Relativity

Study of motion of a particle relative to another particle is called relativity.

Special Relativity (1905)

When we study the relative motion on a flat space, it is called special relativity.

General Relativity (1915)

When we study the relative motion on a curved space, it is called general relativity.

Frame of Refference

There are two types:

1):- Inertial frame of Refference

2):- Non-inertial frame of Refference

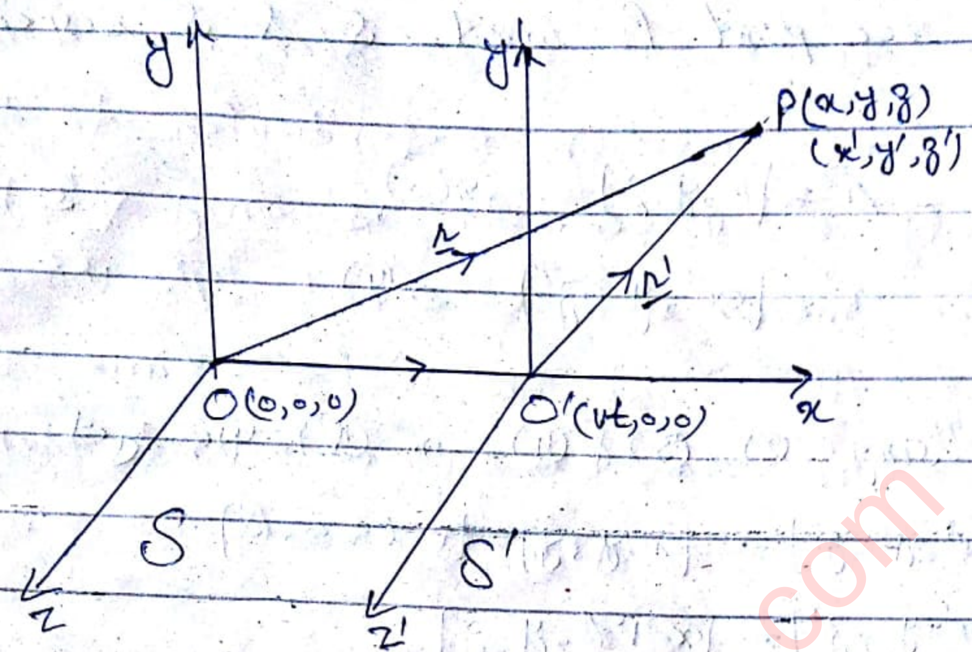
Inertial frames

These frames of refference which are moving with a uniform velocity (i.e non-accelarated) w.r.t each other are called inertial frames.

Non-Inertial frames

Those frames of refference which are moving with non-uniform velocity (i.e accelarated) w.r.t eachother are called non-inertial frames.

Galilean Transformation



Consider two inertial frames S & S' which are initially coincident and then S' moves with a uniform velocity $\underline{v}$ along the direction of common x-axes. After a time t we consider a point P with its position vectors $\underline{r}$ & $\underline{r}'$ w.r.t S & S' respectively.

Then from $\Delta OO'P$ we have

$$\underline{r} = \underline{r}' + \vec{OO'} \longrightarrow ①$$

Since O' moves with a uniform velocity of magnitude v in time t, so

$$|\vec{OO'}| = vt$$

and $\vec{OO'} = [vt, 0, 0] \longrightarrow ②$

If $[x, y, z]$ & $[x', y', z']$ are the coordinates of the point P w.r.t. S & S' respectively then

$$\vec{r} = [x, y, z] \longrightarrow (3)$$

$$r' = [x', y', z'] \longrightarrow (4)$$

Using (2), (3) & (4) in (1) we get

$$[x', y', z'] = [x, y, z] - [vt, 0, 0]$$

$$[x', y', z'] = [x - vt, y, z]$$

$$\Rightarrow \quad x' = x - vt$$

$$y' = y$$

$$z' = z$$

Since initially S & S' were coincident. So, always $t = t'$.

Thus finally we have

$$x' = x - vt$$

$$y' = y$$

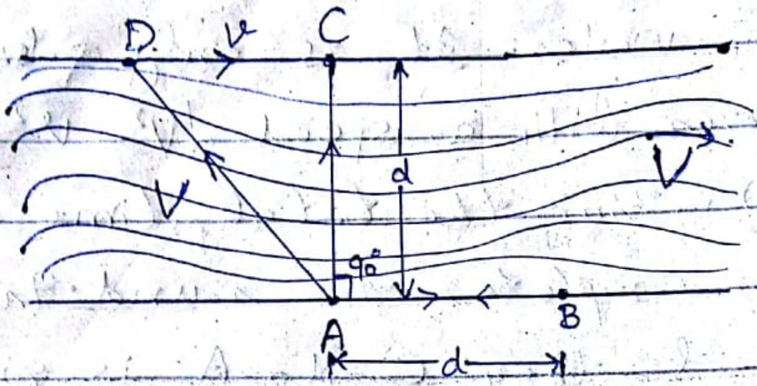
$$z' = z$$

$$t' = t$$

These are called galilean transformation.

Assignment:- If S is an inertial frame then show that S' is also an inertial frame by using galilean transformation.

Effect of velocity of medium on the motion of particle.



Let us consider a stream with it's water flowing with a uniform velocity $\vec{v}$ and 'd' be it's width. A swimmer whose velocity in still water is $\vec{V}$, swims across the stream from point A to B and then from B to A. The distance between A and B is equal to the width d. Then the time t_1 taken by the swimmer to complete one round from A to B and then B to A is given by

$$t_1 = \frac{d}{V+v} + \frac{d}{V-v} \qquad \because S = Vt \Rightarrow t = \frac{d}{V+v}$$

$$t_1 = \frac{Vd - \cancel{vd} + Vd + \cancel{vd}}{V^2 - v^2}$$

$$t_1 = \frac{2Vd}{V^2 - v^2} \longrightarrow (1)$$

Now let the swimmer swims

from A to C and then C to A along the direction perpendicular to the velocity of water. It is only possible if he moves with a speed $\sqrt{V^2 - v^2}$, along AD. Thus the total time t_2 taken to complete one round from A to C & then C to A is given by

$$t_2 = \frac{2d}{\sqrt{V^2 - v^2}} \longrightarrow (2)$$

Therefore the time difference Δt between two intervals is,

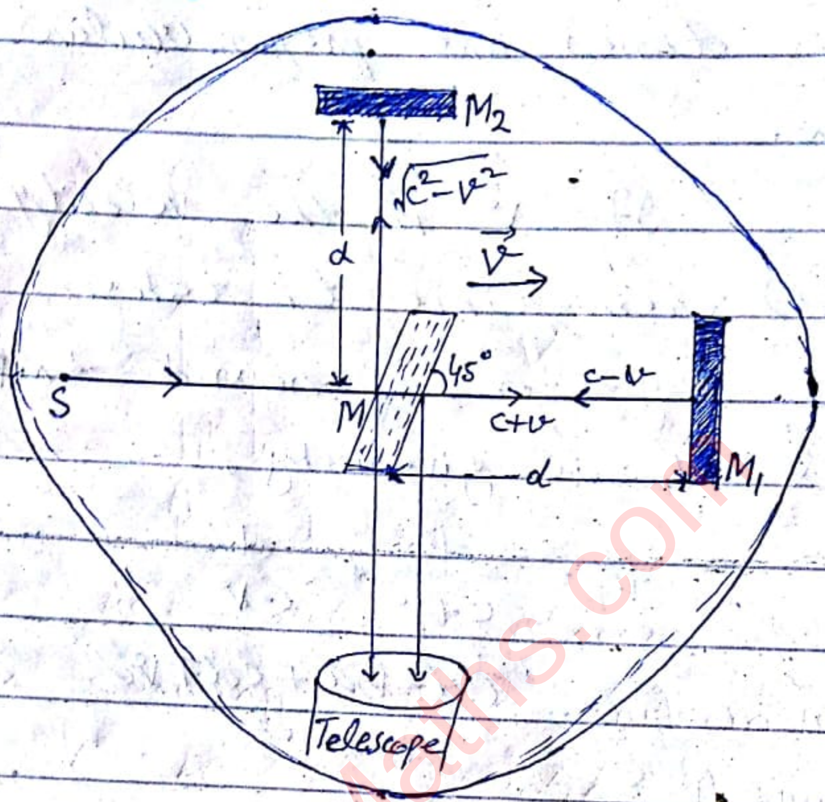
$$\Delta t = t_1 - t_2$$

$$= \frac{2Vd}{V^2 - v^2} - \frac{2d}{\sqrt{V^2 - v^2}}$$

$$= \frac{2d}{V}\left[\frac{1}{1 - \frac{v^2}{V^2}} - \frac{1}{\sqrt{1 - \frac{v^2}{V^2}}}\right]$$

This is the time difference b/w two time intervals.

Michelson–Morley Experiment 13-03-15



This experiment was performed by Michelson and E.W. Morley in 1887. In this experiment a beam of light from a source S is splits into two parts by a thinly silverd glass mirror "M" which inclined at angle 45° to the direction of beam. The two parts of light fall on two mirrors M_1 & M_2 at right angle and a fully reflecting mirrors. The reflected rays are finally enter the telescope.

Further it is assume that the mirrors M_1 & M_2 are placed at equal distance d from the mirror M in such a way that their planes are prependicular to each-other.

If v is the velocity of ether (or wind) then time t_1 taken by the light ray for a round trip b/w M & M_1 is given by,

$$t_1 = \frac{d}{c+v} + \frac{d}{c-v}$$

$$= \frac{cd - \cancel{vd} + cd + \cancel{vd}}{c^2 - v^2}$$

$$= \frac{2cd}{c^2 - v^2}$$

$$= \frac{2d/c}{1 - \frac{v^2}{c^2}} \longrightarrow (1)$$

To travel along the direction prependicular to M_2 the light ray must be aimed such that its resultant velocity is $\sqrt{c^2 - v^2}$ for both journies from M to M_2 and then M_2 to M. So the time t_2 taken for a round trip b/w M and M_2 is given as

$$t_2 = \frac{2d}{\sqrt{c^2 - v^2}}$$

$$t_2 = \frac{2d/c}{\sqrt{1 - \frac{v^2}{c^2}}} \longrightarrow (2)$$

Thus the time difference Δt b/w intervals is

$$\Delta t = t_1 - t_2$$

$$= \frac{2d/c}{1-\left(\frac{v}{c}\right)^2} - \frac{2d/c}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$= \frac{2d}{c}\left[\left(1-\left(\frac{v}{c}\right)^2\right)^{-1} - \left(1-\left(\frac{v}{c}\right)^2\right)^{-1/2}\right]$$

$$= \frac{2d}{c}\left[\left\{1+\left(\frac{v}{c}\right)^2 + \frac{(-1)(-2)}{2!}\left(\frac{v}{c}\right)^4 + \cdots\right\} - \left\{1+\frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{v}{c}\right)^4 + \cdots\right\}\right]$$

$$= \frac{2d}{c}\left[\cancel{1} + \left(\frac{v}{c}\right)^2 + \left(\frac{v}{c}\right)^4 + \cdots - \cancel{1} - \frac{1}{2}\left(\frac{v}{c}\right)^2 - \frac{3}{8}\left(\frac{v}{c}\right)^4 + \cdots\right]$$

As $v \ll c \Rightarrow \left(\frac{v}{c}\right) \ll 1$.

So, neglecting the terms involving $\left(\frac{v}{c}\right)^4$ and its higher powers. We get

$$\Delta t \approx \frac{2d}{c}\left[\frac{v^2}{c^2} - \frac{1}{2}\cdot\frac{v^2}{c^2}\right]$$

$$\Delta t \approx \frac{\cancel{2}d}{c}\left[\frac{v^2}{\cancel{2}c^2}\right]$$

$$\Delta t \approx \frac{v^2 d}{c^3}$$

As c and d are known, so one can find v if Δt is experimently observed. But this experiment failled to give the required results.

