04-03-2015

→ Exponensial Function:-

$$\text{Exp}(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \infty$$

→ Binomial

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \cdots \infty$$

→ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

→ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

→ $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

→ $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots$

→ **Function** = Rule + Input + Output

→ **Domain** = Set of input values.

→ **Range** = Possible values of output.

→ "Sin" function has domain $(-\infty, \infty)$

"Principle Sin" function has domain $(-\pi, \pi)$

→ Factorial Function is also a special function. (given by Christon Cramp 1808)

$\lfloor n = n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

If we say multiply first two integers $= 2! = 2(2-1)$

→ Positive integer (n, k) Positive real number

Double factorial, tripple factorial, ..., n-factorial.

→

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Gamma Function

The Gamma Function is defined as (Weierstrass).

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right) \exp\left(\frac{-z}{n}\right)\right].$$

Where γ is Euler's constant or Mascheroni cons. $\gamma = \lim_{n\to\infty} \left[H_n - \log n\right]$

$\gamma = 0.5772$ where $H_n = \sum_{k=1}^{n} \frac{1}{k}$

Integral form of Gamma Function By Euler is ;

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt , \quad Re(z) > 0$$

Replace z by $z+1$

$$\Gamma(1+z) = \int_0^\infty e^{-t}\, t^{z}\, dt$$

$$= -e^{-t} t^{z}\Big|_0^\infty - \int_0^\infty (-e^{-t})\, z\, t^{z-1}\, dt$$

$$= -\lim_{t\to\infty} e^{-t} t^{z} + \lim_{t\to 0} e^{-t} t^{z} + z\int_0^\infty e^{-t}\, t^{z-1}\, dt$$

$$= -\lim_{t\to\infty} \frac{t^{z}}{e^{t}} + \lim_{t\to 0} \cancel{\frac{t^{z}}{e^{t}}} + z\int_0^\infty e^{-t}\, t^{z-1}\, dt$$

$$= \qquad\qquad + z\,\Gamma(z)$$

$$\Rightarrow \Gamma(1+z) = z\,\Gamma(z)$$

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We know

$$\Gamma(z) = z^{-1} e^{-\gamma z} \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right) \exp\left(\frac{-z}{n}\right)\right]^{-1}$$

Taking "log" on both sides.

$$\log(\Gamma(z)) = -\log z + \log e^{-\gamma z} - \log\left[\prod_{n=1}^{\infty}\left(\left(1+\frac{z}{n}\right)\exp\left(\frac{-z}{n}\right)\right)\right]$$

$$\log(\Gamma(z)) = -\log z - \gamma z - \sum_{n=1}^{\infty}\left[\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right]$$

Taking derivative term by term.

$$\frac{1}{\Gamma(z)} \Gamma'(z) = \frac{-1}{z} - \gamma - \sum_{n=1}^{\infty}\left[\frac{1}{\left(1+\frac{z}{n}\right)}\left(\frac{1}{n}\right) - \frac{1}{n}\right]$$

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma + \sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+z}\right]$$

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma + \sum_{n=1}^{\infty}\left[\frac{z}{n(n+z)}\right]$$

05-03-2015

Singular Point:- A point that contain in a domain of a function, and function not defined on that point.

Isolated Singular point:- If function is analytic in neighbourhood of a singular point then this point is called isolated singular point.

Non-Isolated Singular Point:- Function not defined on singular point and its neighbourhood.

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Removeable Singularity:-

$$f(z) = \frac{\sin z}{z} \quad \text{at } z=0 \text{ is singular}$$

But $$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin z}{z} = 1$$

Essential Singularity:-

$$f(z) = \frac{\cos z}{z} \quad \text{at } z=0$$

Zero's:-

A point at which function becomes zero.

$$f(z) = z-a$$
zero at $z=a$.
$$f'(z) = 1 \neq 0$$ A **simple pole**.

if $$f(z) = (z-a)^2$$
$$f'(z) = 2(z-a)$$
$$f''(z) = 2$$ Its a pole of order 2.

Results from Gamma function

1):- Gamma Function is analytic except non-positive integers $[(0,-1,-2,\dots) \; \& \; \infty]$

2):- It has a simple pole at non-positive integer $(0,-1,-2,\dots\dots)$.

3):- It has an essential singularity at $z=\infty$, a point of condensation of poles.

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4):- $\Gamma(z)$ is never zero. [because $\frac{1}{\Gamma(z)}$ has no poles]

Evaluation of $\Gamma(1)$ & $\Gamma'(1)$:

The Weierstrass definition

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right) e^{-z/n}\right]$$

Put $z=1$

$$\frac{1}{\Gamma(1)} = e^{\gamma} \prod_{n=1}^{\infty} \left[\left(1+\frac{1}{n}\right) e^{-\frac{1}{n}}\right]$$

$$\frac{1}{\Gamma(1)} = e^{\gamma} \lim_{n\to\infty} \prod_{k=1}^{n} \left[\left(\frac{k+1}{k}\right) e^{-1/k}\right]$$

$$= e^{\gamma} \lim_{n\to\infty} \left[\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n+1}{n}\right) \cdot e^{-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}\right]$$

$$= e^{\gamma} \lim_{n\to\infty} \left[(n+1)\, e^{-\sum_{k=1}^{n}\frac{1}{k}}\right]$$

Harmonic Series

$$= e^{\gamma} \lim_{n\to\infty} \left[(n+1)\, e^{-H_n}\right] \qquad \because H_n = \sum_{k=1}^{n} \frac{1}{k}$$

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt, \ \operatorname{Re} z > 0$$

Put $z=1$

$$\Gamma(1) = \int_0^{\infty} e^{-t}\, dt$$

$$\text{"} = -e^{-t}\Big|_0^{\infty}$$

$$\text{"} = -\lim_{t\to\infty} \frac{1}{e^{t}} + e^{0}$$

$$\text{"} = -\frac{1}{e^{\infty}} + 1$$

$$\text{"} = -\frac{1}{\infty} + 1$$

$$= 0 + 1$$

$$\boxed{\Gamma(1) = 1}$$

We know that

$$\gamma = \lim_{n\to\infty} \left[H_n - \log n\right]$$

If we remove lim.

$$\gamma \approx H_n - \log n$$

$$H_n = \gamma + \log n + \varepsilon_n$$

$$\frac{1}{\Gamma(1)} = e^{\gamma} \lim_{n\to\infty} \left[(n+1)\, e^{-(\gamma + \log n + \varepsilon_n)}\right]$$

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$$\frac{1}{\Gamma(1)} = e^{\gamma} \lim_{n\to\infty} \left[(n+1)\, e^{-\gamma} \cdot e^{-\log n} \cdot e^{-\varepsilon_n}\right]$$

$$= \cancel{e^{\gamma}} \cdot \cancel{e^{-\gamma}} \lim_{n\to\infty} \left[(n+1)\left(\frac{1}{n}\right)\right] \lim_{n\to\infty} e^{-\varepsilon_n}$$

We know $\varepsilon_n \to 0$ as $n \to \infty$.

$$= \lim_{n\to\infty} \left[1+\frac{1}{n}\right] \cdot \cancel{e^{0}}\ 1$$

$$= 1 + 0$$

$$\frac{1}{\Gamma(1)} = 1$$

$$\Rightarrow \boxed{\Gamma(1) = 1}$$

Now, we know that,

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma + \sum_{n=1}^{\infty} \left[\frac{z}{n(n+z)}\right]$$

Put $z = 1$ & $\Gamma(1) = 1$

$$\Gamma'(1) = -1 - \gamma + \sum_{n=1}^{\infty} \left[\frac{1}{n(n+1)}\right]$$

$$= -1 - \gamma + \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right]$$ Partial fraction.

$$= -1 - \gamma + \lim_{n\to\infty} \left[\sum_{k=1}^{n} \left[\frac{1}{k} - \frac{1}{k+1}\right]\right]$$

$$= -1 - \gamma + \lim_{n\to\infty} \left[1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cdots + \cancel{\frac{1}{n}} - \frac{1}{n+1}\right]$$

$$= -1 - \gamma + \lim_{n\to\infty} \left[1 - \frac{1}{n+1}\right]$$

$$= -\cancel{1} - \gamma + \cancel{1} - 0$$

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$$\boxed{\Gamma'(1) = -\gamma}$$

Euler's Product for $\Gamma(z)$.

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)\exp\left(\frac{-z}{n}\right)\right]$$

$$\Gamma(z) = \frac{1}{z} e^{-\gamma z} \prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)^{-1}\exp\left(\frac{z}{n}\right)\right]$$

$$z\Gamma(z) = \exp(-\gamma z) \prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)^{-1}\exp\left(\frac{z}{n}\right)\right]$$

$$z\Gamma(z) = \exp(-\gamma z) \lim_{n\to\infty} \prod_{k=1}^{n}\left[\left(1+\frac{z}{k}\right)^{-1}\exp\left(\frac{z}{k}\right)\right] \to (1)$$

Since $\gamma = \lim_{n\to\infty}\left[H_n - \log n\right]$

$$= \lim_{n\to\infty}\left[H_n - \log(n+1)\right] \quad \because \log 1 = 0$$

n is very large So we can write n+1 instead of n

$$= \lim_{n\to\infty}\left[H_n - \sum_{k=1}^{n}\log\left(\frac{k+1}{k}\right)\right]$$

$$= \lim_{n\to\infty}\left[\sum_{k=1}^{n}\left(\frac{1}{k}\right) - \sum_{k=1}^{n}\log\left(\frac{k+1}{k}\right)\right]$$

Multiplying with $-z$ on both sides.

$$-z\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\left(\frac{-z}{k}\right) + z\sum_{k=1}^{n}\log\left(\frac{k+1}{k}\right)\right]$$

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Taking exp on both sides

$$\exp(-z\gamma) = \lim_{n\to\infty} \exp\left[\sum_{k=1}^{n}\left(\frac{-z}{k}\right) + \sum_{k=1}^{n} \log\left(\frac{k+1}{k}\right)^{z}\right]$$

$$= \lim_{n\to\infty} \left[\exp\sum_{k=1}^{n}\left(\frac{-z}{k}\right) \cdot \exp\sum_{k=1}^{n} \log\left(\frac{k+1}{k}\right)^{z}\right]$$

$$= \lim_{n\to\infty} \left[\exp\sum_{k=1}^{n}\log\left(\exp\left(\frac{-z}{k}\right)\right) \cdot \exp\sum_{k=1}^{n} \log\left(\frac{k+1}{k}\right)^{z}\right]$$

Since $\exp\sum_{k=1}^{n}\log(a_k) = \prod_{k=1}^{n}(a_k)$

$$\exp(-\gamma z) = \lim_{n\to\infty}\left[\prod_{k=1}^{n}\left(\exp\left(\frac{-z}{k}\right)\right) \cdot \prod_{k=1}^{n}\left(\frac{k+1}{k}\right)^{z}\right]$$

$$\exp(-\gamma z) = \lim_{n\to\infty}\prod_{k=1}^{n}\left[\exp\left(\frac{-z}{k}\right) \cdot \left(1+\frac{1}{k}\right)^{z}\right]$$

Put in eq. 1.

$$z\Gamma(z) = \lim_{n\to\infty}\prod_{k=1}^{n}\left[\cancel{\exp\left(\frac{-z}{k}\right)}\left(1+\frac{1}{k}\right)^{z} \cdot \left(1+\frac{z}{k}\right)^{-1}\cancel{\exp\left(\frac{z}{k}\right)}\right]$$

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right]$$

By using the Euler product for $\Gamma(z)$ 11/3/2015

we Replace z by $z+1$. To Prove $\Gamma(z+1) = z\Gamma(z)$

$$\Rightarrow \Gamma(z+1) = \left(\frac{1}{z+1}\right)\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z+1}\left(1+\frac{z+1}{n}\right)^{-1}\right]$$

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∴ Now

$$\frac{\Gamma(Z+1)}{\Gamma(Z)} = \left(\frac{Z}{Z+1}\right) \frac{\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{Z+1}\left(1+\frac{Z+1}{n}\right)^{-1}\right]}{\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{Z}\left(1+\frac{Z}{n}\right)^{-1}\right]}$$

$$= \left(\frac{Z}{Z+1}\right) \prod_{n=1}^{\infty}\left[\frac{\left(1+\frac{1}{n}\right)^{Z+1-Z} \cdot \left(1+\frac{Z}{n}\right)}{\left(1+\frac{Z+1}{n}\right)}\right]$$

$$= \left(\frac{Z}{Z+1}\right) \prod_{n=1}^{\infty}\left[\left(\frac{n+1}{n}\right)\cdot\left(\frac{n+Z}{\cancel{n}}\right)\Big/\left(\frac{n+Z+1}{\cancel{n}}\right)\right]$$

$$= \left(\frac{Z}{Z+1}\right) \lim_{n\to\infty} \prod_{K=1}^{n}\left[\left(\frac{K+1}{K}\right)\left(\frac{K+Z}{K+Z+1}\right)\right]$$

$$= \left(\frac{Z}{Z+1}\right) \lim_{n\to\infty}\left[\left(\frac{\cancel{2}}{1}\cdot\frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{4}}{\cancel{3}}\cdot\frac{\cancel{5}}{\cancel{4}}\cdots\frac{n+1}{\cancel{n}}\right)\left(\frac{1+Z}{\cancel{2+Z}}\cdot\frac{\cancel{2+Z}}{\cancel{3+Z}}\cdots\frac{n+Z}{n+Z+1}\right)\right]$$

$$= \left(\frac{Z}{\cancel{Z+1}}\right) \lim_{n\to\infty}\left[\frac{(n+1)(\cancel{1+Z})}{n+Z+1}\right]$$

or

$$\frac{Z}{Z+1}\lim_{n\to\infty}\left[\frac{(n+1)(1+Z)}{n+Z+1}\right]$$

$$= Z\lim_{n\to\infty}\left(\frac{n+1+Z-Z}{n+Z+1}\right)$$

$$= Z\lim_{n\to\infty}\left[1-\frac{Z}{n+Z+1}\right]$$

$$= Z\cdot(1)$$

$$= Z$$

$$= \left(\frac{Z}{\quad}\right) \lim_{n\to\infty}\left[\frac{\left(1+\frac{1}{n}\right)}{1+\frac{Z}{n}+\frac{1}{n}}\right]$$

$$= \left(\frac{Z}{\quad}\right)\cdot\frac{(1+0)}{1+0+0} = Z$$

$$\Rightarrow \Gamma(Z+1) = Z\,\Gamma(Z)$$

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The order Symboles o & O 12-03-15

Let R be a region in complex z-plane.

(1):— If $\lim\limits_{z\to c \text{ in } R} \frac{f(z)}{g(z)} = 0$

we write
$$f(z) = o[g(z)] \quad \text{as } z \to c \text{ in } R.$$

Example:— $f(z) = \sin^2 z$, $g(z) = z$, $z \to 0$

$\lim\limits_{z\to 0} \frac{\sin^2 z}{z} = 0$ as $z \to 0$ in R.

$$\sin^2(z) = o[z]$$

2):— If $\left|\frac{f(z)}{g(z)}\right|$ is bounded as $z \to c$ in R. We write
$$f(z) = O[g(z)] \quad \text{as } z \to c \text{ in } R.$$

Example:—
$$f(z) = \cos z - 4z,$$
$$g(z) = z \quad \text{as } z \to \infty$$

$\left|\frac{\cos z - 4z}{z}\right|$ is bounded as $z \to \infty$.

$\because |\cos z| \le 1$

We write
$$\cos z - 4z = O[z].$$

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Beta function

The function is defined as
$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt \quad , \; Re(p)>0, Re(q)>0$$

Beta function has two types

(i) Complete Beta function: limit $0 \to 1$.

(ii) In-complete Beta function: limit $0 \to x$.

Result or Corollary:-

Let $t = \sin^2\phi$ | Differential: Itself change in 1 variable.

$dt = 2\sin\phi\cos\phi\, d\phi$

When $t \to 0$, $\phi \to 0$

when $t \to 1$, $\phi \to \pi/2$

$$\beta(p,q) = \int_0^{\pi/2} \sin^{2(p-1)}\phi\, \underbrace{(1-\sin^2\phi)}_{\cos^2\phi}{}^{q-1} \cdot 2\sin\phi\cos\phi\, d\phi$$

$$= 2\int_0^{\pi/2} \sin^{2p-1}\phi \, \cos^{2q-1}\phi \, d\phi \longrightarrow (1)$$

Theorem:- If $Re(p)>0$ & $Re(q)>0$ then
$$\beta(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$ Relation b/w Γ & β func.

Proof:- Consider
$$\Gamma(p)\,\Gamma(q) = \int_0^\infty e^{-u} u^{p-1}\, du \int_0^\infty e^{-v} v^{q-1}\, dv$$

Put $u = x^2$, $v = y^2$

$du = 2x\,dx$, $dv = 2y\,dy$

when $u \to 0$, $x \to 0$ | $v \to 0$, $y \to 0$

$u \to \infty$, $x \to \infty$ | $v \to \infty$, $y \to \infty$

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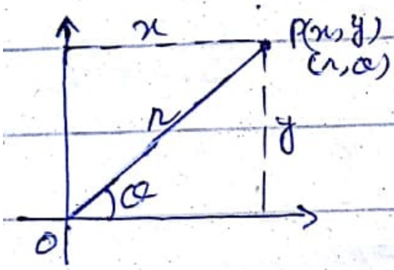
$$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-x^2} x^{2p-2}\, 2x\,dx \cdot \int_0^\infty e^{-y^2} y^{2q-2}\, 2y\,dy$$

$$= 4 \int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2p-1} y^{2q-1}\, dx\,dy$$

Let $x = r\cos\theta$, $y = r\sin\theta$

$$x^2+y^2 = r^2$$

$$dx\,dy = r\,dr\,d\theta$$



Using Jacobian.

$$J = \frac{dx\,dy}{dr\,d\theta} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$dx\,dy = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta$$

$$dx\,dy = r\,dr\,d\theta$$

$0<r<\infty$, $0<\theta<\frac{\pi}{2}$

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty\int_0^{\pi/2} e^{-r^2} r^{2p-1} \cos^{2p-1}\theta \cdot r^{2q-1} \sin^{2q-1}\theta \cdot r\,dr\,d\theta$$

$$= 4\int_0^\infty e^{-r^2} r^{2p+2q-1}\,dr \cdot \int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta\,d\theta \quad \rightarrow (2)$$

Let $r^2 = t \Rightarrow r = \sqrt{t}$

$$dr = \frac{1}{2\sqrt{t}}\,dt$$

When $r \to 0$, $t \to 0$

& $r \to \infty$, $t \to \infty$

Let $\theta = \frac{\pi}{2} - \phi$

$\Rightarrow d\theta = -d\phi$

When $\theta \to 0$, $\phi \to \frac{\pi}{2}$

& $\theta \to \frac{\pi}{2}$, $\phi \to 0$

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$\sin(\frac{\pi}{2}-\phi) = \cos\phi$

$\cos(\frac{\pi}{2}-\phi) = \sin\phi$

Eq. ② becomes,

$$\Gamma(p)\Gamma(q) = \cancel{4}^{2}\int_0^\infty e^{-t}\, t^{p+q-\frac{1}{2}} \left(\frac{dt}{\cancel{2}\sqrt{t}}\right) \cdot \int_{\pi/2}^{0} \sin^{2p-1}\phi \cos^{2q-1}\phi\,(-d\phi)$$

$$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-t}\, t^{p+q-1}\, dt \cdot 2\int_0^{\pi/2} \sin^{2p-1}\phi \cos^{2q-1}\phi\, d\phi$$

using eq ①

$$\Gamma(p)\Gamma(q) = \Gamma(p+q)\cdot B(p,q)$$

$$\Rightarrow B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Corollary:- We know that 17-03-2015

$$B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}\phi \cos^{2q-1}\phi\, d\phi$$

$$\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = 2\int_0^{\pi/2} \sin^{2p-1}\phi \cos^{2q-1}\phi\, d\phi$$

Put $2p-1 = m$, $2q-1 = n$

$\Rightarrow p = \frac{m+1}{2}$, $q = \frac{n+1}{2}$

So,

$$\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)} = \int_0^{\pi/2} \sin^m\phi \cdot \cos^n\phi\, d\phi,$$

$Re(m) > -1$, $Re(n) > -1$

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$\frac{\Gamma(z)\Gamma(1-z)}{\Gamma(z+1-z)} = \beta(z,1-z) = \int_0 t^{z-1}(1-t)^{-z}dt$

$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ If $z=1/2$ $\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}) = \pi$

$\Rightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}$

Theorem 6:- $Re(z)>0$, then

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

Proof:- To prove this theorem we shall establish the following four lemmas.

Lemma 1:- If $0 \le \alpha < 1$, then

$1+\alpha \le \exp(\alpha) \le (1-\alpha)^{-1}$. ← These are equal when $\alpha=0$

Proof:- We compare the following three series.

$$1+\alpha = 1+\alpha$$

$$\exp(\alpha) = 1+\alpha+\frac{\alpha^2}{2!}+\cdots\infty$$

$$(1-\alpha)^{-1} = 1+\alpha+\alpha^2+\cdots\infty$$

So, $1+\alpha \le \exp(\alpha) \le (1-\alpha)^{-1}$ Hence proved.

Lemma 2:- If $0 \le \alpha < 1$, then $(1-\alpha)^n \ge 1-n\alpha$ where n is a positive integer.

Proof:- For $n=1$

$1-\alpha = 1-\alpha$ (True)

Suppose that it is true for $n=K$

i.e $(1-\alpha)^K \ge 1-K\alpha$, ($K$ is a positive integer)

multiplying by $(1-\alpha)$.

$$\Rightarrow (1-\alpha)^{K+1} \ge (1-\alpha)(1-K\alpha)$$

$$\Rightarrow (1-\alpha)^{K+1} \ge 1-\alpha-K\alpha+K\alpha^2$$

$$\Rightarrow (1-\alpha)^{K+1} \ge 1-(K+1)\alpha+K\alpha^2 \quad (\because K\alpha^2>0)$$

$$\Rightarrow (1-\alpha)^{K+1} \ge 1-(K+1)\alpha \quad \text{(True)}$$

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So, it is true for all positive integer n.

Lemma: 3):- V.Imp. If $0 \le t < n$, then

$$0 \le e^{-t} - \left(1 - \frac{t}{n}\right)^n \le \frac{t^2 e^{-t}}{n}$$

where n is a positive integer.

