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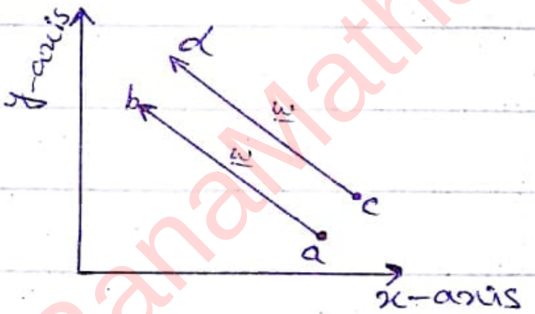
Chap#1:- Points and Vectors

04-03-2015

Points and vector are different mathematicals entities such that a point has no dimension it represents a location in space.

On the other hand a vector has no welldefined location & it's only attributes are direction and magnitude.



This figure shows two pair of points (a,b) & (c,d). Points a & c are different & so are c & d. The vectors b-a & d-c are identical (same).

Example:- Let two points $P_0 = (5,4)$ & $P_1 = (2,6)$. Calculate the difference.

$$P_1 - P_0 = (2-5, 6-4)$$
$$= (-3, 2)$$

The new pair is a vector it

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has a direction and distance. To get the distance from P_0 to P_1 we need to move -3 units in x-direction & to move 2 units in y-direction. There distance between the points

$$P_1P_0 = \sqrt{(-3)^2 + (2)^2} = \sqrt{9+4} = \sqrt{13}$$

If we have two points P_0 & P_2 the expression $P_0 + \alpha(P_2 - P_0)$ is sum of a point and a vector, So it is a point that we can denote by P_1. The vector $P_2 - P_0$ points from P_0 to P_2. So adding it to P_0 produce a point on the line connecting P_0 to P_2. After concluding the result. points are collinear. (P_0, P_1, P_2)

$$\Rightarrow P_1 = P_0 + \alpha(P_2 - P_0)$$
$$= P_0 + \alpha P_2 - \alpha P_0$$
$$= (1-\alpha)P_0 + \alpha P_2$$

showing that P_1 point is a linear combination of P_0 & P_2. In general any of three collinear points can be written as a linear combination of

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other two. (i.e. such points are not independent)

Example:- Given that three points $P_0 = (1,1)$, $P_1 = (2,2.5)$ & $P_2 = (3,4)$, are they collinear?

Sol:- We know

$$P_1 = (1-\alpha)P_0 + \alpha P_2$$

$$\Rightarrow (2,2.5) = (1-\alpha)(1,1) + \alpha(3,4)$$

$$(2,2.5) = (1-\alpha, 1-\alpha) + (3\alpha, 4\alpha)$$

$$(2,2.5) = (1+2\alpha, 1+3\alpha)$$

$$\Rightarrow 2 = 1+2\alpha \quad , \quad 2.5 = 1+3\alpha$$

$$2\alpha = 1 \quad , \quad 3\alpha = 1.5$$

$$\alpha = \frac{1}{2} \quad , \quad \alpha = \frac{1}{2}$$

So, points are collinear.

Barycentric Sum

This is an important case where the sum of points is welldefined. That is called barycentric sum. If we multiply each point by a weight & if the weights addup to 1, then the sum of the weighted points is affinely invarient. (i.e it is a valid point)

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Given that

$$\sum_{i=0}^{n} \omega_i = 1 \quad \text{(By Barycentric Sum)}$$

$1 = \sum_{i=0}^{n} \omega_i$

L.H.S=R.H.S

$$\sum_{i=0}^{n} \omega_i P_i = P_0 + \sum_{i=1}^{n} \omega_i P_i - (1 - \omega_0) P_0$$

$$= P_0 + \omega_1 P_1 + \omega_2 P_2 + \cdots + \omega_n P_n - (\omega_1 + \omega_2 + \cdots + \omega_n) P_0$$

$$= P_0 + \omega_1 (P_1 - P_0) + \omega_2 (P_2 - P_0) + \cdots + \omega_n (P_n - P_0)$$

$$= P_0 + \sum_{i=1}^{n} \omega_i (P_i - P_0)$$

This is the sum of point P_0 and the vector $\sum_{i=1}^{n} \omega_i (P_i - P_0)$ is a Point.

Example:- 11-03-15

Given two points (a,b) & (c,d) we construct the Barycentric sum

$$(x,y) = -0.5(a,b) + 1.5(c,d)$$

If we translate both points by the vector (α, β) then the sum is modified as

$$-0.5(a+\alpha, b+\beta) + 1.5(c+\alpha, d+\beta)$$

$$= -0.5(a,b) + 1.5(c,d) + (\alpha, \beta)$$

$$= (x,y) + (\alpha, \beta)$$

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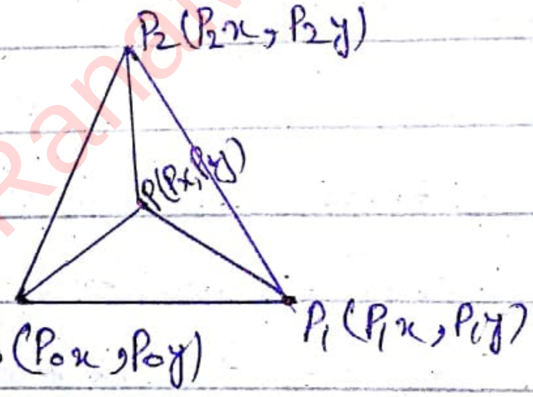
The word Barycentric seems to have first been used in [Dupuy 18]. It is derived from barycenter, meaning "center of gravity", because such weights are used to calculate the center of gravity of an object. Barycentric weights have many uses in geometry, in general & in curve and surface design in particular.

Another useful example is the barycentric coordinates of a two-dimensional point w.r.t the three corners of a triangle.



Any point P inside triangle can be expressed as the weighted combination

$$P = uP_0 + vP_1 + wP_2$$

$$(P_x, P_y) = u(P_{0x}, P_{0y}) + v(P_{1x}, P_{1y}) + w(P_{2x}, P_{2y})$$

$$\Rightarrow \left.\begin{aligned} P_x &= uP_{0x} + vP_{1x} + wP_{2x} \\ P_y &= uP_{0y} + vP_{1y} + wP_{2y} \\ 1 &= u + v + w \end{aligned}\right\} \longrightarrow (1)$$

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The solutions are unique provided that the three equations are independent.

Example:- Let $P_0 = (1,1)$, $P_1 = (2,3)$, $P_2 = (5,1)$ and $P = (2,2)$ using barycentric weight u, v, w and calculate the value of weights.

Solution:- We know that

$(2,2) = u(1,1) + v(2,3) + w(5,1)$

$\Rightarrow$

$2 = u + 2v + 5w \longrightarrow (1)$

$2 = u + 3v + w \longrightarrow (2)$

& $1 = u + v + w \longrightarrow (3)$

By eq. (2) − eq. (1) $\Rightarrow$

$0 = v - 4w$

$\Rightarrow v = 4w \longrightarrow (4)$

By eq. (2) − eq. (3) $\Rightarrow$

$1 = 2v$

$\Rightarrow \boxed{v = \frac{1}{2}}$

eq. (4) $\Rightarrow$ $4w = \frac{1}{2}$

$\Rightarrow \boxed{w = \frac{1}{8}}$

eq. (3) $\Rightarrow$ $u = 1 - \frac{1}{2} - \frac{1}{8} = \frac{8-4-1}{8}$

$\boxed{u = \frac{3}{8}}$

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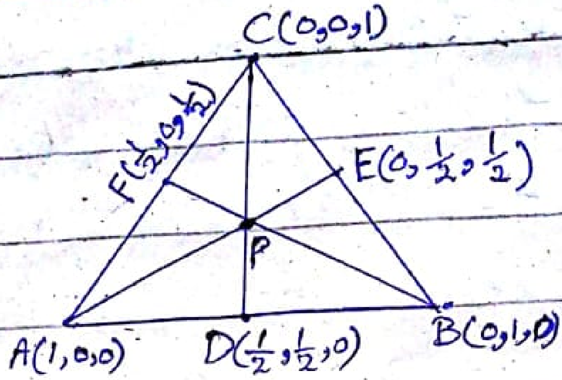
Centroid:-



$$P = \frac{2\left(\frac{1}{2},\frac{1}{2},0\right) + 1(0,0,1)}{2+1}$$

$$= \frac{(1,1,0)+(0,0,1)}{3}$$

$$= \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

Question-1.5: For a given triangle ABC, calculate (x,y,z) coordinates of the point with barycentric coordinates $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$. This point is called centroid of the triangle and is one of many centers that can be defined for a triangle.

$$(x,y,z) = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$$

$$= \frac{1}{3}(1,0,0) + \frac{1}{3}(0,1,0) + \frac{1}{3}(0,0,1)$$

$$= \left(\frac{1}{3},0,0\right) + \left(0,\frac{1}{3},0\right) + \left(0,0,\frac{1}{3}\right)$$

$$= \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

$$\Rightarrow \quad x = \frac{1}{3}, \quad y = \frac{1}{3}, \quad z = \frac{1}{3}$$

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Note: No need to read this page move on next.

~~Note:~~

~~(i):- The difference of two points is a vector.~~

~~(ii):- Multiplying a point by a number produces a point.~~

~~(iii):- The Subtraction of vectors, $P-Q$ is a vector from Q to P.~~

~~(iv):- The product of a real number α by a vector $P(x,y,z)$ is a vector $(\alpha x, \alpha y, \alpha z)$~~

~~(v):- The dot product of two vectors is a scalar.~~

~~(vi):- The cross product of two vectors is a vector. Cross product is not commutative, not associative. But distributive law $P\times(Q\pm T) = P\times Q \pm P\times T$ holds.~~

~~(vii):- Corollary; If the three vectors are coplanar, then the parallelepiped defined by them has zero volume (i.e scalar triple product is zero).~~

~~(viii) Projection of $\underline{a}$ on $\underline{b}$~~

$$= \frac{(\underline{a}\cdot\underline{b})}{|\underline{b}|^2}\underline{b}$$

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(i) Point − point = vector, (ii) Scalar x point = point

(iii) Vector ± Vector = vector

(iv) Scalar x vector = vector

(v) Point + vector = point

(vi) vector • vector = Scalar

(vii) vector x vector = vector

Difference of Points.

The derivative of a point is a vector. $\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t}$

"Parametric Curves"

In practice, curves (and surfaces) are specified by the user in terms of points and vectors which are constructed in an interative process. The user stats by entering the coordinates of points either by scanning a rough image, or by drawing a rough shape on the screen and selecting certain points with a pointing device such as a mouse. After the curve has be drawn, the user may want to modify it's shape by moving, adding or deleting points. Such points can be implied in two different ways:

1):- We may want the curve

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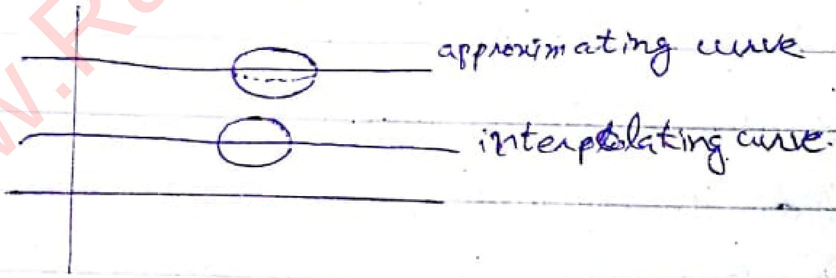
to pass through them. Such points are called data points & the curve is called an interpolating curve.

2):- We may want the points to control the shape of curve by exerting a "pull" on it. A point may pull part of the curve toword it, allowing the user to change the shape of the curve by moving the point. Generally the curve does not pass through the point. Such points are called control point & the curve is called an approximating curve.



Explicit & Implicit Form of Function.

A mathematically function $y=f(x)$ can be plotted as a curve. Such function is called explicit Representation of curve.

(B) The explicit representation is not general, since it cannot represent the verticle lines and is also single valued. For each

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value of x, only a single value of y is normally computed by the function.

The Implicit representation of the curve has the form $F(x,y)=0$, It can be represent multivalued curve.

A common example of an implicit representation is a circle such as,

$$x^2+y^2-r^2=0$$

The curve representation used in practics is called the parametric representation.

A two-dimensional parametric curve has the form $P(t)=(x(t), y(t))$ (or $P(t)=(f(t), g(t))$). Where x&y are the functions of t variables

Example:- $P(t)=(2t-1, t^2)$, $0\le t\le 1$.

$P(0)=?$, $P(1)=?$, $P^{tt}(t)=?$

Sol:- $P(0)=(-1,0)$

$P(1)=(1,1)$

$P^{t}(t)=(2,2t)$

$P^{tt}(t)=(0,2)$

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The slope of the parametric representation of curve is,

$$P(t) = (x(t), y(t))$$

$$\dot{P}(t) = P^t(t) = \frac{dP(t)}{dt} = (\dot{x}(t), \dot{y}(t))$$

$$\text{Slope} = \frac{\dot{y}(t)}{\dot{x}(t)}$$

Example:- Let $P(t) = (x(t), y(t))$
$= (1+\frac{t^2}{2}, t^2)$

Calculate the slope of curve at $t=2$.

Sol:- $P^t(t) = (t, 2t)$

$$\text{Slope} = \frac{2t}{t} = 2$$

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Example:- The following four different parametric representations of a circle with radius R and center at origin.

1):- $P(t) = R[\cos t, \sin t]$ $\quad 0 \le t \le 2\pi$.

2):- By subtituting $t = \tan(\frac{u}{2})$ yeilds

$$P(t) = R\left[\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right] \quad \text{when } 0 \le t \le 1$$

This generates the first quadrant from $(R,0)$ to $(0,R)$

3):- $P(t) = R[t, \pm\sqrt{1-t^2}]$ when

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$0 \le t \le 1$. This generats the first quadrant from $(R,0)$ to $(0,R)$ and simultaneously the third quadrant from $(0,-R)$ to $(-R,0)$.

4):– $P(t) = (0.441, -0.441)t^3 + (-1.485, -0.162)t^2 + (0.044, 1.603)t + (1,0)$

when $0 \le t \le 1$. This generates the first quadrant from $(1,0)$ to $(0,1)$

Here $x(t) = 0.441t^3 - 1.485t^2 + 0.044t + 1$

& $y(t) = -0.441t^3 - 0.162t^2 + 1.603t$

"Properties of parametric curve"

Generally it is impossible to tell much about the behaviour of a parametric curve $P(t) = (x(t), y(t))$ by examining the two components $x(t)$ & $y(t)$ separatly. Each of the two functions may have featurs that do not exist in combination. The reverse is also true – the combined curve may have the featurs not found in any of the two components.

Here, we have an example

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of two smooth curves whose combination is a parametric plane curve with a cusp the following two curves are polynomials in t.

$$x(t) = -18t^2 + 18t + 2$$

$$y(t) = -16t^3 + 24t^2 - 12t + 5$$

where $0 \le t \le 1$. They are smooth, since their derivatives

$$x'(t) = -36t + 18$$

& $y'(t) = -48t^2 + 48t - 12$ are continuous in the range $0 \le t \le 1$. The combined curve is

$$P(t) = (0, -16)t^3 + (-18, 24)t^2 + (18, -12)t + (2, 5)$$

has a sharp corner because it's tangent vector $P^t(t) = 3(0, -16)t^2 + 2(-18, 24)t + (18, -12)$ satisfies $P^t(0.5) = (0, 0)$ This is a cusp.

Exercise 1.16:- Find two curves $x(t)$ and $y(t)$, each with a cusp, such that the combined curve $P(t) = (x(t), y(t))$ is smooth.

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Q:-

$P(t) = (2\cos t + \cos 2t,\ 2\sin t - \sin 2t)$
where $0 \le t \le 2\pi$

at $t = 0$, $t = 0.261799$, $t = 0.523599$

Sol:- $P(t) = (x(t), y(t))$

so $P^t(t) = (\dot{x}(t), \dot{y}(t))$

$\dot{x}(t) = -2\sin t - 2\sin 2t$

$\dot{y}(t) = 2\cos t - 2\cos 2t$

at $t = 0$

$\dot{x}(0) = 0$, $\dot{y}(0) = 0$

So, $P^t(0) = (0, 0)$

$\dot{x}(0.261799) = -0.517637341 - 0.9999$
$= -1.517635998$

$\dot{y}(0.261799) = 1.931851853 - 1.732051$
$= 0.199800267$

So, $P^t(0.261799) = (-1.517635998, 0.199800267)$

Now at $t = 0.523599$

$\dot{x}(0.523599) = -1 - 1.73205$
$= -2.73205$

$\dot{y}(0.523599) = 1.732050 - 1$
$= 0.732050$

So $P^t(0.523599) = (-2.73205, 0.732050)$

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One important feature of curve is independent of coordinates axes. We do not want the curve to change shape when the coordinates exists (or the points defining the curve) are moved rigidly or rotated.

Here we want to discuss one example of how such a thing can happen.

Example: Consider a parametric curve

$$P(t) = (1-t)^3 P_0 + t^3 P_1$$

Let $P_0 = (x_0, y_0)$ & $P_1 = (x_1, y_1)$

$$\Rightarrow P(t) = (1-t)^3 (x_0, y_0) + t^3 (x_1, y_1)$$

$$= ((1-t)^3 x_0, (1-t)^3 y_0) + (t^3 x_1, t^3 y_1)$$

$$= ((1-t)^3 x_0 + t^3 x_1, (1-t)^3 y_0 + t^3 y_1)$$

It is easy to see that $P(0) = P_0$ & $P(1) = P_0$ (The curve passes through the two points). What kind of curve is $P(t)$?

The tangent vector of $P(t)$ is

$$P^t(t) = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) = \left(-3(1-t)^2 x_0 + 3t^2 x_1, -3(1-t)^2 y_0 + 3t^2 y_1\right)$$

To calculate the slope we have

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to select two points. $P_0 = (0,0)$ & $P_1 = (5,6)$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3(1-t)^2 y_0 + 3t^2 y_1}{-3(1-t)^2 x_0 + 3t^2 x_1}$$

$$= \frac{-3(1-t)^2(0) + 3t^2(6)}{-3(1-t)^2(0) + 3t^2(5)}$$

$$= \frac{18t^2}{15t^2}$$

$$= \frac{6}{5}$$

So, the curve is a straight line.

Next we translate both points by the amount $(0,-1)$.

So that the new points are $P_0 = (0,-1)$ & $P_1 = (5,5)$ and the new slope is

$$\text{Slope} = \frac{dy}{dx} = \frac{-3(1-t)^2(-1) + 3t^2(5)}{-3(1-t)^2(0) + 3t^2(5)}$$

$$= \frac{1}{5}(1/t - 1)^2 + 1$$

It is no longer constant. Therefore the curve is no longer a straight line. The curve has change it shape

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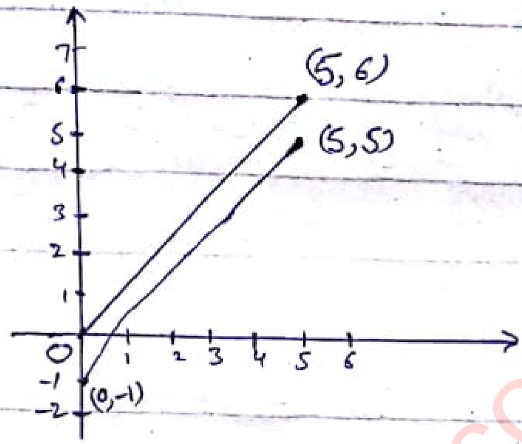
just because its ends points have been moved.



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It turns out a curve of the form $P(t) = \sum_{i=0}^{n} w_i(t) P_i$, is independent of the particular coordinate axes used if $\sum_{i=0}^{n} w_i(t) = 1$. This is arguably the most important property of barycentric weights.

If step size is same.

Uniform & Nonuniform Parametric Curves

Uniform parametric curves are normally easy to calculate and they produce good results when the points are roughly equally spaced. However, when the spacing of the points is very different, a uniform curve may look strange and unnatural, even though it passes through all the data points. This is when a non-uniform parametric curve should be used.

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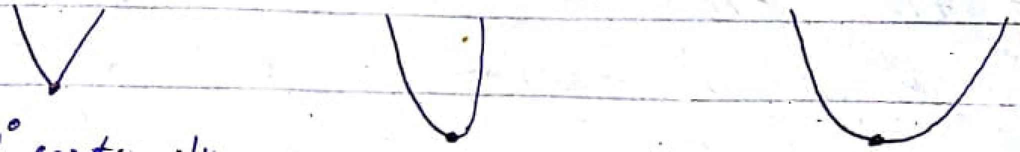
Curve Continuity



(a) G^0 continuity (a sharp corner)

(b) G^1 continuity (A smooth connection)

(c) G^2 continuity (a Tight curve)

C^0 C^1 C^2

A complete curve is often made up of segments, so it is important to understand how individual segments can be connected. There are two types of curve continuities:

1):- Geometric Continuity ($G^0, G^1, G^2, \dots, G^n$)

2):- Parametic Continuity ($C^0, C^1, C^2, \dots, C^n$)

Geometric Continuity

1):- If two consecutive segments meet at a point, the total curve is said to have G^0 geometric Continuity at that Point

2):- If, in addition, the directions

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of the tangent vectors of the two segments are same at that point, the curve has G^1 continuity at that point. The two segments connect smoothly.

3):- In general, a curve has geometric continuity G^n at a join point if every pair of the first n derivatives of the two segments have the same direction at the point.

Parametric Continuity

1):- If the two consecutive segments meet at a point then the combined curve is called has a C^0 continuity at that point.

2):- A curve has parametric continuity C^n at join point if every pair of the first n derivatives of two segments have the same direction as well as the identical magnitude at that point.

We can refer to C^0, C^1 & C^2 as point, tangent and curvature continuities. Generally, high continuity

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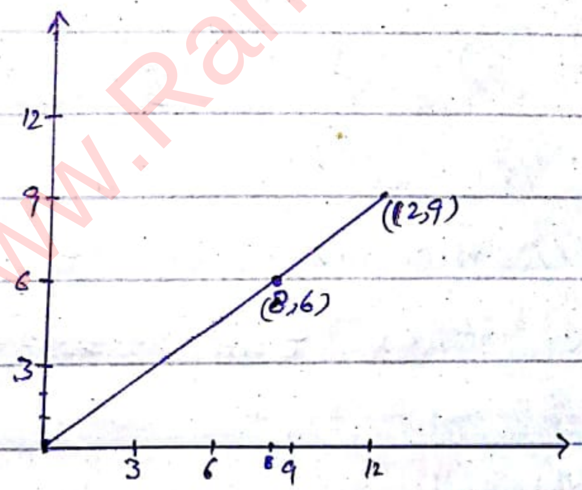
results in a smoother curve.

A C^K continuity is more restrictive than G^K, so a curve that has C^K continuity at a join point also has G^K continuity at that point, but there is an exception.

Example:- Consider the two straight segments $P(t) = (8t, 6t)$ & $Q(t) = (4(t+2), 3(t+2))$, $0 \le t \le 1$

The first segment goes from (0,0) to (8,6) & second from (8,6) to (12,9).



Their tangent vectors are $P^t(t) = (8,6)$ and $Q^t(t) = (4,3)$. The segments connect smoothly at point (8,6) but there tangent vectors are different at that point. Thus the total curve

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has G^1 continuity at (8,6), but not C^1 because the magnitudes are different. It is interesting to note that the unit vectors are equal at the joint.

The magnitudes are

$$|P^t(t)| = \sqrt{(8)^2 + 6^2} = 10$$

$$\& \quad |Q^t(t)| = \sqrt{4^2 + 3^2} = 5$$

The unit tangent vectors of $P^t(t)$ & $Q^t(t)$ are

$$\left(\tfrac{8}{10}, \tfrac{6}{10}\right) \; \& \; \left(\tfrac{4}{5}, \tfrac{3}{5}\right)$$

or $$\left(\tfrac{4}{5}, \tfrac{3}{5}\right) \; \& \; \left(\tfrac{4}{5}, \tfrac{3}{5}\right)$$

Thus the unit tangent vector provides a better measure of the direction of the curve than the tangent vector itself.

G^2 – Continuity

A curve whose tangent vector & curvature vector are everywhere continuous is said to have G^2-continuity (second-order geometric continuity).

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PC curve (Parametric Cubic curve)

Parametric curves used in computer graphics are based on polynomials. A polynomial of degree one has the form $P_1(t) = At + B$ and is therefore, a straight line so it can only be used in limited cases.

A parametric polynomial of degree two has the form

$P_2(t) = At^2 + Bt + C$ and is always a parabola.

A polynomial of degree 3 (cubic) has the form

$P_3(t) = At^3 + Bt^2 + Ct + D$ and is the simplist curve that can have complex shapes and can also be a space curve. It can have atmost one loop and if it does not have a loop, it can have at most two inflection points. $[P'(t) = 0, t \in [a,b]]$

Polynomials of higher degrees are sometimes needed, but they generally wiggle too much and are difficult to control. Also have more coefficients,

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So they require more input data to determine all the coefficients. So, these polynomials are time consuming and very difficult to calculate the coefficients.

As a result, a complete curve is often constructed from segments, each a parametric cubic polynomial (also called P.C).

So parametric cubic polynomials are simple, easy to calculate, time saving and provide us a smooth curve as a result.

A complete curve is a piecewise polynomial curve, sometimes also called a spline.

3-degree (P.C) → 1 loop or 2 wiggles
4-degree Polynomial → 2 loops or 3 wiggles
5 " → 3 " or 4 "

Imp Q:- Why higher degree polynomial curve produce wiggles?

Ans:- In cubic polynomial there are at most two inflection points or

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one loop, which is minimum inflection point or loop. In higher degree polynomials the inflection points are more than 2, so they are not helpful to compute graphic. Because in higher polynomial more wiggles produced.

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Q:- How to represent the parametric cubic curve in matrix form?

Sol:- If we have a cubic polynomial of the form

$$P(t) = At^3 + Bt^2 + Ct + D$$

Where A, B, C & D are unknown coefficients have to be calculated, which requires four equations. The equations must depend on four known quantities, points or vectors, that we denote by G_1 through G_4. The P.C. segment is expressed as the product

$$P(t) = (t^3, t^2, t, 1) \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{pmatrix}$$

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$\Rightarrow P(t) = T(t) \cdot M \cdot G$

Where M is the basis matrix that depends on the method used and G is the Geometry vector, consisting of the four given quantities. The segment can also be written as the weighted sum

$$P(t) = (t^3 m_{11} + t^2 m_{21} + t m_{31} + m_{41}) G_1 +$$
$$+ (t^3 m_{12} + t^2 m_{22} + t m_{32} + m_{42}) G_2 +$$
$$+ (t^3 m_{13} + t^2 m_{23} + t m_{33} + m_{43}) G_3$$
$$+ (t^3 m_{14} + t^2 m_{24} + t m_{34} + m_{44}) G_4$$
$$= B_1(t) G_1 + B_2(t) G_2 + B_3(t) G_3 + B_4(t) G_4$$
$$= B(t) \cdot G$$

Where $B(t) = T(t) \cdot M$ and the $B_i(t)$ $(i = 1,2,3,4)$ are the weights. They are also called the blending functions, since they blend the four given quantities. If any of the quantities being blended are points, their weights should be barycentric. In the case where all four quantities are points,

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this requirement implies that the sum of the elements of matrix M should equal 1.