Justification of Michelson-Morley Experiment

1):- It was explained that the Michelson-Morley experiment failed due to the reason that earth drags ether along the direction of its motion so it is always be stationary w.r.t ether.

2):- The Michelson-Morley experiment was explained by Lorentz and Fitzgereld who made the assumption that all material objects are contracted by a vector $\sqrt{1-\frac{v^2}{c^2}}$.

However this contraction cannot be calculated as it also applies to the measuring rods.

Postulates of Special Relativity 17-03-2015

1):- Speed of light remains constant for all observers.

2):- All laws of physics remains same in all frames (inertial & non-inertial)

But for Galelian transformations

We know that

$$x' = x - vt$$

Diff. w.r.t "t".

$$\frac{dx'}{dt} = \frac{dx}{dt} - v$$

As $t = t'$

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v$$

$$V' = V - v$$

For light case put $V = c$

So, $C' = C - v$

$$\Rightarrow C' \neq C$$

Lorentz Transformations

Since Galelian Transformations violates the postulate of constancy of speed of light. i.e $c' \neq c$.

So, we need another transformation

Consider two frames (x, y, z, t) & (x', y', z', t')

which are coincident initially i.e $t = 0 = t'$ Later on the frame moves along the common x-axis with a uniform velocity v. The other axes remain parallel to eachother. Thus we can write

$$x' \propto x - vt$$

$$\Rightarrow x' = k(x - vt) \longrightarrow (1)$$

where k is constant of proportionality. Eq. (1) shows that a single event in frame S corresponds to a single event in S'. Further for $k=1$ it reduces to Galelian transformations. The inverse relation of eq (1) can be written as

$$x = k'(x' + vt') \longrightarrow (2)$$

Using eq (1) in Eq (2).

$$x = k'(kx - kvt + vt')$$

$$x = k'kx - k'kvt + k'vt'$$

$$k'vt' = k'kvt + x - k'kx$$

$$k'vt' = x(1 - k'k) + k'kvt$$

$$t' = \frac{x(1 - k'k)}{k'v} + kt \longrightarrow (3)$$

To find the value of K & K' we use the postulates of special Relativity.

Let $t=t'=0$, a light signal propagates in these two frames according as

$$x = ct \longrightarrow (i)$$
$$x' = ct' \longrightarrow (ii)$$
$$\Bigg\} \longrightarrow (4)$$

$s=vt$

Using (1) & (3) in (ii)

$$K(x-vt) = c\left[\frac{x(1-KK')}{K'v} + Kt\right]$$

$$x - vt = \frac{cx}{v}\left[\frac{1-KK'}{KK'}\right] + ct$$

Put $x=ct$ by (i)

$$\cancel{ct} - vt = \frac{c^2 t}{v}\left[\frac{1-KK'}{KK'}\right] + \cancel{ct}$$

$$-v^2 \cancel{t} = c^2 \cancel{t}\left[\frac{1}{KK'} - 1\right]$$

$$-\frac{v^2}{c^2} = \frac{1}{KK'} - 1 \longrightarrow (iii)$$

$$\left(1 - \frac{v^2}{c^2}\right) = \frac{1}{KK'}$$

$$KK' = \frac{1}{1-\frac{v^2}{c^2}} \longrightarrow (5)$$

For our convience take $K=K'$

$$K^2 = \frac{1}{1-\frac{v^2}{c^2}}$$

$$\Rightarrow K = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma \text{ (say)}.$$

Eq. (1) becomes

$$x' = \gamma(x - vt)$$

Eq. (3) can be written as:

$$t' = K\left[\frac{x}{v}\left(\frac{1 - KK'}{KK'}\right) + t\right]$$

$$t' = K\left[\frac{x}{v}\left[\frac{1}{KK'} - 1\right] + t\right]$$

By putting the value of K, and put eq. (iii). here

$$t' = \gamma\left[\frac{x}{v}\left(-\frac{v^2}{c^2}\right) + t\right]$$

$$t' = \gamma\left[t - \frac{vx}{c^2}\right]$$

Thus finally we have

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

which are called Lorentz transformation

Assignment:- Show that L.T obeys the postulates of constancy of speed of light i.e $c' = c$.

Sol:- We know

$$x' = \gamma(x - vt) \longrightarrow (i)$$

$$y' = y \longrightarrow (ii)$$

$$z' = z \longrightarrow (iii)$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \longrightarrow (iv)$$

Diff. (i) w.r.t "t".

$$\frac{dx'}{dt} = \gamma\left(\frac{dx}{dt} - v\right)$$

$$\frac{dx'}{dt'} \cdot \frac{dt'}{dt} = \gamma\left(\frac{dx}{dt} - v\right) \longrightarrow (v)$$

Diff. (iv) w.r.t "t".

$$\frac{dt'}{dt} = \gamma\left(1 - \frac{v}{c^2} \cdot \frac{dx}{dt}\right)$$

Put in (v)

$$\frac{dx'}{dt'}\left(\gamma\left(1 - \frac{v}{c^2} \cdot \frac{dx}{dt}\right)\right) = \gamma\left(\frac{dx}{dt} - v\right)$$

Put $\frac{dx'}{dt'} = c'$, $\frac{dx}{dt} = c$

$$c'\left(\gamma\left(1 - \frac{v}{c^2} \cdot c\right)\right) = \gamma(c - v)$$

$$c'\left(1 - \frac{v}{c}\right) = (c - v)$$

$$c'(c - \cancel{v}) = c(c - \cancel{v})$$

$$\Rightarrow \boxed{c' = c}.$$

Inverse Lorentz Transformations 13-3-15

We know that

$$x' = \gamma(x - vt) \longrightarrow (1)$$

&

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \longrightarrow (2)$$

$$(1) \Rightarrow \frac{x'}{\gamma} = x - vt$$

$$\Rightarrow \frac{x'}{\gamma} + vt = x \longrightarrow (3)$$

$$(2) \Rightarrow \frac{t'}{\gamma} = t - \frac{v}{c^2}x$$

$$\Rightarrow t = \frac{t'}{\gamma} + \frac{v}{c^2}x \longrightarrow (4)$$

using (4) in (3) we have

$$x = \frac{x'}{\gamma} + v\left[\frac{t'}{\gamma} + \frac{v}{c^2}x\right]$$

$$x = \frac{x'}{\gamma} + \frac{vt'}{\gamma} + \frac{v^2}{c^2}x$$

$$x - \frac{v^2}{c^2}x = \frac{1}{\gamma}\left[x' + vt'\right]$$

$$x\left(1 - \frac{v^2}{c^2}\right) = \sqrt{1 - \frac{v^2}{c^2}}\,(x' + vt')$$

$$x = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}(x' + vt')$$

$$\Rightarrow x = \gamma(x' + vt')$$

Now using (3) in (4).

$$t = \frac{t'}{\gamma} + \frac{v}{c^2}\left(\frac{x'}{\gamma} + vt\right)$$

$$t = \frac{t'}{\gamma} + \frac{vx'}{c^2\gamma} + \frac{v^2 t}{c^2}$$

$$t - \frac{v^2 t}{c^2} = \frac{1}{\gamma}\left[t' + \frac{v}{c^2}x'\right]$$

$$t\left[1 - \frac{v^2}{c^2}\right] = \frac{1}{\gamma}\left[t' + \frac{v}{c^2}x'\right]$$

$$\frac{t}{\gamma^2} = \frac{1}{\gamma}\left[t' + \frac{v}{c^2}x'\right]$$

$$\Rightarrow \quad t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

So,

$$x = \gamma(x' + vt')$$

$$y = y', \quad z = z'$$

and $t = \gamma\left(t' + \frac{v}{c^2}x'\right)$

are inverse Lorentz transformations.

Q:- Show that $x^2+y^2+z^2-c^2t^2$ remains invariant under L.T.

Sol:- Consider

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = \gamma^2(x-vt)^2 + y^2 + z^2 - c^2\gamma^2\left(t - \frac{vx}{c^2}\right)^2$$

$$= \gamma^2(x^2 + v^2t^2 - 2xvt) + y^2 + z^2 - c^2\gamma^2\left(t^2 + \frac{v^2x^2}{c^4} - 2\frac{tvx}{c^2}\right)$$

$$= \gamma^2x^2 + \gamma^2v^2t^2 - \cancel{2\gamma^2 x v t} + y^2 + z^2 - c^2\gamma^2t^2 - \cancel{c^2}\gamma^2\frac{v^2x^2}{\cancel{c^4}c^2} - \cancel{2tvx\gamma^2}$$

$$= \gamma^2x^2 + \gamma^2v^2t^2 + y^2 + z^2 - c^2\gamma^2t^2 - \frac{\gamma^2v^2x^2}{c^2}$$

$$= \gamma^2x^2\left[1 - \frac{v^2}{c^2}\right] + y^2 + z^2 - \gamma^2t^2\left[c^2 - v^2\right]$$

$$= \cancel{\gamma^2}x^2\left(\frac{1}{\cancel{\gamma^2}}\right) + y^2 + z^2 - \gamma^2t^2c^2\left[1 - \frac{v^2}{c^2}\right]$$

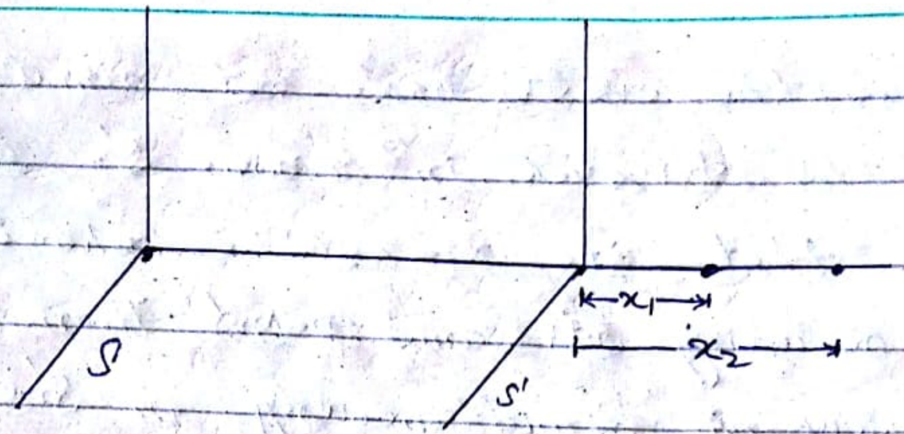
$$= x^2 + y^2 + z^2 - \cancel{\gamma^2}t^2c^2\left(\frac{1}{\cancel{\gamma^2}}\right)$$

$$= x^2 + y^2 + z^2 - t^2c^2$$

Relativity of Simultanity

Remarks:- Two events which are simultaneous to an observer, may not be simultaneous to an other observer who is moving uniformally relative to the first observer.

Consider two frames S & S' at standard configuration. Let two events occur at the same time $t_1 = t_2$ for an observer in S but at different places x_1 & x_2 (i.e $x_1 \neq x_2$)



Then for the observer in S', the time of occuring of the two events is given as (By L.T)

$$t_1' = \gamma\left(t_1 - \frac{v}{c^2}x_1\right) \longrightarrow (1)$$

$$t_2' = \gamma\left(t_2 - \frac{v}{c^2}x_2\right) \longrightarrow (2)$$

as $t_1 = t_2 = t$

So, eq. (1) & eq. (2) becomes

$$t_1' = \gamma\left(t - \frac{v}{c^2}x_1\right) \longrightarrow (3)$$

$$t_2' = \gamma\left(t - \frac{v}{c^2}x_2\right) \longrightarrow (4)$$

The time difference $\Delta t'$ is given as

$$\Delta t' = t_2' - t_1'$$

$$= \gamma\left[t - \frac{v}{c^2}x_2\right] - \gamma\left[t - \frac{v}{c^2}x_1\right]$$

$$= \gamma\left[\cancel{t} - \frac{v}{c^2}x_2 - \cancel{t} + \frac{v}{c^2}x_1\right]$$

$$= \gamma\frac{v}{c^2}\left[x_1 - x_2\right] \longrightarrow (5)$$

which impliese that $\Delta t' \neq 0$ i.e $t_1' \neq t_2'$

Thus the events are not simultaneous for the observer in frame S'. Eq. (5) shows that if $x_1 = x_2$ i.e the events are occuring at same point. Then events are occuring simultaneously in S'.

