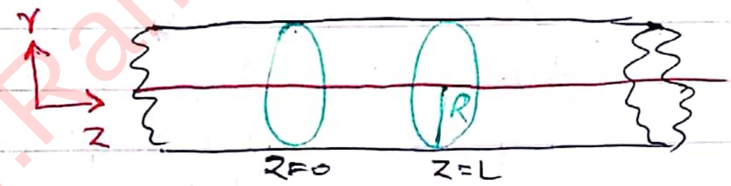
$$\left\{\begin{array}{l}
T_{rr} = 2\mu\frac{\partial u_r}{\partial r}, \quad T_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) \\
T_{zz} = 2\mu\frac{\partial u_z}{\partial z} \\
T_{r\theta} = T_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] \\
T_{rz} = T_{zr} = \mu\left[\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right]
\end{array}\right.$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left[\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right]$$

The only non-zero componet of shear stress are

Pressure Driven Motion in a straight tube of circular cross section.

Closely related to pressure driven unidirectional flow between two parallel plane surfaces is the pressure driven flow in a straight tube of circular cross section. this famous problem studied experimentally as a model of blood flow in arteries by Poiseuille in 1840.



$$\bar{V} = [0, 0, u_z(r)] \longrightarrow (11.17)$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right) = \frac{-G}{\mu} \longrightarrow (11.18)$$

B.C $u_z = 0$ at $r = R$ — wall of tube is stationary

u_z is finite at $r = 0$

Only source of motion is axial-pressure gradient so only a single choice for the characteristic velocity

$$\bar{u}_z = \frac{u_z}{u_c}, \quad u_c = \frac{GR^2}{\mu}$$

and a single obvious choice for characteristic length scale

$$\bar{r} = \frac{r}{l_c} \quad , \quad l_c = R$$

In terms of dimensionless variable

$$\bar{u} = \frac{u_z}{GR^2/\mu} \quad \text{and} \quad \bar{r} = \frac{r}{R} \longrightarrow (11.19)$$

(11.18) becomes

$$\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\left(\bar{r}\frac{d\bar{u}_z}{d\bar{r}}\right) = -1 \longrightarrow (11.20)$$

with $\bar{u}_z = 0$ at $\bar{r} = 1$

$$\frac{d}{d\bar{r}}\left(\bar{r}\frac{du_z}{d\bar{r}}\right) = -\bar{r}$$

$$\bar{r}\frac{du_z}{d\bar{r}} = -\frac{\bar{r}^2}{2} + C_1$$

$$\frac{du_z}{d\bar{r}} = -\frac{\bar{r}}{2} + \frac{C_1}{\bar{r}}$$

$$u_z(\bar{r}) = -\frac{\bar{r}^2}{4} + C_1 \ln\bar{r} + C_2 \longrightarrow (11.21)$$

as $\bar{u}$ is bounded for all $\bar{r}$, $0 \le \bar{r} \le 1$

$$\bar{u}_z = \frac{1}{4}(1-\bar{r}^2) \longrightarrow (11.22)$$

In terms of dimensionless variable

$$u_z = \frac{GR^2}{4\mu}\left(1 - r^2/R^2\right) \longrightarrow (11.23)$$

$\longrightarrow$ Parabolic

Volumetric flux:-

$$Q = 2\pi\int_0^R u_z\, r\, dr$$

$$\Rightarrow \quad Q = \frac{\pi G R^4}{8\mu}$$

Shear stress exerted by the fluid at wall is

$$\tau_w = -\tau_{rz} = -\mu\left(\frac{\partial u_z}{\partial r}\right)$$

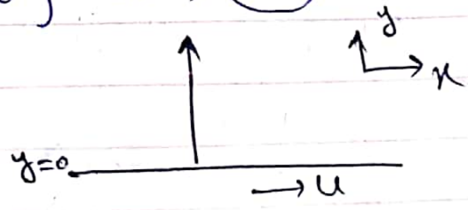
$$= -\mu\left(\frac{GR^2}{4\mu}\right)\left(\frac{-2r}{R^2}\right)$$

$$= \frac{GR}{2}$$

The Rayleigh Problem (Stokes First Problem):-

This famous problem firstly studied by Lord Rayleigh in 1911, in which initially a stationary infinite plate is assumed to being suddenly translating in its own plane with constant velocity through an initially stationary unbounded fluid

$$\bar{V} = [\bar{u}(y,t), 0, 0] \longrightarrow (12.1)$$



In absence of external pressure gradient

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} \quad \text{or} \quad \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \longrightarrow (12.2)$$

IC $\quad u(y,0) = 0 \quad \forall\, y > 0,\ t \le 0 \longrightarrow (12.3)$

$$\left.\begin{aligned} u(0,t) &= U \\ u(y,t) &= 0 \quad y \to \infty \end{aligned}\right\} \longrightarrow (12.4)$$

Equation (12.4) is classical diffusion equation with diffusion coefficients $\nu = [L^2/T]$, Thus ν physically represents diffusion of momentum. At the initial instant, the velocity appears as a step with magnitude arbitrary close to zero everywhere else. As the time increases, however the effect of plate motion propagates further and further out into the fluid as momentum is transferred normal to plate by molecular diffusion.

Solution (By Laplace Transform)

We define the Laplace transform by $\bar{u}(y,s) = \mathcal{L}\{u(y,t)\}$

$$= \int_0^\infty e^{-st} u(y,t)\,dt \longrightarrow (12.5)$$

using (12.5) equation (12.2) becomes

$$s\bar{u}(y,s) - u(y,0) = \nu \frac{d^2\bar{u}}{dy^2}$$

or

$$\frac{d^2\bar{u}}{dy^2} - \frac{s}{\nu}\bar{u} = 0 \longrightarrow (12.6)$$

From (12.3) and (12.4) we get

$$\bar{u}(0,s) = \frac{U}{s} \longrightarrow (12.7)$$

$$\bar{u}(\infty,s) = 0 \longrightarrow (12.8)$$

solving (12.6) we get

$$\bar{u}(y,s) = A e^{\sqrt{s/\nu}\, y} + B e^{-\sqrt{s/\nu}\, y} \longrightarrow (12.9)$$

as $\bar{u}(\infty, s) = 0$ & $\bar{u}(0,s) = U/s$

(12.9) $\Rightarrow$ $A = 0$ | (12.9) $\Rightarrow$ $B = U/s$

$$\Rightarrow \bar{u}(y,s) = \frac{U}{s} e^{-\sqrt{s/\nu}\, y}$$

Taking inverse Laplace we get

$$u(y,t) = U \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right)$$

$$= U\left(1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right) \longrightarrow (12.10)$$

Here $\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\xi^2}\, d\xi$ is

known as error function with

$\operatorname{erf}(0) = 0$ & $\operatorname{erf}(\infty) = 1$

Also $\operatorname{erfc}(\eta) = 1 - \operatorname{erf}(\eta)$

$\underbrace{\qquad\qquad}_{\text{complimentry error function}}$

From (12.10) Let $\eta = \frac{y}{2\sqrt{\nu t}}$, it is clear that with increase in time, value of η decreases for a given / finite value of y, and $\operatorname{erf}(\eta)$ decreases as η decreases, so the velocity u increases at a given y as time increases. of course the value of u never exceeds the value of U at any value of y since

$\operatorname{erf}(0) = 0$

as $t \to \infty$ $\eta \to 0$

$$u(y,t) = U[1 - \mathrm{erf}(\eta)]$$

$\longrightarrow U$

at $y=0$, $\eta=0$, $\mathrm{erf}(0)=0$

$$u(0,t) = U$$

velocity will vary from U to 0 over a distance with time at a rate proportional to $\sqrt{\nu t}$

From (12.10) $\frac{u}{U} = \mathrm{erfc}(\eta)$

$$\frac{u}{U} = F(\eta) \qquad \eta = \frac{y}{2\sqrt{\nu t}}$$

$$[\eta] = \frac{L}{\sqrt{L^2T^{-1}T}} = L^0T^0$$

$\Rightarrow \eta$ is dimensionless

Let $u(y,t) = UF(\eta) \longrightarrow$ (12.11)

$$\eta = \frac{y}{2\sqrt{\nu t}} \longrightarrow (12.12)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{U}{2\sqrt{\nu t}}\frac{dF}{d\eta} \longrightarrow (12.13)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U}{4\nu t}\frac{d^2F}{d\eta^2} \longrightarrow (12.14)$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = U\frac{\partial F}{\partial \eta}\left(\frac{-\eta}{2t}\right) \longrightarrow (12.15)$$

using (12.13) $\longrightarrow$ (12.15) in (12.2)

we get

$$\frac{-\eta U}{2t}\frac{dF}{d\eta} = \nu \frac{U}{4\nu t}\frac{d^2F}{d\eta^2}.$$

or $$\frac{d^2F}{d\eta^2} + 2\eta \frac{dF}{d\eta} = 0 \longrightarrow (12.16)$$

also from (12.3) & (12.4)

$$F(0) = 1 \quad , \quad F(\infty) = 0 \longrightarrow (12.17)$$

from (12.16) $$\frac{d^2F}{d\eta^2} = -2\eta \frac{dF}{d\eta}$$

or $$\frac{d^2F/d\eta^2}{dF/d\eta} = -2\eta$$

integrating we get

$$\ln\left(\frac{dF}{d\eta}\right) = -\eta^2 + \ln C_1$$

$$\Rightarrow \frac{dF}{d\eta} = C_1 e^{-\eta^2} \longrightarrow (12.18)$$

Integrating (12.18) we get

$$F(\eta) = C_1 \int_0^\eta e^{-\xi^2} d\xi + C_2 \longrightarrow (12.19)$$

using $F(0) = 1$, we get from (12.19)

$$1 = C_2$$

$$F(\eta) = C_1 \int_0^\eta e^{-\xi^2} d\xi + 1 \longrightarrow (12.20)$$

$$F(\infty) = 0$$

(12.20) $\Rightarrow$ $$0 = C_1 \int_0^\infty e^{-\xi^2} d\xi + 1$$

$$= C_1 \frac{\sqrt{\pi}}{2} + 1$$

$$C_1 = -2/\sqrt{\pi}$$

$$F(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\xi^2} \, d\xi$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_0^{y/2\sqrt{\nu t}} e^{-\xi^2} \, d\xi$$

$$= 1 - \operatorname{erf}\left(y/2\sqrt{\nu t} \right)$$

$$\frac{u}{U} = 1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}} \right)$$

$$u = U\left[1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}} \right)\right]$$

Similarity Solutions:-

Similarity solutions to PDE are solutions which depend on certain groupings of the indipendent variables, rather on each variable seperately.

In 12.11, 12.12 η is similarity variable and transformations 12.11 & 12.12 are called similarity transformations.

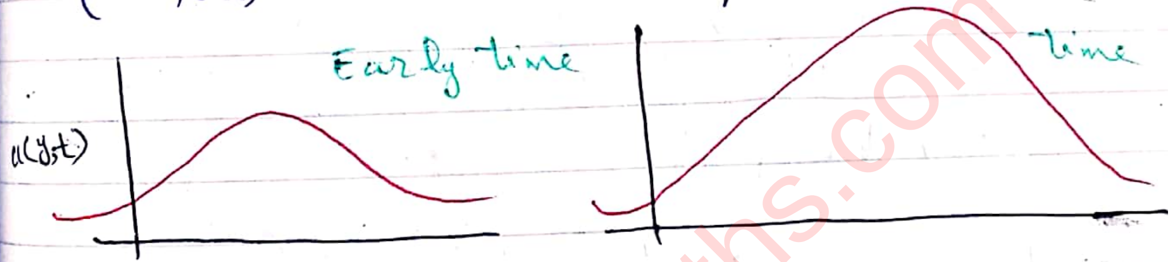
* In these solutions such as $u(y,t)$ in the present problem, that depend on a single dimensionless variable η instead of y and 't' seperately is said to be self similar
* Similarity transformations reduces the number of independent variable by one.
* For self similar problem, the

solution must have the special form

$$u(y,t) = a(t)\, f\left(\frac{y}{b(t)}\right)$$

$a(t)$ changes the amplitude $b(t)$ is time dependent scaling of y-coordinate making velocity profile wider/narrower.

* The choice of similarity variable $\eta = y/b(t)$ is not unique



Energy Equation:-

$$\rho c_p\left[\frac{\partial T}{\partial t} + u_i\frac{\partial T}{\partial x_i}\right] = k\frac{\partial^2 T}{\partial x_i \partial x_i} + \mu \tau_{ij}\frac{\partial u_i}{\partial x_j} \longrightarrow (1)$$

$$\rho c_p\left[\frac{\partial T}{\partial t} + u_i\frac{\partial T}{\partial x_j}\right] = k\frac{\partial^2 T}{\partial x_i^2} + \mu\,\Phi$$

$[T] = \Theta$ = Temp of fluid

$[\rho] = ML^{-3}$ = fluid density

$[c_p] = L^2T^{-2}\Theta^{-1}$ = specific heat

$[k] = MLT^{-3}\Theta^{-1}$ = Thermal conductivity

Here $\Phi = \tau_{ij}\dfrac{\partial u_i}{\partial x_j}$ is the viscous dissipation term

* When the fluid is forced to flow, the velocity as well as kinetic fluid gains some

energy. The viscosity of fluid take that kinetic energy and converts it into internal or thermal energy of the fluid. Consequently, the fluid is heated up and heat transfer occurs. This phenomena is modelled by the energy equation with viscous dissipation effects, For two dimensional flow

$$\underbrace{\rho C_p\left[\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right]}_{\text{convective part}}=\underbrace{K\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right]}_{\text{conductive part}}+\mu\Phi \rightarrow (2)$$

$$\Phi = 2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2$$

equation (2) is second order partial differential equation that is parabolic in nature.