A P.C segment can also be written in the form (for space curve)

$$P(t) = At^3 + Bt^2 + Ct + D$$

$$= (t^3, t^2, t, 1) \begin{pmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \\ D_x & D_y & D_z \end{pmatrix}$$

$$= T(t) \cdot C$$

and tangent vector is

$$P^t(t) = T^t(t) \cdot C$$

$$= (3t^2, 2t, 1, 0)\, C$$

This tangent describes the speed of the curve.

Q:- If we have following basis functions

$$B_0(t) = (1-t)^2 \, , \quad B_1(t) = 2t(1-t)$$

$B_2(t) = t^2$, let P_0, P_1, P_2, are points. How to represent the parametric curve constructing with 3-basis functions in

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a matrix form.

Solution:-

$$P(t) = \sum_{i=0}^{2} B_i(t)\, P_i$$

$$= B_0(t)\, P_0 + B_1(t)\, P_1 + B_2(t)\, P_2$$

$$= (1-t)^2 P_0 + 2t(1-t) P_1 + t^2 P_2$$

$$= (1+t^2-2t) P_0 + (2t-2t^2) P_1 + t^2 P_2$$

$$= (t^2, t, 1) \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$$

For Basis vector.

$$B_i(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!}, \quad i = 0, 1, 2, \ldots, n$$

For degree 3:- Let P_0, P_1, P_2 & P_3 are the points for the cubic polynomial. The basis functions are

$$B_i(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$i = 0, 1, 2, 3$$

$$B_0(t) = \binom{3}{0} t^0 (1-t)^{3-0} = (1-t)^3$$

$$B_1(t) = \binom{3}{1} t^1 (1-t)^{3-1} = 3t(1-t)^2$$

$$B_2(t) = \binom{3}{2} t^2 (1-t)^{3-2} = 3t^2(1-t)$$

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$$B_3(t) = \binom{3}{3} t^3 (1-t)^{3-3} = t^3$$

Let the cubic polynomial

$$P(t) = \sum_{i=0}^{3} B_i(t)\, P_i$$

$$= B_0(t) P_0 + B_1(t) P_1 + B_2(t) P_2 + B_3(t) P_3$$

$$= (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t) P_2 + t^3 P_3$$

$$= (1-3t+3t^2-t^3) P_0 + (3t-6t^2+3t^3) P_1 + (3t^2-3t^3) P_2 + t^3 P_3$$

$$\Rightarrow P(t) = (t^3, t^2, t, 1) \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

For degree 4:-

Let P_0, P_1, P_2, P_3 & P_4 are the points & Basis functions are

$$B_i(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$i = 0, \dots, 4$$

$$B_0(t) = \binom{4}{0} t^0 (1-t)^{4-0} = (1-t)^4$$

$$B_1(t) = \binom{4}{1} t^1 (1-t)^{4-1} = 4t(1-t)^3$$

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$$B_2(t) = \binom{4}{2} t^2 (1-t)^{4-2} = 6t^2(1-t)^2$$

$$B_3(t) = \binom{4}{3} t^3 (1-t)^{4-3} = 4t^3(1-t)$$

$$B_4(t) = \binom{4}{4} t^4 (1-t)^{4-4} = t^4$$

$$P(t) = \sum_{i=0}^{4} B_i(t) P_i$$

$$= B_0(t) P_0 + B_1(t) P_1 + B_2(t) P_2 + B_3(t) P_3 + B_4(t) P_4$$

$$= (1-t)^4 P_0 + 4t(1-t)^3 P_1 + 6t^2(1-t)^2 P_2 + 4t^3(1-t) P_3 + t^4 P_4$$

$$= (1-4t+6t^2-4t^3+t^4) P_0 + (4t-12t^2+12t^3-4t^4) P_1 + (6t^2-12t^3+6t^4) P_2 + (4t^3-4t^4) P_3 + t^4 P_4$$

$$P(t) = (t^4, t^3, t^2, t, 1) \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

For degree 5:- Let P_0, P_1, P_2, P_3, P_4 & P_5 are the points. The basis functions are

$$B_i(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, 1, \dots, 5$$

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$$B_0(t) = \binom{5}{0} t^0 (1-t)^5 = (1-t)^5$$

$$B_1(t) = \binom{5}{1} t^1 (1-t)^{5-1} = 5t(1-t)^4$$

$$B_2(t) = \binom{5}{2} t^2 (1-t)^{5-2} = 10t^2(1-t)^3$$

$$B_3(t) = \binom{5}{3} t^3 (1-t)^{5-3} = 10t^3(1-t)^2$$

$$B_4(t) = \binom{5}{4} t^4 (1-t)^{5-4} = 5t^4(1-t)$$

$$B_5(t) = \binom{5}{5} t^5 (1-t)^{5-5} = t^5$$

$$P(t) = \sum_{i=0}^{5} B_i(t) P_i$$

$$P(t) = B_0(t)P_0 + B_1(t)P_1 + B_2(t)P_2 + B_3(t)P_3 + B_4(t)P_4 + B_5(t)P_5$$

$$P(t) = (1-t)^5 P_0 + 5t(1-t)^4 P_1 + 10t^2(1-t)^3 P_2 + 10t^3(1-t)^2 P_3 + 5t^4(1-t)P_4 + t^5 P_5$$

$$P(t) = (1-5t+10t^2-10t^3+5t^4-t^5)P_0 + (5t-20t^2+30t^3-20t^4+5t^5)P_1 + (10t^2-30t^3+30t^4-10t^5)P_2 + (10t^3-20t^4+10t^5)P_3 + (5t^4-5t^5)P_4 + t^5 P_5$$

$$P(t) = (t^5, t^4, t^3, t^2, t, 1) \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}$$

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Subdividing a Parametric Curve

25-03-15

Consider a cubic parametric curve defined by four non-scalar (points or vectors) entities,

$$P(t) = [t^3, t^2, t, 1] \; M \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \longrightarrow (1)$$

Where parameter $t \in [0,1]$

Where $M = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

We want to construct the two half $P_1(t)$ & $P_2(t)$ of this curve by varying the parameter in the intervals $[0, 0.5]$ & $[0.5, 1]$. Each of the two new curves should have the same shape as half of the original curve. Each half should be written as an expression similar to (1) but based on the new set of entities Q_i computed from the original set P_i. First of all we construct first half $P_1(t)$. We define a new parameter $u = 2t$ where $t \in [0, 0.5]$ & $u \in [0,1]$. The first half can

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be obtained from eq ① by subtituting $t = u/2 \Rightarrow u = 2t$

$$P_1(u) = \left[\frac{u^3}{8}, \frac{u^2}{4}, \frac{u}{2}, 1\right] M \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$= [u^3, u^2, u, 1] \begin{bmatrix} 1/8 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} M \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$= [u^3, u^2, u, 1] \, L \, M \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$= [u^3, u^2, u, 1] \, M \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} \longrightarrow ②$$

From the expression ② we can calculate Q_0, Q_1, Q_2, Q_3 as

$$M \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = L M \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = M^{-1} L M \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

where $M^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1/3 & 1 \\ 0 & 1/3 & 2/3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

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$$M^{-1}LM = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 \\ 1/8 & 3/8 & 3/8 & 1/8 \end{bmatrix}$$

$$\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 \\ 1/8 & 3/8 & 3/8 & 1/8 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$\Rightarrow \quad Q_0 = P_0$$

$$\Rightarrow \quad Q_1 = \frac{1}{2} P_0 + \frac{1}{2} P_1$$

$$Q_1 = \frac{1}{2}(P_0 + P_1)$$

$$\Rightarrow \quad Q_2 = \frac{1}{4} P_0 + \frac{1}{2} P_1 + \frac{1}{4} P_2$$

$$Q_2 = \frac{1}{2}\left[\frac{1}{2}(P_0 + P_1) + \frac{1}{2}(P_1 + P_2)\right]$$

$$Q_3 = \frac{P_0}{8} + \frac{3}{8} P_1 + \frac{3}{8} P_2 + \frac{1}{8} P_3$$

$$= \frac{1}{2}\left\{\frac{1}{2}\left[\frac{1}{2}(P_0+P_1) + \frac{1}{2}(P_1+P_2)\right] + \frac{1}{2}\left[\frac{1}{2}(P_1+P_2) + \frac{1}{2}(P_2+P_3)\right]\right\}$$

The second half, $P_2(t)$ is calculated

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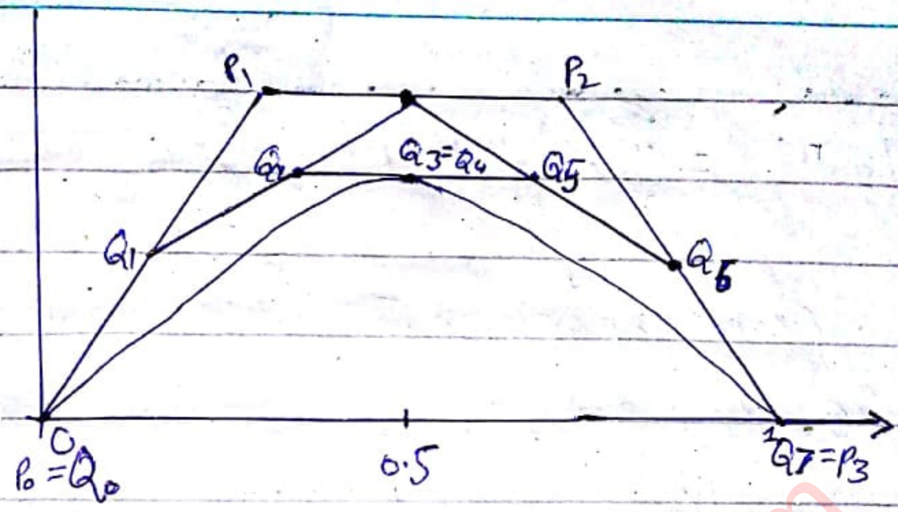

Similarly. We first define a new parameter $u = 2t-1$ where $t \in [0.5, 1]$ & $u \in [0,1]$. The second half of the curve is obtained from expression ① by substituting $t = \frac{u+1}{2}$.

$$P_2(u) = \left[\left(\frac{u+1}{2}\right)^3, \left(\frac{u+1}{2}\right)^2, \frac{u+1}{2}, 1\right] M \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$= \left[\frac{u^3+3u^2+3u+1}{8}, \frac{u^2+2u+1}{4}, \frac{u+1}{2}, 1\right] M \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$= [u^3, u^2, u, 1] \begin{bmatrix} 1/8 & 0 & 0 & 0 \\ \frac{3}{8} & 1/4 & 0 & 0 \\ 3/8 & 1/2 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 & 1 \end{bmatrix} M \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$= [u^3, u^2, u, 1]\, R M \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

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$$\Rightarrow \quad P_2(u) = [u^3, u^2, u, 1]\, M \begin{bmatrix} Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \end{bmatrix} \longrightarrow (3)$$

From expression (3) we can calculate Q_4, Q_5, Q_6, Q_7 as

$$M \begin{bmatrix} Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \end{bmatrix} = RM \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$\begin{bmatrix} Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \end{bmatrix} = M^{-1} R M \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

We know that

$$M = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \& \quad M^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1/3 & 1 \\ 0 & 1/3 & 2/3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$M^{-1} R M = \begin{bmatrix} 1/8 & 3/8 & 3/8 & 1/8 \\ 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So,

$$\begin{bmatrix} Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \end{bmatrix} = \begin{bmatrix} 1/8 & 3/8 & 3/8 & 1/8 \\ 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

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$$\Rightarrow \quad Q_7 = P_3$$

$$\Rightarrow \quad Q_6 = \frac{1}{2} P_2 + \frac{1}{2} P_3$$

$$Q_6 = \frac{1}{2} (P_2 + P_3$$

$$\Rightarrow \quad Q_5 = \frac{1}{4} P_1 + \frac{1}{2} P_2 + \frac{1}{4} P_3$$

$$Q_5 = \frac{1}{2} \left[\frac{1}{2} (P_1 + P_2) + \frac{1}{2} (P_2 + P_3) \right]$$

$$\Rightarrow \quad Q_4 = \frac{1}{8} P_0 + \frac{3}{8} P_1 + \frac{3}{8} P_2 + \frac{1}{8} P_3$$

$$Q_4 = \frac{1}{2} \left\{ \frac{1}{2} \left[\frac{1}{2} (P_0 + P_1) + \frac{1}{2} (P_1 + P_2) \right] + \frac{1}{2} \left[\frac{1}{2} (P_1 + P_2) + \frac{1}{2} (P_2 + P_3) \right] \right\}$$

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Fast computation of a PC   07-04-2015

This method is based on Forward-Differences, together with the Taylor Series representation, to speed up the calculation of a point on a parametric curve $P(t)$. Once this method is implemented, an entire curve can be drawn in a loop where t is incremented from 0 to 1 in small, equal steps of Δ. In iteration $i+1$, a point $P([i+1]\Delta)$ is computed & is connected to the previous point $P(i\Delta)$ by a short, and straight segment.

The principle of forward differences is to find a quantity dP such that $P(t+\Delta) = P(t) + dP \longrightarrow (1)$ for any value of t. If such a dP can be found, then it is enough to calculate $P(0)$, and use forward difference to compute

$t=0 \Rightarrow \quad P(0+\Delta) = P(0) + dP$

$t=\Delta \Rightarrow \quad P(2\Delta) = P(\Delta) + dP.$

$= P(0) + dP + dP$

$= P(0) + 2dP$

$i = 0, 1, 2, \ldots$

$$P([i+1]\Delta) = P(i\Delta) + dP = P(0) + (i+1)dP \longrightarrow (2)$$

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The point is that dP should not depend on t. If dP turns out to depend on t, then as we advance t from 0 to 1, we have to use different values of dP. We calculate dP from the Taylor series representation of the curve. The Taylor series of a function $f(t)$ at a point $f(t+\Delta)$ is the infinite sum,

$$f(t+\Delta) = f(t) + f'(t)\,\Delta + \frac{f''(t)\,\Delta^2}{2!} + \frac{f'''(t)\,\Delta^3}{3!} + \cdots$$

Let we consider general PC curve has the form

$$P(t) = at^3 + bt^2 + ct + d \longrightarrow (3)$$

So only its first three derivatives are non-zero. These derivatives are

$$P^t(t) = 3at^2 + 2bt + c,$$

$$P^{tt}(t) = 6at + 2b,$$

$$P^{ttt}(t) = 6a$$

So Taylor series representation produces

from (1) $\Rightarrow$ $dP = P(t+\Delta) - P(t)$

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By using Taylor series of $P(t+\Delta)$

$$\Rightarrow dP = \cancel{P(t)} + \Delta P^{t}(t) + \frac{\Delta^2 P^{tt}(t)}{2!} + \frac{\Delta^3 P^{ttt}(t)}{3!} - \cancel{P(t)}$$

$$dP = \Delta P^{t}(t) + \frac{P^{tt}(t)\,\Delta^2}{2} + \frac{P^{ttt}(t)\,\Delta^3}{6} \longrightarrow (4)$$

Putting the values of tangents, we obtain

$$dP = (3at^2+2bt+c)\Delta + \frac{(6at+2b)\Delta^2}{2} + \frac{(6a)\,\Delta^3}{6}$$

$$dP = 3at^2\Delta + 2bt\Delta + c\Delta + 3at\Delta^2 + b\Delta^2 + a\Delta^3 \longrightarrow (5)$$

Since dP is a function of t therefore it should be denoted by

$$dP(t) = 3at^2\Delta + 2bt\Delta + c\Delta + 3at\Delta^2 + b\Delta^2 + a\Delta^3$$

Next we want to calculate $ddP(t)$ as

using (1) $\Rightarrow$ $dP(t+\Delta) = dP(t) + ddP(t)$

$$\cancel{dP(t)} + ddP(t) = \cancel{dP(t)} + \Delta dP^{t}(t) + \frac{\Delta^2}{2!} dP^{tt}(t)$$

$$\Rightarrow ddP(t) = \Delta dP^{t}(t) + \frac{\Delta^2}{2} dP^{tt}(t)$$

Since

$$dP^{t}(t) = 6at\Delta + 2b\Delta + 0 + 3a\Delta^2 + 0 + 0$$

$$dP^{t}(t) = 6at\Delta + 2b\Delta + 3a\Delta^2$$

$$\Rightarrow dP^{tt}(t) = 6a\Delta$$

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So,

$$dd\,P(t) = \Delta(6at\Delta + 2b\Delta + 3a\Delta^2) + \frac{\Delta^2}{2}(6a\Delta)$$

$$dd\,P(t) = 6at\Delta^2 + 2b\Delta^2 + 6a\Delta^3 \longrightarrow (6)$$

Similarly we calculate $ddd\,P$ as

$$dd\,P(t+\Delta) = dd\,P(t) + ddd\,P$$

$$dd\,P(t) + ddd\,P = dd\,P(t) + \Delta\, dd\,P^t(t)$$

$$ddd\,P = \Delta\, dd\,P^t(t)$$

$$ddd\,P = \Delta(6a\Delta^2)$$

$$\Rightarrow ddd\,P = 6a\Delta^3 \longrightarrow (7)$$

(a constant i.e. zero degree)

The four quantities involved in the calculation of the curve can be written as

$$P(t) = at^3 + bt^2 + ct + d$$

$$dP(t) = 3at^2\Delta + 2bt\Delta + c\Delta + 3at\Delta^2 + b\Delta^2 + a\Delta^3$$

$$dd\,P(t) = 6at\Delta^2 + 2b\Delta^2 + 6a\Delta^3$$

$$ddd\,P = 6a\Delta^3$$

They have to be caculated at $t=0$ before the loop starts, then each iteration computes the first three quantities from those of the previous iteration ($ddd\,P$ doesn't depend on t).

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$$P(0) = d$$
$$dP(0) = a\Delta^3 + b\Delta^2 + c\Delta$$
$$ddP(0) = 6a\Delta^3 + 2b\Delta^2$$
$$dddP = 6a\Delta^3$$

$$P(\Delta) = \underline{a\Delta^3 + b\Delta^2 + c\Delta} + d$$
$$= P(0) + dP(0)$$

$$dP(\Delta) = 3a\Delta^3 + 2b\Delta^2 + c\Delta + 3a\Delta^3 + b\Delta^2 + a\Delta^3$$
$$= \underline{a\Delta^3 + b\Delta^2 + c\Delta} + \underline{6a\Delta^3 + 2b\Delta^2}$$
$$= dP(0) + ddP(0)$$

$$ddP(\Delta) = \underline{6a\Delta^3 + 2b\Delta^2} + 6a\Delta^3$$
$$= ddP(0) + dddP$$

$$P([i+1]\Delta) = P(i\Delta) + dP(i\Delta) \qquad i = 0, 1, 2, \ldots, n$$
$$dP([i+1]\Delta) = dP(i\Delta) + ddP(i\Delta)$$
$$ddP([i+1]\Delta) = ddP(i\Delta) + dddP$$

Thus, each iteration computes a point $P([i+1]\Delta)$ on the curve by performing six simple operations, three additions and three assignments. No multiplications are needed.

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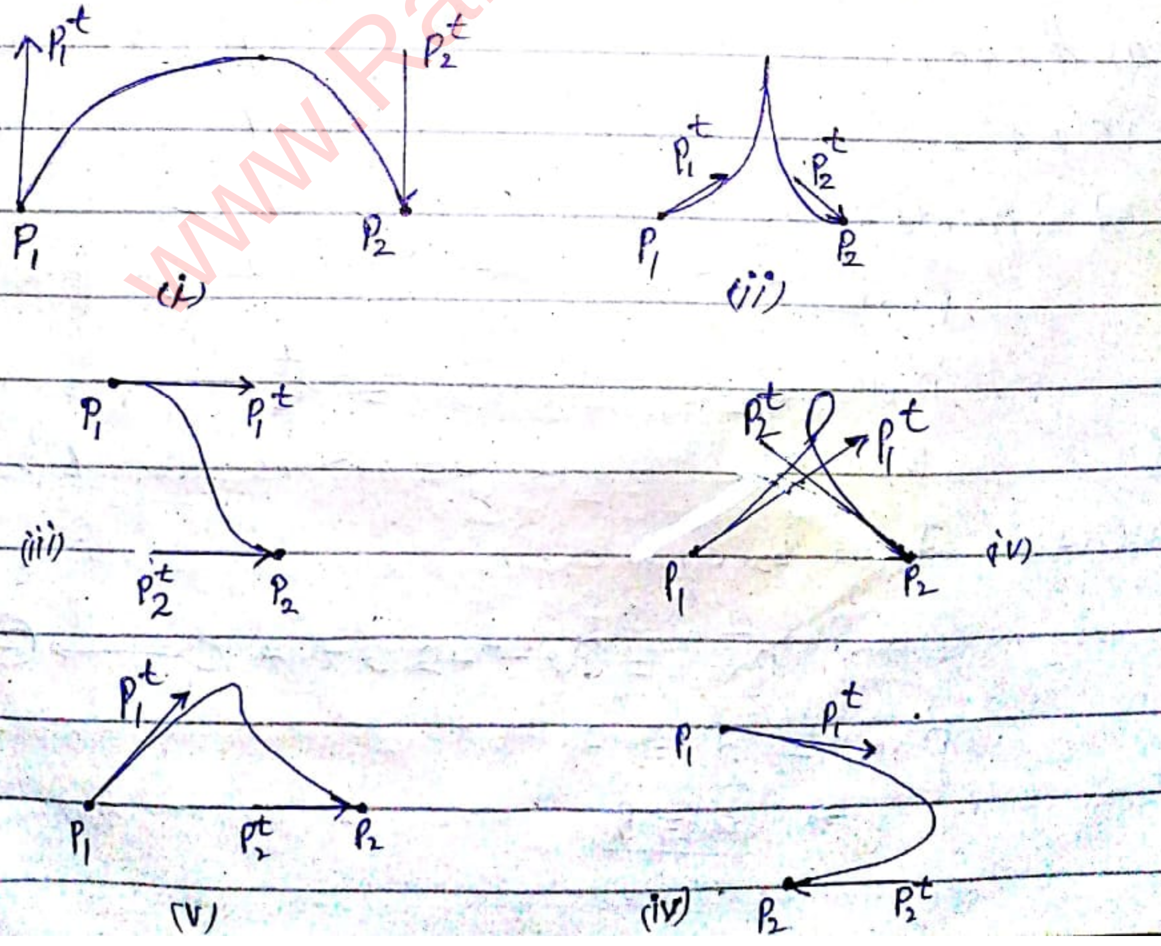
Chap# 4. 08-04-2015

Hermite Interpolation

Hermite interpolation is based on two points P_1 & P_2 and two tangent vectors P_1^t and P_2^t. It computes a curve segment that starts at P_1, going in direction P_1^t and ends at P_2 moving in direction P_2^t.

The method is called Hermite interpolation after Charles Hermite who developed it and derived its blending functions in 1870s, as part of his work on approximation and interpolation.



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The Hermite Curve Segment

The Hermite curve segment is easy to derive. It is a PC curve with four coefficients that depend on the two points and two tangent vectors. The basic equation of a PC curve is

$$P(t) = at^3 + bt^2 + ct + d$$

$$P(t) = (t^3, t^2, t, 1)\ (a, b, c, d)^T$$

$$P(t) = T(t)\ A \longrightarrow (1)$$

This is the algebraic representation of the curve, in which the four coefficients are still unknown. We want to calculate the values of unknowns a, b, c & d in terms of points and tangent vectors.

$$P(0) = P_1 \quad , \quad P^t(0) = P_1^t$$

$$P(1) = P_2 \quad , \quad P^t(1) = P_2^t$$

We use these equations for the solution of a, b, c & d.

from (1) $\Rightarrow$ $P^t(t) = 3at^2 + 2bt + c \longrightarrow (2)$

$$P(0) = d = P_1$$

$$\Rightarrow \boxed{d = P_1}$$

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$P(1) = P_2 = a+b+c+d$

$\Rightarrow \boxed{P_2 = a+b+c+d}$

$P^t(0) = c = P_1^t$

$\Rightarrow \boxed{c = P_1^t}$

$P^t(1) = 3a+2b+c = P_2^t$

$\Rightarrow \boxed{3a+2b+c = P_2^t}$

So,

$$\left.\begin{aligned} d &= P_1 \\ c &= P_1^t \\ a+b+c+d &= P_2 \\ 3a+2b+c &= P_2^t \end{aligned}\right\} \longrightarrow (3)$$

They are easy to calculate the value of a, b, c & d by solving these equation

$\Rightarrow a+b+P_1^t+P_1 = P_2 \longrightarrow (i)$

$\Rightarrow 3a+2b+P_1^t = P_2^t \longrightarrow (ii)$

xing 2 with (i) $\Rightarrow$

$$\begin{array}{r} 2a+\cancel{2b}+2P_1^t+2P_1 = 2P_2 \\ \underline{-3a \mp \cancel{2b} \mp P_1^t \qquad = -P_2^t} \\ -a+P_1^t+2P_1 = 2P_2-P_2^t \end{array}$$

$\Rightarrow a = 2P_1 - 2P_2 + P_1^t + P_2^t$

Put in (i)

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$$2P_1 - 2P_2 + P_1^t + P_2^t + b + P_1^t + P_1 = P_2$$

$$b = P_2 - 2P_1 + 2P_2 - P_1^t - P_2^t - P_1^t - P_1$$

$$b = -3P_1 + 3P_2 - 2P_1^t - P_2^t$$

Substituting these values in Eq. 2.

$$P(t) = (2P_1 - 2P_2 + P_1^t + P_2^t)t^3 + (-3P_1 + 3P_2 - 2P_1^t - P_2^t)t^2 + P_1^t t + P_1$$

which after rearranging, becomes

$$P(t) = \underbrace{(2t^3 - 3t^2 + 1)}_{F_1(t)} P_1 + \underbrace{(-2t^3 + 3t^2)}_{F_2(t)} P_2 + \underbrace{(t^3 - 2t^2 + t)}_{F_3(t)} P_1^t + \underbrace{(t^3 - t^2)}_{F_4(t)} P_2^t$$

$$P(t) = F_1(t) P_1 + F_2(t) P_2 + F_3(t) P_1^t + F_4(t) P_2^t$$

$$P(t) = (F_1(t), F_2(t), F_3(t), F_4(t)) (P_1, P_2, P_1^t, P_2^t)^T$$

$$P(t) = F(t)\, B \longrightarrow (4)$$

Where

$$F_1(t) = 2t^3 - 3t^2 + 1$$

$$F_2(t) = -2t^3 + 3t^2$$

$$F_3(t) = t^3 - 2t^2 + t$$

$$F_4(t) = t^3 - t^2$$

are called Hermite blending basis functions.

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(A) ⇒

$$P(t) = (t^3, t^2, t, 1) \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_1^t \\ P_2^t \end{bmatrix}$$

$$P(t) = T(t)\, H \cdot B$$

Where $H = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ is called Hermite blending basis function matrix.