Assignment:- Show that any material object can't move with a velocity greater than or equal to c.

Sol:- Suppose that a material object is moving along x-axis of an inertial frame S. We know that

$$x' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(x - vt) \longrightarrow (i)$$

and

$$t' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(t - \frac{v}{c^2}x\right) \longrightarrow (ii)$$

If object is moving with velocity $v > c$ w.r.t S the Lorentz transformation would lead to imaginary values of equations (i) & (ii). which is impossible. Hence the velocity of any object can't exceed the speed of light.

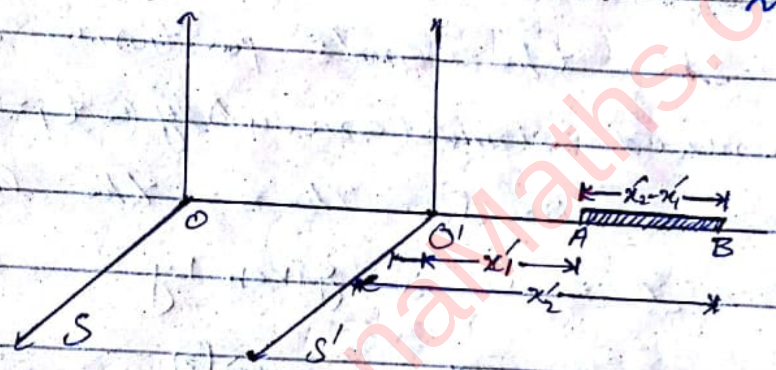
For $v = c$ the denominators of

eq. (i) & eq. (ii) become zero making the expressions indeterminment.

Hence in special relativity the velocity of light is not only a universal constant but is also the upper bound of all physical velocities.

Length Contraction

20-03-2015



Consider two intertial frames S & S' in standard configuration. Let a rod $\overline{AB}$ as measured by an observer in S'. Since the observer in S' is at rest w.r.t rod. The length of the rod can be measured by noting the position of it's ends A & B. If x_1' & x_2' are the positions of A & B respectively then we can write

$$l' = x_2' - x_1' \longrightarrow (1)$$

Let us see what would be the

length of the rod as measured by an observer in frame S.

Since the rod is in motion relative to the observer in S. So the length of rod can be measured correctly only by noting simultaneously the ends positions of the rod. Let x_1 and x_2 be the ends positions of the rod measured simultaneously (i.e $t_1 = t_2$) by an observer in S. Then the length of the rod in S is given by,

$$l = x_2 - x_1 \longrightarrow (2)$$

Using LT we can write

$$x_1' = \gamma (x_1 - vt) \longrightarrow (3)$$

$$x_2' = \gamma (x_2 - vt) \longrightarrow (4)$$

using (3) & (4) in (1) we have

$$l' = \gamma (x_2 - vt) - \gamma (x_1 - vt)$$

$$l' = \gamma (x_2 - \cancel{vt} - x_1 + \cancel{vt})$$

$$l' = \gamma (x_2 - x_1)$$

$$\Rightarrow l' = \gamma l \longrightarrow (5)$$

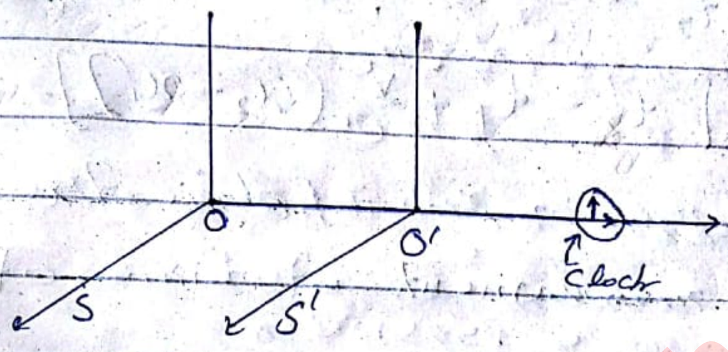
$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} > 1$$

as $\gamma > 1$

So $l' > l$

So, eq. 5 gives the length contraction. It is also known as Lorentz contraction.

Time Dilation



Consider two frames S & S' in standard configuration. Now consider a clock at position x' in S'. Let an observer in S' observe two events at time t_1' & t_2'. The time interval $\Delta t'$ is then given as

$$\Delta t' = t_2' - t_1' \longrightarrow (1)$$

If another observer sitting in frame S observe the same two events at time t_1 & t_2. Then the time interval Δt b/w these two events is given as

$$\Delta t = t_2 - t_1 \longrightarrow (2)$$

Using inverse L.T.

$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$ in eq. (2)

$$\Delta t = \gamma\left(t_2' + \frac{v}{c^2}x_2'\right) - \gamma\left(t_1' + \frac{v}{c^2}x_1'\right)$$

$$\Delta t = \gamma\left[t_2' + \frac{v}{c^2}x_2' - t_1' - \frac{v}{c^2}x_1'\right]$$

$$\Delta t = \gamma\left[t_2' - t_1' + \frac{v}{c^2}(x_2' - x_1')\right] \rightarrow (8)$$

Since the position of clock is fixed during the measurement of time of two events. So, $x_1' = x_2'$

$$\Rightarrow \Delta t = \gamma\left[t_2' - t_1' + 0\right]$$

$$\Rightarrow \Delta t = \gamma \Delta t'$$

as $\gamma > 1$ so $\Delta t > \Delta t'$

which gives the time dilation.

Questions- Define a parameter $v = c \tanh\alpha$ and derive the expression of L.T & I.L.T. 26-03-2015

Solutions-

$$v = c\tanh\alpha \longrightarrow (1)$$

$$\Rightarrow \frac{v^2}{c^2} = \tanh^2\alpha$$

$$\sqrt{1 + \frac{v^2}{c^2}} = \sqrt{1 - \tanh^2\alpha}$$

$$\frac{1}{\gamma} = \operatorname{sech}\alpha$$

$$\Rightarrow \gamma = \cosh\alpha \longrightarrow (2)$$

According to L.T

$$x' = \gamma(x - vt)$$

Using (1) & (2).

$$x' = \cosh\alpha(x - (c\tanh\alpha)t)$$

$$x' = x\cosh\alpha - ct\sinh\alpha$$

Also by L.T.

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

Using (1) & (2)

$$t' = \cosh\alpha\left(t - \frac{c\tanh\alpha}{c^2}x\right)$$

$$t' = t\cosh\alpha - \frac{x}{c}\sinh\alpha$$

Finally we have

$$x' = x\cosh\alpha - ct\sinh\alpha$$

$$y' = y \quad , \quad z' = z$$

$$t' = t\cosh\alpha - \frac{x}{c}\sinh\alpha$$

Now inverse L.T.

$$x = \gamma(x' + vt')$$

Using (1) & (2).

$$x = \cosh\alpha(x' + (c\tanh\alpha)t')$$

$$x = x'\cosh\alpha + ct'\sinh\alpha$$

& $$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

Using (1) & (2).

$$t = \cosh\alpha \left(t' + \frac{c \tanh\alpha}{c^2} x' \right)$$

$$t = t' \cosh\alpha + \frac{x'}{c} \sinh\alpha$$

Finally we have

$$x = x' \cosh\alpha + ct' \sinh\alpha$$

$$y = y' \quad , \quad z = z'$$

$$t = t' \cosh\alpha + \frac{x'}{c} \sinh\alpha$$

Question: Define a parameter

$$\tan\phi = i\frac{v}{c}$$

and derive the expression of LT & inverse LT

Solution: $\tan\phi = i\frac{v}{c}$

$$\Rightarrow \frac{v}{c} = -i \tan\phi$$

$$\Rightarrow v = -ic \tan\phi \longrightarrow (1)$$

$$\frac{v^2}{c^2} = \tan^2\phi$$

$$\Rightarrow \sqrt{1 - \frac{v^2}{c^2}} = \sqrt{1 - \tan^2\phi}$$

$$\Rightarrow \frac{1}{\gamma} = \sec\phi$$

$$\Rightarrow \gamma = \cos\phi \longrightarrow (2)$$

By LT $x' = \gamma(x - tv)$

Using ① & ②

$$x' = \cos\phi\,(x - t(-ic\tan\phi))$$

$$\boxed{x' = x\cos\phi + ict\sin\phi}$$

and $y' = y$ & $z' = z$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

Using ① & ②

$$t' = \cos\phi\left(t - \frac{x}{c^2}(-ic\tan\phi)\right)$$

$$\boxed{t' = t\cos\phi + \frac{ix}{c}\sin\phi}$$

Now inverse L.T

$$x = \gamma(x' + vt)$$

Using ① & ②

$$x = \cos\phi\,(x' + t(-ic\tan\phi))$$

$$\boxed{x = x'\cos\phi - ict\sin\phi}$$

$y = y'$; $z = z'$

and

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

Using ① & ②

$$t = \cos\phi\left(t' + \frac{x'}{c^2}(-ic\tan\phi)\right)$$

$$t = t'\cos\phi - \frac{x'i}{c}\sin\phi$$

Here ϕ is called parameter.

"Velocity addition Formula" 02-04-2015

or "Transformation Law of Velocity"

Consider the motion of a particle in two frames S & S' which are in standard configuration. The space time coordinates in S and S' are (x, y, z, t) & (x', y', z', t') respectively.

Then the velocity of the particle, as measured in S is given as

$$V = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (V_x, V_y, V_z) \longrightarrow (1)$$

Similarly in frame S' the velocity of the particle is given as

$$V' = \left(\frac{dx'}{dt'}, \frac{dy'}{dt'}, \frac{dz'}{dt'}\right) = (V_x', V_y', V_z') \longrightarrow (2)$$

According to L.T we have

$$x' = \gamma(x - vt) \longrightarrow (3)$$

$$\& \quad t' = \gamma\left(t - \frac{v}{c^2}x\right) \longrightarrow (4)$$

Diff. eq. (3) w.r.t t' we have

$$\frac{dx'}{dt'} = \frac{d}{dt'}\left[\gamma(x - vt)\right]$$

$$= \frac{dt}{dt'} \cdot \frac{d}{dt}\left[\gamma(x - vt)\right]$$

$$= \frac{dt}{dt'}\left[\gamma\left(\frac{dx}{dt} - v\right)\right] \longrightarrow (5)$$

~~Diff. eq ④ w.r.t t. we have~~

$$\frac{dt'}{dt} = \gamma\left(1 - \frac{v}{c^2}V_x\right) \longrightarrow ⑥$$

Put ⑥ in ⑤ we have

$$\frac{dx'}{dt'} = \frac{1}{\cancel{\gamma}\left(1-\frac{v}{c^2}V_x\right)}\left[\cancel{\gamma}(V_x - v)\right]$$

$$\boxed{V_x' = \frac{V_x - v}{1 - \frac{v}{c^2}V_x}}$$

Now $y' = y$

Diff. w.r.t t'

$$\frac{dy'}{dt'} = \frac{dt}{dt'} \cdot \frac{d}{dt}(y)$$

using ⑥ ⇒

$$V_y' = \frac{1}{\gamma\left(1-\frac{v}{c^2}V_x\right)} \cdot \frac{dy}{dt}$$

$$\boxed{V_y' = \frac{V_y}{\gamma\left(1-\frac{v}{c^2}V_x\right)}}$$

Similarly

$$\boxed{V_z' = \frac{V_z}{\gamma\left(1-\frac{v}{c^2}V_x\right)}}$$

Remark:- In case $v << c$ i.e $\frac{v^2}{c^2} = 0$

Then $V_x' = V_x - v$

$$V_y' = V_y$$

&

$$V_z' = V_z$$

These represents G.T

P#30

Problem #

The proper length of rod is 10 m.