Proof:- Since from Lemma (1).

$$1 + \alpha \le e^{\alpha} \le (1-\alpha)^{-1}$$

Put $\alpha = \frac{t}{n}$

$$\left(1 + \frac{t}{n}\right) \le e^{t/n} \le \left(1 - \frac{t}{n}\right)^{-1}$$

$$\left(1 + \frac{t}{n}\right)^n \le e^{t} \le \left(1 - \frac{t}{n}\right)^{-n} \longrightarrow (1)$$

Resiprocal.

$$\Rightarrow \left(1 + \frac{t}{n}\right)^{-n} \ge e^{-t} \ge \left(1 - \frac{t}{n}\right)^n \longrightarrow (2)$$

Consider from (2).

$$e^{-t} \ge \left(1 - \frac{t}{n}\right)^n$$

$$\Rightarrow e^{-t} - \left(1 - \frac{t}{n}\right)^n \ge 0$$

$$\Rightarrow 0 \le e^{-t} - \left(1 - \frac{t}{n}\right)^n \longrightarrow (3)$$

Now consider

$$e^{-t} - \left(1 - \frac{t}{n}\right)^n = e^{-t}\left[1 - e^{t}\left(1 - \frac{t}{n}\right)^n\right] \longrightarrow (4)$$

From (1)

$$\left(1 + \frac{t}{n}\right)^n \le e^{t} \quad \text{put in (4)}$$

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$0.6 < 0.8$

$(1-0.8)(4) = 0.8$

$(1-0.6)(4) = 1.6$

(4) becomes

$$e^{-t} - \left(1-\frac{t}{n}\right)^n \le e^{-t}\left[1-\left(1+\frac{t}{n}\right)^n\left(1-\frac{t}{n}\right)^n\right]$$
$$= e^{-t}\left[1-\left(1-\frac{t^2}{n^2}\right)^n\right] \longrightarrow (5)$$

From Lemma (2).

$$(1-\alpha)^n \ge 1-n\alpha$$

Put $\alpha = \frac{t^2}{n^2}$

$$\left(1-\frac{t^2}{n^2}\right)^n \ge \left(1-\frac{t^2}{n}\right)$$

Put in (5)

$$e^{-t} - \left(1-\frac{t}{n}\right)^n \le e^{-t}\left(\cancel{1}-\cancel{1}+\frac{t^2}{n}\right)$$

$$\Rightarrow e^{-t} - \left(1-\frac{t}{n}\right)^n \le \frac{t^2 e^{-t}}{n} \longrightarrow (6)$$

So, from (3) & (6)

$$0 \le e^{-t} - \left(1-\frac{t}{n}\right)^n \le \frac{t^2 e^{-t}}{n}$$

Lemma.4:- If $Re(z) > 0$, then

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt$$

where n is a positive integer.

Proof:- Consider the Integral

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt$$

Put $\frac{t}{n} = \beta \Rightarrow t = n\beta \Rightarrow dt = n\, d\beta$

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when $t \to 0$, $\beta \to 0$

& $t \to n$, $\beta \to 1$

So,

$$= \int_0^1 (1-\beta)^n n^{z-1} \beta^{z-1} n\, d\beta$$

$$= n^z \int_0^1 (1-\beta)^{n+1-1} \beta^{z-1} d\beta$$

$$= \frac{n^z \Gamma(n+1)\Gamma(z)}{\Gamma(n+z+1)}$$

$$= \frac{n^z \cdot n! \cancel{\Gamma(z)}}{(z+n)(z+n-1)\cdots z\cancel{\Gamma(z)}}$$

$$= \frac{n^z n!}{z(z+1)\cdots(z+n)}$$

m is +ve integer

$\Gamma(m+1) = m\Gamma(m)$

$= m(m-1)\Gamma(m)$

$= m(m-1)\cdots 3\cdot 2\cdot 1\, \Gamma(1)$, $\Gamma(1)=1$

$= m!$

If $m=0$

$\Gamma(1) = 0!$

$\Rightarrow 1 = 0!$

Put above

$$= \lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1} dt$$

$$= \lim_{n\to\infty} \frac{n^z n!}{z(z+1)(z+2)\cdots(z+n)}$$

$$= \Gamma(z)$$

219 Proof Lemma 4 by other Method: 18-3-2015

Consider the integral from R.H.S.

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1} dt$$

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Let $\frac{t}{n} = \beta \Rightarrow t = n\beta$

$dt = n\,d\beta$

when $t \to 0$ then $\beta \to 0$

& $t \to n$ then $\beta \to 1$

So, $\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\,dt = \int_0^1 (1-\beta)^n (n\beta)^{z-1}\, n\,d\beta$

$= n^z \int_0^1 (1-\beta)^n \beta^{z-1}\,d\beta$

Consider

$\int_0^1 \underset{I}{(1-\beta)^n} \underset{II}{\beta^{z-1}}\,d\beta = (1-\beta)^n \frac{\beta^{z-1+1}}{z-1+1}\Big|_0^1 - \frac{1}{z}\int_0^1 \beta^z n(1-\beta)^{n-1}(-1)\,d\beta$

$= (0-0) + \frac{n}{z}\int_0^1 \underset{I}{(1-\beta)^{n-1}} \underset{II}{\beta^z}\,d\beta$

$= \frac{n}{z}\left[(1-\beta)^{n-1}\frac{\beta^{z+1}}{z+1}\Big|_0^1 - \frac{1}{z+1}\int_0^1 \beta^{z+1}(n-1)(1-\beta)^{n-2}(-1)\,d\beta\right]$

$= \frac{n}{z}\left\{(0-0) + \frac{n-1}{z+1}\int_0^1 (1-\beta)^{n-2}\beta^{z+1}\,d\beta\right.$

$= \frac{n(n-1)}{z(z+1)}\int_0^1 (1-\beta)^{n-2}\beta^{z+1}\,d\beta$

By continuing the integration we will get,

$= \frac{n(n-1)(n-2)\cdots 3.2.1}{z(z+1)(z+2)\cdots(z+n-1)}\int_0^1 \beta^{z+n-1}\,d\beta$

$= \frac{n(n-1)(n-2)\cdots 3.2.1}{z(z+1)(z+2)\cdots(z+n-1)} \cdot \frac{\beta^{z+n-1+1}}{z+n-1+1}\Big|_0^1$

$= \frac{n(n-1)(n-2)\cdots 3.2.1}{z(z+1)(z+2)\cdots(z+n-1)(z+n)}$

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$$= \frac{n!}{z(z+1)\cdots(z+n)}$$

So,
$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt = \frac{n!\,n^z}{z(z+1)\cdots(z+n)}$$

Taking limit on Both sides, we obtain

$$\lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt = \lim_{n\to\infty} \frac{n^z\,n!}{z(z+1)(z+2)\cdots(z+n)}$$

By eq 4 on P# 12

$$\lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt = \Gamma(z)$$

Now ~~Theorem~~:- If $Re(z) > 0$ then
$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\,dt$$

~~Proof:~~

$$\Rightarrow \int_0^\infty e^{-t}\, t^{z-1}\,dt - \Gamma(z) = 0$$

We know that "If $Re(z)>0$ then
$$\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt\text{"}$$

$$\int_0^\infty e^{-t} t^{z-1}\,dt - \Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,dt - \lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt$$

$$= \lim_{n\to\infty}\left[\int_0^\infty e^{-t}\, t^{z-1}\,dt - \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt\right]$$

$$= \lim_{n\to\infty}\left[\int_0^n e^{-t}\, t^{z-1}\,dt + \int_n^\infty e^{-t}\, t^{z-1}\,dt - \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt\right]$$

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$$= \lim_{n\to\infty}\left[\int_0^n \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{z-1}\,dt + \int_n^\infty e^{-t}\, t^{z-1}\,dt\right]$$

$$= \lim_{n\to\infty}\int_0^n \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{z-1}\,dt + \lim_{n\to\infty}\int_n^\infty e^{-t}\, t^{z-1}\,dt$$

limits are same

$$= \lim_{n\to\infty}\int_0^n \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{z-1}\,dt \longrightarrow (A)$$

Now By Lemma # 3.

"If $0 \le t < n$, then

$$0 \le e^{-t} - \left(1-\frac{t}{n}\right)^n \le \frac{t^2 e^{-t}}{n}$$

where n is positive integer".

& using the fact

$$|t^z| = t^{Re(z)}$$

$$\left|\int_0^n \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{z-1}\,dt\right| \le \int_0^n \frac{t^2 e^{-t}}{n}\, t^{Re(z)-1}\,dt$$

$$= \frac{1}{n}\int_0^n e^{-t}\, t^{Re(z)+1}\,dt$$

Now $\int_0^\infty e^{-t}\, t^{Re(z)+1}\,dt$ converges, $\lim_{n\to\infty} \frac{1}{n} = 0$

So, $\int_0^n e^{-t}\, t^{Re z+1}\,dt$ is bounded.

Therefore

$$\lim_{n\to\infty}\left[\int_0^n \left\{e^{-t} - \left(1-\frac{t}{n}\right)^n\right\} t^{z-1}\,dt\right] = 0$$

Thus equation (A) $\Rightarrow$

$$\int_0^\infty e^{-t}\, t^{z-1}\,dt - \Gamma(z) = 0$$

$$\Rightarrow \Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\,dt, \text{ if } Re(z) > 0$$

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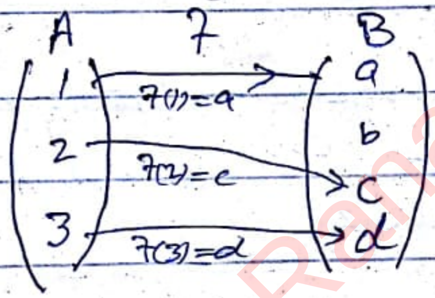
Sequence:-

A special type of function whose domain is subset of N and Range is C or R.

Convergence:- If a sequence approaches to one definite number then converges.

Divergence:- If a sequence approaches to $-\infty$, ∞ or more than one value is called divergence.

In a function Range $\subseteq$ Codomain.



Domain = A , codomain = B

Range = $\{a, c, d\}$

This function is 1-1 but not onto.

If f is onto then Range = Codomain.

If f is 1-1 & onto then inverse (f^{-1}) exist.

Euler or Maschroni Constant γ:- 19-03-2015

$$\gamma = \lim_{n\to\infty} (H_n - \operatorname{Log} n)$$

$$\gamma \approx 0.5772$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$

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Prove that γ exists & $0 \le \gamma < 1$.

Proof:- Let $A_n = H_n - \log n$ be a sequence. We have to show that A_n is decreasing

$$A_{n+1} - A_n = H_{n+1} - \log(n+1) - H_n + \log n$$

$$= H_{n+1} - H_n + \log(n) - \log(n+1)$$

$$= \sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} + \log\left(\frac{n}{n+1}\right)$$

$$= \cancel{\frac{1}{1}} + \cancel{\frac{1}{2}} + \cancel{\frac{1}{3}} + \cdots + \cancel{\frac{1}{n}} + \frac{1}{n+1} - \cancel{1} - \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} - \cdots - \cancel{\frac{1}{n}} + \log\left(\frac{n}{n+1}\right)$$

$$= \frac{1}{n+1} + \log\left(\frac{n+1-1}{n+1}\right)$$

$$= \frac{1}{n+1} + \log\left(1 - \frac{1}{n+1}\right)$$

Machlaurine Series

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \infty$$

Let

$$f(x) = \log(1-x) \quad \text{where } x = \frac{1}{n+1}$$

$$f(0) = \log 1 = 0$$

$$f'(x) = \frac{-1}{1-x} \Rightarrow f'(0) = -1$$

$$f''(x) = \frac{-1}{(1-x)^2} \Rightarrow f''(0) = -1$$

$$f'''(x) = \frac{-2}{(1-x)^3} \Rightarrow f'''(0) = -2$$

$$\Rightarrow f(x) = 0 + \frac{x}{1!}(-1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(-2) + \cdots$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

$$\Rightarrow f\left(\frac{1}{n+1}\right) = \frac{-1}{n+1} - \frac{1}{2(n+1)^2} - \frac{1}{3(n+1)^3} - \cdots$$

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So,

$$A_{n+1} - A_n = \cancel{\frac{1}{n+1}} - \cancel{\frac{1}{n+1}} - \frac{1}{2(n+1)^2} - \frac{1}{3(n+1)^3} - \cdots \infty$$

$$= -\left\{ \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \cdots \infty \right\}$$

$$= -\sum_{K=1}^{\infty} \frac{1}{(K+1)(n+1)^{K+1}} < 0$$

$$\Rightarrow \quad A_{n+1} - A_n < 0$$

$$\Rightarrow \quad A_{n+1} < A_n$$

Hence A_n is decreasing sequence.

Since $\frac{1}{t}$ decreases as t increases.

Therefore,

$$\frac{1}{K} < \int_{K-1}^{K} \frac{1}{t}\,dt < \frac{1}{K-1}, \quad K \geq 2$$

$$\Rightarrow \sum_{K=2}^{n} \frac{1}{K} < \sum_{K=2}^{n} \int_{K-1}^{K} \frac{1}{t}\,dt < \sum_{K=2}^{n} \frac{1}{K-1}$$

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^2 \frac{1}{t}\,dt + \int_2^3 \frac{1}{t}\,dt + \cdots + \int_{n-1}^{n} \frac{1}{t}\,dt < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$$

$$\sum_{K=1}^{n} \frac{1}{K} - 1 < \cancel{\log 2} - \underbrace{\log 1}_{0} + \cancel{\log 3} - \cancel{\log 2} + \cdots + \log n - \cancel{\log n-1} < \sum_{K=1}^{n} \frac{1}{K} - \frac{1}{n}$$

$$H_n - 1 < \log n < H_n - \frac{1}{n}$$

$$-1 < \log n - H_n < -\frac{1}{n}$$

$$1 > H_n - \log n > \frac{1}{n}$$

$$\frac{1}{n} < H_n - \log n < 1$$

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Taking limit $n \to \infty$

$$0 \leq \lim_{n\to\infty} (H_n - \log n) < 1$$

$$\Rightarrow \quad 0 \leq \gamma < 1$$ Hence proved.

Factorial Function

$$(\alpha)_n = \prod_{K=1}^{n} (\alpha + K - 1) \qquad \text{where } n \geq 1$$

$$= \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$$

$$(\alpha)_0 = 1 \quad , \alpha \neq 0$$

This factorial function is called **Pochhamer's Symbol.**

$$(3)_4 = 3(3+1)(3+2)(3+3) = 3\cdot4\cdot5\cdot6$$

$$(-3)_4 = -3(-3+1)(-3+2)(-3+3) = 0$$

$$(-3)_2 = -3(-3+1) = -3\cdot -2 = 6$$

$$(-m)_n = 0 \quad \text{If } m < n$$

Lemma 5:- Prove that

$$(\alpha)_{2n} = 2^{2n} \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n$$

Proof:- We know that

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1) \qquad n \geq 1$$

$$R.H.S = 2^{2n} \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n$$

$$= 2^{2n}\left\{\left(\frac{\alpha}{2}\right)\left(\frac{\alpha}{2}+1\right)\cdots\left(\frac{\alpha}{2}+n-1\right)\right\}\left\{\left(\frac{\alpha+1}{2}\right)\left(\frac{\alpha+1}{2}+1\right)\cdots\left(\frac{\alpha+1}{2}+n-1\right)\right\}$$

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$$= 2^{2n}\left\{\left(\frac{\alpha}{2}\right)\left(\frac{\alpha+2}{2}\right)\cdots\left(\frac{\alpha+2n-2}{2}\right)\right\}\left\{\left(\frac{\alpha+1}{2}\right)\left(\frac{\alpha+1+2}{2}\right)\cdots\left(\frac{\alpha+2n-1}{2}\right)\right\}$$

$$= 2^{2n}\cdot\frac{1}{2^n}\cdot\frac{1}{2^n}\left\{(\alpha)(\alpha+2)\cdots(\alpha+2n-2)\right\}\cdot\left\{(\alpha+1)(\alpha+3)\cdots(\alpha+2n-1)\right\}$$

$$= \frac{\cancel{2^{2n}}}{\cancel{2^{2n}}}\left((\alpha)(\alpha+1)(\alpha+2)(\alpha+3)\cdots(\alpha+2n-2)(\alpha+2n-1)\right)$$

$$= (\alpha)_{2n} = L\cdot H\cdot S$$

V. Imp.

Lemma 6:- If k is a positive integer & n is non-negative integer. Then prove that

$$(\alpha)_{kn} = k^{kn}\left(\frac{\alpha}{k}\right)_n\left(\frac{\alpha+1}{k}\right)_n\cdots\left(\frac{\alpha+k-1}{k}\right)_n$$

Proof:-

$$R\cdot H\cdot S = k^{kn}\left(\frac{\alpha}{k}\right)_n\left(\frac{\alpha+1}{k}\right)_n\cdots\left(\frac{\alpha+k-1}{k}\right)_n$$

$$= k^{kn}\left[\left(\frac{\alpha}{k}\right)\left(\frac{\alpha}{k}+1\right)\cdots\left(\frac{\alpha}{k}+n-1\right)\right]\cdot\left[\left(\frac{\alpha+1}{k}\right)\left(\frac{\alpha+1}{k}+1\right)\cdots\left(\frac{\alpha+1}{k}+n-1\right)\right]\cdots$$
$$\cdots\left[\left(\frac{\alpha+k-1}{k}\right)\left(\frac{\alpha+k-1}{k}+1\right)\cdots\left(\frac{\alpha+k-1}{k}+n-1\right)\right]$$

$$= k^{kn}\left[\left(\frac{\alpha}{k}\right)\left(\frac{\alpha+k}{k}\right)\cdots\left(\frac{\alpha+kn-k}{k}\right)\right]\cdot\left[\left(\frac{\alpha+1}{k}\right)\left(\frac{\alpha+k+1}{k}\right)\cdots\left(\frac{\alpha+1+kn-k}{k}\right)\right]$$
$$\cdots\left[\left(\frac{\alpha+k-1}{k}\right)\left(\frac{\alpha+k+k-1}{k}\right)\cdots\left(\frac{\alpha+\cancel{k}-1+kn-\cancel{k}}{k}\right)\right]$$

$$= k^{kn}\cdot\frac{1}{k^{kn}}\left[(\alpha)(\alpha+k)\cdots(\alpha+kn-k)\right]\left[(\alpha+1)(\alpha+k+1)\cdots(\alpha+1+kn-k)\right]\cdot$$
$$\cdots\left[(\alpha+k-1)(\alpha+2k-1)\cdots(\alpha+kn-1)\right]$$

$$= \left[(\alpha)(\alpha+1)(\alpha+2)\cdots(\alpha+k)(\alpha+k+1)(\alpha+2k)\right.$$
$$\left.(\alpha+2k-1)\cdots(\alpha+kn-1)\right]$$

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$$= (\alpha)_{kn}$$
$$= L.H.S$$

24-03-15

Q. 5 a:- Derive the properties of the Beta function.

$$p\,\beta(p, q+1) = q\,\beta(p+1, q)$$

Proof:- we know that

$$\beta(p, q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

So

$$L.H.S = p\,\beta(p, q+1)$$
$$= \frac{p\,\Gamma(p)\,\Gamma(q+1)}{\Gamma(p+q+1)} \qquad \because \Gamma(z+1) = z\,\Gamma(z)$$
$$\because \Gamma(q+1) = q\,\Gamma(q)$$
$$= \frac{p\,\Gamma(p)\,q\,\Gamma(q)}{\Gamma(p+q+1)}$$
$$= q\,\frac{\Gamma(p+1)\,\Gamma(q)}{\Gamma(p+1+q)}$$
$$= q\,\beta(p+1, q)$$
$$= R.H.S$$

b):- $\beta(p, q) = \beta(p+1, q) + \beta(p, q+1)$

Proof:- $R.H.S = \beta(p+1, q) + \beta(p, q+1)$

$$= \frac{\Gamma(p+1)\,\Gamma(q)}{\Gamma(p+1+q)} + \frac{\Gamma(p)\,\Gamma(q+1)}{\Gamma(p+q+1)}$$
$$= \frac{p\,\Gamma(p)\,\Gamma(q)}{\Gamma(p+q+1)} + \frac{\Gamma(p)\,q\,\Gamma(q)}{\Gamma(p+q+1)}$$

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$$= \frac{p\,\Gamma(p)\,\Gamma(q) + q\,\Gamma(p)\,\Gamma(q)}{\Gamma(p+q+1)}$$

$$= \frac{\cancel{(p+q)}\,\Gamma(p)\,\Gamma(q)}{\cancel{(p+q)}\,\Gamma(p+q)}$$

$$= \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

$$= \beta(p,q) = L.H.S$$

(C):- $(p+q)\,\beta(p,q+1) = q\,\beta(p,q)$

Proof:- $L.H.S = (p+q)\,\beta(p,q+1)$

$$= (p+q) \cdot \frac{\Gamma(p)\,\Gamma(q+1)}{\Gamma(p+q+1)}$$

$$= \frac{\cancel{(p+q)}\,\Gamma(p)\,q\,\Gamma(q)}{\cancel{(p+q)}\,\Gamma(p+q)}$$

$$= q\,\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

$$= q\,\beta(p,q) = R.H.S$$

d):- $\beta(p,q)\,\beta(p+q,r) = \beta(q,r)\,\beta(q+r,p)$

Proof:- $L.H.S = \beta(p,q)\,\beta(p+q,r)$

$$= \frac{\Gamma(p)\,\Gamma(q)}{\cancel{\Gamma(p+q)}} \cdot \frac{\cancel{\Gamma(p+q)}\,\Gamma(r)}{\Gamma(p+q+r)}$$

$$= \frac{\Gamma(p)\,\Gamma(q)\,\Gamma(r)}{\Gamma(p+q+r)} \cdot \frac{\Gamma(q+r)}{\Gamma(q+r)}$$

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$$= \frac{\Gamma(q)\Gamma(r)}{\Gamma(q+r)} \cdot \frac{\Gamma(q+r)\Gamma(p)}{\Gamma(q+r+p)}$$

$$= \beta(q,r) \cdot \beta(q+r, p) = R.H.S$$

Theorem # 9. If n is neither zero nor a negative integer. Then

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

Proof:- Consider

$$\Gamma(\alpha+n) = (\alpha+n-1)\Gamma(\alpha+n-1)$$

$$= (\alpha+n-1)(\alpha+n-2)\Gamma(\alpha+n-2)$$

$$= (\alpha+n-1)(\alpha+n-2)\cdots(\alpha)\Gamma(\alpha)$$

$$\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+n-1)$$

$$\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = (\alpha)_n$$

Lemma # 7. If n is integral & z is not a negative integer. then

$$\lim_{n\to\infty} \frac{(n-1)!\, n^z}{\Gamma(z+n)} = 1$$

Proof:- Since

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n-1)!\, n^z}{z(z+1)\cdots(z+n-1)}$$

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n-1)!\, n^z}{(z)_n}$$

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$$\because (z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}$$

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n-1)!\, n^z}{\frac{\Gamma(z+n)}{\Gamma(z)}}$$

$$\frac{\cancel{\Gamma(z)}}{\cancel{\Gamma(z)}} = \lim_{n\to\infty} \frac{(n-1)!\, n^z}{\Gamma(z+n)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{(n-1)!\, n^z}{\Gamma(z+n)} = 1$$

25-03-15

Q.6:- $\beta(p, n+1) = \frac{n!}{(p)_{n+1}}$

Proof:- $L.H.S = \beta(p, n+1)$

$$= \frac{\Gamma(p)\,\Gamma(n+1)}{\Gamma(p+n+1)} \longrightarrow (A)$$

We know that

$$\Gamma(n+1) = n! \longrightarrow (1)$$

& $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$

$$\Rightarrow (p)_n = \frac{\Gamma(p+n)}{\Gamma(p)}$$

replace n by $n+1$,

$$(p)_{n+1} = \frac{\Gamma(p+n+1)}{\Gamma(p)}$$

$$\frac{1}{(p)_{n+1}} = \frac{\Gamma(p)}{\Gamma(p+n+1)} \longrightarrow (2)$$

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using ① & ② in (A) we get.

$$= \frac{n!}{(P)_{n+1}}$$

$$= R.H.S$$

Q.7:— Evaluate

$$\int_{-1}^{1} (1+x)^{p-1} (1-x)^{q-1} dx$$

Solution:— $\int_{-1}^{1} (1+x)^{p-1} (1-x)^{q-1} dx$

Put $1+x = 2t \Rightarrow 1-2t = -x$

$\Rightarrow dx = 2dt \qquad \Rightarrow 2-2t = 1-x$

when $x \to -1$, $t \to 0$

& $x \to 1$, $t \to 1$

$$= \int_{0}^{1} (2t)^{p-1} (2-2t)^{q-1} \, 2dt$$

$$= 2 \int_{0}^{1} 2^{p-1} (t)^{p-1} \, 2^{q-1} (1-t)^{q-1} dt$$

$$= 2^{1+p-1+q-1} \int_{0}^{1} t^{p-1} (1-t)^{q-1} dt$$

$$= 2^{p+q-1} \int_{0}^{1} t^{p-1} (1-t)^{q-1} dt$$

$$= 2^{p+q-1} \beta(p, q)$$

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The HyperGeometric Function.

