CONTENTS

MATHEMATICS:- Mathematics is a Science of numbers & space.

CALCULAS

- Differential Calculas
  - By Newton
  - English Scientist
- Integral Calculas
  - By Leibniz
  - German Scientist

Arithmatics + Geometery

ماپ + space

↓

Mathematics

0344-8563284

CHAPTER NO.1

REAL NUMBER SYSTEM * * *

* * *

REAL NUMBER:- A number whose square is always non-negative is call a real number. The set of all real numbers is denoted by $\mathbb{R}$.

⇒ Number:- Geometerically each dot represent a number.

⇒ Rational Numbers:-

A number of the form p/q where p and $q \in \mathbb{Z}$ and $q \neq 0$ is called rational numbers. The set of all rational numbers is denoted by $\mathbb{Q}$.

e.g. $\frac{2}{3}$, -7, 0 etc.

⇒ Irrational Numbers:-

A real number which is not rational is called irrational no. The set of all irrational is denoted by $\mathbb{Q}'$. e.g $\sqrt{2}$, $\sqrt{12}$, π, e etc

3.1416:- This is terminating fraction & all terminating fractions are rational.

0.789494....:- All non terminating repeated fractions are also rational numbers.

1.23748939....:- Non terminating non-repeated fractions are Irrational.

★ p/q is a rational number and the greatest common factor of p/q must be one

Q:- Prove that $\sqrt{2}$ is irrational.

Solution

Suppose $\sqrt{2}$ is rational

Then $\exists$ $p, q \in \mathbb{Z}$ ~~such~~ such that

$$\frac{p}{q} = \sqrt{2} \quad , q \neq 0$$

$$\Rightarrow \quad p = \sqrt{2}\, q$$

$$p^2 = 2q^2 \quad \text{——} \ (1)$$

Since R.H.S is divisible by 2

$\Rightarrow$ L.H.S " " " 2

i.e p^2 " " " 2

$\Rightarrow$ p " " " 2

Let $p = 2 \times k$ —— (2) , $k \in \mathbb{Z}$

Put the value of p in equ (1)

$$(2k)^2 = 2q^2$$

$$4k^2 = 2q^2 \Rightarrow 2k^2 = q^2$$

$$q^2 = 2k^2 \quad \text{——} \ (3)$$

Since R.H.S is divisible by 2

$\Rightarrow$ L.H.S " " " 2

i.e q^2 " " " 2

$\Rightarrow$ q " " " 2

Let $q = 2l$ —— (4) , $l \in \mathbb{Z}$

Now $\frac{p}{q} = \frac{2k}{2l}$

$\Rightarrow$ p & q has a common factor $2 > 1$ which is a contradiction. So our supposition is wrong.

$\Rightarrow$ $\sqrt{2}$ is irrational.

Q:- Prove that $\sqrt{7}$ is irrational number.

Solution Suppose $\sqrt{7}$ is rational number

Then $\exists$ $p, q \in \mathbb{Z}$ s.t $q \neq 0$,

$$\frac{p}{q} = \sqrt{7} \quad \Rightarrow \quad p = \sqrt{7}\, q$$

$$p^2 = 7q^2 \quad \text{——— (1)}$$

Since R.H.S is divisible by 7

$\Rightarrow$ L.H.S " " " 7

i.e p^2 " " " 7

$\Rightarrow$ p " " " 7

Let $p = 7k$ ——— (2), $k \in \mathbb{Z}$

Put in equ (1)

$$(7k)^2 = 7q^2 \quad \Rightarrow \quad 49k^2 = 7q^2$$

$$7k^2 = q^2$$

$$q^2 = 7k^2 \quad \text{——— (3)}$$

Since R.H.S is divisible by 7

$\Rightarrow$ L.H.S " " " 7

i.e q^2 " " " 7

$\Rightarrow$ q " " " 7

Let $q = 7l$ ——— (4), $l \in \mathbb{Z}$

Now $$p/q = \frac{7k}{7l}$$

$\Rightarrow$ p & q has a comon factor $7 > 1$ which is a contraduction. So our supposition is wrong.

$\Rightarrow$ $\sqrt{7}$ is irrational number.

Q:- Prove that $\sqrt{m}$ is a irrational where m is a prime number.

Solution: Suppose $\sqrt{m}$ is rational number.
Then $\exists$ $p, q \in \mathbb{Z}$ such that

$$p/q = \sqrt{m} \quad \text{where } m \text{ is prime number, } q \neq 0$$

$$p = \sqrt{m}\, q$$

$$p^2 = m q^2 \quad \text{——} ①$$

Since R.H.S is divisible by m
$\Rightarrow$ L.H.S " " " m
i.e p^2 " " " m
$\Rightarrow$ p " " " m

let $p = mk$ —— ② , $k \in \mathbb{Z}$
Put in eqn ①

$$(mk)^2 = m q^2$$

$$m k^2 = q^2 \quad \text{——} ③$$

Since L.H.S is divisible by m
$\Rightarrow$ R.H.S " " " m
i.e q^2 " " " m
$\Rightarrow$ q " " " m

let $q = ml$ —— ④ , $l \in \mathbb{Z}$

Now $p/q = \dfrac{mk}{ml}$

$\Rightarrow$ p & q has a common factor $m \geq 2 > 1$ ($\because$ m is prime) which is a contradiction.
So our supposition is wrong.
$\Rightarrow$ $\sqrt{m}$ is a irrational number.

Tree Diagram of Number System

Complex Numbers → Imaginary Number

↓

Real Numbers

Rational Numbers — Irrational Numbers

$\frac{P}{Q}$ where $P, Q \in \mathbb{Z}$

Integers

Negative Number — Zero — Positive Number, Natural Number, Even Numbers, Odd Numbers, Prime Numbers

* * *

Q:- Prove that $\sqrt{n}$ is irrational if n is not a perfect square.

solution

We solve this problem in two cases.

CASE 1:- When n has no factor which is a perfect square.

To prove $\sqrt{n}$ is irrational.

Suppose $\sqrt{n}$ is rational.

Then $\exists$ $p, q \in \mathbb{Z}$ ~~s.t~~, $q \neq 0$ s.t

$$\frac{p}{q} = \sqrt{n} \quad \Rightarrow \quad p = \sqrt{n}\, q$$

$$p^2 = n q^2 \quad \text{——} \textcircled{1}$$

Since R.H.S is divisible by n

$\Rightarrow$ L.H.S " " " n

i.e p^2 is divisible by n

$\Rightarrow$ p " " " n

let $p = nk$ —— ②, $k \in \mathbb{Z}$

Put in equ ①

$(nk)^2 = nq^2$

$nk^2 = q^2$ —— ③

Since L.H.S is divisible by n

$\Rightarrow$ R.H.S " " " n

i.e q^2 " " " n

$\Rightarrow$ q " " " n

let $q = nl$ —— ④ , $l \in \mathbb{Z}$

Now p/q has a common factor $n > 1$ ($\because n$ is not perfect square) which is a contradiction. So our supposition is wrong

$\Rightarrow \sqrt{n}$ is irrational number in this case.

CASE 2:- When n has a factor which is a perfect square.

Let $n = n_1^2 \cdot n_2$ —— ⑤

where n_2 has no factor which is a perfect square.

To prove $\sqrt{n}$ is irrational.

From eqn ⑤ we have

$$\sqrt{n} = n_1 \times \sqrt{n_2}$$

By Case 1 $\sqrt{n_2}$ is irrational, also we know that "Product of non-zero rational number & irrational number is a irrational number."

So, $n_1 \times \sqrt{n_2}$ is irrational.

$\Rightarrow \sqrt{n}$ is irrational.

* * *

Q:- Show that sum of two rational numbers is again a rational number.

Solution Let a, b be two Rational numbers

$\therefore a = \dfrac{p_1}{q_1}$ ——— ①, $b = \dfrac{p_2}{q_2}$ ——— ②

where $p_1, p_2, q_1, q_2 \in \mathbb{Z}$ & $q_1, q_2 \neq 0$

$$a + b = \frac{p_1}{q_1} + \frac{p_2}{q_2}$$

$$= \frac{p_1 q_2 + q_1 p_2}{q_1 q_2}$$

$$a + b = p_3/q_3 \quad \text{———} \; ③$$

where $p_3 = p_1 q_2 + q_1 p_2$

& $q_3 = q_1 q_2$

Clearly p_3 & $q_3 \in \mathbb{Z}$ & $q_3 \neq 0$

So from equ ③

$a + b$ is Rational Number.

p_2، p_1، q_2، q_1 سب Integers ہیں۔ ’’Integers کو جمع، ضرب دینے سے اور جمع کرنے سے جواب بھی Integers میں ہی آتا ہے۔ اس لیے p_3، q_3 بھی Integers ہیں اور p_3/q_3 ایک Rational نمبر ہے۔

* * *

Q:- Show that the difference of Two Rational Numbers is again a Rational Number.

Solution

Let a, b be two rational numbers

$\therefore a = p_1/q_1$ ——— ①, $b = p_2/q_2$ ——— ②

where $p_1, p_2, q_1, q_2 \in \mathbb{Z}$ & $q_1, q_2 \neq 0$

$$a - b = p_1/q_1 - \frac{p_2}{q_2}$$

$$= \frac{p_1 q_2 - q_1 p_2}{q_1 q_2}$$

$$a - b = p_3/q_3 \quad ——— ③$$

where $p_3 = p_1 q_2 - q_1 p_2$

& $q_3 = q_1 q_2$

Clearly p_3 & $q_3 \in \mathbb{Z}$ & $q_3 \neq 0$

So from equ ③

a − b is Rational number

Similarly b − a is also a Rational no.

* * *

Q:- Show that the product of two Rational numbers is again a Rational Number.

Solution Let a and b be two rational

numbers $\therefore$ $a = \frac{p_1}{q_1}$ ——— ①

$b = \frac{p_2}{q_2}$ ——— ②

where $p_1, p_2, q_1, q_2 \in \mathbb{Z}$ & $q_1, q_2 \neq 0$

from equ ① & ② we have

$$a.b = \frac{p_1}{q_1} \cdot \frac{p_2}{q_2}$$

$$= \frac{p_1 p_2}{q_1 q_2}$$

$$a.b = \frac{p_3}{q_3} \text{ ——— ③}$$

where $p_3 = p_1 p_2$ & $q_3 = q_1 q_2$

Clearly $p_3, q_3 \in \mathbb{Z}$ & $q_3 \neq 0$

$\Rightarrow$ $a.b$ is a Rational number.

* * *

Q:- Show that the Quotent of ~~two~~ a Rational Number by a non zero ~~Ratio~~ Rational no. is Rational number.

Solution Let a & b be two Rational nos.

$\therefore$ $a = p_1/q_1$ — ①, $b = p_2/q_2$ ——— ②

where $p_1, p_2, q_1, q_2 \in \mathbb{Z}$ & $q_1, q_2, p_2 \neq 0$

Now $\frac{a}{b} = \frac{p_1}{q_1} \Big/ p_2/q_2$

$= \frac{p_1}{q_1} \times \frac{q_2}{p_2}$ $\Rightarrow$ $\frac{a}{b} = \frac{p_1 q_2}{q_1 p_2}$

$a = p_3/q_3$

where $p_3 = p_1 q_2$

& $q_3 = q_1 p_2$

Clearly p_3 & $q_3 \in \mathbb{Z}$ & $q_3 \neq 0$

$\Rightarrow$ a/b is a rational number.

* * *

(1) Irrational + Irrational = Rational

$\sqrt{2} + (-\sqrt{2}) = 0 \notin Q'$

(2) Irrational − Irrational = Rational

$\sqrt{2} - \sqrt{2} = 0 \notin Q'$

(3) Irrational × Irrational $\notin$ Irrational

$\sqrt{2} \times \frac{1}{\sqrt{2}} = 1 \notin Q'$

(4) Irrational ÷ Irrational $\notin$ Irrational

$\sqrt{2}/\sqrt{2} = 1 \notin Q'$

$\Rightarrow$ The combination of Irrational numbers need not to be an Irrational number. In some cases they are Irrational and in some cases they are Rational.

* * *

Q:- Show that the sum of a Rational number and an Irrational number is Irrational.

Solution:-

Let "a" be a rational number and "b" be an irrational number.

As a is Rational

$\Rightarrow a = p/q$ —— ①, $p, q \in \mathbb{Z}$, $q \neq 0$

To prove $c = a + b$ is irrational

Suppose c is Rational

Then $c = p_1/q_1$ —— ② , $p_1, q_1 \in \mathbb{Z}$ & $q_1 \neq 0$

$\Rightarrow a + b = p_1/q_1 \quad (\because c = a+b)$

$\Rightarrow p/q + b = p_1/q_1$ using equ ①

$\Rightarrow b = p_1/q_1 - p/q$

$\Rightarrow b = \dfrac{p_1 q - q_1 p}{q_1 q} \Rightarrow b = p_2/q_2$

where $p_2 = p_1 q - q_1 p$ & $q_2 = q_1 q \neq 0$

Clearly b is rational which is contradiction. So our supposition is wrong
$\Rightarrow c = a + b$ is irrational.

* * *

Q:- Show that the product of a non-zero Rational and an Irrational is an Irrational number.

Solution
Let "a" be a non zero rational number & "b" be an irrational number.

As a is rational

$\Rightarrow a = p/q$, $p, q \in \mathbb{Z} \wedge p \neq 0, q \neq 0$ —— ①

To prove $c = a.b$ is irrational.

Suppose c is Rational

Then $c = p_1/q_1$ —— ② $p_1, q_1 \in \mathbb{Z}$, $q_1 \neq 0$

$a.b = p_1/q_1 \quad (\because c = a.b)$

$\Rightarrow \frac{p}{q} \cdot b = \frac{p_1}{q_1}$ using equ ①

$\Rightarrow b = \frac{p_1}{q_1} \cdot \frac{q}{p} \Rightarrow b = \frac{p_1 q}{q_1 p}$

$\Rightarrow b = \frac{p_2}{q_2}$

where $p_2 = p_1 q$ & $q_2 = q_1 p \neq 0$

Clearly b is rational. which is a contradiction. So our supposition is wrong.

$\Rightarrow$ c is is Irrational number

* * *

Remark:- In real Field mean "$\mathbb{R}$" or $\mathbb{Q}$.

$\Rightarrow$ Ordered Field:- A Field F is said to be an ordered field if $\exists$ a subset P of F s.t

(1) $(P, +)$ & $(P, \cdot)$ are closed

(2) F is partition into $P, -P, \{0\}$

Example 1:- $F = \mathbb{Q}$

$P = \mathbb{Q}^+$, $-P = \mathbb{Q}^-$, $\{0\}$

clerly $(P, +)$ & $(P, \cdot)$ are closed and F is partitions into $P, -P, \{0\}$

Example 2:- $F = \mathbb{R}$

$P = \mathbb{R}^+$, $-P = \mathbb{R}^-$, $\{0\}$

Clearly $(P, +)$ & $(P, \cdot)$ are closed & F is partitions into $P, -P, \{0\}$

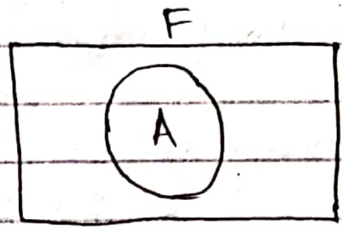
$\Rightarrow$ Ordering Property (Trichotomy Property):- If

$a, b \in \mathbb{R}$ then either $a < b$ or

$a > b$ or $a = b$

Note:- Ordering Property held in set $\mathbb{Q}$ & $\mathbb{R}$.

(11) Definitions:- Let A be a non-empty subset of an ordered field F. Then



⇒ Upper Bound:- An element $\alpha \in F$ is said to be an upper bound of A if $x \leq \alpha \ \forall \ x \in A$.

⇒ Lower Bound:- An element $\beta \in F$ is said to be lower bound of A if $\beta \leq x$ for all $x \in A$.

⇒ Bounded Above:- A is said to be bounded above if A has an upper bound.

⇒ Bounded Below:- A is said to be bounded below if A has a lower bound.

⇒ Bounded Set:- A is said to be bounded set if it is both bounded below and bounded above.

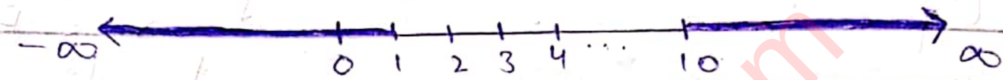
⇒ Infimum or Greatest Lower Bound:- An element $l \in F$ is said to be greatest lower bound of A if

(i) l is a lower bound of A

(ii) $\alpha \leq l$, for any other lower bound α of A.

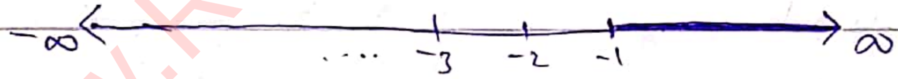
⇒ Supremum or Least Upper Bound:- An element $u \in F$ is said to be least upper bound of A if
(i) u is an upper bound of A
(ii) $u \leq \beta$ for any other upper bound β of A.

Example 1:- $F = \mathbb{R}$
$A = \{1, 2, 3, \ldots 10\}$



1 and all real less than 1 are lower bounds of A. $\text{Inf}(A) = 1$
Also 10 & all real greater than 10 are upper bounds of A. $\text{Sup}(A) = 10$
Clearly A is bounded set.

Example 2:- Let $F = \mathbb{R}$
$A = \{\ldots -3, -2, -1\}$



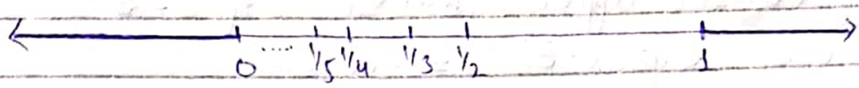
Clearly −1 & all real greater than −1 are upper bounds of A & $\text{Sup}(A) = -1$
& A has no lower bound. So A is bounded above & not bounded below.

Example 3:- $F = \mathbb{R}$
$A = [5, 6)$

5 and all real less than 5 are lower bound of A & $\text{Inf}(A) = 5$,

6 and all reals greater than 6 are upper bounds of A, $Sup(A) = 6 \notin A$

Example 4:- $F = \mathbb{R}$ & $A = \{1, 1/2, 1/3, 1/4, \dots\}$



zero & all reals less than zero are lower bounds of A. But $Inf(A) = 0 \notin A$.
1 & all reals greater than 1 are upper bounds of A & $Sup(A) = 1 \in A$

Remark:- Let A be a non empty bounded ~~set~~ subset of ordered field F

(i) Let $\alpha = Inf(A)$
$\Rightarrow \alpha$ is greatest lower bound of A
$\Rightarrow \alpha + \epsilon$ $(\epsilon > 0)$ is not a lower bound of A
$\Rightarrow \alpha + \epsilon > x$, for some $x \in A$

(ii) Let $\beta = Sup(A)$
$\Rightarrow \beta$ is least upper bound of A
$\Rightarrow \beta - \epsilon$ $(\epsilon > 0)$ is not an upper bound of A
$\Rightarrow \beta - \epsilon < x$, for some $x \in A$.

* * *

Theorem:- Let E be a subset of a complete ordered ~~pair~~ Field F. so that $Inf(E) \in F$. Let $-E = \{-x : x \in E\}$ Then $-E$ is bounded above and
$$Inf(E) = -Sup(-E)$$

Proof:- Let $\alpha = Inf(E)$
$\Rightarrow \alpha \leq x$, $\forall x \in E$

$\Rightarrow -\alpha \geq -x, \ \forall \ -x \in -E$

$\Rightarrow -\alpha$ is an upper bound of $-E$

Next we have to show that $-\alpha$ is least upper bound of $-E$

i.e $Sup(-E) = -\alpha$

As $\alpha = Inf(E)$

$\Rightarrow \alpha + \epsilon$ is not a lower bound of E

$\Rightarrow \alpha + \epsilon > x$, for some $x \in E$

$\Rightarrow -\alpha - \epsilon < -x$, for some $-x \in -E$

$\Rightarrow -\alpha - \epsilon$ is not an upper bound of $-E$

$\Rightarrow -\alpha$ is least upper bound of $-E$

i.e $Sup(-E) = -\alpha$

$\alpha = -Sup(-E)$

$Inf(E) = -Sup(-E)$ Proved.

* * *

⇒Complete Ordered Field:- An ordered field F is said to be complete if every non-empty subset of F, which is bounded below has an Infimum in F or every non empty subset of F which is bounded above has Supremum in F.

* * *

Theorem:- If A is a bounded subset of $\mathbb{R}$ and $b > 0$ Then Prove that

(i) $Inf(bA) = b. Inf A$

(ii) $Sup(bA) = b. Sup A$

Proof :- (i) Let $\alpha = \text{Inf}(A)$

$\Rightarrow \alpha$ is lower bound of A

$\Rightarrow \alpha \leq x \quad \forall x \in A$

$\Rightarrow b\alpha \leq bx \quad \forall bx \in bA$

$\Rightarrow b\alpha$ is lower bound of bA

Next we have to show that $b\alpha$ is greatest lower bound of bA

Again $\alpha = \text{Inf } A$

$\Rightarrow \alpha$ is greatest lower bound of A

$\Rightarrow \alpha + \epsilon$ is not a lower bound of A (for $\epsilon > 0$)

$\Rightarrow \alpha + \epsilon > x$, for some $x \in A$

$\Rightarrow b\alpha + b\epsilon > bx$, " " $bx \in bA$

$\Rightarrow b\alpha + \epsilon' > bx$ " " " " "

where $b\epsilon = \epsilon'$

$\Rightarrow b\alpha + \epsilon'$ is not a lower bound of bA

$\Rightarrow b\alpha$ is greatest lower bound of bA.

i.e $\text{Inf}(bA) = b\alpha$

$\Rightarrow \text{Inf}(bA) = b \cdot \text{Inf } A$

(ii) Let $\beta = \text{Sup}(A)$

$\Rightarrow \beta$ is upper bound of A

$\Rightarrow \beta \geq x, \ \forall x \in A$

$\Rightarrow \beta . b \geq bx \ \forall \ bx \in bA$

$\Rightarrow \beta . b$ is an upper bound of bA

Again $\beta = \text{Sup } A$

$\Rightarrow \beta$ is least upper bound of A

$\Rightarrow$ For $\epsilon > 0$, $\beta - \epsilon$ is not an upper bound of A

$\Rightarrow \beta - \epsilon < x$, for some $x \in A$

$\Rightarrow \beta b - b\epsilon < bx$ for some $bx \in bA$

$\Rightarrow \beta b - \epsilon' < bx$ " " " " "

$\Rightarrow \beta b - \epsilon'$ is not an upper bound of bA

$\Rightarrow \beta b$ is least upper bound of bA

i.e $\text{Sup}(bA) = b\beta$

$\text{Sup}(bA) = b\ \text{Sup}(A)$

* * *

Theorem:- If A is bounded subset of $\mathbb{R}$ & $b \in \mathbb{R}$ then

(i) $\text{Inf}(b+A) = b + \text{Inf } A$

(ii) $\text{Sup}(b+A) = b + \text{Sup } A$

where $b + A = \{b + x : x \in A\}$

Proof:- (i) Let $\alpha = \text{Inf } A$

$\Rightarrow \alpha$ is lower bound of A

$\Rightarrow \alpha \leq x, \forall x \in A$

$\Rightarrow b + \alpha \leq b + x \ \forall \ b + x \in b + A$

$\Rightarrow b + \alpha$ is lower bound of $b + A$

Again $\alpha = \text{Inf } A$

$\Rightarrow \alpha$ is greatest lower bound of A

$\Rightarrow$ For $\epsilon > 0$, $\alpha + \epsilon$ is not a lower bound of A.

$\Rightarrow \alpha + \epsilon > x$ for some $x \in A$

$\Rightarrow (b+\alpha) + \epsilon > b + x$ for some $b+x \in b+A$

$\Rightarrow (b+\alpha) + \epsilon$ is not a lower bound of $b+A$

$\Rightarrow b+\alpha$ is g.l.b of $b+A$

i.e $\operatorname{Inf}(b+A) = b+\alpha$

$\operatorname{Inf}(b+A) = b + \operatorname{Inf} A$

(ii)

Let $\beta = \operatorname{Sup}(A)$

$\Rightarrow \beta$ is upper bound of A

$\Rightarrow \beta \geq x \quad \forall x \in A$

$\Rightarrow b+\beta \geq b+x \quad \forall\ b+x \in b+A$

$\Rightarrow b+\beta$ is upper bound of $b+A$

Again $\beta = \operatorname{Sup}(A)$

$\Rightarrow \beta$ is least upper bound of A

$\Rightarrow$ For $\epsilon > 0$, $\beta - \epsilon$ is not a upper bound of A

$\Rightarrow$ ~~$(b+\beta) - \epsilon \Rightarrow$~~ $\beta - \epsilon < x$ for some $x \in A$

$\Rightarrow (b+\beta) - \epsilon < b+x$ for some $b+x \in b+A$

$\Rightarrow (b+\beta) - \epsilon$ is not a upper bound of $b+A$

$\Rightarrow b+\beta$ is l.u.b of $b+A$

i.e. $\operatorname{Sup}(b+A) = b+\beta$

$\operatorname{Sup}(b+A) = b + \operatorname{Sup} A$

* * *

Theorem:- Let A & B be bounded subsets of $\mathbb{R}$ then prove that

(i) $\operatorname{Inf}(A+B) = \operatorname{Inf} A + \operatorname{Inf} B$

(ii) $\operatorname{Sup}(A+B) = \operatorname{Sup} A + \operatorname{Sup} B$

Proof:- (i) Let $\alpha = \operatorname{Inf}(A)$ & $\beta = \operatorname{Inf}(B)$

$\Rightarrow$ α is ~~least~~ lower bound of A &
β is lower bound of B

$\Rightarrow \alpha \leq x \ \forall \ x \in A$ & $\beta \leq y \ \forall \ y \in B$

$\Rightarrow \alpha + \beta \leq x+y, \ \forall \ x+y \in A+B$

$\Rightarrow \alpha + \beta$ is lower bound of $A+B$

Again $\alpha = \operatorname{Inf}(A)$ & $\beta = \operatorname{Inf}(B)$

$\Rightarrow \alpha$ is g.l.b of A &
β " " " " B

$\Rightarrow$ For $\epsilon > 0$, $\alpha + \epsilon/2$ is not a lower bound of A
& $\beta + \epsilon/2$ " " " " " of B

$\Rightarrow \alpha + \epsilon/2 > x$ for some $x \in A$ &
$\beta + \epsilon/2 > y$ " " $y \in B$

$\Rightarrow (\alpha + \epsilon/2) + (\beta + \epsilon/2) > x+y$ for some $x+y \in A+B$

$\Rightarrow \alpha + \beta + \epsilon > x+y$ for some $x+y \in A+B$

$\Rightarrow \alpha+\beta+\epsilon$ is not a lower bound of $A+B$

$\Rightarrow \alpha+\beta$ is g.l.b of $A+B$

i.e $\text{Inf}(A+B) = \alpha+\beta$

$\text{Inf}(A+B) = \text{Inf} A + \text{Inf} B$

(ii) Let $\alpha = \text{Sup}(A)$ & $\beta = \text{Sup}(B)$

$\Rightarrow \alpha$ is upper bound of A and

β " " " " B

$\Rightarrow \alpha \geq x \ \forall x \in A$ & $\beta \geq y \ \forall y \in B$

$\Rightarrow \alpha+\beta \geq x+y \ \forall x+y \in A+B$

$\Rightarrow \alpha+\beta$ is upper bound of $A+B$

Again $\alpha = \text{Sup}(A)$ & $\beta = \text{Sup}(B)$

$\Rightarrow \alpha$ is l.u.b of A and

β " " " B

For $\epsilon > 0$, $\alpha - \epsilon/2$ is not an upper bound of A &

$\beta - \epsilon/2$ " " " " " " B

$\Rightarrow \alpha - \epsilon/2 < x$ for some $x \in A$ &

$\beta - \epsilon/2 < y$ " " $y \in B$

$\Rightarrow (\alpha - \epsilon/2) + (\beta - \epsilon/2) < x+y$ for some $x+y \in A+B$

$\Rightarrow \alpha+\beta-\epsilon < x+y$ for some $x+y \in A+B$

$\Rightarrow \alpha+\beta-\epsilon$ is not a upper bound of $A+B$

$\Rightarrow \alpha+\beta$ is ~~g~~ l.u.b of $A+B$

i.e $\quad Sup\,(A+B) = \alpha+\beta$

$$Sup\,(A+B) = Sup\,A + Sup\,B.$$

* * *

Theorem:- If A is bounded subset of $\mathbb{R}$ & $b<0$ then showe that

(i) $Inf\,(bA) = b.\,Sup\,(A)$

(ii) $Sup\,(bA) = b.\,Inf\,(A)$

Proof:- (i) Let $\alpha = Sup\,(A)$

$\Rightarrow \alpha$ is an upper bound of A

$\Rightarrow \alpha \geq x$ for all $x \in A$

$\Rightarrow b.\alpha \leq bx \quad \forall\, bx \in bA \quad (\because b<0)$

$\Rightarrow b\alpha$ is lower bound of bA

Again $\alpha = Sup\,(A)$

$\Rightarrow \alpha$ is l.u.b of A

$\Rightarrow$ For $\epsilon>0$, $\alpha-\epsilon$ is not an upper bound of A

$\Rightarrow \alpha-\epsilon < x$ for some $x \in A$

$\Rightarrow b\alpha - b\epsilon > bx$ for some $bx \in bA \quad (\because b<0)$

$\Rightarrow b\alpha + (-b\epsilon) > bx$ " " " " "

$\Rightarrow b\alpha + \epsilon' > bx$ " " " " "

$\Rightarrow b\alpha + \epsilon'$ is not a lower bound of bA

$\Rightarrow b\alpha$ is g.l.b of bA

i.e $\quad Inf\,(bA) = b\alpha$

$$Inf\,(bA) = b\,Sup\,A$$

(ii) Let $\beta = \text{Inf}(A)$

$\Rightarrow \beta$ is a lower bound of A

$\Rightarrow \beta \leq x$ for all $x \in A$

$\Rightarrow b\cdot\beta \geq bx$ " " $bx \in bA$ $(\because b<0)$

$\Rightarrow b\cdot\beta$ is upper bound of bA

Again $\beta = \text{Inf}(A)$

$\Rightarrow \beta$ is g.l.b of A

$\Rightarrow$ For $\epsilon > 0$, $\beta+\epsilon$ is not a lower bound of A

$\Rightarrow \beta + \epsilon > x$ for some $x \in A$

$\Rightarrow b\beta + b\epsilon < bx$ for some $bx \in bA$ $(\because b<0)$

$\Rightarrow b\beta + \epsilon' < bx$ " " " "

$\Rightarrow b\beta + \epsilon'$ is not a upper bound of bA

$\Rightarrow b\beta$ is l.u.b of bA

i.e $\text{Sup}(bA) = b\beta$

$\text{Sup}(bA) = b\cdot\text{Inf}(A)$

* * *

Q:- If $0 < x < y$ then $\frac{1}{y} < \frac{1}{x}$

solution As $x > 0$ & $y > 0$

$\Rightarrow xy > 0$

Also $x < y \Rightarrow \frac{x}{xy} < \frac{y}{xy}$ $(\because xy > 0)$

$\Rightarrow \frac{1}{y} < \frac{1}{x}$

Ans.

Q. (Arithematic Property)

If $a<b$ then $a < \frac{a+b}{2} < b$

Solution Given $a<b$

$\Rightarrow a+b < b+b$

$\Rightarrow a+b < 2b \Rightarrow \frac{a+b}{2} < b$ —— ①

Again $a < b$

$\Rightarrow a+a < a+b$

$\Rightarrow 2a < a+b \Rightarrow a < \frac{a+b}{2}$ —— ②

from equ ① & ②

$a < \frac{a+b}{2} < b$ Proved.

* * *

Theorem (Archemedian Principle)

For any real number x there is an integar n. s.t $x<n$

Proof: Let us consider the set

$S = \{k : k \in \mathbb{Z} \text{ \& } k \leq x\}$

$\Rightarrow x$ is an upper bound of S

$\Rightarrow S$ is bounded above

As $S \subseteq \mathbb{R}$, with $\mathbb{R}$ is complete

$\Rightarrow \sup(S)$ is in $\mathbb{R}$

Let $y = \sup(S)$

$\Rightarrow y$ is l.u.b of S

$y - \frac{1}{2}$ is not an upper bound of S

$\Rightarrow y - \frac{1}{2} < k$ for some $k \in S$

$\Rightarrow k > y - \frac{1}{2}$ " " " " "

$\Rightarrow k + 1 > y - \frac{1}{2} + 1$

$\Rightarrow k + 1 > y + \frac{1}{2} > y$

$\Rightarrow k + 1 > y$

$\Rightarrow k + 1 \notin S \quad (\because y = \text{Sup } S)$

$\Rightarrow k + 1 > x$ ——— ①

Let $n = k + 1 \Rightarrow n \in \mathbb{Z} \quad (\because k \in \mathbb{Z})$

Put in eqn ①

$\therefore n > x$ or $x < n$

* * *

Theorem:- Show that $\mathbb{Q}$ is not a complete ordered field.

Proof:- We have to show that there is a subset of $\mathbb{Q}$ which is bounded above but has no supremum in $\mathbb{Q}$.

Let $E = \{q : q \in \mathbb{Q}, q > 0 \wedge q^2 < 2\}$ reals

Clearly $\sqrt{2}$ & all (rationals) greater than $\sqrt{2}$ are upper bounds of E

$\Rightarrow E$ is bounded above

As E is bounded above so Sup(E) exist

Let $\alpha = \text{Sup } E$

Then there are three possibilities

(i) $\alpha^2 = 2$ (ii) $\alpha^2 > 2$

(iii) $\alpha^2 < 2$

To prove $\alpha \notin \mathbb{Q}$

Suppose $\alpha \in \mathbb{Q}$

Scanned by CamScanner

Case 1:- if $\alpha^2 = 2 \Rightarrow \alpha = \sqrt{2}$

By our supposition $\alpha \in \mathbb{Q}$

$\Rightarrow \sqrt{2} \in \mathbb{Q}$

which is not true $\sqrt{2}$ does not belong to $\mathbb{Q}$ so this case is not possible

i.e $\alpha^2 \neq 2$

Case 2:- If $\alpha^2 > 2$ (greater Than)

$\Rightarrow \alpha^2 - 2 > 0$

$\Rightarrow \dfrac{\alpha^2-2}{\alpha+2} > 0 \Rightarrow -\dfrac{\alpha^2-2}{\alpha+2} < 0$

$\Rightarrow \alpha - \dfrac{\alpha^2-2}{\alpha+2} < \alpha$ ——— ①

Let $\beta = \alpha - \dfrac{\alpha^2-2}{\alpha+2} \Rightarrow \dfrac{\cancel{\alpha^2}+2\alpha-\cancel{\alpha^2}+2}{\alpha+2} = \dfrac{2\alpha+2}{\alpha+2}$

Clearly $\beta \in \mathbb{Q}$ & β is +ve i.e $\beta > 0$

Now $\beta^2 - 2 = \left(\dfrac{2\alpha+2}{\alpha+2}\right)^2 - 2$

$= \dfrac{(2\alpha+2)^2}{(\alpha+2)^2} - 2 \Rightarrow \dfrac{4\alpha^2+4+8\alpha}{(\alpha+2)^2} - 2$

$\Rightarrow \dfrac{4\alpha^2+4+8\alpha-2(\alpha^2+4+4\alpha)}{(\alpha+2)^2}$

$= \dfrac{4\alpha^2+4+\cancel{8\alpha}-2\alpha^2-8-\cancel{8\alpha}}{(\alpha+2)^2}$

$= \dfrac{2\alpha^2-4}{(\alpha+2)^2} = \dfrac{2(\alpha^2-2)}{(\alpha+2)^2}$

$\beta^2 - 2 = \dfrac{2(\alpha^2-2)}{(\alpha+2)^2} > 0$ $\{\because \alpha^2 > 2 \Rightarrow \alpha^2 - 2 > 0$

M

$\beta^2 - 2 > 0 \Rightarrow \beta^2 > 2$

$\Rightarrow \beta \notin E$ but from equ ①

$\beta < \alpha \Rightarrow \beta \in E$

which is not possible. So this case is not possible, i.e $\alpha^2 \ngtr 2$

Case 3:- If $\alpha^2 < 2$

$\alpha^2 - 2 < 0 \Rightarrow \dfrac{\alpha^2 - 2}{\alpha + 2} < 0$

$-\dfrac{\alpha^2 - 2}{\alpha + 2} > 0 \Rightarrow \alpha - \dfrac{\alpha^2 - 2}{\alpha + 2} > \alpha$ ——— ②

Let $\beta = \alpha - \dfrac{\alpha^2 - 2}{\alpha + 2}$

Clearly $\beta \in \mathbb{Q}$, $\beta > 0$ &

$\beta^2 - 2 = \dfrac{2(\alpha^2 - 2)}{(\alpha + 2)^2} < 0$ $\left(\because \alpha^2 < 2 \Rightarrow \alpha^2 - 2 < 0 \right.$

$\Rightarrow \beta \in E$

But from equ ② $\beta > \alpha \Rightarrow \beta \notin E$

which is not possible i.e $\alpha^2 \nless 2$

Hence our supposition is wrong

i.e $\alpha \in \mathbb{Q}$ is wrong

$\Rightarrow \alpha \notin \mathbb{Q} \Rightarrow \operatorname{Sup}(E) \notin \mathbb{Q}$

So $\mathbb{Q}$ is not a complete ordered field.