(i) Calculate the length of rod when it moving with velocity of (a):- 300 Km/s (b):- 0.99c

(ii):- Calculate the change in length of rod when it is moving with velocity of 3 Km/~~hour~~.

Solution:- (i)

$l = 10m$, $V = 300 \, Km/s$

$l' = ?$ $V = 300 \times 1000 m/s$

$\Rightarrow l' = \gamma l$ $V = 3 \times 10^5 m/s$

a):-

$$\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{C^2}}}$$

$$= \frac{1}{\sqrt{1 - \frac{9 \times 10^{10}}{9 \times 10^{16}}}} = \frac{1}{\sqrt{1 - 10^{-6}}}$$

$$= \frac{1}{\sqrt{0.999999}} = 1.0000005$$

$l' = (1.0000005) \times 10$

$l' = 10.000005 \text{ m}$

b):-

$$\gamma = \frac{1}{\sqrt{1 - \frac{(0.99c)^2}{c^2}}}$$

$$\gamma = \frac{1}{\sqrt{1 - 0.9801}} = \frac{1}{\sqrt{0.0199}}$$

$\gamma = 7.0881205$

$l' = \gamma l$

$= 7.0881205 \times 10$

$= 70.881205 \text{ m}$

(ii):- $V = 3 \text{ Km/hour}$

$= 3 \times 1000 \times \frac{1}{3600} \text{ m/s}$

$= \frac{30}{36} \text{ m/s}$

$$\gamma = \frac{1}{\sqrt{1 - \frac{(30/36)^2}{9 \times 10^{16}}}} = \frac{1}{\sqrt{1 - (7.7 \times 10^{-18})}}$$

$\gamma = 1$

$l' = \gamma l$

$= 1(10)$

$\Rightarrow l' = 10$

So, $\Delta l = l' - l$

$\Delta l = 10 - 10$

$\boxed{\Delta l = 0}$

3-04-15

Transformation Law of Acceleration

We know that

$$V_x' = \frac{V_x - v}{1 - \frac{v}{c^2}V_x} \longrightarrow (i)$$

&

$$\frac{dt'}{dt} = \gamma\left(1 - \frac{v}{c^2}V_x\right) \longrightarrow (ii)$$

Diff. (i) w.r.t t'.

$$\frac{d}{dt}(V_x') = \frac{dt}{dt'} \cdot \frac{d}{dt}\left[\frac{V_x - v}{1 - \frac{v}{c^2}V_x}\right]$$

$$a_x' = \frac{\left(1-\frac{v}{c^2}V_x\right)a_x - (V_x - v)\left(0 - \frac{v}{c^2}a_x\right)}{\left(1-\frac{v}{c^2}V_x\right)^2} \cdot \frac{dt}{dt'}$$

$$= \frac{a_x - \cancel{\frac{v}{c^2}V_x a_x} + \cancel{\frac{v}{c^2}V_x a_x} - \frac{v^2}{c^2}a_x}{\left(1-\frac{v}{c^2}V_x\right)^2} \cdot \frac{dt}{dt'}$$

$$= \frac{\left(1-\frac{v^2}{c^2}\right)a_x}{\left(1-\frac{v}{c^2}V_x\right)^2} \cdot \frac{1}{\gamma\left(1-\frac{v}{c^2}V_x\right)}$$

$$= \frac{\left(1-\frac{v^2}{c^2}\right)a_x}{\left(1-\frac{v}{c^2}V_x\right)^3} \cdot \sqrt{1-\frac{v^2}{c^2}} \leftarrow \gamma$$

$$\Rightarrow \boxed{a_x' = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2} a_x}{\left(1-\frac{v}{c^2}V_x\right)^3}}$$

To findout a_y', we know that

$$V_y' = \frac{V_y}{\gamma\left(1-\frac{v}{c^2}V_x\right)}$$

Diff. w.r.t t', we have

$$\frac{d}{dt'}(V_y') = \frac{dt}{dt'} \cdot \frac{d}{dt}\left(\frac{V_y}{\gamma\left(1-\frac{v}{c^2}V_x\right)}\right)$$

$$a_y' = \frac{\left(1-\frac{v}{c^2}V_x\right)a_y - V_y\left(0-\frac{v}{c^2}a_x\right)}{\gamma\left(1-\frac{v}{c^2}V_x\right)^2} \cdot \frac{dt}{dt'}$$

$$= \frac{\cancel{\left(1-\frac{v}{c^2}V_x\right)}\left[a_y + \frac{v/c^2\, a_x V_y}{\left(1-\frac{v}{c^2}V_x\right)}\right]}{\gamma\left(1-\frac{v}{c^2}V_x\right)^{\cancel{2}}} \cdot \frac{1}{\gamma\cancel{\left(1-\frac{v}{c^2}V_x\right)}}$$

$$= \frac{\left[a_y + \frac{v/c^2\, a_x V_y}{\left(1-\frac{v}{c^2}V_x\right)}\right]}{\left(1-\frac{v}{c^2}V_x\right)^2} \cdot \frac{1}{\gamma^2}$$

$$\Rightarrow \boxed{a_y' = \frac{\left[a_y + \frac{v/c^2\, a_x V_y}{1-v/c^2\, V_x}\right]}{\left(1-\frac{v}{c^2}V_x\right)^2} \cdot \left(1-\frac{v^2}{c^2}\right)}$$

Similarly,

$$a_z' = \frac{\left(1-\frac{v^2}{c^2}\right)\left[a_z + \frac{v/c^2\, a_x V_z}{1-\frac{v}{c^2}V_x}\right]}{\left(1-\frac{v}{c^2}V_x\right)^2}$$

Rapidity or Pseudo Velocity

According to special relativity the resultant of two collinear velocities u and v is given by

$$w = \frac{u+v}{1+\frac{v}{c^2}u} \longrightarrow (A)$$

This shows that in special relativity incontract to the classical law for the addition of collinear velocities can not be added algebraically. We can, however define a quantity is called rapidity or Pseudo velocity which is related to the velocity and obeys the classical law for addition of collinear velocities. Now we define the rapidity y_v and y_u corresponding to v & u respectively as

$$y_v = c\tanh^{-1}\frac{v}{c}$$

$$\Rightarrow v = c\tanh\left(\frac{y_v}{c}\right) \longrightarrow (1)$$

and $$y_u = c\tanh^{-1}\left(\frac{u}{c}\right)$$

$$\Rightarrow u = c\tanh\left(\frac{y_u}{c}\right) \longrightarrow (2)$$

& $$y_w = c\tanh^{-1}\left(\frac{w}{c}\right)$$

$$\Rightarrow W = c\tanh\left(\frac{y_w}{c}\right) \longrightarrow (3)$$

Using ①, ② & ③ in Ⓐ.

$$c\tanh\left(\frac{y_w}{c}\right) = \frac{c\tanh\left(\frac{y_u}{c}\right) + c\tanh\left(\frac{y_v}{c}\right)}{1 + \frac{c\tanh(y_u/c)\cdot c\tanh\left(\frac{y_v}{c}\right)}{c^2}}$$

$$\cancel{c}\tanh\left(\frac{y_w}{c}\right) = \cancel{c}\left[\frac{\tan(y_u/c) + \tanh(y_v/c)}{1 + \tanh(y_u/c)\tanh(y_v/c)}\right]$$

$$\tanh\left(\frac{y_w}{c}\right) = \tanh\left(y_u/c + y_v/c\right)$$

$$\frac{y_w}{c} = \frac{y_u}{c} + \frac{y_v}{c}$$

$$\Rightarrow \quad y_w = y_u + y_v$$

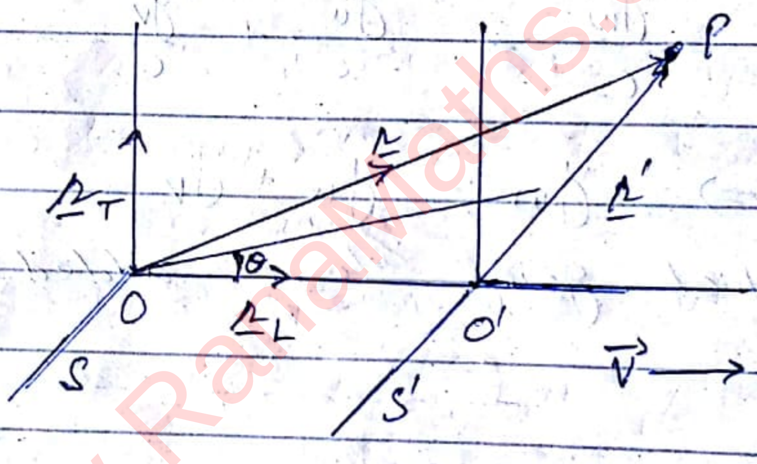
which going to be added algebraically.

14-04-2015

General L.T without Rotation

The previous case of L.T is called restricted L.T as in that case S' was assumed to move along common x-direction. Consider two frames of reference in relative motion such that the origins O & O' are coincident at $t = t' = 0$ and the relative velocity is along the common x-direction this transformation is called the restricted L.T.

Now we consider two frames of reference in relative motion such that there respective axes are parallel to eachother & there relative velocity is parallel to x-axes (say).



Let now we rotate the coordinate axes in these frames through same angle θ as measured in their own frames so as to obtain the coordinate system S & S'. The relative velocity between these two frames will not along x or x'-axes. The frame S' would actually be moving w.r.t S with a velocity $V = (V_x, V_y, V_z)$.

Note: Here we rotate both frames with same angle so we can say that no frame is rotated relative to other.

The corresponding L.T is called G.L.T without rotation.
(General means $\vec{v}$ is not fixed)

It means that by rotating back the coordinates axes through the same angle θ as measured in their respective frames we can bring the x-axes of both frames in the direction of relative velocity. In order to obtain G.L.T without rotation consider an event occuring at a point P with position vector $\bar{r}$ resolve this vector into two components r_L & r_T such that r_L is a component which is parallel to $\vec{v}$ & r_T is the component along the direction prependicular to $\vec{v}$. Then we can write,

$$r = r_L + r_T \longrightarrow ①$$

where
$$r_L = \frac{r \cdot \vec{v}}{v} \hat{v}$$
$$= \frac{r \cdot \vec{v}}{v} \cdot \frac{\vec{v}}{v}$$

$$r_L = \frac{r\cdot\vec{V}}{V^2}\vec{V} \longrightarrow ②$$

Using ② in ①.

$$r = \left(\frac{r\cdot V}{V^2}\right)\vec{V} + r_T$$

$$r_T = r - \left(\frac{r\cdot\vec{V}}{V^2}\right)\vec{V} \longrightarrow ③$$

For the same events in S', if r' be its position vector & r'_L and r'_T as its resolving parts along and prependicular to $\vec{V}$ respectively. Then we can write

$$r' = r'_L + r'_T \longrightarrow ④$$

Since r_L & r'_L moving along $\vec{V}$ so according to lorentz transformation we can write

$$r'_L = \gamma(r_L - Vt) \longrightarrow ⑤$$

Since r_T & r'_T are moving along the direction ⊥ to $\vec{V}$. So

$$r'_T = r_T \longrightarrow ⑥$$

Therefore Eq. ④ becomes

$$r' = \gamma(r_L - Vt) + r_T$$

Using Eq. ③

$$r' = \gamma(r_L - Vt) + r - \frac{r\cdot V}{V^2}\vec{V}$$

$$\Rightarrow \underline{r}' = \underline{r} + \gamma \underline{r}_L - \gamma \underline{V} t - \frac{\underline{r}\cdot\underline{V}}{V^2}\vec{V}$$

Using ②

$$\underline{r}' = \underline{r} + \gamma\left(\frac{\underline{r}\cdot\underline{V}}{V^2}\right)\vec{V} - \left(\frac{\underline{r}\cdot\underline{V}}{V^2}\right)\vec{V} - \gamma \underline{V} t$$

$$\underline{r}' = \underline{r} + \left[(\gamma-1)\frac{\underline{r}\cdot\underline{V}}{V^2} - \gamma t\right]\vec{V} \longrightarrow ⑦$$

We know that 16-04-15

$$t' = \gamma\left(t - \frac{\vec{V}}{c^2} r_L\right)$$

Put $r_L = r - r_T$

$$t' = \gamma\left[t - \frac{\vec{V}}{c^2}\cdot(\underline{r} - \underline{r}_T)\right]$$

$$t' = \gamma\left[t - \frac{1}{c^2}(\vec{V}\cdot\underline{r} - \cancel{\vec{V}\cdot\underline{r}_T})\right] \quad \because \vec{V}\perp \underline{r}_T$$

$$t' = \gamma\left(t - \frac{\vec{V}\cdot\underline{r}}{c^2}\right) \longrightarrow ⑧$$

Eq. ⑦ & ⑧ are called G.L.T without rotation.