From equation (2)

$$\frac{dT}{dt}=\frac{K\nabla^2 T}{\rho C_p}+\frac{\mu}{\rho C_p}\Phi$$

$$\frac{dT}{dt}=\alpha\nabla^2 T+\frac{\mu}{\rho C_p}\Phi \longrightarrow (3)$$

$$\alpha=\frac{K}{\rho C_p} \quad \text{thermal diffusivity}$$

$$[\alpha]=\frac{[K]}{[\rho][C_p]}=\frac{MLT^{-3}\theta^{-1}}{(ML^{-3})(L^2T^{-2}\theta^{-1})}=L^2T^{-1}$$

$\downarrow$ same as ν

$$\left[\rho C_p\frac{\partial T}{\partial t}\right]=[\rho][C_p]\left[\frac{\partial T}{\partial t}\right]=(ML^{-3})(L^2T^{-2}\theta^{-1})\left(\frac{\theta}{T}\right)$$

$$=ML^{-1}T^{-3}$$

$$\left[k\frac{\partial^2 T}{\partial x^2}\right] = [k]\left[\frac{\partial^2 T}{\partial x^2}\right] = (MLT^{-3}\theta^{-1})\left(\frac{\theta}{L^2}\right)$$

$$= ML^{-1}T^{-3}$$

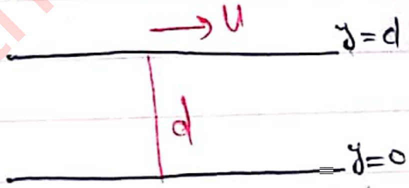
Dimension of each term is energy equ (2) is $ML^{-1}T^{-3}$

From (3) $\left[u\frac{\partial T}{\partial x}\right] = (LT^{-1})\left(\frac{\theta}{L}\right) = \theta T^{-1}$

$$\left[\alpha\frac{\partial^2 T}{\partial x^2}\right] = (L^2T^{-1})\left(\frac{\theta}{L^2}\right) = \theta T^{-1}$$

Heat Transfer Analysis of Unidirectional Flows:-

Recall the plane Coutte flow



$\bar{v} = [u(y), 0, 0]$

$$\frac{d^2u}{dy^2} = 0 \quad , \quad u = 0 \text{ at } y = 0$$
$$u = U \text{ at } y = d$$

$$T = T(y)$$

From energy equation

$$\rho c_p[0+0+0] = k\left[\frac{\partial^2 T}{\partial y^2}\right] + \mu\left(\frac{\partial u}{\partial y}\right)^2$$

i.e $$k\left[\frac{d^2T}{dy^2}\right] + \mu\left(\frac{\partial u}{\partial y}\right)^2 = 0$$

Defining the dimensionless quantities

$$\bar{u} = \frac{u}{U}, \quad \bar{y} = \frac{y}{d}, \quad \bar{T} = \frac{T - T_0}{T_1 - T_0}$$

$$\frac{d^2\bar{u}}{d\bar{y}^2} = 0 \quad , \quad \bar{u}(0) = 0 \;\&\; \bar{u}(d) = 1$$

$$\bar{u}(\bar{y}) = \bar{y}$$

$$\text{or} \quad u(y) = \left(\frac{U}{d}\right) y \longrightarrow (4)$$

For $T = T(y)$ energy equation gives

$$K\left[\frac{d^2T}{dy^2}\right] + \mu\left(\frac{du}{dy}\right)^2 = 0 \longrightarrow (5)$$

$$T = \bar{T}(T_1 - T_0) + T_0$$

$$\frac{dT}{dy} = \frac{dT}{d\bar{T}} \cdot \frac{d\bar{T}}{d\bar{y}} \cdot \frac{d\bar{y}}{dy}$$

$$= \frac{T_1 - T_0}{d} \frac{d\bar{T}}{d\bar{y}}$$

$$\boxed{\frac{du}{dy} = \frac{U}{d}\frac{d\bar{u}}{d\bar{y}}}$$

$$\frac{d^2T}{dy^2} = \frac{(T_1 - T_0)}{d^2} \frac{d^2\bar{T}}{d\bar{y}^2}$$

From (5) $$\frac{K(T_1 - T_0)}{d^2} \frac{\delta^2\bar{T}}{d\bar{y}^2} + \frac{\mu U^2}{d^2}\left(\frac{d\bar{u}}{d\bar{y}}\right)^2 = 0$$

$$\frac{d^2\bar{T}}{d\bar{y}^2} + \frac{\mu U^2}{K(T_1 - T_0)}\left(\frac{d\bar{u}}{d\bar{y}}\right)^2 = 0$$

$$\frac{d^2\bar{T}}{d\bar{y}^2} + \left(\frac{\mu C_p}{K}\right)\left(\frac{U^2}{C_p(T_1 - T_0)}\right)\left(\frac{d\bar{u}}{d\bar{y}}\right)^2 = 0$$

$$\frac{d^2\bar{T}}{d\bar{y}^2} + P_r E_c \left(\frac{d\bar{u}}{d\bar{y}}\right)^2 = 0$$

$$P_r = \frac{\mu C_p}{K}$$ is Prandtl number

$$[P_r] = \frac{[\mu][C_p]}{[K]} = \frac{(ML^{-1}T^{-1})(L^2T^{-2}\theta^{-1})}{MLT^{-3}\theta^{-1}}$$

$$= L^0T^0\theta^0M^0 = [-]$$

$$E_c = \frac{U^2}{C_p(T_1-T_0)}$$ is called Eckert number

$$E_c = \frac{L^2T^{-2}}{(L^2T^{-2}\theta^{-1})(\theta)} = [-]$$

As $P_r = \frac{\mu C_p}{K} = \frac{\mu/\rho}{K/C_p\rho} = \frac{\nu}{\alpha}$

$$P_r = \frac{L^2T^{-1}}{L^2T^{-1}} = [-]$$

$$P_r = \frac{\text{Momentum diffusivity}}{\text{Thermal diffusivity}}$$

$P_r << 1$ means thermal diffusivity dominates (i.e. conductivity is dominant over viscosity

$P_r >> 1$, the momentum diffusivity dominate the behaviour (i.e. viscosity is dominant over conductivity)

as $T = T_0$ at $y = 0$

$\bar{T} = 0$ at $\bar{y} = 0$

$T = T_1$ at $y = d$

$\bar{T} = 1$ at $\bar{y} = 1$

$$\bar{T} = \frac{T - T_0}{T_1 - T_0}$$

$$\frac{d^2\bar{T}}{d\bar{y}^2} + P_r E_c\left(\frac{d\bar{u}}{d\bar{y}}\right)^2 = 0$$

Brinkman number

$$Br = P_r E_c = \frac{\mu U^2}{K(T_1 - T_0)}$$

$$\frac{d^2\bar{T}}{d\bar{y}^2} + Br\left(\frac{d\bar{u}}{d\bar{y}}\right) = 0$$

Derivation of Energy Equation..

Consider the moving fluid element. The statement of first law of thermodynamics when applied to fluid element states that

$$\underbrace{\left\{\begin{array}{l}\text{Rate of change}\\ \text{of energy in-}\\ \text{side the fluid}\\ \text{element}\end{array}\right\}}_{①} = \underbrace{\left\{\begin{array}{l}\text{Rate of work}\\ \text{done on fluid}\\ \text{element by}\\ \text{body \& surface}\\ \text{forces}\end{array}\right\}}_{②} + \underbrace{\left\{\begin{array}{l}\text{Rate at which}\\ \text{heat added}\\ \text{to fluid}\\ \text{element}\end{array}\right\}}_{③}$$

$$[\text{Joule}] = [J] = \left[\frac{Kg\, m^2}{s^2}\right] = ML^2T^{-2}$$

To obtain the rate of change of energy inside the element. The total energy of fluid element per unit mass is the sum of its internal energy 'e' per unit mass and its kinetic energy per unit mass $u_i^2/2$

$$\Rightarrow \left[e + \frac{1}{2}u_i^2\right] \quad \text{Total energy per unit mass}$$

Now for total energy per unit volume (xing by ρ) we get

$$\rho\left[e+\frac{1}{2}u_i^2\right] \longrightarrow = \left[\frac{J}{Kg}\right]\left[\frac{Kg}{m^3}\right] = \left(e+\frac{u_i^2}{2}\right)\rho$$

$\frac{T.E}{Volume} = J/m^3$

Rate of change of total energy

$$= \frac{\partial}{\partial t}\int_V \rho\left(e+\frac{u_i^2}{2}\right)dv \longrightarrow (1)$$

Work done on the fluid element due to body forces. As rate of doing work by a force exerted on moving body is equal to the product of force and velocity at which body force does work

$$= \int_V \rho F_i u_i \, dV \longrightarrow (2)$$

Rate of work done by surface forces

$$\int_S u_i \bar{\bar{\tau}}_{ij} n_j \, ds \longrightarrow (3)$$

The rate at which heat per unit mass added to the fluid

$$\int_V \rho \Phi \, dV \longrightarrow (4)$$

$\hookrightarrow$ source of body force

(Here Φ is thermodynamic equivalant of body force)

Rate of heat flow per unit surface area out of volume

$$= -\int_S K_j n_j \, ds \longrightarrow (5)$$

where K_j is heat flux vector

Leibniz rule is apply here

$$\frac{\partial}{\partial t}\int_V \rho\left(e+\frac{1}{2}u_i^2\right)dV = \int_V \rho F_i u_i \, dv + \int_S u_i \tau_{ij} n_j \, ds + \int_V \rho \Phi \, dV - \int_S K_j n_j \, ds \longrightarrow (6)$$

$$\frac{d}{dt}\left[\int \rho\left(\frac{u_i^2}{2}\right)\right]dv = \int_V \left[\rho u_i \frac{du_i}{dt}\, dv + \frac{d}{dt}(\rho e)\right] dv$$

$$\int_S u_i T_{ij} n_j \, ds = \int_V \frac{\partial}{\partial x_j}(u_i T_{ij})\, dv \quad \left[\text{Gauss divergence Theorem}\right.$$

$$= \int_V \left[u_i \frac{\partial}{\partial x_j}(T_{ij}) + T_{ij}\frac{\partial u_i}{\partial x_j}\right] dv$$

$$\Rightarrow \int_S K_j n_j \, ds = \int_V \frac{\partial}{\partial x_j}(K_j)\, dv$$

Now from ⑥ we have

$$\int_V \rho u_i \frac{du_i}{dt} dv + \int_V \frac{d}{dt}(\rho e) dv = \int_V \rho F_i u_i \, dv + \int_V u_i \frac{\partial}{\partial x_j}(T_{ij}) dv$$

$$+ \int_V T_{ij}\frac{\partial u_i}{\partial x_j} dv + \int_V \rho Q \, dv - \int_V \frac{\partial}{\partial x_j}(K_j) dv$$

we know that

$$\rho \frac{du_i}{dt} = \rho F_i + \frac{\partial}{\partial x_j} T_{ij}$$

$$\Rightarrow \int_V u_i \left\{\cancelto{0}{\rho \frac{du_i}{dt} - \rho F_i - \frac{\partial T_{ij}}{\partial x_j}}\right\} dv + \int_V \frac{d}{dt}(\rho e)\, dv$$

$$= \int_V T_{ij}\frac{\partial u_i}{\partial x_j} dv + \int_V \rho Q \, dv - \int_V \frac{\partial K_j}{\partial x_j} dv$$

$$\frac{d}{dt}(\rho e) = T_{ij}\frac{\partial u_i}{\partial x_j} + \rho Q - \frac{\partial K_j}{\partial x_j} \longrightarrow ⑦$$

Now by Fourier Law

$$K_j = -k\frac{\partial T}{\partial x_j}$$

$$② \Rightarrow \rho \frac{de}{dt} = T_{ij} \frac{\partial u_i}{\partial x_j} + \rho \Phi + k \frac{\partial^2 T}{\partial x_j^2}$$

$$T_{ij} = -p \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$ for incompressible