24-03-15

In the study of second-order linear differential Eq. with three regular singular point, there arises the function

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} \longrightarrow (1) $$

$|z|<1$, $c \neq 0,-1,-2,\ldots$

$$ = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} $$

a, b, c are complex parameters.

z is complex variable.

nth term

$$ A_n = \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} $$

$$ A_{n+1} = \frac{(a)_{n+1} (b)_{n+1}}{(c)_{n+1}} \cdot \frac{z^{n+1}}{(n+1)!} $$

By using Ratio test.

$$ \lim_{n\to\infty} \left| \frac{A_{n+1}}{A_n} \right| = \lim_{n\to\infty} \left| \frac{(a)_{n+1} (b)_{n+1} z^{n+1}}{(c)_{n+1} \cdot (n+1)!} \cdot \frac{(c)_n \cdot n!}{(a)_n (b)_n z^n} \right| $$

We know

$$ (a)_{n+1} = a(a+1) \cdots (a+n+1-1) $$

$$ = a(a+1) \cdots (a+n-1)(a+n) $$

$$ = (a)_n (a+n) $$

$$ \Rightarrow \lim_{n\to\infty} \left| \frac{A_{n+1}}{A_n} \right| = \lim_{n\to\infty} \left| \frac{(a+n)\cancel{(a)_n}(b+n)\cancel{(b)_n} z^{\cancel{n+1-n}} \cdot \cancel{(c)_n}\, \cancel{n!}}{\cancel{(c)_n}(c+n)(n+1)\cancel{n!}\, \cancel{(a)_n (b)_n}} \right| $$

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$$= \lim_{n\to\infty} \left| \frac{(a+n)(b+n)}{(c+n)(n+1)} \cdot z \right|$$

$n^2 \times \& \div$

$$= |z|$$

This series ① will be convergent if $|z|<1$.

Ratio Test Conditions.

$$\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| = l$$

(i) If $l>1$, Series divergent
(ii) If $l<1$, Series convergent
(iii) If $l=1$, Test fails.

Theorem#16. If $|z|<1$ and if $Re(c) > Re(b) > 0$, Then 25-03-2015

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

Proof:- $L.H.S = {}_2F_1(a,b;c;z)$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} \longrightarrow ①$$

Consider

$$\frac{(b)_n}{(c)_n} = \frac{\Gamma(b+n)}{\Gamma(b)} \cdot \frac{\Gamma(c)}{\Gamma(c+n)}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)}$$

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$$= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \cdot \beta(b+n,\ c-b)$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,dt$$

Put in eq. (1).

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \left[\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,dt\right] \frac{z^n}{n!}$$

Since the series is uniformaly convergent. So

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \left[\sum_{n=0}^{\infty} \frac{(a)_n (zt)^n}{n!}\right] dt$$

If Series uniformaly convergent

$$\int_0^2 \sum_{n=0}^{2} x^n\,dx = \sum_{n=0}^{2} \int_0^2 x^n\,dx.$$

$$\int_0^2 (x^2+x+1)\,dx = \sum_{n=0}^{2} \left(\frac{x^{n+1}}{n+1}\Big|_0^2\right)$$

$$\frac{20}{3} = \frac{20}{3}$$

$$\Rightarrow {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt$$

Hence proved.

$$(1-y)^{-\alpha} = 1 + \frac{(-\alpha)}{1!}(-y) + \frac{(-\alpha)(-\alpha-1)}{2!}(-y)^2 + \cdots$$

$$= 1 + \frac{\alpha\, y}{1!} + \frac{\alpha(\alpha+1)}{2!}y^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{3!}y^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n\, y^n}{n!}$$

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Theorem # 18:- If $Re(c-a-b)>0$ & if c is neither zero nor a negative integer, then

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} $$

Proof:- Since

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt $$

Put $z=1$

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t)^{-a}\,dt $$

$$ = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-a-b-1}\,dt $$

$$ = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot B(b,c-a-b) $$

$$ = \frac{\Gamma(c)}{\cancel{\Gamma(b)}\,\Gamma(c-b)} \cdot \frac{\cancel{\Gamma(b)}\,\Gamma(c-b-a)}{\Gamma(\cancel{b}+c-\cancel{b}-a)} $$

$$ = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} $$

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The Hyper Geometric Differential Eq.

Let an operator $\theta = z\frac{d}{dz}$.

Suppose that

$$w = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}$$

is the solution of differential Eq.

Consider

$$\theta(\theta+c-1)w = \theta(\theta+c-1)\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}$$

$\theta(z^n) = z\frac{d}{dz}(z^n)$
$= nz \cdot z^{n-1}$
$= nz^n$
Similarly
$(\theta+c-1)z^n = (n+c-1)z^n$

$$= \sum_{n=0}^{\infty} \frac{n(n+c-1)(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{n(\cancel{n+c-1})(a)_n (b)_n}{(c)_{n-1}(\cancel{c+n-1})} \cdot \frac{z^n}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_{n-1}} \cdot \frac{z^n}{(n-1)!}$$

Shift the index (Replace n by $n+1$)

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+1} (b)_{n+1}}{(c)_n} \cdot \frac{z^{n+1}}{n!}$$

$$= z\sum_{n=0}^{\infty} \frac{(a+n)(a)_n (b+n)(b)_n}{(c)_n} \cdot \frac{z^n}{n!}$$

$$= z(\theta+a)(\theta+b)\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}$$

$$\Rightarrow \theta(\theta+c-1)w = z(\theta+a)(\theta+b)w$$

$$\Rightarrow [\theta(\theta+c-1) - z(\theta+a)(\theta+b)]w = 0$$

$$\theta^2 w + (c-1)\theta w - z\theta^2 w - za\theta w - zb\theta w - zabw = 0$$

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Since $\theta w = z\frac{d}{dz}(w) = zw'$

$$\theta^2 w = \theta(zw')$$
$$= z\frac{d}{dz}(zw')$$
$$= z(1\cdot w' + zw'')$$
$$= zw' + z^2 w''$$

$$\Rightarrow zw' + z^2w'' + (c-1)zw' - z^2w' - z^3w'' - az^2w' - bz^2w' - abzw = 0$$

$\div$ by z $\qquad \because z \neq 0$

$$w' + zw'' + (c-1)w' - zw' - z^2w'' - azw' - bzw' - abw = 0$$

$$\Rightarrow (1-z)zw'' + (\cancel{1}+c\cancel{-1} - z - az - bz)w' - abw = 0$$

$$(1-z)zw'' + [c - (a+b+1)z]w' - abw = 0$$

It is called Hyper Geometric Differential Equation.

17th **Theorem 17:-** 31-03-2015

If c is neither zero nor a negative integer & $Re(c-a-b) > 0$, ${}_2F_1(a,b;c;1)$ is an analytic function of a,b,c.

Proof:- As we know that if $Re(c-a-b) > 0$ then

$${}_2F_1(a,b;c;1) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!}$$ is absolutly convergent.

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Let

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} \text{ is convergent. where } \delta > 0$$

By limit comparison test $\lim \left|\frac{a_n}{b_n}\right|$

Let $Re(c-a-b) \geq 2\delta$

$$\lim_{n\to\infty} \left| \frac{\frac{(a)_n (b)_n}{(c)_n\, n!}}{1/n^{1+\delta}} \right| = \lim_{n\to\infty} \left| \frac{(a)_n (b)_n\, n^{1+\delta}}{(c)_n\, n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(a)_n}{(n-1)!\, n^a} \cdot \frac{(b)_n}{(n-1)!\, n^b} \cdot \frac{(n-1)!\, n^c}{(c)_n} \cdot \frac{(n-1)!\, n^a n^b\, n \cdot n^\delta}{n^c \cdot n!} \right|$$

$$= \frac{1}{\Gamma(a)} \cdot \frac{1}{\Gamma(b)} \cdot \Gamma(c) \lim_{n\to\infty} \left| \frac{\cancel{(n-1)!} \cdot n^a \cdot n^b \cdot \cancel{n} \cdot n^\delta}{n^c \cdot \cancel{n} \cancel{(n-1)!}} \right|$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \lim_{n\to\infty} \left| \frac{1}{n^{c-a-b-\delta}} \right| = 0 \qquad \because \Gamma(z) = \lim_{n\to\infty} \frac{(n-1)!\, n^z}{(z)_n}$$

where $Re(c-a-b-\delta) = Re(c-a-b) - \delta \geq 2\delta - \delta = \delta > 0$

$$\because \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} \text{ is convergent}$$

Therefore $\displaystyle\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}$ is convergent

Hence ${}_2F_1(a, b; c; 1)$ is analytic.

Solution of Differential Eq.

Consider a differential Eq.

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0 \longrightarrow (1)$$

Sol:- Let $y = \displaystyle\sum_{n=0}^{\infty} d_n x^n \longrightarrow (2)$ be a solution of eq (1).

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$$y' = \sum_{n=1}^{\infty} d_n (n x^{n-1})$$

$$y'' = \sum_{n=2}^{\infty} d_n\, n(n-1) x^{n-2}$$

Put in (1).

$$x(1-x)\sum_{n=2}^{\infty} d_n\, n(n-1) x^{n-2} + [c-(a+b+1)x]\sum_{n=1}^{\infty} d_n\, n x^{n-1}$$
$$- ab\sum_{n=0}^{\infty} d_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} d_n\, n(n-1) x^{n-1} - \sum_{n=2}^{\infty} d_n\, n(n-1) x^n + c\sum_{n=1}^{\infty} d_n\, n x^{n-1}$$
$$-(a+b+1)\sum_{n=1}^{\infty} d_n\, n x^n - ab\sum_{n=0}^{\infty} d_n x^n = 0$$

Shift the index to make a same power of x (i.e x^n).

$$\Rightarrow \sum_{n=1}^{\infty} d_{n+1}(n+1)n\, x^n - \sum_{n=2}^{\infty} d_n\, n(n-1) x^n + c\sum_{n=0}^{\infty} d_{n+1}(n+1) x^n$$
$$-(a+b+1)\sum_{n=1}^{\infty} d_n\, n x^n - ab\sum_{n=0}^{\infty} d_n x^n = 0$$

Comparing the coefficient of x^n.

$$d_{n+1}(n^2+n) - d_n(n^2-n) + c\, d_{n+1}(n+1) - (a+b+1) d_n\, n$$
$$- ab\, d_n = 0$$

$$[(n+1)n + c(n+1)]d_{n+1} - [n^2-n+(a+b+1)n+ab]d_n = 0$$

$$[(n+1)(c+n)]d_{n+1} = [n^2 - \cancel{n} + an + nb + \cancel{n} + ab]d_n$$

$$d_{n+1} = \left[\frac{n(n+a)+b(n+a)}{(n+1)(n+c)}\right]d_n$$

$$d_{n+1} = \left[\frac{(n+a)(n+b)}{(n+c)(n+1)}\right]d_n \longrightarrow (2)$$

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Put $n = 0, 1, 2, \ldots$ in Eq. 3.

$$d_1 = \frac{ab}{c} d_0$$

$$d_2 = \frac{(a+1)(b+1)}{(c+1)\cdot 2} d_1 = \frac{(a+1)(b+1)}{(c+1)\cdot 2} \cdot \frac{ab}{c} d_0$$

$$d_2 = \frac{(a)_2 (b)_2}{(c)_2 \, 2!} d_0$$

Similarly

$$d_3 = \frac{(a)_3 (b)_3}{(c)_3 \, 3!} d_0$$

$$\vdots$$

$$d_n = \frac{(a)_n (b)_n}{(c)_n \, n!} d_0 \quad \text{put in } (2).$$

(2) becomes

$$y = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} d_0 x^n$$

$$= d_0 \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} x^n$$

$$= d_0 \, {}_2F_1 [a, b; c; x]$$

Take $d_0 = 1$

$$y = {}_2F_1 [a, b; c; x]$$

This is called solution of H·Geometric differential Eq.

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$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} x^n$$

Put $a=1$, $c=b$

$${}_2F_1[1, b; b; x] = \sum_{n=0}^{\infty} \frac{(1)_n \cancel{(b)_n}}{\cancel{(b)_n} \cancel{n!}} x^n$$

$(1)_n = 1(1+1)\cdots(1+n-1)$
$= n(n-1)\cdots 3\cdot 2\cdot 1$
$= n!$

$$= \sum_{n=0}^{\infty} x^n \quad , |x| < 1$$

Infinite Geometric series (or Power Series)

01-04-2015

(180) Q.1:- $\frac{d}{dx} F[a,b;c;x] = \frac{ab}{c} F[a+1, b+1; c+1; x]$

$\boxed{F\left[\begin{matrix} a, b; \\ c; \end{matrix}\ x\right] = F[a;b;c;x]}$

Sol:- We know that

$${}_2F_1[a,b;c;z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}$$

Now

$$L.H.S = \frac{d}{dx} F[a,b;c;x]$$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{1}{n!} \frac{d}{dx}(x^n)$$

$$= \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{1}{\cancel{n}(n-1)!} \cdot \cancel{n} x^{n-1}$$

Replace n by $n+1$

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+1} (b)_{n+1}}{(c)_{n+1}} \cdot \frac{x^n}{n!} \longrightarrow (1)$$

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Since

$$(a)_{n+1} = a(a+1)(a+2)\cdots(a+n+1-1)$$

$$= a[(a+1)(a+2)\cdots(a+1+n-1)]$$

$$= a(a+1)_n$$

So, (1) becomes

$$= \sum_{n=0}^{\infty} \frac{a(a+1)_n \, b(b+1)_n}{c(c+1)_n} \cdot \frac{x^n}{n!}$$

$$= \frac{ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n} \cdot \frac{x^n}{n!}$$

$$= \frac{ab}{c} F[a+1, b+1; c+1; x]$$

$$= R.H.S$$

Q.4:- ${}_2F_1[-n, b; c; 1] = \dfrac{(c-b)_n}{(c)_n}$

Sol:- We know that

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

Let $a = -n$.

$${}_2F_1(-n, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c+n-b)}{\Gamma(c+n)\,\Gamma(c-b)}$$

$$= \frac{\Gamma(c)}{\Gamma(c+n)} \cdot \frac{\Gamma(c-b+n)}{\Gamma(c-b)}$$

$$= \frac{(c-b)_n}{(c)_n} \qquad \because (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

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Q.5.

$$F[-n, a+n; c; 1] = \frac{(-1)^n (1+a-c)_n}{(c)_n}$$

Sol:- We know that

$$_2F_1(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$

Put $a = -n$ & $b = a+n$ we get.

$$_2F_1[-n, a+n; c; 1] = \frac{\Gamma(c)\,\Gamma(c+n-a-n)}{\Gamma(c+n)\,\Gamma(c-a-n)}$$

$$= \frac{\Gamma(c)}{\Gamma(c+n)} \cdot \frac{\Gamma(c-a)}{\Gamma(c-a-n)} \longrightarrow (1)$$

Since $\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = (\alpha)_n$

and from question 9 of exercise 2.1.

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{(-1)^n}{(\alpha)_n}$$

Here $1-\alpha = c-a$
$\Rightarrow 1-c+a = \alpha$

$$\Rightarrow \frac{\Gamma(c-a-n)}{\Gamma(c-a)} = \frac{(1-c+a)_n}{(-1)^n}$$

So, (1) becomes.

$$= \frac{1}{(c)_n} \cdot \frac{(1+a-c)_n}{(-1)^n}$$

$$= \frac{(-1)^n (1+a-c)_n}{(c)_n}$$

Q.6:-

$$_2F_1[-n, 1-b-n; a; 1] = \frac{(a+b-1)_{2n}}{(a)_n\,(a+b-1)_n}$$

Sol:- We know

$$_2F_1[a, b; c; 1] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$

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Replace a by $-n$,
b by $1-b-n$,
& c by a.

$${}_2F_1\left[-n, 1-b-n; a; 1\right] = \frac{\Gamma(a)\,\Gamma(a+n-1+b+n)}{\Gamma(a+n)\,\Gamma(a-1+b+n)}$$

$$= \frac{\Gamma(a)}{\Gamma(a+n)} \cdot \frac{\Gamma(a+b-1+2n)}{\Gamma(a+b-1+n)} \cdot \frac{\Gamma(a+b-1)}{\Gamma(a+b-1)}$$

$$= \frac{\Gamma(a)}{\Gamma(a+n)} \cdot \frac{\Gamma(a+b-1+2n)}{\Gamma(a+b-1)} \cdot \frac{\Gamma(a+b-1)}{\Gamma(a+b-1+n)}$$

$$= \frac{1}{(a)_n} \cdot (a+b-1)_{2n} \cdot \frac{1}{(a+b-1)_n}$$

$$= \frac{(a+b-1)_{2n}}{(a)_n\,(a+b-1)_n}$$

$$= R.H.S$$

Lemma:- Prove that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k, n-k)$$

Proof:- Consider a series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n)\, t^{n+k} \longrightarrow (1)$$

where $k \geq 0$, $n \geq 0$

Let $k = j$ & $n = m-j$, $m \geq 0$.

$\Rightarrow j \geq 0$, $m - j \geq 0$
$m \geq j$

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$\Rightarrow \quad 0 \le j \le m$

(1) becomes

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n)\, t^{n+k} = \sum_{m=0}^{\infty}\sum_{j=0}^{m} A(j, m-j)\, t^{m-j+j}$$

Put $t=1$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{m=0}^{\infty}\sum_{j=0}^{m} A(j, m-j)$$

Change the dummy indices m by n, j by k.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k, n-k)$$

"Legendre's Duplication Formula" 02-04-2015

Since

$$(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n$$

Put $\alpha = 2z$

$$(2z)_{2n} = 2^{2n}(z)_n\left(z+\frac{1}{2}\right)_n$$

$$\frac{\Gamma(2z+2n)}{\Gamma(2z)} = 2^{2n}\frac{\Gamma(z+n)}{\Gamma(z)}\cdot\frac{\Gamma(z+\frac{1}{2}+n)}{\Gamma(z+\frac{1}{2})}$$

$$\Rightarrow \frac{\Gamma(2z)}{\Gamma(z)\Gamma(z+\frac{1}{2})} = \frac{\Gamma(2z+2n)}{2^{2n}\Gamma(z+n)\Gamma(z+\frac{1}{2}+n)}$$

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Taking $\lim_{n\to\infty}$ on both sides

not depend on "n"

$$\frac{\Gamma(2z)}{\Gamma(z)\,\Gamma(z+1/2)} = \lim_{n\to\infty} \frac{\Gamma(2z+2n)}{2^{2n}\,\Gamma(z+n)\,\Gamma(z+\frac{1}{2}+n)}$$

We know

$$\lim_{n\to\infty} \frac{(n-1)!\, n^z}{\Gamma(z+n)} = 1$$

$$= \lim_{n\to\infty} \left[\frac{\Gamma(2z+2n)}{(2n-1)!\,(2n)^{2z}} \cdot \frac{(n-1)!\, n^z}{\Gamma(z+n)} \cdot \frac{(n-1)!\, n^{z+1/2}}{\Gamma(z+\frac{1}{2}+n)} \cdot \frac{(2n-1)!\,(2n)^{2z}}{2^{2n}\,(n-1)!\, n^z\,(n-1)!\, n^{z+1/2}} \right]$$

$$= \lim_{n\to\infty} \left[\frac{(2n-1)!\; 2^{2z} \cdot \cancel{n^{2z}}}{2^{2n}\,[(n-1)!]^2\, \cancel{n^z \cdot n^z} \cdot n^{1/2}} \right]$$

$$\frac{\Gamma(2z)}{2^{2z}\,\Gamma(z)\Gamma(z+\frac{1}{2})} = \lim_{n\to\infty} \left[\frac{(2n-1)!}{2^{2n}\,[(n-1)!]^2\, n^{1/2}} \right] \longrightarrow (1)$$

As L.H.S is independent of n. So Eq. (1) implies that

$$\frac{\Gamma(2z)}{2^{2z}\,\Gamma(z)\Gamma(z+\frac{1}{2})} = C \text{ (say)} \longrightarrow (2)$$

Put $z=\frac{1}{2}$ in (2).

$$\frac{\Gamma(1)}{2\,\Gamma(\frac{1}{2})\,\Gamma(1)} = C$$

$\because \Gamma(1) = 1$

$$\Rightarrow \quad C = \frac{1}{2\Gamma(\frac{1}{2})}$$

$\because \Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$C = \frac{1}{2\sqrt{\pi}}$$

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$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi}$

Put $z = \frac{1}{2}$

$\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}) = \frac{\pi}{\sin \frac{\pi}{2}}$

$(\Gamma(\tfrac{1}{2}))^2 = \pi$

$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

Put in ②.

$$\frac{\Gamma(2z)}{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})} = \frac{1}{2\sqrt{\pi}}$$

$$\Rightarrow \Gamma(z)\Gamma(z+\tfrac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$$

Example:-

$$F[-\tfrac{1}{2}n, -\tfrac{1}{2}n+\tfrac{1}{2}; b+\tfrac{1}{2}; 1] = \frac{2^n (b)_n}{(2b)_n}$$

Solution:- We know that

$$\boxed{{}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}}$$

$$L.H.S = F[-\tfrac{1}{2}n, -\tfrac{1}{2}n+\tfrac{1}{2}; b+\tfrac{1}{2}; 1]$$

$$= \frac{\Gamma(b+\frac{1}{2})\Gamma(b+\cancel{\frac{1}{2}}+\frac{1}{2}n+\frac{1}{2}n-\cancel{\frac{1}{2}})}{\Gamma(b+\frac{1}{2}+\frac{1}{2}n)\Gamma(b+\cancel{\frac{1}{2}}+\frac{1}{2}n-\cancel{\frac{1}{2}})}$$

$$= \frac{\Gamma(b+\frac{1}{2})\Gamma(b+n)}{\Gamma(b+\frac{1}{2}+\frac{1}{2}n)\Gamma(b+\frac{1}{2}n)} \cdot \frac{\Gamma(b)}{\Gamma(b)}$$

$$= \frac{\Gamma(b)\Gamma(b+\frac{1}{2})}{\Gamma(b+\frac{n}{2})\Gamma(b+\frac{n}{2}+\frac{1}{2})} \cdot \frac{\Gamma(b+n)}{\Gamma(b)}$$

$$= \frac{2^{1-2b}\cancel{\sqrt{\pi}}\,\Gamma(2b)}{2^{1-2(b+\frac{n}{2})}\cancel{\sqrt{\pi}}\,\Gamma(2(b+\frac{n}{2}))} \cdot (b)_n$$

$$= 2^n \frac{\Gamma(2b)}{\Gamma(2b+n)} \cdot (b)_n = \frac{2^n (b)_n}{(2b)_n} = R.H.S$$

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Ex. 2.1

Q. 8 07-04-2015

Show that for $0 \le k \le n$

$$(\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k}$$

Solution:-

$$R.H.S = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k}$$

$$= \frac{(-1)^k \alpha(\alpha+1)\cdots(\alpha+n-1)}{(1-\alpha-n)(1-\alpha-n+1)\cdots(\cancel{1}-\alpha-n+k\cancel{-1})}$$

$$= \frac{(-1)^k \alpha(\alpha+1)\cdots(\alpha+n-1)}{(-\alpha-n+1)(-\alpha-n+2)\cdots(-\alpha-n+k)}$$

$$= \frac{\cancel{(-1)^k} \alpha(\alpha+1)\cdots(\alpha+n-k-1)\cancel{(\alpha+n-k)}\cancel{(\alpha+n-k+1)}\cdots\cancel{(\alpha+n-1)}}{\cancel{(-1)^k}\cancel{(\alpha+n-1)}\cancel{(\alpha+n-2)}\cdots\cancel{(\alpha+n-k+1)}\cancel{(\alpha+n-k)}}$$

$$= \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-k-1)$$

$$= (\alpha)_{n-k}$$

$$= L.H.S$$

special case of this question (8).