Inverse G.L. Transformation without Rotation.

Let r_L' & r_T' are the resolved parts of $\underline{r}'$ along and $\perp$ to $\vec{V}$ respectively.

$$\underline{r}' = \underline{r}_L' + \underline{r}_T' \longrightarrow ①$$

$$\underline{r}_L' = \left(\frac{\underline{r}'\cdot\underline{V}}{V^2}\right)\vec{V} \longrightarrow ②$$

$$① \Rightarrow \quad r'_T = r' - r'_L$$

$$r'_T = r' - \left(\frac{r'\cdot\vec{V}}{V^2}\right)\vec{V} \longrightarrow ③$$

Using inverse L.T (Restricted)

$$r_L = \gamma(r'_L + v t') \longrightarrow ④$$

Since r_T and r'_T are moving along the direction $\perp$ to $\vec{V}$. So

$$r_T = r'_T \longrightarrow ⑤$$

$$\Rightarrow \quad r_T = r' - \left(\frac{r'\cdot\vec{V}}{V^2}\right)\vec{V} \longrightarrow ⑥$$

In frame S we can similarly write

$$r = r_L + r_T$$

using ④ & ⑥

$$r = \gamma(r'_L + \vec{v}\, t') + r' - \frac{r'\cdot\vec{V}}{V^2}\vec{V}$$

using ②

$$r = \gamma\left(\frac{r'\cdot\vec{V}}{V^2}\vec{V} + V t'\right) + r' - \frac{r'\cdot\vec{V}}{V^2}\vec{V}$$

$$r = r' + (\gamma - 1)\frac{r'\cdot\vec{V}}{V^2}\vec{V} + \gamma\vec{V}t'$$

$$r = r' + \left[(\gamma-1)\frac{r'\cdot\vec{V}}{V^2} + \gamma t'\right]\vec{V} \longrightarrow ⑦$$

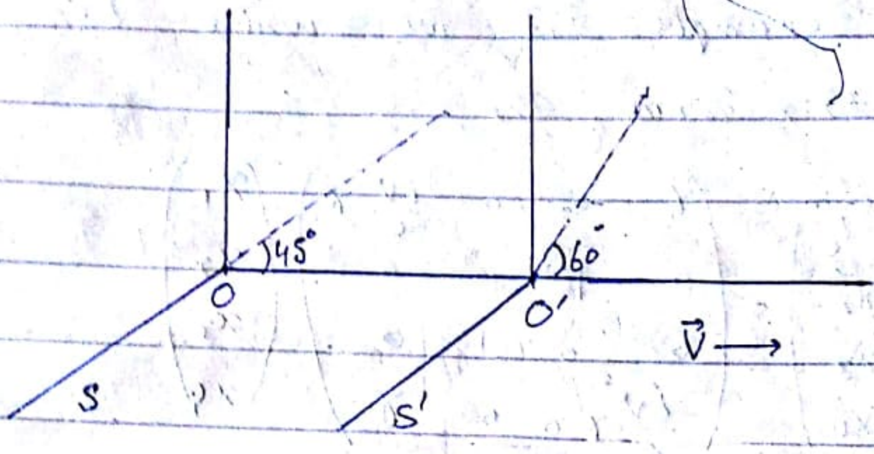
Also $\quad t = \gamma\left(t' + \frac{\vec{V}}{c^2} r'_L\right)$ using ②

$$t = \gamma\left(t' + \frac{\vec{V}}{c^2}\left(\frac{r'\cdot\vec{V}}{V^2}\right)\vec{V}\right)$$

$$t = \gamma\left(t' + \frac{r'\cdot\vec{V}}{c^2}\right) \longrightarrow ⑧$$

⑦ & ⑧ are I.G.L.T without rotation.

General L.T with Rotation



Let us consider the case when the coordinates axes in S & S' do not have the same orientation i.e we say that x- & x'-axes must be rotated through different angles as measured in their own frames in order to brings x-axes in the direction of $\vec{V}$.

To obtain the formula for G.L.T with rotation we need a further transformation through a rotation "R" of one of the frame, say S'. In matrix notation, this rotation is represented by a matrix R with change the vectors $\underline{r}$ and $\underline{v}$ by $R\underline{r}$ & $R\underline{v}$.

But the scalar product will remain unchanged. i.e

$$\underline{r}\cdot\underline{v} = (R\underline{r})\cdot(R\underline{v})$$

For example if we write L.T in matrix form as

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \frac{iv}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{iv}{c}\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\Rightarrow \quad X' = RX$$

Let $\underline{r} = (r_1, r_2, r_3, r_4)$

& $\underline{V} = (V_1, V_2, V_3, V_4)$

Then

$$R\underline{V} = \begin{pmatrix} \gamma & 0 & 0 & \frac{iv}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{iv}{c}\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}$$

$$\Rightarrow \quad R\underline{V} = \begin{pmatrix} \gamma V_1 + i\frac{v}{c}V_4\gamma \\ V_2 \\ V_3 \\ -\frac{i\gamma v}{c}V_1 + \gamma V_4 \end{pmatrix}$$

Similarly

$$R\underline{r} = \begin{pmatrix} \gamma r_1 + \frac{iv\gamma}{c} r_4 \\ r_2 \\ r_3 \\ -\frac{iv\gamma}{c} r_1 + \gamma r_4 \end{pmatrix}$$

Now

$$R_{\underline{v}}\cdot R_{\underline{r}} = \begin{pmatrix}\gamma v_1 + \frac{iv\gamma}{c}v_4\\ v_2\\ v_3\\ -\frac{iv\gamma}{c}v_1+\gamma v_4\end{pmatrix}\cdot\begin{pmatrix}\gamma r_1+\frac{iv\gamma}{c}r_4\\ r_2\\ r_3\\ -\frac{iv\gamma}{c}r_1+\gamma r_4\end{pmatrix}$$

$$= \left(\gamma v_1 + \frac{iv\gamma}{c}v_4\right)\left(\gamma r_1 + \frac{iv\gamma}{c}r_4\right) + r_2 v_2 + r_3 v_3 + \left(-\frac{iv\gamma}{c}v_1+\gamma v_4\right)\left(-\frac{iv\gamma}{c}r_1+\gamma r_4\right)$$

$$= \gamma^2 r_1 v_1 + \cancel{\frac{iv\gamma^2}{c}v_1 r_4} + \cancel{\frac{iv\gamma^2}{c}r_1 v_4} - \frac{v^2\gamma^2}{c^2}r_4 v_4 + r_2 v_2 + r_3 v_3 - \frac{v^2\gamma^2}{c^2}r_1 v_1 - \cancel{\frac{iv\gamma^2}{c}r_4 v_1} - \cancel{\frac{iv\gamma^2}{c}r_1 v_4} + \gamma^2 r_4 v_4$$

$$= r_2 v_2 + r_3 v_3 + \gamma^2\left[r_1 v_1 - \frac{v^2}{c^2}r_4 v_4 - \frac{v^2}{c^2}r_1 v_1 + r_4 v_4\right]$$

$$= r_2 v_2 + r_3 v_3 + \gamma^2\left[1(r_1 v_1 + r_4 v_4) - \frac{v^2}{c^2}(r_1 v_1 + r_4 v_4)\right]$$

$$= r_2 v_2 + r_3 v_3 + \cancel{\frac{1}{\left(1-\frac{v^2}{c^2}\right)}}\left[(r_1 v_1 + r_4 v_4)\cancel{\left(1-\frac{v^2}{c^2}\right)}\right]$$

$$= r_1 v_1 + r_2 v_2 + r_3 v_3 + r_4 v_4$$

$$R_{\underline{v}}\cdot R_{\underline{r}} = \underline{r}\cdot\underline{v}$$

Consequently in this case the G.L.T will becomes:

$$r' = R_{\underline{r}} + \left[(\gamma-1)\left(\frac{R_{\underline{r}}\cdot R_{\underline{v}}}{v^2}\right) - \gamma t\right]R_{\underline{v}}$$

$$= R\left[\underline{r} + \left\{(\gamma-1)\frac{\underline{r}\cdot\underline{v}}{v^2} - \gamma t\right\}\underline{v}\right] \longrightarrow (1)$$

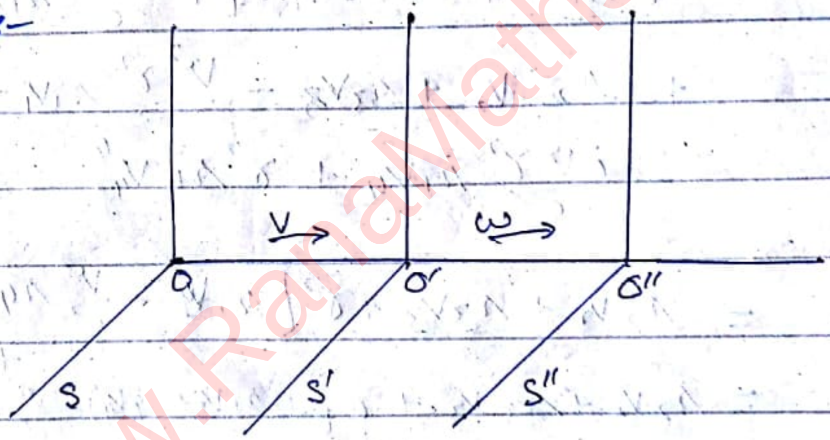
and
$$t' = \gamma\left(t - \frac{R\mathbf{r}\cdot R\mathbf{v}}{c^2}\right)$$
$$t' = \gamma\left(t - \frac{\mathbf{r}\cdot\mathbf{v}}{c^2}\right) \longrightarrow (2)$$

Eq. (1) & (2) gives the G.L.T with rotation.

17-04-2015

V.Imp Q:- Show that the composition of two L.T is also a L.T.