V. imp

14-04-2015

Q:- Express the midpoint $P(0.5)$ of a Hermite segment in terms of the two endpoints and two tangent vectors.

Sol:- The cubic Hermite curve segment is

$$P(t) = F_1(t) P_1 + F_2(t) P_2 + F_3(t) P_1^t + F_4(t) P_2^t$$

$$\Rightarrow P(0.5) = F_1(0.5) P_1 + F_2(0.5) P_2 + F_3(0.5) P_1^t + F_4(0.5) P_2^t$$

We know

$F_1(t) = 2t^3 - 3t^2 + 1$, $\quad F_2(t) = -2t^3 + 3t^2$

$F_1(0.5) = \frac{2}{8} - \frac{3}{4} + 1$, $\quad F_2(0.5) = \frac{-2}{8} + \frac{3}{4}$

$F_1(0.5) = \frac{1}{2}$, $\quad F_2(0.5) = \frac{1}{2}$

& $F_3(t) = t^3 - 2t^2 + t$, $\quad F_4(t) = t^3 - t^2$

$F_3(0.5) = \frac{1}{8} - \cancel{\frac{2}{4}} + \cancel{\frac{1}{2}}$, $\quad F_4(0.5) = \frac{1}{8} - \frac{1}{4}$

$F_3(0.5) = \frac{1}{8}$, $\quad F_4(0.5) = \frac{-1}{8}$

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So,

$$P(0.5) = \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{8}P_1^t - \frac{1}{8}P_2^t$$
$$= \left(\frac{P_1 + P_2}{2}\right) + \frac{1}{8}(P_1^t - P_2^t)$$

After changing the direction of P_2^t.

$$P(0.5) = \frac{P_1 + P_2}{2} + \frac{1}{2}\left(\frac{1}{2}\left(\frac{P_1^t + P_2^t}{2}\right)\right)$$

Hermite Derivatives

The concept of blending can be applied to the calculation of the derivatives of a curve, not just to curve itself. One way to calculate tangent $P^t(t)$ is to differentiate $T(t) = (t^3, t^2, t, 1)$.

Consider the matrix representation of cubic Hermite curve segment,

$$P(t) = T(t)\,HB$$
$$\Rightarrow P^t(t) = T^t(t)\,HB$$
$$= (3t^2, 2t, 1, 0)\,HB$$

A more general method is to use the relation $P(t) = F(t)B$, which implies

$$P^t(t) = F^t(t)B = (F_1^t(t), F_2^t(t), F_3^t(t), F_4^t(t))B$$

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$$= (6t^2-6t, -6t^2+6t, 3t^2-4t+1, 3t^2-2t)B$$

$$= (t^3, t^2, t, 1)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 6 & -6 & 3 & 3 \\ -6 & 6 & -4 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix} B$$

$$= T(t)\ H_t\ B$$

Where H_t is a matrix of first derivative of the cubic Hermite segment.

Now Also (by other Method)

$$P(t) = (t^3, t^2, t, 1)\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_1^t \\ P_2^t \end{bmatrix}$$

$$\Rightarrow P^t(t) = (t^3, t^2, t, 1)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 6 & -6 & 3 & 3 \\ -6 & 6 & -4 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_1^t \\ P_2^t \end{bmatrix}$$

Similarly, the second derivative of the Hermite segment can be expressed as;

$$P^{tt}(t) = (t^3, t^2, t, 1)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & -12 & 6 & 6 \\ -6 & 6 & -4 & -2 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_1^t \\ P_2^t \end{bmatrix}$$

$$= T(t)\ H_{tt}\ B$$

Where H_{tt} is a matrix of second derivative of the cubic Hermite segment.

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Example:-

The two 2-Dimensional points $P_1 = (0,0)$ and $P_2 = (1,0)$ & the two tangents $P_1^t = (1,1)$ and $P_2^t = (0,-1)$ are given to calculate the cubic Hermite curve segment.

$$P(t) = T(t)\, H\, B$$

$$= (t^3, t^2, t, 1)\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} (0,0) \\ (1,0) \\ (1,1) \\ (0,-1) \end{bmatrix}$$

$$= (t^3, t^2, t, 1)\begin{bmatrix} 2(0,0) - 2(1,0) + 1(1,1) + 1(0,-1) \\ -3(0,0) + 3(1,0) - 2(1,1) - 1(0,-1) \\ (0,0) + (0,0) + (1,1) + (0,0) \\ (0,0) + (0,0) + (0,0) + (0,0) \end{bmatrix}$$

$$= (t^3, t^2, t, 1)\begin{bmatrix} (-1, 0) \\ (1, -1) \\ (1, 1) \\ (0, 0) \end{bmatrix}$$

$$= (-1, 0)t^3 + (1, -1)t^2 + (1, 1)t$$

By other method

$$P(t) = F_1(t) P_1 + F_2(t) P_2 + F_3(t) P_1^t + P_2^t F_4(t)$$

$$= (2t^3 - 3t^2 + 1)(0,0)^{0} + (-2t^3 + 3t^2)(1,0) + (t^3 - 2t^2 + t)(1,1)$$

$$+ (t^3 - t^2)(0,-1)$$

$$= t^3\big((-2,0) + (1,1) + (0,-1)\big) + t^2\big((3,0) + (-2,-2) + (0,1)\big)$$

$$+ t(1,1)$$

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$$\Rightarrow P(t) = t^3(-1,0) + t^2(1,-1) + (1,1)t \longrightarrow (A)$$

Q: 4.5 Use above Eq. (A) to show that the segment really passes through points (0,0) & (1,0). Calculate the tangent vectors & use them to show that the segment really starts and ends in the right directions.

Sol:- From Eq. (A)

$$P(t) = (-1,0)t^3 + (1,-1)t^2 + (1,1)t$$

put $t=0$

$$P(0) = (-1,0)(0) + (1,-1)\,0 + (1,1)\,0$$

$$P(0) = (0,0) = P_1$$

Put $t = 1$

$$P(1) = (-1,0)\,1 + (1,-1) + (1,1)$$

$$= (-\cancel{1}+1+\cancel{1},\ 0-\cancel{1}+1)$$

$$= (1,0) = P_2$$

$$\Rightarrow P^t(t) = 3(-1,0)t^2 + 2(1,-1)t + (1,1)$$

$$P^t(t) = (-3,0)t^2 + (2,-2)t + (1,1)$$

Put $t=0$

$$P^t(0) = (-3,0)\cancel{(0)} + (2,-2)\cancel{(0)} + (1,1) = P_1^t$$

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Put $t=1$

$P^t(1) = (-3,0) + (2,-2) + (1,1)$

$= (-3+2+1, -2+1)$

$= (0,-1)$

$= P_2^t$

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4.7

Q#2:- 15-04-2015

Calculate the Hermite curve for two given points P_1 & P_2 assuming that the tangent vectors at the two points are zero. What kind of a curve is this?

Solution:- We know that

$$P(t) = F_1(t)P_1 + F_2(t)P_2 + F_3(t)P_1^t + F_4(t)P_2^t$$

Given that $P_1^t = (0,0) = P_2^t$

$$\Rightarrow P(t) = F_1(t)P_1 + F_2(t)P_2$$

$$P(t) = (2t^3 - 3t^2 + 1)(x_1, y_1) + (-2t^3 + 3t^2)(x_2, y_2)$$

$$\Rightarrow x(t) = 2t^3(x_1 - x_2) + 3t^2(x_2 - x_1) + x_1$$

$$\& \quad y(t) = 2t^3(y_1 - y_2) + 3t^2(y_2 - y_1) + y_1$$

$$\frac{dy}{dx} = \frac{\frac{dy(t)}{dt}}{\frac{dx(t)}{dt}} = \frac{6t^2(y_1 - y_2) + 6t(y_2 - y_1)}{6t^2(x_1 - x_2) + 6t(x_2 - x_1)}$$

$$\frac{dy}{dx} = \frac{t(y_1 - y_2) + (y_2 - y_1)}{t(x_1 - x_2) + (x_2 - x_1)}$$

$$= \frac{ty_1 - ty_2 + y_2 - y_1}{tx_1 - tx_2 + x_2 - x_1}$$

$$= \frac{(t-1)y_1 - y_2(t-1)}{(t-1)x_1 - x_2(t-1)} = \frac{y_1 - y_2}{x_1 - x_2}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

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So, the curve is reduces to a straight line.

Q#4.8 Use the Hermite method to calculate PC segments for the cases where the known quantities are as follows:

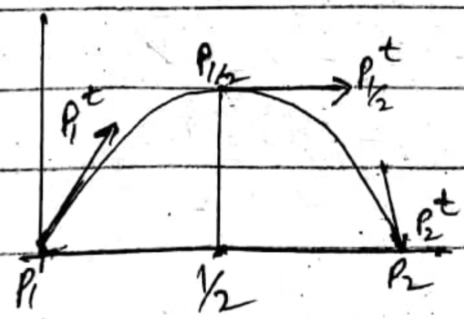
1):- The three tangent vectors at the start, middle & end of the segment.

2):- The two interior points $P(1/3)$ and $P(2/3)$ and the two extreme tangent vectors $P^t(0)$ and $P^t(1)$.

3):- The two extreme points $P(0)$ and $P(1)$ and the two interior tangent vectors $P^t(1/3)$ and $P^t(2/3)$.

Solution:- (1) We know that

$$P(t) = at^3 + bt^2 + ct + d \longrightarrow (1)$$



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Now we have

(i) $P(0) = P_1$ (ii) $P(\tfrac{1}{2}) = P_{1/2}$

$P(\tfrac{1}{2}) = P_{1/2}$ $P(1) = P_2$

$P^t(0) = P_1^t$ $P^t(\tfrac{1}{2}) = P_{1/2}^t$

$P^t(\tfrac{1}{2}) = P_{1/2}^t$ $P^t(1) = P_2^t$

Using (i) in (1)

$P(0) = d = P_1$

$$\boxed{P_1 = d} \longrightarrow (i)$$

$P(\tfrac{1}{2}) = \frac{a}{8} + \frac{b}{4} + \frac{c}{2} + d = P_{1/2}$

$$\frac{a}{8} + \frac{b}{4} + \frac{c}{2} + P_1 = P_{1/2} \longrightarrow (ii)$$

(1) $\Rightarrow P^t(t) = 3at^2 + 2bt + c$

$P^t(0) = c = P_1^t$

$$\Rightarrow \boxed{c = P_1^t} \longrightarrow (iii)$$

$P^t(\tfrac{1}{2}) = \frac{3}{4}a + b + c = P_{1/2}^t \longrightarrow (iv)$

(iv) $\Rightarrow \frac{3}{4}a + b + P_1^t = P_{1/2}^t$

(ii) $\Rightarrow \frac{a}{8} + \frac{b}{4} + \frac{P_1^t}{2} + P_1 = P_{1/2}$

xing by 4.

$$\frac{a}{2} + b + 2P_1^t + 4P_1 = 4P_{1/2} \longrightarrow (v)$$

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By (v) − (vi)

$$\frac{3}{4}a + \cancel{b} + P_1^t = P_{1/2}^t$$
$$\underline{-\frac{a}{2} \pm \cancel{b} \pm 2P_1^t \pm 4P_1 = -4P_{1/2}}$$

$$\left(\frac{3}{4}-\frac{1}{2}\right)a - P_1^t - 4P_1 = P_{1/2}^t - 4P_{1/2}$$

$$\frac{1}{4}a = 4P_1 - 4P_{1/2} + P_{1/2}^t + P_1^t$$

$$\boxed{a = -16P_{1/2} + 16P_1 + 4P_{1/2}^t + 4P_1^t}$$

Put in (vi)

$$8P_1 - 8P_{1/2} + 2P_1^t + 2P_{1/2}^t + b + 2P_1^t + 4P_1 = 4P_{1/2}$$

$$\boxed{b = 12P_{1/2} - 12P_1 - 2P_{1/2}^t - 4P_1^t}$$

Put a, b, c & d in (1)

$$P(t) = (-16P_{1/2} + 16P_1 + 4P_{1/2}^t + 4P_1^t)t^3$$
$$(12P_{1/2} - 12P_1 - 2P_{1/2}^t - 4P_1^t)t^2 + P_1^t t + P_1$$

To check at $t=0$

$$\boxed{P(0) = P_1}$$

at $t = 1$

$$P(1/2) = \frac{1}{8}(-16P_{1/2} + 16P_1 + 4P_{1/2}^t + 4P_1^t) + (12P_{1/2}$$
$$-12P_1 - 2P_{1/2}^t - 4P_1^t)\frac{1}{4} + \frac{P_1^t}{2} + P_1$$

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$P(\tfrac{1}{2}) = -2P_{1/2} + 2P_1 + \frac{1}{2}P^t_{1/2} + \frac{1}{2}P^t_1 + 3P_{1/2}$
$- 3P_1 - \frac{1}{2}P^t_{1/2} - P^t_1 + \frac{1}{2}P^t_1 + P_1$

$= (-2+3)P_{1/2} + \cancel{(2-3+1)}P_1 + \cancel{(\tfrac{1}{2}-\tfrac{1}{2})}P^t_{1/2}$
$+ \cancel{(\tfrac{1}{2}-1+\tfrac{1}{2})}P^t_1$

$\boxed{P(\tfrac{1}{2}) = P_{1/2}}$ also $P^t(0) = P^t_1$
& $P^t(\tfrac{1}{2}) = P^t_{1/2}$

Now using (ii):-

$P(t) = at^3 + bt^2 + ct + d \longrightarrow (1)$

$P(\tfrac{1}{2}) = \frac{a}{8} + \frac{b}{4} + \frac{c}{2} + d = P_{1/2} \longrightarrow (i)$

$P(1) = a + b + c + d = P_2 \longrightarrow (ii)$

$\Rightarrow P^t(t) = 3at^2 + 2bt + c$

$P^t(\tfrac{1}{2}) = \frac{3}{4}a + b + c = P^t_{1/2} \longrightarrow (iii)$

$P^t(1) = 3a + 2b + c = P^t_2 \longrightarrow (iv)$

By (iii) $-$ (iv) $\Rightarrow$

$(\frac{3}{4} - 3)a - b = P^t_{1/2} - P^t_2$

$b = -\frac{9}{4}a + P^t_2 - P^t_{1/2} \longrightarrow (v)$

By (i) $-$ (ii) $(\frac{1}{8}-1)a + (\frac{1}{4}-1)b + (\frac{1}{2}-1)c = P_{1/2} - P_2$

$-\frac{7}{8}a - \frac{3}{4}b - \frac{1}{2}c = P_{1/2} - P_2 \longrightarrow (vi)$

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Now By (vi) + $\frac{(iii)}{2}$

$$-\frac{7}{8}a - \frac{3}{4}b - \cancel{\frac{1}{2}c} = P_{1/2} - P_2$$

$$\frac{3}{8}a + \frac{b}{2} + \cancel{\frac{c}{2}} = \frac{1}{2}P^t_{1/2}$$

$$-\frac{a}{2} - \frac{1}{4}b = P_{1/2} - P_2 + \frac{1}{2}P^t_{1/2}$$

Put Eq.(v) here

$$-\frac{a}{2} - \frac{1}{4}\left(-\frac{9}{4}a + P_2^t - P^t_{1/2}\right) = P_{1/2} - P_2 + \frac{1}{2}P^t_{1/2}$$

$$-\frac{a}{2} + \frac{9}{16}a - \frac{1}{4}P_2^t + \frac{1}{4}P^t_{1/2} = P_{1/2} - P_2 + \frac{1}{2}P^t_{1/2}$$

$$\frac{1}{16}a = P_{1/2} - P_2 + \frac{1}{4}P^t_{1/2} + \frac{1}{4}P_2^t$$

$$\boxed{a = 16P_{1/2} - 16P_2 + 4P^t_{1/2} + 4P_2^t}$$

Put a in (v)

$$-\frac{9}{4}\left[16P_{1/2} - 16P_2 + 4P^t_{1/2} + 4P_2^t\right] + P_2^t - P^t_{1/2} = b$$

$$-36P_{1/2} + 36P_2 - 9P^t_{1/2} - 9P_2^t + P_2^t - P^t_{1/2} = b$$

$$\boxed{b = -36P_{1/2} + 36P_2 - 10P^t_{1/2} - 8P_2^t}$$

Put a & b in (iv)

$$48P_{1/2} - 48P_2 + 12P^t_{1/2} + 12P_2^t - 72P_{1/2} + 72P_2$$

$$-20P^t_{1/2} - 16P_2^t + C - P_2^t = 0.$$

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$$\Rightarrow -24P_{1/2}+24P_2-8P_{1/2}^t-5P_2^t+c=0$$

$$\Rightarrow \boxed{c=24P_{1/2}-24P_2+8P_{1/2}^t+5P_2^t}$$

Put a, b & c in (ii)

$$16P_{1/2}-16P_2+4P_{1/2}^t+4P_2^t-36P_{1/2}+36P_2-10P_{1/2}^t-8P_2^t$$
$$+24P_{1/2}-24P_2+8P_{1/2}^t+5P_2^t+d-P_2=0$$

$$4P_{1/2}-5P_2+2P_{1/2}^t+P_2^t+d=0$$

$$\Rightarrow \boxed{d=-4P_{1/2}+5P_2-2P_{1/2}^t-P_2^t}$$

Put a, b, c & d in ①.

$$P(t)=\left(16P_{1/2}-16P_2+4P_{1/2}^t+4P_2^t\right)t^3$$
$$+\left(-36P_{1/2}+36P_2-10P_{1/2}^t-8P_2^t\right)t^2$$
$$+\left(24P_{1/2}-24P_2+8P_{1/2}^t+5P_2^t\right)t$$
$$-4P_{1/2}+5P_2-2P_{1/2}^t-P_2^t$$

To check put $t=1/2$

$$P(1/2)=\frac{1}{8}\left[16P_{1/2}-16P_2+4P_{1/2}^t+4P_2^t\right]$$
$$+\frac{1}{4}\left[-36P_{1/2}+36P_2-10P_{1/2}^t-8P_2^t\right]$$
$$+\frac{1}{2}\left[24P_{1/2}-24P_2+8P_{1/2}^t+5P_2^t\right]$$
$$-4P_{1/2}+5P_2-2P_{1/2}^t-P_2^t$$

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$$P(\tfrac{1}{2}) = (2-9+12-4)P_{1/2} + \cancel{(-2+9-12+8)P_2}$$

$$\cancel{+(\tfrac{1}{2}-\tfrac{5}{2}+\tfrac{8}{2}-\tfrac{4}{2})P^t_{1/2} + (\tfrac{1}{2}-\tfrac{4}{2}+\tfrac{5}{2}-\tfrac{2}{2})P_2^t}$$

$$P(\tfrac{1}{2}) = P_{1/2}$$

at $t=1$

$$P(1) = 16P_{1/2} - 16P_1 + 4P^t_{1/2} + 4P_2^t$$

$$-36P_{1/2} + 36P_2 - 10P^t_{1/2} - 8P_2^t$$

$$+24P_{1/2} - 24P_2 + 8P^t_{1/2} + 5P_2^t$$

$$-4P_{1/2} + 5P_2 - 2P^t_{1/2} - P_2^t$$

$$\Rightarrow P(1) = P_2$$

Ans-2:- We know

$$P(t) = at^3 + bt^2 + ct + d \longrightarrow (1)$$

Given $P(1/3) = P_{1/3}$, $P(2/3) = P_{2/3}$

$$P^t(0) = P^t_{1/3}, \quad P^t(1) = P^t_{2/3}$$

$$P(1/3) = \frac{a}{27} + \frac{b}{9} + \frac{c}{3} + d = P_{1/3}$$

$$\Rightarrow a + 3b + 9c + 27d = 27P_{1/3} \rightarrow (i)$$

$$P(2/3) = \frac{8}{27}a + \frac{4}{9}b + \frac{2}{3}c + d = P_{2/3}$$

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$\Rightarrow 8a + 12b + 18c + 27d = 27 P_{2/3} \longrightarrow (ii)$

Now

$p^t(t) = 3at^2 + 2bt + c$

$p^t(0) = \boxed{c = P^t_{1/3}} \longrightarrow (iii)$

$p^t(1) = 3a + 2b + c = P^t_{2/3}$

$3a + 2b + P^t_{1/3} = P^t_{2/3}$

$3a + 2b = P^t_{2/3} - P^t_{1/3} \longrightarrow (iv)$

Put c in (i) & (ii) then subtracting.

$a + 3b + 9P^t_{1/3} + \cancel{27d} = 27 P_{1/3}$

$8a + 12b + 18P^t_{1/3} + \cancel{27d} = 27 P_{2/3}$

$-7a - 9b - 9P^t_{1/3} = 27(P_{1/3} - P_{2/3}) \rightarrow (v)$

By $[9 \times (iv)] + [2 \times (v)] \Rightarrow$

$27a + \cancel{18b} = 9P^t_{2/3} - 9P^t_{1/3}$

$-14a - \cancel{18b} = 18P^t_{1/3} + 54P_{1/3} - 54P_{2/3}$

$13a = 9P^t_{1/3} + 9P^t_{2/3} + 54P_{1/3} - 54P_{2/3}$

$$\boxed{a = \frac{54}{13}P_{1/3} - \frac{54}{13}P_{2/3} + \frac{9}{13}P^t_{1/3} + \frac{9}{13}P^t_{2/3}}$$

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Put a in (iv)

$$\frac{162}{13}P_{1/3} - \frac{162}{13}P_{2/3} + \frac{27}{13}P^t_{1/3} + \frac{27}{13}P^t_{2/3} + 2b = P^t_{2/3} - P^t_{1/3}$$

$$2b = -\frac{162}{13}P_{1/3} + \frac{162}{13}P_{2/3} - \frac{27}{13}P^t_{1/3} - P^t_{1/3} - \frac{27}{13}P^t_{2/3} + P^t_{2/3}$$

$$2b = -\frac{162}{13}P_{1/3} + \frac{162}{13}P_{2/3} - \frac{40}{13}P^t_{1/3} - \frac{14}{13}P^t_{2/3}$$

$$\boxed{b = -\frac{81}{13}P_{1/3} + \frac{81}{13}P_{2/3} - \frac{20}{13}P^t_{1/3} - \frac{7}{13}P^t_{2/3}}$$

Put a, b & c in (i).

$$\frac{54}{13}P_{1/3} - \frac{54}{13}P_{2/3} + \frac{9}{13}P^t_{1/3} + \frac{9}{13}P^t_{2/3} - \frac{243}{13}P_{1/3} + \frac{243}{13}P_{2/3}$$
$$- \frac{60}{13}P^t_{1/3} - \frac{21}{13}P^t_{2/3} + 9P^t_{1/3} + 27d = 27P_{1/3}$$

$$\Rightarrow \left(\frac{54}{13} - \frac{243}{13} - 27\right)P_{1/3} + \left(\frac{-54}{13} + \frac{243}{13}\right)P_{2/3}$$
$$+ \left(\frac{9}{13} - \frac{60}{13} + 9\right)P^t_{1/3} + \left(\frac{9}{13} - \frac{21}{13}\right)P^t_{2/3} + 27d = 0$$

$$-\frac{540}{13}P_{1/3} + \frac{189}{13}P_{2/3} + \frac{66}{13}P^t_{1/3} - \frac{12}{13}P^t_{2/3} + 27d = 0$$

$$\Rightarrow 27d = \frac{540}{13}P_{1/3} - \frac{189}{13}P_{2/3} - \frac{66}{13}P^t_{1/3} + \frac{12}{13}P^t_{2/3}$$

$$\Rightarrow \boxed{d = \frac{20}{13}P_{1/3} - \frac{7}{13}P_{2/3} - \frac{22}{117}P^t_{1/3} + \frac{4}{117}P^t_{2/3}}$$

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So, ① ⇒

$$P(t) = \left(\frac{54}{13}P_{1/3} - \frac{54}{13}P_{2/3} + \frac{9}{13}P^{t}_{1/3} + \frac{9}{13}P^{t}_{2/3}\right)t^3$$

$$+\left(-\frac{81}{13}P_{1/3} + \frac{81}{13}P_{2/3} - \frac{20}{13}P^{t}_{1/3} - \frac{7}{13}P^{t}_{2/3}\right)t^2 + P^{t}_{1/3}\,t$$

$$+\frac{20}{13}P_{1/3} - \frac{7}{13}P_{2/3} - \frac{22}{117}P^{t}_{1/3} + \frac{4}{117}P^{t}_{2/3}$$

To check $t = \frac{1}{3}$

$$P\left(\tfrac{1}{3}\right) = \frac{2}{13}P_{1/3} - \frac{2}{13}P_{2/3} + \frac{1}{39}P^{t}_{1/3} + \frac{1}{39}P^{t}_{2/3}$$

$$-\frac{9}{13}P_{1/3} + \frac{9}{13}P_{2/3} - \frac{20}{117}P^{t}_{1/3} - \frac{7}{117}P^{t}_{2/3} + \frac{1}{3}P^{t}_{1/3}$$

$$\frac{20}{13}P_{1/3} - \frac{7}{13}P_{2/3} - \frac{22}{117}P^{t}_{1/3} + \frac{4}{117}P^{t}_{2/3}$$

$$\Rightarrow P\left(\tfrac{1}{3}\right) = P_{1/3}$$

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Ans# 3 We know

$$P(t) = at^3 + bt^2 + ct + d \rightarrow (1)$$

Given

$$P(0) = P_1$$

$$P(1) = P_2$$

$$P^t(1/3) = P^t_{1/3}$$

$$P^t(2/3) = P^t_{2/3}$$

$$P(0) = \boxed{d = P_1}$$

$$P(1) = a + b + c + d = P_2$$

$$\Rightarrow a + b + c + P_1 = P_2$$

$$a + b + c = P_2 - P_1 \longrightarrow (i)$$

$$\Rightarrow P^t(t) = 3at^2 + 2bt + c$$

$$P^t(1/3) = \frac{a}{3} + \frac{2}{3}b + c = P^t_{1/3}$$

$$a + 2b + 3c = 3P^t_{1/3} \longrightarrow (ii)$$

$$P^t(2/3) = \frac{4a}{3} + \frac{4b}{3} + c = P^t_{2/3}$$

$$\Rightarrow 4a + 4b + 3c = 3P^t_{2/3} \rightarrow (iii)$$

By $(ii) - (i) \Rightarrow$

$$b + 2c = 3P^t_{1/3} + P_1 - P_2 \rightarrow (iv)$$

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By $[4\times(ii)] - [(iii)] \Rightarrow$

$$4\cancel{a} + 4\cancel{b} + 4c = 4P_2 - 4P_1$$

$$-\cancel{4a} - \cancel{4b} - 3c = -3P^t_{2/3}$$

$$\boxed{c = -4P_1 + 4P_2 - 3P^t_{2/3}}$$

Put in (iv)

$$b - 8P_1 + 8P_2 - 6P^t_{2/3} = 3P^t_{1/3} + P_1 - P_2$$

$$\boxed{b = 9P_1 - 9P_2 + 3P^t_{1/3} + 6P^t_{2/3}}$$

Put b & c in (i)

$$a + 9P_1 - 9P_2 + 3P^t_{1/3} + 6P^t_{2/3} - 4P_1 + 4P_2 - 3P^t_{2/3} = P_2 - P_1$$

$$a + 5P_1 - 5P_2 + 3P^t_{1/3} + 3P^t_{2/3} = P_2 - P_1$$

$$\boxed{a = -6P_1 + 6P_2 - 3P^t_{1/3} - 3P^t_{2/3}}$$

Put in (1)

$$P(t) = (-6P_1 + 6P_2 - 3P^t_{1/3} - 3P^t_{2/3})t^3 + (9P_1 - 9P_2 + 3P^t_{1/3} + 6P^t_{2/3})t^2 + (-4P_1 + 4P_2 - 3P^t_{2/3})t + P_1$$

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Example:

Given the two 3-dimensional points $P_1(0,0,0)$ and $P_2=(1,1,1)$ and the two tangent vectors $P_1^t=(1,0,0)$ & $P_2^t=(0,1,0)$, the curve segment is the simple cubic polynomial.