* * *

Q:- Let $S = \{x : x^2 < 3\}$, prove that S is bounded above & $\operatorname{Sup} S = \sqrt{3}$

Solution: Clearly $\sqrt{3}$ & all reals greater than $\sqrt{3}$ are upper bounds of S.

So S is bounded above.
As $S \subseteq \mathbb{R}$ & $\mathbb{R}$ is complete. So S has supremum in $\mathbb{R}$

Let $\alpha = \sup(S)$

Then there are three possibilities.

(i) $\alpha > \sqrt{3}$ (ii) $\alpha < \sqrt{3}$

(iii) $\alpha = \sqrt{3}$

Case 1:- If $\alpha > \sqrt{3}$

$\Rightarrow \alpha > \frac{\alpha+\sqrt{3}}{2} > \sqrt{3}$ (By arithematic property)

$\Rightarrow \alpha > \frac{\alpha+\sqrt{3}}{2}$ & $\frac{\alpha+\sqrt{3}}{2} > \sqrt{3}$

$\Rightarrow \frac{\alpha+\sqrt{3}}{2} < \alpha$ & $\frac{\alpha+\sqrt{3}}{2} > \sqrt{3}$

$\Rightarrow \frac{\alpha+\sqrt{3}}{2} \in S$ & $\frac{\alpha+\sqrt{3}}{2} \notin S$

which is not possible.

Case 2:- If $\alpha < \sqrt{3}$

$\Rightarrow \alpha < \frac{\alpha+\sqrt{3}}{2} < \sqrt{3}$ (By arithematic property)

$\Rightarrow \alpha < \frac{\alpha+\sqrt{3}}{2}$ & $\frac{\alpha+\sqrt{3}}{2} < \sqrt{3}$

$\Rightarrow \frac{\alpha+\sqrt{3}}{2} > \alpha$ & $\frac{\alpha+\sqrt{3}}{2} < \sqrt{3}$

$\Rightarrow \frac{\alpha+\sqrt{3}}{2} \notin S$ & $\frac{\alpha+\sqrt{3}}{2} \in S$

Again contradiction

So this case is not possible

ultimately $\alpha = \sqrt{3}$

$\Rightarrow \sup(S) = \sqrt{3}$

* * *

★ Sum of geometric series.

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \quad \text{if } |r| < 1$$

$$= \frac{a(r^n - 1)}{r-1} \quad \text{if } |r| > 1$$

If $r = 1 \Rightarrow S = a + a + a + \cdots + a = na$

★ ★ ★

Theorem:- If $x > 1$ & $0 < r < s$ with $r, s \in Q$ then $\frac{x^r - 1}{r} < \frac{x^s - 1}{s}$, $(n \in N)$

Proof:- Since $x > 1$

$$\Rightarrow x^0 < x^n$$
$$x^1 < x^n$$
$$x^2 < x^n$$
$$\cdots$$
$$x^{n-1} < x^n$$

Adding vertically

$$1 + x + x^2 + x^3 + \cdots + x^{n-1} < nx^n$$

Adding $n(1 + x + x^2 + \cdots + x^{n-1})$ on both sides

$$(1 + x + x^2 + \cdots + x^{n-1}) + n(1 + x + x^2 + \cdots + x^{n-1}) < n(1 + x + \cdots + x^{n-1}) + nx^n$$

$$(1 + x + x^2 + \cdots + x^{n-1})(1+n) < n(1 + x + x^2 + \cdots + x^n)$$

$$\frac{1 + x + x^2 + \cdots + x^{n-1}}{n} < \frac{1 + x + x^2 + \cdots + x^n}{1+n} \quad \text{——— ①}$$

Now $1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$

& $1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$

Put values in eq①

$$\frac{x^n - 1}{n(x-1)} < \frac{x^{n+1} - 1}{(n+1)(x-1)}$$

$$\frac{x^n - 1}{n} < \frac{x^{n+1} - 1}{n+1} \quad \text{——— (2)}$$

As $n \in N \Rightarrow n+1 \in N$ & $n < n+1$.

Now for $m, n \in N$ with $m < n$ we have

$$\frac{x^m - 1}{m} < \frac{x^n - 1}{n} \quad \text{——— (3)}$$

Let $r = p/q$, & $s = p'/q'$

where $p, p', q, q' \in N$.

As $r < s \Rightarrow p/q < p'/q'$.

$\Rightarrow pq' < qp'$

As pq' & $p'q \in N$ with $pq' < p'q$

equality اور Inequality میں Rational عدد کو کراس ملٹی پلائی کر کے انٹیجرز بنایا جا سکتا ہے
i.e $\frac{2}{7} < \frac{3}{5}$
$10 < 14$

So for $y > 1$ (3) $\Rightarrow$

$$\frac{y^{pq'} - 1}{pq'} < \frac{y^{p'q} - 1}{p'q} \quad \text{——— (4)}$$

Let $y = x^{1/qq'}$ clearly $x > 1$

Put in equ (4)

$$\frac{(x^{1/qq'})^{pq'} - 1}{pq'} < \frac{(x^{1/qq'})^{p'q} - 1}{p'q}$$

$$\Rightarrow \frac{x^{p/q} - 1}{pq'} < \frac{x^{p'/q'} - 1}{p'q}$$

$$\Rightarrow q\,\frac{x^{p/q} - 1}{p} < \frac{x^{p'/q'} - 1}{p'} \cdot q'$$

$$\Rightarrow \frac{x^{p/q}-1}{p/q} < \frac{x^{p'/q'}-1}{p'/q'}$$

$$\Rightarrow \frac{x^{r}-1}{r} < \frac{x^{s}-1}{s}$$

* * *

Q: Show that there is a unique real number $x>0$ s.t $x^2=2$

Solution Let x_1 & x_2 be two +ve real numbers s.t

$$x_1^2 = 2 \quad \text{---}(1) \quad \& \quad x_2^2 = 2 \quad \text{---}(2)$$

consider $x_1 - x_2 = \dfrac{(x_1-x_2)(x_1+x_2)}{(x_1+x_2)}$

$$= \frac{x_1^2 - x_2^2}{x_1+x_2}$$

$$= \frac{2-2}{x_1+x_2} = \frac{0}{x_1+x_2}$$

$$= 0$$

$$\Rightarrow x_1 - x_2 = 0$$

$$\Rightarrow x_1 = x_2 \quad \text{Proved.}$$

* * *

Theorem:- (Rational Density Theorem). Show that there is a rational number between every two real numbers.

Proof: Let x & y be two real numbers such that $x<y \Rightarrow y>x$

$$\Rightarrow y-x>0$$

$$\Rightarrow \frac{1}{y-x} > 0$$

As $\frac{1}{y-x}$ is a real number so by Archemedian principle there exist a +ve integer n s.t $\frac{1}{y-x} < n$

$\Rightarrow 1 < ny - nx$ or $1 + nx < ny$ —— ①

As we know that every real number lies between two consective integers so "nx" lies b/w two consective integers say $m-1$ & m

If x is a real no. then ~~m~~ $m-1 \leq x \leq m$

i.e $m-1 < nx < m$ —— ⊛

$\Rightarrow m < 1+nx < 1+m$

$\Rightarrow m < 1+nx$ —— ②

from equ ① & ② we have

$m < 1+nx < ny$ —— ③

$\Rightarrow m < ny$ —— ④ (Transitive property)

from equ ⊛ & ④ we have

$nx < m < ny$

$x < \frac{m}{n} < y$

Let $q = \frac{m}{n} \Rightarrow q \in \mathbb{Q}$

$\Rightarrow x < q < y$ Proved.

* * *

Q: Show that there exist infinite many rational numbers between every two real numbers.

Solution: Let x & y be two real numbers such that $x < y$.

then by rational density theorem there exist a rational number q_1 s.t $x < q_1 < y$

As q_1 is a rational number and every rational number is also a real number.

So again by rational density theorem there exist a rational number b/w x & q_1 say q_2 s.t

$x < q_2 < q_1$

Continuing the above process we can find infinite many rational numbers b/w x & y.

* * *

Q :- Show that there exist a rational number between every two rational numbers.

Solution Let x & y be two rational numbers s.t $x < y$

As x & y are rational numbers & every rational number is also a real number.

So x & y are also real numbers. then by rational density theorem there exist a rational number q_1 b/w x & y s.t $x < q_1 < y$

* * *

Q:- Show that $\exists$ a rational number b/w every two irrational numbers.

Solution Let x & y be two irrational

numbers s.t $x<y$
As x & y are irrational so reals. $\because$ every irrational is real.
So by Rational density theorem $\exists$ a rational number q_1 s.t

$$x < q_1 < y$$

* * *

Q: Show that there exist an irrational number b/w every two real numbers.

Solution:- Let x & y be two real numbers s.t $x<y$

As $x, y \in \mathbb{R} \Rightarrow \sqrt{2}x, \sqrt{2}y \in \mathbb{R}$

As $x<y \Rightarrow \sqrt{2}x < \sqrt{2}y$

Then by Rational density theorem $\exists$ a rational number q b/w $\sqrt{2}x$ & $\sqrt{2}y$ s.t $\sqrt{2}x < q < \sqrt{2}y$

$$\Rightarrow \cancel{\sqrt{2}}x < \frac{q}{\sqrt{2}} < y \quad \text{--- (1)}$$

q is rational
$\frac{1}{\sqrt{2}}$ is irrational
Product $\frac{q}{\sqrt{2}}$ irrational

As we know that product of a rational number & an irrational number is irrational.
So $q_1 = \frac{q}{\sqrt{2}}$ is irrational & from equ (1)

$$x < q_1 < y$$

* * *

Q:- Show that there exist infinite many irrational numbers between two real numbers.

Solution:- Let x & y be two real numbers s.t $x < y$.
Then by well known (Irrational Density) theorem $\exists$ an irrational number q_1 b/w x & y s.t

$$x < q_1 < y$$

As q_1 is an irrational number & every irrational number is also a real number. So q_1 is also a real number.
Again by Irrational Density theorem $\exists$ an other irrational number b/w x & q_1 say q_2 s.t

$$x < q_2 < q_1$$

Continuing this process we can find infinite many irrational numbers between x & y.

* * *

Q:- Show that $\exists$ an irrational number between every two irrational numbers.

Solution Let x & y be two irrational numbers s.t $x < y$.
As x & y be irrational numbers so reals $\because$ every irrational is real.
So by well known (Irrational Density) theorem $\exists$ an irrational number q_1 b/w x & y s.t $x < q_1 < y$

* * *

Q:- Show that there exist a rational number between a rational number and an irrational number.

Solution Let x is a rational number an y is an irrational number s.t

$$x < y$$

As x is rational & y is irrational
So x & y are reals.
So by well known theorem $\exists$ an irration number q' b/w x & y such that $x < q' < y$

* * *

Formula:- If $n \in N$ then

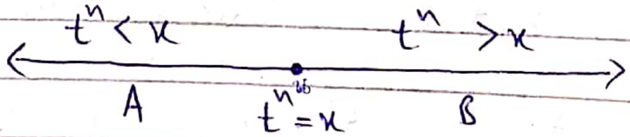
$$a^n - b^n < n(a-b)a^{n-1}$$

* * *

Theorem:- For any $x > 0$ & for every integer $n > 0$ there is one & only one positive real number y s.t $y^n = x$

Proof:- we divide the set $\mathbb{R}$ into two parts (sub sets) A and B such that

$A = \{t : t^n < x\}$



$B = \{t : t^n > x\}$

First we show that A & B are non empty sets.

Since $x > 0 \Rightarrow \dfrac{x}{1+x} > 0$

& $\dfrac{x}{1+x} < x$ & $\dfrac{x}{1+x} < 1$

Let $t_1 = \dfrac{x}{1+x}$

$\therefore t_1 > 0$ & $t_1 < x$ & $t_1 < 1$

Now $t_1^n < t_1$ & $t_1 < x$

$\Rightarrow t_1^n < t_1 < x$

$\because t_1 < 1$ so $t_1^n < t_1$

suppose $t_1 = 1/2$, $n = 10$

$(1/2)^{10} < \frac{1}{2} \Rightarrow \frac{1}{2^{10}} < \frac{1}{2}$

$x^n < x$

$\Rightarrow t_1^n < x$

$\Rightarrow t_1 \in A \quad \therefore A \neq \phi$

If $0 < x < 1$ then $x^n < x$ &

If $x > 1$ then $x^n > x$

Let $t_2 = 1 + x \Rightarrow t_2 > 1$ &

$t_2 > x$

$(4.5)^2 < 25 \quad 5 \quad (5.5)^2 > 25$

$5^2 = 25 \quad \boxed{x = 25}$

As $t_2 > 1 \Rightarrow t_2^n > t_2 > x$

$\Rightarrow t_2^n > x$

$\Rightarrow t_2 \in B \quad \therefore B \neq \phi$

Now for any $t \in A$ & $t' \in B \Rightarrow t < t'$

$\Rightarrow$ Every element of B is an upper bound of A

$\Rightarrow A$ is bounded above.

As A is subset of $\mathbb{R}$, with $\mathbb{R}$ is complete

$\Rightarrow A$ has a supremum in $\mathbb{R}$

Let $y \in \mathbb{R}$ s.t supremum of $A = y$

Then there are three possibilities.

(i) $y^n < x$ (ii) $y^n > x$

(iii) $y^n = x$

If $y \in A \Rightarrow y^n < x$

If $y \in B \Rightarrow y^n > x$

If $y \notin A, B \Rightarrow y^n = x$

Case 1:- If $y^n < x$

$\Rightarrow x > y^n \Rightarrow x - y^n > 0$

Now take a number h s.t

$h < \frac{x - y^n}{n(y+1)^{n-1}}$ for some h b/w 0 & 1

$\Rightarrow nh(y+1)^{n-1} < x - y^n$ ——— ①

Now consider $(y+h)^n - y^n$

$(y+h)^n - y^n < n(y+h-y)(y+h)^{n-1}$

$= nh(y+h)^{n-1}$

$< nh(y+1)^{n-1}$ $(\because h<1)$

$< x - y^n$ using ①

$\Rightarrow (y+h)^n - y^n < x - y^n$

$\Rightarrow (y+h)^n < x$

$\Rightarrow y+h \in A$ But $y = \sup A$

$\Rightarrow y+h \notin A$

So this case is not possible.

i.e $y^n \not< x$

Case II:- If $y^n > x$

$\Rightarrow y^n - x > 0$

Now take a real number k s.t

$k < \dfrac{y^n - x}{ny^{n-1}}$

$\Rightarrow nky^{n-1} < y^n - x$ ———— ②

consider $y^n - (y-k)^n < nk.y^{n-1}$

$\Rightarrow y^n - (y-k)^n < y^n - x$

$\Rightarrow -(y-k)^n < -x$

$\Rightarrow (y-k)^n > x \Rightarrow y-k \in B$

$\Rightarrow y-k$ is an u.b of A

But $y = \text{Sup } A$

i.e y is l.u.b of A

$\Rightarrow y-k \notin B$

So this case is not possible

i.e $y^n \not> x$

ultimately $y^n = x$

Uniqueness:- Let y_1 & y_2 be two numbers such that

$$y_1^n = x \quad \& \quad y_2^n = x$$

$$y_1 = x^{1/n} \quad \& \quad y_2 = x^{1/n}$$

$$\Rightarrow y_1 = y_2 \qquad \text{Proved.}$$

★ ★ ★

Theorem:- Prove that for every real number x. There is a set E of rational numbers s.t $\text{Sup}(E) = x$

Proof:- Let us define $E = \{q : q \in \mathbb{Q} \wedge q < x\}$

clearly x is an upper bound of E

$\Rightarrow E$ is bounded above

As $E \subseteq \mathbb{R}$, with $\mathbb{R}$ is complete so E has supremum in $\mathbb{R}$

Let $y \in \mathbb{R}$ s.t $y = \text{Sup}(E)$

$$\Rightarrow y \leq x$$

If $y = x$ then nothing is left to prove.

If $y < x$ then by rational density theorem $\exists$ a rational number say p s.t $y < p < x$

$\Rightarrow y < p$ & $p < x$

$\Rightarrow p \notin E$ & ~~$p \notin x$~~ $p \in E$

So this case is not possible

ultimately $y = x$

$Sup(E) = x$

* * *

Theorem:- Prove that for every real number x. there exist a subset E of irrationals s.t $Sup\, E = x$

Proof:- Let us define $E = \{q' : q' \in Q' \wedge q' < x\}$

Clearly x is an upper bound of E

$\Rightarrow E$ is bounded above.

As $E \subseteq \mathbb{R}$ with $\mathbb{R}$ is complete. So E has supremum is $\mathbb{R}$.

Let $y \in \mathbb{R}$ s.t $y = Sup(E)$

$\Rightarrow y \leq x$

If $y = x$ then there is nothing left to prove.

If $y < x$ then by well known theorem $\exists$ a irrational number say p s.t

$y < p < x$

$\Rightarrow y < p$ & $p < x$

$\Rightarrow p \notin E$ & $p \in E$

So this case is not possible

ultimately $y = x$

$\Rightarrow Sup(E) = x$

Q. Show that $x^3 - 3x + 1 = 0$ has no rational solution?

Solution

Let y be a rational root of given equation

$$\therefore \quad y^3 - 3y + 1 = 0 \quad \text{——— (1)}$$

Let $y = p/q$, $p, q \in \mathbb{Z}$ and $q \neq 0$

Put in equ (1)

$$\frac{p^3}{q^3} - \frac{3p}{q} + 1 = 0$$

Xing with q^3

$$p^3 - 3pq^2 + q^3 = 0 \quad \text{——— (2)}$$

$$\Rightarrow p^3 = 3pq^2 - q^3$$

$$p^3 = q(3pq - q^2) \quad \text{——— (3)}$$

Since R.H.S is divisible by q

$\Rightarrow$ L.H.S " " " "

i.e p^3 " " " "

$\Rightarrow$ p " " " q

$\Rightarrow$ q is 1 or -1

اگر p divisible ہو by q تو q یا 1 ہوگا یا -1
اسی صورت میں p/q کی Rational
[illegible]

$q = 1$, $p = 1$

$q = -1$, $p = 1$

$q = 1$, $p = -1$

$q = -1$, $p = -1$

$p/q = -1$

$p/q = 1$

Again from equ (2)

$$q^3 = 3pq^2 - p^3$$

$$q^3 = p(3q^2 - p^2) \quad \text{——— (4)}$$

Since R.H.S is divisible by p

$\Rightarrow$ L.H.S " " " p

i.e q^3 " " " "

$\Rightarrow$ q " " " "

$\Rightarrow$ p is 1 or -1

$\Rightarrow y = 1$ or -1

But neither 1 nor -1 satisfy the given equation. which is a contradiction (i.e y is rational). So given condition has no rational root.

* * *

⇒ **Euclidean Space:-** Let $\mathbb{R}$ be the set of all real numbers then

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n factors)

is called Euclidean space of dimension n.

$R^2 = R \times R$
$= R$ ka cartesian product
$= \{(x, y) : x, y \in R\}$

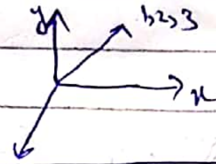
Elements of $\mathbb{R}^n$ are called vectors.

Let $\underline{x}, \underline{y} \in \mathbb{R}^n$

$\therefore x = (x_1, x_2, x_3, \ldots, x_n), x_i \in \mathbb{R}$

& $y = (y_1, y_2, y_3, \ldots, y_n), y_i \in \mathbb{R}$

when point is plotted & position vector is sketched we get vectors



The vector addition & scaler multiplication in $\mathbb{R}^n$ is defined as

$\underline{x} + \underline{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

& For $\lambda \in \mathbb{R}$

$\lambda \underline{x} = \{\lambda x_1, \lambda x_2, \ldots, \lambda x_n\}$

vector space has only 2 properties vector addition & scaler multiplic

⇒ **Norm of A Vector:-**

Let $\underline{x} = \{x_1, x_2, x_3, \ldots, x_n\}$ be a vector in $\mathbb{R}^n$ then norm of x is denoted & defined as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}$$

$$= \sqrt{\sum_{i=1}^{n} x_i^2}$$

$\vec{a} = x\hat{i} + y\hat{j}$
$|a| = \sqrt{x^2 + y^2}$
norm, length, Magnitude

$$\|x\| = \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2}$$

Let $x_i = -2$
$x_i^2 = 4$ Now
$|x_i| = 2$, $|x_i|^2 = 4$
so $x_i^2 = |x_i|^2$

$\vec{a} = (a_1, a_2, a_3)$
$\vec{b} = (b_1, b_2, b_3)$
$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

⇒ **Inner Product:-**

Let $\underline{x}, \underline{y} \in R^n$ with
$\underline{x} = \{x_1, x_2, x_3, \dots, x_n\}$ &
$\underline{y} = \{y_1, y_2, y_3, \dots, y_n\}$

Then Inner Product of x & y is denoted and defined as

$$\langle \underline{x}, \underline{y}\rangle = \underline{x}\cdot\underline{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

Remark:-
$$\begin{aligned}\langle x, x\rangle &= x_1x_1 + x_2x_2 + \dots + x_nx_n \\ &= x_1^2 + x_2^2 + \dots + x_n^2 \\ &= \sum_{i=1}^{n} |x_i|^2 \\ &= \left[\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}\right]^2\end{aligned}$$

[$\langle x, x\rangle = \|x\|^2$ → جب Norm اور Inner Product کو Replace کرنے کے لیے use کریں]

★ ★ ★

Cauchy-Schwarz Inequality:- Let $x, y \in \mathbb{R}^n$ then

$$\underset{(\text{mod})}{|\langle x, y\rangle|} \leq \underset{(\text{norm})}{\|x\|}\ \underset{(\text{norm})}{\|y\|}$$

2 ویکٹرز کا dot پروڈکٹ ایک Scaler ہے
اور سکیلر کا Mod لیتے ہیں Norm نہیں

Proof:- Now for $\lambda \in \mathbb{R}$

consider

$$0 \leq \|\underline{x} - \lambda\underline{y}\|^2$$

2 ویکٹرز یا Parallel ہوں گے یا non-parallel
اگر دو || ہوں تو $\bar{a} = \lambda\bar{b} \Rightarrow \bar{a} - \lambda\bar{b} = 0$
$\Rightarrow |\bar{a} - \lambda\bar{b}| = 0$
اور اگر دو || نہیں ہیں تو $\bar{a} \neq \lambda\bar{b}$
$\Rightarrow \bar{a} - \lambda\bar{b} \neq 0 \Rightarrow \|a - \lambda b\| > 0$

$$\Rightarrow 0 \leq \|\underline{x} - \lambda\underline{y}\|^2 = \langle \underline{x} - \lambda\underline{y}, \underline{x} - \lambda\underline{y}\rangle$$

$$= \langle \underline{x}, \underline{x}\rangle + \langle \underline{x}, -\lambda\underline{y}\rangle + \langle -\lambda\underline{y}, \underline{x}\rangle + \langle -\lambda\underline{y}, -\lambda\underline{y}\rangle \quad \text{using distri law}$$

$$= \langle \underline{x}, \underline{x}\rangle - \lambda\langle \underline{x}, \underline{y}\rangle - \lambda\langle \underline{y}, \underline{x}\rangle + \lambda^2\langle \underline{y}, \underline{y}\rangle$$

$\because \alpha\langle x, y\rangle = \langle \alpha x, y\rangle = \langle x, \alpha y\rangle$

division of vectors & $\sqrt{\ }$ of matrices is not possible generally.

$$0 \leq \|x - \lambda y\|^2 = \|x\|^2 - 2\lambda \langle \underline{x}, \underline{y} \rangle + \lambda^2 \|y\|^2 \quad \text{---} (1)$$

Choose $\lambda = \dfrac{\langle x, y \rangle}{\|y\|^2}$ put in eqn (1)

$$0 \leq \|x - \lambda y\|^2 = \|x\|^2 - 2\frac{[\langle x, y\rangle]^2}{\|y\|^2} + \frac{[\langle x, y\rangle]^2}{\|y\|^4}\|y\|^2$$

$$\Rightarrow \quad 0 \leq \|x\|^2 - 2\frac{[\langle x, y\rangle]^2}{\|y\|^2} + \frac{[\langle x, y\rangle]^2}{\|y\|^2}$$

$$0 \leq \|x\|^2 - \frac{[\langle x, y\rangle]^2}{\|y\|^2}$$

$$0 \leq \|x\|^2 \|y\|^2 - [\langle x, y\rangle]^2 \quad (\text{x}^\text{ing} \text{ by } \|y\|^2 \text{ on b.s})$$

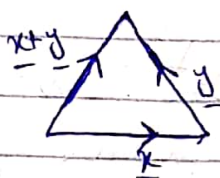
$$\Rightarrow [\langle x, y\rangle]^2 \leq \|x\|^2 \|y\|^2$$

$$\Rightarrow [\langle x, y\rangle]^2 \leq [\|x\| \|y\|]^2$$

$$|\langle x, y\rangle| \leq \|x\| \|y\| \qquad \text{Taking } \sqrt{\ }$$

★ Applications of Cauchy - Schwarz inequality are Triangle inequality.

⇒ In triangle sum of two sides is always greater than or equal to the third side. Then



$$\|x + y\| \leq \|x\| + \|y\|$$

Consider

$$\|x + y\|^2 = \langle \underline{x} + \underline{y}, \underline{x} + \underline{y} \rangle$$

$$= \langle \underline{x}, \underline{x} \rangle + \langle \underline{x}, \underline{y} \rangle + \langle \underline{y}, \underline{x} \rangle + \langle \underline{y}, \underline{y} \rangle$$

$$= \|x\|^2 + 2\langle \underline{x}, \underline{y} \rangle , \|y\|^2$$

$$\|x+y\|^2 = \|x\|^2 + 2\langle \underline{x}, \underline{y} \rangle + \|y\|^2$$

$$\leq \|x\|^2 + 2|\langle \underline{x}, \underline{y} \rangle| + \|y\|^2 \quad \because x \leq |x|$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \because |\langle x, y\rangle| \leq \|x\|\|y\|$$

$$= [\|x\| + \|y\|]^2$$

So $\|x+y\|^2 \leq [\|x\| + \|y\|]^2$

$$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$$

* * *

Theorem:- If $x, y, z \in \mathbb{R}^n$ then

$$\|\underline{x} - \underline{y}\| \leq \|\underline{x} - \underline{z}\| + \|\underline{z} - \underline{y}\|$$

Proof:- Consider

$$\|\underline{x} - \underline{y}\| = \|(\underline{x} - \underline{z}) + (\underline{z} - \underline{y})\|$$

z کو add اور subtract کیا گیا ہے

$$\leq \|x - z\| + \|z - y\| \quad \text{using triange inequality}$$

* * *

Bernoulli's Inequality:- If $x \in \mathbb{R}$ & $x \geq -1$ then $(1+x)^n \geq 1 + nx \quad \forall\, n \in \mathbb{N}$

Proof We prove this result using Mathematical Induction

The all results of natural numbers are proved by Mathematical Induction

Case 1:- For $n = 1$

$$L.H.S = (1+x)^1 = 1+x$$

$$R.H.S = 1 + 1 \cdot x = 1+x$$

$$L.H.S = R.H.S$$

So given result is true for $n = 1$

Case 2:- Suppose given condition is true for $n = k$

i.e $(1+x)^k \geq 1 + kx$

$$\Rightarrow (1+x)^k (1+x) \geq (1+kx)(1+x)$$

given $x \geq -1$ $\Rightarrow x+1 \geq 0$ So inequality does not change

$$\Rightarrow (1+x)^{k+1} \geq 1+x+kx+kx^2$$
$$= 1+(1+k)x+kx^2$$
$$\geq 1+(K+1)x \quad (\because \text{we have neglect } kx^2$$

So given result is true for K+1. Induction is complete so given result is true for all natural numbers.

——— *** ——— *** ——— *** ——— *** ———

Prepared By

M.TAHIR WATTOO

MSc. Math:- Punjab University

MS Math:- COMSATS Islamabad

0344-8563284

CHAPTER NO. 2

DIFFERENTIABILITY ★ ★ ★

⇒ **Limit** :- $\lim_{x \to a} f(x) = \ell$

If for $\epsilon > 0$ there exist a $\delta > 0$ s.t $|f(x) - \ell| < \epsilon$ whenever $0 < |x - a| < \delta$

i.e whenever x is in neighbourhood of a, $f(x)$ is nbhd of "ℓ".

$a-\delta \quad a \quad a+\delta$

$]a-\delta, a+\delta[$

open interval is nbhd of a

$a-\delta < x < a+\delta$

$\Rightarrow |x-a| < \delta$

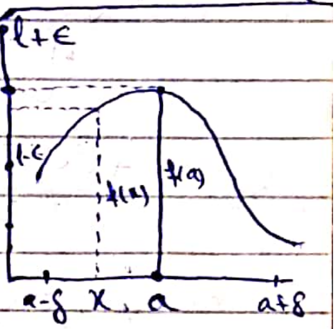
so x is in nbhd of a

Remark :- "ℓ" is finite & unique

eg :- $\lim_{x \to 1} (x^2+2) = 1^2+2 = 3$

ℓ کی nbhd میں 1 کے x جب nbhd کے 3 وہ ہے solution کا x^2+2 میں آتا ہے

x جب nbhd کے a میں $f(x)$ جب nbhd کے ℓ If $y = f(x)$ آئے



⇒ **Derivative** :- A function $f: X \to Y$ is said to be differentiable at $c \in X$ if

(i) f is defined in some nbhd of c

(ii) $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exist

Domain میں، Limit check کرنے Points کے لیے

If this limit exist then it is denoted by $f'(c)$ i.e $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

OR

for $\epsilon > 0$ there exist a $\delta > 0$ s.t

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon \quad \text{whenever}$$

$$0 < |x - c| < \delta$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Put $x = a + \delta x$

$$f'(a) = \lim_{a+\delta x \to a} \frac{f(a+\delta x) - f(a)}{\delta x}$$

$\lim\limits_{\Delta t \to 0} \dfrac{\Delta s}{\Delta t} = \dfrac{ds}{dt} = V$ Instant velocity at Particular point

But $\dfrac{\Delta s}{\Delta t} = V_{av}$ at any point

Remark:-

If $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} = f'(c)$

then $\dfrac{f(x) - f(c)}{x - c} = f'(c) + u(x)$

where $\lim\limits_{x \to c} u(x) = 0$

اگر limit کو ختم کرنا
چاہیں یعنی Instant کو
Average میں کنورٹ کرنا چاہیں
تو ایک ٹرم Add کرنا پڑتا ہے

Example:- Let $f(x) = x^2 - 9$

take $c = 3 \Rightarrow f'(c) = ?$

$f'(x) = 2x$, $f'(c) = 6$

Now $f(x) - f(c) = x^2 - 9 - 0$

$$\Rightarrow \frac{f(x) - f(c)}{x - c} = \frac{x^2 - 9}{x - 3} = x + 3$$

$$= 6 + (x - 3) \quad \text{sub \& add } 3$$

$$\Rightarrow \frac{f(x) - f(c)}{x - c} = f'(c) + u(x)$$

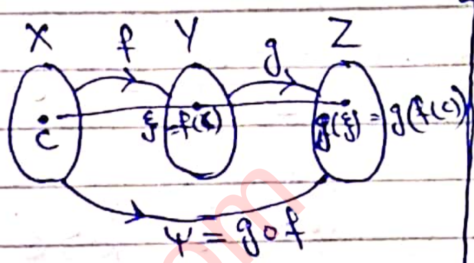
where $u(x) = x - 3 \Rightarrow \lim\limits_{x \to 3} (x - 3) = 0$

Chain Rule:- Suppose "f" is differentiable at point $c \in D_f$ (Domain of f) and g is differentiable at $\xi = f(c)$ then the composition function $\psi = g\circ f = g(f(x))$ is differentiable at c and

$\xi \to$ Zeeta

$$\psi'(c) = g'(f(c))\,f'(c)$$

[why we define composition? $\psi = g\circ f \Rightarrow$ function which link X to Z. Composition is used to connect 1st function with 3rd]



Proof:- Given f is differentiable at Point c.

$$\Rightarrow \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c) \text{ exist}$$

$$\Rightarrow \frac{f(x)-f(c)}{x-c} = f'(c) + u(x)$$

$$\Rightarrow f(x) - f(c) = [(f'(c) + u(x)](x-c) \quad \text{--- (1)}$$

where $\lim_{x \to c} u(x) = 0$

Also given g is differentiable at Point ξ.

$$\Rightarrow \lim_{y \to \xi} \frac{g(y)-g(\xi)}{y-\xi} = g'(\xi)$$

$$\Rightarrow \frac{g(y)-g(\xi)}{y-\xi} = g'(\xi) + v(y)$$

$$\Rightarrow g(y) - g(\xi) = [g'(\xi) + v(y)][y-\xi] \quad \text{--- (2)}$$

where $\lim_{y \to \xi} v(y) = 0$

Let $y = f(x)$ & $\xi = f(c)$ put in eqn ①

$$g(f(x)) - g(f(c)) = [g'(f(c)) + v(f(x))][f(x) - f(c)]$$

$$(g\circ f)(x) - (g\circ f)(c) = [g'(f(c)) + v(f(x))][(u(x) + f'(c))(x - c)]$$ using eqn ①

$$\Rightarrow \frac{\psi(x) - \psi(c)}{x - c} = [g'(f(x)) + v(f(x))][u(x) + f'(c)] \quad ③$$

Applying $\lim_{x\to c}$ on eqn ③

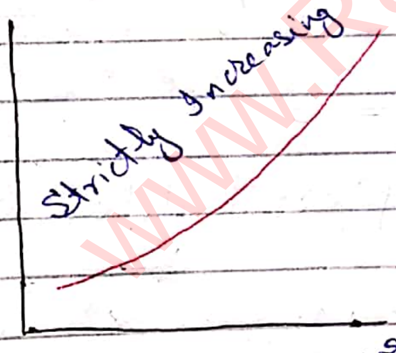
$$\lim_{x\to c} \frac{\psi(x) - \psi(c)}{x - c} = \left[\lim_{x\to c} g'(f(x)) + \lim_{x\to c} v f(x)\right]\left[\lim_{x\to c} u(x) + \lim_{x\to c} f'(c)\right]$$

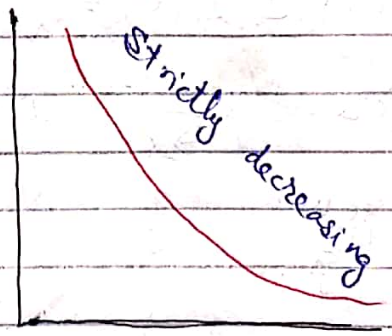
$$\boxed{\begin{array}{l} \text{when } x \to c \Rightarrow f(x) \to f(c) \\ y \to \xi \Rightarrow \lim_{y\to\xi} v(f(x)) = 0 \\ v(y) = 0 \\ \hline \lim_{x\to c} u(x) = 0 \\ \lim_{x\to c} v(y) = 0 \end{array}}$$

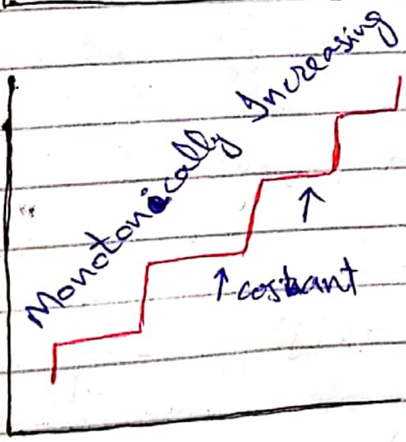
$$\Rightarrow \psi'(c) = [g'(f(c)) + 0][0 + f'(c)]$$

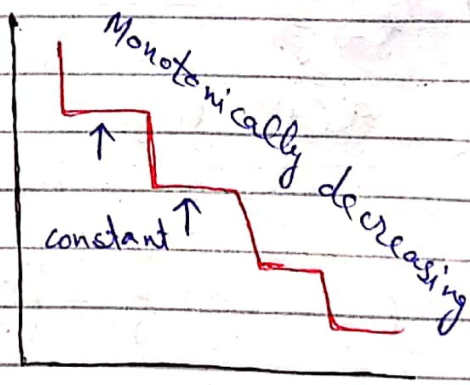
$$\Rightarrow \psi'(c) = g'(f(c))\, f'(c)$$

***









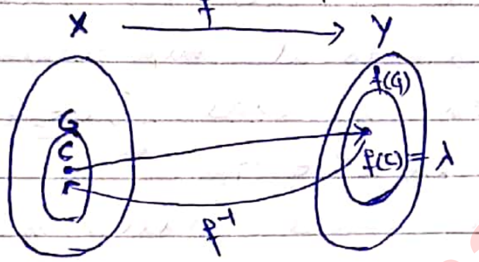
Note:- If f is continuous & one-to-one on G then
(i) f is strictly monotonic
(ii) f^{-1} exist, continuous & one-to-one on $f(G)$

* * *

Theorem Let $f: X \to Y$, $X, Y \subseteq \mathbb{R}$ be one-to-one and is continuous function on an open interval G. Let $c \in G$ & $\lambda \in Y$ s.t $c = f^{-1}(\lambda)$. Suppose f is differentiable at "c" s.t $f'(c) \neq 0$ then f^{-1} is differentiable at λ and

$$(f^{-1})'(\lambda) = \frac{1}{f'(c)}$$

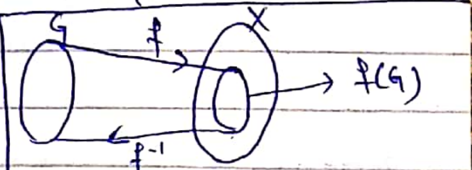
ہے bijective ہو تو Inverse
کا f^{-1} یعنی exist کرے
$\lambda = f(c)$ کو $f(G)$ - Domain



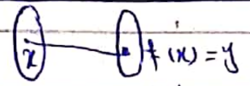
* f is differentiable at c
& f^{-1} " " at λ

Proof:- Since "f" is continuous and one-to-one on G so it is strictly monotonic then also f^{-1} exist & and continuous on $f(G)$ and is also one-to-one.