Sol:-



Let us consider three frames S, S' & S'' such that S' is moving relative to S with a uniform velocity v. While S'' is moving relative to S' with a uniform velocity ω. Now the L.T between S & S' is given as

$$\left.\begin{array}{l} x' = \gamma(x - vt) \longrightarrow (i) \\ y' = y \longrightarrow (ii) \quad z' = z \longrightarrow (iii) \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) \longrightarrow (iv) \end{array}\right\} \longrightarrow (1)$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

Similarly L.T between S' & S'' can be written as

$$\left.\begin{aligned} x'' &= \gamma'(x' - \omega t') \longrightarrow (i) \\ y'' &= y' \longrightarrow (ii) \\ z'' &= z' \longrightarrow (iii) \\ t'' &= \gamma'\left(t' - \frac{\omega}{c^2}x'\right) \longrightarrow (iv) \end{aligned}\right\} \longrightarrow (2)$$

where $\gamma' = \frac{1}{\sqrt{1-\frac{\omega^2}{c^2}}}$

Using Eqs (i) & (iv) of (1) in Eq (i) of (2). We have

$$x'' = \gamma'\left[\gamma(x - vt) - \omega\gamma\left(t - \frac{v}{c^2}x\right)\right]$$

$$x'' = \gamma\gamma'\left[x - vt - \omega t + \frac{v}{c^2}\omega x\right]$$

$$x'' = \gamma\gamma'\left[x\left(1 + \frac{wv}{c^2}\right) - (v + w)t\right] \longrightarrow (3)$$

Now consider

$$\gamma\gamma' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \cdot \frac{1}{\sqrt{1-\frac{\omega^2}{c^2}}}$$

$$= \frac{1}{\sqrt{1 + \frac{v^2\omega^2}{c^4} - \frac{v^2}{c^2} - \frac{w^2}{c^2}}}$$

$$= \frac{1}{\sqrt{(1)^2 + \left(\frac{v\omega}{c^2}\right)^2 + \frac{2v\omega}{c^2} - \frac{v^2}{c^2} - \frac{\omega^2}{c^2} - 2\frac{v\omega}{c^2}}}$$

$$= \frac{1}{\sqrt{\left(1+\frac{vw}{c^2}\right)^2 - \frac{1}{c^2}\left(v^2+w^2+2vw\right)}}$$

$$= \frac{1}{\sqrt{\left(1+\frac{vw}{c^2}\right)^2 - \frac{1}{c^2}\left(v+w\right)^2}}$$

$$\gamma\gamma' = \frac{1}{\left(1+\frac{vw}{c^2}\right)\sqrt{1 - \frac{(v+w)^2/c^2}{\left(1+\frac{vw}{c^2}\right)^2}}} \longrightarrow (4)$$

Using (4) in (3) we have

$$x'' = \frac{\cancel{\left(1+\frac{vw}{c^2}\right)}\left[x - \frac{v+w}{1+\frac{vw}{c^2}}t\right]}{\cancel{\left(1+\frac{vw}{c^2}\right)}\sqrt{1 - \frac{(v+w)^2/c^2}{\left(1+\frac{vw}{c^2}\right)^2}}}$$

$$x'' = \frac{x - \frac{v+w}{1+\frac{vw}{c^2}}t}{\sqrt{1 - \frac{(v+w)^2/c^2}{\left(1+\frac{vw}{c^2}\right)^2}}}$$

Put $u = \dfrac{v+w}{1+\frac{vw}{c^2}}$

$$\Rightarrow x'' = \frac{x-ut}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$x'' = \gamma''(x-ut) \longrightarrow (5)$$

where $\gamma'' = \dfrac{1}{\sqrt{1-\frac{u^2}{c^2}}}$

From (1) & (2) obviously,

$$y'' = y$$

$$\& \quad z'' = z$$

Now using Eqs (i) & (iv) of (1) in Eq. (iv) of (2). we get

$$t'' = \gamma'\left[\gamma\left(t - \frac{V}{c^2}x\right) - \frac{w}{c^2}\gamma(x - Vt)\right]$$

$$t'' = \gamma\gamma'\left[t - \frac{V}{c^2}x - \frac{\omega}{c^2}x + \frac{\omega V}{c^2}t\right]$$

$$t'' = \gamma\gamma'\left[t\left(1 + \frac{V\omega}{c^2}\right) - \frac{x}{c^2}(V + \omega)\right]$$

Using (4)

$$t'' = \frac{\cancel{\left(1 + \frac{V\omega}{c^2}\right)}\left[t - \frac{x}{c^2}\frac{V+\omega}{\left(1 + \frac{V\omega}{c^2}\right)}\right]}{\cancel{\left(1 + \frac{V\omega}{c^2}\right)}\sqrt{1 - \frac{(V+\omega)^2}{\left(1 + \frac{V\omega}{c^2}\right)^2}\cdot\frac{1}{c^2}}}$$

$$\Rightarrow \quad t'' = \frac{t - \frac{u}{c^2}x}{\sqrt{1 - \frac{u^2}{c^2}}} \quad \text{where} \quad u = \frac{V+\omega}{1 + \frac{V\omega}{c^2}}$$

$$\Rightarrow t'' = \gamma''\left(t - \frac{u}{c^2}x\right) \longrightarrow (6)$$

$$\text{where } \gamma'' = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Finally we have

$$\left.\begin{aligned} x'' &= \gamma''(x - ut) \\ y'' &= y \\ z'' &= z \\ t'' &= \gamma''\left(t - \frac{u}{c^2}x\right) \end{aligned}\right\} \longrightarrow (7)$$

Eq. (7) gives the composition

of two L.Ts, which is again a L.T with resultant velocity.

$$\boxed{u = \frac{v+w}{1+\frac{wv}{c^2}}}$$

Question:- Prove that the result of two velocities each of which are less than c, is also less than c.

OR

It is impossible to obtain velocity of light by adding velocities which are less than c.

Answer:- We know that when S' is moving relative to S with a uniform velocity v and S'' is moving relative to S' with a uniform velocity w then S'' will move relative to S with a uniform velocity u, given by

$$u = \frac{v+w}{1+\frac{vw}{c^2}} \longrightarrow (1)$$

Here $v, w < c$

Now consider

$$c - u = c - \frac{v+w}{1+\frac{vw}{c^2}}$$

$$c-u = \frac{1}{1+\frac{vw}{c^2}}\left[c\left(1+\frac{vw}{c^2}\right)-v-w\right]$$

$$c-u = \frac{c}{1+\frac{vw}{c^2}}\left[1+\frac{vw}{c^2}-\frac{v}{c}-\frac{w}{c}\right]$$

$$c-u = \frac{c}{1+\frac{vw}{c^2}}\left[1\left(1-\frac{w}{c}\right)-\frac{v}{c}\left(1-\frac{w}{c}\right)\right]$$

$$c-u = \frac{c}{1+\frac{vw}{c^2}}\left[\left(1-\frac{v}{c}\right)\left(1-\frac{w}{c}\right)\right] \longrightarrow (2)$$

As $v<c$ & $w<c \Rightarrow \frac{v}{c}<1$ & $\frac{w}{c}<1$

$\Rightarrow \left(1-\frac{v}{c}\right)>0$ & $\left(1-\frac{w}{c}\right)>0$

also $\frac{c}{1+\frac{vw}{c^2}}>0$ (always)

So, Eq. (2) implies that

$$c-u>0$$

$$\Rightarrow c>u$$

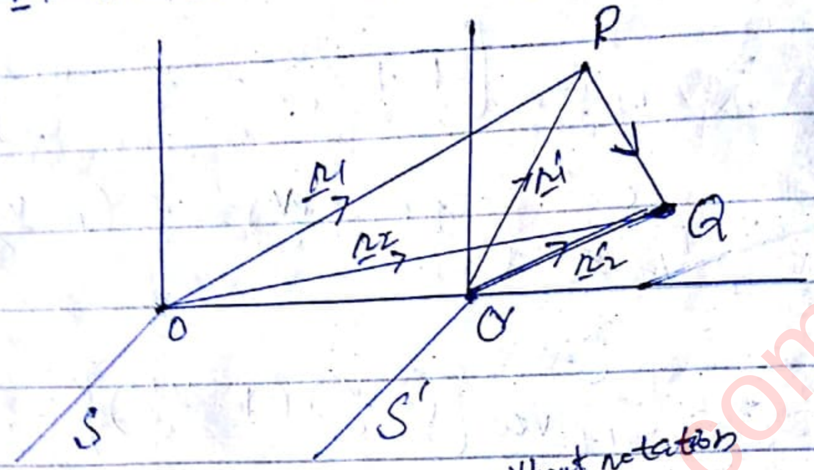
As required.

Q:- 24-04-2015

If two vectors are $\perp$ in S it is not necessary that they are $\perp$ in S'. What are the conditions under which these becomes $\perp$.

Sol:- Let $\vec{r}_1$ & $\vec{r}_2$ be the position vectors of two points P & Q in S &

the corresponding position vectors in S' are $\underline{r}_1'$ & $\underline{r}_2'$ respectively.



Then by G.L.T without rotation we can write

$$\underline{r}_1' = \underline{r}_1 + \left[(\gamma-1)\frac{\underline{r}_1\cdot\underline{V}}{V^2} - \gamma t\right]\vec{V} \longrightarrow (1)$$

&

$$\underline{r}_2' = \underline{r}_2 + \left[(\gamma-1)\frac{\underline{r}_2\cdot\underline{V}}{V^2} - \gamma t\right]\vec{V} \longrightarrow (2)$$

Denoting $\vec{PQ}$ in S $= \underline{x} = \underline{r}_2 - \underline{r}_1 \longrightarrow (3)$

Denoting $\vec{PQ}$ in $S' = \underline{x}' = \underline{r}_2' - \underline{r}_1' \longrightarrow (4)$

Using (1) & (2) in (4) we have

$$\underline{x}' = \underline{r}_2 - \underline{r}_1 + \left[\frac{(\gamma-1)}{V^2}(\underline{r}_2\cdot\underline{V} - \underline{r}_1\cdot\underline{V})\right]\vec{V}$$

using (3)

$$\underline{x}' = \underline{x} + \left[\frac{(\gamma-1)}{V^2}(\underline{r}_2 - \underline{r}_1)\cdot\underline{V}\right]\vec{V}$$

$$\underline{x}' = \underline{x} + \left[\frac{(\gamma-1)}{V^2}\underline{x}\cdot\underline{V}\right]\vec{V} \longrightarrow (5)$$

Let $\underline{x}_1$ & $\underline{x}_2$ be two vectors in S & $\underline{x}_1'$, $\underline{x}_2'$ are corresponding vectors in S'. Then it is given that $\underline{x}_1 \perp \underline{x}_2$

i.e $\underline{x}_1 \cdot \underline{x}_2 = 0 \longrightarrow (6).$

From eq. (5) we can write,

$$\underline{x}_1' = \underline{x}_1 + \left[\frac{\gamma-1}{V^2}\, \underline{x}_1 \cdot \underline{V}\right] \vec{V} \longrightarrow (7)$$

$$\& \quad \underline{x}_2' = \underline{x}_2 + \left[\frac{\gamma-1}{V^2}\, \underline{x}_2 \cdot \underline{V}\right] \vec{V} \longrightarrow (8)$$

Taking dot product of (7) & (8).

$$\underline{x}_1' \cdot \underline{x}_2' = \left\{\underline{x}_1 + \left[\frac{\gamma-1}{V^2}\, \underline{x}_1 \cdot \underline{V}\right] \vec{V}\right\} \cdot \left\{\underline{x}_2 + \left[\frac{\gamma-1}{V^2}\, \underline{x}_2 \cdot \underline{V}\right] \vec{V}\right\}$$

$$= \underbrace{\underline{x}_1 \cdot \underline{x}_2}_{0 \text{ by } (6)} + \frac{\gamma-1}{V^2}(\underline{x}_2 \cdot \underline{V})(\underline{x}_1 \cdot \underline{V}) + \frac{\gamma-1}{V^2}(\underline{x}_2 \cdot \underline{V})(\underline{x}_1 \cdot \underline{V})$$

$$+ \frac{(\gamma-1)^2}{V^4}(\underline{x}_1 \cdot \underline{V})(\underline{x}_2 \cdot \underline{V})\, V^2$$

$$= 2 \cdot \frac{\gamma-1}{V^2}(\underline{x}_1 \cdot \underline{V})(\underline{x}_2 \cdot \underline{V}) + \frac{(\gamma-1)^2}{V^2}(\underline{x}_1 \cdot \underline{V})(\underline{x}_2 \cdot \underline{V})$$

$$= \frac{\gamma-1}{V^2}(\underline{x}_1 \cdot \underline{V})(\underline{x}_2 \cdot \underline{V})\left[2 + \gamma - 1\right]$$

$$= \frac{\gamma-1}{V^2}(\underline{x}_1 \cdot \underline{V})(\underline{x}_2 \cdot \underline{V})\,(\gamma+1)$$

$$= \frac{\gamma^2-1}{V^2}(\underline{x}_1 \cdot \underline{V})(\underline{x}_2 \cdot \underline{V}) \neq 0$$

in General.