As e the the internal energy is given by $e = C_p T \longrightarrow C_p$ specific heat

$$\Rightarrow \rho C_p \frac{dT}{dt} = \left[-p \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] \frac{\partial u_i}{\partial x_j} + k \frac{\partial^2 T}{\partial x_j^2} + \rho \Phi$$

$$\rho C_p \left[\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} \right] = k \frac{\partial^2 T}{\partial x_j^2} + \rho \Phi - \cancel{p \delta_{ij} \frac{\partial u_i}{\partial x_j}}^{\,0} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j}$$

since $\delta_{ij} \frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial x_i} = 0$

$\frac{k}{\rho C_p} \Rightarrow$ Diffusion term

$$\alpha = \frac{k}{\rho C_p} = \frac{MLT^{-3} \theta^{-1}}{ML^{-3} L^2 T^{-2} \theta^{-1}}$$

$$\Rightarrow [\alpha] = L^2 T^{-1}$$

$$\Rightarrow \rho C_p \left[\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} \right] = k \frac{\partial^2 T}{\partial x_j \partial x_j} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} + \rho \Phi$$

MUHAMMAD TAHIR

PhD MATH CU ISLAMABAD

SP18-PMT-005

Stokes Oscillating Plat Problem

$$\bar{V} = [u(y,t), 0, 0]$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \longrightarrow (A)$$

IC $\Rightarrow$ $u(y,0) = 0$

B.Cs $\Rightarrow$ $u(0,t) = U_0 \cos\omega t$

$u(\infty,t) = 0$

Solution

First of all to convert the problem in dimensionless form let we use

$$\bar{u} = \frac{u}{U_0} \Rightarrow \boxed{u = \bar{u} U_0}$$

$$\bar{y} = \frac{y}{\sqrt{\nu/\omega}} \Rightarrow \boxed{y = \bar{y}\sqrt{\nu/\omega}}$$

$$\bar{t} = \frac{t}{1/\omega} \Rightarrow \boxed{t = \frac{\bar{t}}{\omega}}$$

By putting these values in equ (A) we get

$$\frac{\partial \bar{u} U_0}{\partial(\bar{t}/\omega)} = \nu \frac{\partial^2 \bar{u} U_0}{\partial \bar{y}^2 \frac{\nu}{\omega}} \Rightarrow \frac{U_0}{\omega}\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\nu U_0}{\omega}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$\Rightarrow \boxed{\frac{\partial \bar{u}}{\partial \bar{t}} = \nu \frac{\partial^2 \bar{u}}{d\bar{y}^2}}$$

$u(y,0) \Rightarrow \bar{u}(y,0)U_0 \Rightarrow \bar{u}(y,0) = 0$

$u(0,t) \Rightarrow U_0 \cos\omega t \Rightarrow \bar{u}(0,t)U_0 = \cancel{\bar{u}(y,0)} = 0$

$\Rightarrow \bar{u}(0,t) = \cos\omega t$

$\bar{u}(\infty,t) = 0$

Now for solution let

$$\bar{u}(y,t) = Re\left[F(y)e^{i\omega t}\right] \qquad \bar{t} = \omega t$$

$$\bar{u}(y,t) = Re\left[F\left(\bar{y}\sqrt{\tfrac{\nu}{\omega}}\right)e^{i\bar{t}}\right]$$

$$\Rightarrow \frac{\partial \bar{u}(y,t)}{\partial \bar{t}} = Re\left[F\left(\bar{y}\sqrt{\tfrac{\nu}{\omega}}\right)e^{i\bar{t}}\cdot i\right] \longrightarrow ①$$

$$\frac{\partial^2 \bar{u}(y,t)}{\partial \bar{y}^2} = Re\left[\frac{d^2F}{d\bar{y}^2}e^{i\bar{t}}\right] \longrightarrow ②$$

By putting equation ① and ② in equ Ⓐ we get

$$Re\left[F\left(\bar{y}\sqrt{\tfrac{\nu}{\omega}}\right)\right]e^{i\bar{t}}\cdot i = Re\left[\frac{d^2F}{d\bar{y}^2}e^{i\bar{t}}\right]$$

$$\Rightarrow iF(\bar{y}) = \frac{d^2F}{d\bar{y}^2}$$

Now $$\frac{d^2F}{d\bar{y}^2} - iF\left(\bar{y}\sqrt{\tfrac{\nu}{\omega}}\right) = 0$$

To solve ODE

$$\frac{d^2F}{d\bar{y}^2} = -\frac{(1\pm i)^2}{2}F\left(\bar{y}\sqrt{\nu/\omega}\right)$$

$$\Rightarrow F = Ae^{(1+i)\sqrt{\frac{1}{2}}\bar{y}} + Be^{-(1+i)\sqrt{\frac{1}{2}}\bar{y}}$$

$\Rightarrow$ By applying B.Cs we get

$$F = e^{-(1+i)\sqrt{\frac{1}{2}}\bar{y}}$$

$$\Rightarrow \boxed{F = e^{-(1+i)\sqrt{\frac{1}{2}}\bar{y}}}$$

Hence $\bar{u}(\bar{y},\bar{t}) = Re\left[e^{-(1+i)\sqrt{\frac{1}{2}}\bar{y}}\, e^{i\sqrt{\frac{1}{2}}\bar{t}}\right]$

$$\Rightarrow \frac{u}{u_0} = Re\left[e^{-\sqrt{\frac{1}{2}}\bar{y}} \cdot e^{i(\bar{t}-\sqrt{\frac{1}{2}}\bar{y})}\right]$$

$$\Rightarrow u = u_0 Re\left[e^{-\sqrt{\frac{1}{2}}\frac{y}{\sqrt{\nu/\omega}}} \cdot e^{i(\omega t-\sqrt{\frac{1}{2}}\frac{y}{\sqrt{\nu/\omega}})}\right]$$

$$\Rightarrow u = u_0\, Re\left[e^{-\sqrt{\frac{\omega}{2\nu}}} \cdot e^{i(\omega t-\sqrt{\frac{\omega}{\nu}}y)}\right]$$

By Laplace Transform

$$\frac{d\bar{u}}{d\bar{t}} = \frac{d^2\bar{u}}{d\bar{y}^2}$$

$$\Rightarrow s\bar{u}(\bar{y},s) - \bar{u}(\bar{y},0) = \frac{d^2\bar{u}}{d\bar{y}^2}$$

$$\Rightarrow \frac{d^2\bar{u}}{d\bar{y}^2} = s\bar{u} \qquad \because \bar{u}(\bar{y},0)=0$$

$$\Rightarrow \bar{u}(\bar{y},s) = Ae^{\sqrt{s}\,\bar{y}} + Be^{-\sqrt{s}\,\bar{y}}$$

$$\bar{u}(\infty,0)=0 \Rightarrow A=0$$

$$\bar{u}(0,s) = \frac{s}{s^2+1} \Rightarrow B = \frac{s}{s^2+1}$$

$$\Rightarrow \bar{u} = \frac{s}{s^2+1}e^{-\sqrt{s}\,\bar{y}}$$

$$\Rightarrow \bar{u} = \cos(t-1)\, e^{-\sqrt{s}\,\bar{y}}$$

Stoke's Equation For moving Oscillatory Boundary:-

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \longrightarrow (A)$$

$$u(y = C_0 t, 0) = U_0 \cos \omega t$$

$$u(y \to \infty, 0) = 0$$

Solution

For converting the problem from moving frame to stationary frame let us use the transformations

$$u = u(Y, \tau) \; ;$$

where $Y = y - C_0 t$ & $t = \tau$ $\Big\} \longrightarrow (1)$

So
$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial t} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t}$$

$\because$ from (1) $\frac{\partial Y}{\partial t} = -C_0$ & $\frac{\partial \tau}{\partial t} = 1$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} = -C_0 \frac{\partial u}{\partial Y} + \frac{\partial u}{\partial \tau}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial Y}(1) + 0$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial Y^2}}$$

So equation (A) implies

$$\frac{\partial u}{\partial \tau} - c_0 \frac{\partial u}{\partial Y} = \nu \frac{\partial^2 u}{\partial Y^2} \longrightarrow (2)$$

with $u(Y=0,\tau) = U_0 \cos\omega\tau$
$u(Y\to\infty,\tau) = 0$ $\Big]\longrightarrow (3)$

Now to non-dimensionlize the problem

let $\bar{u} = \frac{u}{U_0} \Rightarrow \boxed{u = \bar{u}U_0}$

$\bar{y} = \frac{Y}{y_c} = \frac{c_0 Y}{\nu} \Rightarrow \boxed{Y = \frac{\bar{y}\nu}{c_0}}$

$\bar{\tau} = \frac{\tau}{\nu/c_0^2} \Rightarrow \boxed{\tau = \frac{\bar{\tau}\nu}{c_0^2}}$

Putting these values in equation (2)

$$\frac{\partial(\bar{u}U_0)}{\partial\left(\frac{\bar{\tau}\nu}{c_0^2}\right)} - c_0 \frac{\partial(\bar{u}U_0)}{\partial\left(\frac{\bar{y}\nu}{c_0}\right)} = \nu \frac{\partial^2(\bar{u}U_0)}{\partial(\bar{y}\nu/c_0)^2}$$

$$\Rightarrow \frac{U_0 c_0^2}{\nu}\frac{\partial\bar{u}}{\partial\bar{\tau}} - \frac{c_0^2 U_0}{\nu}\frac{\partial\bar{u}}{\partial\bar{y}} = \frac{\nu U_0 c_0^2}{\nu^2}\frac{\partial^2\bar{u}}{\partial\bar{y}^2}$$

$$\Rightarrow \frac{\partial\bar{u}}{\partial\bar{\tau}} - \frac{\partial\bar{u}}{\partial\bar{y}} = \frac{\partial^2\bar{u}}{\partial\bar{y}^2}$$

$$\Rightarrow \frac{\partial^2\bar{u}}{\partial\bar{y}^2} + \frac{\partial\bar{u}}{\partial\bar{y}} - \frac{\partial\bar{u}}{\partial\bar{\tau}} = 0$$

let $\bar{u} = Re\left[F(\bar{y})\, e^{iT}\right]$

$$\Rightarrow \frac{d^2F}{d\bar{y}^2} + \frac{dF}{d\bar{y}} - iF = 0$$

The characteristic equation will be

$$m^2+m-i=0$$

$$\Rightarrow m=\frac{-1\pm\sqrt{1+4i}}{2} \quad \Rightarrow m=-\frac{1}{2}\pm\sqrt{\frac{1}{4}+i}$$

Let $\alpha+i\beta=\sqrt{\frac{1}{4}+i}$

$$\Rightarrow (\alpha+i\beta)^2=\frac{1}{4}+i \quad \Rightarrow \alpha^2-\beta^2+2\alpha\beta i=\frac{1}{4}+i$$

By comparing real & imaginary parts

$$\Rightarrow \boxed{\alpha^2-\beta^2=\frac{1}{4}} \rightarrow (4) \quad \& \quad \boxed{2\alpha\beta=1} \rightarrow (5)$$

from (5) $\boxed{\beta=\frac{1}{2\alpha}}$ putting in (4)

$$\alpha^2-\frac{1}{4\alpha^2}=\frac{1}{4}$$

$$\Rightarrow \alpha^4-\frac{1}{4}-\frac{\alpha^2}{4}=0 \Rightarrow \alpha^4-\frac{\alpha^2}{4}-\frac{1}{4}=0$$

$$\Rightarrow \alpha^2=\frac{\frac{1}{4}\pm\sqrt{\left(\frac{1}{4}\right)^2-4(1)\left(-\frac{1}{4}\right)}}{2}$$

$$\Rightarrow \alpha^2=\frac{1}{4}\pm\sqrt{\frac{1}{16}+1} \Rightarrow \alpha^2=\frac{1}{4}\pm\frac{\sqrt{17}}{4}$$

$$\Rightarrow \alpha=\left[\frac{1}{4}+\frac{\sqrt{17}}{4}\right]^{1/2}$$

$$\& \quad \beta=\frac{1}{2}\left[\frac{1}{4}\pm\frac{\sqrt{17}}{4}\right]^{-\frac{1}{2}}$$

$$\therefore m=-\frac{1}{2}\pm(\alpha+i\beta)$$

$$\Rightarrow F(\bar{y}) = A_1 e^{\left(-\frac{1}{2}+\alpha+\beta i\right)\bar{y}} + A_2 e^{-\left(\frac{1}{2}+\alpha+i\beta\right)\bar{y}}$$

$F(0) = 1 \quad , \quad F(\infty) = 0$

$\Rightarrow \quad F(0) = 1 \qquad \Rightarrow A_1 + A_2 = 1$

& $\quad F(\infty) = 0 \quad \Rightarrow \quad A_1 = 0 \qquad \Rightarrow \boxed{A_2 = 1}$

$$\Rightarrow F(\bar{y}) = e^{-\left(\frac{1}{2}+\alpha+\beta i\right)\bar{y}}$$

$$\Rightarrow \bar{u} = Re\left[e^{-\left(\frac{1}{2}+\alpha+i\beta\right)\bar{y}} \, e^{i\omega\tau}\right]$$

$$\bar{u} = Re\left[e^{-\left(\frac{1}{2}+\alpha\right)\bar{y}} \, e^{i(\omega\tau - \beta\bar{y})}\right]$$

$$\Rightarrow u = u_0 Re\left[e^{-\left(\frac{1}{2}+\alpha\right)\frac{C_0 y}{\nu}} \cdot e^{i\left(\omega\tau - \beta\frac{C_0 y}{\nu}\right)}\right]$$

$$\Rightarrow u = U_0 Re\left[e^{-\left(\frac{C_0 y}{2\nu} + \frac{\alpha C_0 y}{\nu}\right)} \cdot e^{i\left(\omega\tau - \beta\frac{C_0 y}{\nu}\right)}\right]$$