If $\alpha = 1$

$$(1)_{n-k} = \frac{(-1)^k (1)_n}{(\cancel{1}-\cancel{1}-n)_k}$$

$$(n-k)! = \frac{(-1)^k n!}{(-n)_k}$$

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Q.9-

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{(-1)^n}{(\alpha)_n}$$

Sol:-

$$L.H.S = \frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)}$$

$$= \frac{\Gamma(-\alpha-n+1)}{\Gamma(-\alpha+1)} = \frac{\Gamma(-\alpha-n+1)}{(-\alpha)\,\Gamma(-\alpha)}$$

$$= \frac{\Gamma(-\alpha-n+1)}{(-\alpha)\,\Gamma(-\alpha-1+1)}$$

$$= \frac{\Gamma(-\alpha-n+1)}{(-\alpha)(-\alpha-1)\,\Gamma(-\alpha-1)}$$

$$= \frac{\Gamma(-\alpha-n+1)}{(-\alpha)(-\alpha-1)(-\alpha-2)\,\Gamma(-\alpha-2)}$$

$$= \frac{\cancel{\Gamma(-\alpha-n+1)}}{(-\alpha)(-\alpha-1)(-\alpha-2)\cdots(-\alpha-n+1)\,\cancel{\Gamma(-\alpha-n+1)}}$$

$$= \frac{1}{(-1)^n(\alpha)(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}$$

$$= \frac{(-1)^n}{(\alpha)_n}$$

$$= R.H.S$$

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Ch#4

Theorem: 20.

If $|z|<1$ & $\left|\frac{z}{1-z}\right|<1$ then

$$ {}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix} z\right] = (1-z)^{-a} F\left[\begin{matrix} a, c-b; \\ c; \end{matrix} \frac{-z}{1-z}\right] $$

Proof:-

$$ R.H.S = (1-z)^{-a} F\left[\begin{matrix} a, c-b; \\ c; \end{matrix} \frac{-z}{1-z}\right] $$

$$ = (1-z)^{-a} \sum_{k=0}^{\infty} \frac{(a)_k (c-b)_k}{(c)_k} \left(\frac{-z}{1-z}\right)^k \frac{1}{k!} $$

$$ = \sum_{k=0}^{\infty} \frac{(1-z)^{-(a+k)} (a)_k (c-b)_k (-1)^k (z)^k}{(c)_k \; k!} $$

We know $(1-z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n z^n}{n!}$

$$ = \sum_{k=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{(a+k)_n z^n}{n!}\right] \frac{(a)_k (c-b)_k (-1)^k (z)^k}{(c)_k \; k!} $$

$$ = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (a)_k (c-b)_k (a+k)_n \; z^{n+k}}{(c)_k \; k! \; n!} $$

$$ = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (a)_k (c-b)_k}{(c)_k \; k! \; n!} \cdot \frac{(a)_{n+k}}{(a)_k} z^{n+k} $$

$(\alpha)_{n+k} = (\alpha+n)_k (\alpha)_n$

$\Rightarrow (\alpha+n)_k = \frac{(\alpha)_{n+k}}{(\alpha)_n}$

$$ = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (c-b)_k (a)_{n+k}}{(c)_k \; k! \; n!} z^{n+k} $$

We use here $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n-k)$

Replace n by $n-k$

$$ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (c-b)_k (a)_n \; z^n}{(c)_k \; k! \; (n-k)!} $$

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using special case

$$(n-k)! = \frac{(-1)^k n!}{(-n)_k}$$

$$\Rightarrow \frac{1}{(n-k)!} = \frac{(-n)_k}{(-1)^k n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\cancel{(-1)^k}(c-b)_k (a)_n z^n}{(c)_k \; k!} \cdot \frac{(-n)_k}{\cancel{(-1)^k} n!}$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} \frac{(-n)_k (c-b)_k}{(c)_k} \cdot \frac{1}{k!}\right] \frac{(a)_n z^n}{n!}$$

$$= \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix}-n, c-b; \\ c;\end{matrix} 1\right] \frac{(a)_n z^n}{n!}$$

using

$${}_2F_1\left[\begin{matrix}-n, b; \\ c;\end{matrix} 1\right] = \frac{(c-b)_n}{(c)_n}$$

$$= \sum_{n=0}^{\infty}\left[\frac{(c-(c-b))_n}{(c)_n}\right] \frac{(a)_n z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}$$

$$= {}_2F_1\left[\begin{matrix}a, b; \\ c;\end{matrix} z\right] = L \cdot H \cdot S$$

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41 Q.No:7 08-04-2015

Prove that if $g_n = F(-n, \alpha; 1+\alpha-n; 1)$, and "$\alpha$" is not an integer then

$g_0 = 1$ & $g_n = 0$ for all $n \geq 1$

Sol:- Since ${}_2F_1(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n}$

Replace b by α & c by $1+\alpha-n$.

$$F(-n, \alpha; 1+\alpha-n; 1) = \frac{(1+\alpha-n-\alpha)_n}{(1+\alpha-n)_n}$$

$$\Rightarrow g_n = F(-n, \alpha; 1+\alpha-n; 1) = \frac{(1-n)_n}{(1+\alpha-n)_n}$$

$$\Rightarrow g_n = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \geq 1 \end{cases}$$ $\because (\alpha)_0 = 1$, $\alpha \neq 0$

$(-m)_n = 0$ for $m < n$

Lemma:-

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(n,k) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k, n+k)$$

Sol:- Consider a series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k, n+k)\, t^{n-k}$$

where $n \geq 0$, $k \geq 0$

Let $n = i+j$ & $k = j$

then $i+j \geq 0$, $j \geq 0$

$$\Rightarrow \begin{aligned} 0 &\leq j \leq n \\ 0 &\leq j \leq i+j \\ 0 &\leq j-j \leq i \end{aligned}$$

$\Rightarrow \boxed{0 \leq i}$

$0 \leq j \leq i+j \leq \infty$

$\Rightarrow \boxed{0 \leq j \leq \infty}$

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$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n)\,t^{n-k} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} B(j,i+j)\,t^{i}$$

Let $t=1$ then

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} B(j,i+j) \ldots$$

Replace j by k & i by n

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k)$$

Theorem: 26:- (A theorem due to Kummer.)

If $(1+a-b)$ is neither zero nor a negative integer & $Re(b) \le 1$ for convergence. Then

$${}_2F_1\left[\begin{matrix}a, & b; \\ 1+a-b; \end{matrix} -1\right] = \frac{\Gamma(1+a-b)\,\Gamma(1+\frac{1}{2}a)}{\Gamma(1+\frac{1}{2}a-b)\,\Gamma(1+a)}$$

Proof:- Since

$$(1-z)^{-a}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}a-b; \\ 1+a-b; \end{matrix} \frac{-4z}{(1-z)^2}\right] = {}_2F_1\left[\begin{matrix}a, b; \\ 1+a-b; \end{matrix} z\right] \rightarrow (1)$$

Put $z=-1$ in (1)

$$2^{-a}\,F\left[\begin{matrix}\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}a-b; \\ 1+a-b; \end{matrix} \frac{4}{2^2}=1\right] = {}_2F_1\left[\begin{matrix}a, b; \\ 1+a-b; \end{matrix} -1\right]$$

$$\Rightarrow \frac{2^{-a}\,\Gamma(1+a-b)\,\Gamma(1/2)}{\Gamma(1+\frac{1}{2}a-b)\,\Gamma(\frac{1}{2}+\frac{1}{2}a)} = {}_2F_1\left[\begin{matrix}a, & b; \\ 1+a-b; \end{matrix} -1\right] \rightarrow (2)$$

By Legendre's Duplication Formula.

Let $1-2z=-a$, $2z=1+a$, $z=\frac{1}{2}+\frac{a}{2}$

$$\Gamma(1/2)\,\Gamma(1+a) = 2^{a}\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}a)\,\Gamma(1+\tfrac{1}{2}a)$$

$$\Rightarrow \Gamma(\tfrac{1}{2}+\tfrac{1}{2}a) = \frac{\Gamma(1/2)\,\Gamma(1+a)}{2^{a}\,\Gamma(1+\frac{1}{2}a)} \quad \text{Put in (2).}$$

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(2) becomes

$$\frac{\cancel{2^{a}}\,\Gamma(1+a-b)\,\cancel{\Gamma(\tfrac{1}{2})}}{\Gamma(1+\frac{1}{2}a-b)\cdot\dfrac{\cancel{\Gamma(\frac{1}{2})}\Gamma(1+a)}{\cancel{2^{a}}\,\Gamma(1+\frac{1}{2}a)}} = {}_2F_1\left[\begin{matrix}a, b; \\ 1+a-b;\end{matrix} -1\right]$$

$$\Rightarrow {}_2F_1\left[\begin{matrix}a, b; \\ 1+a-b;\end{matrix} -1\right] = \frac{\Gamma(1+a-b)\,\Gamma(1+\frac{1}{2}a)}{\Gamma(1+\frac{1}{2}a-b)\,\Gamma(1+a)}$$

(B) Theorem # 21 — 09-04-2015

If $|z|<1$ then

$${}_2F_1\left[\begin{matrix}a, b; \\ c;\end{matrix} z\right] = (1-z)^{c-a-b}\,{}_2F_1\left[\begin{matrix}c-a, c-b; \\ c;\end{matrix} z\right]$$

Proof:- Since for $|z|<1$ & $\left|\frac{z}{1-z}\right|<1$

$${}_2F_1\left[\begin{matrix}a, b; \\ c;\end{matrix} z\right] = (1-z)^{-a}\,{}_2F_1\left[\begin{matrix}a, c-b; \\ c;\end{matrix} \frac{-z}{1-z}\right] \longrightarrow (1)$$

Consider

$${}_2F_1\left[\begin{matrix}a, c-b; \\ c;\end{matrix} y\right] = {}_2F_1\left[\begin{matrix}c-b, a; \\ c;\end{matrix} y\right]$$

Using (1) on R.H.S

$${}_2F_1\left[\begin{matrix}a, c-b; \\ c;\end{matrix} y\right] = (1-y)^{-(c-b)}\,{}_2F_1\left[\begin{matrix}c-b, c-a; \\ c;\end{matrix} \frac{-y}{1-y}\right]$$

$$= (1-y)^{-(c-b)}\,{}_2F_1\left[\begin{matrix}c-a, c-b; \\ c;\end{matrix} \frac{-y}{1-y}\right] \longrightarrow (2)$$

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let $\boxed{y = \frac{-z}{1-z}}$

$\Rightarrow 1-y = 1 + \frac{z}{1-z}$

$1-y = \frac{1}{1-z} \Rightarrow (1-y)^{-1} = 1-z$

$\boxed{1-y = (1-z)^{-1}}$ $\quad z = 1 - \frac{1}{1-y}$

$\boxed{z = \frac{-y}{1-y}}$

So, ② becomes

$${}_2F_1\left[\begin{matrix} a, c-b; & \frac{-z}{1-z} \\ c; & \end{matrix}\right] = (1-z)^{c-b} {}_2F_1\left[\begin{matrix} c-a, c-b; & \\ c; & z \end{matrix}\right]$$

xing $(1-z)^{-a}$

$$(1-z)^{-a} {}_2F_1\left[\begin{matrix} a, c-b; & \frac{-z}{1-z} \\ c; & \end{matrix}\right] = (1-z)^{c-a-b} {}_2F_1\left[\begin{matrix} c-a, c-b; & \\ c; & z \end{matrix}\right]$$

Using eq ① on LHS

$${}_2F_1\left[\begin{matrix} a, b; & \\ c; & z \end{matrix}\right] = (1-z)^{c-a-b} {}_2F_1\left[\begin{matrix} c-a, c-b; & \\ c; & z \end{matrix}\right]$$

which is the required result.

Contiguous Function

If we add or subtract 1 in a parameter then the new function (or hypergeometric function) is a contiguous function.

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}$$

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Contiguous function is

$$ {}_2F_1(a+1, b; c; z) = \sum_{n=0}^{\infty} \frac{(a+1)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} $$

Let $\delta_n = \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}$

$$ \Rightarrow F = \sum_{n=0}^{\infty} \delta_n $$

where $F = {}_2F_1(a, b; c; z)$

$$ F(a+) = \sum_{n=0}^{\infty} \frac{(a+1)_n (b)_n}{c_n} \cdot \frac{z^n}{n!} $$

$$ F(a+) = \sum_{n=0}^{\infty} \frac{(a+n)}{a} \delta_n $$

F(a+) is Contiguous function to F.

$$ (a+1)_n = \frac{a(a+1)\cdots(a+n+1-2)(a+n)}{a} = (a)_n \frac{(a+n)}{a} $$

Similarly,

$$ F(b+) = \sum_{n=0}^{\infty} \frac{(b+n)}{b} \delta_n $$

$$ F(c+) = \sum_{n=0}^{\infty} \frac{(c)}{(c+n)} \delta_n $$

$$ F(a-) = \sum_{n=0}^{\infty} \frac{(a-1)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} $$

$$ (a-1)_n = \frac{(a-1)(a)(a+1)\cdots(a-1+n-1)(a+n-1)}{(a+n-1)} $$

$$ = \frac{(a-1)(a)_n}{(a+n-1)} $$

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$$F(a-) = \sum_{n=0}^{\infty} \frac{(a-1)}{a+n-1} \cdot \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}$$

$$F(a-) = \sum_{n=0}^{\infty} \frac{(a-1)}{a+n-1} \delta_n$$

Similarly

$$F(b-) = \sum_{n=0}^{\infty} \frac{b-1}{b+n-1} \delta_n$$

&

$$F(c-) = \sum_{n=0}^{\infty} \frac{c+n-1}{c-1} \delta_n$$

Similarly we can say that $F(a+)$ is contiguous function to $F(a+)$.

2016 Q-8:- Show that 14-04-2015

$$\frac{d^n}{dx^n}\left[x^{a-1+n} F(a,b;c;x)\right] = (a)_n \, x^{a-1} F(a+n,b;c;x)$$

Sol:-

$$L.H.S = \frac{d^n}{dx^n}\left[x^{a-1+n} F(a,b;c;x)\right]$$

$$= \frac{d^n}{dx^n}\left[\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \cdot \frac{x^k}{k!} \cdot x^{a-1+n}\right]$$

$$= \frac{d^n}{dx^n}\left[\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k \, k!} x^{a+k-1+n}\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k \, k!} (a+k-1+n) x^{a+k-2+n}\right]$$

$$= \frac{d^{n-2}}{dx^{n-2}}\left[\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k \, k!} (a+k-1+n)(a+k-2+n) x^{a+k-3+n}\right]$$

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$$= \sum_{K=0}^{\infty} \frac{(a)_K (b)_K}{(c)_K K!} (a+K-1+n)(a+K-2+n) \cdots (a+K-\cancel{n}+\cancel{n})\, x^{a+K-n-1+n}$$

$$= \sum_{K=0}^{\infty} \frac{(a)_K (b)_K}{(c)_K K!} (a+K)(a+K-1) \cdots (a+K-1+n)\, x^{a+K-1}$$

$$= \sum_{K=0}^{\infty} \frac{(a)_K (b)_K}{(c)_K K!} (a+K)_n\, x^{a+K-1} \longrightarrow (A)$$

~~the~~ We know that

$$(a)_{n+K} = (a+K)_n (a)_K$$

$$\Rightarrow (a+K)_n = \frac{(a)_{n+K}}{(a)_K}$$

Therefore Eq. (A) becomes

$$= \sum_{K=0}^{\infty} \frac{\cancel{(a)_K} (b)_K}{(c)_K K!} \cdot \frac{(a)_{n+K}}{\cancel{(a)_K}}\, x^{a+K-1}$$

$$= \sum_{K=0}^{\infty} \frac{(a)_{n+K} (b)_K}{(c)_K K!}\, x^{a+K-1} \longrightarrow (B)$$

We know that

$$(a)_{n+K} = (a)_n (a+n)_K$$

So, (B) becomes

$$= \sum_{K=0}^{\infty} \frac{(a)_n (a+n)_K (b)_K}{(c)_K K!}\, x^{a-1} \cdot x^K$$

$$= (a)_n\, x^{a-1} \sum_{K=0}^{\infty} \frac{(a+n)_K (b)_K}{(c)_K} \cdot \frac{x^K}{K!}$$

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$$= (a)_n \, x^{a-1} \, F(a+n, b; c; x)$$
$$= R.H.S$$

199 **Lemma # 11. (a)**

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(k, n-2k)$$

Proof:– Consider a Series

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n)\, t^{n+2k} \longrightarrow (1)$$

Let $k = j$, $n = i - 2j$

Since $k \geq 0$, $n \geq 0$ for $j = 0$

$\Rightarrow j \geq 0$, $i - 2j \geq 0$ $\boxed{i \geq 0}$

$\Rightarrow \boxed{0 \leq j \leq i/2}$

Since j is integer, So j runs from 0 to the greatest integer in $(\frac{i}{2})$.

$\Rightarrow 0 \leq j \leq [\frac{i}{2}]$

(1) becomes

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n)\, t^{n+2k} = \sum_{i=0}^{\infty} \sum_{j=0}^{[i/2]} A(j, i-2j)\, t^{i}$$

Replace i by n & j by k and let $t = 1$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(k, n-2k).$$

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Q5) Lemma ~~Theorem~~ 11:- (b)

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} B(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n+2k)$$

Proof:- Consider a series

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} B(k,n) t^{n-2k} \longrightarrow (1)$$

Let $k = j$ & $n = i + 2j$

Since $0 \leq k \leq [\frac{n}{2}]$, $n \geq 0$

$\Rightarrow$ $0 \leq j \leq [\frac{n}{2}]$, $i + 2j \geq 0$

$0 \leq j \leq [\frac{n}{2}] \leq \infty$ for $j = 0$

$\boxed{0 \leq j \leq \infty}$ $\boxed{i \geq 0}$

So, (1) becomes

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} B(k,n) t^{n-2k} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} B(j, i+2j) t^{i}$$

Replace j by k & i by n and let $t = 1$.

So,

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} B(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n+2k)$$

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(by Legendre)

15-04-2015

Generating Function

Consider a function $F(x,t)$ which has a formal series expansion in t;

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x)\, t^n \longrightarrow (1)$$

The coefficient of t^n in Eq (1) is, in general, function of x.

The function $F(x,t)$ has generated the set $f_n(x)$ and $F(x,t)$ is called generating function for $f_n(x)$.

e.g.

(i):- $$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Legendre Polynomial

(ii):- $$\exp(2xt-t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!}\, t^n$$

Hermite polynomial

(iii) $$e^t\, {}_0F_1(-;1+\alpha;-xt) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{(1+\alpha)_n}\, t^n$$

Legrray Polynomial.

We can write

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x)\, t^n$$

$$= f_0(x) + f_1(x)t + \dots + f_n(x)t^n + \dots \infty$$

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Theorem:- 44 97

$$G(2xt-t^2)=\sum_{n=0}^{\infty} g_n(x)t^n \longrightarrow (1)$$

then $g_0'(x)=0$ for $n=0$

& $x\,g_n'(x)-n\,g_n(x)=g_{n-1}'(x)$ for $n\geq 1$

Proof:- let

$$F=G(2xt-t^2) \longrightarrow (2)$$

$$\Rightarrow \frac{\partial F}{\partial x}=2t\,G' \qquad G=G(2xt-t^2)$$

$$\qquad G'=G'(2xt-t^2)$$

$$\frac{1}{t}\cdot\frac{\partial F}{\partial x}=2G' \longrightarrow (3)$$

& $\Rightarrow \dfrac{\partial F}{\partial t}=(2x-2t)\,G'$

$$\frac{1}{x-t}\cdot\frac{\partial F}{\partial t}=2G' \longrightarrow (4)$$

from (3) & (4) we can write

$$\frac{1}{t}\frac{\partial F}{\partial x}=\frac{1}{x-t}\cdot\frac{\partial F}{\partial t}$$

$$(x-t)\cdot\frac{\partial F}{\partial x}=t\frac{\partial F}{\partial t}$$

$$(x-t)\frac{\partial F}{\partial x}-t\frac{\partial F}{\partial t}=0 \longrightarrow (5)$$

As $F=\displaystyle\sum_{n=0}^{\infty} g_n(x)\,t^n$

$$\Rightarrow \frac{\partial F}{\partial x}=\sum_{n=0}^{\infty} g_n'(x)\,t^n \longrightarrow (6)$$

& $\Rightarrow \dfrac{\partial F}{\partial t}=\displaystyle\sum_{n=0}^{\infty} n\,g_n(x)\,t^{n-1} \longrightarrow (7)$

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Put ⑥ & ⑦ in Eq. ⑤.