$\Rightarrow$ $\underline{x}_1'$ & $\underline{x}_2'$ are not necessarily be $\perp$. These may be $\perp$ if

(i). $\gamma = 1$ i.e. in Newtonian limit.

(ii) $\underline{x}_1 \cdot \underline{V} = 0$, i.e $\underline{x}_1 \perp \underline{V}$

(iii) $\underline{x}_2 \cdot \underline{V} = 0$ i.e $\underline{x}_2 \perp \underline{V}$.

Now let $\underline{x}_1$ & $\underline{x}_2$ are parallel in S i.e. $\underline{x}_1 \times \underline{x}_2 = 0 \longrightarrow (7)$

$$\Rightarrow \underline{x}_1' \times \underline{x}_2' = \left[\underline{x}_1 + \left[\frac{(\gamma-1)}{V^2}\, \underline{x}_1 \cdot \underline{V}\right]\vec{V}\right] \times \left[\underline{x}_2 + \left[\frac{\gamma-1}{V^2}\, \underline{x}_2 \cdot \underline{V}\right]\vec{V}\right]$$

by (7)

$$= \cancel{\underline{x}_1 \times \underline{x}_2} + \frac{\gamma-1}{V^2}(\underline{x}_2 \cdot \underline{V})(\underline{x}_1 \times \underline{V})$$

$$+ \frac{\gamma-1}{V^2}(\underline{x}_1 \cdot \underline{V})(\underline{V} \times \underline{x}_2) + \frac{(\gamma-1)^2}{V^4}(\underline{x}_1 \cdot \underline{V})(\underline{x}_2 \cdot \underline{V})\underbrace{\cancel{(\vec{V} \times \vec{V})}}_{0}$$

$$= \frac{\gamma-1}{V^2}\left[(\underline{x}_2 \cdot \underline{V})(\underline{x}_1 \times \underline{V}) + (\underline{x}_1 \cdot \underline{V})(\underline{V} \times \underline{x}_2)\right] \neq 0$$

In General

$\Rightarrow$ $\underline{x}_1'$ & $\underline{x}_2'$ are not necessarily be parallel. But These are parallel if

(i) $\gamma = 1$

(ii) $\underline{x}_1$ & $\underline{x}_2$ are perpendicular to $\underline{V}$.

(iii) $\underline{x}_1$ & $\underline{x}_2$ are parallel to $\underline{V}$.

In-Homogeneous L.Ts

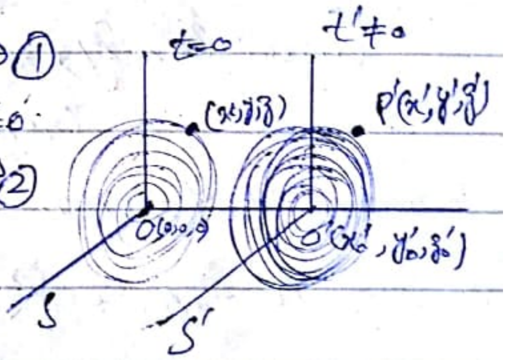
So far we have assume that the origins O & O' coinside at time $t' = t = 0$ so that the L.Ts are homogeneous. Now we shall consider the case when the origins O & O' are not coincident & $t' \neq 0$ at $t = 0$

Let us consider two frames S & S' in standard configuration such that their origins are not coincident & $t' \neq 0$ at $t = 0$.

Suppose that at $t=0$ a light signal is flashed from the origin $O(0,0,0)$ of S. Let the same signal is observed at at t_0' from the point (x_0', y_0', z_0') in S'. Then the light wave front in S & S' are respectively discribed by,

$$x^2+y^2+z^2-c^2t^2=0 \longrightarrow (1)$$

$$(x'-x_0')^2+(y'-y_0')^2+(z'-z_0')^2-c^2(t'-t_0')^2=0 \rightarrow (2)$$



Thus the transformation must yields

$$(x'-x_0')^2+(y'-y_0')^2+(z'-z_0')^2-c^2(t'-t_0')^2 = x^2+y^2+z^2-c^2t^2 \rightarrow (3)$$

Consequently, the L.T corresponding to the motion of S' relative to S along common x-axis with uniform velocity v is given by

$$x'-x_0' = \gamma(x-vt)$$

$$y'-y_0' = y$$

$$z' - z_0' = z$$

$$t' - t_0' = \gamma\left(t - \frac{v}{c^2}x\right)$$

Such transformation is called in-homogeneous L.T.

General inhomogeneous L.T without rotation will takes the form

$$\underline{r}' - \underline{r}_0' = \underline{r} + \left[(\gamma - 1)\frac{\underline{r}\cdot\underline{v}}{v^2} - \gamma t\right]\vec{v}$$

$$\& \quad t' - t_0' = \gamma\left[t - \frac{\underline{r}\cdot\underline{v}}{c^2}\right]$$

where $\underline{r}_0' = (x_0', y_0', z_0')$

Similarly,

General inhomogeneous L.T with rotation can be written as

$$\underline{r}' - \underline{r}_0' = R\left[\underline{r} + \left((\gamma-1)\frac{\underline{r}\cdot\underline{v}}{v^2} - \gamma t\right)\underline{v}\right]$$

$$\& \quad t' - t_0' = \gamma\left(t - \frac{\underline{r}\cdot\underline{v}}{c^2}\right)$$

Mid Term

Best of Luck

Minkowski Spacetime 23-04-2015

We shall now introduce the important geometrical concept of 4-dimensional spacetime continuum which was proposed by Minkowski in 1908.

We denote $x_1 = x$, $x_2 = y$, $x_3 = z$ & $x_4 = ict$ then any event in this spacetime will be expressed by a point (x_1, x_2, x_3, x_4).

If x, y, z, t & x', y', z', t' are the coordinates of some events in frames S & S' respectively. Then we know that

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$$

In Minkowski coordinates if x_1, x_2, x_3, x_4 & x_1', x_2', x_3', x_4' are the coordinates of some events in frames S & S' then it can be verified that

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2$$

Now we will write L.T in terms of Minkowski coordinates

According to L.T,

$$\left.\begin{aligned} x' &= \gamma(x - vt) \longrightarrow (i) \\ y' &= y \longrightarrow (ii) \\ z' &= z \longrightarrow (iii) \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \longrightarrow (iv) \end{aligned}\right\} \longrightarrow (1)$$

Eq (1)-(i) can be written as

$$x_1' = \gamma\left(x_1 - \frac{v}{ic}(ict)\right)$$

$$x_1' = \gamma\left(x_1 + \frac{iv}{c}x_4\right)$$

Eq (1) {(ii) & (iii)} are simply in Minkowski coordinates written as

$$x_2' = x_2$$

$$x_3' = x_3$$

Now Eq (1) (iv) can be written as

xing (ic).

$$ict' = \gamma\left(ict - \frac{v}{c^2}(ic)x\right)$$

$$x_4' = \gamma\left(x_4 - \frac{iv}{c}x_1\right)$$

Thus finally we have L.T in Minkowski coordinates as

$$x_1' = \gamma\left(x_1 + \frac{iv}{c}x_4\right)$$

$$x_2' = x_2$$

$$x_3' = x_3$$

$$x_4' = \gamma\left(x_4 - \frac{iv}{c}x_1\right)$$

In Matrix form

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \frac{iv}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{iv\gamma}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$\Rightarrow \quad \underline{X}' = A\underline{X}$

where $\underline{X}'$ & $\underline{X}$ are 4-vector in frame S & S' respectively and "A" is called matrix of L.T.

We can easily verify $AA^T = I$. Hence A is orthogonal matrix.

Since x_1, x_2, x_3, x_4 locate an event in 4-dimensional space time, so the point (x_1, x_2, x_3, x_4) may be considered as an event. Thus in 4-dim spacetime, a point represents an event occuring at certain time. Such a 4-dim spacetime

is called Minkowski Space.
It is also called world space or spacetime continuum.
The points and curves in this space are called world points & world time respectively.

V. Imp. [5 marks Q must]

05-05-2015

4-vector

A 4-vector A_μ $(\mu = 1,2,3,4)$ in Minkowski space-time continuum is defined as a quantity having four components A_1, A_2, A_3, A_4 which transform under L.T in the same way as the Minkowski coordinates. i.e

$$A_1' = \gamma\left(A_1 + \frac{iv}{c} A_4\right)$$

$$A_2' = A_2$$

$$A_3' = A_3$$

$$A_4' = \gamma\left(A_4 - \frac{iv}{c} A_1\right)$$

For a given observer, the first three components A_1, A_2, A_3 of A_μ will

Note: $i, j = 1, 2, 3$ But $\mu, \nu = 1, 2, 3, 4$

behave like ordinary 3-vector while the fourth component behave like a scalar in 3-dim space. Since under L.T every 4-vector remains invariant. So, we can write

Squaring & adding ⇒ $A_1'^2 + A_2'^2 + A_3'^2 + A_4'^2 = A_1^2 + A_2^2 + A_3^2 + A_4^2$

A 4-vector is said to be space like, time like or light like (null like) according as the square of its length is positive, negative or zero respectively.

$A_\mu \cdot A_\mu = A_1^2 + A_2^2 + A_3^2 + A_4^2$

Interval & Light Cone

Let us now examine the relationship b/w events may bear to each other. Consider two events occuring at points (x_1, y_1, z_1) & (x_2, y_2, z_2) at time t_1 & t_2 respectively in same frame S. Then in 4-dimensional spacetime these events will be represented in Minkowski spacetime by the

$\Delta x^2 \equiv (\Delta x)^2$

points (x_1, y_1, z_1, ict_1) & (x_2, y_2, z_2, ict_2) respectively. The distance b/w these two points in 4-dimensional spacetime is called the interval b/w two events. It is denoted by by ΔS, given by

$$\Delta S^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 + (ict_2 - ict_1)^2$$

$$\Delta S^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 - c^2(t_2-t_1)^2 \longrightarrow (1)$$

If we denote $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, $\Delta z = z_2 - z_1$ & $\Delta t = t_2 - t_1$ then Eq (1) takes the form,

$$\Delta S^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2$$

$$\Delta S^2 = \Delta r^2 - c^2 \Delta t^2 \longrightarrow (2)$$

Where $\Delta r = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$ is a spatial distance & Δt is the time interval b/w two events.

Eq. (2) implies that the interval b/w two events can be classified into following 3-categories.

Case I:- When $\Delta S^2 > 0$ i.e $\Delta r^2 > c^2 \Delta t^2$ or $\Delta r > c\Delta t$

It means the spatial distance Δr between two events is greater than than the distance $c\,\Delta t$, which a light ray can cover during b/w the events time Δt. So, the two events can't be connected causally. In this case, the two events are either so far apart in the coordinate space or occur in such a rapid succession that one of these occurs even before the light signal from the other can reach it. Such intervals are called space like intervals. Since no signal can be propagated with a velocity greater than that of light these can not be connected causally.