Stokes Problem in Porous Plate

$\bar{V} = [u(y,t),\ v(y,t) = 0]$

$\bar{v} = [u, -V_0, 0]$

$\Rightarrow \dfrac{\partial u}{\partial y} = 0 \ , \ \dfrac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad v = -V_0$

Solution

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$\Rightarrow \frac{\partial u}{\partial t} - V_0\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \longrightarrow (A)$$

$\because V_0 < 0$ is for injection

Now to convert the problem in dimensionless form let

$$\bar{y} = \frac{y}{y_c} = \frac{y}{\nu/V_0} \Rightarrow \boxed{y = \frac{\bar{y}\nu}{V_0}}$$

$$\bar{u} = \frac{u}{U_0} \Rightarrow \boxed{u = \bar{u}\, u_0}$$

$$\bar{t} = \frac{t}{t_c} = \frac{t}{\nu/V_0^2} \Rightarrow \boxed{t = \frac{\bar{t}\nu}{V_0^2}}$$

So equation (A) becomes

$$\frac{\partial \bar{u}u_0}{\partial \frac{\bar{t}\nu}{V_0^2}} - V_0 \frac{\partial \bar{u}u_0}{\partial(\bar{y}\frac{\nu}{V_0})} = \nu \frac{\partial^2 \bar{u}u_0}{\partial(\bar{y}\frac{\nu}{V_0})^2}$$

$$\Rightarrow \frac{u_0 V_0^2}{\nu}\frac{\partial \bar{u}}{\partial \bar{t}} - \frac{V_0^2 u_0}{\nu}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\nu V_0^2 u_0}{\nu}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$\Rightarrow \frac{\partial \bar{u}}{\partial \bar{t}} - \frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \longrightarrow (C)$$

Let us assume the solution

$$\bar{u}(\bar{y},\bar{t}) = Re\left[F(\bar{y})e^{i\omega t}\right]$$

Eqn (C) implies

$$\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial \bar{u}}{\partial \bar{y}} - \frac{\partial \bar{u}}{\partial \bar{t}} = 0$$

Hence

$$\frac{d^2 F}{d\bar{y}^2} + \frac{dF}{d\bar{y}} - iF = 0 \longrightarrow (D)$$

$$\bar{u}(0,t) = \cos T \Rightarrow F(0) = 1$$

$$\bar{u}(\infty,t) = 0 \Rightarrow F(\infty) = 0$$

The characteristic equation becomes

$$m^2 + m - i = 0$$

$$\Rightarrow m = \frac{-1 \pm \sqrt{1+4i}}{2} \Rightarrow m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1+4i}$$

$$\Rightarrow m = -\frac{1}{2} \pm \sqrt{\frac{1}{2^2} + i}$$

Let $\alpha + i\beta = \sqrt{\left(\frac{1}{2}\right)^2 + i}$

$$\Rightarrow (\alpha + i\beta)^2 = \left(\frac{1}{2}\right)^2 + i \Rightarrow \alpha^2 - \beta^2 + 2i\alpha\beta = \left(\frac{1}{2}\right)^2 + i$$

By comparing we get

$$\alpha^2 - \beta^2 = \frac{1}{4} \rightarrow (2) \quad \& \quad \boxed{2\alpha\beta = 1} \rightarrow (3)$$

from (3) $\beta = \frac{1}{2\alpha}$ put in (2)

$$\alpha^4 - \frac{1}{4} = \frac{\alpha^2}{4} \Rightarrow \alpha^4 - \frac{\alpha^2}{4} - \frac{1}{4} = 0$$

$$\Rightarrow \alpha^2 = \frac{\frac{1}{4} \pm \sqrt{\left(\frac{1}{4}\right)^2 - 4(1)\left(\frac{-1}{4}\right)}}{2}$$

$$\Rightarrow \alpha^2 = \frac{1}{8} \pm \frac{1}{2}\sqrt{\frac{1}{16} + 1} \Rightarrow \alpha^2 = \frac{1}{8} \pm \frac{\sqrt{17}}{8}$$

$$\Rightarrow \boxed{\alpha = \left[\frac{1}{8} \pm \frac{\sqrt{17}}{8}\right]^{\frac{1}{2}}}$$

$$\Rightarrow \beta = \frac{1}{2\left[\frac{1 \pm \sqrt{17}}{8}\right]^{1/2}}$$

$$\Rightarrow \boxed{\beta = \frac{1}{2}\left[\frac{1 \pm \sqrt{17}}{8}\right]^{-\frac{1}{2}}}$$

So $m = -\frac{1}{2} \pm (\alpha + i\beta)$

$$\Rightarrow F(\bar{y}) = A_1 e^{+(-\frac{1}{2}+\alpha+i\beta)\bar{y}} + A_2 e^{-(\frac{1}{2}+\alpha+i\beta)\bar{y}}$$

Now $F(0) = 1 \Rightarrow A_1 + A_2 = 1$

& $F(\infty) = 0 \Rightarrow A_1 = 0 \Rightarrow \boxed{A_2 = 1}$

$$\therefore \Rightarrow F(\bar{y}) = A_2 e^{-(\frac{1}{2}+\alpha+i\beta)\bar{y}}$$

$$\Rightarrow \bar{u}(\bar{y},\bar{t}) = Re\left[e^{-(\frac{1}{2}+\alpha+i\beta)\bar{y}} e^{i\bar{T}}\right]$$

$$\Rightarrow \frac{u}{U_0} = Re\left[e^{(\alpha-\frac{1}{2})\bar{y}} \cdot e^{i(\bar{T}-\beta\bar{y})}\right]$$

$$\Rightarrow u = U_0 Re\left[e^{(-\alpha-\frac{1}{2})\bar{y}} \cdot e^{i(\omega t - \beta\frac{V_0 y}{\nu})}\right]$$

$$\Rightarrow u = U_0 Re\left[e^{-(\frac{V_0}{2\nu}+\frac{\alpha V_0}{\nu})} \cdot e^{i(\omega t - \beta\frac{V_0}{\nu}y)}\right]$$

Coutte Unsteady Flow

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \longrightarrow (A)$$

$u(0,t) = 0$ & $u(H,t) = U_0$

Answer

Let us assume that

$\bar{u} = \frac{u}{U_0} \Rightarrow \boxed{u = U_0\bar{u}}$, $\bar{t} = \frac{t}{H^2/\nu} = \frac{\nu t}{H^2}$

$\bar{y} = \frac{y}{H} \Rightarrow \boxed{y = \bar{y}H} \Rightarrow \boxed{t = \frac{\bar{t}H^2}{\nu}}$

By putting these values in equation Ⓐ we get

$$\frac{\partial \bar{u} u_0}{\partial \left(\bar{t}\frac{H^2}{\nu}\right)} = \nu \frac{\partial^2 (\bar{u} u_0)}{\partial (\bar{y} H)^2}$$

$$\Rightarrow \frac{u_0 \nu}{H^2} \frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\nu u_0}{H^2} \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$\Rightarrow \frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \longrightarrow ①$$

$$\left.\begin{array}{l} u(0,t) = 0 \Rightarrow \bar{u}(0,t) u_0 = 0 \Rightarrow \bar{u}(0,t) = 0 \\ u(H,t) = U_0 \Rightarrow \bar{u}(H,t) U_0 = U_0 \Rightarrow \bar{u}(H,t) = 1 \end{array}\right\} \longrightarrow ②$$

Let solution be

$$\bar{u}(\bar{y},\bar{t}) = f(\bar{y}) + g(\bar{y},\bar{t}) \longrightarrow ③$$

By putting equ ③ into equ ① we get

$$\frac{\partial g}{\partial \bar{t}} = \left[\frac{\partial^2 f}{\partial \bar{y}^2} + \frac{\partial^2 g}{\partial \bar{y}^2}\right]$$

$$\Rightarrow \frac{\partial g}{\partial \bar{t}} - \frac{\partial^2 g}{\partial \bar{y}^2} - \frac{\partial^2 f}{\partial \bar{y}^2} = 0$$

By comparing we get

$$\frac{\partial g}{\partial \bar{t}} - \frac{\partial^2 g}{\partial \bar{y}^2} = 0 \longrightarrow ④$$

$$\frac{\partial^2 f}{\partial \bar{y}^2} = 0 \longrightarrow ⑤$$

I.C:- $\bar{u}(\bar{y},0) = 0 \Rightarrow \bar{u}(\bar{y},0) = f(\bar{y}) + g(\bar{y},0) = 0$

$$\Rightarrow g(\bar{y},0) = -f(\bar{y})$$

B.Cs :- $\bar{u}(0,t) = 0 \Rightarrow f(0) + g(0,t) = 0$

$$\Rightarrow f(0) = 0$$

$$\& \; g(0,t) = 0$$

$\bar{u}(H,t) = 1 \Rightarrow f(H) + g(H,t) = 1$

$$\Rightarrow f(H) = 1 \;\&\; g(H,t) = 0$$

from eqn ⑤ $\dfrac{d^2 f}{d\bar{y}^2} = 0$

Integrating twice we get

$$\frac{df}{d\bar{y}} = C_1 \Rightarrow \boxed{f = C_1 \bar{y} + C_2}$$

$$f(0) = 0 \Rightarrow \boxed{C_2 = 0}$$

$$f(H) = 1 \Rightarrow C_1 H = 1 \Rightarrow C_1 = \frac{1}{H}$$

$$\Rightarrow \boxed{f(\bar{y}) = \frac{\bar{y}}{H}}$$

Solving equation ④ we get

$$\frac{\partial g}{\partial \bar{t}} - \frac{\partial^2 g}{\partial \bar{y}^2} = 0$$

$$\Rightarrow \frac{\partial g}{\partial \bar{t}} = \frac{\partial^2 g}{\partial \bar{y}^2}$$

$$\Rightarrow T'Y = TY'' = K$$

$$\Rightarrow T' = KT \longrightarrow ⑥ \quad \& \quad Y'' = KY \longrightarrow ⑦$$

from ⑥ $\dfrac{T'}{T} = K$

$$\Rightarrow T = C_1 e^{K\bar{t}}$$

The solution is unbounded for $\bar{t}$

So let $\boxed{k = -m}$

Now $Y'' = kY \Rightarrow Y'' = -mY$

characteristic equation is

$D^2 = -m \Rightarrow D = \pm i m$

$$\Rightarrow Y(\bar{y}) = C_2 \cos\sqrt{m}\,\bar{y} + C_3 \sin\sqrt{m}\,\bar{y} \longrightarrow (8)$$

Now $g(0,t) = 0 \Rightarrow Y(0)T(t) = 0 \Rightarrow Y(0) = 0$

$g(H,t) = 0 \Rightarrow Y(H)T(t) = 0 \Rightarrow Y(H) = 0$

$Y(0) = 0$ implies $\boxed{C_2 = 0}$

$$\Rightarrow Y(\bar{y}) = C_3 \sin\sqrt{m}\,\bar{y}$$

& $Y(H) = 0 \Rightarrow C_3 \sin\sqrt{m}\,H = 0$

$$\Rightarrow \sin\sqrt{m}\,H = 0 \Rightarrow \sqrt{m}\,H = n\pi$$

$$\Rightarrow m = \frac{n^2\pi^2}{H^2}$$

$$\Rightarrow Y(\bar{y}) = \sum_{n=1}^{\infty} C_3 \left[\sin \frac{n\pi}{H}\bar{y}\right]$$

To find the value of c_3

$$\int_0^1 \sin(mx)\sin(nx)\,dx = \begin{cases} 1/2 & m = n \\ 0 & m \neq n \end{cases}$$

Let $\frac{y}{H} = \xi$

$y = 0 \Rightarrow \xi = 0$ & $y = H \Rightarrow \xi = 1$

$dy = H\,d\xi$

writing $g(\bar{y}, 0) = \frac{\bar{y}}{H}$

$$\frac{\bar{y}}{H} = \sum_{n=1}^{\infty} C_3 \sin\left(\frac{n\pi}{H}\bar{y}\right) \longrightarrow ⓐ$$

$$\int_0^H \frac{\bar{y}}{H} \sin\left(\frac{n\pi}{H}\bar{y}\right) d\bar{y} = \int_0^H \sum_{n=1}^{\infty} C_3\left(\frac{n\pi}{H}\bar{y}\right)\sin\left(\frac{m\pi}{H}\bar{y}\right) d\bar{y}$$

$$\Rightarrow \int_0^1 \xi \sin(m\pi\xi)\, d\xi = \sum_{n=1}^{\infty} C_3 \int_0^1 \sin(n\pi\xi)\sin(m\pi\xi)\, d\xi$$

$$\Rightarrow \int_0^1 \xi \sin(m\pi)\xi\, d\xi = \frac{C_3}{L}$$

$$\frac{C_3}{L} = \left[\xi \frac{\cos m\pi\xi}{m\pi}\Big|_0^1 - \int_0^1 \frac{\cos m\pi\xi}{n\pi} d\xi\right]$$

$$= \left[\frac{1}{m\pi}(0-1) - \frac{1}{n^2\pi^2}\sin n\pi\xi\Big|_0^1\right]$$

$$\Rightarrow \frac{C_3}{L} = -\frac{1}{m\pi} \Rightarrow \boxed{C_3 = -\frac{L}{m\pi}}$$

$$\Rightarrow g(\bar{y},\bar{t}) = -\frac{L}{m\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{H}\bar{y}\; e^{-\frac{n^2\pi^2}{H^2}\bar{t}}$$

Now $\bar{u}(\bar{y},\bar{t}) = f(\bar{y}) + g(\bar{y},\bar{t})$

$$\Rightarrow u = u_0\frac{y}{H} - \frac{2}{m\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{H^2}\bar{y}\; e^{-\frac{n^2\pi^2}{H^2}\bar{t}}$$

Boundary Layer Flow:- Navier-Stokes equations can be simplified using boundary layer concept. Under special conditions certain terms in equations can be neglected.