Solution:- We know that the cubic Hermite curve segments in terms of two points & two tangent vectors is

$$P(t)=(t^3,t^2,t,1)\begin{bmatrix}2 & -2 & 1 & 1\\ -3 & 3 & -2 & -1\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}P_1\\ P_2\\ P_1^t\\ P_2^t\end{bmatrix}$$

$$P(t)=(t^3,t^2,t,1)\begin{bmatrix}2 & -2 & 1 & 1\\ -3 & 3 & -2 & -1\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}(0,0,0)\\ (1,0,1)\\ (1,0,0)\\ (0,1,0)\end{bmatrix}$$

After some simplification we obtain

$$x(t)=-t^3+t^2+t$$

$$y(t)=-t^3+2t^2$$

$$z(t)=-2t^3+3t^2$$ which are 3-parametric Eqns of cubic polynomial in three dimensional space.

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Hermite Segments with Tension

We create a Hermite curve segment under tension by employing a non-uniform Hermite segment. Such a segment is obtained when the parameter "t" varies in the interval $[0, \Delta]$, where Δ can be any real positive number. The derivation of this case is follows as:

$$P(t) = at^3 + bt^2 + ct + d \longrightarrow (A)$$

where $t \in [0, \Delta]$ and

$$P(0) = P_1 \quad , \quad P(\Delta) = P_2$$

$$P^t(0) = P_1^t, \quad P^t(\Delta) = P_2^t$$

So,

$$P(0) = \boxed{d = P_1} \longrightarrow (1)$$

$$P(\Delta) = a\Delta^3 + b\Delta^2 + c\Delta + d = P_2 \longrightarrow (2)$$

$$P^t(0) = \boxed{c = P_1^t} \longrightarrow (3)$$

$$P^t(\Delta) = \boxed{3a\Delta^2 + 2b\Delta + c = P_2^t} \longrightarrow (4)$$

Put c & d in Eq. (2) & (4).

$$(2) \Rightarrow \quad a\Delta^3 + b\Delta^2 = P_2 - \Delta P_1^t - P_1 \longrightarrow (5)$$

$$(4) \Rightarrow \quad 3a\Delta^2 + 2b\Delta = P_2^t - P_1^t \longrightarrow (6)$$

Multiplying 2 with (5).

$$2a\Delta^3 + 2b\Delta^2 = -2P_1 + 2P_2 - 2\Delta P_1^t \rightarrow (7)$$

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By [$\Delta \times$ Eq (6)] $-$ Eq (7) -

$$3a\Delta^3 + 2b\Delta^2 = -\Delta P_1^t + \Delta P_2^t$$

$$2a\Delta^3 + 2b\Delta^2 = -2P_1 + 2P_2 - 2\Delta P_1^t$$

$$a\Delta^3 = 2P_1 - 2P_2 + \Delta P_1^t + \Delta P_2^t$$

$$\boxed{a = \frac{2}{\Delta^3} P_1 - \frac{2}{\Delta^3} P_2 + \frac{1}{\Delta^2} P_1^t + \frac{1}{\Delta^2} P_2^t}$$

Now by $3 \times$ Eq (5) $- \Delta \times$ Eq (6).

$3 \times$ Eq (5) $\Rightarrow$ $3a\Delta^3 + 3b\Delta^2 = -3P_1 + 3P_2 - 3\Delta P_1^t$

$\Delta \times$ Eq (6) $\Rightarrow$ $3a\Delta^3 + 2b\Delta^2 = -\Delta P_1^t + \Delta P_2^t$

$$b\Delta^2 = -3P_1 + 3P_2 - 2\Delta P_1^t - \Delta P_2^t$$

$$\boxed{b = \frac{-3}{\Delta^2} P_1 + \frac{3}{\Delta^2} P_2 - \frac{2}{\Delta} P_1^t - \frac{1}{\Delta} P_2^t}$$

So (A) becomes,

$$P(t) = \left(\frac{2}{\Delta^3} P_1 - \frac{2}{\Delta^3} P_2 + \frac{1}{\Delta^2} P_1^t + \frac{1}{\Delta^2} P_2^t\right) t^3$$
$$+ \left(\frac{-3}{\Delta^2} P_1 + \frac{3}{\Delta^2} P_2 - \frac{2}{\Delta} P_1^t - \frac{1}{\Delta} P_2^t\right) t^2$$
$$+ \left(P_1^t\right) t + P_1$$

In matrix form,

$$P(t) = (t^3, t^2, t, 1) \begin{bmatrix} \frac{2}{\Delta^3} & \frac{-2}{\Delta^3} & \frac{1}{\Delta^2} & \frac{1}{\Delta^2} \\ \frac{-3}{\Delta^2} & \frac{3}{\Delta^2} & \frac{-2}{\Delta} & \frac{-1}{\Delta} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_1^t \\ P_2^t \end{bmatrix}$$

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PC Conic Approximations 16-4-15

Hermite interpolation can be applied to compute the conic sections (circle, ellipse, parabola, hyperbola). Given three point P_0, P_1 & P_2 and a scalar α, we construct the 4-tuple

$$B = (P_0, P_2, 4\alpha(P_1 - P_0), 4\alpha(P_2 - P_1))$$

where $0 \leq \alpha \leq 1$,

to become our two points and two extreme tangent vectors and compute a segment that approximates a conic section. Here we want to discuss the following cases

(i):- We obtain an ellipse when $0 \leq \alpha < 0.5$

(ii):- A parabola when $\alpha = 0.5$

(iii):- A Hyperbola when $0.5 < \alpha \leq 1$

The tangent vectors at the two ends points are $P^t(0) = 4\alpha(P_1 - P_0)$ and $P^t(1) = 4\alpha(P_2 - P_1)$. The tangent vector at mid point is

$P^t(0.5) = (1.5 - \alpha)(P_2 - P_0)$ it is parallel to the vector $P_2 - P_0$.

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V. Imp.

Exercise 4.9

We know that any three points P_0, P_1 & P_2 define a unique parabola. Use Hermite interpolation to calculate the parabola from P_0 to P_2 whose start and end tangents go in the directions from P_0 to P_1 and from P_1 to P_2, respectively.

(may be changed)

Solution:- We know that for parabola $\alpha = 0.5$. Now, the cubic Hermite curve segment in terms of two points and two tangents is

$$P(t) = (t^3, t^2, t, 1)\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_1^t \\ P_2^t \end{bmatrix}$$

Using the given values, we get

$$P(t) = (t^3, t^2, t, 1)\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_2 \\ 4(0.5)(P_1 - P_0) \\ 4(0.5)(P_2 - P_1) \end{bmatrix}$$

$$P(t) = (t^3, t^2, t, 1)\begin{bmatrix} \cancel{2P_0} - 2P_2 + \cancel{2P_1} - \cancel{2P_0} + \cancel{2P_2} - \cancel{2P_1} \\ -3P_0 + 3P_2 - 4P_1 + 4P_0 - 2P_2 + 2P_1 \\ 0 + 0 + 2P_1 - 2P_0 + 0 \end{bmatrix}$$

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$$P(t) = (t^3, t^2, t, 1)\begin{bmatrix} 0 \\ P_0 - 2P_1 + P_2 \\ 2P_1 - 2P_0 \\ P_0 \end{bmatrix}$$

$$P(t) = (0)t^3 + (P_0 - 2P_1 + P_2)t^2 + (2P_1 - 2P_0)t + P_0$$

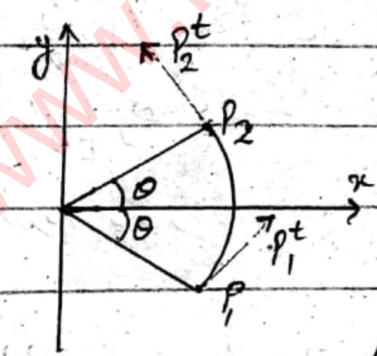
$$P(t) = P_0(t^2 - 2t + 1) + P_1(-2t^2 + 2t) + P_2(t^2)$$

$$P(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2$$

This is the parabola.

Q. How to construct approximate circles and circular arcs by using Hermite interpolation?

Sol:-



← a circular arc of unit radius about the origin.

We assume that an arc spanning an angle 2θ is needed and we place it's two endpoints P_1 & P_2 at locations $(\cos\theta, -\sin\theta)$ and $(\cos\theta, \sin\theta)$ respectively. This arc is symmetric about the x-axis. Since

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a circle is always prependicular to its radius, we select as our start and end tangents two vectors that are perpendicular to P_1 & P_2. They are

$$P_1^t = a(\sin\theta, \cos\theta)$$

$$\& \quad P_2^t = a(-\sin\theta, \cos\theta)$$

where "a" is a parameter to be determined.

Since cubic Hermite curve segment in terms of two points and two tangent vectors is

$$P(t) = (t^3, t^2, t, 1)\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_1^t \\ P_2^t \end{bmatrix}$$

Using the given values we have,

$$P(t) = (t^3, t^2, t, 1)\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} (\cos\theta, -\sin\theta) \\ (\cos\theta, \sin\theta) \\ a(\sin\theta, \cos\theta) \\ a(-\sin\theta, \cos\theta) \end{bmatrix}$$

$$P(t) = \left[2t^3-3t^2+1 \quad -2t^3+3t^2 \quad t^3-2t^2+t \quad t^3-t^2\right]\begin{bmatrix} (\cos\theta, -\sin\theta) \\ (\cos\theta, \sin\theta) \\ a(\sin\theta, \cos\theta) \\ a(-\sin\theta, \cos\theta) \end{bmatrix}$$

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$$P(t) = (2t^3-3t^2+1)(\cos\theta, -\sin\theta) + (-2t^3+3t^2)(\cos\theta, \sin\theta)$$
$$+ (t^3-2t^2+t)a(\sin\theta, \cos\theta) + (t^3-t^2)a(-\sin\theta, \cos\theta)$$
$\longrightarrow (1)$

The curve segment passes through the circular arc at its center i.e $P(0.5) = (1,0)$ so

(1) $\Rightarrow$

$$(1,0) = P(0.5) = \left(\frac{2}{8} - \frac{3}{4} + 1\right)(\cos\theta, -\sin\theta)$$
$$+ \left(-\frac{2}{8} + \frac{3}{4}\right)(\cos\theta, \sin\theta)$$
$$+ \left(\frac{1}{8} - \frac{2}{4} + \frac{1}{2}\right) a(\sin\theta, \cos\theta)$$
$$+ \left(\frac{1}{8} - \frac{1}{4}\right) a(-\sin\theta, \cos\theta)$$

$$(1,0) = \frac{1}{2}(\cos\theta, -\sin\theta) + \frac{1}{2}(\cos\theta, \sin\theta)$$
$$+ \frac{a}{8}(\sin\theta, \cos\theta) - \frac{a}{8}(-\sin\theta, \cos\theta)$$

$$(1,0) = \left(\left(\frac{1}{2} + \frac{1}{2}\right)\cos\theta + \left(\frac{a}{8} + \frac{a}{8}\right)\sin\theta, 0\right)$$

$$\Rightarrow 1 = \cos\theta + \frac{a}{4}\sin\theta$$

$$\frac{a}{4}\sin\theta = 1 - \cos\theta$$

$$\boxed{a = \frac{4(1-\cos\theta)}{\sin\theta}}$$

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The curve can now be written in the form

$$P(t) = (t^3, t^2, t, 1)\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} (\cos\theta, -\sin\theta) \\ (\cos\theta, \sin\theta) \\ \left(4(1-\cos\theta), \frac{4(1-\cos\theta)}{\tan\theta}\right) \\ \left(-4(1-\cos\theta), \frac{4(1-\cos\theta)}{\tan\theta}\right) \end{bmatrix}$$

This curve provides an excellent approximation to a circular arc, even for angles θ as large as 90°.

Degree-4 Hermite Interpolation

Since

$$P(t) = at^4 + bt^3 + ct^2 + dt + e \longrightarrow (A)$$

Given $P(0) = P_1$, $P(1) = P_2$

$P^t(0) = P_1^t$, $P^t(1) = P_2^t$

$P^{tt}(0) = P_1^{tt}$

$$(A) \Rightarrow P(0) = \boxed{e = P_1} \longrightarrow (1)$$

$$P(1) = a + b + c + d + e = P_2 \longrightarrow (2)$$

$$P^t(0) = \boxed{d = P_1^t} \longrightarrow (3)$$

$$P^t(1) = 4a + 3b + 2c + d = P_2^t \longrightarrow (4)$$

$$P^{tt}(0) = 2c = P_1^{tt}$$

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$$\Rightarrow \boxed{c = \frac{1}{2} P_1^{tt}} \longrightarrow (5)$$

Put (1), (3) & (5) in (2) & (4)

$$(2) \Rightarrow \quad a + b = P_2 - P_1 - P_1^t - \frac{1}{2} P_1^{tt} \longrightarrow (6)$$

$$(4) \Rightarrow \quad 4a + 8b = P_2^t - P_1^t - P_1^{tt} \longrightarrow (7)$$

By $[4 \times (6)] - (7)$

$$\cancel{4a} + 4b = -4P_1 + 4P_2 - 4P_1^t - 2P_1^{tt}$$

$$\underline{\pm \cancel{4a} \pm 8b = \mp P_1^t \pm P_2^t \mp P_1^{tt}}$$

$$b = -4P_1 + 4P_2 - 3P_1^t - P_2^t - P_1^{tt}$$

Put b in (6).

$$a = P_2 - P_1 - P_1^t - \frac{1}{2} P_1^{tt} + 4P_1 - 4P_2 + 3P_1^t + P_2^t + P_1^{tt}$$

$$\Rightarrow \boxed{a = 3P_1 - 3P_2 + 2P_1^t + P_2^t + \frac{1}{2} P_1^{tt}}$$

Put a, b, c, d & e in Eq. (A).

$$P(t) = \left(3P_1 - 3P_2 + 2P_1^t + P_2^t + \frac{1}{2} P_1^{tt}\right) t^4$$

$$+ \left(-4P_1 + 4P_2 - 3P_1^t - P_2^t - P_1^{tt}\right) t^3 + \frac{1}{2} P_1^{tt} t^2$$

$$+ P_1^t \, t + P_1$$

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Degree-5 Hermite Interpolation

$$P(t) = at^5 + bt^4 + ct^3 + dt^2 + et + f \longrightarrow (A)$$

Conditions are

$P(0) = P_1$, $P(1) = P_2$, $P^t(0) = P_1^t$, $P^t(1) = P_2^t$

$P^{tt}(0) = P_1^{tt}$, $P^{tt}(1) = P_2^{tt}$

$$(A) \Rightarrow P(0) = \boxed{f = P_1} \longrightarrow (1)$$

$$P(1) = a + b + c + d + e + f = P_2 \longrightarrow (2)$$

$$P^t(t) = 5at^4 + 4bt^3 + 3ct^2 + 2dt + e$$

$$P^t(0) = \boxed{e = P_1^t} \longrightarrow (3)$$

$$P^t(1) = 5a + 4b + 3c + 2d + e = P_2^t \longrightarrow (4)$$

$$P^{tt}(t) = 20at^3 + 12bt^2 + 6ct + 2d$$

$$P^{tt}(0) = 2d = P_1^{tt}$$

$$\Rightarrow \boxed{d = \frac{1}{2} P_1^{tt}} \longrightarrow (5)$$

$$P^{tt}(1) = 20a + 12b + 6c + 2d = P_2^{tt} \longrightarrow (6)$$

Put d, e & f in (2), (4) & (6)

$$(2) \Rightarrow a + b + c = P_2 - P_1 - P_1^t - \frac{1}{2} P_1^{tt} \longrightarrow (7)$$

$$(4) \Rightarrow 5a + 4b + 3c = P_2^t - P_1^t - P_1^{tt} \longrightarrow (8)$$

$$(6) \Rightarrow 20a + 12b + 6c = P_2^{tt} - P_1^{tt} \longrightarrow (9)$$

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$5 \times (7) \Rightarrow \quad 5a + 5b + 5c = -5P_1 + 5P_2 - 5P_1^t - \frac{5}{2}P_1^{tt}$

$(8) \Rightarrow \quad \cancel{5a} + 4b + 3c = \quad -P_1^t - P_1^{tt} + P_2^t$

$b + 2c = -5P_1 + 5P_2 - 4P_1^t - \frac{3}{2}P_1^{tt} - P_2^t \quad \rightarrow (10)$

$4 \times (8) \Rightarrow \quad 20a + 16b + 12c = -4P_1^t + 4P_2^t - 4P_1^{tt}$

$(9) \Rightarrow \quad 20a + 12b + 6c = \quad P_1^{tt} + P_2^{tt}$

$4b + 6c = -4P_1^t + 4P_2^t - 3P_1^{tt} - P_2^{tt} \quad \rightarrow (11)$

$(10) \times 4 \Rightarrow \quad 4b + 8c = -20P_1 + 20P_2 - 16P_1^t - 6P_1^{tt} - 4P_2^t$

$(11) \Rightarrow \quad 4b + 6c = \quad -4P_1^t + 4P_2^t - 3P_1^{tt} - P_2^{tt}$

$2c = -20P_1 + 20P_2 - 12P_1^t - 8P_2^t - 3P_1^{tt} + P_2^{tt} \quad \rightarrow (12)$

$(12) \Rightarrow \boxed{c = -10P_1 + 10P_2 - 6P_1^t - 4P_2^t - \frac{3}{2}P_1^{tt} + \frac{1}{2}P_2^{tt}}$

Put (12) in (10)

$(10) \Rightarrow$

$b = -5P_1 + 5P_2 - 4P_1^t - P_2^t - \frac{3}{2}P_1^{tt} + 20P_1 - 20P_1$
$\quad + 12P_1^t + 8P_2^t + 3P_1^{tt} - P_2^{tt}$

$\Rightarrow \boxed{b = 15P_1 - 15P_2 + 8P_1^t + 7P_2^t + \frac{3}{2}P_1^{tt} - P_2^{tt}}$

Put b & c in (7).

$(7) \Rightarrow a = P_2 - P_1 - P_1^t - \frac{1}{2}P_1^{tt} - 15P_1 + 15P_2 - 8P_1^t - 7P_2^t - \frac{3}{2}P_1^{tt} + P_2^{tt}$
$\quad + 10P_1 - 10P_2 + 6P_1^t + 4P_2^t + \frac{3}{2}P_1^{tt} - \frac{1}{2}P_2^{tt}$

$\Rightarrow \boxed{a = -6P_1 + 6P_2 - 3P_1^t - \frac{1}{2}P_1^{tt} + \frac{1}{2}P_2^{tt}}$

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Put a, b, c, d, e & f in Eq. (A)

$$P(t) = \left(-6P_1 + 6P_2 - 3P_1^t - 3P_2^t - \frac{1}{2}P_1^{tt} + \frac{1}{2}P_2^{tt}\right)t^5$$
$$+ \left(15P_1 - 15P_2 + 8P_1^t + 7P_2^t + \frac{3}{2}P_1^{tt} - P_2^{tt}\right)t^4$$
$$+ \left(-10P_1 + 10P_2 - 6P_1^t - 4P_2^t - \frac{3}{2}P_1^{tt} + \frac{1}{2}P_2^{tt}\right)t^3$$
$$+ \left(\frac{1}{2}P_1^{tt}\right)t^2 + \left(P_1^t\right)t + P_1$$

In matrix form,

$$P(t) = (t^5, t^4, t^3, t^2, t, 1)\begin{bmatrix} -6 & 6 & -3 & -3 & -\frac{1}{2} & \frac{1}{2} \\ 15 & -15 & 8 & 7 & \frac{3}{2} & -1 \\ -10 & 10 & -6 & -4 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_1^t \\ P_2^t \\ P_1^{tt} \\ P_2^{tt} \end{bmatrix}$$

which is our required result.

Controlling the Hermite Segment 22-04-15

In this section we study about the controling curve of cubic Hermite segment. The amount of editing and controlling that can be achieved by varying the magnitudes of tangent vectors.

We start with Hermite segment defined by the two end points

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$P_1 = (0,0)$ & $P_2 = (2,1)$ & by two tangent vectors.

$$P^t(0) = P_1^t = (1,1)$$

$$P^t(1) = P_2^t = (1,0)$$

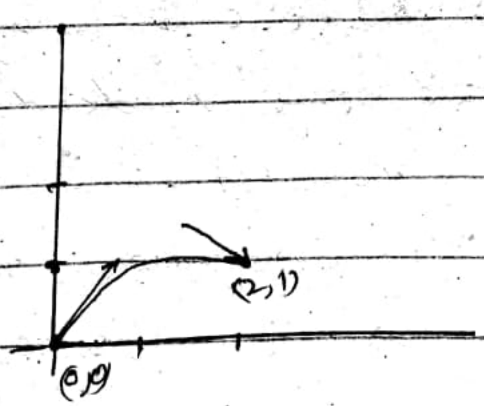
The curve is easy to calculate. Its expression is

$$P(t) = (t^3, t^2, t, 1)\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} (0,0) \\ (2,1) \\ (1,1) \\ (1,0) \end{bmatrix}$$

$$P(t) = (t^3, t^2, t, 1)\begin{bmatrix} (0,0)+(-4,-2)+(1,1)+(1,0) \\ (0,0)+(6,3)+(-2,-2)+(-1,0) \\ (0,0)+(0,0)+(1,1)+(0,0) \\ (0,0)+(0,0)+(0,0)+(0,0) \end{bmatrix}$$

$$P(t) = (t^3, t^2, t, 1)\begin{bmatrix} (-2,-1) \\ (3,1) \\ (1,1) \\ (0,0) \end{bmatrix}$$

$$P(t) = -(2,1)t^3 + (3,1)t^2 + (1,1)t \longrightarrow (1)$$



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Suppose that the designer wants to raise the curve a bit, but also keep the same start and end directions & endpoints. The only way to edit the curve is to change the magnitudes of the tangents.

Consider two new tangent vectors of the form

$$P_1^t = (a, a)$$
$$\& \quad P_2^t = (b, 0)$$

where a, b are new parameters that have to calculate.

To raise the curve, we go through the following steps:-

1):- Calculate the midpoint of the curve. This is $P(0.5) = (1, 5/8)$

2):- Decide by how much to raise it. Let's say we decide to raise the midpoint to $(1, 1)$.

3):- Construct a new curve $Q(t)$, based on the tangents (a, a) and $(b, 0)$.

4):- Require that the new

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curve pass through (1,1) as it's mid point and determine "a" & "b" from above steps.

The general form of new curve $Q(t)$ is

$$Q(t) = (t^3, t^2, t, 1)\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} (0,0) \\ (2,1) \\ (a,a) \\ (b,0) \end{bmatrix}$$

$$Q(t) = (t^3, t^2, t, 1)\begin{bmatrix} (0,0)+(-4,-2)+(a,a)+(b,0) \\ (0,0)+(6,3)+(-2a,-2a)+(-b,0) \\ (0,0)+(0,0)+(a,a)+(0,0) \\ (0,0)+(0,0)+(0,0)+(0,0) \end{bmatrix}$$

$$Q(t) = (t^3, t^2, t, 1)\begin{bmatrix} (a+b-4,\ a-2) \\ (-2a-b+6,\ 3-2a) \\ (a,a) \\ (0,0) \end{bmatrix}$$

$$Q(t) = (a+b-4,\ a-2)t^3 + (-2a-b+6,\ -2a+3)t^2 + (a,a)t \longrightarrow (2)$$

★ Since $Q(0.5) = (1,1)$ such that

$$(1,1) = (a+b-4,\ a-2)\frac{1}{8} + (-2a-b+6,\ -2a+3)\frac{1}{4} + \frac{1}{2}(a,a)$$

$$\Rightarrow (1,1) = \left(\frac{a+b-4}{8} + \frac{-2a-b+6}{4} + \frac{a}{2},\ \frac{a-2}{8} + \frac{-2a+3}{4} + \frac{a}{2}\right)$$

$$\Rightarrow (1,1) = \left(\frac{a-b+8}{8},\ \frac{a+4}{8}\right) \longrightarrow (A)$$

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So,
$$1 = \frac{a-b+8}{8}$$
& $$1 = \frac{a+4}{8}$$
$$\Rightarrow a+4=8$$
$$\boxed{a=4}$$
So, $$1 = \frac{4-b+8}{8}$$
$$\Rightarrow 8 = 12-b$$
$$\Rightarrow \boxed{b=4}$$

Subtituting these values in Eq ②.
we get,

$$Q(t) = (4,2)\,t^3 + (-6,-5)\,t^2 + (4,4)\,t$$

To check:-
$$Q(0) = (0,0)$$
$$Q(1) = (4-6+4,\ 2-5+4)$$
$$\Rightarrow Q(1) = (2,1)$$
$$Q^t(t) = 3(4,2)t^2 + 2(-6,-5)t + (4,4)$$
$$Q^t(0) = (4,4)$$
& $$Q^t(1) = (12-12+4,\ 6-10+4)$$
$$Q^t(1) = (4,0)$$

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★ If $Q(0.5) = (1, \frac{5}{8})$

then Eq. (A) becomes.

$$\Rightarrow (1, \tfrac{5}{8}) = \left(\frac{a-b+8}{8}, \frac{a+4}{8}\right)$$

$$\Rightarrow \quad 1 = \frac{a-b+8}{8}, \quad \frac{5}{\cancel{8}} = \frac{a+4}{\cancel{8}}$$

$$\cancel{8} = a-b+\cancel{8}, \quad a = 5-4$$

$$a = b, \quad \boxed{a = 1}$$

$$\Rightarrow \boxed{b = 1}$$

So, we get the same curve as perivious $P(t)$

By Putting $a = b = 1$ in Eq (2).

$$Q(t) = t^3(-2,-1) + (3,1)t^2 + (1,1)t$$

★ Let us assume that we raise the mid point from $(1, \frac{5}{8})$ to $(1, \frac{5}{8} + \alpha)$. Where α is any real number.