Eq ⑤ $\Rightarrow$ $$(x-t)\sum_{n=0}^{\infty} g_n'(x)\,t^n - t\left(\sum_{n=1}^{\infty} n\,g_n(x)\,t^{n-1}\right)=0$$

$$\sum_{n=0}^{\infty} x\,g_n'(x)\,t^n - \sum_{n=0}^{\infty} g_n'(x)\,t^{n+1} - \sum_{n=1}^{\infty} n\,g_n(x)\,t^n = 0$$

Shift the index (from n to $n-1$) in second series.

$$\sum_{n=0}^{\infty} x\,g_n'(x)\,t^n - \sum_{n=1}^{\infty} g_{n-1}'(x)\,t^n - \sum_{n=1}^{\infty} n\,g_n(x)\,t^n = 0 \quad \longrightarrow ⑧$$

Comparing the coefficient of t^0 in Eq. ⑧.

we can write second series $\sum_{n=0}^{\infty} g_{n-1}'(x)\,t^n = 0 + \sum_{n=1}^{\infty} g_{n-1}'(x)\,t^n$

$$x\,g_0'(x) - 0 - 0 = 0$$

$\Rightarrow x \neq 0$ So, $g_0'(x)=0$ for $n=0$

Comparing the coefficient of t^n in Eq ⑧

$$x\,g_n'(x) - g_{n-1}'(x) - n\,g_n(x) = 0$$

$$\Rightarrow x\,g_n'(x) - n\,g_n(x) = g_{n-1}'(x) \quad \text{for } n \geq 1$$

Sequence $\Rightarrow$

Recurrence Relation:- It is a sequence in which a term is given along with n-th term, to find the next term. e.g $a_{n+1} = 2a_n + 3$

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Theorem:- 45 97 — 16-04-2015

$$e^t \psi(xt) = \sum_{n=0}^{\infty} \sigma_n(x) t^n$$

then $\sigma_0'(x) = 0$ for $n=0$

& $x\sigma_n'(x) - n\sigma_n(x) = -\sigma_{n-1}(x)$ for $n \geq 1$

Proof:- let

$$F = e^t \psi(xt)$$

$$\Rightarrow \frac{\partial F}{\partial x} = t e^t \psi'(xt) \longrightarrow (1)$$

$$\& \quad \frac{\partial F}{\partial t} = e^t \psi(xt) + x e^t \psi'(xt) \longrightarrow (2)$$

By $[x \times (1)] - [t \times (2)]$

OR take the value of $e^t \psi'(xt)$ from (1) and put in (2).

$$x\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = \cancel{xt e^t \psi'(xt)} - t e^t \psi(xt) - \cancel{xt e^t \psi'(xt)}$$

$$\Rightarrow x\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = -tF \longrightarrow (3) \qquad \because F = e^t \psi(xt)$$

Now $F = \sum_{n=0}^{\infty} \sigma_n(x) t^n$

$$\Rightarrow \frac{\partial F}{\partial x} = \sum_{n=0}^{\infty} \sigma_n'(x) t^n$$

$$\& \quad \frac{\partial F}{\partial t} = \sum_{n=0}^{\infty} n\sigma_n(x) t^{n-1}$$

Put in (3) we get

$$x\sum_{n=0}^{\infty} \sigma_n'(x) t^n - t\sum_{n=0}^{\infty} n\sigma_n(x) t^{n-1} = -t\sum_{n=0}^{\infty} \sigma_n(x) t^n$$

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$$\Rightarrow x\sum_{n=0}^{\infty}\sigma_n'(x)t^n-\sum_{n=0}^{\infty}n\,\sigma_n(x)t^n=-\sum_{n=0}^{\infty}\sigma_n(x)t^{n+1}$$

Shift index in R.H.S (Replace n by $n-1$)

$$\Rightarrow x\sum_{n=0}^{\infty}\sigma_n'(x)t^n-\sum_{n=0}^{\infty}n\,\sigma_n(x)t^n=-\sum_{n=1}^{\infty}\sigma_{n-1}(x)t^n \longrightarrow (4)$$

Comparing the coefficient of t^0 in Eq (4).

$$x\sigma_n'(x)-0=0$$

Since $x\neq 0$, so $\sigma_n'(x)=0$ for $n=0$

Comparing the coefficient of t^n in (4).

$$x\,\sigma_n'(x)-n\,\sigma_n(x)=-\sigma_{n-1}(x)$$

for $n\geq 1$

Imp. Theorem: ~~46~~ 97

$$e^t\psi(x-t)=\sum_{n=0}^{\infty}\sigma_n(x)t^n$$

and $\psi(u)=\sum_{n=0}^{\infty}\gamma_n u^n$ then for arbitrary c,

$$(1-t)^{-c}F\left(\frac{xt}{1-t}\right)=\sum_{n=0}^{\infty}(c)_n\,\sigma_n(x)t^n \longrightarrow (A)$$

where $F(u)=\sum_{n=0}^{\infty}(c)_n\,\gamma_n u^n$

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Proof:- Since

$$\psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n$$

Put $u = xt$

$$\Rightarrow \psi(xt) = \sum_{n=0}^{\infty} \gamma_n x^n t^n \longrightarrow (1)$$

Since $\sum_{n=0}^{\infty} \sigma_n(x) t^n = e^t \psi(xt)$

Using (1) here & also $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$

$$\sum_{n=0}^{\infty} \sigma_n(x) t^n = \left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty} \gamma_k x^k t^k\right)$$

$$\sum_{n=0}^{\infty} \sigma_n(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^n}{n!} \gamma_k x^k t^k$$

$$\Rightarrow \sum_{n=0}^{\infty} \sigma_n(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\gamma_k x^k t^{n+k}}{n!}$$

$$\sum_{n=0}^{\infty} \sigma_n(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\gamma_k x^k t^n}{(n-k)!}$$

By using Lemma
Replace n by $n-k$

Comparing the coefficient of t^n.

$$\sigma_n(x) = \sum_{k=0}^{n} \frac{\gamma_k x^k}{(n-k)!} \longrightarrow (2)$$

Consider the R.H.S of (A).

$$\sum_{n=0}^{\infty} (c)_n \sigma_n(x) t^n = \sum_{n=0}^{\infty} (c)_n \left[\sum_{k=0}^{n} \frac{\gamma_k x^k}{(n-k)!}\right] t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(c)_n \gamma_k x^k t^n}{(n-k)!}$$ using (2)

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$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(c)_{n+k}\, \gamma_k\, x^k}{n!}\, t^{n+k}$$

Using lemma replace n by n+k

Using

$$(c)_{n+k} = (c+k)_n\, (c)_k$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(c+k)_n\, (c)_k\, \gamma_k\, x^k}{n!}\, t^n \cdot t^k$$

$$= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} \frac{(c+k)_n\, t^n}{n!}\right) (c)_k\, \gamma_k\, (xt)^k$$

$$= \sum_{k=0}^{\infty} (1-t)^{-(c+k)}\, (c)_k\, \gamma_k\, (xt)^k$$

Using $\sum_{n=0}^{\infty} \frac{(\alpha)_n t^n}{n!} = (1-t)^{-\alpha}$

So,

$$\sum_{n=0}^{\infty} (c)_n\, \sigma_n(x)\, t^n = (1-t)^{-c} \sum_{k=0}^{\infty} (1-t)^{-k} (c)_k\, \gamma_k\, (xt)^k$$

$$= (1-t)^{-c} \sum_{k=0}^{\infty} (c)_k\, \gamma_k \left(\frac{xt}{1-t}\right)^k \longrightarrow (3)$$

Since

$$F(u) = \sum_{n=0}^{\infty} (c)_n\, \gamma_n\, u^n$$

Put $u = \left(\frac{xt}{1-t}\right)$ & $n = k$

$$F\left(\frac{xt}{1-t}\right) = \sum_{k=0}^{\infty} (c)_k\, \gamma_k \left(\frac{xt}{1-t}\right)^k$$

so Put in (3):

$$\sum_{n=0}^{\infty} (c)_n\, \sigma_n(x)\, t^n = (1-t)^{-c}\, F\left(\frac{xt}{1-t}\right)$$

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Generating Functions for Legendre Polynomials

22-04-2015

We define Legendre Polynomials $P_n(x)$ by generating relation (function).

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n \longrightarrow (1)$$

in which $(1-2xt+t^2)^{-1/2}$ denotes the particular branch and $(1-2xt+t^2)^{-1/2} \longrightarrow 1$ as $t \to 0$

First show that $P_n(x)$ is a polynomial of degree precisely "n" in variable x.

Consider

$$\sum_{n=0}^{\infty} P_n(x) t^n = (1-2xt+t^2)^{-1/2} = \left[1-(2xt-t^2)\right]^{-1/2}$$

$$\sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (2xt-t^2)^n}{n!} \qquad \because (1-t)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\frac{1}{2})_n}{\cancel{n!}} \cdot \frac{(-1)^k \cancel{n!} (2xt)^{n-k} (t^2)^k}{k!\,(n-k)!}$$

$$\because (a+b)^n = \sum_{k=0}^{n} \frac{n!\, a^{n-k} b^k}{k!\,(n-k)!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (\frac{1}{2})_n (2x)^{n-k} t^{n+k}}{k!\,(n-k)!}$$

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$$\therefore \sum_{n=0}^{\infty}\sum_{K=0}^{\infty} A(K,n) = \sum_{n=0}^{\infty}\sum_{K=0}^{n} A(K,n-K)$$

&

$$\sum_{n=0}^{\infty}\sum_{K=0}^{n} A(K,n) = \sum_{n=0}^{\infty}\sum_{K=0}^{[n/2]} A(K,n-\cancel{K})$$

So Replace n by $n-K$

$$\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty}\sum_{K=0}^{[n/2]} \frac{(-1)^K \left(\frac{1}{2}\right)_{n-K} (2x)^{n-2K}\, t^n}{K!\,(n-2K)!} \longrightarrow (2)$$

By comparing the coefficient of t^n, we have

$$P_n(x) = \sum_{K=0}^{[n/2]} \frac{(-1)^K \left(\frac{1}{2}\right)_{n-K} (2x)^{n-2K}}{K!\,(n-2K)!} \longrightarrow (3)$$

It is called Legendre Polynomial of degree precisely "n" in variable x.

Eq (3) can be written as,

$$P_n(x) = \frac{\left(\frac{1}{2}\right)_n (2x)^n}{n!} + \sum_{K=1}^{[n/2]} \frac{(-1)^K \left(\frac{1}{2}\right)_{n-K} (2x)^{n-2K}}{K!\,(n-2K)!}$$

$$= \frac{\left(\frac{1}{2}\right)_n (2x)^n}{n!} + \Pi_{n-2}$$

Where Π_{n-2} is a polynomial of degree precisely $n-2$ in variable x.

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Properties of Legendre Polynomial

1):- If we replace x by $-x$, t by $-t$ in (1). Then there is no change in L.H.S.

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(-x)(-t)^n$$

$$\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty}(-1)^n P_n(-x)\, t^n$$

By comparing the coefficient of t^n.

$$\Rightarrow \quad P_n(x) = (-1)^n P_n(-x)$$

If $x = +1$

$$P_n(-1) = (-1)^n \underbrace{P_n(1)}_{1}$$

$$\boxed{P_n(-1) = (-1)^n}$$

2):- If $x = 0$ then Eq (1) becomes

$$\sum_{n=0}^{\infty} P_n(0)\, t^n = (1+t^2)^{-1/2}$$

$$\sum_{n=0}^{\infty} P_n(0)\, t^n = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n t^{2n}}{n!}$$

$\because \sum_{n=0}^{\infty} \frac{(\alpha)_n t^n}{n!} = (1-t)^{-\alpha}$

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3:-

If $x=1$ then (1) becomes

$$\sum_{n=0}^{\infty} P_n(1)\, t^n = (1-2t+t^2)^{-1/2}$$

$$\text{"} \quad = ((1-t)^2)^{-1/2} = (1-t)^{-1}$$

$$\sum_{n=0}^{\infty} P_n(1)\, t^n = \sum_{n=0}^{\infty} \frac{(1)_n\, t^n}{\cancel{n!}} \qquad \because (1)_n = n!$$

$$\sum_{n=0}^{\infty} P_n(1)\, t^n = \sum_{n=0}^{\infty} t^n$$

Comparing the coefficient of t^n.

$$P_n(1) = 1$$

23-04-2015

Since

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (2x)^{n-2k}}{k!\,(n-2k)!}$$

Put $\boxed{n=0}$

$$P_0(x) = \sum_{k=0}^{0} \frac{(-1)^k \left(\frac{1}{2}\right)_{-k} (2x)^{-2k}}{k!\,(-2k)!}$$

$$P_0(x) = \frac{(-1)^0 \left(\frac{1}{2}\right)_0 (2x)^0}{0!\,(0)!}$$

$$P_0(x) = 1$$

Put $\boxed{n=1}$

$$P_1(x) = \sum_{k=0}^{[1/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{1-k} (2x)^{1-2k}}{k!\,(1-2k)!}$$

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$$P_1(x) = \sum_{k=0}^{0} \frac{(-1)^k \left(\frac{1}{2}\right)_{1-k} (2x)^{1-2k}}{k!\,(1-2k)!} \qquad \left[\frac{1}{2}\right] = [0.5] = 0$$

$$P_1(x) = \frac{(-1)^0 \left(\frac{1}{2}\right)_1 (2x)^1}{0!\,(1)!}$$

$$P_1(x) = x$$

For $n=2$

$$P_2(x) = \sum_{k=0}^{[1]} \frac{(-1)^k \left(\frac{1}{2}\right)_{2-k} (2x)^{2-2k}}{k!\,(2-2k)!}$$

$$= \frac{(-1)^0 \left(\frac{1}{2}\right)_{2-0} (2x)^{2-0}}{0!\,(2-0)!} + \frac{(-1)^1 \left(\frac{1}{2}\right)_{2-1} (2x)^{2-2}}{1!\,(2-2)!}$$

$$= \frac{\frac{1}{2}\left(\frac{1}{2}+1\right) 4x^2}{2} + \frac{(-1)\left(\frac{1}{2}\right)(2x)^0}{0!}$$

$$= \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) 4x^2}{2} - \frac{1}{2}$$

$$= \frac{3x^2}{2} - \frac{1}{2}$$

$$= \frac{1}{2}\left(3x^2 - 1\right)$$

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For $n=3$.

$$P_3(x) = \sum_{k=0}^{[\frac{3}{2}]} \frac{(-1)^k (\frac{1}{2})_{3-k} (2x)^{3-2k}}{k!\,(3-2k)!}$$

$$P_3(x) = \sum_{k=0}^{1} \frac{(-1)^k (\frac{1}{2})_{3-k} (2x)^{3-2k}}{k!\,(3-2k)!}$$

$$P_3(x) = \frac{(-1)^0 (\frac{1}{2})_3 (2x)^3}{0!\;3!} + \frac{(-1)^1 (\frac{1}{2})_2 (2x)^1}{1!\;1!}$$

$$P_3(x) = \frac{(\frac{1}{2})(\frac{3}{2})(\frac{5}{2}) 2^3 x^3}{6} - \frac{(\frac{1}{2})(\frac{3}{2})(2x)}{1}$$

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

$$P_3(x) = \frac{1}{2}[5x^3 - 3x]$$

For $n=4$

$$P_4(x) = \sum_{k=0}^{2} \frac{(-1)^k (\frac{1}{2})_{4-k} (2x)^{4-2k}}{k!\,(4-2k)!}$$

$$= \frac{(-1)^0 (\frac{1}{2})_4 (2x)^4}{0!\;4!} + \frac{(-1)^1 (\frac{1}{2})_3 (2x)^2}{1!\;2!} + \frac{(-1)^2 (\frac{1}{2})_2 (2x)^0}{2!\;0!}$$

$$= \frac{(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})(\frac{7}{2}) 2^4 x^4}{4\cdot 3\cdot 2} - \frac{(\frac{1}{2})(\frac{3}{2})(\frac{5}{2}) 2^2 x^2}{2} + \frac{(\frac{1}{2})(\frac{3}{2})}{2}$$

$$= \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$$

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For $n=5$

$$P_5(x)=\sum_{k=0}^{[5/2]\to 2}\frac{(-1)^k\left(\frac{1}{2}\right)_{5-k}(2x)^{5-2k}}{k!\,(5-2k)!}$$

$$=\frac{(-1)^0\left(\frac{1}{2}\right)_5(2x)^5}{0!\;5!}+\frac{(-1)^1\left(\frac{1}{2}\right)_4(2x)^3}{1!\;3!}+\frac{(-1)^2\left(\frac{1}{2}\right)_3(2x)^1}{2!\;1!}$$

$$=\frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)\left(\frac{9}{2}\right)\cdot 2^5 x^5}{5\cdot 4\cdot 3\cdot 2\cdot 1}-\frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)2^3x^3}{3\cdot 2\cdot 1}$$

$$+\frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\cdot 2\cdot x}{2\cdot 1}$$

$$=\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x$$

For $n=6$

$$P_6(x)=\sum_{k=0}^{[6/2]\to 3}\frac{(-1)^k\left(\frac{1}{2}\right)_{6-k}(2x)^{6-2k}}{k!\,(6-2k)!}$$

$$=\frac{(-1)^0\left(\frac{1}{2}\right)_6(2x)^6}{0!\;6!}+\frac{(-1)^1\left(\frac{1}{2}\right)_5(2x)^4}{1!\;4!}+\frac{(-1)^2\left(\frac{1}{2}\right)_4(2x)^2}{2!\;2!}$$

$$+\frac{(-1)^3\left(\frac{1}{2}\right)_3(2x)^0}{3!\;0!}$$

$$=\frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)\left(\frac{9}{2}\right)\left(\frac{11}{2}\right)\cdot 2^6x^6}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}-\frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)\left(\frac{9}{2}\right)2^4x^4}{4\cdot 3\cdot 2\cdot 1}$$

$$+\frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)2^2\cdot x^2}{2\cdot 2}-\frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{3\cdot 2\cdot 1}$$

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$$P_6(x) = \frac{231}{16}x^6 - \frac{315}{16}x^4 + \frac{105}{16}x^2 - \frac{5}{16}$$

$$P_6(x) = \frac{1}{16}\left[231x^6 - 315x^4 + 105x^2 - 5\right]$$

The Rodrigues Formula 06-05-2015

Since

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (2x)^{n-2k}}{k!\,(n-2k)!} \longrightarrow (1)$$

As we know that

$$(\alpha)_{2m} = 2^{2m} \left(\frac{\alpha}{2}\right)_m \left(\frac{\alpha+1}{2}\right)_m$$

Put $\alpha = 1$

$$(1)_{2m} = 2^{2m} \left(\frac{1}{2}\right)_m (1)_m$$

$$\Rightarrow (2m)! = 2^{2m} \left(\frac{1}{2}\right)_m m!$$

$$\frac{(2m)!}{m!} = 2^{2m} \left(\frac{1}{2}\right)_m$$

Put $m = n-k$

$$\frac{(2n-2k)!}{(n-k)!} = 2^{2n-2k} \left(\frac{1}{2}\right)_{n-k}$$

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OR $$2^{n-2k}\left(\frac{1}{2}\right)_{n-k} = \frac{(2n-2k)!}{2^n (n-k)!}$$

Put in (1).

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{k!\,(n-2k)!} \frac{x^{n-2k}}{2^n (n-k)!}$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k D^n (x^{2n-2k})}{k!\, 2^n (n-k)!}$$

$D = \frac{d}{dx}$

$D^s(x^m) = \frac{m!\, x^{m-s}}{(m-s)!}$

$s = n,\ m = 2n-2k$

$D^n(x^{2n-2k}) = \frac{(2n-2k)!\, x^{n-2k}}{(n-2k)!}$

$$= \frac{D^n}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k x^{2n-2k}}{k!\,(n-k)!}$$

$$= \frac{D^n}{2^n n!} \sum_{k=0}^{[n/2]} \frac{n!}{k!\,(n-k)!} (-1)^k x^{2n-2k}$$

nc_k

$\because C_{n,k} = \frac{n!}{k!\,(n-k)!}$

$\binom{n}{k}$

$$= \frac{D^n}{2^n n!} \sum_{k=0}^{[n/2]} C_{n,k}\, x^{2n-2k} (-1)^k$$

for $\left[\frac{n}{2}\right] < k \le n$

$$\Rightarrow D^n\left[x^{2n-2k}\right] = 0$$

So,

$$P_n(x) = \frac{D^n}{2^n n!} \sum_{k=0}^{[n/2]} C_{n,k}\, x^{2n-2k} (-1)^k + \frac{D^n}{2^n n!} \sum_{k=[n/2]+1}^{n} C_{n,k}\, x^{2n-2k} (-1)^k$$

$$P_n(x) = \frac{D^n}{2^n n!} \sum_{k=0}^{n} C_{n,k} (x^2)^{n-k} (-1)^k$$

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$$\Rightarrow \quad P_n(x) = \frac{1}{n!\,2^n} D^n \cdot (x^2-1)^n$$

Binomial formula.

$$\because \ (a+b)^n = \sum_{k=0}^{n} C_{n,k}\, a^{n-k} b^k$$

It is called Rodrigues formula.

Since

$$P_n(x) = \frac{1}{n!\,2^n} \cdot D^n\,(x^2-1)^n$$

$$\Rightarrow \quad P_n(x) = \frac{1}{n!\,2^n} D^n\left[(x-1)^n (x+1)^n\right]$$

By using Leibnitz Rule. i.e

$$D^n(uv) = \sum_{k=0}^{n} C_{n,k}\, D^k(u)\, D^{n-k}(v)$$

So,

$$P_n(x) = \frac{1}{n!\,2^n}\left[\sum_{k=0}^{n} C_{n,k}\, D^k (x-1)^n\, D^{n-k}(x+1)^n\right]$$

$$= \frac{1}{n!} \sum_{k=0}^{n} \frac{C_{n,k}}{2^{n-k+k}} \frac{n!\,(x-1)^{n-k}}{(n-k)!} \cdot \frac{n!\,(x+1)^k}{k!}$$

$$\because \ D^s x^m = \frac{m!\, x^{m-s}}{(m-s)!}$$

$$= \sum_{k=0}^{n} C_{n,k} \frac{n!}{k!\,(n-k)!} \frac{(x-1)^{n-k}}{2^{n-k}} \cdot \frac{(x+1)^k}{2^k}$$

$$= \sum_{k=0}^{n} C_{n,k}^2 \left(\frac{x-1}{2}\right)^{n-k} \left(\frac{x+1}{2}\right)^k$$

It is also called Rodrigues formula for L.P

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184

Differential Recurrence Relations 7/15

We know that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n \longrightarrow (1)$$

Let

$$F = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

$$\Rightarrow \quad F = (1-2xt+t^2)^{-1/2}$$

$$\frac{\partial F}{\partial x} = (\cancel{-}\tfrac{1}{\cancel{2}})(1-2xt+t^2)^{-3/2}(\cancel{-2}t)$$

$$\frac{1}{t}\cdot\frac{\partial F}{\partial x} = (1-2xt+t^2)^{-3/2} \longrightarrow (2)$$

&

$$\frac{\partial F}{\partial t} = (-\tfrac{1}{2})(1-2xt+t^2)^{-3/2}(-2x+2t)$$

$$\frac{\partial F}{\partial t} = (\cancel{-}\tfrac{1}{\cancel{2}})(1-2xt+t^2)^{-3/2}(\cancel{-2})(x-t)$$

$$\frac{1}{x-t}\cdot\frac{\partial F}{\partial t} = (1-2xt+t^2)^{-3/2} \longrightarrow (3)$$

Comparing (2) & (3).

$$\frac{1}{t}\cdot\frac{\partial F}{\partial x} = \frac{1}{x-t}\cdot\frac{\partial F}{\partial t}$$

$$(x-t)\cdot\frac{\partial F}{\partial x} = t\cdot\frac{\partial F}{\partial t}$$

$$(x-t)\cdot\frac{\partial F}{\partial x} - t\cdot\frac{\partial F}{\partial t} = 0 \longrightarrow (4)$$

Since $F = \sum_{n=0}^{\infty} P_n(x)\, t^n$

$$\Rightarrow \quad \frac{\partial F}{\partial x} = \sum_{n=0}^{\infty} P_n'(x)\, t^n \longrightarrow (5)$$

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& $\frac{\partial F}{\partial t} = \sum_{n=0}^{\infty} n P_n(x) t^n \longrightarrow (6)$

Put (5) & (6) in (4).