Boundary Layer:- German scientist Ludwig Prandtl in 1904 showed for the first time, usually the viscosity (Friction μ) plays an important role in a thin layer (along a solid boundary) called a boundary layer or shear layer.

The viscosity of fluid can not be neglected in all the regions. Flow at the surface of the body is at rest relative to that body. At a certain distance from the body, the viscosity of flow again can be neglected. This very ~~closed~~ thin layer closed to that body in which effect of viscosity are important is the boundary layer.

→ Friction is not important ~~article~~ outside this region and solution to the outer flow field is closed to a frictionless solution.

In this layer of fluid tangential component of velocity of fluid relative to body increases from zero at the surface to the free stream velocity at some distance from the surface.

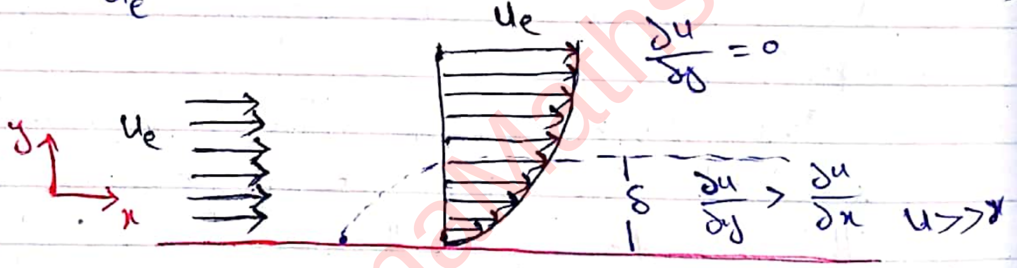
Therefore, in the boundary layer region strong gradient of velocity occurs (i.e. velocity varies rapidly from zero to main stream velocity)

This thin layer of fluid thickness is usually denoted by δ.

$\delta \rightarrow$ normal distance from the wall where

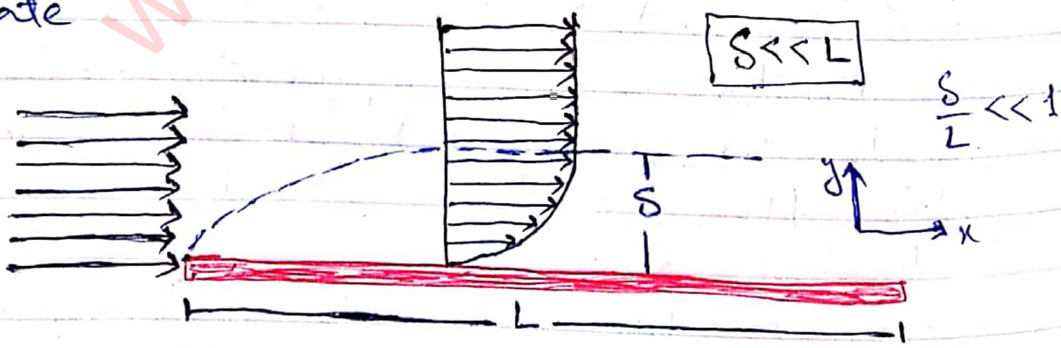
$$u = 0.99\, u_e \quad \text{or} \quad 0.999\, u_e \quad \text{or} \quad 0.9999\, u_e$$

i.e $\dfrac{u}{u_e} \approx 1$



Derivation of Boundary Layer Equations:-

Consider the flow of fluid parallel to a thin flat plate



For two dimensional flows, governing continuity and momentum equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \longrightarrow (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \rightarrow (2)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \rightarrow (3)$$

$$\frac{\partial u}{\partial x} = \frac{U_e}{L} \quad , \quad \frac{\partial u}{\partial y} = \frac{U_e}{\delta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{U_e}{L^2} \quad , \quad \frac{\partial^2 u}{\partial y^2} = \frac{U_e}{\delta^2}$$

as $\delta << L$, from above equations we observe that

$$\frac{\partial u}{\partial x} << \frac{\partial u}{\partial y} , \quad \frac{\partial^2 u}{\partial x^2} << \frac{\partial^2 u}{\partial y^2}$$

The continuity equation gives us an estimation of vertical component of velocity V_e inside the boundary layer.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{U_e}{L} + \frac{V_e}{\delta} \approx 0 \quad \Rightarrow V_e \approx \frac{\delta}{L}U_e \rightarrow (4)$$

(4) shows that inside the boundary layer v component of velocity is much smaller than u component (i.e $u >> v$ or $v << u$)

From (2) we now estimate different terms

$$u\frac{\partial u}{\partial x} \approx U_e \frac{U_e}{L} = \frac{U_e^2}{L_2} \longrightarrow (5)$$

$$v\frac{\partial u}{\partial y} \approx V_e \frac{U_e}{\delta} = \frac{U_e}{L} \longrightarrow (6) \qquad \because V_e = \frac{\delta}{L} U_e$$

(5) and (6) shows that both convective terms are of the same order and none can be ignored.

The size of viscous term can be approximated as

$$\nu \frac{\partial^2 u}{\partial x^2} \approx \nu \frac{U_e}{L^2} \longrightarrow (7)$$

$$\nu \frac{\partial^2 u}{\partial y^2} \approx \nu \frac{U_e}{\delta^2} \longrightarrow (8)$$

(7) and (8) shows that

$$\nu \frac{\partial^2 u}{\partial x^2} << \nu \frac{\partial^2 u}{\partial y^2}$$

i.e we can ignore the term $\nu \frac{\partial^2 u}{\partial x^2}$ as compared to the viscous term $\nu \frac{\partial^2 u}{\partial y^2}$.

The pressure forces in priciple can not be ignored. So they must be of same order as convective term

$$\frac{1}{\rho}\frac{\partial p}{\partial x} \approx \frac{U_e^2}{L}$$

Thus x variation of pressure must be of order

$$\Delta_x P \approx \rho U_e^2 \longrightarrow (9)$$

Now from (3) y component of momentum equation

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \longrightarrow (10)$$

$$u\frac{\partial v}{\partial x}\approx U_e\frac{V_e}{L}=\frac{U_e^2}{L}\frac{\delta}{L} \longrightarrow (10.1)$$

$$v\frac{\partial v}{\partial x}\approx V_e\frac{V_e}{\delta}=\frac{U_e^2}{L}\frac{\delta}{L} \longrightarrow (10.2)$$

$$\nu\frac{\partial^2 v}{\partial x^2}\approx\nu\frac{V_e}{L^2}=\nu\frac{U_e}{L^2}\frac{\delta}{L} \longrightarrow (10.3)$$

$$\nu\frac{\partial^2 v}{\partial y^2}\approx\nu\frac{V_e}{\delta^2}=\nu\frac{U_e}{\delta^2}\frac{\delta}{L} \longrightarrow (10.4)$$

From (10.1) to (10.4) one can notice that y component equation is δ/L time smaller than x-component form

$$\frac{1}{\rho}\frac{\partial P}{\partial y}\approx\frac{U_e^2}{L}\frac{\delta}{L}$$

$$\Delta_y P\approx\rho U_e^2\frac{\delta^2}{L^2} \quad \text{i.e } \Delta_y P\approx\Delta_x P\left(\frac{\delta}{L}\right)^2$$

as a consequence, the vertical pressure gradient can be neglected

$$\frac{\partial P}{\partial y}\approx\frac{\partial P}{\partial x}\frac{\delta}{L} \longrightarrow 0$$

This implies that pressure is constant across the boundary layer and $P=P(x)$. Finally we reached at the conclusion that under boundary layer approximation governing equations becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

* The convective term is of same order of magnitude as the viscous term

$$u\frac{\partial u}{\partial x} \approx \nu\frac{\partial^2 u}{\partial y^2}$$

$$u_e\left(\frac{u_e}{L}\right) \approx \nu\frac{u_e}{\delta^2} \Rightarrow \left(\frac{\delta}{L}\right)^2 \approx \frac{\nu}{u_e L}$$

$$\text{i.e. } \frac{\delta}{L} \approx \sqrt{\frac{\nu}{u_e L}} \Rightarrow \frac{\delta}{L} \approx \frac{1}{\sqrt{Re_L}}$$

Note that Reynolds number is based on length L and consequently it is denoted by Re_L. Reynolds number is typically based on characteristic dimension of the object.

————

Gradient of a Scaler:-

Let ϕ be a scaler, then

$$\text{grad}\,\phi = \frac{\hat{e}_r}{(g_{11})^{1/2}}\frac{\partial\phi}{\partial x^1} + \frac{\hat{e}_\theta}{(g_{22})^{1/2}}\frac{\partial\phi}{\partial x^2} + \frac{\hat{e}_z}{(g_{33})^{1/2}}\frac{\partial\phi}{\partial x^3}$$

For cylydrical coordinate system

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where g_{ij} is the metric tensor

so $g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = 1$

$$(x^1, x^2, x^3) = (r, \theta, z)$$

$$\Rightarrow \text{grad}\,\phi = \frac{\hat{e}_r}{1}\frac{\partial\phi}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial\phi}{\partial\theta} + \frac{e_z}{1}\frac{\partial\phi}{\partial z}$$

$$\bar{\nabla}\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right)$$

So $\bar{\nabla} = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial z}\right)$

Gradient of a vector:-

Let $\bar{V}$ be a vector, then

$$(\text{grad}\,V)_{jk} = \frac{1}{\sqrt{g_{kk}}}\frac{\partial V_j}{\partial x^k} - \Gamma^i_{jk} V_i$$

As the gradient of a vector is a tensor where Γ^i_{jk} are the christofell symbols and

$$g_{ij}\, g^{ij} = I \quad \Gamma^i_{jk} = \frac{1}{2} g^{il}\left(\Gamma^1_{11} V_1 + \Gamma^2_{11} V_2 + \Gamma^3_{11} V_3\right)$$

As $x^1 = r$, $x^2 = \theta$, $x^3 = z$

$V_1 = u$, $V_2 = v$, $V_3 = \omega$

$g^{11} = 1$, $\Gamma_{11}^1 = 0 = \Gamma_{11}^2 = \Gamma_{11}^3$

So $\boxed{(\text{grad } V)_{1,1} = \dfrac{\partial u}{\partial r}}$

Now
$$(\text{grad } V)_{1,2} = \frac{1}{g^{22}} \frac{\partial V_1}{\partial x^2} - \left(\Gamma_{12}^1 V_1 + \Gamma_{12}^2 V_2 + \Gamma_{12}^3 V_3\right)$$

$g^{22} = r$, $V_1 = u$, $x^2 = \theta$

$\Gamma_{12}^1 = 0$, $\Gamma_{12}^2 = \dfrac{1}{r}$, $\Gamma_{12}^3 = 0$

So
$$(\text{grad } V)_{1,2} = \frac{1}{r}\frac{\partial u}{\partial \theta} - \left(0 + \frac{1}{r} v + 0\right)$$

$$\Rightarrow \boxed{(\text{grad } V)_{1,2} = \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}}$$

$$(\text{grad } V)_{1,3} = \frac{1}{g^{33}} \frac{\partial V_1}{\partial x^3} - \left(\Gamma_{13}^1 V_1 + \Gamma_{13}^2 V_2 + \Gamma_{13}^3 V_3\right)$$

$g^{33} = 1$, $x^3 = z$, $V_1 = u$

$\Gamma_{13}^1 = 0$, $\Gamma_{13}^2 = 0$, $\Gamma_{13}^3 = 0$

$$\Rightarrow \boxed{(\text{grad } V)_{1,3} = \frac{\partial u}{\partial z}}$$

$$(\text{grad } V)_{2,1} = \frac{1}{g^{11}} \frac{\partial V_2}{\partial x^1} - \left(\Gamma_{21}^1 V_1 + \Gamma_{21}^2 V_2 + \Gamma_{21}^3 V_3\right)$$