Then Eq (A) becomes

$$(1, \tfrac{5}{8} + \alpha) = \left(\frac{a-b+8}{8}, \frac{a+4}{8}\right)$$

$$\Rightarrow 1 = \frac{a-b+8}{8} \quad \& \quad \frac{5+8\alpha}{\cancel{8}} = \frac{a+4}{\cancel{8}}$$

$$\Rightarrow \cancel{8} = a-b+\cancel{8} \quad \& \quad 5+8\alpha = a+4$$

$$a = b \quad \Rightarrow \boxed{a = 1+8\alpha}$$

$$\Rightarrow \boxed{b = 1+8\alpha}$$

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So, we get

$$Q(t) = (t^3, t^2, t, 1)\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} (0,0) \\ (2,1) \\ (1+8\alpha)(1,1) \\ (1+8\alpha, 0) \end{bmatrix}$$

$$Q(t) = (t^3, t^2, t, 1)\begin{bmatrix} (0,0)+(-4,-2)+(1+8\alpha, 1+8\alpha)+(1+8\alpha, 0) \\ (0,0)+(6,3)+(-2-16\alpha, -2-16\alpha)+(-1-8\alpha, 0) \\ (0,0)+(0,0)+(1+8\alpha, 8\alpha+1)+(0,0) \\ (0,0)+(0,0)+(0,0)+(0,0) \end{bmatrix}$$

$$Q(t) = (-2+16\alpha, -1+8\alpha)t^3 + (3-24\alpha, 1-16\alpha)t^2 + (1+8\alpha, 1+8\alpha)t$$

This means that α can be vary without limit. When α is positive then curve is pulled up. Negative values of α push the curve down.

Where $\alpha = -\frac{1}{8}$ is a special case it implies that $a = b = 0$ the result in the curve

$$Q(t) = (-4, -2)t^3 + (6, 3)t^2$$

$$Q(t) = (-4t^3 + 6t^2, -2t^3 + 3t^2)$$

$$Q(t) = (2(-2t^3 + 3t^2), -2t^3 + 3t^2)$$

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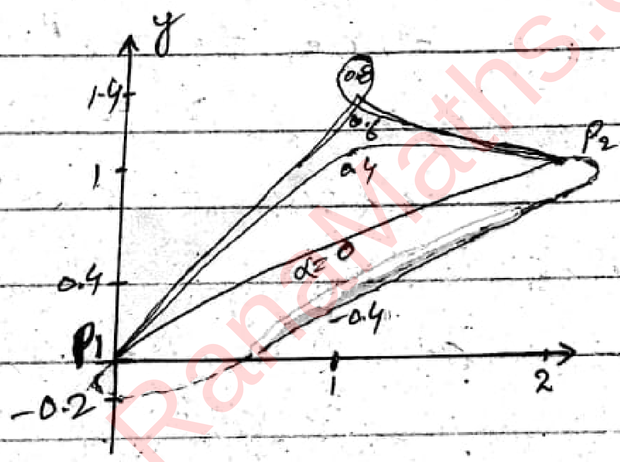
97 $u = -2t^3 + 3t^2$

$\Rightarrow Q(u) = (2u, u) \qquad u \in [0, 1]$

The curve is a straight line from $(0,0)$ to $(2,1)$. Its mid point is $(1, \frac{1}{2})$.

Effect of parameter "α" 5-5-15



The fig. shows that the effect of parameter α raising the curve is done by increasing the size of tangent vector. This idea forces the curve to continue longer in the initial and final directions. This is also the reason why too much raising causes undesirable effects. For $\alpha = 0$ the curve is orignal Hermite curve & we want to the effects of increasing α.

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For $\alpha = 0.4$, the curve is raised and still has a reasonable shape. However, for longer values of α, the curve gets tight, & show a cusp (a kink) for the value $\alpha = 0.6$ & then starts looping on itself. It is easy to see that when $\alpha = 5/8$, the tangent vector becomes indefinite at the midpoint $(t = 0.5)$. To show this effect,

Consider

$$Q(t) = (a+b-4, a-2)t^3 + (-2a-b+6, 3-2a)t^2 + (a, a)t$$

$$\Rightarrow \quad Q^t(t) = 3(a+b-4, a-2)t^2 + 2(-2a-b+6, 3-2a)t + (a, a)$$

Since $a = b = 1 + 8\alpha$ & $\alpha = 5/8$

$$a = b = 1 + 8(5/8)$$

$$a = b = 6$$

$$\Rightarrow \quad Q^t(t) = 3(6+6-4, 6-2)t^2 + 2(-12-6+6, 3-12)t + (6, 6)$$

$$Q^t(t) = 3(8, 4)t^2 + 2(-12, -9)t + (6, 6)$$

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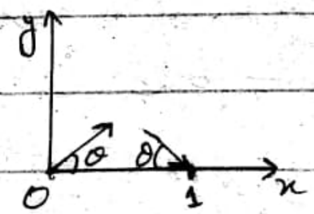
Put $t = 0.5$

$$Q^t(0.5) = 3(2,1) + (-12,-9) + (6,6)$$

$$Q^t(0.5) = (0,0)$$

Hence at the midpoint tangent vector becomes zero so the curve has a cusp.

V.Imp Exercise:- 4.13 Given the two endpoints $P_1 = (0,0)$ & $P_2 = (1,0)$ and the two tangent vectors $P_1^t = \alpha(\cos\theta, \sin\theta)$ & $P_2^t = \alpha(\cos\theta, -\sin\theta)$. Calculate the value of "α" for which the Hermite segment from P_1 to P_2 has a cusp.



Solution:- Consider a cubic curve segment in the terms of two end points & two tangent vectors is

$$P(t) = (t^3, t^2, t, 1)\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} (0,0) \\ (1,0) \\ \alpha(\cos\theta, \sin\theta) \\ \alpha(\cos\theta, -\sin\theta) \end{bmatrix}$$

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$$P(t) = (t^3, t^2, t, 1)\begin{bmatrix} (0,0)+(-2,0)+(\alpha\cos\theta, \alpha\sin\theta)+(\alpha\cos\theta, -\alpha\sin\theta) \\ (0,0)+(3,0)+(-2\alpha\cos\theta, -2\alpha\sin\theta)+(-\alpha\cos\theta, \alpha\sin\theta) \\ (\alpha\cos\theta, \alpha\sin\theta) \\ (0,0) \end{bmatrix}$$

$$P(t) = (-2+2\alpha\cos\theta, 0)t^3 + (3-3\alpha\cos\theta, -\alpha\sin\theta)t^2 + (\alpha\cos\theta, \alpha\sin\theta)t$$

$$\Rightarrow P^t(t) = 3(-2+2\alpha\cos\theta, 0)t^2 + 2(3-3\alpha\cos\theta, -\alpha\sin\theta)t + (\alpha\cos\theta, \alpha\sin\theta)$$

Since the cubic Hermite curve segment has a cusp at midpoint

so, $P^t(0.5) = (0,0)$

$$\Rightarrow (0,0) = \frac{3}{4}(-2+2\alpha\cos\theta, 0) + (3-3\alpha\cos\theta, -\alpha\sin\theta) + (\alpha\cos\theta, \alpha\sin\theta)$$

$$\Rightarrow 0 = \frac{-3+3\alpha\cos\theta}{2} + 3 - 3\alpha\cos\theta + \alpha\cos\theta$$

$$0 = -3+3\alpha\cos\theta + 6 - 4\alpha\cos\theta$$

$$\alpha\cos\theta = 3$$

$$\Rightarrow \alpha = \frac{3}{\cos\theta}$$

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$$\Rightarrow P(t) = \left(-2+2\left(\frac{3}{\cos\theta}\right)\cos\theta, 0\right)t^3 + \left(3-3\left(\frac{3}{\cos\theta}\right)\cos\theta, -\frac{3}{\cos\theta}\sin\theta\right)t$$

$$+ \left(\left(\frac{3}{\cos\theta}\right)\cos\theta, \left(\frac{3}{\cos\theta}\right)\sin\theta\right)t.$$

$$P(t) = (4,0)t^3 + (-6, -3\tan\theta)t^2 + (3, 3\tan\theta)t$$

$$\Rightarrow P^t(t) = 3(4,0)t^2 + 2(-6,-3\tan\theta)t + (3, 3\tan\theta)$$

put $t = 0.5$

$$P^t(0.5) = \frac{3}{4}(4,0) + (-6, -3\tan\theta) + (3, 3\tan\theta)$$

$$= (3-6+3, 0-3\tan\theta+3\tan\theta)$$

$$= (0,0)$$

06-05-2015

Special & Degenerate Hermite Segments

The following special cases result in Hermit curve segments that are either especially simple (degenerate) or especially interesting.

1):- The case when $P_1 = P_2$ & $P_1^t = P_2^t = (0,0)$. Since the cubic Hermite curve segment is

$$P(t) = F_1(t)P_1 + F_2(t)P_2 + F_3(t)\cancel{P_1^t} + F_4(t)\cancel{P_2^t}$$

$P_1 = P_2$

$$P(t) = (F_1(t) + F_2(t))P_1$$

$F_1(t) + F_2(t) = 2t^3 - 3t^2 + 1 - 2t^3 + 3t^2$

$$\Rightarrow P(t) = P_1 \qquad \text{since } F_1(t) + F_2(t) = 1$$

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2:- The case when $P_1^t = P_2^t = P_2 - P_1$;

The two tangents point in the same direction, from P_1 to P_2. So, the cubic Hermite curve segment reduces to straight line,

$$P(t) = F_1(t)P_1 + F_2(t)P_2 + F_3(t)P_1^t + F_4(t)P_2^t$$

$$= (2t^3-3t^2+1)P_1 + (-2t^3+3t^2)P_2 + (F_3(t)+F_4(t))(P_2-P_1)$$

$$= (2t^3-3t^2+1)P_1 + (-2t^3+3t^2)P_2 + (2t^3-3t^2+t)(P_2-P_1)$$

$$= (\cancel{2t^3}-\cancel{3t^2}+1-\cancel{2t^3}+\cancel{3t^2}-t)P_1 + (\cancel{-2t^3}+\cancel{3t^2}+\cancel{2t^3}-\cancel{3t^2}+t)P_2$$

$$= P_1 - tP_1 + tP_2$$

$$= (P_2-P_1)t + P_1$$

3:- The case when $P_1 = P_2$.

$$P(t) = F_1(t)P_1 + F_2(t)P_2 + F_3(t)P_1^t + F_4(t)P_2^t$$

$$P(t) = \underbrace{(F_1(t)+F_2(t))}_{1}P_1 + (t^3-2t^2+t)P_1^t + (t^3-t^2)P_2^t$$

$$P(t) = (P_1^t+P_2^t)t^3 + (-2P_1^t-P_2^t)t^2 + P_1^t\,t + P_1$$

Put $t=0 \Rightarrow \boxed{P(0) = P_1}$

Put $t=1 \Rightarrow P(1) = P_1^t + \cancel{P_2^t} - 2P_1^t - \cancel{P_2^t} + P_1^t + P_1$

$$P(1) = \cancel{2P_1^t} - \cancel{2P_1^t} + P_1$$

$$\Rightarrow \boxed{P(1) = P_1}$$

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So, it is a closed curve (but is not a circle).

4):- The case when $P_1^t = P_2^t = (x_2-x_1, y_2-y_1, 0)$

$$\Rightarrow P(t) = F_1(t)P_1 + F_2(t)P_2 + F_3(t)P_1^t + F_4(t)P_2^t$$

$$= (2t^3-3t^2+1)P_1 + (-2t^3+3t^2)P_2 + (F_3(t)+F_4(t))(x_2-x_1, y_2-y_1, 0)$$

$$= (2t^3-3t^2+1)(x_1, y_1, z_1) + (-2t^3+3t^2)(x_2, y_2, z_2) + (2t^3-3t^2+t)(x_2-x_1, y_2-y_1, 0)$$

$$= t^3[(2x_1, 2y_1, 2z_1) + (-2x_2, -2y_2, -2z_2) + (2x_2-2x_1, 2y_2-2y_1, 0)] + t^2[(-3x_1, -3y_1, -3z_1) + (3x_2, 3y_2, 3z_2) + (-3x_2+3x_1, -3y_2+3y_1, 0)] + t(x_2-x_1, y_2-y_1, 0) + (x_1, y_1, z_1)$$

$$= t^3[\cancel{2x_1} - \cancel{2x_2} + \cancel{2x_2} - \cancel{2x_1}, \cancel{2y_1} - \cancel{2y_2} + \cancel{2y_2} - \cancel{2y_1}, 2z_1 - 2z_2] + t^2[\cancel{-3x_1} + \cancel{3x_2} - \cancel{3x_2} + \cancel{3x_1}, \cancel{-3y_1} + \cancel{3y_2} - \cancel{3y_2} + \cancel{3y_1}, -3z_1 + 3z_2] + t(x_2-x_1, y_2-y_1, 0) + (x_1, y_1, z_1)$$

$$= t^3(0, 0, 2(z_1-z_2)) + t^2(0, 0, -3(z_1-z_2)) + t(x_2-x_1, y_2-y_1, 0) + (x_1, y_1, z_1)$$

$$= \left(x_1 + (x_2-x_1)t,\ y_1 + (y_2-y_1)t,\ z_1 + (z_2-z_1)(3t^2-2t^3)\right)$$

The x & y coordinates of this curve are linear functions of t, so

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its tangent vector has the form $(\alpha, \beta, z(t))$. Its x & y components, so, it always points in the same plane. Thus, the curve is planar.

Hermite Straight Segments

This section describes variations on Hermite straight segments. to look the detail where two extreme tangents are given in the same direction but have different magnitudes.

We denote them by

$$P_1^t = \alpha(P_2 - P_1)$$

$$\& \quad P_2^t = \beta(P_2 - P_1)$$

where α & β can be any real numbers.

$$P(t) = (P_2 - P_1)t + P_1 \longrightarrow (a)$$

This Eq (a). is obtained in the special case $\alpha = \beta = 1$ (same as in case 2 in perviours section).

The Hermite cubic curve segment in the terms of basis

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function is expressed as

$$P(t) = F(t)\, B$$

$$P(t) = \left(F_1(t), F_2(t), F_3(t), F_4(t)\right) \begin{bmatrix} P_1 \\ P_2 \\ \alpha(P_2 - P_1) \\ \beta(P_2 - P_1) \end{bmatrix}$$

$$P(t) = F_1(t) P_1 + F_2(t) P_2 + \alpha F_3(t)(P_2 - P_1) + \beta F_4(t)(P_2 - P_1)$$

$$P(t) = F_1(t) P_1 + \underline{F_2(t) P_1 - F_2(t) P_1} + F_2(t) P_2 + (\alpha F_3(t) + \beta F_4(t)) \cdot (P_2 - P_1)$$

$$P(t) = \underbrace{(F_1(t) + F_2(t))}_{1} P_1 + (P_2 - P_1) F_2(t) + (\alpha F_3(t) + \beta F_4(t))(P_2 - P_1)$$

$$P(t) = P_1 + (F_2(t) + \alpha F_3(t) + \beta F_4(t))(P_2 - P_1)$$

$$P(t) = P_1 + \left[(-2t^3 + 3t^2) + \alpha(t^3 - 2t^2 + t) + \beta(t^3 - t^2)\right](P_2 - P_1)$$

$$P(t) = P_1 + \left[(\alpha + \beta - 2)t^3 + (-2\alpha - \beta + 3)t^2 + \alpha t\right](P_2 - P_1)$$

This has the form $P(t) = P_1 + G(t)(P_2 - P_1)$

Where $G(t) = F_2(t) + \alpha F_3(t) + \beta F_4(t)$

or $G(t) = (\alpha + \beta - 2)t^3 + (-2\alpha - \beta + 3)t^2 + \alpha t$

This Eq. shows that all the points of the curve $P(t)$ lie on the straight line that passes through P_1 & has the tangent vector $(P_2 - P_1)$.

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We have to discuss,

Case	1	2	3	4	5	6	7	8	9
α	1	0	-1	>0	<0	>0	<0	≤ 0	≥ 0
β	1	0	-1	>0	<0	≤ 0	≥ 0	>0	<0

$$P(t) = P_1 + [(\alpha+\beta-2)t^3 + (-2\alpha-\beta+3)t^2 + \alpha t](P_2 - P_1) \rightarrow (A)$$

Case 1: when $\alpha = \beta = 1$

$$Eq(A) \Rightarrow P(t) = P_1 + [\cancel{(1+1-2)t^3} + \cancel{(-2-1+3)t^2} + t](P_2 - P_1)$$

$$P(t) = P_1 + (P_2 - P_1)t$$

This is a straight segment from P_1 to P_2.

Case 2:- When $\alpha = \beta = 0$.

$$Eq(A) \Rightarrow P(t) = P_1 + [(0+0-2)t^3 + (0+0+3)t^2 + 0t](P_2 - P_1)$$

$$P(t) = P_1 + (-2t^3 + 3t^2)(P_2 - P_1)$$

This is also a straight segment from P_1 to P_2 but moving at a variable speed. It accelerates up to point $P(0.5)$, then decelerates.

Case 3:- when $\alpha = \beta = -1$

$$Eq. (A) \Rightarrow P(t) = P_1 + [(-1-1-2)t^3 + (2+1+3)t^2 - t](P_2 - P_1)$$

$$P(t) = P_1 + (-4t^3 + 6t^2 - t)(P_2 - P_1)$$

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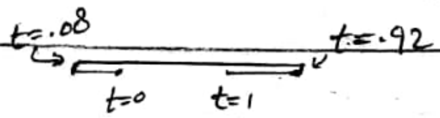
which is a curve shown as



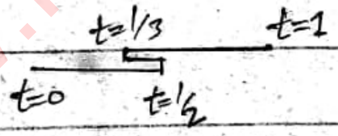
It consists of three straight segments.

Case:4:- When $\alpha > 0$, $\beta > 0$. As an example, we try the values $\alpha = 2$ & $\beta = 4$

Eq. (A) $\Rightarrow$ $P(t) = P_1 + [(2+4-2)t^3 + (-4-4+3)t^2 + 2t](P_2 - P_1)$

$$P(t) = P_1 + (4t^3 - 5t^2 + 2t)(P_2 - P_1)$$

This curve also consists of three straight segments, but it behaves differently.



Case: 5:- When $\alpha < 0$, $\beta < 0$. As an example, we try the values $\alpha = -2$ & $\beta = -4$.

Eq. (A) $\Rightarrow$ $P(t) = P_1 + [(-2-4-2)t^3 + (4+4+3)t^2 - 4t](P_2 - P_1)$

$$P(t) = P_1 + (-8t^3 + 11t^2 - 4t)(P_2 - P_1)$$

This curve again consists of three straight segments.

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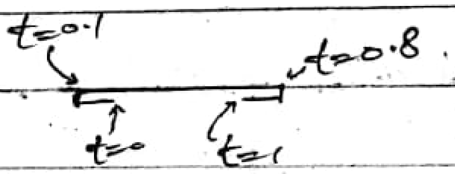

Case: 6 = When $\alpha > 0$, $\beta \leq 0$. As an example we try the values $\alpha = 2$ and $\beta = -4$.

$$\text{Eq.}(A) \Rightarrow p(t) = P_1 + [(\cancel{2}-4-\cancel{2})t^3 + (-\cancel{4}+\cancel{4}+3)t^2 + 2t](P_2 - P_1)$$

$$p(t) = P_1 + (-4t^3 + 3t^2 + 2t)(P_2 - P_1)$$

Case 7:– when $\alpha < 0$, $\beta \geq 0$.
As an example we try the values $\alpha = -2$ and $\beta = 4$.

$$\text{Eq.}(A) \Rightarrow p(t) = P_1 + [\cancel{(-2+4-2)}t^3 + (\cancel{4}-\cancel{4}+3)t^2 - 2t](P_2 - P_1)$$

$$p(t) = P_1 + (3t^2 - 2t)(P_2 - P_1)$$

Case 8:– When $\alpha \leq 0$, $\beta > 0$. As an example we try the values $\alpha = 0$ & $\beta = 4$

$$\text{Eq.}(A) \Rightarrow p(t) = P_1 + [(4-2)t^3 + (0-4+3)t^2 + 0t](P_2 - P_1)$$

$$p(t) = P_1 + (2t^3 - t^2)(P_2 - P_1)$$

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Case 9:- When $\alpha \geq 0$, $\beta < 0$

Let $\alpha = 0$, $\beta = -1$

Eq (A) $\Rightarrow$

$$P(t) = P_1 + \left[(0-1-2)t^3 + (0+1+3)t^2 + 0\,t\right](P_2 - P_1)$$

$$P(t) = P_1 + (-3t^3 + 4t^2)(P_2 - P_1)$$

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Chap# 5 Spline Interpolation 12-05-2015

Definition:-

A spline is a set of polynomials of degree K that are smoothly connected at certain data points. At each data point, two polynomials connect and their first derivatives (tangent vectors) have the same values. The definition also requires that all their derivatives up to the $(K-1)$th be the same at the point.

The Cubic Spline Curve

The cubic spline was originally introduced by James Ferguson. Given "n" data points that are numbered P_1 through P_n, there $n-1$ curves that pass through all the points in order of their numbers, but the eye often tends to trace one imaginary smooth curve through the points.

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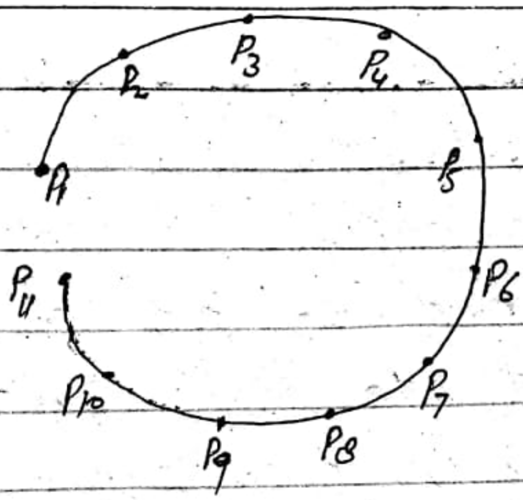
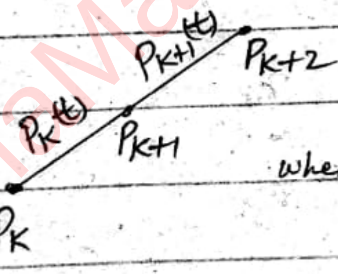


Given the "n" data points P_1, P_2 through P_n $(n=11)$, we look $n-1$ parametric cubic curves $P_1(t), P_2(t), \dots, P_{n-1}(t)$ such that $P_K(t)$ is the polynomial segment from point P_K to point P_{K+1} & $P_{K+1}(t)$ is another polynomial segment from point P_{K+1} to point P_{K+2}. where $k = 1, 2, \dots, n-2$.



where $K = 1, 2, \dots, n-2$.

The PCs will have to be smoothly connected at the $n-2$ interior points $(P_2, P_3, \dots, P_{n-1})$ which means that their first derivatives will have to match at every interior point. The definition of a spline requires that their second derivatives math too.

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The principle of cubic splines is to divide the set of n points into $n-1$ overlapping pairs of two points ~~sand~~ and to fit a Hermite curve segment to each pair. The pairs are (P_1, P_2), (P_2, P_3), $\cdots \to (P_{n-1}, P_n)$.

Recall that a Hermite curve segment is specified by two points and two tangent vectors. In order for curve segments $P_K(t)$ and $P_{K+1}(t)$ to connect smoothly at interior point P_{K+1}. For this case points are given but tangent vectors are unknown.

By using the result, the end tangent of $P_K(t)$ has to equal the start tangent of $P_{K+1}(t)$ at point P_{K+1}. Let

$$P_K(0) = P_K \quad , \quad P_K(1) = P_{K+1}$$

$$\& \quad P_K^t(0) = P_K^t \quad , \quad P_K^t(1) = P_{K+1}^t$$

Since the cubic curve segment in terms of 4-unknowns is

$$P_K(t) = a_K t^3 + b_K t^2 + c_K t + d_K \longrightarrow (A)$$

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$\Rightarrow \quad P_k(0) = d_k$

$$\boxed{P_k = d_k} \longrightarrow (1)$$

$\Rightarrow \quad P_k(1) = a_k + b_k + c_k + d_k$

$$P_{k+1} - P_k = a_k + b_k + c_k \longrightarrow (2)$$

$\Rightarrow \quad P_k^t(t) = 3a_k t^2 + 2b_k t + c_k$

$\Rightarrow \quad P_k^t(0) = c_k$

$$\boxed{P_k^t = c_k} \longrightarrow (3)$$

$\Rightarrow \quad P_k^t(1) = 3a_k + 2b_k + c_k$

$\Rightarrow \quad P_{k+1}^t - P_k^t = 3a_k + 2b_k \longrightarrow (4)$

Eq (2) $\Rightarrow \quad P_{k+1} - P_k = a_k + b_k + P_k^t$

$$a_k + b_k = P_{k+1} - P_k - P_k^t$$

$$b_k = P_{k+1} - P_k - P_k^t - a_k \longrightarrow (5)$$

Put (5) in (4).

$$3a_k + 2P_{k+1} - 2P_k - 2P_k^t - 2a_k = P_{k+1}^t - P_k^t$$

$$\Rightarrow \boxed{a_k = 2P_k - 2P_{k+1} + P_k^t + P_{k+1}^t}$$

use in (5).

$$b_k = P_{k+1} - P_k - P_k^t - 2P_k + 2P_{k+1} - P_k^t - P_{k+1}^t$$

$$\Rightarrow \boxed{b_k = -3P_k + 3P_{k+1} - 2P_k^t - P_{k+1}^t}$$

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Put these values in Eq. (A).

$$P_k(t) = (2(P_k - P_{k+1}) + P_k^t + P_{k+1}^t)t^3 + (3(P_{k+1} - P_k) - 2P_k^t - P_{k+1}^t)t^2 + P_k^t\, t + P_k \longrightarrow (B)$$

For the graphically display of Eq. (B) still tangent vectors are unknown.