$(x-t) \sum_{n=0}^{\infty} P_n'(x) t^n - t \sum_{n=0}^{\infty} n P_n(x) t^{n-1} = 0$

$\sum_{n=0}^{\infty} x P_n'(x) t^n - \sum_{n=0}^{\infty} P_n'(x) t^{n+1} - \sum_{n=0}^{\infty} n P_n(x) t^n = 0 \longrightarrow (7)$

Replace n by $n-1$ in 2nd term.

$\sum_{n=0}^{\infty} x P_n'(x) t^n - \sum_{n=1}^{\infty} P_{n-1}'(x) t^n - \sum_{n=0}^{\infty} n P_n(x) t^n = 0$

$\sum_{n=0}^{\infty} x P_n'(x) t^n = \sum_{n=0}^{\infty} n P_n(x) t^n + \sum_{n=1}^{\infty} P_{n-1}'(x) t^n$

Comparing the coefficients of t^n.

$x P_n'(x) = n P_n(x) + P_{n-1}'(x) \longrightarrow (8)$

Now, as we know that

$F = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$

$\frac{\partial F}{\partial x} = t(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} P_n'(x) t^n$

$(1-2xt+t^2)^{-3/2} = \frac{1}{t} \sum_{n=0}^{\infty} P_n'(x) t^n$

$(1-2xt+t^2)^{-3/2} = \sum_{n=1}^{\infty} P_n'(x) t^{n-1} \longrightarrow (9)$

& $\frac{\partial F}{\partial t} = (x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=1}^{\infty} n P_n(x) t^{n-1}$

$(x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=1}^{\infty} n P_n(x) t^{n-1} \longrightarrow (10)$

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Multiplying (9) by $(1-t^2)$ & (10) by $2t$.
We have

$$(1-t^2)(1-2xt+t^2)^{-3/2} = (1-t^2)\sum_{n=1}^{\infty} P_n'(x)\, t^{n-1} \longrightarrow (11)$$

&

$$(2\,t)(x-t)(1-2xt+t^2)^{-3/2} = 2t\sum_{n=1}^{\infty} n P_n(x)\, t^{n-1} \longrightarrow (12)$$

Subtracting (12) from (11), we obtain

$$(1-2xt+t^2)^{-3/2}\left[1-t^2-2xt+2t^2\right] = (1-t^2)\sum_{n=1}^{\infty} P_n'(x)\, t^{n-1} - 2\sum_{n=1}^{\infty} n P_n(x)\, t^n$$

$$(1-2xt+t^2)^{-3/2}(1-2xt+t^2) = \sum_{n=1}^{\infty} P_n'(x)\, t^{n-1} - \sum_{n=1}^{\infty} P_n'(x)\, t^{n+1} - 2\sum_{n=1}^{\infty} n P_n(x)\, t^n$$

$$(1-2xt+t^2)^{-1/2} = \sum_{n=1}^{\infty} P_n'(x)\, t^{n-1} - \sum_{n=1}^{\infty} P_n'(x)\, t^{n+1} - 2\sum_{n=1}^{\infty} n P_n(x)\, t^n$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x)\, t^n = \sum_{n=1}^{\infty} P_n'(x)\, t^{n-1} - \sum_{n=1}^{\infty} P_n'(x)\, t^{n+1} - 2\sum_{n=1}^{\infty} n P_n(x)\, t^n$$

Replace n by $n+1$ in first term & n by $n-1$ in second term in R.H.S of the above equation.

$$\sum_{n=0}^{\infty} P_n(x)\, t^n = \sum_{n=0}^{\infty} P_{n+1}'(x)\, t^n - \sum_{n=2}^{\infty} P_{n-1}'(x)\, t^n - 2\sum_{n=1}^{\infty} n P_n(x)\, t^n$$

By comparing the coefficient of t^n.

$$P_n(x) = P_{n+1}'(x) - P_{n-1}'(x) - 2n P_n(x)$$

$$(2n+1)\, P_n(x) = P_{n+1}'(x) - P_{n-1}'(x) \longrightarrow (13)$$

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Eq. (8) & (13) are independent of differential recurrence relations. From Eq. (8) & (13) another recurrence relation may be obtained.

From (13) we eliminate $P'_{n-1}(x)$.

$$P'_{n-1}(x) = P'_{n+1}(x) - (2n+1)P_n(x) \longrightarrow (14)$$

Put in Eq. (8) i.e $xP'_n(x) = nP_n(x) + P'_{n-1}(x)$

$$\Rightarrow xP'_n(x) = nP_n(x) + P'_{n+1}(x) - (2n+1)P_n(x)$$

$$xP'_n(x) = P'_{n+1}(x) - (n+1)P_n(x) \longrightarrow (15)$$

This is another recurrence relation.

Now change the index n by $n-1$ in Eq. (15), we have

$$xP'_{n-1}(x) = P'_n(x) - nP_{n-1}(x)$$

$$\Rightarrow P'_{n-1}(x) = \frac{1}{x}P'_n(x) - \frac{n}{x}P_{n-1}(x) \longrightarrow (16)$$

Put in Eq. (8) we have

$$xP'_n(x) = nP_n(x) + \frac{P'_n(x)}{x} - \frac{n}{x}P_{n-1}(x)$$

xing by x,

$$x^2P'_n(x) = nxP_n(x) + P'_n(x) - nP_{n-1}(x)$$

$$(x^2-1)P'_n(x) = nxP_n(x) - nP_{n-1}(x) \longrightarrow (17)$$

This is our required result.

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175 The Pure Recurrence Relation 12-05-2015

We know that

(8) ⇒ $$x P_n'(x) = n P_n(x) + P_{n-1}'(x) \rightarrow (1)$$

&

(17) ⇒ $$(x^2-1) P_n'(x) = nx P_n(x) - n P_{n-1}(x) \rightarrow (2)$$

Multiply the Eq. (1) by (x^2-1) on both sides, we have

$$x(x^2-1) P_n'(x) = n(x^2-1) P_n(x) + (x^2-1) P_{n-1}'(x) \rightarrow (3)$$

Using Eq. (2) in Eq. (3) on L.H.S

$$x\left[nx P_n(x) - n P_{n-1}(x)\right] = nx^2 P_n(x) - n P_n(x) + (x^2-1) P_{n-1}'(x)$$

Using Eq (2) Replacing n by $n-1$

$$\cancel{nx^2 P_n(x)} - nx P_{n-1}(x) = \cancel{nx^2 P_n(x)} - n P_n(x) + (n-1)x P_{n-1}(x) - (n-1) P_{n-2}(x)$$

$$-nx P_{n-1}(x) = -n P_n(x) + nx P_{n-1}(x) - x P_{n-1}(x) - n P_{n-2}(x) + P_{n-2}(x)$$

$$n P_n(x) = nx P_{n-1}(x) + nx P_{n-1}(x) - x P_{n-1}(x) - (n-1) P_{n-2}(x)$$

$$n P_n(x) = (2n-1)x P_{n-1}(x) - (n-1) P_{n-2}(x) \quad \text{for } n \geq 2$$

This is pure recurrence Relation.

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176 ## Differential Eq. for Legendre Polynomials

We know that

(8) ⇒ $$x P_n'(x) = n P_n(x) + P_{n-1}'(x) \longrightarrow (1)$$

(15) ⇒ & $$x P_n'(x) = P_{n+1}'(x) - (n+1) P_n(x) \longrightarrow (2)$$

Replace n by $n-1$ in (2).

$$x P_{n-1}'(x) = P_n'(x) - n P_{n-1}(x)$$

Diff. w.r.t 'x'.

$$x P_{n-1}''(x) + P_{n-1}'(x) = P_n''(x) - n P_{n-1}'(x)$$

$$x P_{n-1}''(x) = P_n''(x) - n P_{n-1}'(x) - P_{n-1}'(x)$$

$$x P_{n-1}''(x) = P_n''(x) - (n+1) P_{n-1}'(x) \longrightarrow (3)$$

From (1) we elliminate factor $P_{n-1}'(x)$

i.e $$P_{n-1}'(x) = x P_n'(x) - n P_n(x) \longrightarrow (4)$$

Diff. w.r.t. "x".

$$P_{n-1}''(x) = x P_n''(x) + P_n'(x) - n P_n'(x)$$

$$P_{n-1}''(x) = x P_n''(x) + (1-n) P_n'(x) \longrightarrow (5)$$

Using (4) & (5) in (3).

$$x\left[x P_n''(x) + (1-n) P_n'(x)\right] = P_n''(x) - (n+1)x\left[x P_n'(x) - n P_n(x)\right]$$

$$x^2 P_n''(x) + x P_n'(x) - n x \cancel{P_n'(x)} = P_n''(x) - n x \cancel{P'(x)} + n^2 P_n(x) - x P_n'(x) + n P_n(x)$$

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$$0 = -x^2 P_n''(x) - x P_n'(x) + P_n''(x) + n^2 P_n(x) - x P_n'(x) + n P_n(x)$$

$$(1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0$$

i.e required Eq. in $P_n(x)$.

V.Imp. 195+187

Orthogonality:-

Consider a set of polynomials (real polynomials) $\phi_n(x)$, then $\exists$ a function $w(x) > 0$ in the interval $a < x < b$ & let m & n are non-negative integers. Then

$$\int_a^b w(x)\, \phi_n(x)\, \phi_m(x)\, dx = 0, \quad m \neq n$$

V.Imp. **Theorem:-** The zero's of $P_n(x)$ are distinct, and all lie in the open interval $-1 < x < 1$.

Proof:- We know that the Legendre polynomial $P_n(x)$ satisfies the differential Eq.

$$(1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0 \rightarrow (1)$$

We can also write this Eq. as

$$[(1-x^2) P_n'(x)]' + n(n+1) P_n(x) = 0 \rightarrow (2)$$

Replace n by m.

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$$\left[(1-x^2)P_m'(x)\right]' + m(m+1)P_m(x) = 0 \longrightarrow (3)$$

Now multiplying Eq. (2) by $P_m(x)$ & Eq (3) by $P_n(x)$.

$$\left[P_m(x) \times Eq(2)\right] \Rightarrow \left[(1-x^2)P_n'(x)\right]' P_m(x) + n(n+1)P_m(x)P_n(x) = 0 \longrightarrow (4)$$

$$\left[P_n(x) \times Eq.(3)\right] \Rightarrow \left[(1-x^2)P_m'(x)\right]' P_n(x) + m(m+1)P_m(x)P_n(x) = 0 \longrightarrow (5)$$

Now subtracting Eq. (5) from Eq. (4).

$$\left[(1-x^2)P_n'(x)\right]' P_m(x) - \left[(1-x^2)P_m'(x)\right]' P_n(x) + n(n+1)P_m(x)P_n(x) - m(m+1)P_m(x)P_n(x) = 0$$

$$\left[(1-x^2)\left(P_m(x)P_n'(x) - P_m'(x)P_n(x)\right)\right]' + \left[n^2+n-m^2-m\right]P_m(x)P_n(x) = 0$$

$$\left[n^2-m^2+n-m\right]P_m(x)P_n(x) = -\left[(1-x^2)\left(P_m(x)P_n'(x) - P_m'(x)P_n(x)\right)\right]'$$

$$\left[(n+m)(n-m)+(n-m)\right]P_m(x)P_n(x) = \left[(1-x^2)\left(P_m'(x)P_n(x) - P_m(x)P_n'(x)\right)\right]'$$

Since $\left[(1-x^2)\left(P_m(x)P_n'(x) - P_m'(x)P_n(x)\right)\right]'$

$= \left[(1-x^2)P_m(x)P_n'(x)\right]' - \left[(1-x^2)P_m'(x)P_n(x)\right]'$

$= (1-x^2)P_m'(x)P_n'(x) + P_m(x)\left[(1-x^2)P_n'(x)\right]' - (1-x^2)P_m'(x)P_n'(x) - P_n(x)\left[(1-x^2)P_m'(x)\right]'$

$= \left[(1-x^2)P_n'(x)\right]'P_m(x) - \left[(1-x^2)P_m'(x)\right]'P_n(x)$

$$(n-m)(n+m+1)P_m(x)P_n(x) = \left[(1-x^2)\left(P_m'(x)P_n(x) - P_m(x)P_n'(x)\right)\right]'$$

Integrating w.r.t "x".

$$(n-m)(n+m+1)\int_a^b P_m(x)P_n(x)\,dx = \int_a^b \left[(1-x^2)\left(P_m'(x)P_n(x) - P_m(x)P_n'(x)\right)\right]' dx$$

$$(n-m)(n+m+1)\int_a^b P_m(x)P_n(x)\,dx = \left[(1-x^2)\left(P_m'(x)P_n(x) - P_m(x)P_n'(x)\right)\right]_a^b$$

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at $x = \pm 1$, the factor $(1-x^2)$ in R.H.S will be vanished. So we can write

$$(n-m)(n+m+1)\int_{-1}^{1} P_m(x) P_n(x)\,dx = 0$$

As $m \neq n$ & m, n are non-negative

So,

$$(n-m) \neq 0, \quad (m+n+1) \neq 0$$

& then $\int_{-1}^{1} 1 \cdot P_m(x) P_n(x)\,dx = 0$

Hence the set of polynomials $P_n(x)$ are orthogonal with respect to the weight function unity (i.e $w(x) = 1$) on the interval $-1 < x < 1$.

Note:- As a result of this theorem "The Legendre polynomials therefore possess the properties held by all orthogonal polynomials."

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Bateman's Generating Function 13-05-2015

Since

$$P_n(x) = \sum_{k=0}^{n} C^2_{n,k} \left(\frac{x-1}{2}\right)^{n-k} \left(\frac{x+1}{2}\right)^{k}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{P_n(x)}{(n!)^2} t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{C^2_{n,k}}{(n!)^2} \left(\frac{x-1}{2}\right)^{n-k} \left(\frac{x+1}{2}\right)^{k} t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\cancel{(n!)^2} \cdot \left[\frac{1}{2}(x-1)\right]^{n-k} \left(\frac{1}{2}(x+1)\right)^{k} t^n}{(k!)^2 ((n-k)!)^2 \cancel{(n!)^2}}$$

By using lemma

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n+k)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\left[\frac{1}{2}(x-1)\right]^{n} \left[\frac{1}{2}(x+1)\right]^{k} t^{n+k}}{(k!)^2 (n!)^n}$$

$$= \sum_{n=0}^{\infty} \frac{\left[\frac{1}{2}(x-1)t\right]^{n}}{n! \cdot n!} \cdot \sum_{k=0}^{\infty} \frac{\left[\frac{1}{2}(x+1)t\right]^{k}}{k!\, k!}$$

$$= \sum_{n=0}^{\infty} \frac{\left[\frac{1}{2}(x-1)t\right]^{n}}{(1)_n \, n!} \cdot \sum_{k=0}^{\infty} \frac{\left[\frac{1}{2}(x+1)t\right]^{k}}{(1)_k \, k!}$$

$$= {}_0F_1\left(-;1;\frac{1}{2}(x-1)t\right) \cdot {}_0F_1\left(-;1;\frac{1}{2}(x+1)t\right)$$

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P+201+194

Generating Functions for Hermite Polynomials

Chap# 11

We define the Hermite polynomials $H_n(x)$ by means of the relation

$$\exp(2xt-t^2)=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n \longrightarrow (1)$$

valid for all finite x & t.

Since

$$\exp(2xt-t^2)=e^{2xt}\cdot e^{-t^2}$$

$$=\sum_{n=0}^{\infty}\frac{(2xt)^n}{n!}\cdot\sum_{k=0}^{\infty}\frac{(-t^2)^k}{k!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(2xt)^n(-t^2)^k}{n!\,k!}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(2x)^n(-1)^k t^{n+2k}}{n!\,k!}$$

Since $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}A(k,n-2k)$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{(2x)^{n-2k}(-1)^k t^n}{(n-2k)!\,k!}$$

From Eq. (2) L.H.S of this Eq. will becomes

$$\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{(-1)^k(2x)^{n-2k}}{k!(n-2k)!}t^n$$

Comparing the coefficients of t^n, we have

$$\frac{H_n(x)}{n!}=\sum_{k=0}^{[n/2]}\frac{(-1)^k(2x)^{n-2k}}{k!\,(n-2k)!}$$

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$$\Rightarrow H_n(x) = \sum_{K=0}^{[n/2]} \frac{(-1)^K n! (2x)^{n-2K}}{K! (n-2K)!} \longrightarrow (2)$$

Eq. (2) shows that $H_n(x)$ is a polynomial of degree precisely n in variable x.

$$H_n(x) = \frac{(-1)^0 (2x)^n \cancel{n!}}{0! \cancel{n!}} + \sum_{K=1}^{[n/2]} \frac{(-1)^K (2x)^{n-2K} n!}{K! (n-2K)!}$$

$$= 2^n x^n + \Pi_{n-2} \longrightarrow (3)$$

in which Π_{n-2} is a polynomial of degree $(n-2)$ in variable x.

It follows that $H_n(x)$ is an even function of "x" for even "n", an odd function of "x" for odd n.

Replace x by $-x$ in Eq. (2).

$$H_n(-x) = \sum_{K=0}^{[n/2]} \frac{(-1)^K n! (-2x)^{n-2K}}{K! (n-2K)!}$$

$$H_n(-x) = \sum_{K=0}^{[n/2]} \frac{(-1)^K n! (2x)^{n-2K}}{K! (n-2K)!} \cdot (-1)^{n-2K}$$

$$H_n(-x) = (-1)^n H_n(x)$$

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For $n=0$ Eq. ② becomes.

$$H_0(x) = \sum_{K=0}^{0} \frac{(-1)^K \, 0! \, (2x)^{0-2K}}{K! \, (0-2K)!}$$

$$H_0(x) = \frac{(-1)^0 (2x)^0}{0! \; 0!}$$

$$\boxed{H_0(x) = 1}$$

For $n=1$.

$$H_1(x) = \sum_{K=0}^{\cancel{[1/2]} \to 0} \frac{(-1)^K \, 1! \, (2x)^{1-2K}}{K! \, (1-2K)!}$$

$$H_1(x) = \frac{(-1)^0 (2x)^1}{0! \; 1!}$$

$$\boxed{H_1(x) = 2x}$$

For $n=2$

$$H_2(x) = \sum_{K=0}^{\cancel{[2/2]} \to 1} \frac{(-1)^K \, 2! \, (2x)^{2-2K}}{K! \, (2-2K)!}$$

$$H_2(x) = \frac{(-1)^0 \cancel{2!} \cdot (2x)^2}{0! \; \cancel{2!}} + \frac{(-1)^1 2 \cdot (2x)^{2-2}}{1! \, (2-2)!}$$

$$H_2(x) = 4x^2 - 2$$

$$\boxed{H_2(x) = 2(2x^2-1)}$$

For $n=3$

$$H_3(x) = \sum_{K=0}^{\cancel{[3/2]} \to 1} \frac{(-1)^K \, 3! \, (2x)^{3-2K}}{K! \, (3-2K)!}$$

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$$H_3(x) = \frac{(-1)^0\, 3!\,(2x)^{3-0}}{0!\, 3!} + \frac{(-1)^1\, 3!\,(2x)^{3-2}}{1!\,(3-2)!}$$

$$H_3(x) = 8x^3 - 6\times 2x$$

$$\boxed{H_3(x) = 4(2x^3 - 3x)}$$

14-5-15

203. Differential & Pure Recurrence Relations

Since we know that the generating function

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!} \longrightarrow (A)$$

Let $F = \exp(2xt - t^2)$

$$\frac{\partial F}{\partial x} = 2t\exp(2xt - t^2)$$

$$\Rightarrow \frac{1}{t}\frac{\partial F}{\partial x} = 2\exp(2xt - t^2) \longrightarrow (1)$$

Now $\frac{\partial F}{\partial t} = (2x - 2t)\exp(2xt - t^2)$

$$\frac{1}{x-t}\cdot\frac{\partial F}{\partial t} = 2\exp(2xt - t^2) \longrightarrow (2)$$

Comparing (1) & (2) we have

$$\frac{1}{t}\cdot\frac{\partial F}{\partial x} = \frac{1}{x-t}\cdot\frac{\partial F}{\partial t}$$

$$(x-t)\frac{\partial F}{\partial x} = t\frac{\partial F}{\partial t} \qquad \because F = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

$$(x-t)\frac{\partial}{\partial x}\left[\sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!}\right] = t\frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!}\right]$$

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$$(x-t)\sum_{n=0}^{\infty}\frac{H_n'(x)t^n}{n!} = t\sum_{n=1}\frac{nH_n(x)\,t^{n-1}}{n!}$$

$$x\sum_{n=0}^{\infty}\frac{H_n'(x)t^n}{n!}-\sum_{n=0}^{\infty}\frac{H_n'(x)}{n!}t^{n+1}=\sum_{n=1}^{\infty}\frac{n\,H_n(x)t^n}{n!}$$

Replace n by $n-1$ in 2nd term of above Eq.

$$\sum_{n=0}^{\infty}\frac{x H_n'(x)}{n!}t^n-\sum_{n=1}^{\infty}\frac{H_{n-1}'(x)}{(n-1)!}t^n=\sum_{n=1}^{\infty}\frac{n\,H_n(x)t^n}{n!}$$

Comparing the coefficients of t^n

$$\frac{x H_n'(x)}{n!}-\frac{H_{n-1}'(x)}{(n-1)!}=\frac{n\,H_n(x)}{n!}$$

xing $n!$

$$x H_n'(x)-\frac{n\cancel{(n-1)!}\,H_{n-1}'(x)}{\cancel{(n-1)!}}=n\,H_n(x)$$

$$\Rightarrow x H_n'(x) = n H_{n-1}'(x) + n\,H_n(x) \longrightarrow (3)$$

for $n \geq 1$

Also the relation

$$\exp(2xt-t^2)=\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!}\longrightarrow (4)$$

Diff. (4) w.r.t. x

$$2t\,\exp(2xt-t^2)=\sum_{n=0}^{\infty}\frac{H_n'(x)\,t^n}{n!}$$

dividing by $2t$.

$$\exp(2xt-t^2)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n'(x)\,t^{n-1}}{n!}\longrightarrow (5)$$

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Comparing (5) with Eq (4)

We can write

$$\sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n'(x)}{n!} t^{n-1}$$

Replace n by $n-1$ in L.H.S of the above relation.

$$\sum_{n=1}^{\infty} \frac{H_{n-1}(x)}{(n-1)!} t^{n-1} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n'(x)}{n!} t^{n-1}$$

By comparing the coefficients of t^{n-1}, we obtain

$$\frac{H_{n-1}(x)}{(n-1)!} = \frac{1}{2} \cdot \frac{H_n'(x)}{n!}$$

$$2n! \cdot \frac{H_{n-1}(x)}{(n-1)!} = H_n'(x)$$

$$\frac{2n\cancel{(n-1)!}\; H_{n-1}(x)}{\cancel{(n-1)!}} = H_n'(x)$$

$$\Rightarrow H_n'(x) = 2n\, H_{n-1}(x) \longrightarrow (6)$$

for $n \geq 1$

Now put (6) in (3).

$$x\left[2n H_{n-1}(x)\right] = n H_{n-1}'(x) + n H_n(x)$$

dividing by n.

$$2x H_{n-1}(x) = H_{n-1}'(x) + H_n(x) \quad , n \geq 1$$

$$H_n(x) = 2x H_{n-1}(x) - H_{n-1}'(x) \longrightarrow (7)$$

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Pure Recurrence Relation ⇒

Eqns (6) & (7) are independent differential recurrence relations.