$g^{11} = 1$, $x^1 = r$, $V_2 = v$

$\Gamma_{21}^1 = 0 = \Gamma_{21}^2 = \Gamma_{21}^3$

$$\Rightarrow \boxed{(\text{grad } V)_{2,1} = \frac{\partial v}{\partial r}}$$

$$(\text{grad } V)_{2,2} = \frac{1}{g^{22}} \frac{\partial V_2}{\partial x^2} - \left(\Gamma_{22}^{1} V_1 + \Gamma_{22}^{2} V_2 + \Gamma_{22}^{3} V_3\right)$$

$$g^{22} = r, \quad \Gamma_{22}^{1} = -\frac{1}{r}, \quad \Gamma_{22}^{2} = 0 = \Gamma_{22}^{3}$$

$$(\text{grad } V)_{2,2} = \frac{1}{r} \frac{\partial v}{\partial \theta} - \left(-\frac{1}{r} u + 0 + 0\right)$$

$$\Rightarrow \boxed{(\text{grad} V)_{2,2} = \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r}}$$

$$(\text{grad } V)_{2,3} = \frac{1}{g^{33}} \frac{\partial V_2}{\partial x^3} - \left(\Gamma_{23}^{1} V_1 + \Gamma_{23}^{2} V_2 + \Gamma_{23}^{3} V_3\right)$$

$$g^{33} = 1, \quad \Gamma_{23}^{1} = 0 = \Gamma_{23}^{2} = \Gamma_{23}^{3}$$

$$\Rightarrow \boxed{(\text{grad } V)_{2,3} = \frac{\partial v}{\partial z}}$$

$$(\text{grad } V)_{3,1} = \frac{1}{g^{11}} \frac{\partial V_3}{\partial x^1} - \left(\Gamma_{31}^{1} V_1 + \Gamma_{31}^{2} V_2 + \Gamma_{31}^{3} V_3\right)$$

$$g^{11} = 1, \quad \Gamma_{31}^{1} = 0 = \Gamma_{31}^{2} = \Gamma_{31}^{3}$$

$$\boxed{(\text{grad} V)_{3,1} = \frac{\partial w}{\partial r}}$$

$$(\text{grad } V)_{3,2} = \frac{1}{g^{22}} \frac{\partial V_3}{\partial x^2} - \left(\Gamma_{31}^{1} V_1 + \Gamma_{32}^{2} V_2 + \Gamma_{32}^{3} V_3\right)$$

$$g^{22} = r, \quad \Gamma_{32}^{1} = 0 = \Gamma_{32}^{2} = \Gamma_{32}^{3}$$

$$\boxed{(\text{grad } V)_{3,2} = \frac{1}{r} \frac{\partial w}{\partial \theta}}$$

$$(\text{grad } V)_{3,3} = \frac{1}{g^{33}} \frac{\partial V_3}{\partial x^3} - \left(\Gamma_{33}^{1} V_1 + \Gamma_{32}^{2} V_3 + \Gamma_{33}^{3} V_3\right)$$

$g^{33} = 1, \; \Gamma_{32}^{1} = 0 = \Gamma_{32}^{2} = \Gamma_{32}^{3}$

$$\boxed{(\text{grad } V)_{3,3} = \frac{\partial w}{\partial z}}$$

Thus

$$\text{grad } V = \begin{bmatrix} (\text{grad} V)_{1,1} & (\text{grad} V)_{1,2} & (\text{grad} V)_{1,3} \\ (\text{grad} V)_{2,1} & (\text{grad} V)_{2,2} & (\text{grad} V)_{2,3} \\ (\text{grad} V)_{3,1} & (\text{grad} V)_{3,2} & (\text{grad} V)_{3,3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial r} & \frac{1}{r}\frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial z} \end{bmatrix}$$

★ Navier-Stoke's Equations in Cylindrical Coordinates:-

The law of conservation of momentum is

$$\rho \frac{d\bar{V}}{dt} = \text{div}\,\bar{T} + \rho b \longrightarrow (1)$$

Where $\bar{V}$ is the velocity field, ρ is the density of fluid, b the body force per unit mass and T the Cauchy stress tensor

→ Cauchy stress tensor for a viscous fluid is given by

$$\bar{T} = -P\bar{I} + \mu A_1 \longrightarrow (2)$$

with P as the pressure, $\bar{I}$ the

identity tensor and A_1 is the first Rivlin-Ericksen tensor and is given by

$$\bar{A}_1 = L + L^T = (\nabla\bar{V}) + (\nabla\bar{V})^T \longrightarrow (3)$$

Let the velocity field is given by

$$\bar{V} = [0, 0, \omega(r,t)] \longrightarrow (4)$$

Now

$$(\nabla\bar{V}) = \begin{bmatrix} \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} & \frac{\partial v}{\partial z} \\ \frac{\partial \omega}{\partial r} & \frac{1}{r}\frac{\partial \omega}{\partial \theta} & \frac{\partial \omega}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial \omega}{\partial r} & 0 & 0 \end{bmatrix} \longrightarrow (5)$$

$$L^T = (\nabla\bar{V})^T = \begin{bmatrix} 0 & 0 & \frac{\partial \omega}{\partial r} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow (6)$$

Thus

$$\bar{A}_1 = (\nabla\bar{V}) + (\nabla\bar{V})^T$$

$$\Rightarrow A_1 = \begin{bmatrix} 0 & 0 & \frac{\partial \omega}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial \omega}{\partial r} & 0 & 0 \end{bmatrix} \longrightarrow (7)$$

Also $\bar{V}.\bar{\nabla} = (u, v, \omega)\cdot\left[\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right]$

$$= \left[\cancel{u}\cancel{\frac{\partial}{\partial r}}^{0} + \cancel{\frac{v}{r}}\frac{\partial}{\partial \theta}^{0} + \omega\frac{\partial}{\partial z}\right]$$

$$= \omega\frac{\partial}{\partial z}$$

And

$$(\bar{V}\cdot\bar{\nabla})\bar{V} = \left(\omega\frac{\partial}{\partial z}\right)(0,0,\omega)$$

$$= \omega\frac{\cancel{\partial\omega}^{\,0}}{\cancel{\partial z}} = 0 \longrightarrow (8)$$

Now as $\bar{T} = -P\bar{I} + \mu\bar{A}_1$

$$\bar{T} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} + \mu\begin{bmatrix} 0 & 0 & \frac{\partial\omega}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial\omega}{\partial r} & 0 & 0 \end{bmatrix}$$

$$T = -P\bar{I} + \bar{\tau}$$

As
$$[\mathrm{div}\,\bar{T}]_r = -\frac{\partial P}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(rT_{rr}) + \frac{1}{r}\frac{\partial}{\partial\theta}(T_{\theta r}) + \frac{\partial}{\partial z}(T_{zr}) - \frac{T_{\theta\theta}}{r}$$

$$= -\frac{\partial P}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(rT_{rr}) - \frac{T_{\theta\theta}}{r} \longrightarrow (*)$$

As
$$\tau = \begin{bmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mu\frac{\partial\omega}{\partial r} \\ 0 & 0 & 0 \\ \mu\frac{\partial\omega}{\partial r} & 0 & 0 \end{bmatrix} \longrightarrow (**)$$

using (**) & (*)

$$[\mathrm{div}\,\bar{T}]_r = -\frac{\partial P}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(r,0) - \frac{0}{r}$$

$$= -\frac{\partial P}{\partial r} \longrightarrow (10)$$

&

$$[\mathrm{div}\,\bar{T}]_\theta = -\frac{1}{r}\frac{\partial P}{\partial\theta} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cancel{\tau_{r\theta}}^{\,0}) + \frac{1}{r}\frac{\partial}{\partial\theta}(\cancel{\tau_{\theta\theta}}^{\,0}) + \frac{\partial}{\partial z}(\tau_{\theta z}) + \frac{\tau_{\theta r} - \tau_{r\theta}}{r}$$

$$\Rightarrow [\operatorname{div} \bar{T}]_\theta = -\frac{1}{r}\frac{\partial p}{\partial \theta} \longrightarrow (11)$$

$$[\operatorname{div} \bar{T}]_z = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r T_{rz}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\cancel{\tau_{\theta z}})^{0}$$

$$+ \frac{\partial}{\partial z}(\tau_{zz})$$

$$= -\frac{\partial p}{\partial z} + \frac{\partial}{\partial r}\tau_{rz} + \frac{1}{r}\tau_{rz}$$

$$= -\frac{\partial p}{\partial z} + \frac{\partial}{\partial r}\left[\mu \frac{\partial w}{\partial r}\right] + \frac{1}{r}\left[\mu \frac{\partial w}{\partial r}\right]$$

$$= -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right] \longrightarrow (12)$$

from 1

$$\rho\left[\frac{\partial \bar{V}}{\partial t} + \underbrace{\bar{V}\cdot\bar{\nabla}\bar{V}}_{}{}^{0}\right] = \operatorname{div}\bar{T} \longrightarrow (13)$$

In component form equ (13) is

$$\rho\left[\frac{\partial \bar{V}}{\partial t}\right]_r = [-\bar{\nabla} p]_r + [\operatorname{div}\bar{\tau}]_r \longrightarrow (14)$$

$$\rho\left[\frac{\partial \bar{V}}{\partial t}\right]_\theta = [-\bar{\nabla} p]_\theta + [\operatorname{div}\bar{\tau}]_\theta \longrightarrow (15)$$

$$\rho\left[\frac{\partial \bar{V}}{\partial t}\right]_z = [-\nabla p]_z + [\operatorname{div}\bar{\tau}]_z \longrightarrow (16)$$

substituting values we arrive at

$$0 = -\frac{\partial p}{\partial r}, \longrightarrow (17)$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} \longrightarrow (18)$$

$$\rho \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial z}\right] \longrightarrow (19)$$

Equation (17) to (19) are known as Navier-Stoke's equations for one dimensional flow in cylindrical coordinates.

Problem:- Derive the Navier-Stokes Equations in cylindrical coordinates by taking
$\bar{V} = [0, v(r,t), 0]$

Energy Equation in Cylindrical Coordinates:-

The law of conservation of energy is

$$\rho \frac{de}{dt} = \bar{\tau}.\bar{L} - \operatorname{div}\vec{q} + \rho \gamma \longrightarrow (1)$$

In above equation ρ is the density of the fluid, e ($= C_p T$, C_p being specific heat) is the speific thermal energy, T is the temperature, $\bar{\tau}$ is the cauchy stress tensor, $\bar{L}$ is the velocity gradient, $\bar{q}$ ($= k \operatorname{div} T$, k beig the thermal conductivity) is the heat flux vector and γ is the internal heat generation taken to be zero here.

The cauchy stress tensor for a Newtonian fluid is

$$\bar{\tau} = -p\bar{I} + \mu A_1 \longrightarrow (2)$$

where $\bar{p}$ is the pressure, I is the identity tensor and $\bar{A}_1$ is the first Rivlin-Ericksen tensor and is given by

$$\bar{A}_1 = L + L^T \;;\; L = \bar{\nabla}\bar{V} \longrightarrow (3)$$

Let us consider the velocity and temperature fields of the form

$$\bar{V} = (0, v(r,t), 0) \;;\; T = T(r,t) \longrightarrow (4)$$

Now

$$\bar{\nabla}\bar{V} = \begin{bmatrix} \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} & \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial r} & \frac{1}{r}\frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{v}{r} & 0 \\ \frac{\partial v}{\partial r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow (5)$$

$$\bar{A}_1 = L + L^T = \begin{bmatrix} 0 & -\frac{v}{r} & 0 \\ \frac{\partial v}{\partial r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{\partial v}{\partial r} & 0 \\ -\frac{v}{r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{\partial v}{\partial r} - \frac{v}{r} & 0 \\ \frac{\partial v}{\partial r} - \frac{v}{r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow (6)$$

As $\bar{A}\cdot\bar{B} = \text{trace}$

$$\bar{\tau}\cdot\bar{L} = -p\bar{I}\cdot\bar{L} + \mu\bar{A}_1\cdot\bar{L}$$

Now

$$-p\bar{I}\cdot\bar{L} = tr\left[\begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}\begin{pmatrix} 0 & -\frac{v}{r} & 0 \\ \frac{\partial v}{\partial r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$\Rightarrow -P\bar{I}\cdot L = tr\begin{bmatrix} 0 & p\frac{v}{r} & 0 \\ p\frac{\partial v}{\partial r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \longrightarrow (7)$$

trace

and

$$A_1\cdot L = tr\left[\begin{pmatrix} 0 & \frac{\partial v}{\partial r}-\frac{u}{r} & 0 \\ \frac{\partial v}{\partial r}-\frac{v}{r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\frac{v}{r} & 0 \\ \frac{\partial v}{\partial r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$= tr\begin{bmatrix} \frac{\partial v}{\partial r}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right) & 0 & 0 \\ 0 & -\frac{v}{r}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{\partial v}{\partial r}\left[\frac{\partial v}{\partial r}-\frac{v}{r}\right]-\frac{v}{r}\left[\frac{\partial v}{\partial r}-\frac{v}{r}\right]$$

$$= \left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)^2 \longrightarrow (8)$$