In similar way we can calculate next segment $P_{k+1}(t)$ (from P_{k+1} to P_{k+2}) as

$$P_{k+1}(t) = [2(P_{k+1} - P_{k+2}) + P_{k+1}^t + P_{k+2}^t]t^3 + [3(P_{k+2} - P_{k+1}) - 2P_{k+1}^t - P_{k+2}^t]t^2 + P_{k+1}^t\, t + P_{k+1} \longrightarrow (C)$$

Case 1:

$$\boxed{P_k(1) = P_{k+1}(0) \quad (C^0)}$$

This Eq. can easily be satisfied.

$$a_k + b_k + c_k + d_k = d_{k+1}$$

by putting the values, we get

$$2P_k - 2P_{k+1} + \cancel{P_k^t} + \cancel{P_{k+1}^t} + 3P_{k+1} - 3P_k - 2P_k^t - \cancel{P_{k+1}^t} + P_k^t + P_k = P_{k+1}$$

$$\cancel{3P_k} - \cancel{3P_k} + 3P_{k+1} - 2P_{k+1} + \cancel{2P_k^t} - \cancel{2P_k^t} = P_{k+1}$$

$$\Rightarrow P_{k+1} = P_{k+1}$$

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Case. 2:-

$$P_k^t(1) = P_{k+1}^t(0) \quad (C^1)$$

This Eq. can easily satisfied.

from (A) $P_k^t(t) = 3a_k t^2 + 2b_k t + c_k$

So, $3a_k + 2b_k + c_k = c_{k+1}$

$$3[2P_k - 2P_{k+1} + P_k^t + P_{k+1}^t] + 2[3P_{k+1} - 3P_k - 2P_k^t - P_{k+1}^t] + P_k^t = P_{k+1}^t$$

$$\cancel{6P_k} - \cancel{6P_{k+1}} + 3P_k^t + 3P_{k+1}^t + \cancel{6P_{k+1}} - \cancel{6P_k} - 4P_k^t - 2P_{k+1}^t + P_k^t = P_{k+1}^t$$

$$\cancel{-4P_k^t} + \cancel{4P_k^t} + 3P_{k+1}^t - 2P_{k+1}^t = P_{k+1}^t$$

$$\Rightarrow P_{k+1}^t = P_{k+1}^t$$

Case 3:- $P_k^{tt}(1) = P_{k+1}^{tt}(0) \quad (C^2)$

We use case 3 in which the second derivatives of the two curve segments be equal at interior points such that $P_k^{tt}(1) = P_{k+1}^{tt}(0)$

$P_k^{tt}(t) = 6a_k t + 2b_k$

$$\Rightarrow 6a_k + 2b_k = 2b_{k+1}$$

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$$3a_k + b_k = b_{k+1}$$

$$3[2P_k - 2P_{k+1} + P_k^t + P_{k+1}^t] + 3P_{k+1} - 3P_k - 2P_k^t - P_{k+1}^t = 3P_{k+2} - 3P_{k+1} - 2P_{k+1}^t - P_{k+2}^t$$

$$6P_k - 6P_{k+1} + 3P_k^t + 3P_{k+1}^t - 2P_k^t - P_{k+1}^t + 2P_{k+1}^t + P_{k+2}^t = 3P_{k+2} - 3P_{k+1} - 3P_{k+1} + 3P_k$$

$$P_k^t + 4P_{k+1}^t + P_{k+2}^t = 3P_{k+2} - \cancel{6P_{k+1}} + 3P_k - 6P_k + \cancel{6P_{k+1}}$$

$$P_k^t + 4P_{k+1}^t + P_{k+2}^t = 3(P_{k+2} - P_k) \longrightarrow (D)$$

where $k = 1, 2, 3, \ldots, n-2$

we can write it in matrix form as

$$n-2\left\{\underbrace{\begin{bmatrix} 1 & 4 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & & 1 & 4 & 1 \end{bmatrix}}_{n}\right. \begin{bmatrix} P_1^t \\ P_2^t \\ \vdots \\ P_n^t \end{bmatrix} = \begin{bmatrix} 3(P_3 - P_1) \\ 3(P_4 - P_2) \\ \vdots \\ 3(P_n - P_{n-2}) \end{bmatrix} \longrightarrow (D)$$

Eq. (D) is the system of $n-2$ equations in the n-unknowns P_1^t, $P_2^t, \ldots, P_n^t$. A practical approach to the solution is to let the user specify the values of two extreme tangents P_1^t.

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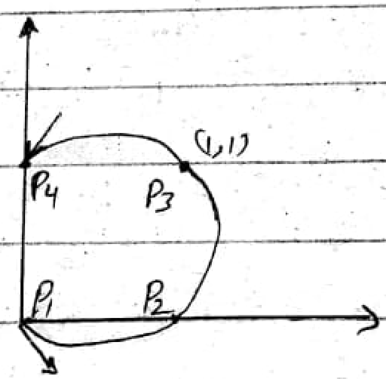
14-05-15

and P_n^t. Once these values have been substituted in Eq.(D) it is easy to solve it and obtain values for the remaining $n-2$ tangent vectors, P_2^t through P_{n-1}^t. These tangent vectors are now used to calculate the original coefficients a, b, c and d of each segment.

We use these tangents $n-1$ times, once for each segment of the spline.

Example:- Given the four points $P_1=(0,0)$, $P_2=(1,0)$, $P_3=(1,1)$ & $P_4=(0,1)$.



We are looking for three Hermite curve segments $P_1(t)$, $P_2(t)$ & $P_3(t)$ that will connect smoothly at the two interior points P_2 & P_3

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and will constitute the spline with given initial tangent vector $P_1^t = (1,-1)$ and final ~~tang~~ tangent vector $P_4^t = (-1,-1)$.

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \end{pmatrix}\begin{pmatrix} P_1^t \\ P_2^t \\ P_3^t \\ P_4^t \end{pmatrix} = \begin{bmatrix} 3(P_3 - P_1) \\ 3(P_4 - P_2) \end{bmatrix}$$

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \end{pmatrix}\begin{pmatrix} (1,-1) \\ P_2^t \\ P_3^t \\ (-1,-1) \end{pmatrix} = \begin{pmatrix} 3[(1,1)-(0,0)] \\ 3[(0,1)-(1,0)] \end{pmatrix}$$

$$\begin{pmatrix} (1,-1) + 4P_2^t + P_3^t \\ P_2^t + 4P_3^t + (-1,-1) \end{pmatrix} = \begin{pmatrix} 3(1,1) \\ 3(-1,1) \end{pmatrix}$$

$$\Rightarrow (1,-1) + 4P_2^t + P_3^t = (3,3) \longrightarrow (i)$$

$$\& \quad P_2^t + 4P_3^t + (-1,-1) = (-3,3) \longrightarrow (ii)$$

$$(i) \Rightarrow \quad 4P_2^t + P_3^t = (2,4) \longrightarrow (iii)$$

$$(ii) \Rightarrow \quad P_2^t + 4P_3^t = (-2,4) \longrightarrow (iv)$$

By $(iii) - [4 \times (iv)]$

$$\begin{aligned} \cancel{4P_2^t} + P_3^t &= (2,4) \\ \cancel{4P_2^t} + 16P_3^t &= (-8,16) \\ \hline -15P_3^t &= (10,-12) \end{aligned}$$

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$$\Rightarrow \boxed{P_3^t = \left(\frac{-2}{3}, \frac{4}{5}\right)}$$

Put in Eq (iii)

$$Eq(iii) \Rightarrow 4P_2^t = (2,4) - \left(\frac{-2}{3}, \frac{4}{5}\right)$$

$$4P_2^t = \left(2+\frac{2}{3}, 4-\frac{4}{5}\right)$$

$$P_2^t = \frac{1}{4}\left(\frac{8}{3}, \frac{16}{5}\right)$$

$$\boxed{P_2^t = \left(\frac{2}{3}, \frac{4}{5}\right)}$$

For first Hermite curve segment.

$$P_1(t) = (t^3, t^2, t, 1)\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_1^t \\ P_2^t \end{bmatrix}$$

$$P_1(t) = (t^3, t^2, t, 1)\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} (0,0) \\ (1,0) \\ (1,-1) \\ (2/3, 4/5) \end{bmatrix}$$

$$P_1(t) = (t^3, t^2, t, 1)\begin{bmatrix} (0,0)+(-2,0)+(1,-1)+\left(\frac{2}{3}, \frac{4}{5}\right) \\ (0,0)+(3,0)+(-2,2)+\left(\frac{-2}{3}, \frac{-4}{5}\right) \\ (1,-1) \\ (0,0) \end{bmatrix}$$

$$P_1(t) = \left(\frac{-1}{3}, \frac{-1}{5}\right)t^3 + \left(\frac{1}{3}, \frac{6}{5}\right)t + (1,-1)t$$

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For $P_2(t)$:-

$$P_2(t) = (t^3, t^2, t, 1) \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_2 \\ P_3 \\ P_2^t \\ P_3^t \end{bmatrix}$$

$$P_2(t) = (t^3, t^2, t, 1) \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} (1,0) \\ (1,1) \\ (2/3, 4/5) \\ (-2/3, 4/5) \end{bmatrix}$$

$$P_2(t) = (t^3, t^2, t, 1) \begin{bmatrix} (2,0)+(-2,-2)+(\frac{2}{3}, \frac{4}{5})+(\frac{-2}{3}, \frac{4}{5}) \\ (-3,0)+(3,3)+(-4/3, -8/5)+(2/3, -4/5) \\ (\frac{2}{3}, 4/5) \\ (1,0) \end{bmatrix}$$

$$P_2(t) = (t^3, t^2, t, 1) \begin{bmatrix} (0, -2/5) \\ (-2/3, 3/5) \\ (2/3, 4/5) \\ (1,0) \end{bmatrix}$$

$$\Rightarrow P_2(t) = (0, \frac{-2}{5})t^3 + (\frac{-2}{3}, \frac{3}{5})t^2 + (\frac{2}{3}, 4/5)t + (1,0)$$

For $P_3(t)$:-

$$P_3(t) = T(t)\ M \begin{bmatrix} P_3 \\ P_4 \\ P_3^t \\ P_4^t \end{bmatrix}$$

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$$P_3(t) = T(t)\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} (1,1) \\ (0,1) \\ (\frac{-2}{3}, \frac{4}{5}) \\ (-1,-1) \end{bmatrix}$$

$$P_3(t) = T(t)\begin{bmatrix} (2,2)+(0,-2)+(\frac{-2}{3}, \frac{4}{5})+(-1,-1) \\ (-3,-3)+(0,3)+(\frac{4}{3}, \frac{-8}{5})+(1,1) \\ (\frac{-2}{3}, \frac{4}{5}) \\ (1,1) \end{bmatrix}$$

$$P_3(t) = T(t)\begin{bmatrix} (\frac{1}{3}, \frac{-1}{5}) \\ (\frac{-2}{3}, \frac{-3}{5}) \\ (\frac{-2}{3}, \frac{4}{5}) \\ (1,1) \end{bmatrix}$$

$$\Rightarrow P_3(t) = (\tfrac{1}{3}, \tfrac{-1}{5})t^3 + (\tfrac{-2}{3}, \tfrac{-3}{5})t^2 + (\tfrac{-2}{3}, \tfrac{4}{5})t + (1,1)$$

Relaxed Cubic Splines

19-05-15

The original approach to the cubic spline curve is for the user to specify the two extreme tangent vectors. This approach is

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known as the clamped end condition. It is possible to have different end conditions, and the one discribed in this section is based on the simple idea of setting (choosing) the two extreme second derivatives of the curve $P_1^{tt}(0)$ and $P_{n-1}^{tt}(1)$ to zero.

If we think of the second derivative as the acceleration of the curve, then this end condition implies constant speeds and therefore small curvatures at both ends of the curve. This is why this end condition is called relaxed."

It is easy to calculate the relaxed cubic spline. The second derivative of the parametric cubic $P(t)$ is

$$P^{tt}(t) = 6at + 2b$$

Using the first end condition

$$P_1^{tt}(0) = 0 = 6a_1(0) + 2b_1$$

$$\Rightarrow \quad b_1 = 0$$

from perivious lecture

Since $\quad b_K = 3(P_{K+1} - P_K) - 2P_K^t - P_{K+1}^t$

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as $b_1 = 0$ so

$$3P_2 - 3P_1 - 2P_1^t - P_2^t = 0$$

$$2P_1^t = 3P_2 - 3P_1 - P_2^t$$

$$\boxed{P_1^t = \frac{3}{2}(P_2 - P_1) - \frac{1}{2}P_2^t}$$

Using the other end condition

$$P_{n-1}^{tt}(1) = 0$$

$$\Rightarrow 6a_{n-1} + 2b_{n-1} = 0$$

$$3a_{n-1} + b_{n-1} = 0$$

$$3(2P_{n-1} - 2P_n + P_{n-1}^t + P_n^t) + (-3P_{n-1} + 3P_n - 2P_{n-1}^t - P_n^t) = 0$$

$$6P_{n-1} - 6P_n + 3P_{n-1}^t + 3P_n^t - 3P_{n-1} + 3P_n - 2P_{n-1}^t - P_n^t = 0$$

$$2P_n^t + 3P_{n-1} - 3P_n + P_{n-1}^t = 0$$

$$\Rightarrow \boxed{P_n^t = \frac{3}{2}[P_n - P_{n-1}] - \frac{1}{2}P_{n-1}^t}$$

Substituting these values in Eq (1)

$$\begin{bmatrix} 1 & 4 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 1 & 4 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2}(P_2 - P_1) - \frac{1}{2}P_2^t \\ P_2^t \\ \vdots \\ P_{n-1}^t \\ \frac{3}{2}(P_n - P_{n-1}) - \frac{1}{2}P_{n-1}^t \end{bmatrix} = \begin{bmatrix} 3(P_3 - P_1) \\ 3(P_4 - P_2) \\ \vdots \\ 3(P_{n-1} - P_{n-3}) \\ 3(P_n - P_{n-2}) \end{bmatrix}$$

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This is a system of $n-2$ Equation in the $n-2$ unknowns $P_2^t, P_3^t, \ldots, P_{n-1}^t$. So it is easy to calculate the relaxed cubic spline by using any numerical approach.

Example:- Given that the same four points of perivious given example ($P_1=(0,0)$, $P_2=(1,0)$, $P_3=(1,1)$, $P_4=(0,1)$) to calculate the relaxed cubic spline and compare the result with cubic spline at $t=0.5$.

Solution:-

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2}(P_2-P_1)-\frac{1}{2}P_2^t \\ P_2^t \\ P_3^t \\ \frac{3}{2}(P_4-P_3)-\frac{1}{2}P_3^t \end{pmatrix} = \begin{pmatrix} 3(P_3-P_1) \\ 3(P_4-P_2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2}(1,0)-\frac{1}{2}P_2^t \\ P_2^t \\ P_3^t \\ \frac{3}{2}(-1,0)-\frac{1}{2}P_3^t \end{pmatrix} = \begin{pmatrix} 3(1,1) \\ 3(-1,1) \end{pmatrix}$$

$$\left(\frac{3}{2},0\right)-\frac{1}{2}P_2^t+4P_2^t+P_3^t=(3,3) \longrightarrow (i)$$

$$P_2^t+4P_3^t+\left(\frac{-3}{2},0\right)-\frac{1}{2}P_3^t=(-3,3) \longrightarrow (ii)$$

$$(i) \Rightarrow \quad \frac{7}{2}P_2^t+P_3^t=\left(\frac{3}{2},3\right) \longrightarrow (iii)$$

$$(ii) \Rightarrow \quad P_2^t+\frac{7}{2}P_3^t=\left(\frac{-3}{2},3\right) \longrightarrow (iv)$$

$\frac{7}{2}\times(iii) \Rightarrow \quad \frac{49}{4}P_2^t + \frac{7}{2}P_3^t = \left(\frac{21}{4}, \frac{21}{2}\right)$

$\quad - P_2^t \pm \cancel{\frac{7}{2}P_3^t} = \left(\frac{-3}{2}, 3\right)$

$$\frac{45}{4}P_2^t = \left(\frac{27}{4}, \frac{15}{2}\right)$$

$$P_2^t = \left(\frac{27}{45}, \frac{30}{45}\right)$$

$$\boxed{P_2^t = \left(\frac{3}{5}, \frac{2}{3}\right)}$$

Put in Eq. (iii)

$$\frac{7}{2}\left(\frac{3}{5}, \frac{2}{3}\right) + P_3^t = \left(\frac{3}{2}, 3\right)$$

$$\left(\frac{21}{10}, \frac{7}{3}\right) + P_3^t = \left(\frac{3}{2}, 3\right)$$

$$P_3^t = \left(\frac{3}{2} - \frac{21}{10}, 3 - \frac{7}{3}\right)$$

$$P_3^t = \left(\frac{-6}{10}, \frac{2}{3}\right)$$

$$\boxed{P_3^t = \left(\frac{-3}{5}, \frac{2}{3}\right)}$$

Now $\quad P_1^t = \frac{3}{2}(P_2 - P_1) - \frac{1}{2}P_2^t$

$$P_1^t = \frac{3}{2}(1,0) - \frac{1}{2}\left(\frac{3}{5}, \frac{2}{3}\right)$$

$$P_1^t = \frac{1}{2}\left[(3,0) - \left(\frac{3}{5}, \frac{2}{3}\right)\right]$$

$$P_1^t = \frac{1}{2}\left[\left(\frac{12}{5}, \frac{-2}{3}\right)\right]$$

$$\boxed{P_1^t = \left(\frac{6}{5}, \frac{-1}{3}\right)}$$

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Now

$$P_4^t = \frac{3}{2}(P_4 - P_3) - \frac{1}{2} P_3^t$$

$$= \frac{3}{2}(-1, 0) - \frac{1}{2}\left(-\frac{3}{5}, \frac{2}{3}\right)$$

$$= \frac{1}{2}\left[(-3, 0) + \left(\frac{3}{5}, -\frac{2}{3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{-12}{5}, \frac{-2}{3}\right)$$

$$\boxed{P_4^t = \left(\frac{-6}{5}, -\frac{1}{3}\right)}$$

Now the value of all four tangent vectors are known and it is easy to calculate the three curve segments of relaxed cubic spline using the information.

For $P_1(t)$:-

$$P_1(t) = T(t)\, M \begin{pmatrix} P_1 \\ P_2 \\ P_1^t \\ P_2^t \end{pmatrix}$$

$$P_1(t) = T(t) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (0,0) \\ (1,0) \\ (6/5, -1/3) \\ (3/5, 2/3) \end{pmatrix}$$

$$P_1(t) = T(t) \begin{pmatrix} (0,0) + (-2,0) + (6/5, -1/3) + (3/5, 2/3) \\ (0,0) + (3,0) + \left(\frac{-12}{5}, 2/3\right) + (-3/5, -2/3) \\ (6/5, -1/3) \\ (0,0) \end{pmatrix}$$

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$$\Rightarrow P_1(t) = (t^3, t^2, t, 1)\begin{pmatrix} (-1/5, 1/3) \\ (0, 0) \\ (6/5, -1/3) \\ (0, 0) \end{pmatrix}$$

$$P_1(t) = (-1/5, 1/3)t^3 + (6/5, -1/3)t$$

For $P_2(t)$:-

$$P_2(t) = T(t)\, M \begin{pmatrix} P_2 \\ P_3 \\ P_2^t \\ P_3^t \end{pmatrix}$$

$$P_2(t) = T(t)\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} (1, 0) \\ (1, 1) \\ (3/5, 2/3) \\ (-3/5, 2/3) \end{pmatrix}$$

$$P_2(t) = T(t)\begin{pmatrix} (2,0) + (-2,-2) + (\frac{3}{5}, 2/3) + (\frac{-3}{5}, 2/3) \\ (-3,0) + (3,3) + (\frac{-6}{5}, -4/3) + (3/5, -2/3) \\ (3/5, 2/3) \\ (1, 0) \end{pmatrix}$$

$$P_2(t) = (t^3, t^2, t, 1)\begin{pmatrix} (0, -2/3) \\ (-3/5, 1) \\ (3/5, 2/3) \\ (1, 0) \end{pmatrix}$$

$$P_2(t) = (0, \tfrac{-2}{3})t^3 + (\tfrac{-3}{5}, 1)t^2 + (\tfrac{3}{5}, \tfrac{2}{3})t + (1, 0)$$

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For $P_3(t)$:-

$$P_3(t) = T(t)\, M \begin{pmatrix} P_3 \\ P_4 \\ P_3^t \\ P_4^t \end{pmatrix}$$

$$P_3(t) = T(t) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (1,1) \\ (0,1) \\ (-3/5, 2/3) \\ (-6/5, -1/3) \end{pmatrix}$$

$$P_3(t) = T(t) \begin{pmatrix} (2,2) + (0,-2) + (-3/5, 2/3) + (-6/5, -1/3) \\ (-3,-3) + (0,3) + (6/5, -4/3) + (6/5, 1/3) \\ (-3/5, 2/3) \\ (1,1) \end{pmatrix}$$

$$P_3(t) = (t^3, t^2, t, 1) \begin{pmatrix} (1/5, 1/3) \\ (-\frac{3}{5}, -1) \\ (-3/5, 2/3) \\ (1,1) \end{pmatrix}$$

$$P_3(t) = (1/5, 1/3)t^3 - (3/5, 1)t^2 + (\frac{-3}{5}, 2/3)t + (1,1)$$

Cyclic Cubic Splines 20-05-15

The cyclic end condition is ideal for a closed cubic spline and also for a periodic cubic spline. The condition is that the tangent vectors be equal at the two extremes

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of the curve (i.e $P_1^t = P_n^t$) and the same for the second derivatives $P_1^{tt} = P_n^{tt}$. Notice that the curve doesn't have to be closed i.e a curve segment from P_n to P_1 is not required.

We know that the cubic spline curve is

$$P(t) = at^3 + bt^2 + ct + d \longrightarrow ①$$

$$P^t(t) = 3at^2 + 2bt + c \longrightarrow ②$$

Applying the given condition of cyclic cubic spline

$$P_1^t(0) = P_{n-1}^t(1)$$

$$c_1 = 3a_{n-1} + 2b_{n-1} + c_{n-1} \longrightarrow ③$$

Applying second condition of cyclic cubic spline

$$P_1^{tt}(0) = P_{n-1}^{tt}(1) \qquad \because P^{tt}(t) = 6at + 2b$$

$$2b_1 = 6a_{n-1} + 2b_{n-1} \longrightarrow ④$$

Subtracting ④ from ③ we get.

$$c_1 - 2b_1 = -3a_{n-1} + c_{n-1} \longrightarrow ⑤$$

For values see page 102.

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Eq. (5) can be written as

$$P_1^t - 2\left[3(P_2-P_1) - 2P_1^t - P_2^t\right] = -3\left[2(P_{n-1}-P_n) + P_{n-1}^t + P_n^t\right] + P_{n-1}^t$$

$$P_1^t - 6P_2 + 6P_1 + 4P_1^t + 2P_2^t = -6P_{n-1} + 6P_n - 3P_{n-1}^t - 3P_n^t + P_{n-1}^t$$

$$5P_1^t + 3P_n^t = 6P_2 - 6P_1 + 6P_n - 6P_{n-1} - 2P_2^t - 2P_{n-1}^t$$

Since $P_1^t = P_n^t$

$$P_1^t = P_n^t = \frac{3}{4}(P_2 - P_1 + P_n - P_{n-1}) - \frac{1}{4}(P_2^t + P_{n-1}^t)$$

The system of Eq's (5) given in Eq. (D) becomes as

$$\begin{bmatrix} 1 & 4 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & & 1 & 4 & 1 & 0 \\ 0 & \cdots & & 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{4}(P_2-P_1+P_n-P_{n-1}) - \frac{1}{4}(P_2^t - P_{n-1}^t) \\ P_2^t \\ \vdots \\ P_{n-1}^t \\ \frac{3}{4}(P_2-P_1+P_n-P_{n-1}) - \frac{1}{4}(P_2^t - P_{n-1}^t) \end{bmatrix} = \begin{bmatrix} 3(P_3-P_1) \\ 3(P_4-P_2) \\ \vdots \\ 3(P_{n-1}-P_{n-3}) \\ 3(P_n-P_{n-2}) \end{bmatrix}$$

This system $n-2$ Equation and $n-2$ unknowns. It is easy to solve the system by using any numerical approach.

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Example:-

Given five points $P_1 = P_5 = (0,-1)$, $P_2 = (1,0)$, $P_3 = (0,1)$ & $P_4 = (-1,0)$.