As $f'(c) \neq 0 \Rightarrow \frac{1}{f'(c)}$ be defined.



$f(G)$ is domain for f^{-1}
G " " " f



$y = f(x) \Rightarrow f^{-1}(y) = x$

Also $c = f^{-1}(\lambda)$

$\Rightarrow f(c) = \lambda$

Also let $f^{-1}(y) = x$

$\Rightarrow f(x) = y$

$\lim\limits_{x \to c} f(x) = f(c)$

$\Rightarrow \lim\limits_{x \to c} y = \lambda$

So when $x \to c$, $y \to \lambda$

Now $(f^{-1})'(\lambda) = \lim_{y \to \lambda} \dfrac{f^{-1}(y) - f^{-1}(\lambda)}{y - \lambda}$

$$= \lim_{y \to \lambda} \frac{x - c}{f(x) - f(c)}$$

$$= \lim_{y \to \lambda} \frac{1}{\dfrac{f(x) - f(c)}{x - c}}$$

$$= \frac{1}{f'(c)}$$

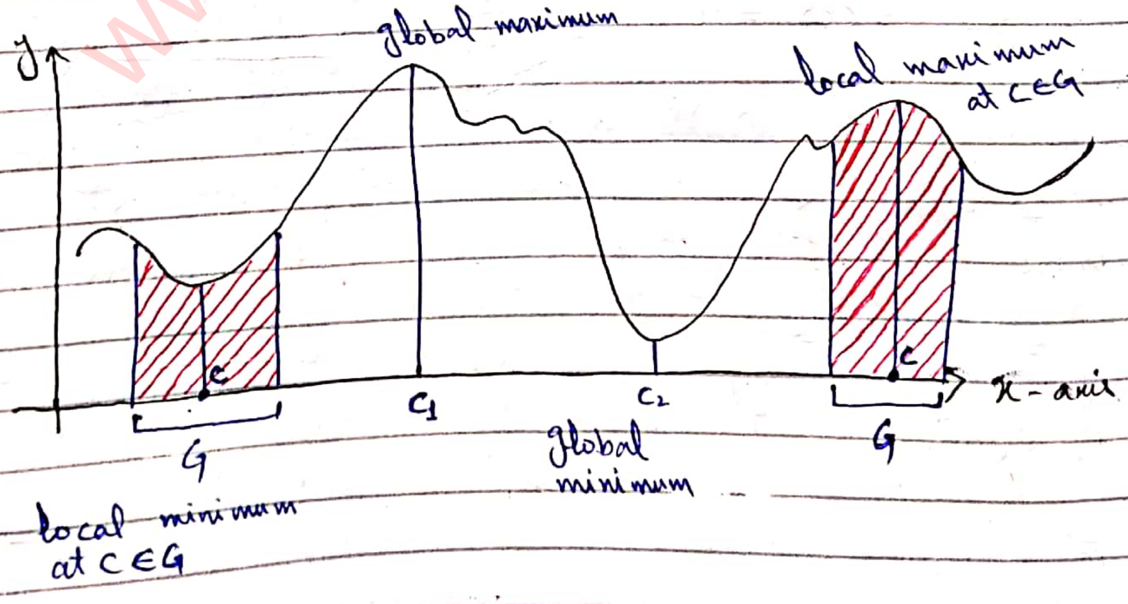
* * *

⇒ **Definitions:-** Let $X, Y \subseteq \mathbb{R}$ and $f : X \to Y$ then

(i) f is said to have a local minimum at c if there exist an open interval G, $c \in G$ such that $f(c) \leq f(x)$ $\forall$ $x \in G$

(ii) f is said to have a local maximum at c if there exist an open interval G, $c \in G$ s.t $f(x) \leq f(c)$ $\forall$ $x \in G$

(iii) f is said to have a local extremum at c if f has either local maximum or local minimum at c



When we talk about the local extremas at some point "c" we will consider its height only in the neighbourhood of "c" i.e open interval G it should be minimum or maximum than all other heights at different point of G.

And when we talk about the global ~~minima~~ extrema its mean this height is maximum or minimum from all the heights.

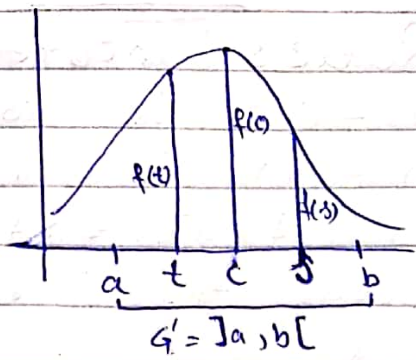
Geomatrically derivative represents the slope. We draw tangent tangent at extrema that is parallel to x-axis and any line which is parallel to x-axis have slope 0.

* * *

Theorem:- Suppose f is function $f: G \to Y$ & $G, Y \subseteq \mathbb{R}$. G is an open interval containing c and f has a local extremum at c. If f is differentiable at c then $f'(c) = 0$

Proof:- Without any loss of generality we suppose that f has local maximum at $x = c$ then theire exist an open interval say $G' =]a, b[$ containg c s.t

$f(x) \leq f(c) \quad \forall\, x \in G'$



Let $t, s \in G'$ s.t

$a < t < c < s < b$

Now

$t < c$

$\Rightarrow t - c < 0$

But $f(t) \leq f(c)$ ($\because f$ has l. max at $x = c$)

$\Rightarrow f(t) - f(c) \leq 0$

$$\Rightarrow \frac{f(t)-f(c)}{t-c} \geq 0 \qquad \because \frac{-ve}{-ve} = +ve \text{ or } \frac{0}{-ve} = 0$$

$$\Rightarrow \lim_{t\to c} \frac{f(t)-f(c)}{t-c} \geq 0$$

$$\Rightarrow f'(c) \geq 0 \quad \text{——— (1)}$$

Again

$$c < s$$

$$\Rightarrow c - s > 0 \qquad s - c > 0$$

Also $f(s) \leq f(c)$ ($\because f$ has local max at c)

$$\Rightarrow f(s) - f(c) \leq 0$$

$$\Rightarrow \frac{f(s)-f(c)}{s-c} \leq 0 \qquad \because \frac{-ve}{+ve} = -ve \text{ or } \frac{-ve}{0} = 0$$

$$\Rightarrow \lim_{s\to c} \frac{f(s)-f(c)}{s-c} \leq 0$$

$$\Rightarrow f'(c) \leq 0 \quad \text{——— (2)}$$

from eqn (1) & (2)

$$0 \leq f'(c) \leq 0 \Rightarrow f'(c) = 0$$

* * *

⇒ **Rolle's Theorem:-** Suppose $f(x)$ be s.t

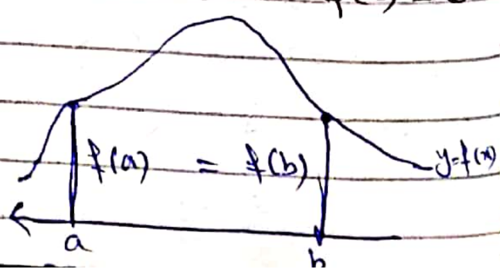
(i) $f(x)$ is continuous on $[a, b]$

(ii) $f(x)$ is differentiable on $]a, b[$

(iii) $f(a) = f(b)$

Then there exist a point c s.t $f'(c) = 0$

Proof We suppose that $f(x)$ has a



local maximum at $c \in]a, b[$. then by previous theorem

$$f'(c) = 0$$

***

⇒ Lagrange Mean Value Theorem:-

Suppose $f : \mathbb{R} \to \mathbb{R}$ be such that

(i) $f(x)$ is continuous on $[a, b]$

(ii) $f(x)$ is differentiable on $]a, b[$

Then there exist a point $c \in]a, b[$ s.t

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof:- Let us consider a function

$F(x) = f(x) + Ax$ where A is constant to be determined such that $F(b) = F(a)$

Then Clearly (i) $F(x)$ is continuous on $[a, b]$

(ii) $F(x)$ is differentiable on $]a, b[$

(iii) Also $F(a) = F(b)$

Then $F(x)$ satisfies all the conditions of Rolle's theorem. So by Rolle's theorem there exist a point $c \in]a, b[$ s.t

$$F'(c) = 0 \quad \text{——— (1)}$$

> always ~ so [illegible] Ax
> [illegible] diff اور contin

As $F(x) = f(x) + Ax$

$\Rightarrow F'(x) = f'(x) + A$

$\Rightarrow F'(c) = f'(c) + A$

> F is diff at all x Points on $]a, b[$ so $c \in]a, b[$ $F'(c) = f'(c) + A$

Put in eqn (1)

$$f'(c) + A = 0 \Rightarrow f'(c) = -A \quad \text{——— (2)}$$

As $F(b) = F(a)$

$f(b) + Ab = f(a) + Aa$

$\Rightarrow f(b) - f(a) = -Ab + Aa$

$\Rightarrow f(b) - f(a) = -A(b - a)$

$$\Rightarrow \frac{f(b)-f(a)}{b-a} = A \quad \text{--- (3)}$$

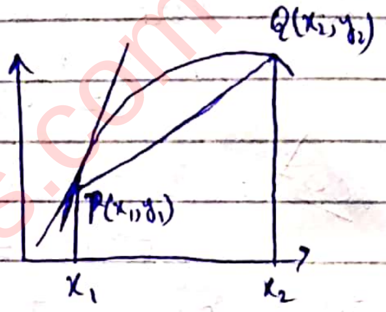
From equ (2) & (3)

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

* Lagrange M.V.T is generilized form of Rolle's theor–

***

Explanation of Lagrang M.V.T:–



$y = f(x)$

$\frac{dy}{dx}$ = Rate of change of y w.r.t x

$\frac{dy}{dx}$ = Slope of tangent

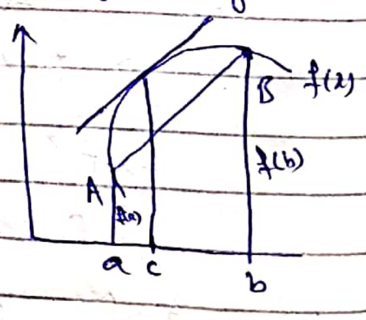
$P(x_1, y_1)$, $Q(x_2, y_2)$

Slope of Chord $PQ = \frac{y_2 - y_1}{x_2 - x_1}$

When $x_2 \to x_1$ line PQ become tangent at P

$\lim_{x_2 \to x_1} \frac{y_2 - y_1}{x_2 - x_1}$ represent slope of tangent.

$$\boxed{\lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_1)} \to \text{at pt } x_1$$



When $f'(x) \parallel AB$

slope of $AB = \frac{f(b)-f(a)}{b-a}$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

ہم کہتے ہیں کہ جب $x_2 \to x_1$ تو یہ x_1 پر Tangent کی equation کو ظاہر کرتا ہے۔

اور جب یہ x_1 پر $\overline{PQ}$ کسی ایک جگہ پر ملتا ہے تو اس نقطہ c پر جا کے ملتا ہے پر

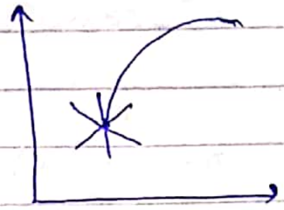
~~equation~~ Tangent $\overline{PQ}$ کے Parallel ہو جاتا ہے تو $f'(c)$ اور $\overline{PQ}$ کی

Slope برابر ہو جاتی ہے۔

Note :- Polynomial, cos, sin functions are always continuous and differentiable.

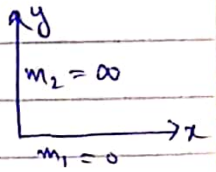
Slope of line is always unique.

Function is always differentiable on open interval $]a, b[$. Because as slope of line is unique. But at end points so many tangents can be drawn so that the slope will not be unique which is a contradiction.



★ Two lines are perpendicular if $(m_1)(m_2) = -1$

But in case of co-ordinate axis $\longrightarrow$

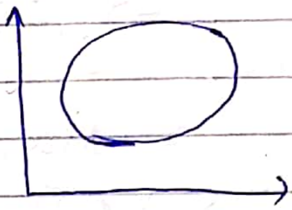


So $m_1 \times m_2 = 0 \times \infty \neq -1$

So the above condition is for lines which are not parallel to axis.

★ Equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



In ellipse & circle only the boundry points are included. Ellipse will represent circle when $a = b$.

★ ★ ★

$\Rightarrow$ **Cauchy's Mean Value Theorem:-** Suppose f and g are such that

(i) f & g are continuous on $[a, b]$

(ii) f & g are differentiable on $]a, b[$

further suppose that $g'(x) \neq 0 \ \forall x \in]a, b[$

then their exist a point $c \in]a, b[$ s.t

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof: Define a function

$$F(x) = f(x) + A g(x)$$

where A is a constant to be determined such that $F(b) = F(a)$

The clearly $F(x)$ satisfied all the conditions of Rolle's theorem. So by Rolle's theorem $\exists$ a point $c \in]a, b[$ s.t

$$F'(c) = 0 \quad \text{——— } ①$$

As $F(x) = f(x) + Ag(x)$

$$\Rightarrow F'(x) = f'(x) + Ag'(x)$$

$$F'(c) = f'(c) + Ag'(c) \quad \underline{\text{Put in } ①}$$

$$f'(c) + Ag'(c) = 0$$

$$\Rightarrow f'(c) = -Ag'(c)$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = -A \quad \text{——— } ②$$

Also $F(b) = F(a)$

$$\Rightarrow f(b) + Ag(b) = f(a) + Ag(a)$$

$$\Rightarrow f(b) - f(a) = -Ag(b) + Ag(a)$$

$$\Rightarrow f(b) - f(a) = -A[g(b) - g(a)] \quad \text{——— } ③$$

Next we show that $g(b) - g(a) \neq 0$

Suppose that $g(b) - g(a) = 0$

$$\Rightarrow g(b) = g(a)$$

Then by Rolle's theorem on $g(x)$ their exist a point $c_1 \in]a, b[$

s.t $g'(c_1) = 0$

which is contradiction because $g'(x) \neq 0$ $\forall\ x \in]a, b[$ (By given condition)

So our supposition is wrong

$\therefore\ g(b) - g(a) \neq 0$

Therefore equ ③ implies

$$\frac{f(b) - f(a)}{g(b) - g(a)} = -A \quad \text{——— ④}$$

from equ ③ & ④ we have

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad \text{Proved.}$$

* * *

Q:- Derive Lagrange's M.V.T by Cauchy's M.V.T

Solution Suppose "f" and "g" are two functions such that

(i) f & g are continuous on $[a, b]$

(ii) f & g are differentiable on $]a, b[$

Then by Cauchy's M.V.T there exist a point $c \in]a, b[$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \quad \text{——— ①}$$

Let $g(x) = x \Rightarrow g(b) = b \Rightarrow g(a) = a$

$\Rightarrow g'(x) = 1 \Rightarrow g'(c) = 1$

Put in equ ①

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Which is Lagrange's Mean Value theorem.

★ Cauchy's Mean Value Theorem is the generalized form of Lagrange's Mean Value Theorem.

* * *

★ Application of Rolle's and M.V.T

Q:- Show that $|\sin b - \sin a| \leq |b-a|$

Solution Define a function $f(x) = \sin x$ on $[a,b]$
Clearly (i) $f(x)$ is continuous on $[a,b]$
(ii) $f(x)$ is diff on $]a,b[$
So by Lagrange M.V.T there exist a point $c \in]a,b[$ s.t

$$\frac{\sin b - \sin a}{b-a} = f'(c)$$

$$\Rightarrow \frac{\sin b - \sin a}{b-a} = \cos c$$

$$\Rightarrow \sin b - \sin a = \cos c\,(b-a)$$

$b-a>0$
$b-a$ always $+ve$

$$\Rightarrow |\sin b - \sin a| = |\cos c|\,|b-a|$$

$$\Rightarrow |\sin b - \sin a| \leq |b-a| \quad \because \cos c \leq 1$$

★ ★ ★

Q:- Show that $\frac{x-t}{1+x^2} < \tan^{-1}x - \tan^{-1}t < \frac{x-t}{1+t^2}$

Solution Define a function

$$f(\theta) = \tan^{-1}(\theta) \quad \text{on } [t,x]$$

Clearly (i) $f(x)$ is continuous on $[t,x]$
(ii) As $f'(\theta) = \frac{1}{1+\theta^2}$

$$\Rightarrow f'(\theta) \text{ is defined on } [t,x]$$

$$\Rightarrow f'(\theta) \text{ is diff on }]a,b[$$

Then by M.V.T there exist a point

$c \in]t, x[$ s.t

M.V.T ⟹ Lagrang M.V.T
& For Cauchy M.V.T
used full name

$$\frac{f(x) - f(t)}{x - t} = f'(c)$$

$$\Rightarrow \frac{\tan^{-1}x - \tan^{-1}t}{x - t} = \frac{1}{1+c^2}$$

$$\Rightarrow \tan^{-1}x - \tan^{-1}t = \frac{x-t}{1+c^2} \quad \text{--- (1)}$$

As $c \in]t, x[\Rightarrow t < c < x$

$$\Rightarrow t^2 < c^2 < x^2$$

$$\Rightarrow 1+t^2 < 1+c^2 < 1+x^2$$

$$\Rightarrow \frac{1}{1+t^2} > \frac{1}{1+c^2} > \frac{1}{1+x^2}$$

$$\Rightarrow \frac{x-t}{1+t^2} > \frac{x-t}{1+c^2} > \frac{x-t}{1+x^2}$$

or $$\frac{x-t}{1+x^2} < \frac{x-t}{1+c^2} < \frac{x-t}{1+t^2}$$

$$\Rightarrow \frac{x-t}{1+x^2} < \tan^{-1}x - \tan^{-1}t < \frac{x-t}{1+t^2} \quad \text{using (1)}$$

Proved.

★ ★ ★

⟹ **Constant Function:-** A function $f: X \to Y$ is said to be constant function if $f(x_1) = f(x_2) \;\; \forall \; x_1, x_2 \in X$

★ ★ ★

Theorem:- Let "f" be differentiable on $]a, b[$ then $f'(x) = 0 \;\; \forall \; x \in]a, b[$ if & only if "f" is a constant function.

Proof:- Let f be a constant function i.e $f(x) = K$

To prove $f'(x) = 0 \quad \forall\, x \in]a,b[$

Let $c \in]a,b[$

As $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

$$= \lim_{x \to c} \frac{K - K}{x - c}$$

$$= \lim_{x \to c} \frac{0}{x - c} = \lim_{x \to c} 0$$

$$f'(c) = 0$$

As c is orbitrary so $f'(x) = 0 \ \forall\, x \in]a,b[$

Conversely suppose $f'(x) = 0 \ \forall\, x \in]a,b[$

To prove $f(x) =$ constant

Let $x_1, x_2 \in]a,b[$ s.t $x_2 > x_1$

Clearly

(i) $f(x)$ is continuous on $[x_1, x_2]$

(ii) $f(x)$ " diff on $]x_1, x_2[$

Then by M.V.T $\exists$ a point $c \in]x_1, x_2[$ s.t

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \quad (\because f'(x) = 0 \ \forall\, x \in]x_1, x_2[$$

$$\Rightarrow f(x_2) - f(x_1) = 0$$

$$\Rightarrow f(x_2) = f(x_1)$$

$$\Rightarrow f(x) = \text{constant.}$$

* * *

Q:- Let f be a function defined on $\mathbb{R}$ s.t $|f(x) - f(y)| \leq |x-y|^2$ then show that f is a constant function.

Solution Given $|f(x) - f(y)| \leq |x-y|^2$

$$\Rightarrow \left|\frac{f(x) - f(y)}{x-y}\right| \leq |x-y|$$

$$\Rightarrow \left|\lim_{x\to y} \frac{f(x) - f(y)}{x-y}\right| \leq \left|\lim_{x\to y} (x-y)\right|$$

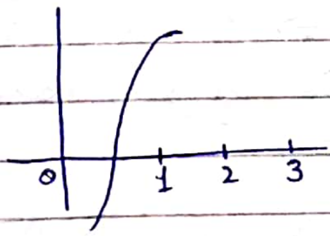
$$\Rightarrow \quad |f'(y)| \leq 0 \quad \Rightarrow f'(y) = 0$$

$\Rightarrow$ f is constant.

***

Note:- Consider $f(x) = x^2 - 1$

$f(1) = 1^2 - 1 = 0$



Root satisfy the given equation. Graphically the curve cuts the x-axis at root. If root lies b/w 0 & 1 the the curve cuts the x-axis b/w 0 & 1.

Now consider $f(-1) = 0$, $f(0) = -1$, $f(-2) = 3$, $f(2) = 3$.

Now if the value of function has been converted from +ve to −ve or from −ve to +ve its mean there is root b/w them.

***

Q:- If $c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \frac{c_3}{4} + \cdots + \frac{c_n}{n+1} = 0$, for $c_i \in \mathbb{R}$ & $i = 0, 1, 2, \ldots, n$. Then show that the equation $c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0$ has atleast 1 real root b/w 0 & 1.

Solution:- Define a function

$$f(x) = c_0 x + \frac{c_1 x^2}{2} + \frac{c_2 x^3}{3} + \dots + \frac{c_n x^{n+1}}{n+1}$$

Clearly (i) $f(x)$ is continuous in $[0,1]$
(ii) $f(x)$ is diff in $]0,1[$

Now $f(0) = 0$

& $f(1) = c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \dots + \frac{c_n}{n+1} = 0$ By given condition

$\Rightarrow f(0) = f(1)$

Then by Rolle's theorem there exist a point "a" $\in]0,1[$ s.t $f'(a) = 0$ —— ①

As $f'(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n$

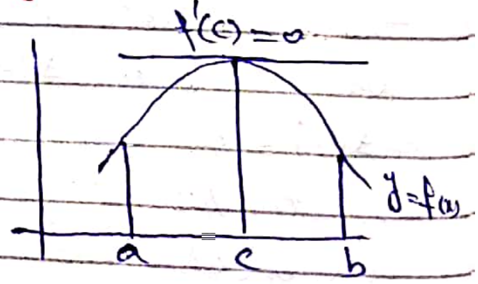
$f'(a) = c_0 + c_1 a + c_2 a^2 + \dots + c_n a^n$ Put in eq ①

$c_0 + c_1 a + c_2 a^2 + c_2 a^2 + \dots c_n a^n = 0$

$\Rightarrow$ "a" is the root of given condition.
So given condition has atleast one real root between 0 and 1.

* * *

$\Rightarrow$ Geomatrical Interpretation of M.V.T's.

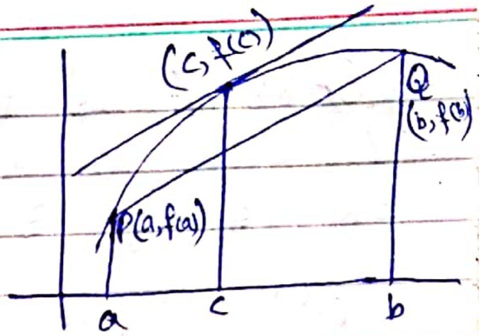


* **Rolle's Theorem:-** Since at "c" f has a local extrema. So tangent to the curve $[y = f(x)]$ at point $(c, f(c))$ is parallel to the x-axis.

So $f'(c)$ = slope of tangent
= slope of x-axis
= 0

i.e $f'(c) = 0$

★ Lagrange's M.V.T :-



$P(a, f(a))$ & $Q(b, f(b))$ be the two points on curve $y = f(x)$

$\therefore$ Slope of Chord PQ $= \dfrac{f(b) - f(a)}{b - a}$

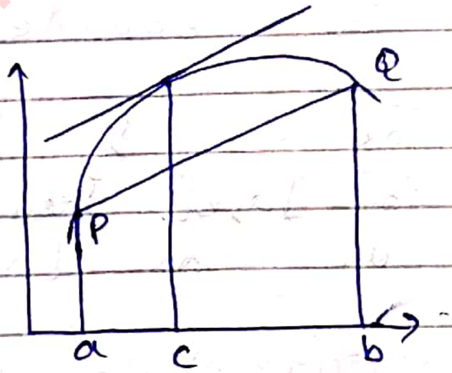
Slope of tangent at $(c, f(c)) = f'(c)$

If chord PQ & tangent at $(c, f(c))$ are parallel then

Slope of tangent at $(c, f(c))$ = Slope of chord $\overline{PQ}$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

★ Cauchy's M.V.T :-



Let $h(x)$ be a curve whose parametric equations are

$x = f(t)$ & $y = g(t)$

Let $P(f(a), g(a))$ & $Q(f(b), g(b))$ be two points on the curve then at point $f(c), g(c)$ we have

$$\frac{g'(c)}{f'(c)} = \frac{g(b) - g(a)}{f(b) - f(a)}$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

★ ★ ★

Lagrange $\Rightarrow$ when curve in Cartesian form.

Cauchy $\Rightarrow$ when curve in parametric form.

⇒ **Continuous Function:-** A function $f: X \to Y$ is said to be continuous at point c if for $\epsilon > 0$ $\exists$ a $\delta > 0$ s.t

$|f(x) - f(c)| < \epsilon$ whenever $0 \leq |x - c| < \delta$

If δ depends on ϵ only then the continuity is said to be uniform continuity.

⇒ **Bounded Function:-** A function $f: X \to Y$ is said to be bounded if there exist a +ve M s.t $|f(x)| < M$, $\forall x \in X$.

i.e A function is bounded if its range is bounded.

***

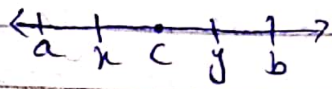
Theorem:- If $f'(x)$ exist and is bounded on $]a,b[$ then f is uniformly continuous on $[a,b]$

⇒ **Proof:-** Since $f'(x)$ is bounded on $]a,b[$

$\Rightarrow |f'(x)| < M$ ——— ① $\forall x \in]a,b[$

Let $x, y \in]a,b[$ s.t $x < y$

Clearly (i) $f(x)$ is continuous on $[x,y]$



(ii) $f(x)$ is differentiable on $]x,y[$

Then by Lagrange M.V.T there exist a point $c \in]x,y[$ s.t

$$\frac{f(y) - f(x)}{y - x} = f'(c)$$

$$\Rightarrow |f(y) - f(x)| = |f'(c)||y - x|$$

$$\Rightarrow |f(y) - f(x)| < M|y - x| \quad \text{using ①}$$

Now for $\epsilon > 0$ we can find $\delta = \epsilon/M > 0$ s.t
$|f(y) - f(x)| < \epsilon$ whenever $|y - x| < \delta$

If $|y-x| < \delta$
$M|y-x| < M\delta = \epsilon$
$M\delta = \epsilon \Rightarrow \delta = \frac{\epsilon}{M}$

Since δ depends on ϵ therefore continuity is uniform continuity.

***

Theorem,- If f is differentiable at c then show that $\lim\limits_{x \to c} \frac{x f(x) - c f(c)}{x - c} = c f'(c) + f(c)$

Proof:- L.H.S $= \lim\limits_{x \to c} \frac{x f(x) - c f(c)}{x - c}$

$$= \lim_{x \to c} \frac{x f(x) - x f(c) + x f(c) - c f(c)}{x - c}$$

$$= \lim_{x \to c} \frac{x[f(x) - f(c)] + f(c)[x - c]}{x - c}$$

$$= \lim_{x \to c} \frac{x[f(x) - f(c)]}{x - c} + \lim_{x \to c} \frac{f(c)[x - c]}{x - c}$$

$$= \lim_{x \to c} \left[\frac{x[f(x) - f(c)]}{x - c} + f(c) \right]$$

$$= c f'(c) + f(c) = R.H.S$$

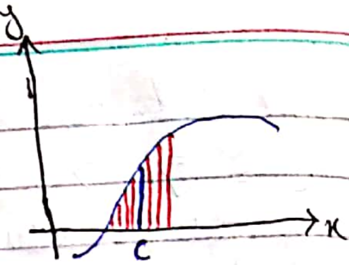
$\because \lim\limits_{x \to c} x = c$

***

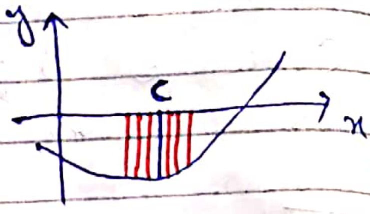
Remarks:- (i) If $f(c) > 0$ then $\exists$ an open interval G containing c s.t $f(x) > 0 \; \forall x \in G$

(ii) If $f(c) < 0$ then there exist an open interval G containing c s.t $f(x) < 0 \; \forall x \in G$

~~(iii)~~ $\Rightarrow$ If height is +ve at c then there lies an interval in neighbourhood of c where the height at each point is +ve.



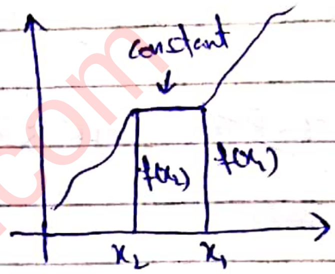
And if height is negative at c then there lies an interval in neighbourhood of c where the height at each point is −ve.



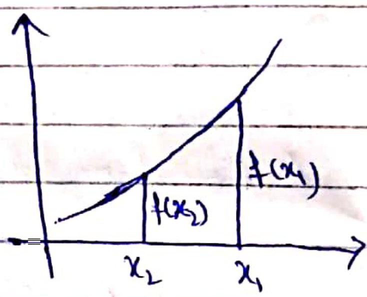
★ ★ ★

⇒ Definitions:-

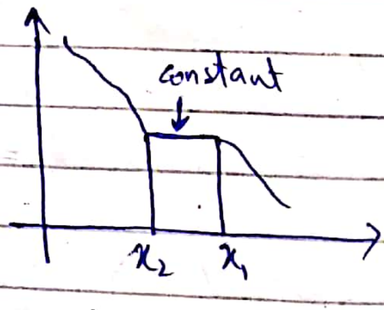
★ **Increasing Function:-** f is said to be increasing function if $f(x_1) \geq f(x_2)$ whenever $x_1 > x_2$.



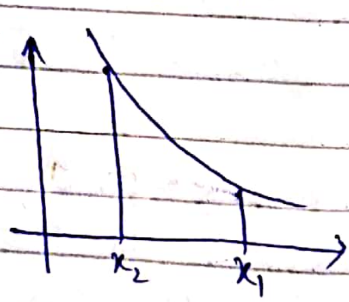
★ **Strictly Increasing Function:-** f is said to be strictly increasing function if $f(x_1) > f(x_2)$ whenever $x_1 > x_2$



★ **Decreasing Function:-** f is said to be decreasing function if $f(x_1) \leq f(x_2)$ whenever $x_1 > x_2$



★ **Strictly Decreasing Function:-** f is said to be strictly decreasing function if $f(x_1) < f(x_2)$ whenever $x_1 > x_2$.



★ ★ ★

⇒ As we know that derivative represent the slope of line. If slope is −ve, function is decreasing, & if slope is

+ve then the function is increasing.

★ ★ ★

Theorem:- Let function $f: X \to Y$, $X, Y \subseteq \mathbb{R}$. Suppose f is differentiable at $c \in X$ then

(i) If $f'(c) > 0$ then f is strictly increasing on an interval G containing c.

(ii) If $f'(c) < 0$ then f is strictly decreasing on an interval G containing c.

Proof:- (i) If $f'(c) > 0$ then there exist an open interval G containing c s.t $f'(x) > 0$ —— ①
for all $x \in G$.

Let $x_1, x_2 \in G$ such that $x_1 < x_2$

Clearly $f(x)$ is continuous on $[x_1, x_2]$ and $f(x)$ is differentiable on $]x_1, x_2[$

Then by Lagrange's M.V.T there exist a point $c \in]x_1, x_2[$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

$$\Rightarrow f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0$$ using ① & $x_2 - x_1 = +ve$

$$\Rightarrow f(x_2) - f(x_1) > 0$$

$$\Rightarrow f(x_2) > f(x_1) \quad \forall \; x_1, x_2 \in G$$

$\Rightarrow f(x)$ is strictly increasing on G.

(ii) If $f'(c) < 0$ then there exist an open interval G containing c s.t $f'(x) < 0$ —— ②
for all $x \in G$

Let $x_1, x_2 \in G$ s.t $x_1 < x_2$

Clearly $f(x)$ is continuous on $[x_1, x_2]$ and $f(x)$ is diff on $]x_1, x_2[$

Then by Lagrang M.V.T there exist a Point $c \in]x_1, x_2[$ s.t

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

$\Rightarrow f(x_2) - f(x_1) = f'(c)(x_2 - x_1) < 0$ using ②

$\Rightarrow f(x_2) - f(x_1) < 0$

$\Rightarrow f(x_2) < f(x_1)$

$\Rightarrow f(x)$ is strictly decreasing because $f(x_2) < f(x_1)$ when $x_2 > x_1$

***

Q:- Using Rolle's theorem show that the equation $10x^4 - 6x + 1 = 0$ has atleast one real root b/w 0 & 1

Solution Let $f(x) = 2x^5 - 3x^2 + x$ on $[0, 1]$

clearly $f(x)$ is continuous on $[0,1]$

and $f(x)$ is diff on $]0,1[$

& $f(0) = 0$

$f(1) = 2 - 3 + 1 = 0$

$\Rightarrow f(0) = f(1)$

So by Rolle's theorem there exist a point $c \in]0,1[$ s.t $f'(c) = 0$ —— ①

As $f'(x) = 10x^4 - 6x + 1$

$\Rightarrow f'(c) = 10c^4 - 6c + 1$ Put in ①

$10c^4 - 6c + 1 = 0$

So $c \in]0,1[$ is a root of given equation.