Thus

$$\bar{\tau}\cdot L = \mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)^2 \longrightarrow (9)$$

As

$$\bar{q} = -k\nabla T$$

$$= -k\left[\frac{\partial T}{\partial r}, \frac{1}{r}\frac{\partial T}{\partial \theta}, \frac{\partial T}{\partial z}\right]$$

$$= -k\left[\frac{\partial T}{\partial r}, 0, 0\right] \longrightarrow (10)$$

Since;

$$\bar{\nabla}\cdot\bar{V} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}$$

So $$\text{div}\,\bar{q} = -k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \longrightarrow (11)$$

substituting values in (1) we get

$$\rho C_p \frac{dT}{dt} = \bar{\tau}\cdot\bar{I} - \text{div}\,\bar{q}$$

$$\Rightarrow \rho C_p\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \theta} + w\frac{\partial T}{\partial z}\right]$$
$$= \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)^2 - \left(-\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right)$$

$$\Rightarrow \rho C_p\frac{\partial T}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)^2 \rightarrow (12)$$

which is the required form

Flow between Rotating Cocentric Cylinders.- Consider the steady flow contained between two concentric cylinders by steady angular velocity of one or both cylinders. Let the inner cylinder have radius r_0, angular velocity ω_0, temperature T_0, while the outer cylinder have r_1, ω_1 and T_1 respectively. The geometry is such that only non-zero velocity component is v and the variable v, T and p must be function only of ~~motion in polar coordinates~~

radius r. The equation of motion in polar coordinates reduces to

Continuity $\frac{\partial v}{\partial \theta} = 0 \longrightarrow ①$

r momentum $\frac{dp}{dr} = \frac{\rho v^2}{r} \longrightarrow ②$

θ momentum $\frac{d^2 v}{dr^2} + \frac{d}{dr}\left(\frac{v}{r}\right) = 0 \longrightarrow ③$

Energy $0 = \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \mu\left(\frac{dv}{dr} - \frac{v}{r}\right)^2 \longrightarrow ④$

with boundary conditions of each cylinder

$$\left.\begin{array}{ll} \text{At } r = r_0: & v = r_0\omega_0,\ T = T_0,\ P = P_0 \\ \text{At } r = r_1: & v = r_1\omega_1,\ T = T_1 \end{array}\right\} \rightarrow ⑤$$

The solution to the θ momentum equation has the form

$$v = C_1 r + \frac{C_2}{r} \longrightarrow ⑥$$

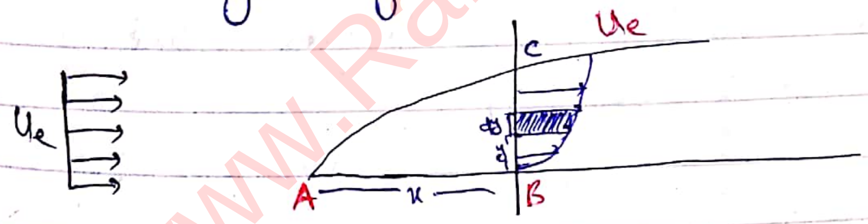
MUHAMAD TAHIR WATTOO

SP18-PMT-005

0344-8563284

Boundary Layer Thickness:- (δ) is defined as the distance from the boundary of solid body to the point where the velocity reaches 99% free stream velocity U_e i.e. at δ, $u = 0.99 U_e$

Displacement thickness:- (δ^*) It is defined as the thickness of a layer of fluid of velocity U_e for which mass flow rate is equal to mass flow rate deficit (decrease) that exist within the boundary layer (i.e. the distance perpendicular to the boundary by which free stream is displaced due to the formation of boundary layer)



Distance $BC = \delta$ (Boundary Layer thickness)

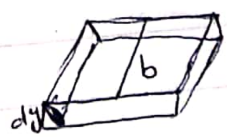
Let y = distance of elementry strip from plate

$dy \rightarrow$ thickness of elemental strip

$u \rightarrow$ velocity of fluid at strip

$b \rightarrow$ width of plate

Area of strip $= dA = b\,dy$



Mass flow rate / mass of fluid

per second flowing through strip

$= \rho(\text{velocity})(\text{Area of strip})$

$= \rho u b\,dy \longrightarrow ①$

$(\rho)(\text{velocity})(\text{Area})$
$(kg/m^3)(m/s)\ m^2$
$= kg/s$

<u>When No Plate</u>-

Mass of fluid flowing per sec flowing through elemental strip would be

$= \rho U_e b\,dy \longrightarrow ②$

∴ The reduction in mass/sec flowing through elemental strip

$= \rho U_e b\,dy - \rho U b\,dy$

$= \rho b(U_e - U)\,dy$

∴ Total reduction in mass of fluid flow through boundary condition due to plate will be

$$= \int_0^{\delta} \rho b(U_e - U)\,dy$$

$$= \rho b \int_0^{\delta} (U_e - U)\,dy \longrightarrow ③$$

Let the ~~plate~~ fluid is displaced by a distance (δ^*) and the velocity of fluid flow for distance δ^* is equal to free stream velocity the loss of the mass of fluid per sec will be at δ^*

$= \rho U_e \delta^* b \longrightarrow ④$

From ③ and ④

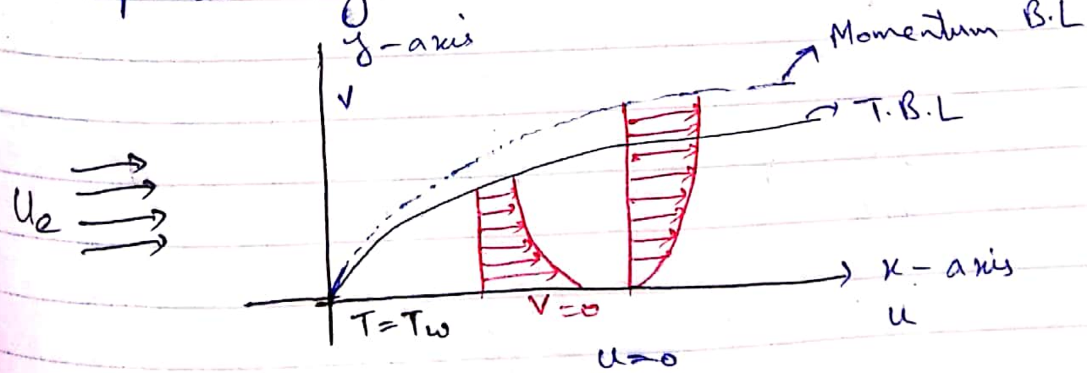
$$\rho b \int_0^{\delta} (u_e - u)\,dy = \rho b u_e \delta^*$$

$$\text{or} \quad u_e \delta^* = \int_0^{\delta} (u_e - u)\,dy$$

$$\delta^* = \frac{1}{u_e} \int_0^{\delta} (u_e - u)\,dy$$

$$\delta^* = \int_0^{\delta} \left(1 - \frac{u}{u_e}\right) dy$$

Blasius Flow over flat Plate with heat Transfer:- Consider two dimensional boundary layer flow of an incompressible and viscous fluid over a flat plate. It is assumed that plate is placed in the fluid flowing over the plate with uniform free stream velocity U_e. T_w and T_∞ are the constant temperature of plate and the fluid in free stream region respectively.



Governing Boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \longrightarrow (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \longrightarrow (2)$$

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 \longrightarrow (3)$$

Boundary Conditions

$$\left.\begin{array}{l} u=0, \quad v=0, \quad T=T_w \quad \text{at } y=0 \\ u=u_e, \quad T\to T_\infty \quad \text{at } y\to\infty \end{array}\right\} \longrightarrow (4)$$

Similarity transformations:-

$$\eta = y\sqrt{\frac{u_e}{x\nu}}, \quad u = u_e f', \quad v = \frac{1}{2}\sqrt{\frac{u_e \nu}{x}}(\eta f' - f)$$

$$\theta = \frac{T-T_\infty}{T_w - T_\infty} \longrightarrow (5)$$

$$\frac{\partial u}{\partial y} = u_e f''\sqrt{\frac{u_e}{x\nu}}, \quad \frac{\partial u}{\partial x} = -u_e \frac{\eta}{2x} f''$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_e^2}{x\nu} f'''$$

$$T = (T_w - T_\infty)\theta + T_\infty$$

$$\frac{\partial T}{\partial x} = (T_w - T_\infty)\theta'\left(\frac{-\eta}{2x}\right)$$

$$\frac{\partial T}{\partial y} = (T_w - T_\infty)\theta'\sqrt{\frac{u_e}{x\nu}}$$

$$\frac{\partial^2 T}{\partial y^2} = (T_w - T_\infty)\frac{u_e}{x\nu}\theta''$$

using similarity transformations ⑤ Equ (1−4) in dimensionless form becomes

$$f''' + \frac{1}{2} f f'' = 0 \longrightarrow ⑥$$

$$\frac{1}{P_r}\theta'' + f\theta' + E_c f''^2 = 0 \longrightarrow ⑦$$

$$\left.\begin{array}{l} f(0)=0\ ,\ f'(0)=0\ ,\ f'(\infty)=1 \\ \theta(0)=1\ ,\ \theta(\infty)=0 \end{array}\right\} \rightarrow ⑧$$

$$P_r = \frac{\mu c_p}{K} \quad \text{Prandtl number}$$

$$E_c = \frac{U_e^2}{c_p(T_w - T_\infty)} \quad \text{Eckert number}$$

Equ ⑥ & ⑦ are coupled non-linear equations, so exact solution is not possible

* Momentum thickness is defined as "The thickness of layer of fluid of velocity u_e for which the momentum flux is equal to the deficit of momentum flux i.e the momentum, that exists in the boundary layer.

* Momentum of fluid flowing per second through strips

$$= [\rho (u)(b dy)] u$$

$$= [\rho u b dy] u$$

When there is no boundary layer. Momentum of fluid flowing per second through the elementry strip with stream velocity

$$= [\rho u b dy] u_e$$

The total reduction in fluid flowing per second

$$= [\rho u b dy] u_e - [\rho u b dy] u$$

$$= \rho b [u_e - u] u dy$$

$$= \rho b \int^{\delta} (u_e - u) u dy$$

Let the fluid be displaced by distance δ_m would result reduction in momentum flow rate of $\rho u_e^2 \delta_m b$

$$\cancel{\rho} u_e^2 \delta_m \cancel{b} = \cancel{\rho}\cancel{b} \int_0^{\delta} (u_e - u) u dy$$

$$\Rightarrow \delta_m = \frac{u}{u_e^2}\int_0^{\delta}(u_e - u)\,dy$$

$$\boxed{\delta_m = \frac{u}{u_e}\int_0^{\delta}\left[1 - \frac{u}{u_e}\right]dy}$$