Calculate the cubic spline with the cyclic end condition for these points. Notice that the curve is closed since $P_1 = P_5$.

$$\begin{bmatrix} 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{4}(P_2 - P_1 + P_5 - P_4) - \frac{1}{4}(P_2^t + P_4^t) \\ P_2^t \\ P_3^t \\ P_4^t \\ \frac{3}{4}(P_2 - P_1 + P_5 - P_4) - \frac{1}{4}(P_2^t + P_4^t) \end{bmatrix} = \begin{bmatrix} 3(P_3 - P_1) \\ 3(P_4 - P_2) \\ 3(P_5 - P_3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{4}((1,0) - (-1,0)) - \frac{1}{4}(P_2^t + P_4^t) \\ P_2^t \\ P_3^t \\ P_4^t \\ \frac{3}{4}(2,0) - \frac{1}{4}(P_2^t + P_4^t) \end{bmatrix} = \begin{bmatrix} 3(0,2) \\ 3(-2,0) \\ 3(0,-2) \end{bmatrix}$$

$$\Rightarrow (\tfrac{3}{2},0) - \tfrac{1}{4}(P_2^t + P_4^t) + 4P_2^t + P_3^t = (0,6) \longrightarrow (i)$$

$$P_2^t + 4P_3^t + P_4^t = (-6,0) \longrightarrow (ii)$$

$$P_3^t + 4P_4^t + (\tfrac{3}{2},0) - \tfrac{1}{4}P_2^t - \tfrac{1}{4}P_4^t = (0,-6) \longrightarrow (iii)$$

$$(i) \Rightarrow \quad \tfrac{15}{4}P_2^t + P_3^t - \tfrac{1}{4}P_4^t = (\tfrac{-3}{2}, 6)$$

xing by 4 $\quad 15P_2^t + 4P_3^t - P_4^t = (-6, 24) \longrightarrow (iv)$

$$(ii) \Rightarrow \quad P_2^t + 4P_3^t + P_4^t = (-6, 0) \longrightarrow (v)$$

$$(iii) \Rightarrow \quad -\tfrac{1}{4}P_2^t + P_3^t + \tfrac{15}{4}P_4^t = (\tfrac{-3}{2}, -6)$$

-4 xing $\quad P_2^t - 4P_3^t - 15P_4^t = (6, 24) \longrightarrow (vi)$

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By adding (iii) & (v)

$$16P_2^t + 8P_3^t = (-12, 24)$$

$\div$ ing by 4 $\Rightarrow$ $4P_2^t + 2P_3^t = (-3, 6) \longrightarrow$ (vii)

$15\times(v) \Rightarrow$ $15P_2^t + 60P_3^t + \cancel{15P_4^t} = (-90, 0)$

(vi) $\Rightarrow$ $P_2^t - 4P_3^t - \cancel{15P_4^t} = (6, 24)$

$$16P_2^t + 56P_3^t = (-84, 24)$$

$\div$ ing by 4 $\Rightarrow$ $4P_2^t + 14P_3^t = (-21, 6) \longrightarrow$ (viii)

By Eq. (vii) $-$ Eq. (viii)

$$-12P_3^t = (18, 0)$$

$$\boxed{P_3^t = \left(\frac{-3}{2}, 0\right)}$$

Put in Eq. (vii)

$$4P_2^t + (-3, 0) = (-3, 6)$$

$$P_2^t = \frac{1}{4}(0, 6)$$

$$\boxed{P_2^t = (0, 3/2)}$$

Put in Eq.

$$\left(0, \frac{3}{2}\right) + 4\left(\frac{-3}{2}, 0\right) + P_4^t = (-6, 0)$$

$$\left(0, \frac{3}{2}\right) + (\cancel{-6}, 0) + P_4^t = (\cancel{-6}, 0)$$

$$\boxed{P_4^t = (0, -3/2)}$$

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Now

$$P_1^t = P_5^t = \frac{3}{4}(P_2 - \cancel{P_1} + \cancel{P_5} - P_4) - \frac{1}{4}(P_2^t + P_4^t)$$

$$P_1^t = P_5^t = \frac{3}{4}(2,0) - \frac{1}{4}\left[(0,\cancel{\tfrac{3}{2}}) + (0,\cancel{-\tfrac{3}{2}})\right]$$

$$P_1^t = P_5^t = \left(\frac{3}{2}, 0\right)$$

For $P_1(t)$:-

$$P_1(t) = T(t)\, M \begin{bmatrix} P_1 \\ P_2 \\ P_1^t \\ P_2^t \end{bmatrix}$$

$$P_1(t) = T(t) \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} (0,-1) \\ (1,0) \\ (3/2,0) \\ (0,3/2) \end{bmatrix}$$

$$P_1(t) = T(t) \begin{bmatrix} (0,-2)+(-2,0)+(\frac{3}{2},0)+(0,\frac{3}{2}) \\ (0,3)+(3,0)+(-3,0)+(0,-\frac{3}{2}) \\ (3/2,0) \\ (0,-1) \end{bmatrix}$$

$$P_1(t) = (t^3, t^2, t, 1) \begin{bmatrix} (-1/2, -1/2) \\ (0, 3/2) \\ (3/2, 0) \\ (0,-1) \end{bmatrix}$$

$$P_1(t) = -(1/2, 1/2)t^3 + (0, \tfrac{3}{2})t^2 + (\tfrac{3}{2}, 0)t + (0,-1)$$

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For $P_2(t)$:-

$$P_2(t) = T(t)\, M \begin{pmatrix} P_2 \\ P_3 \\ P_2^t \\ P_3^t \end{pmatrix}$$

$$P_2(t) = T(t) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} (1,0) \\ (0,1) \\ (0,\frac{3}{2}) \\ (-\frac{3}{2},0) \end{bmatrix}$$

$$P_2(t) = T(t) \begin{bmatrix} (2,0)+(0,-2)+(0,\frac{3}{2})+(-\frac{3}{2},0) \\ (-3,0)+(0,3)+(0,-3)+(\frac{3}{2},0) \\ (0,\frac{3}{2}) \\ (1,0) \end{bmatrix}$$

$$P_2(t) = (t^3, t^2, t, 1) \begin{bmatrix} (\frac{1}{2},-\frac{1}{2}) \\ (-\frac{3}{2},0) \\ (0,\frac{3}{2}) \\ (1,0) \end{bmatrix}$$

$$P_2(t) = \left(\frac{1}{2},-\frac{1}{2}\right)t^3 + \left(-\frac{3}{2},0\right)t^2 + \left(0,\frac{3}{2}\right)t + (1,0).$$

For $P_3(t)$:-

$$P_3(t) = T(t)\, M \begin{bmatrix} P_3 \\ P_4 \\ P_3^t \\ P_4^t \end{bmatrix}$$

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$$P_3(t) = T(t)\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} (0,1) \\ (-1,0) \\ (-\frac{3}{2},0) \\ (0,-\frac{3}{2}) \end{pmatrix}$$

$$P_3(t) = T(t)\begin{bmatrix} (0,2)+(2,0)+(-\frac{3}{2},0)+(0,-\frac{3}{2}) \\ (0,-3)+(-3,0)+(3,0)+(0,\frac{3}{2}) \\ (-\frac{3}{2},0) \\ (0,1) \end{bmatrix}$$

$$P_3(t) = (t^3, t^2, t, 1)\begin{bmatrix} (\frac{1}{2},\frac{1}{2}) \\ (0,-\frac{3}{2}) \\ (-\frac{3}{2},0) \\ (0,1) \end{bmatrix}$$

$$P_3(t) = (\tfrac{1}{2},\tfrac{1}{2})t^3 + (0,\tfrac{-3}{2})t^2 + (\tfrac{-3}{2},0)t + (0,1)$$

For $P_4(t)$:-

$$P_4(t) = T(t)\, M \begin{bmatrix} P_4 \\ P_5 \\ P_4^t \\ P_5^t \end{bmatrix}$$

$$P_4(t) = T(t)\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} (-1,0) \\ (0,-1) \\ (0,-\frac{3}{2}) \\ (\frac{3}{2},0) \end{pmatrix}$$

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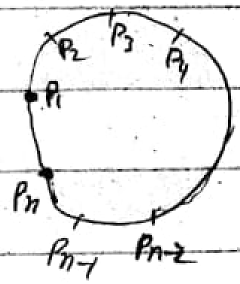
$$P_4(t) = T(t)\begin{bmatrix} (-2,0)+(0,2)+(0,-\frac{3}{2})+(\frac{3}{2},0) \\ (3,0)+(0,-3)+(0,3)+(-\frac{3}{2},0) \\ (0,-\frac{3}{2}) \\ (-1,0) \end{bmatrix}$$

$$P_4(t) = (t^3, t^2, t, 1)\begin{bmatrix} (-\frac{1}{2}, \frac{1}{2}) \\ (\frac{3}{2}, 0) \\ (0, -\frac{3}{2}) \\ (-1, 0) \end{bmatrix}$$

$$P_4(t) = (-\tfrac{1}{2}, \tfrac{1}{2})t^3 + (\tfrac{3}{2}, 0)t^2 + (0, -\tfrac{3}{2})t + (-1, 0)$$

Closed Cubic Spline 26-05-2015

A closed cubic spline has an extra curve segment from P_n to P_1 that closes the curve. In such a curve every point is interior, so system of Eqs (D) becomes a system of "n" equations in the same "n" unknowns.



No use input is needed, which implies that the only way to control or modify such a curve is to move, add or delete points. It is convenient

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to define the two additional points.

$$P_{n+1} \overset{\text{def.}}{=} P_1 \quad \& \quad P_{n+2} \overset{\text{def.}}{=} P_2$$

We know that

$$P_k^t + 4P_{k+1}^t + P_{k+2}^t = 3(P_{k+2} - P_k)$$

where $k = 1, 2, \ldots, n-2$

But we want to add two points in this system so the variation of k is $1, 2, \ldots, n$.

For $k = n-1$

$$P_{n-1}^t + 4P_n^t + P_{n+1}^t = 3(P_{n+1} - P_{n-1}) \longrightarrow ①$$

For $k = n$

$$P_n^t + 4P_{n+1}^t + P_{n+2}^t = 3(P_{n+2} - P_n) \longrightarrow ②$$

Note:- If two points are equal then their tangent vector are also equal But converse may or may not be true.

So,

$$P_{n+1} = P_1 \quad \& \quad P_{n+2} = P_2$$

$$\Rightarrow P_{n+1}^t = P_1^t \quad \& \quad P_{n+2}^t = P_2^t$$

Eq. ① $\Rightarrow$ $P_{n-1}^t + 4P_n^t + P_1^t = 3(P_1 - P_{n-1})$

$$P_1^t + 0P_2^t + 0P_3^t + \cdots + P_{n-1}^t + 4P_n^t = 3(P_1 - P_{n-1})$$

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$$② \Rightarrow P_n^t + 4P_1^t + P_2^t = 3(P_2 - P_n)$$

$$4P_1^t + P_2^t + 0P_3^t + \cdots + 0P_{n-1}^t + P_n^t = 3(P_2 - P_n)$$

By adding these values in the system ①.

$$\begin{bmatrix} 1 & 4 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & & \vdots \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1\;4 & 1 \\ 1 & 0 & 0 & \cdots & 0\;1 & 4 \\ 4 & 1 & 0 & \cdots & 0\;0 & 1 \end{bmatrix} \begin{bmatrix} P_1^t \\ P_2^t \\ \vdots \\ P_{n-1}^t \\ P_n^t \end{bmatrix} = \begin{bmatrix} 3(P_3 - P_1) \\ 3(P_4 - P_2) \\ \vdots \\ 3(P_1 - P_{n-1}) \\ 3(P_2 - P_n) \end{bmatrix}$$

This system has 'n' equations & same 'n' unknowns. So it is easy to solve by using any numerical approach.

Note: In this spline number of curve segments is equal to the number of points. But previous all splines have $n-1$ curve segments.

Example:- Given $P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (1,1)$ & $P_4 = (0,1)$. We are looking looking for 4-Hermite segments $P_1(t)$, $P_2(t)$, $P_3(t)$ & $P_4(t)$ that would connect smoothly at four

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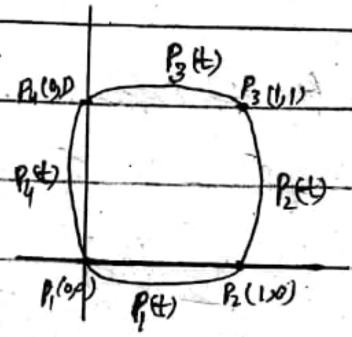
points.



We can write,

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 1 & 0 & 1 & 4 \\ 4 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_1^t \\ P_2^t \\ P_3^t \\ P_4^t \end{pmatrix} = \begin{bmatrix} 3(P_3 - P_1) \\ 3(P_4 - P_2) \\ 3(P_1 - P_3) \\ 3(P_2 - P_4) \end{bmatrix}$$

$$\begin{pmatrix} P_1^t + 4P_2^t + P_3^t \\ P_2^t + 4P_3^t + P_4^t \\ P_1^t + P_3^t + 4P_4^t \\ 4P_1^t + P_2^t + P_4^t \end{pmatrix} = \begin{bmatrix} 3(1,1) \\ 3(-1,1) \\ 3(-1,-1) \\ 3(1,-1) \end{bmatrix}$$

$\Rightarrow P_1^t + 4P_2^t + P_3^t = (3,3) \longrightarrow (1)$

$P_2^t + 4P_3^t + P_4^t = (-3,3) \longrightarrow (2)$

$P_1^t + P_3^t + 4P_4^t = (-3,-3) \longrightarrow (3)$

$4P_1^t + P_2^t + P_4^t = (3,-3) \longrightarrow (4)$

By $4 \times (1) - (2)$

$$\begin{aligned} 4P_1^t + 16P_2^t + 4P_3^t &= (12,12) \\ \pm P_2^t \pm 4P_3^t \pm P_4^t &= (-3,3) \\ \hline 4P_1^t + 15P_2^t - P_4^t &= (15,9) \longrightarrow (5) \end{aligned}$$

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By $4\times③ - ②$

$$4P_1^t + \cancel{4P_3^t} + 16P_4^t = (-12,-12)$$
$$\pm P_2^t \quad \pm \cancel{4P_3^t} \pm P_4^t = \mp(-3,3)$$

$$4P_1^t - P_2^t \quad +15P_4^t = (-9,-15) \rightarrow ⑥$$

By $⑤ - ④$

$$\cancel{4P_1^t} + 15P_2^t - P_4^t = (15,9)$$
$$\pm \cancel{4P_1^t} \pm P_2^t \pm P_4^t = \mp(3,-3)$$

$$14P_2^t - 2P_4^t = (12,12)$$

$$\Rightarrow 7P_2^t - P_4^t = (6,6)$$

$$\Rightarrow P_4^t = 7P_2^t - (6,6) \rightarrow ⑦$$

By $⑥ - ④$

$$\cancel{4P_1^t} - P_2^t + 15P_4^t = (-9,-15)$$
$$\pm \cancel{4P_1^t} \pm P_2^t \pm P_4^t = \pm(3,-3)$$

$$-2P_2^t + 14P_4^t = (-12,-12)$$

$$\Rightarrow P_2^t - 7P_4^t = (6,6) \rightarrow ⑧$$

Put ⑦ in ⑧

$$P_2^t - 49P_2^t + (42,42) = (6,6)$$

$$(42-6, 42-6) = 48P_2^t$$

$$(36,36) = 48P_2^t$$

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$$P_2^t = \left(\frac{36}{48}, \frac{36}{48}\right)$$

$$\boxed{P_2^t = \left(\frac{3}{4}, \frac{3}{4}\right)}$$

Put in (7)

$$P_4^t = 7\left(\frac{3}{4}, \frac{3}{4}\right) - (6, 6)$$

$$P_4^t = \left(\frac{21}{4} - 6, \frac{21}{4} - 6\right)$$

$$\boxed{P_4^t = \left(\frac{-3}{4}, \frac{-3}{4}\right)}$$

Put P_2^t & P_4^t in Eq. (4)

$$4P_1^t + \left(\frac{3}{4}, \cancel{\frac{3}{4}}\right) + \left(\cancel{\frac{-3}{4}}, \frac{-3}{4}\right) = (3, -3)$$

$$\Rightarrow \boxed{P_1^t = \left(\frac{3}{4}, \frac{-3}{4}\right)}$$

Put P_2^t & P_4^t in Eq. (2)

$$\left(\frac{3}{4}, \cancel{\frac{3}{4}}\right) + 4P_3^t + \left(\cancel{\frac{-3}{4}}, \frac{-3}{4}\right) = (-3, 3)$$

$$\boxed{P_3^t = \left(\frac{-3}{4}, \frac{3}{4}\right)}$$

For $P_1(t)$:-

$$P_1(t) = T(t)\, M \begin{bmatrix} P_1 \\ P_2 \\ P_1^t \\ P_2^t \end{bmatrix}$$

must be

$$P_1(t) = -\frac{1}{3}(P_1^t + P_2^t)t^3 + P_2^t t^2 + P_2^t t + P_1$$

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$$P_1(t) = T(t)\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{bmatrix} (0,0) \\ (1,0) \\ (\frac{3}{4},\frac{-3}{4}) \\ (\frac{3}{4},\frac{3}{4}) \end{bmatrix}$$

$$P_1(t) = T(t)\begin{bmatrix} (0,0)+(-2,0)+(\frac{3}{4},\frac{-3}{4})+(\frac{3}{4},\frac{3}{4}) \\ (0,0)+(3,0)+(\frac{-3}{2},\frac{3}{2})+(\frac{-3}{4},\frac{-3}{4}) \\ (\frac{3}{4},\frac{-3}{4}) \\ (0,0) \end{bmatrix}$$

$$P_1(t) = (t^3, t^2, t, 1)\begin{bmatrix} (-\frac{1}{2},0) \\ (\frac{3}{4},\frac{3}{4}) \\ (\frac{3}{4},\frac{-3}{4}) \\ (0,0) \end{bmatrix}$$

$$P_1(t) = (-\tfrac{1}{2},0)t^3 + (\tfrac{3}{4},\tfrac{3}{4})t^2 + (\tfrac{3}{4},\tfrac{-3}{4})t$$

For $P_2(t)$:

$$P_2(t) = T(t)\, M \begin{bmatrix} P_2 \\ P_3 \\ P_2^t \\ P_3^t \end{bmatrix}$$

$$P_2(t) = T(t)\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} (1,0) \\ (1,1) \\ (\frac{3}{4},\frac{3}{4}) \\ (\frac{-3}{4},\frac{3}{4}) \end{pmatrix}$$

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$$P_2(t) = T(t)\begin{bmatrix} (2,0)+(-2,-2)+(\frac{3}{4},\frac{3}{4})+(\frac{-3}{4},\frac{3}{4}) \\ (-3,0)+(3,3)+(\frac{-3}{2},\frac{-3}{2})+(\frac{3}{4},\frac{-3}{4}) \\ (\frac{-3}{4},\frac{3}{4}) \\ (1,0) \end{bmatrix}$$

$$P_2(t) = (t^3, t^2, t, 1)\begin{bmatrix} (0,-\frac{1}{2}) \\ (\frac{-3}{4},\frac{3}{4}) \\ (\frac{3}{4},\frac{3}{4}) \\ (1,0) \end{bmatrix}$$

$$P_2(t) = (0,-\tfrac{1}{2})t^3 + (\tfrac{-3}{4},\tfrac{3}{4})t^2 + (\tfrac{3}{4},\tfrac{3}{4})t + (1,0)$$

For $P_3(t)$:-

$$P_3(t) = T(t)M\begin{bmatrix} P_3 \\ P_4 \\ P_3^t \\ P_4^t \end{bmatrix}$$

$$P_3(t) = T(t)\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} (1,1) \\ (0,1) \\ (\frac{-3}{4},\frac{3}{4}) \\ (\frac{-3}{4},\frac{-3}{4}) \end{pmatrix}$$

$$P_3(t) = T(t)\begin{bmatrix} (2,2)+(0,-2)+(\frac{-3}{4},\frac{3}{4})+(\frac{-3}{4},\frac{-3}{4}) \\ (-3,-3)+(0,3)+(\frac{3}{2},\frac{-3}{2})+(\frac{3}{4},\frac{3}{4}) \\ (\frac{-3}{4},\frac{3}{4}) \\ (1,1) \end{bmatrix}$$

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$$P_3(t) = (t^3, t^2, t, 1)\begin{bmatrix} (1/2, 0) \\ (-3/4, -9/4) \\ (-3/4, 3/4) \\ (1, 1) \end{bmatrix}$$

$$P_3(t) = (\tfrac{1}{2}, 0)t^3 + (\tfrac{-3}{4}, \tfrac{-3}{4})t^2 + (\tfrac{-3}{4}, \tfrac{3}{4})t + (1, 1)$$

For $P_4(t)$:-

$$P_4(t) = T(t)\, M \begin{bmatrix} P_4 \\ P_1 \\ P_4^t \\ P_1^t \end{bmatrix}$$

$$P_4(t) = T(t)\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{bmatrix} (0,1) \\ (0,0) \\ (-3/4, -3/4) \\ (3/4, -3/4) \end{bmatrix}$$

$$P_4(t) = T(t)\begin{bmatrix} (0,2) + (0,0) + (\tfrac{-3}{4}, \tfrac{-3}{4}) + (\tfrac{3}{4}, \tfrac{-3}{4}) \\ (0,3) + (0,0) + (\tfrac{3}{2}, \tfrac{3}{2}) + (\tfrac{-3}{4}, \tfrac{3}{4}) \\ (\tfrac{-3}{4}, \tfrac{-3}{4}) \\ (0,1) \end{bmatrix}$$

$$P_4(t) = (t^3, t^2, t, 1)\begin{bmatrix} (0, \tfrac{1}{2}) \\ (3/4, -3/4) \\ (-3/4, -3/4) \\ (0,1) \end{bmatrix}$$

$$P_4(t) = (0, \tfrac{1}{2})t^3 + (\tfrac{3}{4}, \tfrac{-3}{4})t^2 + (\tfrac{-3}{4}, \tfrac{-3}{4})t + (0,1)$$

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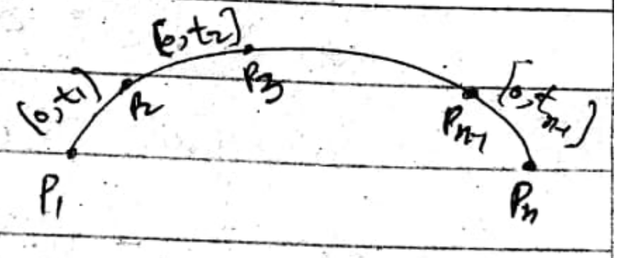
Non-uniform Cubic Splines 27-05-2015

All the different types of cubic spline discussed so far assume that the parameter "t" varies in the interval [0,1] in every segment. These types of cubic spline are therefore uniform or normalized. The non-uniform cubic spline is obtained by adding another parameter t_k to every spline segment and letting t vary in the interval $[0, t_k]$. Since there are $n-1$ spline segments connecting the n data points, this adds $n-1$ parameters to the curve, which makes it easier to fine-tune the shape of the curve.



The calculation of non-uniform cubic spline is based on that of uniform version. Since we know that

$$P_k(t) = a_k t^3 + b_k t^2 + c_k t + d_k \longrightarrow (1)$$

$$\& \quad t \in [0, t_k] \qquad k = 1, 2, \cdots, n-2$$

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$P_k(0) = d_k$

$\Rightarrow \boxed{P_k = d_k} \longrightarrow (2)$ $\quad \because P_k(0) = P_k$

$P_k(t_k) = a_k t_k^3 + b_k t_k^2 + c_k t_k + d_k$

$P_{k+1} - P_k = a_k t_k^3 + b_k t_k^2 + c_k t_k \longrightarrow (3)$ $\quad \because P_k(t_k) = P_{k+1}$ & $P_k = d_k$

Also

$P_k^t(t) = 3a_k t^2 + 2b_k t + c_k$

$P_k^t(0) = c_k$

$\Rightarrow \boxed{c_k = P_k^t} \longrightarrow (4)$

$\Rightarrow P_k^t(t_k) = 3a_k t_k^2 + 2b_k t_k + c_k$

$P_{k+1}^t - P_k^t = 3a_k t_k^2 + 2b_k t_k \longrightarrow (5)$

~~After~~ Using (4) in (3)

$P_{k+1} - P_k - P_k^t t_k = a_k t_k^3 + b_k t_k^2 \longrightarrow (6)$

By $t_k \times (5) - 2 \times (6)$

$t_k P_{k+1}^t - t_k P_k^t = 3a_k t_k^3 + 2b_k t_k^2$

$\pm 2P_{k+1} \mp 2P_k \mp 2P_k^t t_k = \pm 2a_k t_k^3 \pm 2b_k t_k^2$

$2P_k - 2P_{k+1} + P_k^t t_k + P_{k+1}^t t_k = a_k t_k^3$

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$$\Rightarrow \boxed{a_k = \frac{2(P_k - P_{k+1})}{t_k^3} + \frac{P_k^t}{t_k^2} + \frac{P_{k+1}^t}{t_k^2}.}$$

Put in (5).

$$P_{k+1}^t - P_k^t = 3t_k^2\left[\frac{2P_k}{t_k^3} - \frac{2P_{k+1}}{t_k^3} + \frac{P_k^t}{t_k^2} + \frac{P_{k+1}^t}{t_k^2}\right] + 2b_k t_k$$

$\div$ ing by t_k.

$$\frac{P_{k+1}^t}{t_k} - \frac{P_k^t}{t_k} = \frac{6P_k}{t_k^2} - \frac{6P_{k+1}}{t_k^2} + \frac{3P_k^t}{t_k} + \frac{3P_{k+1}^t}{t_k} + 2b_k$$

$$2b_k = \frac{6P_{k+1}}{t_k^2} - \frac{6P_k}{t_k^2} - \frac{4P_k^t}{t_k} - \frac{2P_{k+1}^t}{t_k}.$$

$$\boxed{b_k = \frac{3(P_{k+1} - P_k)}{t_k^2} - \frac{2P_k^t}{t_k} - \frac{P_{k+1}^t}{t_k}}$$

So, Eq. (1) becomes

$$P_k(t) = \left[\frac{2(P_k - P_{k+1})}{t_k^3} + \frac{P_k^t}{t_k^2} + \frac{P_{k+1}^t}{t_k^2}\right]t^3 + \left[\frac{3(P_{k+1} - P_k)}{t_k^2} - \frac{2P_k^t}{t_k} - \frac{P_{k+1}^t}{t_k}\right]t^2$$

$$+ P_k^t\, t + P_k \longrightarrow (7)$$

By using the Condition

$$P_{k+1}^{tt}(0) = P_k^{tt}(t_k) \longrightarrow (8) \quad (C^2)$$

$$2b_{k+1} = 6a_k t_k + 2b_k$$

$$b_{k+1} = 3a_k t_k + b_k$$

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Using the values of a_k, b_k & b_{k+1} we get

$$3\left[\frac{2(P_k-P_{k+1})}{t_k^3}+\frac{P_k^t}{t_k^2}+\frac{P_{k+1}^t}{t_k^2}\right]t_k+\frac{3(P_{k+1}-P_k)}{t_k^2}-\frac{2P_k^t}{t_k}-\frac{P_{k+1}^t}{t_k}$$

$$=\frac{3(P_{k+2}-P_{k+1})}{t_{k+1}^2}-\frac{2P_{k+1}^t}{t_{k+1}}-\frac{P_{k+2}^t}{t_{k+1}}$$

$$\frac{6P_k}{t_k^2}-\frac{6P_{k+1}}{t_k^2}+\frac{3P_k^t}{t_k}+\frac{3P_{k+1}^t}{t_k}+\frac{3P_{k+1}}{t_k^2}-\frac{3P_k}{t_k^2}-\frac{2P_k^t}{t_k}-\frac{P_{k+1}^t}{t_k}$$

$$=\frac{3P_{k+2}}{t_{k+1}^2}-\frac{3P_{k+1}}{t_{k+1}^2}-\frac{2P_{k+1}^t}{t_{k+1}}-\frac{P_{k+2}^t}{t_{k+1}}$$

$$\frac{P_k^t}{t_k}+\frac{2P_{k+1}^t}{t_k}+\frac{2P_{k+1}^t}{t_{k+1}}+\frac{P_{k+2}^t}{t_{k+1}}=\frac{3P_{k+2}}{t_{k+1}^2}-\frac{3P_{k+1}}{t_{k+1}^2}+\frac{3P_{k+1}}{t_k^2}-\frac{3P_k}{t_k^2}$$

x'ing $t_k t_{k+1}$ on both sides.

$$t_{k+1}P_k^t+2t_{k+1}P_{k+1}^t+2t_kP_{k+1}^t+t_kP_{k+2}^t=\frac{3t_kP_{k+2}}{t_{k+1}}-\frac{3t_kP_{k+1}}{t_{k+1}}+\frac{3t_{k+1}P_{k+1}}{t_k}-\frac{3t_{k+1}P_k}{t_k}$$

$$t_{k+1}P_k^t+2(t_k+t_{k+1})P_{k+1}^t+t_kP_{k+2}^t$$

$$=\frac{3}{t_kt_{k+1}}\left[t_k^2(P_{k+2}-P_{k+1})+t_{k+1}^2(P_{k+1}-P_k)\right]$$

This is again a system of Eqs. So

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We can write

$$\begin{bmatrix} t_2 & 2(t_1+t_2) & t_1 & 0 & 0 & \cdots & 0 \\ 0 & t_3 & 2(t_2+t_3) & t_2 & 0 & \cdots & 0 \\ \vdots & & & & \ddots & & \vdots \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & & t_{n-1} & 2(t_{n-1}+t_{n-2}) & t_{n-2} \end{bmatrix} \begin{bmatrix} P_1^t \\ P_2^t \\ \vdots \\ \vdots \\ P_n^t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{t_1 t_2}\left[t_1^2(P_3-P_2)+t_2^2(P_2-P_1)\right] \\ \frac{3}{t_2 t_3}\left[t_2^2(P_4-P_3)+t_3^2(P_3-P_2)\right] \\ \vdots \\ \frac{3}{t_{n-2}t_{n-1}}\left[t_{n-2}^2(P_n-P_{n-1})+t_{n-1}^2(P_{n-1}-P_{n-2})\right] \end{bmatrix}$$

This is again a system of $n-2$ Eq.s in the "n" unknowns T.vectors $P_1^t, P_2^t, \cdots, P_n^t$. To solve this system we need two values of extreme tangent vectors P_1^t & P_n^t, then it is easy to solve the system for remaining $n-2$ tangent vectors. After getting the values of remaining tangent vectors we substitute thes values of tangent vectors in Eq. and it yeilds $(n-1)$ spline curve segments.