Now replace n by $n-1$ in Eq. (6) i.e $H_n'(x) = 2n\, H_{n-1}(x)$

$$\Rightarrow H_{n-1}'(x) = 2(n-1)\, H_{n-2}(x) \longrightarrow (8)$$

Put (8) in (7), we have

$$H_n(x) = 2x\, H_{n-1}(x) - 2(n-1)\, H_{n-2}(x)$$

This is pure recurrence relation.

20th Hermit's Differential Equation

We know that

(6) ⇒ $$H_n'(x) = 2n\, H_{n-1}(x) \longrightarrow (1)$$

&

(3) ⇒ $$x\, H_n'(x) = n\, H_{n-1}'(x) + n\, H_n(x) \longrightarrow (2)$$

Diff. (1) w.r.t x.

$$H_n''(x) = 2n\, H_{n-1}'(x) \longrightarrow (3)$$

Eq. (2) ⇒ $n H_{n-1}'(x) = x\, H_n'(x) - n\, H_n(x)$

$$H_{n-1}'(x) = \frac{x H_n'(x) - n\, H_n(x)}{n} \longrightarrow (4)$$

Put Eq. (4) in Eq. (3).

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$$H_n''(x) = 2\cancel{n}\left[\frac{x H_n'(x) - n H_n(x)}{\cancel{n}}\right]$$

$$H_n''(x) = 2x H_n'(x) - 2n H_n(x)$$

$$H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0$$

This is Hermite's differential Eq.

2/2 Rodriguess Formula for Hermite Polynomials.

Since

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

In the light of Maclaurin's Theorem

$$H_n(x) = \left|\frac{d^n}{dt^n} \exp(2xt - t^2)\right|_{t=0}$$

The function $\exp(-x^2)$ is independent of t, so we may write

$$\exp(-x^2) H_n(x) = \frac{d^n}{dt^n} \left|\exp(2xt - t^2) \cdot \exp(-x^2)\right|_{t=0}$$

$$= \left|\frac{d^n}{dt^n} \exp(2xt - t^2 - x^2)\right|_{t=0}$$

$$= \left|\frac{d^n}{dt^n} \exp(-(x^2 + t^2 - 2xt))\right|_{t=0}$$

$$= \left|\frac{d^n}{dt^n} \exp(-(x - t)^2)\right|_{t=0}$$

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Now, we put $w = x - t$, we may write as ($w = x$ at $t = 0$)

$$\exp(-x^2) H_n(x) = (-1)^n \left[\frac{d^n}{dw^n} \exp(-w^2) \right]_{w=x}$$

It is ridiculous to diff. w.r.t w a function of w alone & afterward to put $w = x$. The w is superfluous. Therefore, we can write

meaningless. / un-necessary

$$\exp(-x^2) H_n(x) = (-1)^n D^n \exp(-x^2) \qquad D = \frac{d}{dx}$$

$$H_n(x) = (-1)^n \exp(x^2) D^n \exp(-x^2)$$

a formula of the same nature as Rodriguess formula for Legendre polynomials.

Topic : Orthogonality on page 98

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191 18-05-15

The polynomial $L_n^{(\alpha)}(x)$:-

Let us consider naturally terminating ${}_1F_1$. We define, for "n" a non-negative integer,

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_1F_1(-n; 1+\alpha; x) \longrightarrow (1)$$

is called Lequerre polynomials, generalized Lequerre or Sonine polynomials.

Special case:- for $\alpha = 0$

$$L_n^{(0)}(x) = \frac{(1+0)_n}{n!}\, {}_1F_1(-n; 1+0; x)$$

$$L_n(x) = \frac{\cancel{n!}}{\cancel{n!}}\, {}_1F_1(-n; 1; x)$$

$L_n(x) = {}_1F_1(-n; 1; x)$ is called simple Lequerre polynomial.

$$(1) \Rightarrow L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k\, x^k}{(1+\alpha)_k\, k!}$$

P# 47
S.C. Q8

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (-n)_k\, x^k}{n!\, (1+\alpha)_k\, k!}$$

$\because (n-k)! = \frac{(-1)^k n!}{(-n)_k}$

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (-1)^k \cancel{n!}\, x^k}{\cancel{n!}\, (1+\alpha)_k\, k!\, (n-k)!}$$

$(-n)_k = \frac{(-1)^k n!}{(n-k)!}$

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n\, x^k}{k!\, (n-k)!\, (1+\alpha)_k} \longrightarrow (2)$$

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For $n=0$

$$L_0^{(\alpha)}(x) = \sum_{k=0}^{0} \frac{(-1)^k \cancel{(1+\alpha)_0} \, x^k}{k!\,(-k)!\,(1+\alpha)_k} \quad \xrightarrow{} 1$$

where $\alpha \neq -1$

$$L_0^{(\alpha)}(x) = \frac{(-1)^0 \, x^0}{0!\;0!\;(1+\alpha)_0}$$

$$\boxed{L_0^{(\alpha)}(x) = 1}$$

For $n=1$

$$L_1^{(\alpha)}(x) = \sum_{k=0}^{1} \frac{(-1)^k (1+\alpha)_1 \, x^k}{k!\,(1-k)!\,(1+\alpha)_k}$$

$$L_1^{(\alpha)}(x) = \frac{(-1)^0 (1+\alpha)\, x^0}{0!\,(1-0)!\,(1+\alpha)_0} + \frac{(-1)^1 \cancel{(1+\alpha)_1}\, x^1}{1!\,(1-1)!\,\cancel{(1+\alpha)_1}}$$

$$\boxed{L_1^{\alpha}(x) = (1+\alpha) - x}$$

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Generating Function:-

Since

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n x^k}{k!\,(n-k)!\,(1+\alpha)_k} \longrightarrow (2)$$

We can write,

$$\Rightarrow \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{(1+\alpha)_n} t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-x)^k\, t^n}{k!\,(n-k)!\,(1+\alpha)_k}$$

Using Lemma on R.H.S.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n+k)$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{(1+\alpha)_n} t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-x)^k\, t^{n+k}}{k!\,n!\,(1+\alpha)_k}$$

$$\sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{(1+\alpha)_n} t^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(-xt)^k}{k!\,(1+\alpha)_k}$$

$$\sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)\, t^n}{(1+\alpha)_n} = e^t\, {}_0F_1(-;(1+\alpha);-xt)$$

2.13 Orthogonality for Hermite Polynomials

Theorem:- Hermite polynomials are orthogonal with weight function $w(x) = e^{-x^2}$ over the interval $(-\infty, +\infty)$.

Proof:- We know that by Hermite differential equation;

$$H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0$$

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Multiplying by $\exp(-x^2)$ on both sides.

$H_n''(x)\exp(-x^2) - 2x H_n'(x)\exp(-x^2) + 2n H_n(x)\exp(-x^2) = 0$

We can write it as

$$\left[\exp(-x^2)\,.\,H_n'(x)\right]' + 2n\,H_n(x)\exp(-x^2) = 0 \longrightarrow (1)$$

Replace n by m.

$$\left\{\exp(-x^2)\cdot H_m'(x)\right\}' + 2m\,H_m(x)\exp(-x^2) = 0 \longrightarrow (2)$$

Now By $\left[(1) \times H_m(x)\right] - \left[(2) \times H_n(x)\right]$

$$H_m(x)\left\{\exp(-x^2)H_n'(x)\right\}' + 2n H_n(x) H_m(x)\exp(-x^2)$$
$$- H_n(x)\left[\exp(-x^2)H_m'(x)\right]' + 2m H_n(x) H_m(x)\exp(-x^2) = 0$$

$$2(n-m)\exp(-x^2) H_n(x) H_m(x) = H_n(x)\left[\exp(-x^2)H_m'(x)\right]' - H_m(x)\left[\exp(-x^2)H_n'(x)\right]'$$

$$2(n-m)\exp(-x^2)H_n(x)H_m(x) = \left[\exp(-x^2)\left\{H_n(x)H_m'(x) - H_n'(x)H_m(x)\right\}\right]'$$

On integrating w.r.t. 'x'.

$$2(n-m)\int_a^b \exp(-x^2) H_n(x) H_m(x)\,dx = \exp(-x^2)\left\{H_n(x)H_m'(x) - H_n'(x)H_m(x)\right\}\Big|_a^b$$

When $x \longrightarrow \pm\infty$ then $\exp(-x^2)$ vanishes.

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$\frac{1}{e^{x^2}} = \frac{1}{e^{(-\infty)^2}} = \frac{1}{e^{\infty}} = 0$

$$\Rightarrow 2(n-m)\int_{-\infty}^{\infty} \exp(-x^2)\, H_n(x)\, H_m(x)\, dx = 0$$

as $m \neq n \Rightarrow n-m \neq 0$

So

$$\int_{-\infty}^{\infty} \exp(-x^2)\, H_n(x)\, H_m(x)\, dx = 0.$$

which shows that $H_n(x)\, H_m(x)$ are orthogonal with weight function $w(x) = \exp(-x^2)$ over the interval $(-\infty, +\infty)$.

196 The Rodriguess formula (for L.P)

Since

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n\, x^k}{k!\,(n-k)!\,(1+\alpha)_k}$$

$$= \frac{1}{x^{\alpha}\, n!} \sum_{k=0}^{n} \frac{(-1)^k\, n!}{k!\,(n-k)!} \cdot \frac{(1+\alpha)_n\, (x)^{k+\alpha}}{(1+\alpha)_k}$$

$$= \frac{x^{-\alpha}}{n!} \sum_{k=0}^{n} (-1)^k\, C_{n,k}\, D^{n-k} \cdot x^{n+\alpha}$$

Since $C_{n,k} = \frac{n!}{k!\,(n-k)!}$ $\quad \boxed{(n+\alpha)! = \alpha(\alpha+1)_n}$

& $D^{n-k}\, x^{n+\alpha} = \frac{\alpha(1+\alpha)_n\, x^{k+\alpha}}{\alpha\, (1+\alpha)_k}$

Now

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}}{n!} \sum_{k=0}^{n} \frac{(-1)^k e^{-x}}{e^{-x}}\, C_{n,k}\, D^{n-k}\, x^{n+\alpha}$$

$$= \frac{x^{-\alpha} e^{x}}{n!} \sum_{k=0}^{n} C_{n,k}\, D^{n-k}\, x^{n+\alpha}\, D^{k}\, e^{-x}$$

$\because D^k e^{-x} = (-1)^k e^{-x}$

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By using Liebnitz Rule,

$$D^n(uv) = \sum_{k=0}^{n} C_{n,k} D^{n-k}(u) D^k(v)$$

So

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} e^{x}}{n!} D^n\left[e^{-x} x^{n+\alpha}\right]$$

This is called Rodriguess formula

192 Recurrence Relation 20-05-2015
for Leguerre Polynomial

We know that

$$e^t \psi(xt) = \sum_{n=0}^{\infty} \sigma_n(x) t^n$$

$$\Rightarrow \sigma_0(x) = 0 \quad \text{for } n=0$$

&

$$x\sigma_n'(x) - n\sigma_n(x) = -\sigma_{n-1}(x) \text{ for } n \geq 1$$

By definition of Leguerre

$$e^t {}_0F_1(-; 1+\alpha; -xt) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)\, t^n}{(1+\alpha)_n} \longrightarrow (A)$$

Using

$$\frac{xDL_n^{(\alpha)}(x)}{(1+\alpha)_n} - \frac{nL_n^{\alpha}(x)}{(1+\alpha)_n} = -\frac{L_{n-1}^{(\alpha)}(x)}{(1+\alpha)_{n-1}}$$

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$$\frac{1}{(1+\alpha)_n}\left[xD L_n^{(\alpha)}(x) - n L_n^{(\alpha)}(x)\right] = -\frac{L_{n-1}^{(\alpha)}(x)\,(\alpha+n)}{(1+\alpha)_n}$$

$\because (1+\alpha)_n = (1+\alpha)_{n-1}(\alpha+n)$

$$\frac{1}{(1+\alpha)_{n-1}} = \frac{(\alpha+n)}{(1+\alpha)_n}$$

$$xD L_n^{(\alpha)}(x) - n L_n^{(\alpha)}(x) = -(\alpha+n)\, L_{n-1}^{(\alpha)}(x)$$

This is differential recurrence relation for Laguerre Polynomials.

Now, we know that

$$A(t)\exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} y_n(x)\, t^n$$

It follows that $y_0'(x) = 0$ for $n=0$

& $y_n'(x) = y_{n-1}'(x) - y_{n-1}(x)$ for $n \geq 1$

& $y_n'(x) = -\sum_{k=0}^{n-1} y_k(x)$

Since

$$(1-t)^{-1-\alpha}\exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n$$

The choice $A(t) = (1-t)^{-1-\alpha}$ yeilds

$y_n(x) = L_n^{(\alpha)}(x)$, we have shown

the Laguerre polynomials satisfy for $n \geq 1$

$$D L_n^{(\alpha)}(x) = D L_{n-1}^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x)$$

& $$D L_n^{(\alpha)}(x) = -\sum_{k=0}^{n-1} L_k^{(\alpha)}(x)$$

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192

Pure Recurrence Relation for L.P.

We have

$$x D L_n^{(\alpha)}(x) = n L_n^{(\alpha)}(x) - (\alpha+n) L_{n-1}^{(\alpha)}(x) \rightarrow ①$$

$$D L_n^{(\alpha)}(x) = D L_{n-1}^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x) \longrightarrow ②$$

$$D L_n^{(\alpha)}(x) = -\sum_{k=0}^{n-1} L_k^{(\alpha)}(x) \longrightarrow ③$$

xing ② by x & subtract it from ①.

$$x D L_n^{(\alpha)}(x) = n L_n^{(\alpha)}(x) - (\alpha+n) L_{n-1}^{(\alpha)}(x)$$

$$\underline{-\; x D L_n^{(\alpha)}(x) = x D L_{n-1}^{(\alpha)}(x) - x L_{n-1}^{(\alpha)}(x)}$$

$$0 = n L_n^{(\alpha)}(x) - x D L_{n-1}^{(\alpha)}(x) - (\alpha+n) L_{n-1}^{(\alpha)}(x) + x L_{n-1}^{(\alpha)}(x) \rightarrow Ⓑ$$

Replace n by $n-1$ in Eq ①.

$$x D L_{n-1}^{(\alpha)}(x) = (n-1) L_{n-1}^{(\alpha)}(x) - (\alpha+n-1) L_{n-2}^{(\alpha)}(x) \rightarrow ④$$

put ④ in Ⓑ.

$$0 = n L_n^{(\alpha)}(x) - (n-1) L_{n-1}^{(\alpha)}(x) + (\alpha+n-1) L_{n-2}^{(\alpha)}(x) - (\alpha+n) L_{n-1}^{(\alpha)}(x) + x L_{n-1}^{(\alpha)}(x)$$

$$n L_n^{(\alpha)}(x) = (n-1) L_{n-1}^{(\alpha)}(x) - (\alpha+n-1) L_{n-2}^{(\alpha)}(x) + (\alpha+n) L_{n-1}^{(\alpha)}(x) - x L_{n-1}^{(\alpha)}(x)$$

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$$nL_n^{(\alpha)}(x) = (n-1+\alpha+n-x)L_{n-1}^{(\alpha)}(x) + (\alpha+n-1)L_{n-2}^{(\alpha)}(x)$$

$$nL_n^{(\alpha)}(x) = (2n+\alpha-1-x)L_{n-1}^{(\alpha)}(x) + (\alpha+n-1)L_{n-2}^{(\alpha)}(x)$$

This is the pure recurrence relation.

198 Orthogonality of Laguerre Polynomials

Since

$$xD^2L_n^{(\alpha)}(x) + (1+\alpha-x)DL_n^{(\alpha)}(x) + nL_n^{(\alpha)}(x) = 0 \quad \rightarrow (1)$$

Multiplying by $x^{\alpha}e^{-x}$ on both sides.

$$x^{\alpha}e^{-x}xD^2L_n^{(\alpha)}(x) + x^{\alpha}e^{-x}(1+\alpha-x)DL_n^{(\alpha)}(x) + x^{\alpha}e^{-x}nL_n^{(\alpha)}(x) = 0$$

$$x^{\alpha+1}e^{-x}D^2L_n^{(\alpha)}(x) + \left((1+\alpha)e^{-x}x^{\alpha} - x^{\alpha+1}e^{-x}\right)DL_n^{\alpha}(x) + x^{\alpha}e^{-x}nL_n^{(\alpha)}(x) = 0$$

$$D\left[x^{\alpha+1}e^{-x}DL_n^{(\alpha)}(x)\right] + x^{\alpha}e^{-x}nL_n^{(\alpha)}(x) = 0 \quad \rightarrow (2)$$

Replace n by m.

$$D\left[x^{\alpha+1}e^{-x}DL_m^{(\alpha)}(x)\right] + mx^{\alpha}e^{-x}L_m^{(\alpha)}(x) = 0 \quad \rightarrow (3)$$

By $\left[(2) \times L_m^{(\alpha)}(x)\right] - \left[(3)\, L_n^{(\alpha)}(x)\right]$

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$$L_m^{(\alpha)}(x)\, D\left[x^{\alpha+1} e^{-x} D L_n^{(\alpha)}(x)\right] + n x^{\alpha} e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)$$
$$- L_n^{(\alpha)}(x)\, D\left[x^{\alpha+1} e^{-x} D L_m^{(\alpha)}(x)\right] - m x^{\alpha} e^{-x} L_m^{(\alpha)}(x) L_m^{(\alpha)}(x) = 0$$

$$(m-n) e^{-x} x^{\alpha} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) = L_m^{(\alpha)}(x)\, D\left[x^{\alpha+1} e^{-x} D L_n^{(\alpha)}(x)\right]$$
$$- L_n^{(\alpha)}(x)\, D\left[x^{\alpha+1} e^{-x} D L_m^{(\alpha)}(x)\right]$$

$$(m-n) x^{\alpha} e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) = D\left[x^{\alpha+1} e^{-x} \left\{ L_m^{(\alpha)}(x) D L_n^{(\alpha)}(x) - L_n^{(\alpha)}(x) D L_m^{(\alpha)}(x) \right\}\right]$$

On integrating

$$(m-n)\int_a^b x^{\alpha} e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\, dx = x^{\alpha+1} e^{-x} \left\{ L_m^{(\alpha)}(x) D L_n^{(\alpha)}(x) - L_n^{(\alpha)}(x) D L_m^{(\alpha)}(x) \right\}\Big|_a^b$$

In the R.H.S the factor $x^{\alpha+1} e^{-x} \to 0$ for $Re(\alpha) > -1$

& $0 < x < \infty$

$$\Rightarrow (m-n)\int_0^{\infty} x^{\alpha} e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\, dx = 0$$

Since $m \neq n \Rightarrow m-n \neq 0$

Hence $\displaystyle\int_0^{\infty} x^{\alpha} e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\, dx = 0$

Hence Laguerre polynomial are orthogonal with weight function $w = x^{\alpha} e^{-x}$ over the interval $(0, \infty)$.

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193

Differential Eq. for L.P.

Since

Page 103 ① ⇒
$$xDL_n^{(\alpha)}(x) = nL_n^{(\alpha)}(x) - (\alpha+n)L_{n-1}^{(\alpha)}(x) \longrightarrow ①$$

& ② ⇒
$$DL_n^{(\alpha)}(x) = DL_{n-1}^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x) \longrightarrow ②$$

Eq. ① ⇒
$$(\alpha+n)L_{n-1}^{(\alpha)}(x) = nL_n^{(\alpha)}(x) - xDL_n^{(\alpha)}(x)$$

$$\Rightarrow L_{n-1}^{(\alpha)}(x) = \frac{1}{(\alpha+n)}\left[nL_n^{(\alpha)}(x) - xDL_n^{(\alpha)}(x)\right] \longrightarrow ③$$

Diff. ③ w.r.t "x".

$$DL_{n-1}^{(\alpha)}(x) = \frac{1}{(\alpha+n)}\left[nDL_n^{(\alpha)}(x) - xD^2L_n^{(\alpha)}(x) - DL_n^{(\alpha)}(x)\right]$$

$$DL_{n-1}^{(\alpha)}(x) = \frac{1}{(\alpha+n)}\left[(n-1)DL_n^{(\alpha)}(x) - xD^2L_n^{(\alpha)}(x)\right] \longrightarrow ④$$

Put Eq. ④ & ③ in Eq. ②.

$$② \Rightarrow DL_n^{(\alpha)}(x) = \frac{1}{\alpha+n}\left[(n-1)DL_n^{(\alpha)}(x) - xD^2L_n^{(\alpha)}(x)\right] - \frac{1}{(\alpha+n)}\left[nL_n^{(\alpha)}(x) - xDL_n^{(\alpha)}(x)\right]$$

×ing $(\alpha+n)$.

$$\Rightarrow (\alpha+n)DL_n^{(\alpha)}(x) = (n-1)DL_n^{(\alpha)}(x) - xD^2L_n^{(\alpha)}(x) - nL_n^{(\alpha)}(x) + xDL_n^{(\alpha)}(x)$$

$$xD^2L_n^{(\alpha)}(x) + (\alpha+n)DL_n^{(\alpha)}(x) - (n-1)DL_n^{(\alpha)}(x) - xDL_n^{(\alpha)}(x) + nL_n^{(\alpha)}(x) = 0$$

$$xD^2L_n^{(\alpha)}(x) + (1+\alpha-x)DL_n^{(\alpha)}(x) + nL_n^{(\alpha)}(x) = 0$$

This is called differential Eq. for L.P.