$x^1 = r$, $x^2 = \theta$ & $x^3 = z$

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}\left[\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right]$$

$$= \frac{1}{2} g^{il}\left[g_{jl,k} + g_{lk,j} - g_{jk,l}\right]$$

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \;\&\; g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \Gamma^1_{11} = \frac{1}{2} g^{1l}\left[g_{1l,1} + g_{l1,1} - g_{11,l}\right]$$

$$= \frac{1}{2} g^{11}\left[\cancel{g_{11,1}}^{0} + \cancel{g_{11,1}}^{0} - \cancel{g_{11,1}}^{0}\right] \qquad \because g_{12} = 0 = g_{13} \;\&\; \frac{\partial 1}{\partial r} = 0.$$

$$\Rightarrow \boxed{\Gamma^1_{11} = 0}$$

$$\Gamma^2_{11} = \frac{1}{2} g^{2l}\left[g_{1l,1} + g_{l1,1} - g_{11,l}\right]$$

$$= \frac{1}{2} g^{22}\left[g_{12,1} + g_{21,1} - g_{11,2}\right] \qquad \because g_{21} = 0 = g_{23}$$

$$= \frac{1}{2}\left(\frac{1}{r^2}\right)(0 + 0 - 0)$$

$$\Rightarrow \boxed{\Gamma^2_{11} = 0}$$

$$\Gamma_{11}^{3} = \frac{1}{2} g^{3l}\left[g_{1l,1} + g_{l1,1} - g_{11,l}\right]$$

$$= \frac{1}{2} g^{33}\left[\cancel{g_{13,1}}^{0} + \cancel{g_{31,1}}^{0} - \cancel{g_{11,3}}^{0}\right]$$

$$\Rightarrow \boxed{\Gamma_{11}^{3} = 0}$$

$$\Gamma_{12}^{1} = \frac{1}{2} g^{1l}\left[g_{1l,2} + g_{l2,1} - g_{12,l}\right]$$

$$= \frac{1}{2} g^{11}\left[\cancel{g_{11,2}}^{0} + \cancel{g_{12,1}}^{0} - \cancel{g_{12,1}}^{0}\right]$$

$$\Rightarrow \boxed{\Gamma_{12}^{1} = 0}$$

$$\Gamma_{12}^{2} = \frac{1}{2} g^{2l}\left[g_{1l,2} + g_{l2,1} - g_{12,l}\right]$$

$$= \frac{1}{2} g^{22}\left[\cancel{g_{12,2}}^{0} + g_{22,1} - \cancel{g_{12,2}}^{0}\right]$$

$$= \frac{1}{2}\left(\frac{1}{r^2}\right)\left[0 + \frac{\partial}{\partial r}(r^2) - 0\right)$$

$$= \frac{1}{2r^2}(2r)$$

$$\Rightarrow \boxed{\Gamma_{12}^{2} = \frac{1}{r}}$$

$$\Gamma_{12}^{3} = \frac{1}{2} g^{3l}\left[g_{1l,2} + g_{l2,1} - g_{12,l}\right]$$

$$= \frac{1}{2} g^{33}\left[\cancel{g_{13,2}}^{0} + \cancel{g_{32,1}}^{0} - \cancel{g_{12,3}}^{0}\right]$$

$$\Rightarrow \boxed{\Gamma_{12}^{3} = 0}$$

$$\Gamma^{1}_{13} = \frac{1}{2} g^{1l}\left[g_{1l,3} + g_{l3,1} - g_{13,l}\right]$$

$$= \frac{1}{2} g^{11}\left[\cancel{g_{11,3}}^{0} + \cancel{g_{13,1}}^{0} - \cancel{g_{13,1}}^{0}\right]$$

$$\Rightarrow \boxed{\Gamma^{1}_{13} = 0}$$

$$\Gamma^{2}_{13} = \frac{1}{2} g^{2l}\left[g_{1l,3} + g_{l3,1} - g_{13,l}\right]$$

$$= \frac{1}{2} g^{22}\left[\cancel{g_{12,3}}^{0} + \cancel{g_{23,1}}^{0} - \cancel{g_{13,2}}^{0}\right]$$

$$= \frac{1}{2}\left(\frac{1}{r^2}\right)(0)$$

$$\Rightarrow \boxed{\Gamma^{2}_{13} = 0}$$

$$\Gamma^{3}_{13} = \frac{1}{2} g^{3l}\left[g_{1l,3} + g_{l3,1} - g_{13,l}\right]$$

$$= \frac{1}{2} g^{33}\left[\cancel{g_{13,3}}^{0} + \cancel{g_{33,1}}^{0} - \cancel{g_{13,3}}^{0}\right]$$

$$\Rightarrow \boxed{\Gamma^{3}_{13} = 0}$$

$$\Gamma^{1}_{21} = \frac{1}{2} g^{1l}\left[g_{2l,1} + g_{l1,2} - g_{21,l}\right]$$

$$= \frac{1}{2} g^{11}\left[\cancel{g_{21,1}}^{0} + \cancel{g_{11,2}}^{0} - \cancel{g_{21,1}}^{0}\right]$$

$$\Rightarrow \boxed{\Gamma^{1}_{21} = 0}$$

$$\Gamma^{2}_{21} = \frac{1}{2} g^{2l}\left[g_{2l,1} + g_{l2,1} - g_{21,l}\right]$$

$$\Rightarrow \Gamma_{21}^{2} = \frac{1}{2} g^{2l}\left[\cancel{g_{2l,1}} + \cancel{g_{l1,2}} - \cancel{g_{21,l}}\right]$$

$$\Rightarrow \boxed{\Gamma_{21}^{2} = 0}$$

$$\Gamma_{21}^{3} = \frac{1}{2} g^{3l}\left[g_{2l,1} + g_{l1,2} - g_{21,l}\right]$$

$$= \frac{1}{2} g^{33}\left[\cancel{g_{23,1}} + \cancel{g_{31,2}} - \cancel{g_{21,3}}\right]$$

$$\Rightarrow \boxed{\Gamma_{21}^{3} = 0}$$

$$\Gamma_{22}^{1} = \frac{1}{2} g^{1l}\left[g_{2l,2} + g_{l2,2} - g_{22,l}\right]$$

$$= \frac{1}{2} g^{11}\left[\cancel{g_{21,2}} + \cancel{g_{12,2}} - g_{22,1}\right]$$

$$= \frac{1}{2}\left(\frac{1}{1}\right)\left[0 + 0 - \frac{\partial}{\partial r}(r^2)\right]$$

$$= \frac{1}{\cancel{2}}(-\cancel{2}r)$$

$$\Rightarrow \boxed{\Gamma_{22}^{1} = -r}$$

$$\Gamma_{22}^{2} = \frac{1}{2} g^{2l}\left[g_{2l,2} + g_{l2,2} - g_{22,l}\right]$$

$$= \frac{1}{2} g^{22}\left[\cancel{g_{22,2}} + \cancel{g_{22,2}} - \cancel{g_{22,2}}\right]$$

$$= \frac{1}{2}\left(\frac{1}{r^2}\right)(0)$$

$$\Rightarrow \boxed{\Gamma_{22}^{2} = 0}$$

$$\Gamma^3_{22} = \frac{1}{2} g^{3l}[g_{2l,2} + g_{l2,2} - g_{22,l}]$$

$$= \frac{1}{2} g^{33}[\cancel{g_{23,2}}^{0} + \cancel{g_{32,2}}^{0} - \cancel{g_{22,3}}^{0}]$$

$$\Rightarrow \boxed{\Gamma^3_{22} = 0}$$

$$\Gamma^1_{23} = \frac{1}{2} g^{1l}[g_{2l,3} + g_{l3,2} - g_{23,l}]$$

$$= \frac{1}{2} g^{11}[\cancel{g_{21,3}}^{0} + \cancel{g_{13,2}}^{0} - \cancel{g_{23,1}}^{0}]$$

$$\Rightarrow \boxed{\Gamma^1_{23} = 0}$$

$$\Gamma^2_{23} = \frac{1}{2} g^{2l}[g_{2l,3} + g_{l3,2} - g_{23,l}]$$

$$= \frac{1}{2} g^{22}[\cancel{g_{22,3}}^{0} + \cancel{g_{23,2}} - \cancel{g_{23,2}}]$$

$$\Rightarrow \boxed{\Gamma^2_{23} = 0}$$

$$\Gamma^3_{23} = \frac{1}{2} g^{3l}[g_{2l,3} + g_{l3,2} - g_{23,l}]$$

$$= \frac{1}{2} g^{33}[\cancel{g_{23,3}} + \cancel{g_{33,2}}^{0} - \cancel{g_{23,3}}]$$

$$\Rightarrow \boxed{\Gamma^3_{23} = 0}$$

$$\Gamma^1_{31} = \frac{1}{2} g^{1l}[g_{3l,1} + g_{l1,3} - g_{31,l}]$$

$$= \frac{1}{2} g^{11}[\cancel{g_{31,1}}^{0} + \cancel{g_{11,3}}^{0} - \cancel{g_{31,1}}^{0}]$$

$$\Rightarrow \boxed{\Gamma_{31}^{1} = 0}$$

$$\Gamma_{31}^{2} = \frac{1}{2} g^{2\ell} \left[g_{3\ell,1} + g_{\ell 1,3} - g_{31,\ell} \right]$$

$$= \frac{1}{2} g^{22} \left[\cancel{g_{32,1}}^{0} + \cancel{g_{21,3}}^{0} - \cancel{g_{31,2}}^{0} \right]$$

$$= \frac{1}{2} \left(\frac{1}{r^2} \right) (0)$$

$$\Rightarrow \boxed{\Gamma_{31}^{2} = 0}$$

$$\Gamma_{31}^{3} = \frac{1}{2} g^{3\ell} \left[g_{3\ell,1} + g_{\ell 1,3} - g_{31,\ell} \right]$$

$$= \frac{1}{2} g^{33} \left[\cancel{g_{33,1}}^{0} + \cancel{g_{31,3}} - \cancel{g_{31,3}} \right]$$

$$\Rightarrow \boxed{\Gamma_{31}^{3} = 0}$$

$$\Gamma_{32}^{1} = \frac{1}{2} g^{1\ell} \left[g_{3\ell,2} + g_{\ell 2,3} - g_{32,\ell} \right]$$

$$= \frac{1}{2} g^{11} \left[\cancel{g_{31,2}}^{0} + \cancel{g_{12,3}}^{0} - \cancel{g_{32,1}}^{0} \right]$$

$$\Rightarrow \boxed{\Gamma_{32}^{1} = 0}$$

$$\Gamma_{32}^{2} = \frac{1}{2} g^{2\ell} \left[g_{3\ell,2} + g_{\ell 2,3} - g_{32,\ell} \right]$$

$$= \frac{1}{2} g^{22} \left[\cancel{g_{32,2}} + \cancel{g_{22,3}}^{0} - \cancel{g_{32,2}} \right]$$

$$\Rightarrow \boxed{\Gamma_{32}^{2} = 0}$$

$$\Gamma_{32}^{3} = \frac{1}{2} g^{3l}\left[g_{3l,2} + g_{l2,3} - g_{32,l}\right]$$

$$= \frac{1}{2} g^{33}\left[\cancel{g_{33,2}}^{0} + \cancel{g_{32,3}} - \cancel{g_{32,3}}\right]$$

$$\Rightarrow \quad \boxed{\Gamma_{32}^{3} = 0}$$

$$\Gamma_{33}^{1} = \frac{1}{2} g^{1l}\left[g_{3l,3} + g_{l3,3} - g_{33,l}\right]$$

$$= \frac{1}{2} g^{11}\left[\cancel{g_{31,3}}^{0} + \cancel{g_{13,3}}^{0} - \cancel{g_{33,3}}^{0}\right]$$

$$\Rightarrow \boxed{\Gamma_{33}^{1} = 0}$$

$$\Gamma_{33}^{2} = \frac{1}{2} g^{2l}\left[g_{3l,3} + g_{l3,3} - g_{33,l}\right]$$

$$= \frac{1}{2} g^{22}\left[\cancel{g_{32,3}}^{0} + \cancel{g_{23,3}}^{0} - \cancel{g_{33,2}}^{0}\right]$$

$$\Rightarrow \boxed{\Gamma_{33}^{2} = 0}$$

$$\Gamma_{33}^{3} = \frac{1}{2} g^{3l}\left[g_{3l,3} + g_{l3,3} - g_{33,l}\right]$$

$$= \frac{1}{2} g^{33}\left[g_{33,3} + \cancel{g_{33,3}} - \cancel{g_{33,3}}\right]$$

$$= \frac{1}{2}(0)$$

$$\Rightarrow \boxed{\Gamma_{33}^{3} = 0}$$

Now we calculate Gradient

$$(\text{grad}\,\bar{V})_{j,k} = \frac{1}{\sqrt{g_{kk}}}\frac{\partial V_j}{\partial x^k} - \Gamma^i_{jk} V_i$$

$$\Rightarrow (\text{grad}\,\bar{V})_{1,1} = \frac{1}{\sqrt{g_{11}}}\frac{\partial V_1}{\partial x^1} - \Gamma^i_{11} V_i$$

$$= \frac{1}{\sqrt{1}}\frac{\partial u}{\partial r} - \left[\Gamma^1_{11}V_1 + \Gamma^2_{11}V_2 + \Gamma^3_{11}V_3\right]$$

$$= \frac{\partial u}{\partial r} - (0+0+0)$$

$$\Rightarrow \boxed{(\text{grad}\,\bar{V})_{1,1} = \frac{\partial u}{\partial r}}$$

$$(\text{grad}\,\bar{V})_{1,2} = \frac{1}{\sqrt{g_{22}}}\frac{\partial V_1}{\partial x^2} - \Gamma^i_{12} V_i$$

$$= \frac{1}{\sqrt{r^2}}\frac{\partial u}{\partial \theta} - \left[\Gamma^1_{12}V_1 + \Gamma^2_{12}V_2 + \Gamma^3_{12}V_3\right]$$

$$= \frac{1}{r}\frac{\partial u}{\partial \theta} - \left[0 + \frac{1}{r}(v) + 0\right]$$

$$\Rightarrow \boxed{(\text{grad}\,\bar{V})_{1,2} = \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}}$$

$$(\text{grad}\,\bar{V})_{1,3} = \frac{1}{\sqrt{g_{33}}}\left(\frac{\partial V_1}{\partial x^3}\right) - \Gamma^i_{13} V_i$$

$$= \frac{1}{\sqrt{1}}\frac{\partial u}{\partial z} - \left(\Gamma^1_{13}V_1 + \Gamma^2_{13}V_2 + \Gamma^3_{13}V_3\right)$$

$$\Rightarrow \boxed{(\text{grad}\,\bar{V})_{1,3} = \frac{\partial u}{\partial z}}$$