***

L' Hospital Rule:- Suppose f and g are differentiable on $]a,b[$ and that

(i) $\lim\limits_{x\to b^-} f(x) = \lim\limits_{x\to b^-} g(x) = 0$

(ii) As $x \to b^-$ $f(x) \to \infty$, $g(x) \to \infty$

If $g'(x) \neq 0$ on some interval $]x,b[$

then $\lim\limits_{x\to b^-} \dfrac{f'(x)}{g'(x)} = l \Rightarrow \lim\limits_{x\to b^-} \dfrac{f(x)}{g(x)} = l$

Proof:- Given that $\lim\limits_{x\to b^-} \dfrac{f'(x)}{g'(x)} = l$

To prove $\lim\limits_{x\to b^-} \dfrac{f(x)}{g(x)} = l$

As f & g are differentiable on $]a,b[$

$\Rightarrow$ f and g are continuous on $[a,b]$

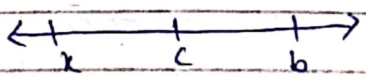
$\Rightarrow$ $f(a), f(b), g(a), g(b)$ are defined.

(i) Define $f(b) = g(b) = 0$ $\boxed{\text{for } \tfrac{0}{0} \text{ form}}$

As $\dfrac{f(x) - f(b)}{g(x) - g(b)} = \dfrac{f(x)}{g(x)}$ $\because$ $f(b)$ & $g(b)$ both are equal to zero.

or $\dfrac{f(x)}{g(x)} = \dfrac{f(x) - f(b)}{g(x) - g(b)} = \dfrac{f'(c)}{g'(c)}$ $x < c < b$ by Cauchy M.V.T

$\Rightarrow \dfrac{f(x)}{g(x)} = \dfrac{f'(c)}{g'(c)}$ ——— ①



Note that as $x \to b^- \Rightarrow c \to b^-$ so ① implies

$$\lim_{x\to b^-} \frac{f(x)}{g(x)} = \lim_{c\to b^-} \frac{f'(c)}{g'(c)}$$

$= l$ By given condition

ویری ایبل شفٹ کر نے سے فرق نہیں پڑتا

$\Rightarrow \lim\limits_{x\to b^-} \dfrac{f(x)}{g(x)} = l$ Proved for $\frac{0}{0}$ form

(ii) let $\phi(x) = \dfrac{1}{f(x)}$, $\psi(x) = \dfrac{1}{g(x)}$ $\quad \dfrac{\infty}{\infty} \to \dfrac{0}{0}$

Not that $\lim_{x\to b^-} \frac{f(x)}{g(x)} = \lim_{x\to b^-} \frac{\frac{1}{g(x)}}{1/f(x)}$ $\frac{0}{0}$ form

$$= \lim_{x\to b^-} \frac{-g'(x)/[g(x)]^2}{-f'(x)/[f(x)]^2} \quad \text{using Part } ①$$

$$= \lim_{x\to b^-} \left[\frac{f(x)}{g(x)}\right]^2 \frac{g'(x)}{f'(x)}$$

$$\lim_{x\to b^-} \frac{f(x)}{g(x)} = \left[\lim_{x\to b^-} \frac{f(x)}{g(x)}\right]^2 \lim_{x\to b^-} \frac{g'(x)}{f'(x)} \quad \text{——— ②}$$

Let $\lim_{x\to b^-} \frac{f(x)}{g(x)} = l'$ then there are three possibilities

(i) $l' \neq 0,\ l' \neq \infty$ (ii) $l' = 0$

(iii) $l' = \infty$

Case 1:- If $l' \neq 0,\ l' \neq \infty$

Equ ② becomes $l' = [l']^2 \lim_{x\to b^-} \frac{g'(x)}{f'(x)}$

$$\frac{1}{[l']^2} \lim_{x\to b^-} \frac{f'(x)}{g'(x)} = \frac{1}{l'}$$

$$\lim_{x\to b^-} \frac{f'(x)}{g'(x)} = \frac{[l']^2}{l'}$$

$$\lim_{x\to b^-} \frac{f'(x)}{g'(x)} = l'$$

Case 2:- If $l' = 0$

$\Rightarrow 1 = l' + 1$

$\Rightarrow \lim_{x\to b^-} \frac{f(x)}{g(x)} + 1 = 1$

$\Rightarrow \lim_{x\to b^-} \left[\frac{f(x)}{g(x)} + 1\right] = 1$

$$\Rightarrow \lim_{x \to b^-} \left[\frac{f(x)+g(x)}{g(x)}\right] = 1$$

$$\Rightarrow \lim_{x \to b^-} \left[\frac{f'(x)+g'(x)}{g'(x)}\right] = 1$$ By Case 1 $\because$ R.H.S $\neq 0, \infty$

$$\Rightarrow \lim_{x \to b^-} \left[\frac{f'(x)}{g'(x)} + \frac{g'(x)}{g'(x)}\right] = 1$$

$$\Rightarrow \lim_{x \to b^-} \frac{f'(x)}{g'(x)} + 1 = 1$$

$$\Rightarrow \lim_{x \to b^-} \frac{f'(x)}{g'(x)} = 0 = \ell'$$

Case 3:- let $\ell' = \infty$

$$\Rightarrow \lim_{x \to b^-} \frac{f(x)}{g(x)} = \infty$$

$$\Rightarrow \lim_{x \to b^-} \frac{g(x)}{f(x)} = 0$$

$$\Rightarrow \lim_{x \to b^-} \frac{g'(x)}{f'(x)} = 0$$ by case 2 $\because$ R.H.S $= 0$

$$\Rightarrow \lim_{x \to b^-} \frac{f'(x)}{g'(x)} = \infty$$

So in all possible cases

$$\lim_{x \to b^-} \frac{f'(x)}{g'(x)} = \lim_{x \to b^-} \frac{f(x)}{g(x)}$$

* * *

Q:- If f is twice differentiable at x_0 then prove that there exist a number c, $c \in \,]x_0 - h, x_0 + h[$ such that

$$\lim_{h \to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = f''(c)$$

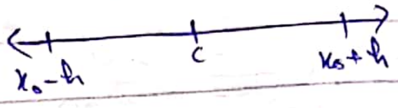
Solution $$\lim_{h \to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}$$ form $\frac{0}{0}$

$$= \lim_{h \to 0} \frac{f'(x_0+h) - 0 - f'(x_0-h)}{2h} \quad \text{form } \frac{0}{0}$$

$$= \lim_{h \to 0} \frac{f''(x_0+h) + f''(x_0-h)}{2} \quad \text{———} \circledast$$

As $c \in]x_0 - h, x_0 + h[$

$\Rightarrow x_0 - h < c < x_0 + h$



Clearly when $h \to 0 \Rightarrow x_0 \to c$ i.e $(x_0 = c)$

equ $\circledast$ implies $= \dfrac{f''(c) + f''(c)}{2}$

$$= \frac{2f''(c)}{2} = f''(c)$$

$$\Rightarrow f''(c) = R.H.S$$

***

Q:- If f and g are twice differentiable at x_0 then for $c \in]x_0 - h, x_0 + h[$ then show that

$$\lim_{h \to 0} \frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{g(x_0+h) + g(x_0-h) - 2g(x_0)} = \frac{f''(c)}{g''(c)}$$

Solution As $c \in]x_0 - h, x_0 + h[$

$\Rightarrow x_0 - h < c < x_0 + h$

$\Rightarrow$ when $h \to 0 \Rightarrow x_0 < c < x_0 \Rightarrow x_0 = c$

$\Rightarrow x_0 \to c$

$$L.H.S = \lim_{h \to 0} \frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{g(x_0+h) + g(x_0-h) - 2g(x_0)} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{f'(x_0+h) + f'(x_0-h)(-1) - 0}{g'(x_0+h) + g'(x_0-h)(-1) - 0}$$

$$L.H.S = \lim_{h\to 0} \frac{f'(x_0+h) - f'(x_0-h)}{g'(x_0+h) - g'(x_0-h)} \quad \left(\frac{0}{0}\right)$$

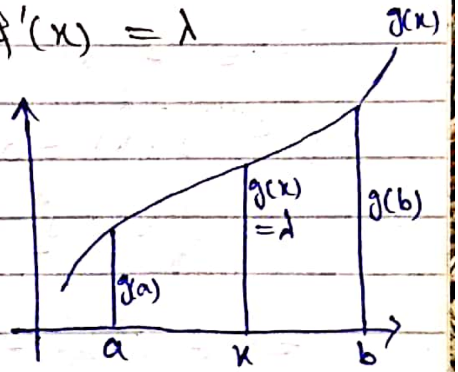
$$= \lim_{h\to 0} \frac{f''(x_0+h) + f''(x_0-h)}{g''(x_0+h) + g''(x_0-h)}$$

$$\Rightarrow \frac{f''(c+0) + f''(c-0)}{g''(c+0) + g''(c-0)} = \frac{2f''(c)}{2g''(c)}$$

$$= \frac{f''(c)}{g''(c)} = R.H.S \quad \text{Proved.}$$

***

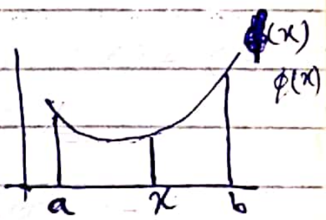
Theorem:- If f is differentiable on $]a,b[$ and $f'(a) < \lambda < f'(b)$ then there exist a point $x \in]a,b[$ such that $f'(x) = \lambda$



Proof:- Define a function

$$\phi(x) = f(x) - \lambda x$$

Clearly $\phi(x)$ is differentiable on $]a,b[$ &



$$\phi'(x) = f'(x) - \lambda \quad \text{———} \ (*)$$

$$\Rightarrow \phi'(a) = f'(a) - \lambda \quad \text{———} \ (1)$$

$$\& \ \phi'(b) = f'(b) - \lambda \quad \text{———} \ (2)$$

Given $f'(a) < \lambda < f'(b)$

$$\Rightarrow f'(a) - \lambda < 0 < f'(b) - \lambda \Rightarrow f'(a) - \lambda < 0 \ \& \ f'(b) - \lambda > 0$$

$\therefore$ equ (1) & (2) implies

$$\phi'(a) < 0 \quad \& \quad \phi'(b) > 0$$

Since $\phi'(a) < 0$ implies ϕ is decreasing in some nbhd of a & $\phi'(b) > 0$

is increasing in some nbhd of b

$\Rightarrow \phi$ has a local minimum at some point say $x \in]a, b[$

$\Rightarrow \phi'(x) = 0$

$\Rightarrow f'(x) - \lambda = 0$ using ⊛

$\Rightarrow f'(x) = \lambda$ proved.

***

Q:- If $x > 0$ then show that $\log x \leq x - 1$

Solution Define a function $f(x) = \log x - x + 1$

$\Rightarrow f'(x) = 1/x - 1$

$\Rightarrow f''(x) = -1/x^2$

put $f'(x) = 0 \Rightarrow \frac{1}{x} - 1 = 0$

$\Rightarrow \frac{1}{x} = 1 \Rightarrow x = 1$

$\Rightarrow f''(1) = -1/(1)^2 = -1$

$\Rightarrow f(x)$ has a local maximum at 1

$\Rightarrow f(x) \leq f(1)$

$\Rightarrow \log x - x + 1 \leq \log 1 - 1 + 1$

$\Rightarrow \log x - x + 1 \leq 0$

$\Rightarrow \log x \leq x - 1$

***

⇒ Lemma Theorem:- Let a and b be non-negative real numbers with $b \neq 0$ and $p, q > 1$ and $1/p + \frac{1}{q} = 1$ then show that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$\boxed{p \text{ \& } q \text{ are conjugate numbers of each other}}$

Proof:- Define a function $f(x) = x^\lambda - \lambda x$, $0 < \lambda < 1$, $x > 0$.
Clearly f is continuous and differentiable every where and

$$f'(x) = \lambda x^{\lambda-1} - \lambda$$

$$f'(x) = \lambda(x^{\lambda-1} - 1)$$

Then for x lies b/w 0 & 1.

$$f'(x) > 0 \Rightarrow f(x) \text{ is increasing in } [0,1]$$

& for $x > 1 \Rightarrow f'(x) < 0$

$\Rightarrow f(x)$ is decreasing if $x > 1$

Also $f(1) = 1 - \lambda$ ——— ①

Clearly $f(x)$ is continuous on $[u, 1]$ & $f(x)$ is diff on $]u, 1[$. Then by Lagrange M.V.T there exist a point $c_1 \in]u, 1[$ s.t

$0 \quad u \quad 1 \quad \infty$

$$\frac{f(1) - f(u)}{1 - u} = f'(c_1)$$

$$f(1) - f(u) = f'(c_1)(1 - u)$$

$$> 0 \quad (\because f'(x) > 0 \text{ in }]0,1[)$$

$$\Rightarrow f(1) - f(u) > 0 \Rightarrow f(u) < f(1)$$

$$\Rightarrow u^\lambda - \lambda u < 1 - \lambda \quad ——— ②$$

Also for any $v > 1$ by M.V.T on $[1, v]$ we have

$$\frac{f(v)-f(1)}{v-1} = f'(c_2) \quad , c_2 \in]1, v[$$

$$\Rightarrow f(v) - f(1) = f'(c_2)(v-1)$$

$$\Rightarrow f(v) - f(1) < 0 \quad (\because f'(x) < 0 \text{ for } x > 1)$$

$$\Rightarrow f(v) < f(1)$$

$$\Rightarrow v^{\lambda} - \lambda v < 1 - \lambda \quad \text{---} ③$$

from equ ①, ② & ③ we have

$$x^{\lambda} - \lambda x \leq 1 - \lambda \quad \forall x > 0$$

$$\Rightarrow x^{\lambda} \leq (1-\lambda) + \lambda x$$

Let $\lambda = 1/p$ & $1-\lambda = 1/q$

$\therefore$ we have $x^{1/p} \leq 1/q + (1/p)x$ ——— ③

Put $x = a^p/b^q$ in ③

$$\left(a^p/b^q\right)^{1/p} \leq 1/q + 1/p \, \frac{a^p}{b^q}$$

Xing both sides with b^q

$$\frac{ab^q}{b^{q/p}} \leq \frac{b^q}{q} + \frac{a^p}{p}$$

$$ab^{q-q/p} \leq \frac{b^q}{q} + \frac{a^p}{p}$$

$$ab^{q(1-1/p)} \leq a^p/p + \frac{b^q}{q}$$

$$\Rightarrow ab^{q(1/q)} \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \because \left(\frac{1}{p} + \frac{1}{q} = 1\right)$$

$$\Rightarrow ab \leq a^p/p + b^q/q$$

***

Remarks:- $C^n[a,b]$ = space of all functions defined on $[a,b]$ s.t $f, f', f'', \dots, f^{(n)}$ are continuous

★ $C^n[a,b] = \{f : [a,b] \to R$ s.t $f, f', f'', \dots, f^{(n)}$ are continuous

★ $C[a,b] = \{f : [a,b] \to R$ s.t f is continuous.

★ ★ ★

⇒ **Taylor's Theorem:-** Suppose that $f \in C^n[a,b]$ and $f^{(n+1)}$ exist on $]a,b[$ then for every choice of x and x_0 there exist a real number c, $x_0 < c < x$ s.t $f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \dots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + R_n(x)$

where

(i) $R_n(x) = \frac{(x-c)^n (x-x_0)}{n!} f^{(n+1)}(c)$

Lagrange form of remainder

(ii) $R_n(x) = \frac{(1+\theta)^n (x-x_0)^{n+1}}{n!} f^{(n+1)}(x_0 - \theta(x-x_0))$

$0 < \theta < 1$

Cauchy form of remainder.

Proof Case 1 If $x = x_0$

Then nothing is left to prove.

Case 2 If $x \neq x_0$, we suppose $x_0 < x$

Now for "t" s.t $x_0 < t < x$ define

$$\phi(t) = \phi_n(t) - \frac{(x-t)^{n+1}}{(x-x_0)^{n+1}} \phi_n(x_0)$$

where $\phi_n(t) = f(x) - f(t) - (x-t) f'(t) - \frac{(x-t)^2}{2!} f''(t)$

$$- \dots - \frac{(x-t)^n}{n!} f^{(n)}(t)$$

Now $\phi_n'(t) = 0 - f'(t) - \{(x-t) f''(t) + f'(t)(-1)\} - \{\frac{(x-t)^2}{2!} f'''(t)$

$+ \frac{f''(t) 2(x-t)(-1)}{2!}\} \dots - \{\frac{(x-t)^n}{n!} f^{(n+1)}(t) + \frac{f^{(n)}(t) n (x-t)^{n-1}}{n!}(-1)$

$$= -\cancel{f'(t)} - (x-t)\cancel{f''(t)} + \cancel{f'(t)} - \frac{(x-t)^2}{2!} f'''(t)$$

$$+ (x-t)\cancel{f''(t)} \cdots - \frac{(x-t)^n}{n!} f^{(n+1)}(t) + \frac{f^{(n)}(t)(x-t)^{n-1}}{(n-1)!}$$

$$\phi_n'(t) = -\frac{(x-t)^n}{n!} f^{(n+1)}(t) \quad \text{———} \; (*)$$

Clearly $\phi_n(t)$ is continuous on $[x_0, x]$
($\because f \in C^n[a,b]$)
and $\phi_n(t)$ is differentiable on $]x_0, x[$
($\because f^{(n+1)}$ exists)

Then (i) $\phi(t)$ is continuous on $[x_0, x]$
(ii) $\phi(t)$ is differentiable on $]x_0, x[$

Also $\phi(x_0) = \phi_n(x_0) - \frac{(x-x_0)^{n+1}}{(x-x_0)^{n+1}} \phi_n(x_0) = 0$

& $\phi(x) = \phi_n(x) - 0 = 0 - 0 = 0$

(iii) $\phi(x_0) = \phi(x)$

So ϕ satisfy all the conditions of Roll's theorem then there exist a point $c \in]x_0, x[$ such that

$$\phi'(c) = 0 \quad (1)$$

$\Rightarrow$ As

$$\phi(t) = \phi_n(t) - \frac{(x-t)^{n+1}}{(x-x_0)^{n+1}} \phi_n(x_0)$$

$$\phi'(t) = \phi_n'(t) - \left\{ \frac{(n+1)(x-t)^n(-1)}{(x-x_0)^{n+1}} \phi_n(x_0) \right\}$$

$$\phi'(t) = \phi_n'(t) + \frac{(n+1)(x-t)^n}{(x-x_0)^{n+1}} \phi_n(x_0)$$

$$\phi'(c) = \phi_n'(c) + \frac{(n+1)(x-c)^n}{(x-x_0)^{n+1}} \phi_n(x_0)$$

$$0 = \frac{-(x-c)^n}{n!} f^{(n+1)}(c) + \frac{(n+1)(x-c)^n}{(x-x_0)^{n+1}} \phi_n(x_0)$$

using (1) & (*)

$$\frac{(n+1)(x-c)^n}{(x-x_0)^{n+1}}\left[\phi_n(x_0)-\frac{(x-x_0)^{n+1}}{(n+1)n!}f^{(n+1)}(c)\right]=0$$

As $\frac{(n+1)(x-c)^n}{(x-x_0)^{n+1}} \neq 0$ $\quad \left\{ \begin{array}{l} x_0<c<x \\ x-c\neq 0 \text{ \& } x-x_0\neq 0 \end{array}\right.$

$$\Rightarrow \phi_n(x_0)-\frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}(c)=0$$

$$\Rightarrow f(x)-f(x_0)-(x-x_0)f'(x_0)-(x-x_0)^2f''(x_0)\cdots-\frac{(x-x_0)^n}{n!}f^{(n)}(x_0)$$
$$-\frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}(c)=0$$

$$\Rightarrow f(x)=f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+\cdots$$
$$+\frac{(x-x_0)^n f^{(n)}(x_0)}{n!}+R_n(x)$$

where $R_n(x)=\frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}(c)$

Which is Lagrange form of Remainder

Next we prove Taylor Theorem with Cauchy form of Remainder.

Note that ϕ_n is continuous on $[x_0,x]$

and ϕ_n is differentiable on $]x_0,x[$

Then by M.V.T there exist a point $c\in]x_0,x[$ such that

$$\frac{\phi_n(x)-\phi_n(x_0)}{x-x_0}=\phi_n'(c)$$

$$\Rightarrow \phi_n(x)-\phi_n(x_0)=\phi_n'(c)(x-x_0)$$

$$\Rightarrow 0-\phi_n(x_0)=(x-x_0)\phi_n'(c)$$

$$\Rightarrow \phi_n(x_0)=-(x-x_0)\phi_n'(c)$$

$$\Rightarrow f(x)-f(x_0)-(x-x_0)f'(x_0)-\frac{(x-x_0)^2}{2!}f''(x_0)-\cdots$$
$$\cdots\frac{(x-x_0)^n f^{(n)}(x_0)}{n!}=-(x-x_0)\left\{-\frac{(x-c)^n}{n!}f^{(n+1)}(c)\right\}$$

$$\Rightarrow f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \dots$$
$$+ \frac{(x-x_0)^n f^{(n)}(x_0)}{n!} + R_n(x)$$

where $R_n(x) = \frac{(x-x_0)(x-c)^n}{n!} f^{(n+1)}(c)$

Now Put $c = x_0 - \theta(x-x_0)$

$$R_n(x) = \frac{(x-x_0)}{n!}\left[(x-x_0)+\theta(x-x_0)\right]^n f^{(n+1)}(x_0-\theta(x-x_0))$$
$$= \frac{(x-x_0)}{n!}\left[(x-x_0)(1+\theta)\right]^n f^{(n+1)}(x_0-\theta(x-x_0))$$
$$= \frac{(x-x_0)^{n+1}(1+\theta)^n}{n!} f^{(n+1)}(x_0-\theta(x-x_0))$$
$$0<\theta<1$$

As $c \in]a, a+h[$

$0 < \theta h < h \Rightarrow a < a+\theta h < a+h$

$\Rightarrow c = (a+\theta h)$

***

Theorem:- Suppose that $f \in C^n[a,b]$ s.t $f^{(n)}(x_0) \neq 0$ for some $x_0 \in]a,b[$ & $f^{(K)}(x_0) = 0$ for $K = 1,2,3,\dots(n-1)$ then

(i) If n is even & $f^{(n)}(x_0) < 0$ then f has a local maximum at x_0

(ii) If n is even & $f^{(n)}(x_0) > 0$ then f has a local minimum at x_0

Proof Consider Taylor Theorem with Lagrange form of Remainder

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \dots$$
$$+ \frac{(x-x_0)^{n-1}}{(n-1)!} f^{(n-1)}(x_0) + \frac{(x-x_0)^n}{n!} f^{(n)}(c)$$
$$x_0 < c < x$$

$$f(x) = f(x_0) + \frac{(x-x_0)^n}{n!} f^n(c) \quad \because f^{(k)}(x_0)=0 \text{ for } k=1 \text{ to } n-1$$

$$f(x) - f(x_0) = \frac{(x-x_0)^n}{n!} f^{(n)}(c) \quad \text{——— ①}$$

Given $f^{(n)}(x_0) \neq 0$

$\Rightarrow$ $f^{(n)}(x_0)$ & $f^{(n)}(c)$ has same sign in neighbourhood of x_0

(i) given n is even and $f^{(n)}(x_0) < 0$ so equ ① implies

$$f(x) - f(x_0) < 0 \quad \text{in nbhd of } x_0$$

$$\Rightarrow f(x) < f(x_0)$$

$\Rightarrow$ f has local maximum at x_0

(ii) given n is even and $f^{(n)}(x_0) > 0$ so equ ① implies

$$f(x) - f(x_0) > 0$$

$$\Rightarrow f(x) > f(x_0)$$

$\Rightarrow$ f has local minimum at x_0

* * *

PREPARED BY

MUHAMMAD TAHIR WATTOO

M.Sc. MATHEMATICS:- Punjab University

M.S. MATHEMATICS:- Comsats University.

0344-8563284

CHAPTER NO. 3

REIMANN INTEGRAL ★★★

★ $y = f(x)$

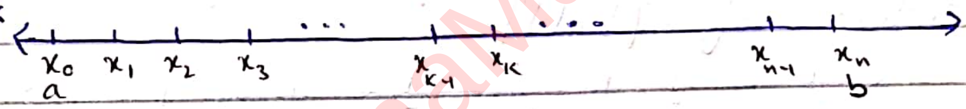
$\Rightarrow \frac{dy}{dx} = f'(x)$ gives the slope of curve

$\int f'(x)\,dx = f(x)$ Again gives the curve.

★ Reimann integral gives area under curve

⇒ **Partition:-** A finite set $P = \{a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b\}$ of real numbers is said to be partition of closed interval $[a, b]$ if $a = x_0 < x_1 < x_2 < \ldots < x^{k-1} < x^k < \ldots < x_n = b$

Here $[x^{k-1}, x^k]$ called kth sub interval of $[a, b]$

★ 

Suppose

$P_1 = \{1, 2, 5, 6\}$ $\qquad$ $P_2 = \{1, 3, 4, 5, 6\}$

infinite partitions are Possible for one set because there are infinite numbers b/w every two real numbers

$[1, 3]$ finite interval $\because$ end points are finite

$[1, \infty[,]-\infty, 1]$, $]-\infty, \infty[$ are infinite intervals $\because$ end points are not finite

But all are infinite sets

★ All sub intervals from this partition★ are $[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{k-1}, x_k], \ldots, [x_{n-1}, x_n]$

$P[a, b]$ or $P(a, b)$ is set of all partitions of $[a, b]$ i.e $P = \{P : P$ is a partition of $[a, b]\}$

★ If P_1 & P_2 are two partitions of $[a, b]$ s.t $P_1 \subset P_2$ then P_2 is called finer than P_1 or P_2 is said to be refinement of P_1.

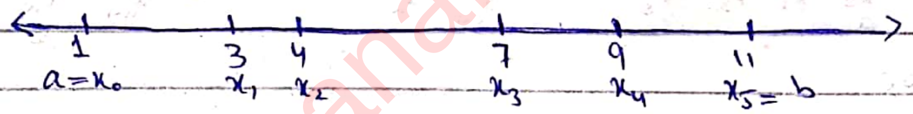
If $P_1 = \{1, 2, 5\}$, $P_2 = \{1, 2, 3, 5\}$ are two partitions of $[1, 5]$ interval then P_2 is the refinement of P_1

⇒ **Norm of a Partition:-** If $P = \{a = x_0, x_1, x_2, \ldots, x_{k-1}, x_k, \ldots x_n = b\}$ is a partition of $[a, b]$ then norm of P is denoted & defined as

$$\|P\| = \max_{k=1}^{n} \Delta x_k$$

where $\Delta x_k = x_k - x_{k-1}$ = Length of Kth sub-interval

Example:- Let $P = \{1, 3, 4, 7, 9, 11\}$ be the partition of $[1, 11]$



(i)

$$\Delta x_1 = x_1 - x_0 = 3 - 1 = 2$$
$$\Delta x_2 = x_2 - x_1 = 4 - 3 = 1$$
$$\Delta x_3 = x_3 - x_2 = 7 - 4 = 3$$
$$\Delta x_4 = x_4 - x_3 = 9 - 7 = 2$$
$$\Delta x_5 = x_5 - x_4 = 11 - 9 = 2$$

(ii) $$\|P\| = \max_{k=1}^{5} \Delta x_k = \max\{2, 1, 3, 2, 2\} = 3$$

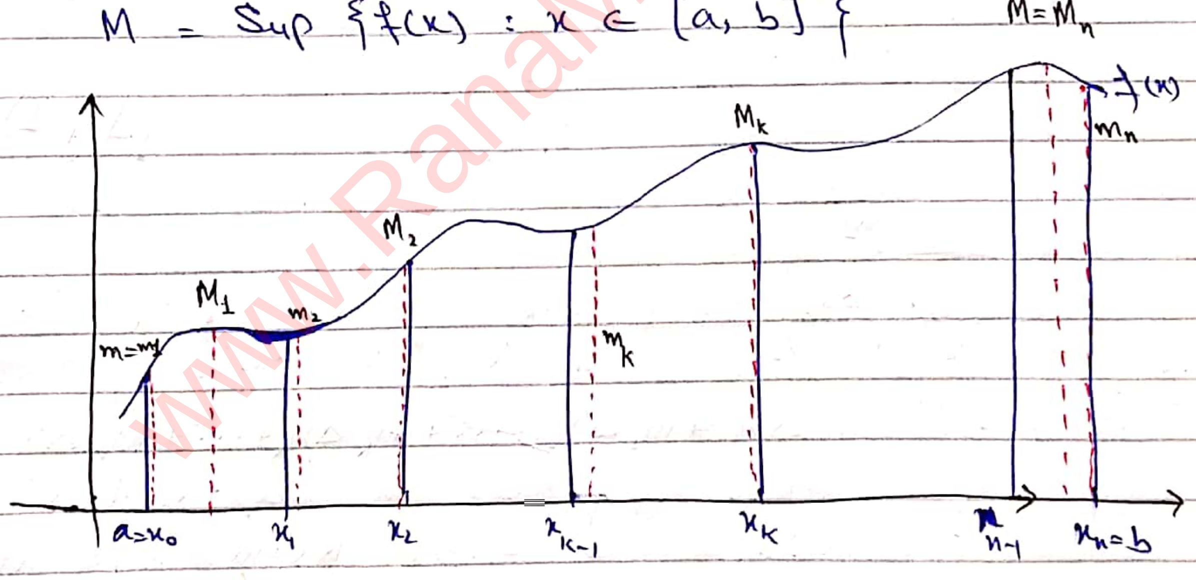
(iii)
$$\sum_{k=1}^{5} \Delta x_k = \Delta x_1 + \Delta x_2 + \Delta x_3 + \Delta x_4 + \Delta x_5$$
$$= 2 + 1 + 3 + 2 + 2$$
$$= 10$$
$$= 11 - 1$$
$$= x_n - x_0 = b - a$$

$C[a,b] \to$ space
$P[a,b] \to$ set ...
$B[a,b] \to$ space ...

$$\Rightarrow \sum_{k=1}^{n} \Delta x_k = b-a$$

* Definition :- $B[a,b]$ or $B(a,b)$ is the space of all bounded real valued functions defined on $[a,b]$
i.e $B[a,b] = \{f:[a,b] \to R \text{ \& } f \text{ is bounded}\}$
* Function is bounded when range of function is bounded $\Rightarrow |f(x)| < M$
* Definition :- If $f \in B[a,b]$ and $P \in P[a,b]$
s.t $P = \{a = x_0, x_1, x_2, \dots x_{k-1}, x_k, \dots x_{n-1}, x_n = b\}$

then we can define
$m_k = \text{Inf}\{f(x) : x \in [x_{k-1}, x_k]\}$
$M_k = \text{Sup}\{f(x) : x \in [x_{k-1}, x_k]\}$
$m = \text{Inf}\{f(x) : x \in [a,b]\}$
$M = \text{Sup}\{f(x) : x \in [a,b]\}$



Clearly $m \le m_k \le M_k \le M$

* To calculate Reimann integral f must be bounded.
$[a,b]$ must be finite interval
$\int_0^\infty f(x)\,dx$ is not reimann integral

as range is not finite

($\int_0^1 \frac{1}{x}\,dx \Rightarrow f(x) = \frac{1}{x}$ It is again improper because $f(x)$ is not bounded. when $x \to 0$, $f(x) \to \infty$

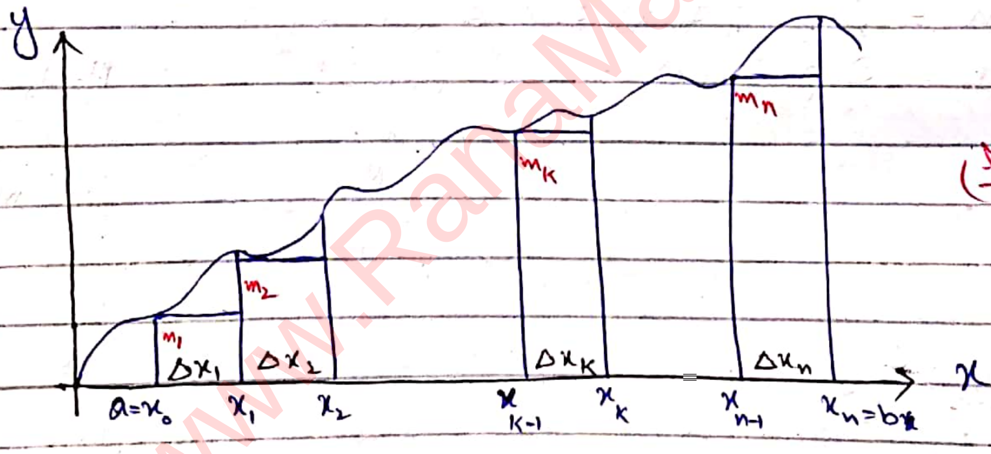
⇒ Lower & Upper Reimann Sums:-

If $f \in B[a,b]$ and $P \in P(a,b)$ then the values

$$S(f,P) = \sum_{K=1}^{n} M_K \Delta x_K$$

$$s(f,P) = \sum_{K=1}^{n} m_K \Delta x_K$$

are respectively called Upper Reimann sums & Lower Reimann sums of f corresponding to P.



(fig 1)

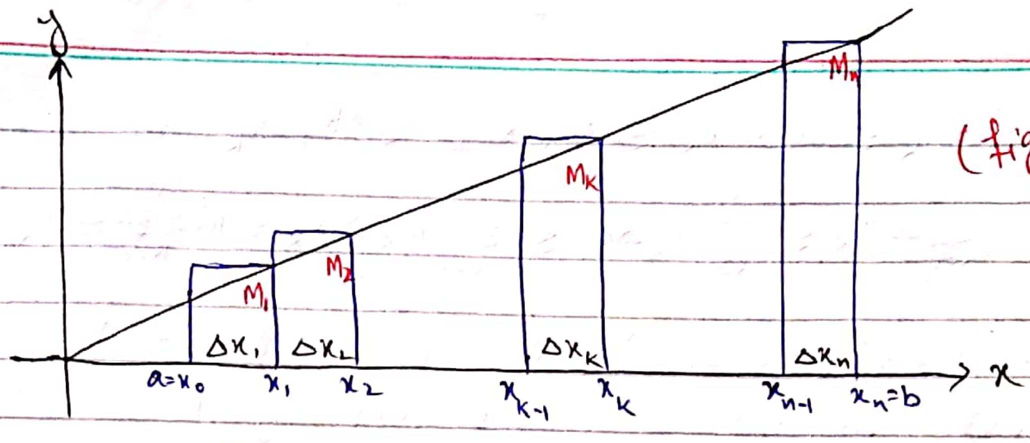
Then $s(f,P) = m_1 \Delta x_1 + m_2 \Delta x_2 + \dots + m_K \Delta x_K + \dots + m_n \Delta x_n$

Sum of area of all the rectangles under the curve is called Lower Reimann sum. Area calculated in this way is less than the original are under the curve. These areas are calculated by taking the infimum heights of all the intervals.

If we consider (fig 2) then

$S(f,P) = M_1 \Delta x_1 + M_2 \Delta x_2 + \dots + M_K \Delta x_K + \dots + M_n \Delta x_n$

these are as are calculated by taking the



(fig 2)

supremum heights of each interval. Area calculated in this way is greater than the original area under the curve.

When the number of points $n \to \infty$ in partitions then area calculated by U.R.S or L.R.S is approximately equal to the area under the curve. When the points in any partitions will be increased mean partition is going to be finer. U.R.S will be decreased and L.R.S will be increased. When $n \to \infty$ the sum will be approached to the integral.