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Example:- 28-05-2015

Given points

$P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (1,1)$, $P_4 = (0,1)$

& $P_1^t = (1,-1)$, $P_4^t = (-1,-1)$

Where $t_k = 2$

$$\begin{bmatrix} t_2 & 2(t_1+t_2) & t_1 & 0 \\ 0 & t_3 & 2(t_2+t_3) & t_2 \end{bmatrix} \begin{bmatrix} (1,-1) \\ P_2^t \\ P_3^t \\ (-1,-1) \end{bmatrix} = \begin{bmatrix} \frac{3}{t_1 t_2}\left[t_1^2(P_3-P_2)+t_2^2(P_2-P_1)\right] \\ \frac{3}{t_2 t_3}\left[t_2^2(P_4-P_3)+t_3^2(P_3-P_2)\right] \end{bmatrix}$$

Using the values. $t_1 = t_2 = t_3 = 2$

$$\begin{bmatrix} 2 & 8 & 2 & 0 \\ 0 & 2 & 8 & 2 \end{bmatrix} \begin{bmatrix} (1,-1) \\ P_2^t \\ P_3^t \\ (-1,-1) \end{bmatrix} = \begin{bmatrix} \frac{3}{4}\left[4(0,1)+4(1,0)\right] \\ \frac{3}{4}\left[4(-1,0)+4(0,1)\right] \end{bmatrix}$$

$$\begin{bmatrix} (2,-2)+8P_2^t+2P_3^t \\ 2P_2^t+8P_3^t+(-2,-2) \end{bmatrix} = \begin{bmatrix} 3(1,1) \\ 3(-1,1) \end{bmatrix}$$

$\Rightarrow (2,-2)+8P_2^t+2P_3^t = (3,3) \longrightarrow (1)$

& $2P_2^t+8P_3^t+(-2,-2) = (-3,3) \longrightarrow (2)$

$(1) \Rightarrow \quad 8P_2^t+2P_3^t = (1,5) \longrightarrow (3)$

$(2) \Rightarrow \quad 2P_2^t+8P_3^t = (-1,5) \longrightarrow (4)$

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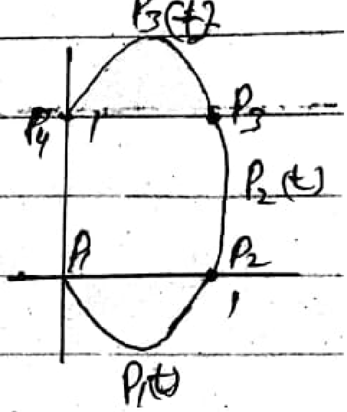
By $4 \times$ Eq. ③ $-$ Eq. ④

$$32P_2^t + \cancel{8P_3^t} = (4, 20)$$
$$-2P_2^t + \cancel{8P_3^t} = (-1, 5)$$

$$30P_2^t = (5, 15)$$

$$\boxed{P_2^t = \left(\frac{1}{6}, \frac{1}{2}\right)}$$



Put in ④.

$$\left(\frac{1}{3}, 1\right) + 8P_3^t = (-1, 5)$$

$$8P_3^t = \left(-\frac{4}{3}, 4\right)$$

$$\boxed{P_3^t = \left(-\frac{1}{6}, \frac{1}{2}\right)}$$

For $P_1(t)$:-

$$P_1(t) = T(t)\begin{bmatrix} 2/t_1^3 & \frac{-2}{t_1^3} & 1/t_1^2 & 1/t_1^2 \\ \frac{-3}{t_1^2} & 3/t_1^2 & \frac{-2}{t_1} & -1/t_1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} (0,0) \\ (1,0) \\ (1,-1) \\ (1/6, 1/2) \end{bmatrix}$$

$\because t_1 = 2$

$$P_1(t) = T(t)\begin{bmatrix} 1/4 & -1/4 & 1/4 & 1/4 \\ \frac{-3}{4} & 3/4 & -1 & -1/2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} (0,0) \\ (1,0) \\ (1,-1) \\ (1/6, 1/2) \end{bmatrix}$$

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$$P_1(t) = T(t)\begin{bmatrix} (0,0)+(-\frac{1}{4},0)+(\frac{1}{4},-\frac{1}{4})+(\frac{1}{24},\frac{1}{8}) \\ (0,0)+(\frac{3}{4},0)+(-1,1)+(-\frac{1}{12},-\frac{1}{4}) \\ (1,-1) \\ (0,0) \end{bmatrix}$$

$$P_1(t) = (t^3, t^2, t, 1)\begin{bmatrix} (\frac{1}{24}, -\frac{1}{8}) \\ (-\frac{1}{3}, \frac{3}{4}) \\ (1,-1) \\ (0,0) \end{bmatrix}$$

$$P_1(t) = (\frac{1}{24}, -\frac{1}{8})t^3 + (-\frac{1}{3}, \frac{3}{4})t^2 + (1,-1)t$$

For $P_2(t)$:- $\quad t_2 = 2$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1/4 & -1/4 & 1/4 & 1/4 \\ -\frac{3}{4} & \frac{3}{4} & -1 & -1/2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} P_2 \\ P_3 \\ P_2^t \\ P_3^t \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1/4 & -1/4 & 1/4 & 1/4 \\ -3/4 & 3/4 & -1 & -1/2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} (1,0) \\ (1,1) \\ (1/6, 1/2) \\ (-1/6, 1/2) \end{pmatrix}$$

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$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{bmatrix} (1/4, 0) + (-1/4, -1/4) + \left(\frac{1}{24}, \frac{1}{8}\right) + \left(\frac{-1}{24}, \frac{-1}{8}\right) \\ \left(\frac{-3}{4}, 0\right) + \left(\frac{3}{4}, 3/4\right) + \left(\frac{-1}{6}, \frac{-1}{2}\right) + \left(\frac{1}{12}, -1/4\right) \\ (1/6, 1/2) \\ (1, 0) \end{bmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{bmatrix} (0, 0) \\ (-1/12, 0) \\ (1/6, 1/2) \\ (1, 0) \end{bmatrix}$$

So, $a = (0,0)$, $b = (-1/12, 0)$, $c = (1/6, 1/2)$
& $d = (1, 0)$

So,

$$P_2(t) = \left(\frac{-1}{12}, 0\right)t^2 + (1/6, 1/2)t + (1, 0)$$

For $P_3(t)$:-

$\because t_3 = 2$

$$P_3(t) = T(t)\begin{pmatrix} 1/4 & -1/4 & 1/4 & 1/4 \\ -3/4 & 3/4 & -1 & -1/2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} (1,1) \\ (0,1) \\ (-1/6, 1/2) \\ (-1,-1) \end{pmatrix}$$

$$P_3(t) = T(t)\begin{bmatrix} (1/4, 1/4) + (0, -1/4) + \left(\frac{-1}{24}, \frac{1}{8}\right) + \left(\frac{-1}{4}, \frac{-1}{4}\right) \\ \left(\frac{-3}{4}, \frac{-3}{4}\right) + (0, 3/4) + (1/6, -1/2) + (1/2, 1/2) \\ (-1/6, 1/2) \\ (1, 1) \end{bmatrix}$$

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$$P_3(t) = (t^3, t^2, t, 1)\begin{bmatrix} (-\frac{1}{24}, -\frac{1}{8}) \\ (-\frac{1}{12}, 0) \\ (-\frac{1}{6}, \frac{1}{2}) \\ (1, 1) \end{bmatrix}$$

$$P_3(t) = -(\frac{1}{24}, \frac{1}{8})t^3 + (\frac{-1}{12}, 0)t^2 + (\frac{1}{6}, \frac{1}{2})t + (1, 1)$$

The Quadratic Spline

The cubic spline curve is useful in certain practical applications; which raises the question of splines of different degrees based on the same concepts. It turns out that splines of degrees higher than 3 are useful only for special applications because they are more computationally intensive and tend to have many undesirable inflection points (i.e they tend to wiggle excessively). Splines of degree 1 are of course, just straight segments connected to form a polyline, but quadratic (degree-2) splines can be useful in some applications. Such a spline is easy to

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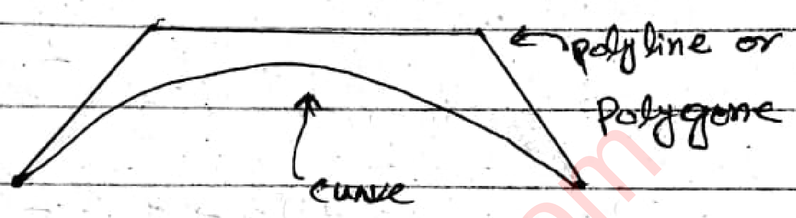
Polyline:- In computer graphics a line is composed of one or more line segments by specifying end points of each segment.



⇒ derive and to computationaly economical. Each spline segment is a quadratic polynomial i.e a parabolic arc or circular arc, so it results in fewer oscillations in the curve. On the other hand, quadratic spline segments connect with at most C^1 continuity because their second derivative is a constant.

Since $P(t) = at^2 + bt + c \longrightarrow ①$

$P^t(t) = 2at + b \longrightarrow ②$

Given for quadratic curve

$P(0) = P_1$, $P(1) = P_2$ & $P^t(0) = P_1^t$

① ⇒ $P(0) = \boxed{P_1 = c}$

& $P(1) = P_2 = a + b + c \longrightarrow ③$

② ⇒ $P^t(0) = \boxed{P_1^t = b} \longrightarrow ④$

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So, ③ $\Rightarrow P_2 = a + P_1^t + P_1$

$\Rightarrow \boxed{a = P_2 - P_1 - P_1^t} \longrightarrow$ ⑤

Using these values in ①.

$$P(t) = (P_2 - P_1 - P_1^t)t^2 + P_1^t t + P_1$$

$$P(t) = (-t^2+1)P_1 + t^2 P_2 + (-t^2+t)P_1^t$$

$$P(t) = (t^2, t, 1)\begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_1^t \end{bmatrix}$$

$$P(t) = T(t)\ H B$$

Where H is 3×3 matrix.

Now $P^t(1) = P_2^t$

sine $P^t(t) = 2at + b$

$\Rightarrow P_2^t = 2a + b$

$$P_2^t = 2(P_2 - P_1 - P_1^t) + P_1^t$$

$P_1^t(1) \Rightarrow P_2^t = 2(P_2 - P_1) - P_1^t \longrightarrow$ ⑥

The quadratic spline curve is derived in this section based on the variant Hermite segment. Each segment $P_i(t)$ is therefore a quadratic polynomial defined by its two endpoints P_i & P_{i+1}

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and by its start tangent vector P_i^t. The end ~~point~~ tangent of such a segment is $P_i^t(1) = 2(P_{i+1} - P_i) - P_i^t$. The first two spline segments are

$$P_1(t) = (P_2 - P_1 - P_1^t)t^2 + P_1^t t + P_1$$

&

$$P_2(t) = (P_3 - P_2 - P_2^t)t^2 + P_2^t t + P_2$$

At their joint point P_2 they have the tangent vectors $P_1^t(1) = 2(P_2 - P_1) - P_1^t$ and $P_2^t(0) = P_2^t$. In order to achieve C^1 - continuity we have the boundary condition

$$P_1^t(1) = P_2^t(0) \text{ or } 2(P_2 - P_1) - P_1^t = P_2^t$$

This Eq. can be written as

$$P_1^t + P_2^t = 2(P_2 - P_1)$$

and when duplicated $n-1$ times for the point P_1 through P_{n-1} the result is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} P_1^t \\ P_2^t \\ \vdots \\ P_n^t \end{bmatrix} = \begin{bmatrix} 2(P_2 - P_1) \\ 2(P_3 - P_2) \\ \vdots \\ 2(P_n - P_{n-1}) \end{bmatrix}$$

As with the cubic spline,

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there are more unknowns than equations (n unknowns and $n-1$ Eqs) and the standard technique is to ask the user to provide a value for one of the unknown tangent vectors, normally P_1^t.

Example:- 02-06-2015

Given that four points $P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (1,1)$ & $P_4 = (0,1)$ we are looking a quadratic Hermite curve segment with start extreme tangent vector $P_1^t = (1,-1)$.

For $n = 4$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} P_1^t \\ P_2^t \\ P_3^t \\ P_4^t \end{bmatrix} = \begin{bmatrix} 2(P_2 - P_1) \\ 2(P_3 - P_2) \\ 2(P_4 - P_3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} (1,-1) \\ P_2^t \\ P_3^t \\ P_4^t \end{bmatrix} = \begin{bmatrix} (2,0) \\ (0,2) \\ (-2,0) \end{bmatrix}$$

$\Rightarrow (1,-1) + P_2^t = (2,0)$

$\Rightarrow \boxed{P_2^t = (1,1)}$

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$$\Rightarrow P_2^t + P_3^t = (0,2)$$

$$P_3^t = (0,2) - (1,1)$$

$$\boxed{P_3^t = (-1,1)}$$

$$\Rightarrow P_3^t + P_4^t = (-2,0)$$

$$P_4^t = (-2,0) - (-1,1)$$

$$\boxed{P_4^t = (-1,-1)}$$

The three spline segments

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Is this curve achieve C^1 continuity?

Ans= Yes.

$$P_1^t(t) = (1, 2t-1)$$

& $P_2^t(t) = (-2t+1, 1)$

& $P_3^t(t) = (-1, -2t+1)$

$$P_1^t(1) = (1,1) = P_2^t(0)$$

$$P_2^t(1) = (-1,1) = P_3^t(0)$$

Hence the joint has a C^1 - continuity.

Periodic Cubic Splines

A periodic function $f(x)$ is one that repeats itself. If p is the period of the function then

$$f(x+p) = f(x), \forall x$$

A two-dimensional cubic spline is periodic if it has the same extrem tangent vectors and if its two extreme points have the same

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y coordinates. If the curve satisfies these conditions then we can place consecutive copies of it side by side & the result would look like a single periodic curve.

Example:- The following parametric Eqs

$$P(t) = (\cos t, \sin t, t)$$

describes a helix. If we modify this equation as

$$P(t) = (0.05t + \cos t, \sin t, 0.1t)$$
$$t \in [0, 10\pi]$$

Creates a helix moves in x-direction as it raises up in z-direction.

$$P^t(t) = (0.05 - \sin t, \cos t, 0.1)$$
$$t \in [0, 10\pi]$$

$$P^t(0) = (0.05, 1, 0.1)$$

$$P^t(10\pi) = (0.05, 1, 0.1)$$

Hence the extreme tangents are equal. We need the y-coordinates of the 2-extreme

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points are same.

For this

$P(0) = (1, 0, 0)$

$P(10\pi) = ((0.05 \times 10\pi) + 1, 0, 0.1(10\pi))$

Hence the y-coordinates are same.

Hence this Helix is Quadratic spline.

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Bezier Approximation

~~The Bezier curve is a parametric curve P(t) that is a poly~~

Bezier methods for curves are popular, are mostly used in practical work & are described here in detail. Two approaches to the design of a Bezier curve are described.

1) Using Bernstein polynomials
2) Using the mediation operator.

The Bezier Curve

The Bezier curve is a parametric curve $P(t)$ that is a polynomial function of the parameter t. The degree of polynomial depends on the number of points used to defined the curve. The method employs control points & produces an approximating curve. The curve does not pass through the interior points but is attached by them.

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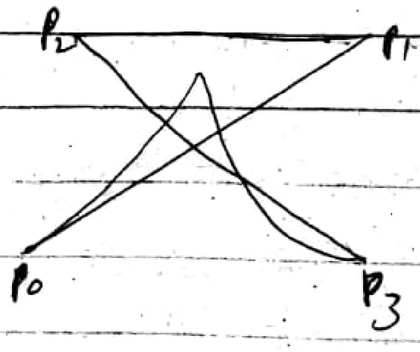

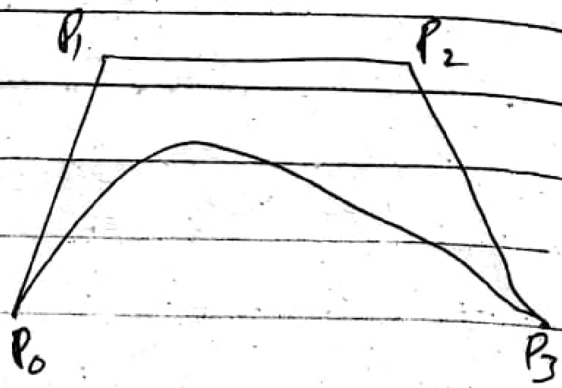

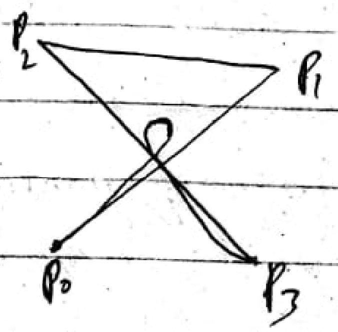

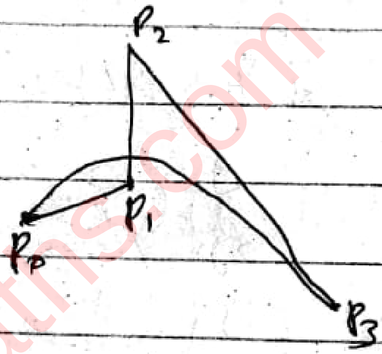

Control polygon

The Control polygon of the Bezier curve is the polygon obtained when the control points are connected, in their natural order, ~~in~~ with straight segments

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The Bernstein Form of the Bezier Curve.

$$P(t) = \sum_{i=0}^{n} P_i \, B_{n,i}(t) \qquad 0 \le t \le 1$$

This is a Bezier curve of degree n with $n+1$ control points in parameter t.

where

$$B_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

where $\binom{n}{i} = \dfrac{n!}{i!(n-i)!}$ are ~~$B_{n,i}(t)$ are~~ the binomial coefficients.

& $B_{n,i}(t)$ is called Bernstein polynomial.

For $n=1$

$$P(t) = \sum_{i=0}^{1} P_i \, B_{1,i}(t)$$

$$= P_0 B_{1,0}(t) + P_1 B_{1,1}(t)$$

$$B_{1,0}(t) = \binom{1}{0} t^0 (1-t)^{1-0} = 1-t$$

$$B_{1,1}(t) = \binom{1}{1} t^1 (1-t)^{1-1} = t$$

So,

$$P(t) = P_0(1-t) + P_1 \times t$$

This is linear Bezier curve.

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For $n=2$

$$B_{2,0}(t) = \binom{2}{0} t^0 (1-t)^{2-0} = (1-t)^2$$

$$B_{2,1}(t) = \binom{2}{1} t^1 (1-t)^{2-1} = 2t(1-t)$$

$$B_{2,2}(t) = \binom{2}{2} t^2 (1-t)^{2-2} = t^2$$

So,

$$P(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2$$

$$= \left((1-t)^2, 2t(1-t), t^2\right)\left(P_0, P_1, P_2\right)^T$$

$$P(t) = (t^2, t, 1)\begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} P_0 \\ P_1 \\ P_2 \end{pmatrix}$$

For $n=3$

$$B_{3,0}(t) = \binom{3}{0} t^0 (1-t)^{3-0} = (1-t)^3$$

$$B_{3,1}(t) = \binom{3}{1} t^1 (1-t)^{3-1} = 3t(1-t)^2$$

$$B_{3,2}(t) = \binom{3}{2} t^2 (1-t)^{3-2} = 3t^2(1-t)$$

$$B_{3,3}(t) = \binom{3}{3} t^3 (1-t)^{3-3} = t^3$$

So,

$$P(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3$$

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Computation of Bezier curve By shift operator-

The Bezier curve can also be represented in a very compact & elegant way as

$$P(t) = (1-t+tE)^n P_0$$

where E is the shift operator defined by

$$EP_i = P_{i+1}$$

$$EP_0 = P_1$$

or $$E^2P_0 = E(EP_0)$$
$$= EP_1$$
$$= P_2$$

So, $$E^iP_0 = P_i$$

The B-curve of degree "n" is

$$P(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} P_i$$

$$= \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} E^i P_0$$

$$= \sum_{i=0}^{n} \binom{n}{i} (Et)^i (1-t)^{n-i} P_0$$

$$= (tE + 1-t)^n P_0$$ (By Binomial Theory)

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~~For n=1~~

$$P(t) = (1 - t + Et)^1 P_0$$

$$= (1-t) P_0 + t E P_0$$

$$= (1-t) P_0 + t P_1$$

$$= \sum_{i=0}^{1} B_{1,i}(t) P_i$$

This is linear B. curve.

For n=2

$$P(t) = (1 - t + tE)^2 P_0$$

$$= ((1-t)^2 + 2(tE)(1-t) + t^2 E^2) P_0$$

$$= (1-t)^2 P_0 + 2t(1-t) E P_0 + t^2 E^2 P_0$$

$$= (1-t)^2 P_0 + 2t(1-t) P_1 + t^2 P_2$$

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Pascal Triangle 8-06-2015

The pascal triangle & Binomial theorem are related ~~both~~ because both employ the same numbers. The pascal triangle is an infinite triangular matrix that's built from the edges inside

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮

We first fill the left & right edges with ones, then compute each interior element as the sum of the two elements directly above it. As can be expected, it is not hard to obtain an explicit expression for the general element of the pascal triangle.

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Also we can calculate the interior elements of triangle as

$$
\begin{array}{ccccccccccccc}
1 & & 1 & & & & & & & & & & \\
 & & 1 & & 1 & & & & & & & & \\
\hline
1 & & 2 & & 1 & & & & & & & & \\
 & & 1 & & 2 & & 1 & & & & & & \\
\hline
1 & & 3 & & 3 & & 1 & & & & & & \\
 & & 1 & & 3 & & 3 & & 1 & & & & \\
\hline
1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
\hline
1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\hline
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

This shows that the element of the triangle satisfy the following:

$$\binom{i}{0} = \binom{i}{i} = 1, \quad i = 0, 1, 2, \dots$$

$$\binom{i}{j} = \binom{i-1}{j-1} + \binom{i-1}{j}, \quad i = 2, 3, \dots, j = 1, 2, \dots, i-1$$

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$$R.H.S = \binom{i-1}{j-1} + \binom{i-1}{j}$$

$$= \frac{(i-1)!}{(j-1)!\,(i-1-j+1)!} + \frac{(i-1)!}{j!\,(i-1-j)!}$$

$$= (i-1)!\left[\frac{1}{(j-1)!\,(i-j)(i-j-1)!} + \frac{1}{j(j-1)!(i-j-1)!}\right]$$

$$= \frac{(i-1)!}{(j-1)!\,(i-j-1)!}\left[\frac{1}{i-j} + \frac{1}{j}\right]$$

$$= \frac{i\,(i-1)!}{j\,(j-1)!\,(i-j)\,(i-j-1)!}$$

$$= \frac{i!}{j!\,(i-j)!} = \binom{i}{j} = L.H.S$$

The Binomial theorem states as

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i\, b^{n-i}$$

Remarks:

When Bezier started searching for such functions (Bernstein polynomial) in the

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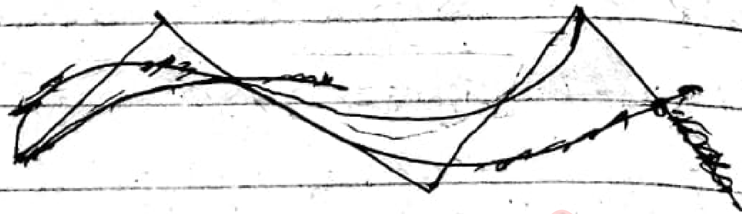
early 1860s, he set the following requirement.

(ii)



(i)

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Computation of Bezier Curve by Recursive Formula

Given $n+1$ control points P_0 through P_n, we can represent the Bezier curve for the points by $P_n^{(n)}(t)$, where the quantity

$$P_i^{(j)}(t) = \begin{cases} (1-t)P_{i-1}^{(j-1)}(t) + t P_i^{(j-1)}(t) & \text{for } j>0 \\ P_i, & \text{for } j=0 \end{cases}$$

Calculation of different degrees of Bezier curve.

For $n=1$

$$P_1^{(1)}(t) = (1-t)P_0^{(0)}(t) + t P_1^{(0)}(t) \longrightarrow ①$$

$$P_0^{(0)}(t) = P_0$$

$$P_1^{(0)}(t) = P_1 \;\; \ldots \;\; P_n^{(0)}(t) = P_n$$

$$① \Rightarrow P_1^{(1)}(t) = (1-t)P_0 + t P_1 \longrightarrow ②$$

$$= \sum_{i=0}^{1} B_{1,i}(t) \; P_i$$

is called B curve of degree 1.

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For $n=2$ using ①

$$P_2^{(2)}(t) = (1-t)\,P_1^{(1)}(t) + t\,P_2^{(1)}(t) \longrightarrow ①$$

$$P_2^{(1)}(t) = (1-t)\,P_1^{(0)}(t) + t\,P_2^{(0)}(t)$$

$$P_2^{(1)}(t) = (1-t)\,P_1 + t\,P_2$$

② $\Rightarrow$

$$P_2^{(2)}(t) = (1-t)\big((1-t)P_0 + tP_1\big) + t\big((1-t)P_1 + tP_2\big)$$

$$= (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2\}$$

$$= \sum_{i=0}^{2} B_{2,i}(t)\,P_i$$

On the familiar way we are able to calculate the Bezier curve of remaining any degree using recursive formula.

Properties of curves.

① unity

The weight (B.F) are add upto 1 (B-property) guarantee to

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To provide the shape of the curve independent of coordinate system.

$$1 = 1^n$$
$$= (1-t+t)^n$$
$$= \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i}$$
$$= \sum_{i=0}^{n} B_{n,i}(t)$$

(2) End point condition

Always the curve passes through the start & final points.

The B curve of degree n is

$$P(t) = \sum_{i=0}^{n} B_{n,i}(t) P_i$$
$$= B_{n,0}(t) P_0 + B_{n,1}(t) P_1 + \cdots + B_{n,n}(t) P_n \quad \text{(1)}$$

at $t=0$

$$P(0) = B_{n,0}(0) P_0 + B_{n,1}(0) P_1 + \cdots + B_{n,n}(0) P_n$$

$$B_{n,0}(t) = \binom{n}{0} t^0 (1-t)^{n-0} = \binom{n}{0} (1-t)^n$$

$$\Rightarrow B_{n,0}(0) = 1$$

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$B_{n,1}(t) = \binom{n}{1} t^1 (1-t)^{n-1}$

$\Rightarrow B_{n,1}(0) = \binom{n}{1} (0)^1 (1-t)^{n-1} = 0$

$P(0) = 1\, P_0 + 0 + 0 + \cdots + 0$

$\Rightarrow P(0) = P_0$

For $t = 1$

$P(1) = B_{n,0}(1) P_0 + B_{n,1}(1) P_1 + \cdots + B_{n,n}(1) P_n$

$B_{n,0}(t) = \binom{n}{0} (1-t)^n$

$\Rightarrow B_{n,0}(1) = \binom{n}{0} (1-1)^n = 0$

$B_{n,1}(t) = \binom{n}{1} t (1-t)^{n-1}$

$\Rightarrow B_{n,1}(1) = 0$

for $B_{n,n}(t) = \binom{n}{n} t^n (1-t)^{n-n} = t^n$

$B_{n,n}(1) = 1$

So,

$P(1) = P_n$

Hence proved the result.

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