Case II:- When $\Delta s^2 < 0$

i.e $\Delta r^2 < c^2 \Delta t^2$ or $\Delta r < c\Delta t$

In this case the spatial distance b/w two events can be covered by a light signal during the time Δt therefore one of the events can be a cause of the other i.e the two events can be connected causally, such intervals b/w two events are said to be *time like intervals.*

Case III:- When $\Delta S^2 = 0$

i.e $\Delta r^2 = c^2 \Delta t^2$ or $\Delta r = c \Delta t$

It means the spatial distance b/w two events can be just covered by a light signal in time Δt and hence a causual relationship b/w two events can be set up. Such intervals are called *light like or null-like intervals.*

It is noted here that the path of light ray, in 4-dim. space time is always be represented by a null vector.

Since the length of 4-vector in the world space (M.S) is an invariant quantity, so the interval Δs b/w two events will remain invariant. Thus although the spatial & temporal intervals transform under L.T. So, there is no L.T. which can transform a space like interval into time like interval.

07-05-15

and vice versa.

It may be noted that the most significant difference b/w the metrics of space & space time given respectively as

$$(\Delta r)^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \longrightarrow (1)$$

$$(\Delta S)^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 \longrightarrow (2)$$

is that of signature.

The space metric, irrespective of the choice of the coordinate system i.e all the metric coefficients have positive signs which we represent by $(+ + +)$. In contrast, the signature of the Minkowski space-time is $(+ + + -)$ or equally we can take $(- - - +)$. A signature of that kind is called indefinite. Consequently, whereas the interval between two distinct points in space is always positive, in space-time it can be positive, negative or zero. Such a metric is sometimes called a pseudo metric.

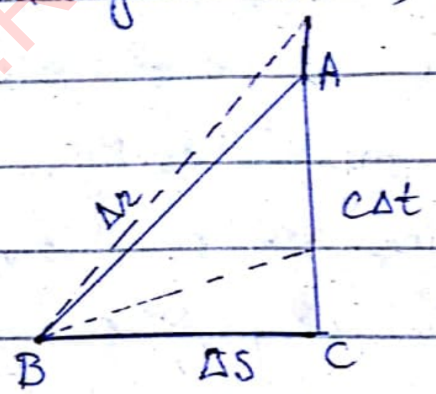
We shall now give a geometrical representation of the relation among the interval Δs between two events and the corresponding spatial and temporal intervals. We know that

$$\Delta s^2 = \Delta r^2 - c^2 \Delta t^2 \longrightarrow (3)$$

or $$\Delta r^2 = \Delta s^2 + c^2 \Delta t^2 \longrightarrow (4)$$

★ If $\Delta s^2 > 0$ i.e. the interval b/w two events is spacelike. Then Eq (4) represent pythagoras law as the quantities Δr^2 & $c^2 \Delta t^2$ are always positive.

Thus we can express these three quantities along the sides of a right angled triangle ABC, given as



In such a way that Δr is along hypotenuse $\overline{AB}$ & the other two intervals are along $\overline{AC}$ & $\overline{BC}$. Since Δs is invariant under L.T so the base

19-05-15

$\overline{BC}$ would remain fixed in all inertial frames. While the other two sides vary.

From ΔABC it is clear that, for a fixed ΔS the quantities Δr and $c\Delta t$ can either increase or decrease together; It is not possible to increase one of these & decrease the other or vice versa. Also it is not possible to change one quantity by keeping the other constant.

★ For the case of time-like interval Eq. (3) can be written as

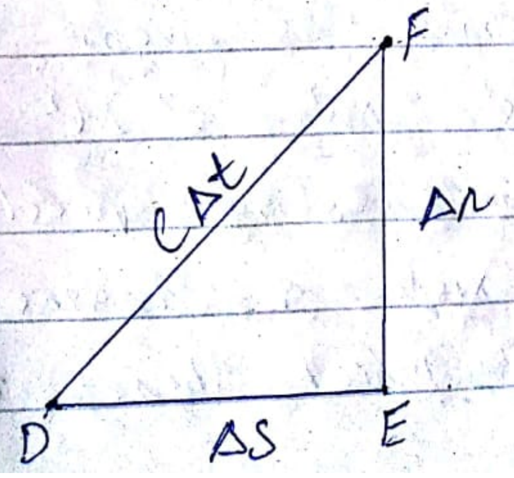
$$+\Delta S'^2 = \Delta r^2 - c^2 \Delta t^2$$

$\Delta S'^2 = -\Delta S^2$

$$-\Delta S^2 = \Delta r^2 - c^2 \Delta t^2$$

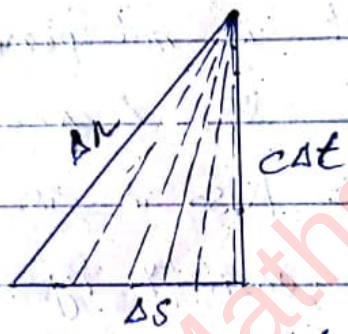
$$c^2 \Delta t^2 = \Delta r^2 + \Delta S^2$$

which can be expressed as the sides of right angle triangle CEF, given as



From ΔCEF it is clear that, for a fixed ΔS the quantities Δr & $c\Delta t$ can either increase or decrease together.

★ In the case of null interval Eq. ③ implies that $\Delta S^2 = 0$ i.e $\Delta r = c\Delta t$



Then the right-angled triangle collapses into a single vertical straight line equal to both Δr & $c\Delta t$.

Let us again examine two events which occur at different time and at different places in a frame of reference S. The interval ΔS b/w these two events is given by

$$\Delta S^2 = \Delta r^2 - c^2 \Delta t^2 \longrightarrow ①$$

An interesting question arises: Does there exist a frame of reference in which these two events occur simultaneously? If such a frame of reference, say S', does exist, then we must have

$$\Delta s'^2 = \Delta r'^2 - c^2 \Delta t'^2 = \Delta r'^2$$

$$\Delta s'^2 = \Delta r'^2 \rightarrow (2) \quad (\because \Delta t' = 0)$$

But always $\Delta r'^2 > 0$ b/c Δr is real. So Eq. (2) implies that $\Delta s'^2 > 0$.

Since interval b/w two events in space-time is invariant, so we can write $\Delta s^2 > 0$.

This shows that it is possible to find a frame of reference in which two given events occur at the same time (or simultaneously) provided that $\Delta s^2 > 0$ i.e if the interval between the two given events is space-like. For a time-like or a null interval such a frame would not exist.

Another interesting question is: Does there exist a frame of reference in which these two events occur in the reverse order? OR

In other words, is it possible to find a frame of reference say S'', in which an event A occuring earlier than another event B in a frame S, occurs later than B?

If this were possible, the time interval $\Delta t''$ between these two events as observed in the frame S'' would be negative:

$$\Delta t'' < 0 \longrightarrow (1)$$

According to L.T. we can connect $\Delta t''$ & Δt as

$$\Delta t'' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \longrightarrow (2)$$

From eq (1) & Eq (2) it is clear that

$$\gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) < 0$$

Since always $\gamma > 0$ so

$$\Delta t - \frac{v}{c^2}\Delta x < 0$$

$$\Delta t < \frac{v}{c^2}\Delta x$$

$$\frac{c^2}{v}\Delta t < \Delta x$$

$$\Rightarrow \Delta x > \frac{c^2}{v} \Delta t$$

Squaring

$$\Delta x^2 > \frac{c^2}{v^2} c^2 \Delta t^2$$

Since $\frac{c^2}{v^2} > 1 \Rightarrow \Delta x^2 > c^2 \Delta t^2$

$$\Delta x^2 - c^2 \Delta t^2 > 0$$

Since $\Delta y^2 \geq 0$ & $\Delta z^2 \geq 0$

Combining the last three relations, we get

$$\Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 > 0$$

$$\Delta r^2 - c^2 \Delta t^2 > 0$$

$$\Delta s^2 > 0$$

This shows that it is possible to find frames of reference in which two given events occur in the reverse order provided that the interval between the two events is space-like. For a time-like or null interval such frames would not exist.

Assignment:-

Show that for the two events occuring at different times and places, there exist a frame of reference in which these two events occur at the same place, provided the interval between the events is time-like.

Sol: If this is possible then

$\Delta x'' < 0$

According to L.T we can connect $\Delta x''$ & Δx as

$$\Delta x'' = \gamma(\Delta x - V\Delta t)$$

$$\Rightarrow \quad \gamma(\Delta x - V\Delta t) < 0$$

Since always $\gamma > 0$ so

$$\Delta x - V\Delta t < 0$$

$$\Delta x < V\Delta t$$

Squaring of both sides

$$\Delta x^2 < V^2 \Delta t^2$$

multiplying by $\frac{c^2}{V^2}$

$$\frac{c^2}{V^2}\Delta x^2 < c^2 \Delta t^2$$

Since $\frac{c^2}{V^2} > 1 \Rightarrow \Delta x^2 < c^2 \Delta t^2$

$$\Delta x^2 - c^2 \Delta t^2 < 0$$

Since $\Delta y^2 \geq 0$ & $\Delta z^2 \geq 0$

& Let $\Delta y^2 + \Delta z^2 < -(x^2 - c^2 \Delta t^2)$

So,

$$\cancel{\Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 < 0}$$

$$\cancel{\Delta s^2 < 0}$$

$$\Delta S^2 = \Delta r^2 - c^2 \Delta t^2$$

put $\Delta r^2 = 0$

$$\Delta s^2 = -c^2 \Delta t^2$$

$\Rightarrow$ $\Delta S^2 < 0$

Time like

DEPARTMENT OF MATHEMATICS UNIVERSITY OF SARGODHA

MSc-IV(R+SS)+BS-VIII (Mid Term Examination), April 30, 2015

Course Title: Special Relativity

Total Time: 60 min

ROLL NO: 177

Total Marks: 20

Q 1. Show that the resultant of two velocities which are less than c is less than c. (04)

Q 2. Define rapidity and show that the rapidities associated with the collinear velocities are added algebraically. (05)

Q 3. An event was observed at x= $3.2\times10^{8}\,m$ and t= 1.5 s in frame **S**. Find its respective coordinates in the frame **S′**, moving with velocity $\frac{1}{\sqrt{2}}$ c. (04)

G.

Q 4. Derive the Lorentz transformation without rotation. (07)

-: Good Luck :-

DEPARTMENT OF MATHEMATICS UNIVERSITY OF SARGODHA

MSc-IV(R+SS)+BS-VIII (Final Term Examination)

Course Title: Special Relativity

Total Time: 2 hrs

ROLL NO: 177

Total Marks: 60

SUBJECTIVE

(15-06-2015)

Q1. Answer the following short questions. $(4+4+2=10)$

i). Show that $p^2 + m_0^2 c^2 = m^2 c^2$, where all terms have usual meanings.

ii). Show that the 4-velocity $\vec{V_\mu}.\vec{V_\mu} = -c^2$.

iii). Define binding energy with example.

Q 2. Derive inhomogeneous L.T and G.L.T with and without rotation. (10)

Q 3. (a): Define proper time and derive its relation with ordinary time. (05)

(b): If $p=(0.5c, 0, 0)$ the 3-momentum of a particle whom mass in frame **S** is 25 kg. Find its mass in frame S', where S' is moving with velocity 0.5c relative to the frame S. (05)

Q 4. (a): Define 4-force and derive relativistic laws of motion. (05)

(b): Prove that if the vectors are perpendicular in a frame S may not be perpendicular in frame S'. (05)

Q 5. (a): Derive the energy mass relation. (07)

(b): Define time-like interval and its graphical representation. (03)

Q 6. (a): Show that the quantity $c^2\sqrt{1-V^2/c^2}\,dm_0/dt$ may be regarded as the energy taken from an external source, were m_0 is the variable proper mass of a particle moving with velocity **V**. (07)

(b): Obtain transformation law of total energy by using $E = mc^2$. (03)

-:(Good Luck):-

-:(Dr. Jamil Amir):-