***

Lemma:- Let $f \in B[a,b]$ & $P_1, P_2 \in P(a,b)$ s.t $P_1 \subset P_2$ then

$$m(b-a) \leq s(f,P_1) \leq s(f,P_2) \leq S(f,P_2) \leq S(f,P_1) \leq M(b-a)$$

Proof:- Let $P_1 = \{a = x_0, x_1, x_2, \cdots, x_{K-1}, x_K, \cdots x_n = b\}$ be a partition of interval $[a,b]$ also let

$$m_K = \inf\{f(x) : x_{K-1} \leq x \leq x_K\}$$

$$M_K = \sup\{f(x) : x_{K-1} \leq x \leq x_K\}$$

$$m = \inf\{f(x) : a \leq x \leq b\}$$

$$M = \sup\{f(x) : a \leq x \leq b\}$$

$$m \leq m_K \leq M_K \leq M$$

$$\Rightarrow m\Delta x_k \leq m_k \Delta x_k \leq M_k \Delta x_k \leq M\Delta x_k$$

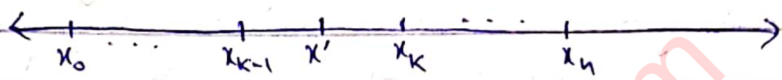
$$\Rightarrow m \sum_{k=1}^{n} \Delta x_k \leq \sum_{k=1}^{n} m_k \Delta x_k \leq \sum_{k=1}^{n} M_k \Delta x_k \leq M \sum_{k=1}^{n} \Delta x_k$$

$$\Rightarrow m(b-a) \leq s(f,P_1) \leq S(f,P_1) \leq M(b-a) \quad \text{——— (1)}$$

next defined a refinement of P_1

$$P_2 = \{a = x_0, x_1, x_2, \dots, x_{k-1}, x', x_k, \dots, x_n = b\}$$

Now define

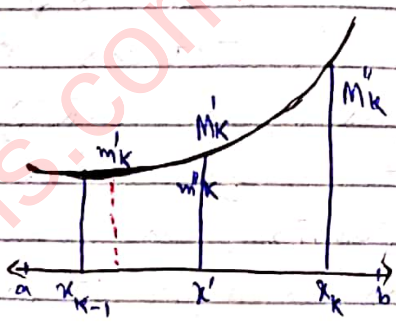


$m_k' = \text{Inf}\{f(x) : x \in [x_{k-1}, x']\}$

$M_k' = \text{Sup}\{f(x) : x \in [x_{k-1}, x']\}$

$m_k'' = \text{Inf}\{f(x) : x \in [x', x_k]\}$

$M_k'' = \text{Sup}\{f(x) : x \in [x', x_k]\}$



Clearly $m_k \leq m_k'$ & $m_k \leq m_k''$ &

$M_k \geq M_k'$ & $M_k \geq M_k''$

Consider $S(f,P_1) - S(f,P_2)$

$= [M_1\Delta x_1 + M_2\Delta x_2 + \dots + M_k\Delta x_k + \dots + M_n\Delta x_n] -$
$[M_1\Delta x_1 + M_2\Delta x_2 + \dots + M_{k-1}\Delta x_{k-1} + M_k'(x' - x_{k-1}) +$
$M_k''(x_k - x_k') + \dots + M_n\Delta x_n]$

$= M_k\Delta x_k - M_k'(x' - x_{k-1}) - M_k''(x_k - x')$

$= M_k(x_k - x_{k-1}) - M_k'(x' - x_{k-1}) - M_k''(x_k - x')$

$= M_k x_k - M_k x_{k-1} - M_k' x' + M_k' x_{k-1} - M_k'' x_k + M_k'' x'$
$+ M_k x' - M_k x'$ {Add & sub $M_k x'$}

$$= (M_K - m_K')(x' - x_{K-1}) + (M_K - m_K'')(x_K - x')$$

$$\geq 0$$

As $M_K \geq m_K'$
$\Rightarrow M_K - m_K' \geq 0$
& $M_K \geq m_K''$
$\Rightarrow M_K - m_K'' \geq 0$

$\Rightarrow \bar{S}(f, P_1) - \bar{S}(f, P_2) \geq 0$

$\Rightarrow \bar{S}(f, P_1) \geq \bar{S}(f, P_2)$

OR $\bar{S}(f, P_2) \leq \bar{S}(f, P_1)$ —— ②

Similarly we can prove

$$s(f, P_1) \leq s(f, P_2) \text{ —— ③}$$

from equ ② & ③

$$s(f, P_1) \leq s(f, P_2) \leq \bar{S}(f, P_2) \leq \bar{S}(f, P_1) \text{ —— ④}$$

from equ ① & ④

$$m(b-a) \leq s(f, P_1) \leq s(f, P_2) \leq \bar{S}(f, P_2) \leq \bar{S}(f, P_1) \leq M(b-a)$$

Remark:- If $\{P_n\}$ is a sequence of partitions s.t $P_n \subset P_{n+1}$. Then $\{\bar{S}(f, P_n)\}$ & $\{s(f, P_n)\}$ are monotonocally decreasing & increasing sequences respectively.

$\Rightarrow$ **Reimann Integral:-** Let $f \in B(a,b)$ then

$$\bar{\int_a^b} f(x)\,dx = \operatorname{Inf}\{\bar{S}(f, P) : P \in P(a,b)\}$$ and

$$\underline{\int_a^b} f(x)\,dx = \operatorname{Sup}\{s(f, P) : P \in P(a,b)\}$$

are respectively called upper Reimann Integral and Lower Reimann Integral. f is Reimann Integrable on $[a,b]$ if

$$\bar{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$ and we write

$$\bar{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$$

Example:- Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational in } [0,1] \\ 0 & \text{if } x \text{ is irrational in } [0,1] \end{cases}$$

Show that f is not reimann integrable on $[0,1]$

Solution:- Let $P = \{x_0 = 0, x_1, x_2, \dots, x_n = 1\}$ be a partition of $[0,1]$.

In Kth sub Interval

$m_K = 0$ & $M_K = 1$

$$\Rightarrow s(f,P) = \sum_{K=1}^{n} m_K \Delta x_K$$

$$s(f,P) = 0 \quad , \forall P \in P(0,1)$$

$$\Rightarrow \underline{\int_0^1} f = \text{Sup}\{s(f,P) : P \in P(0,1)\}$$

$$= \text{Sup}\{0\}$$

$$\Rightarrow \underline{\int_0^1} f = 0 \quad \text{——— (1)}$$

$$S(f,P) = \sum_{K=1}^{n} M_K \Delta x_K$$

$$= \sum_{K=1}^{n} \Delta x_K \qquad \because M_K = 1$$

$$S(f,P) = 1 - 0 = 1, \forall P \in P(0,1)$$

$$\Rightarrow \overline{\int_0^1} f = 1 \quad \text{——— (2)}$$

from equ (1) & (2) we have

$$\underline{\int_0^1} f \neq \overline{\int_0^1} f$$

So f is not Reimann Integrable on $[0,1]$

* * *

⇒ Give an example which shows that not every function is Reimann Integrable

Solution:- Above Example.

⇒ **Cauchy's Theorem:-** Let $f \in B(a,b)$ then f is R.I (Reimann Integrable) on $[a,b]$ iff for every $\epsilon > 0$ there is a partition P of $[a,b]$ s.t $S(f,P) - s(f,P) < \epsilon$.

Proof:- Suppose f is R.I on $[a,b]$

$$\Rightarrow \int_a^{\bar{b}} f = \int_{\underline{a}}^{b} f = \int_a^b f \quad \text{———} \circledast$$

To prove for $\epsilon > 0$ there is a partition P of $[a,b]$ s.t

$$S(f,P) - s(f,P) < \epsilon$$

Now for $\epsilon > 0$ $\exists$ partitions P_1 & P_2 of $[a,b]$ s.t

$$S(f,P_1) < \int_a^{\bar{b}} f + \epsilon/2$$

By Definition of sup & infimum

$$s(f,P_2) > \int_{\underline{a}}^{b} f - \epsilon/2$$

using $\circledast$ we have

$$S(f,P_1) < \int_a^b f + \epsilon/2 \quad \text{———} ①$$

$$s(f,P_2) > \int_a^b f - \epsilon/2 \quad \text{———} ②$$

Let $P = P_1 + P_2 \Rightarrow P_1 \subseteq P$ & $P_2 \subseteq P$

$$\Rightarrow S(f,P) \leq S(f,P_1) \quad \text{———} ③$$

$$\& \ s(f,P) \geq s(f,P_2) \quad \text{———} ④$$

from eqn ① & ③ we have

$$S(f,P) < \int_a^b f + \epsilon/2 \quad \text{———} ⑤$$

from eqn ② & ④ we have

$$s(f,P) > \int_a^b f - \epsilon/2$$

$$\Rightarrow -s(f,P) < -\int_a^b f + \epsilon/2 \quad \text{———} ⑥$$

Adding eqn ⑤ & ⑥

$$\bar{S}(f,P) - s(f,P) < \epsilon$$

Conversely suppose that for $\epsilon > 0$ there is a partition P of (a,b) s.t $\bar{S}(f,P) - s(f,P) < \epsilon$

To Prove f is R.I.

As $\bar{\int_a^b} f \leq \bar{S}(f,P)$ ——— ⑦

& $\underline{\int_a^b} f \geq s(f,P)$

$\Rightarrow -\underline{\int_a^b} f \leq -s(f,P)$ ——— ⑧

Add respective sides of eqn ⑦ & ⑧

$$\bar{\int_a^b} f - \underline{\int_a^b} f \leq \bar{S}(f,P) - s(f,P)$$

$$\bar{\int_a^b} f - \underline{\int_a^b} f < \epsilon \quad \text{By given condition.}$$

$$\Rightarrow \bar{\int_a^b} f - \underline{\int_a^b} f = 0$$

If $b-a < \epsilon$, $\forall \epsilon$ & $b > a$ then $b-a = 0$

$$\Rightarrow \bar{\int_a^b} f = \underline{\int_a^b} f$$

$\Rightarrow f$ is R.I on $[a,b]$

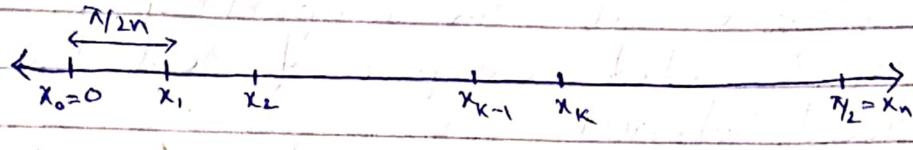
★★★

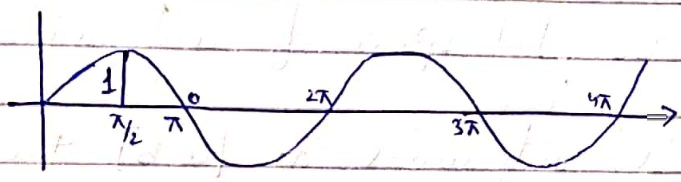
Q1:- Show that $f(x) = \sin x$ is R.I over $[0, \pi/2]$

Solution Divide $[0, \pi/2]$ into n sub intervals each of length

$$\Delta x = \frac{b-a}{n} = \frac{\pi/2 - 0}{n} = \frac{\pi}{2n}$$

$\therefore$ partition is $P = \{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \frac{3\pi}{2n}, \dots, \frac{n\pi}{2n} = \frac{\pi}{2}\}$





Sin x

$x_1 = x_0 + \Delta x$
$= 0 + \frac{\pi}{2n} = \frac{\pi}{2n}$

$x_2 = x_1 + \Delta x$
$= \frac{\pi}{2n} + \frac{\pi}{2n} = \frac{2\pi}{2n}$

Although x is angle but in radian measurments we get real numbers which can be plot on Real line.

As $\sin x$ in interval $[0, \pi/2]$ is an increasing function so in kth sub interval

$$M_k = f(x_k) = f\left(\frac{k\pi}{2n}\right) = \sin\frac{k\pi}{2n}$$

$$m_k = f(x_{k-1}) = f\left(\frac{(k-1)\pi}{2n}\right) = \sin\frac{(k-1)\pi}{2n}$$

$$S(f,P) - s(f,P) = \sum_{k=1}^{n} M_k \Delta x_k - \sum_{k=1}^{n} m_k \Delta x_k$$

$$= \sum_{k=1}^{n} (M_k - m_k)\Delta x_k$$

$$= \sum_{k=1}^{n}\left[\sin\frac{k\pi}{2n} - \sin\frac{(k-1)\pi}{2n}\right] \times \frac{\pi}{2n}$$

$$\le \frac{\pi}{2n} < \varepsilon \text{ for } n > \frac{\pi}{2\varepsilon}$$

$\pi/2n < \varepsilon$
$\Rightarrow \varepsilon > \pi/2n \Rightarrow n > \frac{\pi}{2\varepsilon}$

$\Rightarrow S(f,P) - s(f,P) < \varepsilon$

$\Rightarrow$ f is R.I by cauchy Theorem.

* * *

Note:- Heine Boral Theorem:-

① Every closed & bounded subspace of real line is compact.

② Any continuous function defined on

R.I

Continuous $\longleftarrow$ Differentiable

a compact space is uniformly continuous.
Converse is true only in compact space.

③ Closed & bounded subspace of real line is some closed interval.
$[-1, 5]$ closed & bounded space of real line.

---

v.v.I

Theorem:- If f is continuous on $[a,b]$ then f is reimann integrable on $[a,b]$

Solution

Since f is continuous on $[a,b]$ & $[a,b]$ is compact so f is uniformly continuous. then for every $\epsilon > 0$ there exist a δ
$\delta = \delta(\epsilon) > 0$ s.t

$$|f(x_2) - f(x_1)| < \epsilon/b-a \quad \text{when } 0 \le |x_2 - x_1| < \delta \quad (1)$$

In particular take a partition P of (a,b) s.t $\|P\| < \delta$

Therefo equ (1) implies

$$|M_K - m_K| < \epsilon/b-a$$

$$\Rightarrow M_K - m_K < \epsilon/b-a$$

$$\Rightarrow \sum_{K=1}^{n} M_K \Delta x_K - \sum_{K=1}^{n} m_K \Delta x_K < \frac{\epsilon}{b-a} \sum_{K=1}^{n} \Delta x_K$$

$$S(f,P) - s(f,P) < \frac{\epsilon}{b-a}(b-a)$$

$$S(f,P) - s(f,P) < \epsilon$$

$\Rightarrow f$ is R.I on $[a,b]$

Example:- Show that $f(x) = x^{100} - 100x + 5$ is R.I on $[a,b]$

Solution:- Since $f(x)$ is a polynomial so is continuous. Therefore by previous theorem $f(x)$ is Reimann Integrable on $[a,b]$

★★★

Theorem:- Let $\alpha \in \mathbb{R}$ and f is R.I on $[a,b]$ then so is αf and $\int_a^b \alpha f = \alpha \int_a^b f$

Proof:- Since f is R.I on $[a,b]$

$$\Rightarrow \underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f \quad \text{———} \; (*)$$

To prove αf is R.I on $[a,b]$. We divide the prove into three cases.

Case 1:- If $\alpha = 0$

Then nothing is left to prove

Case 2:- If $\alpha > 0$

Let $M_K = \text{Sup}\{f(x) : x \in [x_{K-1}, x_K]\}$

$m_K = \text{Inf}\{f(x) : x \in [x_{K-1}, x_K]\}$

Then $\alpha M_K = \text{Sup}\{(\alpha f)(x) : x \in [x_{K-1}, x_K]\}$

$\alpha m_K = \text{Inf}\{(\alpha f)(x) : x \in [x_{K-1}, x_K]\}$

As $$S(\alpha f, P) = \sum_{K=1}^{n} (\alpha M_K) \Delta x_K$$

$$= \alpha \sum_{K=1}^{n} M_K \Delta x_K$$

$$S(\alpha f, P) = \alpha S(f, P) \quad \text{———} \; ①$$

Similarly $s(\alpha f, P) = \alpha s(f, P)$ ——— ②

Now $$\overline{\int_a^b} \alpha f = \text{Inf}\{S(\alpha f, P) : P \in P(a,b)\}$$

$$= \text{Inf}\{\alpha S(f, P) : P \in P(a,b)\}$$

$$= \alpha \, \text{Inf}\{S(f, P) : P \in P(a,b)\} \quad \because \text{Inf}\, \alpha A = \alpha \, \text{Inf} A \text{ if } \alpha > 0$$

$$\overline{\int_a^b} \alpha f = \alpha \overline{\int_a^b} f$$

$$\Rightarrow \overline{\int_a^b} \alpha f = \alpha \int_a^b f \quad \text{— ③ using } (*)$$

$$\underline{\int_a^b} \alpha f = \operatorname{Sup}\{s(\alpha f, P) : P \in \mathcal{P}(a,b)\}$$

$$= \operatorname{Sup}\{\alpha s(f,P) : P \in \mathcal{P}(a,b)\} \quad \text{using equ ②}$$

$$= \alpha \operatorname{Sup}\{s(f,P) : P \in \mathcal{P}(a,b)\} \quad \because \operatorname{Sup}\alpha A = \alpha \operatorname{Sup} A \text{ if } \alpha > 0$$

$$\Rightarrow \underline{\int_a^b} \alpha f = \alpha \underline{\int_a^b} f$$

$$\Rightarrow \underline{\int_a^b} \alpha f = \alpha \int_a^b f \quad \text{— ④ using } (*)$$

from equ ③ & ④ we have

$$\underline{\int_a^b} \alpha f = \overline{\int_a^b} \alpha f$$

$\Rightarrow \alpha f$ is R.I on $[a,b]$

Case 3:- If $\alpha < 0$

Let $M_k = \operatorname{Sup}\{f(x) : x \in [x_{k-1}, x_k]\}$

$m_k = \operatorname{Inf}\{f(x) : x \in [x_{k-1}, x_k]\}$

$\Rightarrow \alpha M_k = \operatorname{Inf}\{(\alpha f)(x) : x \in [x_{k-1}, x_k]\} \quad \because \alpha < 0$

& $\alpha m_k = \operatorname{Sup}\{(\alpha f)(x) : x \in [x_{k-1}, x_k]\} \quad \because \alpha < 0$

Now $S(\alpha f, P) = \sum_{k=1}^{n} \alpha m_k \Delta x_k$

$$= \alpha \sum_{k=1}^{n} m_k \Delta x_k$$

S2

$$\bar{S}(\alpha f, P) = \alpha\, s(f,P) \quad \text{——} \; (5)$$

Similarly $s(\alpha f, P) = \alpha\, \bar{S}(f,P)$ —— (6)

Now
$$\bar{\int_a^b} \alpha f = \inf\{\bar{S}(\alpha f, P) : P \in P(a,b)\}$$
$$= \inf\{\alpha\, s(f,P) : P \in P(a,b)\} \quad \text{using eq}^n\ (5)$$
$$= \alpha \sup\{s(f,P) : P \in P(a,b)\} \quad \because \inf \alpha A = \alpha \sup A \text{ if } \alpha < 0$$

$$\Rightarrow \bar{\int_a^b} \alpha f = \alpha \underline{\int_a^b} f$$

$$\Rightarrow \bar{\int_a^b} \alpha f = \alpha \int_a^b f \quad \text{——} \; (7) \quad \text{using } (\star)$$

Similarly we have
$$\underline{\int_a^b} \alpha f = \alpha \int_a^b f \quad \text{——} \; (8)$$

from equation (7) & (8) we obtain
$$\underline{\int_a^b} \alpha f = \bar{\int_a^b} \alpha f$$

$\Rightarrow \alpha f$ is Reimann Integrable on $[a,b]$
$\Rightarrow \alpha f$ is R.I in all possible cases
Since f & αf both or R.I on $[a,b]$
So relation (8) implies
$$\int_a^b \alpha f = \alpha \int_a^b f$$

$\because \alpha f$ is R.I on $[a,b]$
so $\underline{\int_a^b} \alpha f = \int_a^b \alpha f$

***

Theorem:- Let $f : B(a,b)$. Then f is R.I on $[a,b]$ iff for every $\epsilon > 0$ there exist a $\delta = \delta(\epsilon) > 0$ s.t

$$\left| \sum_{k=1}^{n} f(t_k)\Delta x_k - \int_a^b f \right| < \epsilon \quad \text{——} \; (1)$$

for every partition P of (a,b) & for every choice of point $t_k \in [x_{k-1}, x_k]$

Proof) Given f is R.I on $[a,b]$

$$\Rightarrow \overline{\int_a^b} f = \underline{\int_a^b} f = \int_a^b f \quad \text{——} \; (A)$$

To prove relation (1) holds

As $t_k \in [x_{k-1}, x_k]$

$$\Rightarrow m_k \leq f(t_k) \leq M_k$$

$$\Rightarrow \sum_{k=1}^{n} m_k \Delta x_k \leq \sum_{k=1}^{n} f(t_k)\Delta x_k \leq \sum_{k=1}^{n} M_k \Delta x_k$$

$$\Rightarrow s(f,P) \leq \sum_{k=1}^{n} f(t_k)\Delta x_k \leq S(f,P) \quad \text{——} \; (2)$$

Again for every $\epsilon > 0$ there exist a partition $P \in P(a,b)$ s.t

$$S(f,P) < \int_a^b f + \epsilon \quad \text{——} \; (3)$$

$$\& \; s(f,P) > \int_a^b f - \epsilon \quad \text{——} \; (4)$$

$$\Rightarrow \int_a^b f - \epsilon < s(f,P) \quad \text{——} \; (5)$$

from equation (2), (3) & (5) we have

$$\int_a^b f - \epsilon < s(f,P) \leq \sum_{k=1}^{n} f(t_k)\Delta x_k \leq S(f,P) < \int_a^b f + \epsilon$$

$$\Rightarrow \int_a^b f - \epsilon < \sum_{k=1}^{n} f(t_k)\Delta x_k < \int_a^b f + \epsilon$$

$$\Rightarrow -\epsilon < \sum_{k=1}^{n} f(t_k)\Delta x_k - \int_a^b f < \epsilon$$

$$\Rightarrow \left| \sum_{K=1}^{n} f(t_K)\Delta x_K - \int_a^b f \right| < \epsilon$$

Conversely suppose that for every $\epsilon > 0$ there exist a $\delta > 0$ s.t

$$\left| \sum_{K=1}^{n} f(t_K)\Delta x_K - \int_a^b f \right| < \epsilon/4 \quad \text{——} (6)$$

For every choice of $t_K \in [x_{K-1}, x_K]$ & $\|P\| < \delta$

To prove f is R.I

Let $t_K, t_K' \in [x_{K-1}, x_K]$

Then eqn (6) implies $\left| \sum_{K=1}^{n} f(t_K')\Delta x_K - \int_a^b f \right| < \epsilon/4$ —— (7)

Also $M_K - m_K = \operatorname{Sup}\{ f(t_K) - f(t_K') : t_K, t_K' \in [x_{K-1}, x_K]\}$

Now for $\epsilon > 0$ we have

$$M_K - m_K - \frac{\epsilon}{2(b-a)} < f(t_K) - f(t_K')$$

$$\Rightarrow M_K - m_K < f(t_K) - f(t_K') + \frac{\epsilon}{2(b-a)}$$

$$\sum_{K=1}^{n} M_K \Delta x_K - \sum_{K=1}^{n} m_K \Delta x_K < \sum_{K=1}^{n} \left[f(t_K) - f(t_K') + \frac{\epsilon}{2(b-a)} \right] \Delta x_K$$

$$\Rightarrow S(f,P) - s(f,P) < \sum_{K=1}^{n} f(t_K)\Delta x_K - \sum_{K=1}^{n} f(t_K')\Delta x_K + \frac{\epsilon}{2(b-a)} \sum_{K=1}^{n} \Delta x_K$$

$$= \left[\sum_{K=1}^{n} f(t_K)\Delta x_K - \int_a^b f \right] + \left[\int_a^b f - \sum_{K=1}^{n} f(t_K')\Delta x_K \right] + \frac{\epsilon}{2}$$

$$\leq \left| \sum_{K=1}^{n} f(t_K)\Delta x_K - \int_a^b f \right| + \left| \int_a^b f - \sum_{K=1}^{n} f(t_K')\Delta x_K \right| + \frac{\epsilon}{2}$$

$$< \epsilon/4 + \epsilon/4 + \epsilon/2$$

$= \epsilon$ using eqn (6) & (7)

$\Rightarrow S(f,P) - s(f,P) < \epsilon \rightarrow f$ is R.I

Example:- If $f(x) = x^2$ & $g(x) = 1/x$ on $[1,3]$ = kth sub in

$M'_k = \text{Sup of } f = (3)^2 = 9$ Increasing function so Sup is at last point

$M''_k = \text{Sup of } g = 1/1 = 1$ Decreasing function so Sup is at 1st point

$\Rightarrow M'_k + M''_k = 9 + 1 = 10$ —— ①

Let $h = f + g$

$h = x^2 + 1/x$

$M_k = \text{Sup of } h = (3)^2 + \frac{1}{3} = 9 + \frac{1}{3}$ —— ②

from eqn ① & ②

$M_k < M'_k + M''_k$

Remark:- If f, g & $h = f + g$ are functions then in kth sub interval

$M_k \leq M'_k + M''_k$ —— ①

$m_k \geq m'_k + m''_k$ —— ②

Where M'_k, M''_k & M_k are Supremums of f, g & $f+g$. Similarly m'_k, m''_k, m_k are Infimums of f, g & $f+g$.

★ If both functions are decreasing or increasing then in these cases equality hold.

★ ★ ★

Theorem:- If f and g are R.I on $[a,b]$ then so is $f+g$ & $\int_a^b f+g = \int_a^b f + \int_a^b g$

Proof:- Since f & g are R.I on $[a,b]$ so for $\epsilon > 0$ there exist partitions P_1, P_2 of $[a,b]$

s.t $\bar{S}(f,P_1) - s(f,P_1) < \epsilon/2$ ——— ①

& $\bar{S}(g,P_2) - s(g,P_2) < \epsilon/2$ ——— ②

Let M_k', M_k'', M_k be the supremums of f, g & $f+g$ in $[x_{k-1}, x_k]$ similarly
m_k', m_k'' & m_k be the infimums of f, g & $f+g$ in $[x_{k-1}, x_k]$ respectively

Then clearly

$$M_k \le M_k' + M_k'' \quad \text{——— ③}$$

$$\& \quad m_k \ge m_k' + m_k'' \quad \text{——— ④}$$

Let $P = P_1 + P_2$ be the refinement of P_1 & P_2

from eqn ③
$$\sum_{k=1}^{n} M_k \Delta x_k \le \sum_{k=1}^{n} M_k' \Delta x_k + \sum_{k=1}^{n} M_k'' \Delta x_k$$

$$\bar{S}(f+g,P) \le \bar{S}(f,P) + \bar{S}(g,P)$$

$\le \bar{S}(f,P_1) + \bar{S}(g,P_2)$ $\because$ P is refinement of P_1 & P_2

$< s(f,P_1) + \epsilon/2 + s(g,P_2) + \epsilon/2$ using ① & ②

$\le s(f,P) + s(g,P) + \epsilon$

$\le s(f+g,P) + \epsilon$ using eqn ④

$\Rightarrow \bar{S}(f+g,P) < s(f+g,P) + \epsilon$

$\Rightarrow \bar{S}(f+g,P) - s(f+g,P) < \epsilon$

$\Rightarrow f+g$ is R.I

Next we have to show that

$$\int_a^b f+g = \int_a^b f + \int_a^b g$$

As $\int_a^b f+g \le \bar{S}(f+g,P)$

$$\int_a^b f+g \leq S(f,P)+S(g,P)$$
$$\leq S(f,P_1)+S(g,P_2)$$
$$< s(f,P_1)+s(g,P_2)+\epsilon$$
$$\leq s(f,P)+s(g,P)+\epsilon$$
$$< \int_a^b f + \int_a^b g + \epsilon$$

$$\Rightarrow \int_a^b f+g < \int_a^b f + \int_a^b g + \epsilon$$

Since ϵ is arbitrary

$$\therefore \int_a^b f+g \leq \int_a^b f + \int_a^b g \quad \text{--- (5)}$$

Similarly we can prove

$$\int_a^b f+g \geq \int_a^b f + \int_a^b g \quad \text{--- (6)}$$

from eqn (5) & (6) we have

$$\int_a^b f+g = \int_a^b f + \int_a^b g$$

***

Theorem: Let f be R.I on $[a,b]$ if $f(x) \geq 0$ $\forall\, x \in [a,b]$ then $\int_a^b f(x)\,dx \geq 0$

Proof: Since $f(x) \geq 0$, $\forall x \in [a,b]$

$$\Rightarrow M_K \geq 0 \quad \& \quad m_K \geq 0$$

$$\Rightarrow m_K \geq 0$$

$$\Rightarrow m_K \, \Delta x_K \geq 0 \quad \Rightarrow \quad \sum_{K=1}^{n} m_K \Delta x_K \geq 0$$

$$\Rightarrow s(f,P) \geq 0$$

As $\int_a^b f(x)\,dx \geq s(f,P) \geq 0$

$$\Rightarrow \int_a^b f(x)\,dx \geq 0$$

***

Theorem:- If f & g are R.I on $[a,b]$ & $f(x) \leq g(x)$, $\forall x \in [a,b]$ then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

Proof:- Since $f(x) \leq g(x)$ i.e $f \leq g$

$$\Rightarrow g \geq f \quad \Rightarrow g - f \geq 0$$

$$\Rightarrow \int_a^b (g-f) \geq 0 \quad \Rightarrow \int_a^b g - \int_a^b f \geq 0$$

$$\Rightarrow \int_a^b g \geq \int_a^b f \quad \text{or} \quad \int_a^b f \leq \int_a^b g$$

***

Theorem:- If f is R.I on $[a,b]$ then so $|f|$ & $\left|\int_a^b f\right| \leq \int_a^b |f|$

Proof:- Since f is R.I on $[a,b]$ so for $\epsilon > 0$ there exist a partition P of $[a,b]$ s.t

$$S(f,P) - s(f,P) < \epsilon \qquad \text{———} \ (1)$$

Let $M_k = \text{Sup}\{f(x) : x \in [x_{k-1}, x_k]\}$

$m_k = \text{Inf}\{f(x) : x \in [x_{k-1}, x_k]\}$

$M_k' = \text{Sup}\{|f(x)| : x \in [x_{k-1}, x_k]\}$

$m_k' = \text{Inf}\{|f(x)| : x \in [x_{k-1}, x_k]\}$

eg $M_k = 6$, $m_k = -5$
$M_k - m_k = 6-(-5) = 11$
$M_k' = 6$, $m_k' = 5$
$M_k' - m_k' = 6-5 = 1$

Then clearly $M_k' - m_k' \leq M_k - m_k$

$$\Rightarrow \sum_{k=1}^{n} M_k'\Delta x_k - \sum_{k=1}^{n} m_k'\Delta x_k \leq \sum_{k=1}^{n} M_k\Delta x_k - \sum_{k=1}^{n} m_k\Delta x_k$$

$$\Rightarrow S(|f|,P) - s(|f|,P) \leq S(f,P) - s(f,P)$$

$\Rightarrow S(|f|, P) - s(|f|, P) < \epsilon$ using eqn ①

$\Rightarrow |f|$ is R.I on $[a, b]$

Next we prove $\left|\int_a^b f\right| \leq \int_a^b |f|$.

As $f \leq |f|$ $\quad\quad \because f \leq g \Rightarrow \int_a^b f \leq \int_a^b g$

$\Rightarrow \int_a^b f \leq \int_a^b |f|$ ——— ②

Also $-f \leq |f|$

$\Rightarrow -\int_a^b f \leq \int_a^b |f|$ ——— ③

from eqn ② & ③ we have

$\left|\int_a^b f\right| \leq \int_a^b |f|$ $\quad \because$ If $x \leq a$ & $-x \leq a$ $\Rightarrow |x| \leq a$

* * *

Remark:- The converse of the above theorem is not true in general i.e if $|f|$ is R.I on $[a, b]$ then f is may or may not.

Example:- Let $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational on } [0,1] \\ -1 & \text{if } x \text{ is irrational on } [0,1] \end{cases}$

Then $|f|(x) = 1$

$\Rightarrow |f|$ is R.I on $[0,1]$ ($\because$ every constant function is R.I)

But f is not R.I on $[0,1]$ already proved.

* * *

Theorem:- (Mean Value theorem):- If f is continuous on $[a, b]$ & if $g > 0$ is integrable on $[a, b]$ then there is a number c, $a \leq c \leq b$ s.t $\int_a^b f.g = f(c) \int_a^b g$.

Proof:- Since $f(x)$ is continuous on $[a, b]$

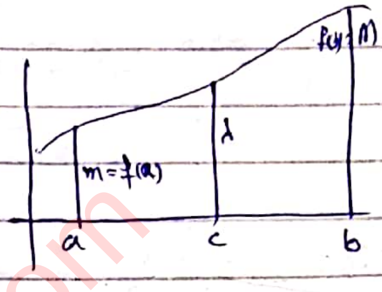
$\therefore m = \inf\{f(x) : x \in [a, b]\}$

$M = \text{Sup}\{f(x) : x \in [a,b]\}$ exists and

$m \leq f(x) \leq M \quad \Rightarrow g(x)m \leq f(x)g(x) \leq g(x)M \ , \forall x \in [a,b]$

$\Rightarrow m\int_a^b g(x)dx \leq \int_a^b f(x)\,g(x)dx \leq M\int_a^b g(x)\,dx$

$$\Rightarrow m \leq \frac{\int_a^b f(x)\,g(x)\,dx}{\int_a^b g(x)\,dx} \leq M$$



$$\text{Let } \lambda = \frac{\int_a^b f(x)\,g(x)\,dx}{\int_a^b g(x)\,dx}$$

$\therefore \ m \leq \lambda \leq M$ ——— ②

Then by intermediate theorem there exist a number c, $a \leq c \leq b$ s.t $f(c) = \lambda$ ($\because f$ is continuous)

$$\Rightarrow f(c) = \frac{\int_a^b f(x)\,g(x)dx}{\int_a^b g(x)\,dx}$$

$$\Rightarrow f(c)\int_a^b g(x)\,dx = \int_a^b f(x)\,g(x)\,dx$$

* * *

Theorem:- If f is R.I on $[a,b]$ then so is f^2

Proof:- Let $M_k = \text{Sup}\{f(x) : x \in [x_{k-1}, x_k]\}$

$m_k = \text{Inf}\{f(x) : x \in [x_{k-1}, x_k]\}$

$M = \text{Sup}\{|f|(x) : x \in [a,b]\}$

Then clearly $M_k + m_k \leq 2M$ $\quad \because M_k \leq M$ & $m_k \leq M$

Then clearly M_k^2, m_k^2 are the supremums & the Infimums of f^2 on $[x_{k-1}, x_k]$

Since f is R.I on $[a,b]$

So for $\epsilon > 0 \ \exists$ a partition P on $[a,b]$

s.t $S(f,P) - s(f,P) < \epsilon/2M$ ——— ①

As $M_k^2 - m_k^2 = (M_k + m_k)(M_k - m_k)$

$\leq 2M(M_k - m_k)$

$\Rightarrow \sum_{k=1}^{n} M_k^2 \Delta x_k - \sum_{k=1}^{n} m_k^2 \Delta x_k \leq 2M\left[\sum_{k=1}^{n} M_k \Delta x_k - \sum_{k=1}^{n} m_k \Delta x_k\right]$

$\Rightarrow S(f^2; P) - s(f^2; P) \leq 2M(S(f, P) - s(f, P))$

$< 2M \cdot \epsilon/2M$ using 1

$\Rightarrow S(f^2; P) - s(f^2; P) < \epsilon$

$\Rightarrow f^2$ is R.I on $[a, b]$

***

Theorem:- If f and g are R.I on $[a,b]$ then
(i) $f-g$ is R.I on $[a,b]$
(ii) $f \cdot g$ is R.I on $[a,b]$

Proof:- i) Since g is R.I on $[a,b]$
$\Rightarrow \alpha g$ is R.I on $[a,b]$, $\forall \alpha \in \mathbb{R}$
In particular $\alpha = -1$
$\Rightarrow -g$ is R.I on $[a,b]$
Also $f - g = f + (-g)$
$\Rightarrow f-g$ is R.I on $[a,b]$ ($\because$ sum of two R.I functions is R.I)

(ii) As $fg = \frac{1}{4}((f+g)^2 - (f-g)^2)$
Clearly R.H.S is R.I on $[a,b]$ because sum, difference & square of R.I functions is R.I.
So fg is R.I on $[a,b]$

***

Theorem:- Let f be R.I on $[a,b]$ & $c \in]a,b[$ then f is R.I on $[a,c]$ and $[c,b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof:- Since f is R.I on $[a,b]$. so for

$\epsilon > 0$ there exist a partition P of $[a,b]$ s.t

$$S(f,P) - s(f,P) < \epsilon \quad \text{——— ①}$$

Let $P_1 = \{a = x_0, x_1, x_2, \ldots, x_{k-1}, c\}$

$P_2 = \{c, x_k, x_{k+1}, \ldots x_n = b\}$ be partitions of $[a,c]$ and $[c,b]$ respectively.