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Additional Generating Function. 26-05-2015

Since

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n \longrightarrow ①$$

Consider

$$(1-2xt+t^2)^{-1/2} = \left[1-2xt+x^2t^2-x^2t^2+t^2\right]^{-1/2}$$

$$= \left[(1-xt)^2 - t^2(x^2-1)\right]^{-1/2}$$

$$= (1-xt)^{-1}\left[1 - \frac{t^2(x^2-1)}{(1-xt)^2}\right]^{-1/2}$$

By ① $\Rightarrow$

$$\sum_{n=0}^{\infty} P_n(x) t^n = (1-xt)^{-1} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k}{k!} \left[\frac{(x^2-1)t^2}{(1-xt)^2}\right]^k$$

$$= \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (x^2-1)^k t^{2k}}{k! \, (1-xt)^{2k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (x^2-1)^k t^{2k}}{k!} (1-xt)^{-(2k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (x^2-1)^k t^{2k}}{k!} \sum_{n=0}^{\infty} \frac{(2k+1)_n}{n!} (xt)^n$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (x^2-1)^k x^n t^{2k+n}}{n! \, k!} (2k+1)_n$$

$$\left[\because (2k+1)_n = \frac{(n+2k)!}{(2k)!}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (x^2-1)^k x^n t^{2k+n} (n+2k)!}{n! \, k! \, (2k)!}$$

Using Lemma

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A(k, n-2k)$$

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Replace n by $n-2k$.

$$\sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(1/2)_k (x^2-1)^k x^{n-2k} t^n n!}{k! (n-2k)! (2k)!}$$

$$\sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{n! (x^2-1)^k x^{n-2k} t^n}{k! (n-2k)! k! 2^{2k}}$$

$\because (\alpha)_{2k} = 2^{2k} \left(\frac{\alpha}{2}\right)_k \left(\frac{\alpha+1}{2}\right)_k$

put $\alpha = 1$

$(2k)! = 2^{2k} \left(\frac{1}{2}\right)_k k!$

$\frac{1}{k!\, 2^{2k}} = \frac{(1/2)_k}{(2k)!}$

By comparing the coefficient of t^n.

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{n! (x^2-1)^k x^{n-2k}}{(k!)^2 2^{2k} (n-2k)!}$$

Multiplying by $\frac{(c)_n t^n}{n!}$ and then taking $\sum_{n=0}^{\infty}$ on both sides

$$\sum_{n=0}^{\infty} \frac{(c)_n P_n(x) t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(c)_n (x^2-1)^k x^{n-2k} t^n}{2^{2k} (k!)^2 (n-2k)!}$$

Using Lemma

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n+2k)$$

$$'' = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c)_{n+2k} (x^2-1)^k x^n t^{n+2k}}{2^{2k} (k!)^2 n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c+2k)_n (c)_{2k} (x^2-1)^k (x t)^n t^{2k}}{2^{2k} (k!)^2 n!}$$

$$= \sum_{k=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{(c+2k)_n (xt)^n}{n!} \right] \frac{(c)_{2k} (x^2-1)^k t^{2k}}{2^{2k} k! \, k!}$$

$(c)_{2k} = 2^{2k} \left(\frac{c}{2}\right)_k \left(\frac{c+1}{2}\right)_k$

$$= \sum_{k=0}^{\infty} (1-xt)^{-(c+2k)} \frac{(c/2)_k \left(\frac{c}{2}+\frac{1}{2}\right)_k}{(1)_k \, k!} \left((x^2-1)t^2\right)^k$$

$\frac{(c)_{2k}}{2^{2k}} = \left(\frac{c}{2}\right)_k \left(\frac{c+1}{2}\right)_k$

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$$\sum_{n=0}^{\infty} \frac{(c)_n P_n(x) t^n}{n!} = (1-xt)^{-c} \sum_{k=0}^{\infty} \frac{\left(\frac{c}{2}\right)_k \left(\frac{c+1}{2}\right)_k}{(1)_k \, k!} \left[\frac{(x^2-1)t^2}{(1-xt)^2}\right]^k$$

$$= (1-xt)^{-c} \, {}_2F_1\left[\begin{matrix} \frac{c}{2}, \frac{c+1}{2}; \\ 1 ; \end{matrix} \ \frac{(x^2-1)t^2}{(1-xt)^2}\right]$$

Proved.

10-3-19 Hyper Geometric Forms of $P_n(x)$. 27-05-2015

Since we know that

$$(1-xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n \longrightarrow (1)$$

Consider

$$(1-2xt+t^2)^{-1/2} = (1-2xt+t^2-2t+2t)^{-1/2}$$

$$= (1-2t+t^2-2xt+2t)^{-1/2}$$

$$= \left((1-t)^2-2t(x-1)\right)^{-1/2}$$

$$= (1-t)^{-1}\left[1-\frac{2t(x-1)}{(1-t)^2}\right]^{-1/2}$$

$\because (1-z)^{-\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k z^k}{k!}$

$$\sum_{n=0}^{\infty} P_n(x) t^n = (1-t)^{-1} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!} \left[\frac{2t(x-1)}{(1-t)^2}\right]^k$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!} (2t)^k (x-1)^k (1-t)^{-(2k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!} 2^k t^k (x-1)^k \sum_{n=0}^{\infty} \frac{(2k+1)_n t^n}{n!}$$

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$\because (2k+1)_n = \frac{(n+2k)!}{(2k)!}$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k 2^k (x-1)^k t^{n+k}}{k!\; n!} \cdot \frac{(n+2k)!}{(2k)!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(x-1)^k t^{n+k} (n+2k)!}{k!\, n!} \cdot \frac{(\frac{1}{2})_k 2^k}{(2k)!}$$

$$\because \quad (\alpha)_{2k} = 2^{2k} \left(\frac{\alpha}{2}\right)_k \left(\frac{\alpha+1}{2}\right)_k$$

$$\alpha = 1 \Rightarrow \quad (2k)! = 2^{2k} \left(\tfrac{1}{2}\right)_k \; k!$$

We can write $\dfrac{1}{2^k\, k!} = \dfrac{(\frac{1}{2})_k\, 2^k}{(2k)!}$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(x-1)^k t^{n+k} (n+2k)!}{2^k\, k!\, k!\, n!}$$

Using lemma

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k, n-k)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(x-1)^k (n+k)!\, t^n}{2^k (k!)^2 (n-k)!}$$

P# 47 S.C Q.8

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(x-1)^k (n+k)!\, (-1)^k (-n)_k\, t^n}{2^k (k!)^2\, n!}$$

$(n-k)! = \dfrac{(-1)^k n!}{(-n)_k}$

$\dfrac{1}{(n-k)!} = \dfrac{(-1)^k (-n)_k}{n!}$

$(n+1)_k = (n+1)(n+2)\cdots(n+k+k-1)$

$$= \frac{(n+1)(n+2)\cdots(n+k)\, n!}{n!}$$

$$= \frac{(n+k)!}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1(x-1))^k (-n)_k (n+1)_k\, t^n}{2^k\, k!\, k!}$$

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$$\sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{(1)_k \, k!} \left(\frac{1-x}{2}\right)^k t^n$$

$$\sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix} -n, & n+1; \\ & 1; \end{matrix} \left(\frac{1-x}{2}\right)\right] t^n$$

By comparing the coefficients of t^n

$$P_n(x) = {}_2F_1\left[\begin{matrix} -n, & n+1; \\ & 1; \end{matrix} \frac{1-x}{2}\right],$$

$$\Rightarrow P_n(-x) = {}_2F_1\left[\begin{matrix} -n, & n+1; \\ & 1; \end{matrix} \frac{1+x}{2}\right]$$

Since $P_n(-x) = (-1)^n P_n(x)$

$$\Rightarrow P_n(x) = (-1)^n P_n(-x)$$

$$P_n(x) = (-1)^n {}_2F_1\left[\begin{matrix} -n, & n+1; \\ & 1; \end{matrix} \frac{1+x}{2}\right]$$

Theorem:- 47

28-05-15

$$A(t) \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} y_{n-1}(x) t^n$$

then $y_0'(x) = 0$, $y_n'(x) = y_{n-1}'(x) - y_{n-1}(x)$

& $y_n'(x) = -\sum_{k=0}^{n-1} y_k(x)$ for $n \geq 1$

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Solution:-

let

$$F = A(t)\exp\left(\frac{-xt}{1-t}\right), \quad F = \sum_{n=0}^{\infty} y_n(x)\, t^n$$

Diff. w.r.t 'x'.

$$\frac{\partial F}{\partial x} = \sum_{n=0}^{\infty} y_n'(x)\, t^n$$

$$\frac{\partial F}{\partial x} = A(t)\exp\left(\frac{-xt}{1-t}\right)\cdot\left(\frac{-t}{1-t}\right)$$

$$(1-t)\frac{\partial F}{\partial x} = -t\, F \longrightarrow (1)$$

We can write

$$(1-t)\sum_{n=0}^{\infty} y_n'(x)\, t^n = -t\sum_{n=0}^{\infty} y_n(x)\, t^n$$

$$\sum_{n=0}^{\infty} y_n'(x)\, t^n - \sum_{n=0}^{\infty} y_n'(x)\, t^{n+1} = -\sum_{n=0}^{\infty} y_n(x)\, t^{n+1}$$

By comparing the coefficient of t^0

$$\Rightarrow y_0'(x) = 0$$

Now Replace n by $n-1$ in 2nd term & R.H.S.

$$\sum_{n=0}^{\infty} y_n'(x)\, t^n - \sum_{n=1}^{\infty} y_{n-1}'(x)\, t^n = -\sum_{n=1}^{\infty} y_{n-1}(x)\, t^n$$

By comparing the coefficient of t^n,

$$\Rightarrow y_n'(x) - y_{n-1}'(x) = -y_{n-1}(x)$$

or $\quad y_n'(x) = y_{n-1}'(x) - y_{n-1}(x)$ for $n \geq 1$

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Again from (1)

$$\frac{\partial F}{\partial x} = \frac{-t}{(1-t)} F$$

$$\frac{\partial F}{\partial x} = -t(1-t)^{-1} F \qquad (1-y)^{-\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} y^k$$

$$\sum_{n=0}^{\infty} y_n'(x) t^n = -t \sum_{n=0}^{\infty} t^n \sum_{k=0}^{\infty} y_k(x) t^k$$

$$= -\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} y_k(x) t^{n+k+1}$$

By Using Lemma

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n-k)$$

$$\Rightarrow \sum_{n=0}^{\infty} y_n'(x) t^n = -\sum_{n=0}^{\infty} \sum_{k=0}^{n} y_k(x) t^{n+1}$$

Replace n by $n-1$ on R.H.S.

$$\sum_{n=0}^{\infty} y_n'(x) t^n = -\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} y_k(x) t^n$$

By comparing the coefficients of t^n.

$$y_n'(x) = -\sum_{k=0}^{n-1} y_k(x)$$

for $n \geq 1$

Identity Eq. $\Rightarrow$ Coefficients of like powers are same.

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Theorem:- 04-06-2015

97

$$\psi(u)=\sum_{n=0}^{\infty}\gamma_n u^n, \quad \gamma_0\neq 0$$

and

$$(1-t)^{-c}\,\psi\left(\frac{-4xt}{(1-t)^2}\right)=\sum_{n=0}^{\infty}f_n(x)\,t^n$$

then prove that

(i):-
$$f_n(x)=\frac{(c)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k\,(c+n)_k\,\gamma_k\,x^k}{(\frac{1}{2}c)_k\,(\frac{1}{2}c+\frac{1}{2})_k}$$

(ii):-
$$x^n=\frac{(c)_{2n}}{2^{2n}\gamma_n}\sum_{k=0}^{n}\frac{(-1)^k\,(c+2k)\,f_k(x)}{(n-k)!\,(c)_{n+k+1}}$$

Proof:- (i):- Consider

$$\sum_{n=0}^{\infty}f_n(x)\,t^n=(1-t)^{-c}\,\psi\left(\frac{-4xt}{(1-t)^2}\right)\longrightarrow (1)$$

Put $u=\frac{-4xt}{(1-t)^2}$ in above equation.

$$\psi\left(\frac{-4xt}{(1-t)^2}\right)=\sum_{k=0}^{\infty}\gamma_k\left(\frac{-4xt}{(1-t)^2}\right)^k$$

Put in Eq. (1)

Eq (1) $\Rightarrow$
$$\sum_{n=0}^{\infty}f_n(x)\,t^n=(1-t)^{-c}\sum_{k=0}^{\infty}\gamma_k\,\frac{(-4xt)^k}{(1-t)^{2k}}$$

$$=\sum_{k=0}^{\infty}\gamma_k\,(-1)^k(4x)^k\,t^k\,(1-t)^{-(2k+c)}$$

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$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \gamma_k \, 2^{2k} x^k t^k \frac{(2k+c)_n \, t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k 2^{2k} x^k (2k+c)_n}{n!} t^{n+k}$$

$$\because (c+2k)_n = \frac{(c)_{n+2k}}{(c)_{2k}}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k 2^{2k} x^k (c)_{n+2k} t^{n+k}}{n! \, (c)_{2k}}$$

By using Lemma.

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n-k)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k \gamma_k 2^{2k} x^k (c)_{n+k}}{(n-k)! \, (c)_{2k}} t^n$$

By comparing the coefficients of t^n

$$f_n(x) = \sum_{k=0}^{n} \frac{(-1)^k 2^{2k} (c)_{n+k} \gamma_k x^k}{(n-k)! \, (c)_{2k}}$$

$$\because (c)_{2k} = 2^{2k} \left(\tfrac{c}{2}\right)_k \left(\tfrac{c+1}{2}\right)_k$$

$$\Rightarrow \frac{1}{(\frac{1}{2}c)_k (\frac{1}{2}c+\frac{1}{2})_k} = \frac{2^{2k}}{(c)_{2k}}$$

& $(n-k)! = \dfrac{(-1)^k n!}{(-n)_k}$

$$\Rightarrow \frac{(-1)^k}{(n-k)!} = \frac{(-n)_k}{n!}$$

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& $(c)_{k+n} = (c)_n \, (c+n)_k$

So

$$z_n(x) = \sum_{k=0}^{n} \frac{(-n)_k \, (c)_n \, (c+n)_k \, \gamma_k \, x^k}{n! \, (\frac{1}{2}c)_k \, (\frac{1}{2}c+\frac{1}{2})_k}$$

$$z_n(x) = \frac{(c)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k \, (c+n)_k \, \gamma_k \, x^k}{(\frac{1}{2}c)_k \, (\frac{1}{2}c+\frac{1}{2})_k}$$

10-06-15

Proof:- (ii):- Since

$$\psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n \longrightarrow (1)$$

&

$$(1-t)^{-c} \, \psi\left(\frac{-4xt}{(1-t)^2}\right) = \sum_{n=0}^{\infty} z_n(x) \, t^n$$

$$\psi\left(\frac{-4xt}{(1-t)^2}\right) = (1-t)^c \sum_{n=0}^{\infty} z_n(x) \, t^n \longrightarrow (2)$$

Let $\frac{-4t}{(1-t)^2} = v \Rightarrow \frac{4t}{(1-t)^2} = -v$

$$1 + \frac{4t}{(1-t)^2} = 1-v$$

$$\frac{1-2t+t^2+4t}{(1-t)^2} = 1-v$$

$$\frac{(1+t)^2}{(1-t)^2} = 1-v$$

$$\frac{1+t}{1-t} = \sqrt{1-v}$$

-1

$1-t \,)\, 1+t$

$\pm 1 \mp t$

2

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$$-1+\frac{2}{1-t}=\sqrt{1-v}$$

$$\frac{2}{1-t}=1+\sqrt{1-v}$$

$$\frac{2}{1+\sqrt{1-v}}=1-t \longrightarrow (3)$$

$$t=1-\frac{2}{1+\sqrt{1-v}}$$

$$t=\frac{1+\sqrt{1-v}-2}{1+\sqrt{1-v}}$$

$$t=\frac{\sqrt{1-v}-1}{\sqrt{1-v}+1}\times\frac{1+\sqrt{1-v}}{1+\sqrt{1-v}}$$

$$t=\frac{1-v-1}{(1+\sqrt{1-v})^2}=\frac{-v}{(1+\sqrt{1-v})^2} \longrightarrow (4)$$

Eq (2) $\Rightarrow$ (3) (4)

$$\psi(x,v)=\left(\frac{2}{1-\sqrt{1-v}}\right)^c\sum_{k=0}^{\infty}Z_k(x)\left(\frac{-v}{(1+\sqrt{1-v})^2}\right)^k$$

$$\psi(x,v)=\left(\frac{2}{1-\sqrt{1-v}}\right)^c\sum_{k=0}^{\infty}Z_k(x)\frac{(-1)^k v^k}{(1+\sqrt{1-v})^{2k}}$$

Multiplying & dividing by 2^{2k} on the R.H.S of above equation, we have

$$\psi(x,v)=\sum_{k=0}^{\infty}\left(\frac{2}{1+\sqrt{1-v}}\right)^{c+2k}\frac{Z_k(x)(-1)^k v^k}{2^{2k}} \longrightarrow (5)$$

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Since

$$\left(\frac{2}{1+\sqrt{1-V}}\right)^{2\gamma-1} = {}_2F_1\left[\begin{matrix}\gamma, \gamma-\frac{1}{2}; \\ 2\gamma;\end{matrix}\ V\right]$$

$$2\gamma-1 = 2k+c$$

$$2\gamma = 2k+c+1$$

$$\gamma = \frac{2k+c+1}{2} \quad \text{put in above Eq.}$$

$$\left(\frac{2}{1+\sqrt{1-V}}\right)^{2k+c} = {}_2F_1\left[\begin{matrix}\frac{2k+c+1}{2}, \frac{2k+c}{2}; \\ 2k+c+1;\end{matrix}\ V\right]$$

Put in Eq. (5).

$$\psi(x,V) = \sum_{k=0}^{\infty} \frac{\gamma_k(x)(-1)^k V^k}{2^{2k}}\ {}_2F_1\left[\begin{matrix}\frac{2k+c+1}{2}, \frac{2k+c}{2}; \\ 2k+c+1;\end{matrix}\ V\right]$$

$$\psi(x,V) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\gamma_k(x)(-1)^k V^k}{2^{2k}} \frac{\left(\frac{2k+c+1}{2}\right)_n \left(\frac{2k+c}{2}\right)_n}{(2k+c+1)_n} \cdot \frac{V^n}{n!}$$

$$\frac{(\alpha)_{2n}}{2^{2n}} = \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k\, \gamma_k(x) \,.. V^{n+k}}{2^{2k+2n}\, n!} \frac{(2k+c)_{2n}}{(2k+c+1)_n}$$

$$(\alpha)_{n+k} = (\alpha+k)_n (\alpha)_k$$

$$(\alpha+k)_n = \frac{(\alpha)_{n+k}}{(\alpha)_k}$$

$$\begin{matrix} n=1 \\ k=2k \end{matrix} \Rightarrow (c)_{1+2k} = (c+2k)_1 (c)_{2k}$$

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$$(c+2k+1)_n = \frac{(c)_{n+2k+1}}{(c)_{2k+1}}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k v^{n+k} \, \gamma_k(x) \, (c)_{2n+2k} \, \cancel{(c)_{2k}} \, (c+2k)}{2^{2k+2n} \, n! \, \cancel{(c)_{2k}} \, (c)_{n+2k+1}}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k v^{n+k} \, \gamma_k(x) \, (c)_{2n+2k} \, (c+2k)}{2^{2k+2n} \, n! \, (c)_{n+2k+1}}$$

Using lemma,

$$\psi(xv) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k \gamma_k(x) \, v^n \, (c)_{2n} \, (c+2k)}{2^{2n} (n-k)! \, (c)_{n+k+1}} \longrightarrow (A)$$

$$\psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n$$

$$u = xv$$

$$\psi(xv) = \sum_{n=0}^{\infty} \gamma_n x^n v^n \longrightarrow (B)$$

From (A) & (B)

$$\sum_{n=0}^{\infty} \gamma_n x^n v^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k \gamma_k(x) \, (c)_{2n} (c+2k)}{2^{2n}(n-k)! \, (c)_{n+k+1}} v^n$$

Comparing the coefficient of v^n.

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$$\gamma_n x^n = \sum_{k=0}^{n} \frac{(-1)^k (c)_{2n} (c+2k)\, z_k(x)}{2^{2n} (n-k)!\, (c)_{n+k+1}}$$

$$x^n = \frac{(c)_{2n}}{2^{2n}\gamma_n} \sum_{k=0}^{n} \frac{(-1)^k (c+2k)\, z_k(x)}{(n-k)!\, (c)_{n+k+1}}$$

Hence proved.

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UNIVERSITY OF SARGODHA, SARGODHA
DEPARTMENT OF MATHEMATICS

Paper: Special Functions MATH-450 **Time:** 60 Minutes
Date: 27-04-2015 **Class: BS-VIII** **Max. Marks:** 20

1. Find the value of $\Gamma(1)$ by using the definition of Weierstrass gamma function. [5]
2. State and prove the Legendre's duplication formula. [5]
3. If $|z|<1$ and $|z/(1-z)|<1$, then ${}_2F_1(a,b;c;z)=(1-z)^{-a}\,{}_2F_1(a,c-b;c;-z/(1-z))$. [5]
4. What do you meant by contiguous functions? If $|x|<1$ and $\operatorname{Re}(r)>\operatorname{Re}(q)>0$, prove that
$$ {}_2F_1(p,q;r;x)=\frac{\Gamma(r)}{\Gamma(q)\Gamma(r-q)}\int_0^1 s^{q-1}(1-s)^{r-q-1}(1-xs)^{-p}\,ds. $$ [5]

May Allah Bless You, Best of Luck *Dr. Shahid Mubeen*

UNIVERSITY OF SARGODHA, SARGODHA
DEPARTMENT OF MATHEMATICS

Paper: Special Functions **MATH-450** **Time:** **2 Hours**
Date: 17-06-2015 **Class: BS-VIII (R+SS1+SS2)** **Max. Marks:** **60**

1. Define the beta function and hence prove that $B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ for $\operatorname{Re}(p)>0,\operatorname{Re}(q)>0$. [12]
2. Define the Orthogonality of simple set of real polynomials and also prove that the Laguarre polynomials form an orthogonal set over the interval $(0,\infty)$ with weight function $x^{\alpha}e^{-x}$ for $\operatorname{Re}(\alpha)>-1$, and $m\neq n$. [12]
3. If $\psi(u)=\sum_{n=0}^{\infty}\gamma_n u^n,\gamma_0\neq 0$ and $(1-t)^{-c}\psi\left(\frac{-4xt}{(1-t)^2}\right)=\sum_{n=0}^{\infty}f_n(x)t^n$, then $x^n=\frac{(c)_{2n}}{2^{2n}\gamma_n}\sum_{k=0}^{\infty}\frac{(-1)^k(c+2k)f_k(x)}{(n-k)!(c)_{n+k+1}}$. [12]
4. Let $P_n(x)=\sum_{m=0}^{[n/2]}\frac{(-1)^m\left(\frac{1}{2}\right)_{n-m}(2x)^{n-2m}}{m!(n-2m)!}$ be the Legendre polynomials, then show that the Rodrigues formula for Legendre polynomials as $P_n(x)=\sum_{m=0}^{n}C_{n,m}^2\left(\frac{x-1}{2}\right)^{n-m}\left(\frac{x+1}{2}\right)^m$. [12]
5. Derive the Hermite differential equation $H_n''(x)-2xH_n'(x)+2nH_n(x)=0$ for the generating function
$$ \exp(2xt-t^2)=\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!}. $$ [12]

May Allah Bless You, Best of Luck *Dr. Shahid Mubeen*

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