Let $M_c = \text{Sup}\{f(x) : x \in [x_{k-1}, c]\}$

$m_c = \text{Inf}\{f(x) : x \in [x_{k-1}, c]\}$

Now consider

$$S(f,P_1) - s(f,P_1) = [M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_{k-1} \Delta x_{k-1} + M_c(c - x_{k-1})]$$
$$- [m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_{k-1} \Delta x_{k-1} - m_c(c - x_{k-1})]$$

$$S(f,P_1) - s(f,P_1) = \sum_{i=1}^{k-1} (M_i - m_i)\Delta x_i + (M_c - m_c)(c - x_{k-1})$$

$$< \sum_{i=1}^{n} (M_i - m_i)\Delta x_i$$

$$= \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i$$

$$= S(f,P) - s(f,P) < \epsilon \quad \text{using eqn ①}$$

$\Rightarrow S(f,P_1) - s(f,P_1) < \epsilon$

$\Rightarrow f$ is R.I on $[a,c]$

Similarly we can prove f is R.I on $[c,b]$

Next we prove $\int_a^b f = \int_a^c f + \int_c^b f$

As $\int_a^c f \le S(f,P_1)$ &

$\int_c^b f \le S(f,P_2)$

$$\Rightarrow \int_a^c f + \int_c^b f \le S(f,P_1) + S(f,P_2) \quad \text{——— ②}$$

As f is R.I on $[a,b] \Rightarrow S(f,P) < \int_a^b f + \epsilon/2$ ——— ③

& $s(f,P) > \int_a^b f - \epsilon/2$ ——— ④

Also $S(f,P_1) + S(f,P_2) = S(f, P_1 \cup P_2)$

$S(f,P_1) + S(f,P_2) = S(f,P)$ $\because P = P_1 \cup P_2$ is a partition of $[a,b]$

$< \int_a^b f + \epsilon/2$ using ③

$\Rightarrow S(f,P_1) + S(f,P_2) < \int_a^b f + \epsilon/2$ —— ⑤

Combining eqn ② & ⑤ we obtain

$$\int_a^c f + \int_c^b f < \int_a^b f + \epsilon/2$$

Since ϵ is arbitrary $\Rightarrow \int_a^c f + \int_c^b f \leq \int_a^b f$ —— ⑥

Similarly considering Lower R. sums we obtain $\int_a^c f + \int_c^b f \geq \int_a^b f$ —— ⑦

Combining eqn ⑥ & ⑦ we have

$$\int_a^c f + \int_c^b f = \int_a^b f$$

* * *

$\Rightarrow$ **Holder Inequality:-** If f and g are R.I on $[a,b]$ then $\int_a^b fg \leq \left(\int_a^b f^2\right)^{1/2} \left(\int_a^b g^2\right)^{1/2}$

Proof:- Since f & g are R.I on $[a,b]$ so f^2, g^2 and fg are also R.I on $[a,b]$

consider $(f - \lambda g)^2 \geq 0$

$$f^2 + \lambda^2 g^2 - 2\lambda fg \geq 0$$

$$\Rightarrow \int_a^b f^2 + \lambda^2 \int_a^b g^2 - 2\lambda \int_a^b fg \geq 0$$

Let $\lambda = \int_a^b fg \Big/ \int_a^b g^2$

$$\Rightarrow \int_a^b f^2 + \frac{\left(\int_a^b fg\right)^2}{\left(\int_a^b g^2\right)^{\cancel{2}}} \cancel{\left(\int_a^b g^2\right)} - 2\,\frac{\int_a^b fg}{\int_a^b g^2} \int_a^b fg \geq 0$$

$$\Rightarrow \left(\int_a^b f^2\right)\left(\int_a^b g^2\right) + \left(\int_a^b fg\right)^2 - 2\left(\int_a^b fg\right)^2 \geq 0$$

$$\Rightarrow \left(\int_a^b f^2\right)\left(\int_a^b g^2\right) - \left(\int_a^b fg\right)^2 \geq 0$$

$$\Rightarrow \left(\int_a^b f^2\right)\left(\int_a^b g^2\right) \geq \left(\int_a^b fg\right)^2$$

$$\Rightarrow \int_a^b fg \leq \left(\int_a^b f^2\right)^{1/2}\left(\int_a^b g^2\right)^{1/2}$$

***

⇒ Minkowski's Inequality:- If f and g are R.I on $[a,b]$ then

$$\left[\int_a^b (f+g)^2\right]^{1/2} \leq \left(\int_a^b f^2\right)^{1/2} + \left(\int_a^b g^2\right)^{1/2}$$

concept

$\boxed{|a+b| \leq |a| + |b|}$

Proof:- Since f & g are R.I on $[a,b]$

so f^2, g^2 & $(f+g)^2$ are also R.I on $[a,b]$

Consider $(f+g)^2 = f^2 + g^2 + 2fg$

$$\int_a^b (f+g)^2 = \int_a^b f^2 + \int_a^b g^2 + 2\int_a^b fg$$

$$\leq \int_a^b f^2 + \int_a^b g^2 + 2\left(\int_a^b f^2\right)^{1/2}\left(\int_a^b g^2\right)^{1/2}$$

$$= \left[\left(\int_a^b f^2\right)^{1/2}\right]^2 + \left[\left(\int_a^b g^2\right)^{1/2}\right]^2 + 2\left(\int_a^b f^2\right)^{1/2}\left(\int_a^b g^2\right)^{1/2}$$

$$\Rightarrow \int_a^b (f+g)^2 \leq \left[\left(\int_a^b f^2\right)^{1/2} + \left(\int_a^b g^2\right)^{1/2}\right]^2$$

$$\Rightarrow \left[\int_a^b (f+g)^2\right]^{1/2} \leq \left(\int_a^b f^2\right)^{1/2} + \left(\int_a^b g^2\right)^{1/2}$$

***

R.I $\leftarrow$ Contin $\leftarrow$ Diff

113

1st Fundamental Theorem of Calculus:-

Suppose f is R.I on $[a,b]$ and $F(x) = \int_a^x f(x)\,dx$ then $F(x)$ is continuous on $[a,b]$. Further if f is continuous on $[a,b]$ then F is differentiable on $[a,b]$ and $F'(x) = f(x)$, $\forall x \in [a,b]$

Proof:- Since f is R.I on $[a,b]$

$\Rightarrow f$ is bounded on $[a,b]$

$\Rightarrow |f(x)| < \lambda$ ——— ① $\forall x \in [a,b]$

To prove F is continuous on $[a,b]$

Let $x_0 \in [a,b]$

Consider

$$F(x) - F(x_0) = \int_a^x f(x)\,dx - \int_a^{x_0} f(x)\,dx$$

$$= \int_a^x f(x)\,dx + \int_{x_0}^a f(x)\,dx$$

$$= \int_{x_0}^x f(x)\,dx$$

$$|F(x) - F(x_0)| = \left|\int_{x_0}^x f(x)\,dx\right|$$

$$\leq \int_{x_0}^x |f(x)|\,|dx| \quad \because \left|\int f\right| \leq \int |f|$$

$$< \int_{x_0}^x \lambda\,|dx| \quad \text{using ①}$$

$$= \lambda\,|x - x_0|$$

$\Rightarrow |F(x) - F(x_0)| < \lambda\,|x - x_0|$

For $\epsilon > 0$ we have choose $\delta = \epsilon/\lambda$ s.t

$|F(x) - F(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$

$\Rightarrow$ F is continuous on x_0.
Since x_0 is orbitrary therefore F is continuous for all $x \in [a,b]$

Now suppose f is continuous on $[a,b]$
To prove F is differentiable on $[a,b]$ &
$F'(x) = f(x)$
Let $x_0 \in]a,b[$

Consider $F(x) - F(x_0) = \int_{x_0}^{x} f(x)\,dx$

$$\Rightarrow \frac{F(x)-F(x_0)}{x-x_0} = \frac{1}{x-x_0}\int_{x_0}^{x} f(x)\,dx$$

$$\Rightarrow \frac{F(x)-F(x_0)}{x-x_0} - f(x_0) = \frac{1}{x-x_0}\left(\int_{x_0}^{x} f(x)\,dx\right) - f(x_0)$$

$$= \left[\frac{1}{x-x_0}\int_{x_0}^{x} f(x)\,dx\right] - \frac{f(x_0)}{x-x_0}\int_{x_0}^{x} dx \quad \boxed{\int_{x_0}^{x} dx = x-x_0}$$

$$\frac{F(x)-F(x_0)}{x-x_0} - f(x_0) = \frac{1}{x-x_0}\int_{x_0}^{x}\left(f(x)-f(x_0)\right)dx \quad \because f(x_0) \text{ is constant}$$

$$\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| = \left|\frac{1}{x-x_0}\int_{x_0}^{x}\left(f(x)-f(x_0)\right)dx\right|$$

$$\leq \frac{1}{|x-x_0|}\int_{x_0}^{x}|f(x)-f(x_0)|\,|dx| \quad \because \left|\int f\right| \leq \int |f|$$

$$< \frac{1}{|x-x_0|}\int_{x_0}^{x} \epsilon\,|dx| \quad \text{whenever } |x-x_0|<\delta$$

$$= \epsilon$$

$$\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| < \epsilon \quad \text{whenevere } |x-x_0|<\delta$$

$$\Rightarrow F'(x_0) = f(x_0)$$

Since x_0 is orbitrary $\Rightarrow F'(x) = f(x)\ \forall x \in]a,b[$

***

Remark:- The function $F(x) = \int_a^x f(x)\,dx$ is called the anti-derivative or Primitive of f on $[a,b]$

***

Theorem:- Suppose F is a primitive of a continuous function f on $[a,b]$. A function G defined on $[a,b]$ is also a premitive of f on $[a,b]$ if & only if for some constant c, $G(x) = F(x) + c$.

Proof: Since F and G are premitives of f on $[a,b]$ $\Rightarrow$ $F'(x) = f(x)$ ——— ①

And $G'(x) = f(x)$ ——— ②

Subtracting equ ① from ②

$$\Leftrightarrow G'(x) - F'(x) = 0$$

$$\Leftrightarrow \frac{d}{dx}[G(x) - F(x)] = 0$$

$$\Leftrightarrow G(x) - F(x) = c \quad \text{for some constant "c"}$$

$$\Leftrightarrow G(x) = F(x) + c$$

***

$\Rightarrow$ 2nd Fundamental Theorem of Calculas:-

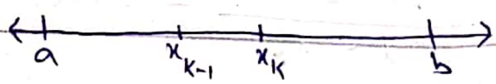
If f is R.I on $[a,b]$ and F is anti-derivative of f on $[a,b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof: Given $F'(x) = f(x)$

$\Rightarrow F$ is diff on $]a,b[$

$\Rightarrow F$ is diff on $]x_{k-1}, x_k[$



$\Rightarrow F$ is continuous on $[x_{k-1}, x_k]$

Then by M.V.T there exist a point $t_k \in]x_{k-1}, x_k[$ s.t

$$\frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = F'(t_k)$$

$$\Rightarrow F(x_k) - F(x_{k-1}) = f(t_k)\,\Delta x_k \quad \left\{ \begin{array}{l} \because F'(t_k) = f(t_k) \\ \text{\& } x_k - x_{k-1} = \Delta x_k \end{array} \right.$$

$$\Rightarrow \sum_{k=1}^{n} [F(x_k) - F(x_{k-1})] = \sum_{k=1}^{n} f(t_k)\Delta x_k$$

$$\Rightarrow F(b) - F(a) = \sum_{k=1}^{n} f(t_k)\Delta x_k \quad \text{——— ①}$$

Given f is R.I on $[a,b]$ so for every $\epsilon > 0$ there exist a $\delta > 0$ s.t

$$\left| \int_a^b f(x)\,dx - \sum_{k=1}^{n} f(t_k)\Delta x_k \right| < \epsilon \quad \text{——— ②}$$

$\forall$ partition P of $[a,b]$ s.t $\|P\| < \delta$ and for every choice of point t_k in $]x_{k-1}, x_k[$

Using equ ① & ② we have

$$\left| \int_a^b f(x)\,dx - [F(b) - F(a)] \right| < \epsilon$$

Since this result is true for every ϵ. So $\left| \int_a^b f(x)\,dx - [F(b) - F(a)] \right| = 0$

$$\Rightarrow \int_a^b f(x)\,dx - [F(b) - F(a)] = 0$$

$$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$$

* * *

$\Rightarrow$ **Change of Variable :-** Suppose that

(i) g has continuous derivative on $[c,d]$

(ii) f is continuous in a interval containing $g[c,d]$ Then $\int_c^d f(g(t))\,g'(t)\,dt = \int_{g(c)}^{g(d)} f(x)\,dx$

$\boxed{\begin{array}{l} g[c,d] \\ \text{mean} \\ g(c)\ \&\ g(d) \end{array}}$

Proof Since g is continuous on $[c,d]$ and $[c,d]$ is closed. So $g[c,d]$ is closed

Let $g[c,d] = [a,b]$

Also we know that the composition and product of continuous functions is continuous

Let $F(y) = \int_a^y f(u)\,du$, $a \le y \le b$

& $G(x) = \int_c^x f(g(t))g'(t)\,dt$ $c \le x \le d$ —— ⊛

Domain of F is $[a,b]$ & of G is $[c,d]$

Also let $H(x) = F(g(x))$

Then $H'(x) = F'(g(x))\,g'(x)$

$H'(x) = f(g(x))\,g'(x)$ —— ①

From ⊛ $G'(x) = f(g(x))\,g'(x)$ ∵ 1st Fundamental —— ②

From equ ① & ② we have

$H'(x) - G'(x) = 0$

$\Rightarrow \frac{d}{dx}[H(x) - G(x)] = 0$

$\Rightarrow H(x) - G(x) = k$ some constant k

$\Rightarrow H(x) = G(x) + k$ —— ③

Consider $H(d) - H(c) = G(d) + k - [G(c) + k]$

$H(d) - H(c) = G(d) - G(c)$

$F(g(d)) - F(g(c)) = G(d) - G(c)$ ∵ $H(x) = F(g(x))$

$\int_a^{g(d)} f(u)\,du - \int_a^{g(c)} f(u)\,du = \int_c^d f(g(t))g'(t)\,dt$ — 0

$\Rightarrow \int_{g(c)}^{a} f(u)\,du + \int_a^{g(d)} f(u)\,du = \int_c^d f(g(t))\,g'(t)\,dt$

$$\Rightarrow \int_{g(c)}^{g(d)} f(u)\,du = \int_c^d f(g(t))\,g'(t)\,dt$$

$$\Rightarrow \int_{g(c)}^{g(d)} f(x)\,dx = \int_c^d f(g(t))\,g'(t)\,dt$$

***

Theorem:- Prove that $\log e = 1$

Proof: As $\log x = \int_1^x \frac{1}{x}\,dx$ ——— ①

$$\boxed{\begin{aligned}\log x &= \int_1^x \tfrac{1}{x}\,dx\\ &= [\log x]_1^x\\ &= \log x - \log 1\\ &= \log x\end{aligned}}$$

By M.V.T of integration on the interval $\left[\frac{1}{n+1}, \frac{1}{n}\right]$, we have $c_n \in \left]\frac{1}{n+1}, \frac{1}{n}\right[$ s.t

$$\int_{\frac{1}{n+1}}^{\frac{1}{n}} \log x\,dx = \log c_n \int_{\frac{1}{n+1}}^{\frac{1}{n}} dx \qquad \because \int_a^b f(x)g(x)\,dx = g(c)\int_a^b f(x)\,dx$$

$$\int_{\frac{1}{n+1}}^{\frac{1}{n}} \log x\,dx = (\log c_n)\left[\frac{1}{n} - \frac{1}{n+1}\right] \qquad \because \int_{\frac{1}{n+1}}^{\frac{1}{n}} dx = [x]_{\frac{1}{n+1}}^{\frac{1}{n}} \Rightarrow \left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$\int_{\frac{1}{n+1}}^{\frac{1}{n}} \log x\,dx = \frac{1}{n(n+1)} \log c_n \text{ ——— ②}$$

Again consider $\int_{\frac{1}{n+1}}^{\frac{1}{n}} \log x\,dx = [\log x \cdot x]_{\frac{1}{n+1}}^{\frac{1}{n}} - \int_{\frac{1}{n+1}}^{\frac{1}{n}} x \cdot \frac{1}{x}\,dx$

$$\int_{\frac{1}{n+1}}^{\frac{1}{n}} \log x\,dx = \left[\frac{1}{n}\log\frac{1}{n}\right] - \left[\frac{1}{n+1}\log\frac{1}{n+1}\right] - \left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$= \frac{1}{n(n+1)}\left[(n+1)\log\frac{1}{n} - n\log\frac{1}{n+1} - 1\right] \qquad \because \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$$

$$= \frac{1}{n(n+1)}\left[\left(\log\frac{1}{n}\right)^{n+1} - \left(\log\frac{1}{n+1}\right)^{n} - 1\right]$$

$$= \frac{1}{n(n+1)}\left[\log\frac{(1/n)^{n+1}}{\left(\frac{1}{n+1}\right)^{n}} - 1\right]$$

$$\int_{\frac{1}{n+1}}^{1/n} \log x\,dx = \frac{1}{n(n+1)}\left[\log \frac{(n+1)^n}{n^{n+1}} - 1\right] \quad \text{——} ③$$

From equ ② & ③

$$\frac{1}{\cancel{n(n+1)}}\left[\log \frac{(n+1)^n}{n^{n+1}} - 1\right] = \frac{1}{\cancel{n(n+1)}} \log C_n$$

$$\Rightarrow \log \frac{(n+1)^n}{n^{n+1}} - \log C_n = 1$$

$$\Rightarrow \log \frac{(n+1)^n}{C_n . n^{n+1}} = 1$$

$$\Rightarrow \log x = 1 \quad \text{——} ④ \qquad \text{where } x = \frac{(n+1)^n}{C_n . n^{n+1}}$$

$$x = \frac{(n+1)^n}{nC_n . n^n} = \frac{1}{nC_n}\left[1+\frac{1}{n}\right]^n$$

$$\boxed{\frac{(n+1)^n}{n^n} = \left[1+\frac{1}{n}\right]^n = \left[\frac{n+1}{n}\right]^n}$$

As $\quad C_n \in \left]\frac{1}{n+1}, \frac{1}{n}\right[$

$$\Rightarrow \frac{1}{n+1} < C_n < \frac{1}{n}$$

$$\Rightarrow n+1 > \frac{1}{C_n} > n$$

$$\Rightarrow \frac{n+1}{n} > \frac{1}{nC_n} > 1$$

when $n \to \infty$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right) > \lim_{n\to\infty} \frac{1}{nC_n} > 1$$

$$1 > \lim_{n\to\infty} \frac{1}{nC_n} > 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{nC_n} = 1 \qquad \text{By Sandwitch theorem}$$

Appling limit on x

$$\lim_{n\to\infty} x = \lim_{n\to\infty} \frac{1}{nC_n} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$x = 1.e = e$ Put in (iv)

$\log e = 1$

* * *

Theorem:- Let f be R.I over $[a,b]$. If $P_1, P_2 \in P[a,b]$ such that $P_1 \subset P_2$, where P_2 contains l extra points then prove that

$S(f,P_1) \leq S(f,P_2) + 2l\lambda\delta$ and

$s(f,P_2) \leq s(f,P_1) + 2l\lambda\delta$

where $\|P_1\| < \delta$ and $|f(x)| < \lambda$

Proof) Since f is Reimann Integrable over $[a,b]$

$\Rightarrow f$ is bounded on $[a,b]$

$\Rightarrow |f(x)| < \lambda$ —① λ is the constant

Let $P_1 = \{a = x_0, x_1, x_2, \dots x_{k-1}, x_k, \dots x_n = b\}$

and $P_2 = \{a = x_0, x_1, \dots, x_{k-1}, \xi_1, \xi_2, \dots \xi_l, x_k, \dots x_n = b\}$

Also let,

$P_2' = P_1 \cup \{\xi_1\}$

$P_2'' = P_2' \cup \{\xi_2\}$

$P_2''' = P_2'' \cup \{\xi_3\}$

$\vdots$

$P_2^{l} = P_2^{l-1} \cup \{\xi_l\} = P_2$

let $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$

$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$

$M_k' = \sup\{f(x) : x \in [x_{k-1}, \xi_1]\}$

$M_k'' = \sup\{f(x) : x \in [\xi_1, x_k]\}$

Then clearly $M'_K \leq M_K$

$M''_K \leq M_K$

Now consider

$$S(f,P_1) - S(f,P'_1) = M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots M_K \Delta x_K + \cdots + M_n \Delta x_n$$
$$-(M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_{K-1} \Delta x_{K-1} +$$
$$M'_K(\xi_1 - x_{K-1}) + M''_K(x_K - \xi_1) + \cdots + M_n \Delta x_n$$
$$= M_K \Delta x_K - M'_K(\xi_1 - x_{K-1}) - M''_K(x_K - \xi_1)$$
$$= M_K(x_K - x_{K-1}) - M'_K \xi_1 + M'_K x_{K-1} -$$
$$M''_K x_K + M'' \xi_1$$
$$= M_K x_K - M_K x_{K-1} - M'_K \xi_1 + M'_K x_{K-1} - M''_K x_K$$
$$+ M''_K \xi_1 + M_K \xi_1 - M_K \xi_1$$
$$= M_K(x_K - \xi_1) + M_K(\xi_1 - x_{K-1})$$
$$- M'_K(\xi_1 - x_{K-1}) - M''_K(x_K - \xi_1)$$

$$S(f,P_1) - S(f,P'_1) = (M_K - M''_K)(x_K - \xi_1) + (M_K - M'_K)(\xi_1 - x_{K-1}) \quad \text{--- (2)}$$

From equ (1) $|f(x)| < \lambda \Rightarrow -\lambda < f(x) < \lambda \quad \forall x \in [a,b]$

$\Rightarrow -\lambda < M_K < \lambda$

and $-\lambda < M'_K < \lambda$

All heights lies b/w λ & $-\lambda$
So M_K, M'_K also lies b/w λ & $-\lambda$

$\Rightarrow M_K < \lambda$ & $M'_K > -\lambda \Rightarrow -M'_K < \lambda$

$\Rightarrow M_K - M'_K < 2\lambda$ similarly $M_K - M''_K < 2\lambda$

Put in equ (2)

$$S(f,P_1) - S(f,P'_2) < 2\lambda(x_K - \xi_1) + 2\lambda(\xi_1 - x_{K-1})$$
$$= 2\lambda\{x_K - x_{K-1}\}$$
$$= 2\lambda \Delta x_K$$
$$\leq 2\lambda \|P_1\|$$

$\bar{S}(f,P_1) - \bar{S}(f,P_2') < 2\lambda\delta \qquad \because \|P_1\| < \delta$

$\Rightarrow \bar{S}(f,P_1) < \bar{S}(f,P_2') + 2\lambda\delta$

As $P_1 \subset P_2'$, $P_2' \subset P_2''$, $P_2'' \subset P_2'''$

Similarly repeating the process on partitions

$\{P_2', P_2''\}, \{P_2'', P_2'''\}, \dots, \{P_2^{l-1}, P_2^{l}\}$

We have

$\bar{S}(f,P_2') < \bar{S}(f,P_2'') + 2\lambda\delta$

$\bar{S}(f,P_2'') < \bar{S}(f,P_2''') + 2\lambda\delta$

$\vdots$

$\bar{S}(f,P_2^{l-1}) < \bar{S}(f,P_2^{l}) + 2\lambda\delta$

$\bar{S}(f,P_1) < \bar{S}(f,P_2) + 2\lambda\delta + 2\lambda\delta + 2\lambda\delta + \dots + 2\lambda\delta$. l factors

$\Rightarrow \bar{S}(f,P_1) < \bar{S}(f,P_2) + 2l\lambda\delta$

Similarly $\underline{s(f,P_2) < s(f,P_1) + 2l\lambda\delta}$ ***

$\Rightarrow$ **Darboux Theorem:-** If f is R.I on $[a,b]$ then for $\varepsilon > 0$ there exist a $\delta > 0$ such that

(i) $\bar{S}(f,P) < \int_a^{\bar{b}} f + \varepsilon$ (ii) $s(f,P) > \int_{\underline{a}}^{b} f - \varepsilon$

$\forall$ partition P of $[a,b]$ s.t $\|P\| < \delta$

Proof) To prove this theorem first we have to prove the previous theorem then Darboux theorem.

Since $f(x)$ is R.I on $[a,b]$ then f is bounded on $[a,b]$

$\Rightarrow |f(x)| < \lambda$

As $\int_a^{\bar{b}} f = \operatorname{Inf}\{\bar{S}(f,P) : P \in P(a,b)\}$

& $\int_{\underline{a}}^{b} f = \operatorname{Sup}\{s(f,P) : P \in P(a,b)\}$

then for $\epsilon > 0$ there exist $P_1, P_2 \in P(a,b)$

$$\overline{S}(f,P_1) < \overline{\int_a^b} f + \epsilon/2 \quad \text{——— (1)}$$

$$s(f,P_2) > \underline{\int_a^b} f - \epsilon/2 \quad \text{——— (2)}$$

$2\lambda l \delta_1 = \epsilon/2$, Let $P \in P(a,b)$ s.t

Let $\delta_1 > 0$ be such that $\|P\| < \delta_1$. Then if the partition $P \cup P_1$ contains l additional points then by previous theorem

$$\overline{S}(f,P) < \overline{S}(f, P\cup P_1) + 2\lambda l \delta_1$$

$$\leq \overline{S}(f,P_1) + 2\lambda l \delta_1$$

$$< \overline{\int_a^b} f + \epsilon/2 + \epsilon/2$$

$$= \overline{\int_a^b} f + \epsilon$$

$$\overline{S}(f,P) < \overline{\int_a^b} f + \epsilon$$

Similarly $s(f,P) > \underline{\int_a^b} f - \epsilon$

* * *

Question:- Show that $f(x) = \frac{1}{1+x}$, $x \in [0,1]$ is R.I on $[0,1]$

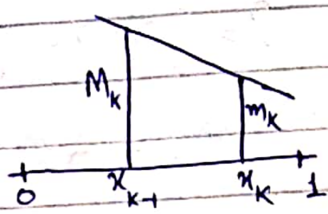
Solution As $f(x) = \frac{1}{1+x} = (1+x)^{-1}$

$$f'(x) = \frac{-1}{(1+x)^2} < 0 \quad \text{on } [0,1]$$

$\Rightarrow f(x)$ is decreasing on $[0,1]$
consider the partition P of $[0,1]$ with length of each sub-interval

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$P = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{k-1}{n}, \frac{k}{n}, \ldots \frac{n}{n} = 1\}$



The maximum and minimum heights of kth sub-interval will be at the point k-1 and k because function is decreasing.

$$M_k = f(x_{k-1}) = f\left(\frac{k-1}{n}\right) = \frac{1}{1+\frac{k-1}{n}} = \frac{n}{n+k-1}$$

$$m_k = f(x_k) = f\left(\frac{k}{n}\right) = \frac{1}{1+k/n} = \frac{n}{n+k}$$

Consider $S(f,P) - s(f,P) = \sum_{k=1}^{n} M_k \Delta x_k - \sum_{k=1}^{n} m_k \Delta x_k$

$$= \sum_{k=1}^{n} (M_k - m_k)\Delta x_k$$

$$= \sum_{k=1}^{n} \left[\frac{n}{n+k-1} - \frac{n}{n+k}\right] \times \frac{1}{n}$$

$$= \frac{n}{n} \sum_{k=1}^{n} \left[\frac{1}{n+k-1} - \frac{1}{n+k}\right]$$

$$= \left[\frac{1}{n} - \cancel{\frac{1}{n+1}}\right] + \left[\cancel{\frac{1}{n+1}} - \cancel{\frac{1}{n+2}}\right] + \left[\cancel{\cdot} \cancel{\cdot}\right] + \cdots$$

$$+ \left[\cancel{\frac{1}{2n-1}} - \frac{1}{2n}\right]$$

$$= \frac{1}{n} - \frac{1}{2n} = \frac{2-1}{2n} = \frac{1}{2n}$$

$S(f,P) - s(f,P) = \frac{1}{2n} < \epsilon \quad$ for $n > \frac{1}{2\epsilon}$

$\Rightarrow S(f,P) - s(f,P) < \epsilon \Rightarrow f$ is R.I on $[0,1]$

* * *

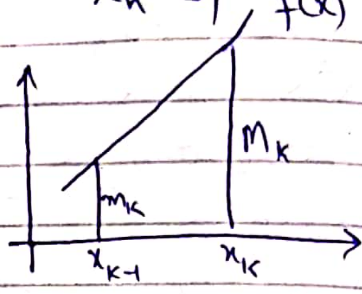
Question:- Show that $f(x) = e^x$ is R.I on $[a,b]$

Solution $f(x) = e^x \Rightarrow f'(x) = e^x > 0$

$f(x)$ is increasing on $[a,b]$

Consider the partition P of $[a,b]$,

$$P = \{a = x_0, x_1, x_2, \dots x_{k-1}, x_k, \dots x_n = b\}$$



Now $m_k = f(x_{k-1}) = e^{x_{k-1}}$

$M_k = f(x_k) = e^{x_k}$

Consider

$$S(f,P) - s(f,P) = \sum_{k=1}^{n} M_k \Delta x_k - \sum_{k=1}^{n} m_k \Delta x_k$$

$$= \sum_{k=1}^{n} [M_k - m_k] \Delta x_k$$

$$= \sum_{k=1}^{n} [e^{x_k} - e^{x_{k-1}}] \Delta x_k$$

$$= \sum_{k=1}^{n} [e^{x_k} - e^{x_{k-1}}] \frac{b-a}{n}$$

Length of each interval is $\Delta x = \frac{b-a}{n}$

So $$S(f,P) - s(f,P) = \sum_{k=1}^{n} [f(b) - f(a)] \frac{b-a}{n}$$

$$= \frac{b-a}{n} [f(b) - f(a)]$$

$$= \frac{b-a}{n} [e^b - e^a]$$

$$< \epsilon$$

$S(f,P) - s(f,P) < \epsilon$ for $n > \frac{(b-a)}{\epsilon}[e^b - e^a]$

$\Rightarrow f$ is R.I on $[a,b]$

***

CHAPTER NO. 4

REIMANN STEILJES INTEGRAL.

★★★★

★ Definition:- Let $f \in B(a,b)$ & $P \in P(a,b)$ where
$P = \{a = x_0, x_1, x_2, \dots, x_{k-1}, x_k, \dots, x_n = b\}$
Suppose that α is a function defined and monotonically increasing on $[a,b]$.
Denote $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$ Then
$\bar{S}(f,P,\alpha) = \sum_{k=1}^{n} M_k \Delta\alpha_k$ & $s(f,P,\alpha) = \sum_{k=1}^{n} m_k \Delta\alpha_k$
where

$$M_k = \text{Sup}\{f(x) : x \in [x_{k-1}, x_k]\}$$
$$m_k = \text{Inf}\{f(x) : x \in [x_{k-1}, x_k]\}$$

are called upper and lower Reimann Stieljes ~~Integral~~. sums for "f" corresponding to the partition P.

★ Definition:- $\bar{\int_a^b} f\,d\alpha = \text{Inf}\{\bar{S}(f,P,\alpha) : P \in P(a,b)\}$

$$\underline{\int_a^b} f\,d\alpha = \text{Sup}\{s(f,P,\alpha) : P \in P(a,b)\}$$

are called Upper and Lower Reimann Steiljes Integral. f is said to be Reimann Steiljes Integrable w.r.t α if

$$\bar{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$

and common value is denoted by $\int_a^b f\,d\alpha$

★ Changes:- $\Delta x_k \to \Delta\alpha_k$, $\sum_{k=1}^{n} \Delta\alpha_k = \alpha(b) - \alpha(a)$

★ Remark:- If $\alpha(x) = x$ then $\Delta\alpha_k = \Delta x_k$
$\Rightarrow \bar{S}(f,P,\alpha) = \sum_{k=1}^{n} M_k \Delta\alpha_k = \sum_{k=1}^{n} M_k \Delta x_k = \bar{S}(f,P)$
Similarly

$$s(f,P,\alpha) = s(f,P)$$

Then Reimann Steiljes Integral becomes R.I.

⇒ Reiman Integral is the special case of Reimann Steiljes Integral.

⇒ Reimann Steiljes integral is stranger critaria than Reimann integral.

* * *

⇒ **Lemma Theorem:-** If $f \in B(a,b)$ and α is monotonically increasing on $[a,b]$ and if $P_1, P_2 \in P(a,b)$ such that $P_1 \subseteq P_2$ then

$$m(\alpha(b)-\alpha(a)) \leq s(f,P_1,\alpha) \leq s(f,P_2,\alpha) \leq S(f,P_2,\alpha) \leq S(f,P_1,\alpha) \leq M(\alpha(b)-\alpha(a))$$

Proof) Suppose $P_1 = \{a = x_0, x_1, x_2, \dots, x_{k-1}, x_k, \dots x_n = b\}$

Let

$$m_k = \text{Inf}\{f(x) : x \in [x_{k-1}, x_k]\}$$

$$M_k = \text{Sup}\{f(x) : x \in [x_{k-1}, x_k]\}$$

$$m = \text{Inf}\{f(x) : x \in [a,b]\}$$

$$M = \text{Sup}\{f(x) : x \in [a,b]\}$$

Then $m \leq m_k \leq M_k \leq M$

$$\Rightarrow m\Delta\alpha_k \leq m_k\Delta\alpha_k \leq M_k\Delta\alpha_k \leq M\Delta\alpha_k$$

$$\Rightarrow m\sum_{k=1}^{n}\Delta\alpha_k \leq \sum_{k=1}^{n} m_k\Delta\alpha_k \leq \sum_{k=1}^{n} M_k\Delta\alpha_k \leq \sum_{k=1}^{n} M\Delta\alpha_k$$

$$\Rightarrow m[\alpha(b)-\alpha(a)] \leq s(f,P_1,\alpha) \leq S(f,P_1,\alpha) \leq M[\alpha(b)-\alpha(a)] \quad \text{---} ①$$

Consider the Partition P_2

$$P_2 = \{a = x_0, x_1, x_2, \dots, x_{k-1}, x', x_k, \dots x_n = b\}$$

Clearly $P_1 \subset P_2$. i.e P_2 is finer than P_1

Now denote

$$m'_k = \text{Inf}\{f(x) : x \in [x_{k-1}, x']\}$$

$$m''_k = \text{Inf}\{f(x) : x \in [x', x_k]\}$$

$$M'_k = \text{Sup}\{f(x) : x \in [x_{k-1}, x']\}$$

$$M''_k = \text{Sup}\{f(x) : x \in [x', x_k]\}$$

Now consider

$$S(f,P_1,\alpha)' - S(f,P_2,\alpha) = \{M_1\Delta\alpha_1 + M_2\Delta\alpha_2 + \dots + M_K\Delta\alpha_K + \dots + M_n\Delta\alpha_n\}$$
$$- \{M_1\Delta\alpha_1 + M_2\Delta\alpha_2 + \dots + M_K'(\alpha(x') - \alpha(x_{k-1}))$$
$$+ M_K''(\alpha(x_k) - \alpha(x')) + \dots + M_n\Delta\alpha_n\}$$
$$= M_K\Delta\alpha_K - M_K'(\alpha(x') - \alpha(x_{k-1})) - M_K''(\alpha(x_k) - \alpha(x'))$$
$$= M_K(\alpha(x_k) - \alpha(x_{k-1})) - M_K'(\alpha(x') - \alpha(x_{k-1}))$$
$$- M_K''(\alpha(x_k) - \alpha(x'))$$
$$= M_K\alpha(x_k) - M_K\alpha(x_{k-1}) - M_K'\alpha(x') + M_K'\alpha(x_{k-1})$$
$$- M_K''\alpha(x_k) + M_K''\alpha(x') + M_K\alpha(x') - M_K\alpha(x')$$
$$= M_K[\alpha(x_k) - \alpha(x')] + M_K[\alpha(x') - \alpha(x_{k-1})]$$
$$- M_K'[\alpha(x_k') - \alpha(x_{k-1})] - M_K''[\alpha(x_k) - \alpha(x')]$$
$$S(f,P_1,\alpha) - S(f,P_2,\alpha) = (M_K - M_K'')(\alpha(x_k) - \alpha(x')) + (M_K - M_K')(\alpha(x') - \alpha(x_{k-1})) \quad \text{——— ②}$$

As $m_K \leq m_K'$ & $m_K \leq m_K''$

$M_K' \leq M_K \Rightarrow M_K' - M_K \leq 0 \Rightarrow M_K - M_K' \geq 0$

also $M_K'' \leq M_K \Rightarrow M_K'' - M_K \leq 0 \Rightarrow M_K - M_K'' \geq 0$

So equ ② implies $S(f,P_1,\alpha) - S(f,P_2,\alpha) \geq 0$

$$\Rightarrow S(f,P_1,\alpha) \geq S(f,P_2,\alpha)$$
$$\Rightarrow S(f,P_2,\alpha) \leq S(f,P_1,\alpha) \quad \text{——— ③}$$

Similarly we can prove

$$s(f,P_1,\alpha) \leq s(f,P_2,\alpha) \quad \text{——— ④}$$

By equ ③ & ④ $s(f,P_1,\alpha) \leq s(f,P_2,\alpha) \leq S(f,P_2,\alpha) \leq S(f,P_2,\alpha)$ ——— ⑤

By equ ① & ⑤

$$m[\alpha(b) - \alpha(a)] \leq s(f,P_1,\alpha) \leq s(f,P_2,\alpha) \leq S(f,P_2,\alpha) \leq S(f,P_2,\alpha)$$
$$\leq M[\alpha(b) - \alpha(a)]$$

——— ★ ★ ★ ———

⇒ **Cauchy's Theorem:-** Let $f \in B(a,b)$ and α be increasing on $[a,b]$ then f is $R S \alpha$ w.r.t α i.e $f \in R S \alpha$ on $[a,b] \Leftrightarrow$ for $\epsilon > 0$ there exist a partition P of $[a,b]$ such that

$$\bar{S}(f,P,\alpha) - \underline{s}(f,P,\alpha) < \epsilon$$

Proof) Suppose f is Reimann Steiljes Integrable on $[a,b]$ then

$$\int_a^{\bar{b}} f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha \quad \text{——— (1)}$$

To prove for $\epsilon > 0$ there is partition P of $[a,b]$ s.t

$$\bar{S}(f,P,\alpha) - \underline{s}(f,P,\alpha) < \epsilon$$

Now for $\epsilon > 0$ there are partition P_1 and P_2 of $[a,b]$ such that

$$\bar{S}(f,P_1,\alpha) < \int_a^b f\,d\alpha + \epsilon/2 \quad \text{——— (2)}$$

and
$$\underline{s}(f,P_2,\alpha) > \int_a^b f\,d\alpha - \epsilon/2 \quad \text{——— (3)}$$

Let $P = P_1 \cup P_2$, $P_1 \subset P$, $P_2 \subset P$

Then P is finer than P_1 & P_2

$\Rightarrow \bar{S}(f,P,\alpha) \leq \bar{S}(f,P_1,\alpha)$

& $\underline{s}(f,P,\alpha) \geq \underline{s}(f,P_2,\alpha)$

Now $\bar{S}(f,P,\alpha) \leq \bar{S}(f,P_1,\alpha) < \int_a^b f\,d\alpha + \epsilon/2$ —— by (2)

$$\Rightarrow \bar{S}(f,P,\alpha) < \int_a^b f\,d\alpha + \epsilon/2 \quad \text{——— (4)}$$

Also $\underline{s}(f,P,\alpha) \geq \underline{s}(f,P_2,\alpha) > \int_a^b f\,d\alpha - \epsilon/2$ by (3)

$$\Rightarrow \underline{s}(f,P,\alpha) > \int_a^b f\,d\alpha - \epsilon/2 \quad \text{——— (5)}$$

$\Rightarrow -s(f,P,\alpha) < -\int_a^b f\,d\alpha + \epsilon/2$ — (5)

Adding (4) & (5)

$$\bar{S}(f,P,\alpha) - s(f,P,\alpha) < \epsilon$$

Conversely given that for every $\epsilon > 0$ there is partition P of $[a,b]$ such that

$$\bar{S}(f,P,\alpha) - s(f,P,\alpha) < \epsilon$$

To Prove $f \in RS_\alpha$

For this we have to show $\int_a^{\bar{b}} f\,d\alpha = \int_{\underline{a}}^{b} f\,d\alpha$

Now as for any partition and in particular for partition P

$\int_a^{\bar{b}} f\,d\alpha \leq \bar{S}(f,P,\alpha)$ — (6) & $\int_{\underline{a}}^{b} f\,d\alpha \geq s(f,P,\alpha)$ — (7)

Adding equ (6) & (7)

$$\int_a^{\bar{b}} f\,d\alpha - \int_{\underline{a}}^{b} f\,d\alpha \leq \bar{S}(f,P,\alpha) - s(f,P,\alpha) < \epsilon$$

$$\Rightarrow \int_a^{\bar{b}} f\,d\alpha - \int_{\underline{a}}^{b} f\,d\alpha < \epsilon$$

Since ϵ is arbitrary so $\int_a^{\bar{b}} f\,d\alpha - \int_{\underline{a}}^{b} f\,d\alpha = 0$

$$\Rightarrow \int_a^{\bar{b}} f\,d\alpha = \int_{\underline{a}}^{b} f\,d\alpha \Rightarrow f \in RS_\alpha$$

* * *

Theorem:- Suppose that α is increasing on $[a,b]$ & f is continuous on $[a,b]$ then $f \in RS_\alpha$

Proof) Since f is continuous on $[a,b]$ and $[a,b]$ is compact. So f is uniformly continuous on $[a,b]$ then for $\epsilon > 0$ there exist a $\delta > 0$ such that

$$|f(x_2) - f(x_1)| < \epsilon \quad \text{whenever } |x_2 - x_1| < \delta$$

In particular take partition P of $[a,b]$ s.t $\|P\| < \delta$ then

$$|M_k - m_k| < \epsilon/\alpha(b)-\alpha(a)$$

$$\Rightarrow M_k - m_k < \frac{\epsilon}{\alpha(b)-\alpha(a)}$$

$$\Rightarrow \sum_{k=1}^{n} M_k \Delta\alpha_k - \sum_{k=1}^{n} m_k \Delta\alpha_k < \frac{\epsilon}{\alpha(b)-\alpha(a)} \sum_{k=1}^{n} \Delta\alpha_k$$

$$\Rightarrow S(f,P,\alpha) - s(f,P,\alpha) < \frac{\epsilon}{\cancel{\alpha(b)-\alpha(a)}} \left[\cancel{\alpha(b)-\alpha(a)}\right]$$

$$\Rightarrow S(f,P,\alpha) - s(f,P,\alpha) < \epsilon$$

$$\Rightarrow f \in RS\alpha$$

★ Every continuous function is $RS\alpha$ but converse is not true

★ ★ ★

Theorem:- Let $f \in B(a,b)$ and α be increasing on $[a,b]$ then $f \in RS\alpha$ on $[a,b] \Leftrightarrow$ for $\epsilon > 0$ $\exists$ a $\delta > 0$ s.t

$$\left|\sum_{k=1}^{n} f(t_k)\Delta\alpha_k - \int_a^b f\right| < \epsilon \quad \text{——— (1)}$$

Proof) Suppose $f \in RS\alpha$

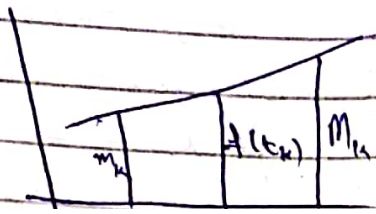
To prove for every $\epsilon > 0$ $\exists$ a $\delta > 0$ s.t inequality (1) holds.

Since $f \in RS\alpha$ over $[a,b]$ so

$$\Rightarrow \underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha \quad \text{——— (*)}$$

Now for every partition P and for every choice of $t_k \in [x_{k-1}, x_k]$

$$m_k \le f(t_k) \le M_k$$



$$\Rightarrow \sum_{K=1}^{n} m_K \Delta\alpha_K \leq \sum_{K=1}^{n} f(t_K)\Delta\alpha_K \leq \sum_{K=1}^{n} M_K \Delta\alpha_K$$

$$s(f,P,\alpha) \leq \sum_{K=1}^{n} f(t_K)\Delta\alpha_K \leq S(f,P,\alpha) \quad \text{—— (2)}$$

Again for every $\epsilon > 0$ there exist a $\delta > 0$ and $P \in P(a,b)$ such that

$$S(f,P,\alpha) < \int_a^b f\,d\alpha + \epsilon$$

& $$s(f,P,\alpha) > \int_a^b f\,d\alpha - \epsilon$$

Now using (*)

$$S(f,P,\alpha) < \int_a^b f\,d\alpha + \epsilon \quad \text{—— (3)}$$

$$s(f,P,\alpha) > \int_a^b f\,d\alpha - \epsilon$$

$$\int_a^b f\,d\alpha - \epsilon < s(f,P,\alpha) \quad \text{—— (4)}$$

from eqn (2), (3) & (4)

$$\int_a^b f\,d\alpha - \epsilon < s(f,P,\alpha) \leq \sum_{K=1}^{n} f(t_K)\Delta\alpha_K \leq S(f,P,\alpha) < \int_a^b f\,d\alpha + \epsilon$$

$$\Rightarrow \int_a^b f\,d\alpha - \epsilon < \sum_{K=1}^{n} f(t_K)\Delta\alpha_K < \int_a^b f\,d\alpha + \epsilon$$

$$\left|\sum_{K=1}^{n} f(t_K)\Delta\alpha_K - \int_a^b f\,d\alpha\right| < \epsilon \qquad \left[\because a-\epsilon < x < a+\epsilon \Rightarrow |x-a| < \epsilon\right.$$

Conversely suppose that for every $\epsilon > 0$ $\exists\, \delta > 0$ s.t

$$\left|\sum_{K=1}^{n} f(t_K)\Delta\alpha_K - \int_a^b f\,d\alpha\right| < \epsilon \quad \text{—— (5)}$$

For every choice of point $t_K \in [x_{K-1}, x_K]$ and for every partition $P \in P(a,b)$ s.t $\|P\| < \delta$

To prove $f \in RS\alpha$

Let $t_K, t'_K \in [x_{K-1}, x_K]$

Then by given condition

$$\left|\sum_{K=1}^{n} f(t_k)\Delta\alpha_k - \int_a^b f\,d\alpha\right| < \epsilon/4 \quad \text{——} ⑥$$

$$\left|\sum_{K=1}^{n} f(t'_k)\Delta\alpha_k - \int_a^b f\,d\alpha\right| < \epsilon/4 \quad \text{——} ⑦$$

Also

$$M_k - m_k = \text{Sup}\{f(t) - f(t') : t, t' \in [x_{k-1}, x_k]\}$$

Then by the definition of Sup for $\epsilon > 0$ we can find a pair $t_k, t'_k \in [x_{k-1}, x_k]$ such that

$$(M_k - m_k) - \frac{\epsilon}{2[\alpha(b)-\alpha(a)]} < f(t_k) - f(t'_k)$$

Now

$$S(f,P,\alpha) - s(f,P,\alpha) = \sum_{K=1}^{n} M_k\Delta\alpha_k - \sum_{K=1}^{n} m_k\Delta\alpha_k$$

$$= \sum_{K=1}^{n} (M_k - m_k)\Delta\alpha_k$$

$$< \sum_{K=1}^{n}\left[f(t_k) - f(t'_k) + \frac{\epsilon}{2[\alpha(b)-\alpha(a)]}\right]\Delta\alpha_k$$

$$= \sum_{K=1}^{n} f(t_k)\Delta\alpha_k - \sum_{K=1}^{n} f(t'_k)\Delta\alpha_k + \sum_{K=1}^{n}\frac{\epsilon}{2\cancel{[\alpha(b)-\alpha(a)]}}\cancel{[\alpha(b)-\alpha(a)]}$$

$$= \sum_{K=1}^{n} f(t_k)\Delta\alpha_k - \int_a^b f\,d\alpha + \int_a^b f\,d\alpha - \sum_{K=1}^{n} f(t'_k)\Delta\alpha_k + \epsilon/2$$

$$= \left[\sum_{K=1}^{n} f(t_k)\Delta\alpha_k - \int_a^b f\,d\alpha\right] + \left[\int_a^b f\,d\alpha - \sum_{K=1}^{n} f(t'_k)\Delta\alpha_k\right] + \epsilon/2$$

$$< \left|\left(\sum_{K=1}^{n} f(t_k)\Delta\alpha_k - \int_a^b f\,d\alpha\right) + \left(\int_a^b f\,d\alpha - \sum_{K=1}^{n} f(t'_k)\Delta\alpha_k\right) + \epsilon/2\right|$$

$$\leq \left| \sum_{k=1}^{n} f(t_k)\Delta\alpha_k - \int_a^b f\,d\alpha \right| + \left| \int_a^b f\,d\alpha - \sum_{k=1}^{n} f(t_k')\Delta\alpha_k \right| + \left| \epsilon/2 \right|$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2}$$

$$= \epsilon$$

$\Rightarrow S(f,P,\alpha) - s(f,P,\alpha) < \epsilon \Rightarrow f \in RS\alpha$

Theorem:- Let $\beta \in \mathbb{R}$, if $f \in RS\alpha$ on $[a,b]$ then $\beta f \in RS\alpha$ on $[a,b]$ and

$$\int_a^b \beta f\,d\alpha = \beta \int_a^b f\,d\alpha$$

Proof: Given $f \in RS\alpha$. So

$$\int_a^b f\,d\alpha = \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha \quad \text{———} \; (*)$$

To prove $\beta f \in RS\alpha$

Case 1:- If $\beta = 0$ then nothing is left to prove

Case 2:- If $\beta > 0$

Let $M_k = \operatorname{Sup}\{f(x) : x \in [x_{k-1}, x_k]\}$

$m_k = \operatorname{Inf}\{f(x) : x \in [x_{k-1}, x_k]\}$

Then

$\beta M_k = \operatorname{Sup}\{(\beta f)(x) : x \in [x_{k-1}, x_k]\}$

$\beta m_k = \operatorname{Inf}\{(\beta f)(x) : x \in [x_{k-1}, x_k]\}$

Now

$$\underline{\int_a^b} \beta f\,d\alpha = \operatorname{Sup}\{s(\beta f, P, \alpha) : P \in P(a,b)\}$$

$$= \operatorname{Sup}\{\beta s(f, P, \alpha) : P \in P(a,b)\}$$

$$= \beta \operatorname{Sup}\{s(f, P, \alpha) : P \in P(a,b)\}$$

$\because \operatorname{Sup}(\alpha A) = \alpha \operatorname{Sup}(A) \; (\alpha > 0)$

$$\Rightarrow \underline{\int_a^b} \beta f \, d\alpha = \beta \underline{\int_a^b} f \, d\alpha \quad \text{—— ①}$$

Now $\overline{\int_a^b} \beta f \, d\alpha = \text{Inf} \{ S(\beta f, P, \alpha) : P \in P(a,b) \}$

$= \text{Inf} \{ \beta S(f, P, \alpha) : P \in P(a,b) \}$

$= \beta \, \text{Inf} \{ S(f, P, \alpha) : P \in P(a,b) \}$

$\because \text{Inf}(\beta A) = \beta \, \text{Inf}(A) \text{ s.t } \beta > 0$

$$\Rightarrow \overline{\int_a^b} \beta f \, d\alpha = \beta \overline{\int_a^b} f \, d\alpha \quad \text{—— ②}$$

From ⓐ $\underline{\int_a^b} \beta f \, d\alpha = \overline{\int_a^b} \beta f \, d\alpha$

$$\Rightarrow \beta \underline{\int_a^b} f \, d\alpha = \beta \overline{\int_a^b} f \, d\alpha$$

$$\Rightarrow \underline{\int_a^b} \beta f \, d\alpha = \overline{\int_a^b} \beta f \, d\alpha \quad \text{from eq}^n \text{ ① \& ②}$$

$$\Rightarrow \beta f \in R_\alpha$$

Case 3 :- If $\beta < 0$

Let $M_k = \text{Sup} \{ f(x) : x \in [x_{k-1}, x_k] \}$

$m_k = \text{Inf} \{ f(x) : x \in [x_{k-1}, x_k] \}$

$\beta M_k = \text{Inf} \{ (\beta f)(x) : x \in [x_{k-1}, x_k] \}$

$\beta m_k = \text{Sup} \{ (\beta f)(x) : x \in [x_{k-1}, x_k] \}$

As $\underline{\int_a^b} \beta f \, d\alpha = \text{Sup} \{ s(\beta f, P, \alpha) : P \in P(a,b) \}$

$= \text{Sup} \{ \beta S(f, P, \alpha) : P \in P(a,b) \}$

$= \beta \, \text{Inf} \{ S(f, P, \alpha) : P \in P(a,b) \}$

$\because \text{Inf}(\beta A) = \beta \, \text{Sup} A, \; \beta < 0$

$$\int_a^b \beta f\, d\alpha = \beta \int_a^{\bar{b}} f\, d\alpha \quad \text{——— (3)}$$

Also

$$\int_a^{\bar{b}} \beta f\, d\alpha = \inf\{ S(\beta f, P, \alpha) : P \in P(a,b)\}$$

$$= \inf\{\beta\, s(f, P, \alpha) : P \in P(a,b)\}$$

$$= \beta \sup\{s(f, P, \alpha) : P \in P(a,b)\}$$

$$\because \inf(\beta A) = \beta \sup A, \ \beta < 0$$

$$\int_a^{\bar{b}} \beta f\, d\alpha = \beta \int_{\underline{a}}^{b} f\, d\alpha \quad \text{——— (4)}$$

Now from (*) $\int_{\underline{a}}^{b} f\, d\alpha = \int_a^{\bar{b}} f\, d\alpha$

$$\Rightarrow \beta \int_{\underline{a}}^{b} f\, d\alpha = \beta \int_a^{\bar{b}} f\, d\alpha$$

$$\Rightarrow \int_a^{\bar{b}} \beta f\, d\alpha = \int_{\underline{a}}^{b} \beta f\, d\alpha \quad \text{by (3) \& (4)}$$

$$\Rightarrow \beta f\, d\alpha \in R\$_\alpha$$

Hence in all cases $\beta f\, d\alpha \in R\$_\alpha$ over $[a,b]$

From eqn (4)

$$\int_a^{\bar{b}} \beta f\, d\alpha = \beta \int_{\underline{a}}^{b} f\, d\alpha$$

As $f\, d\alpha$ & $\beta f\, d\alpha$ both are Reimann Steiljes integral so

$$\int_a^b \beta f\, d\alpha = \beta \int_a^b f\, d\alpha$$

* * *

Theorem:- If $f, g \in R\$_\alpha$ then $f+g \in R\$_\alpha$ and

$$\int_a^b (f+g)\, d\alpha = \int_a^b f\, d\alpha + \int_a^b g\, d\alpha$$

Proof since $f, g \in R\$_\alpha$ over $[a,b]$ So for $\epsilon > 0$

There exist partitions P_1 & P_2 of $[a,b]$ s.t

$$\bar{S}(f, P_1, \alpha) - s(f, P_1, \alpha) < \epsilon \quad \text{—— ①}$$

$$\bar{S}(g, P_2, \alpha) - s(g, P_2, \alpha) < \epsilon \quad \text{—— ②}$$

Let M'_K, M''_K, M_K be the supremums of f, g & $f+g$ respectively in $[x_{K-1}, x_K]$

Similarly let m'_K, m''_K, m_K be the infimums of f, g & $f+g$ respectively

$$M_K \leq M'_K + M''_K \quad \text{—— ③}$$

$$m'_K + m''_K \leq m_K \quad \text{—— ④}$$

Now let $P = P_1 \cup P_2$ then P is finer than P_1 & P_2. As

$$M_K \leq M'_K + M''_K$$

$$\sum_{K=1}^{n} M_K \Delta\alpha_K \leq \sum_{K=1}^{n} M'_K \Delta\alpha_K + \sum_{K=1}^{n} M''_K \Delta\alpha_K$$

$$\bar{S}(f+g, P, \alpha) \leq \bar{S}(f, P, \alpha) + \bar{S}(g, P, \alpha)$$

$$\leq \bar{S}(f, P_1, \alpha) + \bar{S}(g, P_2, \alpha) \quad \because P \text{ is finer}$$

$$< s(f, P_1, \alpha) + \epsilon/2 + s(g, P_2, \alpha) + \epsilon/2$$

by ① & ②

$$= s(f, P_1, \alpha) + s(g, P_2, \alpha) + \epsilon$$

$$< s(f, P, \alpha) + s(g, P, \alpha) + \epsilon$$

$$\leq s(f+g, P, \alpha) + \epsilon \quad \text{using 4}$$

$$\bar{S}(f+g, P, \alpha) - s(f+g, P, \alpha) < \epsilon$$

$$\Rightarrow f+g \in R\bar{S}\alpha$$

Now we have to show

$$\int_a^b (f+g)\,d\alpha = \int_a^b f\,d\alpha + \int_a^b g\,d\alpha$$

As $\int_a^b (f+g)\,d\alpha \leq \bar{S}(f+g, P, \alpha)$

$$\leq \bar{S}(f,P,\alpha)+\bar{S}(g,P,\alpha)$$

$$\leq \bar{S}(f,P_1,\alpha)+\bar{S}(g,P_2,\alpha)$$

$$< s(f,P_1,\alpha)+\frac{\epsilon}{2}+s(g,P_2,\alpha)+\epsilon/2$$

by ① & ②

$$\leq s(f,P,\alpha)+s(g,P,\alpha)+\epsilon$$

$$< \int_a^b f\,d\alpha+\int_a^b g\,d\alpha+\epsilon$$

$$\int_a^b (f+g)\,d\alpha < \int_a^b f\,d\alpha+\int_a^b g\,d\alpha+\epsilon$$

Since ϵ is arbitrary so we can also prove that

$$\int_a^b (f+g)\,d\alpha > \int_a^b f\,d\alpha+\int_a^b g\,d\alpha$$

$$\Rightarrow \int_a^b (f+g)\,d\alpha = \int_a^b f\,d\alpha+\int_a^b g\,d\alpha$$

* * *

Theorem:– Let $f \in R\,\S\alpha$ on $[a,b]$ if $f \geq 0$ on $[a,b]$ then $\int_a^b f\,d\alpha \geq 0$

Proof) Since $f(x) \geq 0$ on $[a,b]$

$$\Rightarrow M_k \geq 0 \quad , m_k \geq 0$$

As $m_k \geq 0 \Rightarrow \sum_{k=1}^{n} m_k \Delta\alpha_k \geq 0$

$$\Rightarrow s(f,P,\alpha) \geq 0$$

Aso $\int_a^b f\,d\alpha \geq s(f,P,\alpha) \geq 0$

$$\Rightarrow \int_a^b f\,d\alpha \geq 0$$

* * *

Theorem:- If $f, g \in R\mathfrak{S}\alpha$ & $f \leq g$ on $[a,b]$ then $\int_a^b f\,d\alpha \leq \int_a^b g\,d\alpha$

Proof)

Given $f \leq g$

$\Rightarrow g \geq f \quad \Rightarrow g - f \geq 0$

$\Rightarrow \int_a^b (g-f)\,d\alpha \geq 0$

$\Rightarrow \int_a^b [g + (-f)]\,d\alpha \geq 0$

$\Rightarrow \int_a^b g\,d\alpha + \int_a^b (-f)\,d\alpha \geq 0 \qquad \because \int_a^b (f+g)\,d\alpha = \int_a^b f\,d\alpha + \int_a^b g\,d\alpha$

$\Rightarrow \int_a^b g\,d\alpha - \int_a^b f\,d\alpha \geq 0 \qquad \because \int_a^b \beta f\,d\alpha = \beta \int_a^b f\,d\alpha$

$\Rightarrow \int_a^b g\,d\alpha \geq \int_a^b f\,d\alpha$

$\Rightarrow \int_a^b f\,d\alpha \leq \int_a^b g\,d\alpha$

***

Theorem:- If $f \in R\mathfrak{S}\alpha$ ~~on [a,b] s.o for ∈>0 there is partition P of [a,b] ⇒~~ s.o is $|f|$ and

$$\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$$

Proof) Since $f \in R\mathfrak{S}\alpha$ on $[a,b]$ so for $\epsilon > 0$ there is partition P of $[a,b]$ s.t

$$\mathfrak{S}(f,P,\alpha) - s(f,P,\alpha) < \epsilon \quad \text{——} \circledast$$

Let $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$

$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$

$M'_K = \operatorname{Sup}\{|f(x)| : x \in [x_{k-1}, x_k]\}$

$m'_K = \operatorname{Inf}\{|f(x)| : x \in [x_{k-1}, x_k]\}$

Then clearly $M'_K - m'_K \leq M_K - m_K$

$\Rightarrow \sum_{K=1}^{n} M'_K \Delta\alpha_K - \sum_{K=1}^{n} m'_K \Delta\alpha_K \leq \sum_{K=1}^{n} M_K \Delta\alpha_K - \sum_{K=1}^{n} m_K \Delta\alpha_K$

$\Rightarrow S(|f|, P, \alpha) - s(|f|, P, \alpha) \leq S(f, P, \alpha) - s(f, P, \alpha)$

$\Rightarrow S(|f|, P, \alpha) - s(|f|, P, \alpha) < \epsilon$ using (B)

$\Rightarrow |f| \in RS\alpha$

Next we show that

$$\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$$

As $-f \leq |f| \Rightarrow \int_a^b -f\,d\alpha \leq \int_a^b |f|\,d\alpha$

$\Rightarrow -\int_a^b f\,d\alpha \leq \int_a^b |f|\,d\alpha$ ——— (1)

Also $f \leq |f| \Rightarrow \int_a^b f\,d\alpha \leq \int_a^b |f|\,d\alpha$ ——— (2)

By eqn (1) & (2) $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$

* * *

Theorem:- If $f \in RS\alpha$ on $[a,b]$ then so is f^2

Proof Let $M_K = \operatorname{Sup}\{f(x) : x \in [x_{k-1}, x_k]\}$

$m_K = \operatorname{Inf}\{f(x) : x \in [x_{k-1}, x_k]\}$

and $M = \operatorname{Sup}\{|f(x)| : x \in [x_{k-1}, x_k]\}$

Then clearly $M_K + m_K \leq 2M$ ——— (1)

It is clear that M_K^2, m_K^2 are supremums & infimums of f^2 on $[x_{k-1}, x_k]$

Since f is Reimann Stieljes Integrable on $[a,b]$ so for $\epsilon>0$ there exist a portition P of $[a,b]$ such that

$$S(f,P,\alpha)-s(f,P,\alpha)<\frac{\epsilon}{2M} \quad \text{——— ②}$$

Now

$$M_k^2-m_k^2=(M_k+m_k)(M_k-m_k)$$

$$\leq 2M(M_k-m_k) \quad \text{by ①}$$

$$\sum_{K=1}^{n} M_k^2\Delta\alpha_k-\sum_{K=1}^{n} m_k^2\Delta\alpha_k \leq 2M\left[\sum_{K=1}^{n} M_k\Delta\alpha_k-\sum_{K=1}^{n} m_k\Delta\alpha_k\right]$$

$$\Rightarrow S(f^2,P,\alpha)-s(f^2,P,\alpha)\leq 2M[S(f,P,\alpha)-s(f,P,\alpha)]$$

$$<2M\left[\epsilon/2M\right]=\epsilon$$

$$\Rightarrow S(f^2,P,\alpha)-s(f^2,P,\alpha)<\epsilon$$

$\Rightarrow f^2$ is Reimann Stieljes Integrable.

* * *

Theorem:- If f is continuous on $[a,b]$ and $g\geq 0$, $g\in RS_\alpha$ on $[a,b]$ then there is a number $c\in[a,b]$ such that

$$\int_a^b fg\,d\alpha=f(c)\int_a^b g\,d\alpha$$

Proof: Since f is continuous on $[a,b]$ then

$$m=\inf\{f(x):x\in[a,b]\}$$
$$M=\sup\{f(x):x\in[a,b]\}$$

exists and $\quad m\leq f(x)\leq M$

$$\Rightarrow m\,g(x)\leq f(x)\,g(x)\leq M\,g(x) \quad \text{——— ①}$$

Since f is continuous on $[a,b]$ so f is R.S.I on $[a,b]$. Also $g\in RS_\alpha$ so $fg\in RS_\alpha$

$$① \Rightarrow \int_a^b m\,g(x)\,d\alpha\leq\int_a^b f(x)g(x)\,d\alpha\leq\int_a^b M g(x)\,d\alpha$$

$$\Rightarrow m\int_a^b g(x)\,d\alpha \leq \int_a^b f(x)g(x)\,d\alpha \leq M\int_a^b g(x)\,d\alpha$$

$$\Rightarrow \quad m \leq \frac{\int_a^b f(x)\,g(x)\,d\alpha}{\int_a^b g(x)\,d\alpha} \leq M$$

Let $\lambda = \dfrac{\int_a^b f(x)\,g(x)\,d\alpha}{\int_a^b g(x)\,d\alpha}$

$\therefore \quad m \leq \lambda \leq M$

Since f is continuous on $[a,b]$ so by intermediate value theorem $\exists\ c \in [a,b]$ s.t

$$f(c) = \lambda$$

$$\Rightarrow f(c) = \frac{\int_a^b f(x)\,g(x)\,d\alpha}{\int_a^b g(x)\,d\alpha}$$

$$\Rightarrow f(c)\int_a^b g(x)\,d\alpha = \int_a^b f(x)g(x)\,d\alpha$$

$$\Rightarrow \int_a^b fg\,d\alpha = f(c)\int_a^b g\,d\alpha$$

* * *

Theorem: Let $f \in R\$\alpha$ on $[a,b]$ and $c \in]a,b[$ then $f \in R\$\alpha$ on $[a,c]$ and $[c,b]$.

Also $\displaystyle\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^d f\,d\alpha$

Proof: Since $f \in R\$\alpha$ on $[a,b]$ so for $\epsilon > 0$ there is a partition P of (a,b) s.t

$$S(f,P,\alpha) - s(f,P,\alpha) < \epsilon \quad \text{——} \ (1)$$

Let $P_1 = \{a = x_0, x_1, x_2, \ldots x_{k-1}, c\}$

and $P_2 = \{c, x_k, x_{k+1}, \ldots, x_n = b\}$

be the partitions of $[a,c]$ and $[c,b]$ respect

Also note that

$$x_{k-1} < c < x_k$$

Let $M_c = \sup\{f(x) : x \in [x_{k-1}, c]\}$

$m_c = \inf\{f(x) : x \in [x_{k-1}, c]\}$

Now
$$\bar{S}(f, P_1, \alpha) - \underline{s}(f, P_1, \alpha) = \sum_{i=1}^{k-1} M_i \Delta\alpha_i + M_c(c - x_{k-1}) - \sum_{i=1}^{k-1} m_i \Delta\alpha_i - m_c(c - x_{k-1})$$
$$= \sum_{i=1}^{k-1} (M_i - m_i)\Delta\alpha_i + (M_c - m_c)(c - x_{k-1})$$
$$< \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$
$$= \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$$
$$= \bar{S}(f, P, \alpha) - \underline{s}(f, P, \alpha)$$
$$< \epsilon \quad \text{by } ①$$

$\Rightarrow \bar{S}(f, P_1, \alpha) - \underline{s}(f, P_1, \alpha) < \epsilon$

$\Rightarrow f \in RS\alpha$ on $[a, c]$

Similarly $f \in RS\alpha$ on $[c, b]$

Again as $f \in RS\alpha$ so

$$\bar{S}(f, P, \alpha) < \int_a^b f\,d\alpha + \epsilon/2$$
$$\underline{s}(f, P, \alpha) > \int_a^b f\,d\alpha - \epsilon/2$$

Now $\bar{S}(f, P_1, \alpha) + \bar{S}(f, P_2, \alpha) = \bar{S}(f, P_1 \cup P_2, \alpha)$
$$= \bar{S}(f, P, \alpha)$$

$$\bar{S}(f, P_1, \alpha) + \bar{S}(f, P_2, \alpha) < \int_a^b f\,d\alpha + \epsilon/2 \quad \text{——— } ②$$

Now
$$\int_a^c f\,d\alpha \leq \bar{S}(f, P_1, \alpha)$$
$$\int_c^b f\,d\alpha \leq \bar{S}(f, P_2, \alpha)$$

by adding above two we have

$$\int_a^c f\,d\alpha + \int_c^b f\,d\alpha \leq S(f,P_1,\alpha) + S(f,P_2,\alpha)$$

$$< \int_a^b f\,d\alpha + \epsilon/2 \quad \text{by } ②$$

Since ϵ is arbitrary so

$$\int_a^c f\,d\alpha + \int_c^b f\,d\alpha \leq \int_a^b f\,d\alpha \quad \text{———} ③$$

similary for lower Reimann Sums we can prove

$$\int_a^c f\,d\alpha + \int_c^b f\,d\alpha \geq \int_a^b f\,d\alpha \quad \text{———} ④$$

by ~~eq~~ inequalties ③ & ④

$$\int_a^c f\,d\alpha + \int_c^b f\,d\alpha = \int_a^b f\,d\alpha.$$

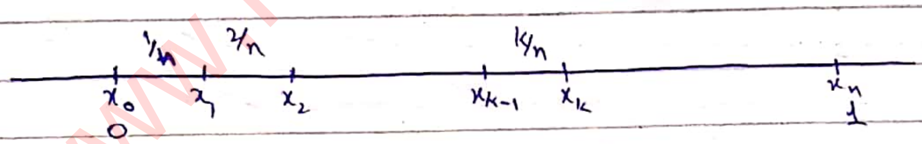
* * *

Question:- Let $\alpha(x) = x^3$ $\quad 0 < x < 1$

then show that $f(x) = x$ is $RS\alpha$

Solution Divide $[0,1]$ into n subintervals each of length Δx i.e

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$



Then $P = \{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, 1\}$

$$x_k = k/n \qquad x_{k-1} = \frac{k-1}{n}$$

As $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$

$$= \alpha\left(\frac{k}{n}\right) - \alpha\left(\frac{k-1}{n}\right)$$

$$= \left(\frac{k}{n}\right)^3 - \left(\frac{(k-1)}{n}\right)^3$$

$$M_k = \operatorname{Sup}\{f(x) : x \in [x_{k-1}, x_k]\}$$

$$= f(x_k) \qquad \because f(x) \text{ is increasing}$$

$$= f\left(\frac{k}{n}\right) \qquad \text{so } M_k = f(x_k)$$

$\Rightarrow \quad M_k = \frac{k}{n}$

$$\bar{S}(f,P,\alpha) = \sum_{k=1}^{n} M_k \Delta\alpha_k = \sum_{k=1}^{n} \frac{k}{n} \Delta\alpha_k$$

Similarly $s(f,P,\alpha) = \sum_{k=1}^{n} m_k \Delta\alpha_k = \sum_{k=1}^{n} \frac{k-1}{n} \Delta\alpha_k$

$$\bar{S}(f,P,\alpha) - s(f,P,\alpha) = \sum_{k=1}^{n} \frac{k}{n} \Delta\alpha_k - \sum_{k=1}^{n} \frac{k-1}{n} \Delta\alpha_k$$

$$= \sum_{k=1}^{n} \left(\frac{k}{n} - \frac{k-1}{n}\right) \Delta\alpha_k$$

$$= \sum_{k=1}^{n} \left(\frac{k-k+1}{n}\right) \Delta\alpha_k$$

$$= \frac{1}{n} \sum_{k=1}^{n} \Delta\alpha_k = \frac{1}{n}[\alpha(1) - \alpha(0)]$$

$$= \frac{1}{n}[1-0] = \frac{1}{n}$$

$\boxed{\frac{1}{n} < \epsilon \quad n > \frac{1}{\epsilon}}$

$\bar{S}(f,P,\alpha) - s(f,P,\alpha) < \epsilon$ for $n > \frac{1}{\epsilon}$

$\Rightarrow f \in RS_\alpha$ by Cauchy critaria.

***

Question:- Let $\alpha(x) = x^5 \quad 0 < x < 1$ & $f(x) = x^2 + 1$. Show that $f \in RS_\alpha$

Solution) Divide $[0,1]$ into n subintervals each of length Δx i.e

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

Then $P = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{k-1}{n}, \frac{k}{n}, \dots, 1\}$

$$x_k = \frac{k}{n}, \quad x_{k-1} = \frac{k-1}{n}$$

As $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$

$$= \alpha\left(\frac{k}{n}\right) - \alpha\left(\frac{k-1}{n}\right)$$

$$= \left(\frac{k}{n}\right)^5 - \left(\frac{k-1}{n}\right)^5$$

$M_k = \operatorname{Sup}\{f(x) : x \in [x_{k-1}, x_k]\}$

$= f(x_k) = f\left(\frac{k}{n}\right) = \frac{k^2}{n^2} + 1 = \frac{k^2+n^2}{n^2}$ $\because f(x)$ incr

$$\bar{S}(f,P,\alpha) = \sum_{k=1}^{n} M_k \Delta\alpha_k = \sum_{k=1}^{n} \frac{k^2+n^2}{n^2} \Delta\alpha_k$$

$$s(f,P,\alpha) = \sum_{K=1}^{n} m_K \Delta\alpha_K = \sum_{K=1}^{n} \frac{(K+1)^2+n^2}{n^2} \Delta\alpha_K$$

$$S(f,P,\alpha) - s(f,P,\alpha) = \sum_{K=1}^{n} \frac{K^2+n^2}{n^2}\Delta\alpha_K - \sum_{K=1}^{n} \frac{(K+1)^2+n^2}{n^2}$$

$$= \sum_{K=1}^{n} \left[\frac{K^2+n^2}{n^2} - \frac{(K+1)^2+n^2}{n^2}\right]\Delta\alpha_K$$

$$= \sum_{K=1}^{n} \left[\frac{\cancel{K^2}+\cancel{n^2}-\cancel{K^2}-1-2K-\cancel{n^2}}{n^2}\right]\Delta\alpha_K$$

$$= \sum_{K=1}^{n} \left[\frac{2K-1}{n^2}\right]\Delta\alpha_K$$

$$= \frac{1}{n^2} \qquad \because K=1$$

S ~~how~~ this function is continuous. As every continuous function is $R\S\alpha$

N/A

Question:- Let $\alpha(x) = x^5$ $\quad 0 < x < 1$
and $f(x) = \frac{1}{x+1}$ show that $f \in R\S\alpha$

Solution

$$f(x) = \frac{1}{x+1} \Rightarrow f'(x) = \frac{1}{(x+1)^2}$$

$f(x)$ is decreasing on $[0,1]$

Divide $[0,1]$ into n subintervals each of length Δx

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

then
$$P = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{k-1}{n}, \frac{k}{n}, \dots, 1\right\}$$

$$x_k = \frac{k}{n}, \quad x_{k-1}, \quad \frac{k-1}{n}$$

Now
$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$$
$$= f(x_{k-1}) \quad \because f(x) \text{ is decreasing}$$
$$= f\left(\frac{k-1}{n}\right)$$
$$= \frac{1}{\left(\frac{k-1}{n}\right)+1} = \frac{n}{k-1+n}$$

$$m_k = f(x_k) = f\left(\frac{k}{n}\right) = \frac{1}{\frac{k}{n}+1} = \frac{n}{k+1}$$

Now
$$\bar{S}(f,P,\alpha) - \underline{s}(f,P,\alpha) = \sum_{k=1}^{n} M_k \Delta\alpha_k - \sum_{k=1}^{n} m_k \Delta\alpha_k$$
$$= \sum_{k=1}^{n}\left(\frac{n}{k-1+n}\right)\Delta\alpha_k - \sum_{k=1}^{n}\left(\frac{n}{k+1}\right)\Delta\alpha_k$$
$$= \sum_{k=1}^{n}\left[\frac{n}{k-1+n} - \frac{n}{k+n}\right]\Delta\alpha_k$$
$$= n\left\{\left[\frac{1}{n} - \frac{1}{n+1}\right] + \left[\frac{1}{n+1} - \frac{1}{n+2}\right] + \dots + \left[\frac{1}{2n-1} - \frac{1}{2n}\right]\right\}[\alpha(1) - \alpha(0)]$$
$$= n\left[\frac{1}{n} - \frac{1}{2n}\right](1)$$
$$= \frac{\cancel{n}}{\cancel{n}}\left[1 - \frac{1}{2}\right] = \frac{1}{2}$$

$\bar{S}(f,P,\alpha) - \underline{s}(f,P,\alpha) < \epsilon$ $\quad$ we choose $\epsilon > \frac{1}{2}$

$\Rightarrow f \in R\S\alpha$

Question:- If f is continuous on $[a,b]$ and α is defined by

$$\alpha(x) = \begin{cases} \lambda & , a \le x < c \\ \mu & , c \le x \le b \end{cases}$$

where $\lambda < \mu$ and $c \in]a,b[$ then $f \in RS\alpha$ on $[a,b]$ and $\int_a^b f\,d\alpha = f(c)[\mu - \lambda]$

Solution, Since f is continuous on $[a,b]$ so is continuous at $c \in [a,b]$

Then for $\epsilon > 0$ there exist a $\delta > 0$ s.t

$$|f(x) - f(c)| < \frac{\epsilon}{2(\mu-\lambda)} \text{ whenever } |x-c| < \delta$$

$$\Rightarrow f(c) - \frac{\epsilon}{2(\mu-\lambda)} < f(x) < f(c) + \frac{\epsilon}{2(\mu-\lambda)} \quad \text{——— (1)}$$

Consider a partition $P = \{a = x_0, x_1, x_2, x_3 = b\}$

let $m = \inf\{f(x) : x \in [a,b]\}$

$M = \sup\{f(x) : x \in [a,b]\}$

Then clearly (1) $\Rightarrow$

$$f(c) - \frac{\epsilon}{2(\mu-\lambda)} < m \le M < f(c) + \frac{\epsilon}{2(\mu-\lambda)} \quad \text{——— (2)}$$

Consider $S(f,P,\alpha) = \sum_{k=1}^{3} M_k \Delta\alpha_k$

$\le \sum_{k=1}^{3} M \Delta\alpha_k \quad \because M_k \le M$

$= M \sum_{k=1}^{3} \Delta\alpha_k$

$= M[\alpha(b) - \alpha(a)]$

$= M[\mu - \lambda]$ by def of α

$\Rightarrow S(f,P,\alpha) \le M[\mu - \lambda]$

$< f(c)[\mu - \lambda] + \frac{\epsilon}{2(\mu-\lambda)}[\mu - \lambda]$ by (2)

$$\bar{S}(f,P,\alpha) < f(c)[\mu-\lambda] + \frac{\epsilon}{2} \quad \text{---} \quad (3)$$

Similarly we have

$$s(f,P,\alpha) > f(c)[\mu-\lambda] - \epsilon/2$$

$$-s(f,P,\alpha) < -f(c)[\mu-\lambda] + \epsilon/2 \quad \text{---} \quad (4)$$

Adding (3) & (4)

$$\bar{S}(f,P,\alpha) - s(f,P,\alpha) < \epsilon$$

$\Rightarrow f \in RS\alpha$ by Cauchy Criteria

(ii) As we know that

$$\int_a^b f d\alpha \leq \bar{S}(f,P,\alpha)$$

$$< f(c)[\mu-\lambda] + \epsilon/2 \quad \text{by } (3)$$

since ϵ is arbitrary so

$$\int_a^b f d\alpha \leq f(c)[\mu-\lambda]$$

Similarly $\int_a^b f d\alpha \geq f(c)(\mu-\lambda)$

$$\Rightarrow \int_a^b f d\alpha = f(c)[\mu-\lambda]$$

* * *

Theorem:- Let $f \in RS\alpha_1$ & $f \in RS\alpha_2$ then $f \in RS_{\alpha_1+\alpha_2}$ and

$$\int_a^b f\, d(\alpha_1+\alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$$

Proof) Since $f \in RS\alpha_1$ & $f \in RS\alpha_2$

$$\Rightarrow \underline{\int_a^b} f\, d\alpha_1 = \overline{\int_a^b} f\, d\alpha_1 = \int_a^b f\, d\alpha_1 \quad \text{--- (1)}$$

& $$\underline{\int_a^b} f\, d\alpha_2 = \overline{\int_a^b} f\, d\alpha_2 = \int_a^b f\, d\alpha_2 \quad \text{--- (2)}$$

Consider $$S(f, P, \alpha_1+\alpha_2) = \sum_{k=1}^{n} M_k \Delta(\alpha_1+\alpha_2)_k$$

$$= \sum_{k=1}^{n} M_k [\Delta(\alpha_1)_k + \Delta(\alpha_2)_k]$$

$$= \sum_{k=1}^{n} M_k \Delta(\alpha_1)_k + \sum_{k=1}^{n} M_k \Delta(\alpha_2)_k$$

$$S(f, P, \alpha_1+\alpha_2) = S(f, P, \alpha_1) + S(f, P, \alpha_2)$$

taking infimums on both sides w.r.t $P \in P(a,b)$

$$\Rightarrow \text{Inf}\{S(f, P, \alpha_1+\alpha_2) : P \in P(a,b)\} = \text{Inf}\{S(f, P, \alpha_1) : P \in P(a,b)\} + \text{Inf}\{S(f, P, \alpha_2) : P \in P(a,b)\}$$

$$\Rightarrow \overline{\int_a^b} f\, d(\alpha_1+\alpha_2) = \overline{\int_a^b} f\, d\alpha_1 + \overline{\int_a^b} f\, d\alpha_2$$

$$\Rightarrow \overline{\int_a^b} f\, d(\alpha_1+\alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2 \quad \text{--- (3)}$$

by (1) & (2)

Similarly $$\underline{\int_a^b} f\, d(\alpha_1+\alpha_2) = \underline{\int_a^b} f\, d\alpha_1 + \underline{\int_a^b} f\, d\alpha_2$$

$$\Rightarrow \underline{\int_a^b} f\, d(\alpha_1+\alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2 \quad \text{--- (4)}$$

from equ ③ & ④

$$\overline{\int_a^b} f\, d(\alpha_1+\alpha_2) = \underline{\int_a^b} f\, d(\alpha_1+\alpha_2)$$

$$\Rightarrow f \in RS(\alpha_1+\alpha_2)$$

As $f \in RS(\alpha_1+\alpha_2) \Rightarrow \int_a^b f\, d(\alpha_1+\alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$

* * *

Theorem:- If $f \in RS\alpha$ & $c>0$ then $f \in RS c\alpha$ and $\int_a^b f\, d(c\alpha) = c\int_a^b f\, d\alpha$

Proof) Since $f \in RS\alpha$

$$\Rightarrow \overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha = \int_a^b f\, d\alpha \quad \text{——— ①}$$

Consider

$$S(f,P,c\alpha) = \sum_{k=1}^{n} M_k \Delta(c\alpha)_k$$

$$= \sum_{k=1}^{n} M_k [(c\alpha)(x_k) - (c\alpha)(x_{k-1})]$$

$$= \sum_{k=1}^{n} M_k [c\alpha(x_k) - c\alpha(x_{k-1})] \quad \because (cf)x = cf(x)$$

$$= \sum_{k=1}^{n} M_k [c\Delta\alpha_k]$$

$$= c\sum_{k=1}^{n} M_k \Delta\alpha_k$$

$$S(f,P,c\alpha) = cS(f,P,\alpha)$$

Taking Infimums on both sides w.r.t $P \in P[a,b]$

$$\overline{\int_a^b} f\, d(c\alpha) = c\overline{\int_a^b} f\, d\alpha$$

$$\Rightarrow \overline{\int_a^b} f\, d(c\alpha) = c\int_a^b f\, d\alpha \quad \text{using eqn ①} \quad \text{——— ②}$$

Similarly we can prove

$$\int_a^b f\,d(c\alpha) = c\int_a^b f\,d\alpha \quad \text{—— (3)}$$

from equ (2) & (3) we have

$$\underline{\int_a^b} f\,d(c\alpha) = \overline{\int_a^b} f\,d(c\alpha) \Rightarrow f \in R\S c\alpha$$

since $f \in R\S c\alpha \Rightarrow \underline{\int_a^b} f\,d(c\alpha) = \overline{\int_a^b} f\,d(c\alpha) = \int_a^b f\,d(c\alpha)$

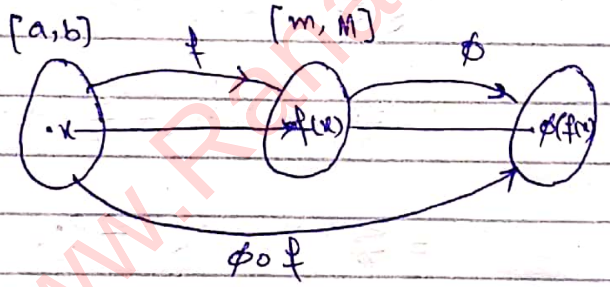
so equ (2) becomes

$$\int_a^b f\,d(c\alpha) = c\int_a^b f\,d\alpha$$

* * *

Theorem:- If $f \in R\S\alpha$ with $m \le f(x) \le M$ and $h(x) = (\phi \circ f)(x) = \phi(f(x))$ where ϕ is continuous on $[m, M]$ then $h \in R\S\alpha$ on $[a,b]$

Proof)



Since ϕ is continuous on $[m, M]$ and $[m, M]$ is compact $\Rightarrow \phi$ is uniformly continuous on $[m, M]$ then for $\epsilon > 0$ there exist a $\delta = \delta(\epsilon) > 0$ s.t

$$|\phi(s) - \phi(t)| < \epsilon \quad \text{whenever } |s - t| < \delta$$

Again as $f \in R\S\alpha$ so for $\epsilon = \delta^2 > 0$ $\exists$ a partition P of $[a, b]$ s.t

$$S(f, P, \alpha) - s(f, P, \alpha) < \delta^2$$

Let us define $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$$

$M_k^* = \sup\{h(x) : x \in [x_{k-1}, x_k]\}$

$m_k^* = \inf\{h(x) : x \in [x_{k-1}, x_k]\}$

$K = \max|h(x)| \quad , \forall\ x \in [a,b]$

Then clearly $M_k^* - m_k^* \leq 2K$

Now consider two subsets A, B s.t $A, B \subset P$

& $M_k - m_k < \delta$ for A

$M_k - m_k \geq \delta$ for B

Then clearly $A \cup B = P$ & $A \cap B = \phi$

Further $M_k^* - m_k^* < \epsilon$ for A

$M_k^* - m_k^* \leq 2K$ for B

Now consider $\delta \sum_B \Delta\alpha_k = \sum_B \delta \Delta\alpha_k$

$\leq \sum_B (M_k - m_k)\Delta\alpha_k$

$\leq \sum_B (M_k - m_k)\Delta\alpha_k + \sum_A (M_k - m_k)\Delta\alpha_k$

$= \sum_P (M_k - m_k)\Delta\alpha_k$

$= \sum_P M_k \Delta\alpha_k - \sum_P m_k \Delta\alpha_k$

$= S(f,P,\alpha) - s(f,P,\alpha) < \delta^2$

$\delta \sum_B \Delta\alpha_k < \delta^2$

$\sum_B \Delta\alpha_k < \delta$

Consider $S(h,P,\alpha) - s(h,P,\alpha) = \sum_P M_k^* \Delta\alpha_k - \sum_P m_k^* \Delta\alpha_k$

$= \sum_P (M_k^* - m_k^*)\Delta\alpha_k$

$$= \sum_A (M_k^* - m_k^*)\Delta\alpha_k + \sum_B (M_k^* - m_k^*)\Delta\alpha_k$$

$$< \left[\sum_A \epsilon \Delta\alpha_k + \sum_B 2k\Delta\alpha_k\right] = \epsilon'$$

$$\Rightarrow \bar{S}(h,P,\alpha) - s(h,P,\alpha) < \epsilon'$$

$$\Rightarrow h \in R\bar{S}_\alpha$$

* * *

Theorem:- If f is monotonic on $[a,b]$ and α is continuous and increasing on $[a,b]$ then prove that $f \in R\bar{S}_\alpha$

Proof) Since α is continuous and increasing

$$\Rightarrow \alpha(a) \leq \alpha(x) \leq \alpha(b) \quad \forall\ x \in [a,b]$$

$$\Rightarrow \alpha(b) > \alpha(a)$$

$$\Rightarrow \alpha(b) - \alpha(a) > 0$$

Divide $[a,b]$ into n subintervals such that

$$\Delta\alpha_k = \frac{\alpha(b) - \alpha(a)}{n}$$

Also f is monotonic on $[a,b]$.

So without any loss of generality suppose that f is increasing on $[a,b]$ so on $[x_{k-1}, x_k]$

$$M_k = f(x_k) \quad \& \quad m_k = f(x_{k-1})$$

Now

$$\bar{S}(f,P,\alpha) - s(f,P,\alpha) = \sum_{k=1}^n M_k\Delta\alpha_k - \sum_{k=1}^n m_k\Delta\alpha_k$$

$$= \sum_{k=1}^n [M_k - m_k]\Delta\alpha_k$$

$$= \sum_{k=1}^n [f(x_k) - f(x_{k-1})]\left[\frac{\alpha(b) - \alpha(a)}{n}\right]$$

$$= \left[\frac{\alpha(b) - \alpha(a)}{n}\right]\{\cancel{f(x_1)} - f(x_0) + f(x_2) - \cancel{f(x_1)} + \dots + f(x_n) - f(x_{n-1})\}$$

$$= \frac{\alpha(b) - \alpha(a)}{n}[f(b) - f(a)]$$

$$< \epsilon \quad \text{for } n > \frac{\alpha(b) - \alpha(a)}{\epsilon}[f(b) - f(a)]$$

$\Rightarrow \bar{S}(f,P,\alpha) - \underline{s}(f,P,\alpha) < \epsilon$

$\Rightarrow f \in R\,\S\,\alpha$

* * *

Theorem:- If $f \in B(a,b)$ & $f \in C(a,b)$

$\alpha(x) = I(x-t)$ then

$\int_a^b f\,d\alpha = f(t)$ where I is unit step function defined by

$$I(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$$

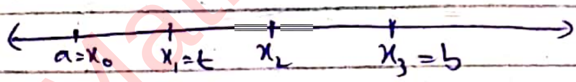
Proof Consider the partition

$P = \{a = x_0, x_1 = t, x_2, x_3 = b\}$

If f is continuous and $x_n \to x$ then $f(x_n) \to f(x)$

Then

$\alpha(x) = I(x - x_1)$



$\Rightarrow \alpha(x_0) = I(x_0 - x_1) = 0$

$\Rightarrow \alpha(x_1) = I(x_1 - x_1) = 0$

$\Rightarrow \alpha(x_2) = I(x_2 - x_1) = 1$

$\Rightarrow \alpha(x_3) = I(x_3 - x_1) = 1$

$x_0 < x_1$

$x_0 - x_1 < 0$ $(-ve)$

$\Rightarrow I(x_0 - x_1) = 0$

Consider $\bar{S}(f,P,\alpha) = \sum_{k=1}^{3} M_k \Delta\alpha_k$

$= M_1\Delta\alpha_1 + M_2\Delta\alpha_2 + M_3\Delta\alpha_3$

$= M_1[\alpha(x_1) - \alpha(x_0)] + M_2[\alpha(x_2) - \alpha(x_1)]$

$+ M_3[\alpha(x_3) - \alpha(x_2)]$

$= M_1[0-0] + M_2[1-0] + M_3[1-1]$

$\bar{S}(f,P,\alpha) = M_2$

Similarly $\underline{s}(f,P,\alpha) = m_2$

when $n \to \infty$

$\bar{S}(f,P,\alpha) = \int_a^b f\,d\alpha$ —— ①

$$s(f,P,\alpha) = \int_a^b f\,d\alpha \quad \text{———— ②}$$

Also $x_2 \to x_1 = t$

$\Rightarrow M_2 \to f(t)$

$\Rightarrow m_2 \to f(t)$

$\Rightarrow S(f,P,\alpha) = f(t)$

and $s(f,P,\alpha) = f(t)$

$n \to \infty$ the points will become so closer that they are joined and all points approches to one anoth
And all heights at all points will be equal to $f(x_1) = f(t)$
$\Rightarrow M_2$ & m_2 also $\to f(t)$

So equ ① & ② becomes

$$\overline{\int_a^b} f\,d\alpha = f(t) \quad \& \quad \underline{\int_a^b} f\,d\alpha = f(t)$$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$

$$\Rightarrow f \in R(\alpha)$$

Since $f \in R(\alpha)$ So From equ①

$$S(f,P,\alpha) = \int_a^b f\,d\alpha = f(t)$$

$$\Rightarrow \int_a^b f\,d\alpha = f(t)$$

* * *

Theorem:- If $c_n \geq 0$ for $n \in \mathbb{N} = \{1,2,3,\ldots\}$ and $\sum_{n=1}^{\infty} c_n$ converges and t_n is a sequence of distinct points in $[a,b]$ and $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x-t_n)$ where $I(x-t_n)$ is unit step function then prove that

$$\int_a^b f(x)\,d\alpha = \sum_{n=1}^{\infty} c_n f(t_n)$$ where f is continuous on $[a,b]$

$c_n \geq 0 \Rightarrow c_n$ is positive term series.

Proof: Since $c_n \geq 0$ & $\sum_{n=1}^{\infty} c_n$ is convergent so for $\varepsilon > 0$ $\exists$ $m \in N$ s.t

$$\sum_{n=m+1}^{\infty} c_n < \epsilon$$

(If series is convergent then its advantage is that it can be summed up and we can find single number)

Now as given that f is continuous therefore f is bounded. then $\exists\ M > 0$ s.t

$$|f(x)| \leq M \quad \forall\ x \in [a,b]$$

Now consider

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x-t_n)$$

$$= \sum_{n=1}^{m} c_n I(x-t_n) + \sum_{n=m+1}^{\infty} c_n I(x-t_n)$$

$$= \alpha_1(x) + \alpha_2(x)$$

where $\alpha_1(x) = \sum_{n=1}^{m} c_n I(x-t_n)$ & $\alpha_2(x) = \sum_{n=m+1}^{\infty} c_n I(x-t_n)$

Again consider

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x-t_n)$$

$$= c_1 I(x-t_1) + c_2 I(x-t_2) + c_3 I(x-t_3) + \cdots$$

$$\leq c_1 + c_2 + c_3 + \cdots \qquad \because I(x-t_1) \leq 1$$

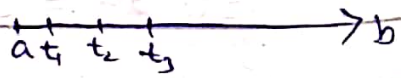
$$= \sum_{n=1}^{\infty} c_n$$

$$\Rightarrow \alpha(x) \leq \sum_{n=1}^{\infty} c_n$$

Since $\sum_{n=1}^{\infty} c_n$ is convergent so by B.C.T

$\alpha(x)$ is convergent

As $\alpha_2(x) = \sum_{n=m+1}^{\infty} c_n I(x-t_n)$



$$\alpha_2(a) = \sum_{n=m+1}^{\infty} c_n I(a-t_n)$$

$$= \sum_{n=m+1}^{\infty} c_n(0) \qquad (\because t_n > a \Rightarrow a - t_n < 0)$$

$$= 0$$

and $\alpha_2(b) = \sum_{n=m+1}^{\infty} C_n I(b-t_n)$

$= \sum_{n=m+1}^{\infty} C_n (1) = \sum_{n=m+1}^{\infty} C_n \quad \because b > t_n \Rightarrow b-t_n > 0$

Now consider $\int_a^b f\,d\alpha_1 = \int_a^b f\,d\left[\sum_{n=1}^{m} C_n I(x-t_n)\right]$

$= \int_a^b f\,d\left[C_1 I(x-t_1) + C_2 I(x-t_2) + \cdots + C_m I(x-t_m)\right]$

$= \int_a^b C_1 I(x-t_1) + \int_a^b f\,d\,C_2 I(x-t_2) + \cdots + \int_a^b f\,d\,C_m I(x-t_m)$

$= C_1 \int_a^b f\,d\,I(x-t_1) + C_2 \int_a^b f\,d\,I(x-t_2) + \cdots + C_m \int_a^b f\,d\,I(x-t_m)$

$= C_1 f(t_1) + C_2 f(t_2) + \cdots + C_m f(t_m) \quad \because \int_a^b f\,d\,I(x-t) = f(t)$

$$\int_a^b f\,d\alpha_1 = \sum_{n=1}^{m} C_n f(t_n) \quad \text{——— (1)}$$

Now consider

$\left|\int_a^b f\,d\alpha - \sum_{n=1}^{\infty} C_n f(t_n)\right| = \left|\int_a^b f\,d(\alpha_1+\alpha_2) - \sum_{n=1}^{m} C_n f(t_n) - \sum_{n=m+1}^{\infty} C_n f(t_n)\right|$

$= \left|\int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 - \sum_{n=1}^{m} C_n f(t_n) - \sum_{n=m+1}^{\infty} C_n f(t_n)\right|$

$= \left|\int_a^b f\,d\alpha_2 - \sum_{n=m+1}^{\infty} C_n f(t_n)\right|$ using (1)

$\leq \left|\int_a^b f\,d\alpha_2\right| - \left|\sum_{n=m+1}^{\infty} C_n f(t_n)\right|$

$\leq \int_a^b |f|\,d\alpha_2 + \left|\sum_{n=m+1}^{\infty} C_n f(t_n)\right|$

$$< \int_a^b M \, d\alpha_2 + \epsilon \quad \because |f| \le M \ \& \ \sum_{n=m+1}^{\infty} c_n < \epsilon$$

$$= M \int_a^b d\alpha_2 + \epsilon$$

$$= M[\alpha_2(b) - \alpha_2(a)] + \epsilon$$

$$= M\left[\sum_{n=m+1}^{\infty} c_n - 0\right] + \epsilon$$

$$< M\epsilon + \epsilon = \epsilon'$$

$$\left| \int_a^b f \, d\alpha - \sum_{n=1}^{\infty} c_n f(t_n) \right| < \epsilon'$$

Since ϵ' is arbitrary so

$$\left| \int_a^b f \, d\alpha - \sum_{n=1}^{\infty} c_n f(t_n) \right| = 0$$

$$\Rightarrow \int_a^b f \, d\alpha - \sum_{n=1}^{\infty} c_n f(t_n) = 0$$

$$\Rightarrow \int_a^b f \, d\alpha = \sum_{n=1}^{\infty} c_n f(t_n)$$

* * *

Theorem:- Suppose that derivative α' exist on $[a,b]$. If f & α' are Reimann Integrable on $[a,b]$ then $f \in RS_\alpha$ on $[a,b]$ and

$$\int_a^b f(x) \, d\alpha = \int_a^b f(x) \alpha'(x) \, dx$$

Proof) Since α' exists on $[a,b]$

$\Rightarrow \alpha$ is differentiable on $]x_{k-1}, x_k[$

$\Rightarrow \alpha$ is continuous on $[x_{k-1}, x_k]$

Then by M.V.T $\exists$ $c_k \in]x_{k-1}, x_k]$ such that

$$\frac{\alpha(x_k) - \alpha(x_{k-1})}{x_k - x_{k-1}} = \alpha'(c_k)$$

$$\Rightarrow \alpha(x_k) - \alpha(x_{k-1}) = \alpha'(c_k)\Delta x_k$$

$$\Rightarrow \Delta\alpha_k = \alpha'(c_k)\Delta x_k \quad \text{—— (A)}$$

Also since f & α' are Reimann Integrable
$\Rightarrow f\alpha'$ is R.I (Because Product of two R.I is also R.I)

Then for $\epsilon > 0$ $\exists$ a $\delta > 0$ such that

$$\left|\sum_{k=1}^{n} f(t_k)\alpha'(t_k)\Delta x_k - \int_a^b f(x)\alpha'(x)\,dx\right| < \frac{\epsilon}{2} \quad \text{—— (1)}$$

for every $P \in P(a,b)$ such that $\|P\| < \delta$

$\left\{ \because (f\alpha')(t_k) = f(t_k)\alpha'(t_k) \right.$
$\left. \text{If } f \text{ is R.I} \Rightarrow \left|\sum_{k=1}^{n} f(t_k)\Delta x_k - \int_a^b f\right| < \epsilon \right\}$

and for every choice of $t_k \in [x_{k-1}, x_k]$

since f is bounded
$\Rightarrow |f(x)| < \lambda$ for some $\lambda > 0$ $\quad$ (A')

Let $M_k' = \operatorname{Sup}\{\alpha'(x) : x \in [x_{k-1}, x_k]\}$

$m_k' = \operatorname{Inf}\{\alpha'(x) : x \in [x_{k-1}, x_k]\}$

As α' is Reimann Integrable so for $\epsilon > 0$ there exist a partition P of $[a,b]$ such that

$$S(\alpha', P) - s(\alpha', P) < \epsilon/2\lambda \quad \text{—— (2)}$$

Also for any $t_k, c_k \in [x_{k-1}, x_k]$

$$|\alpha'(t_k) - \alpha'(c_k)| \leq M_k' - m_k' \quad \text{—— (3)}$$

$$\Rightarrow \left|\sum_{k=1}^{n}\alpha'(t_k)\Delta x_k - \sum_{k=1}^{n}\alpha'(c_k)\Delta x_k\right| \leq \sum_{k=1}^{n} M_k'\Delta x_k - \sum_{k=1}^{n} m_k'\Delta x_k$$

$$= S(\alpha', P) - s(\alpha', P)$$

$$\Rightarrow \left|\sum_{k=1}^{n}\alpha'(t_k)\Delta x_k - \sum_{k=1}^{n}\alpha'(c_k)\Delta x_k\right| < \epsilon/2\lambda \quad \text{—— (4)}$$

Now consider

$$\left|\sum_{K=1}^{n} f(t_K)\Delta\alpha_K - \int_a^b f(x)\alpha'(x)\,dx\right|$$

$$= \left|\sum_{K=1}^{n} f(t_K)\alpha'(c_K)\Delta x_K - \int_a^b f(x)\alpha'(x)\,dx\right| \quad \text{using } *$$

$$= \left|\sum_{K=1}^{n} f(t_K)\alpha'(c_K)\Delta x_K - \sum_{K=1}^{n} f(t_K)\alpha'(t_K)\Delta x_K + \sum_{K=1}^{n} f(t_K)\alpha'(t_K)\Delta x_K - \int_a^b f(x)\alpha'(x)\,dx\right|$$

$$\leq \left|\sum_{K=1}^{n} f(t_K)\alpha'(c_K)\Delta x_K - \sum_{K=1}^{n} f(t_K)\alpha'(t_K)\Delta x_K\right| + \left|\sum_{K=1}^{n} f(t_K)\alpha'(t_K)\Delta x_K - \int_a^b f(x)\alpha'(x)\,dx\right|$$

$$= \left|\sum_{K=1}^{n} f(t_K)\left[\alpha'(c_K) - \alpha'(t_K)\right]\Delta x_K\right| + \left|\sum_{K=1}^{n} f(t_K)\alpha'(t_K)\Delta x_K - \int_a^b f(x)\alpha'(x)\,dx\right|$$

$$\leq \sum_{K=1}^{n} |f(t_K)|\,|\alpha'(c_K) - \alpha'(t_K)|\,\Delta x_K + \left|\sum_{K=1}^{n} f(t_K)\alpha'(t_K)\Delta x_K - \int_a^b f(x)\alpha'(x)\,dx\right|$$

$$< \lambda \cdot \frac{\epsilon}{2\lambda} + \frac{\epsilon}{2} = \epsilon \quad \text{using } *', ①, ④$$

$$\Rightarrow \left|\sum_{K=1}^{n} f(t_K)\Delta\alpha_K - \int_a^b f(x)\alpha'(x)\,dx\right| < \epsilon \quad \text{——— ⑤}$$

Since this result is true for all $P \in P(a,b)$ s.t $\|P\| < \delta$. In particular this result is also hold for the partition for which

$$\sum_{K=1}^{n} f(t_K)\Delta\alpha_K = \int_a^b f\,d\alpha$$

$\therefore$ equ ⑤ implies $\left|\int_a^b f\,d\alpha - \int_a^b f(x)\alpha'(x)\,dx\right| < \epsilon$

Since ϵ is orbitrary

$$\Rightarrow \int_a^b f\,d\alpha - \int_a^b f(x)\alpha'(x)\,dx = 0$$

$$\Rightarrow \int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx$$

MUHAMMAD TAHIR

0344-8563284

CHAPTER NO. 5

IMPROPER INTEGRAL★★

In Reimann Integrals
(i) Function is bounded i.e $f(x)$ is bounded in $[a, b]$
(ii) Interval is finite i.e $[a,b]$ is finite

Improper Integrals deals with the Integrals in which
(i) Function is bounded but interval is infinite
(ii) Function is unbounded but interval is finite
(iii) Unbounded function with infinite intervals.

Remark:- ① If a function is bounded and interval is infinite then the integral is called improper integral of 1st kind. for example

$$\int_0^{\infty} \sin x \, dx \, , \int_1^{\infty} \frac{1}{x^2+1} dx$$

② If function is unbounded and interval is finite then the integral is called improper integral of 2nd kind. for example

$$\int_0^1 \frac{1}{x} dx \, , \int_{-2}^{2} \frac{1}{x-1} dx$$

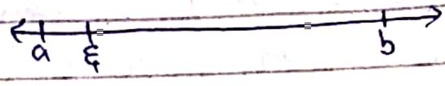
③ If both the conditions of Reimann Integrals are violated i.e If function is unbounded & interval is infite then the integral is called improper integral of 3rd kind. for example

$$\int_0^{\infty} \frac{1}{x} dx \, , \int_0^{\infty} \frac{\sin x}{x} dx$$

⇒ **2nd Kind of Improper Integrals:-** Let a function f be defined on closed interval $[a, b]$ everywhere except at finite number of points then.....

1) **Convergence at the Left End:** Let "a" be the only point of infinite discontinuity of "f" in $[a,b]$ then we have

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^{b} f(x)\,dx$$



$f(x)$ is R.I on $[a+\epsilon, b]$

2) **Convergence at the Right End:** Let "b" be the only point of infinite discontinuity of f in $[a,b]$ then we have

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \int_a^{b-\epsilon} f(x)\,dx$$

3) **Convergence at the Both Ends:** Let the end points "a" and "b" be the only points of infinite discontinuity of f in $[a,b]$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

فنکشن کو درمیان کے کسی pt
پر توڑ دیا جاتا ہے جس سے 2 disc
Pts End پر چلی جاتی ہیں

$$= \lim_{\epsilon \to 0} \int_{a+\epsilon}^{c} f(x)\,dx + \lim_{\epsilon \to 0} \int_c^{b-\epsilon} f(x)\,dx$$

Remark:- If c is the only point of infinite discontinuity of "f" in $[a,b]$ and $c \in]a,b[$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{\epsilon \to 0} \left[\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^{b} f(x)\,dx \right]$$

* * *

Questions:- Test the convergence of the following improper integrals.

1) $\displaystyle \int_0^1 \frac{1}{x^2}\,dx = \lim_{\epsilon \to 0} \int_{0+\epsilon}^{1} \frac{1}{x^2}\,dx$

$$= \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \frac{1}{x^2} \, dx$$

$$= \lim_{\epsilon \to 0} \left[\frac{+x^{-1}}{-1} \right]_{\epsilon}^{1}$$

$$= - \lim_{\epsilon \to 0} \left[\frac{1}{x} \right]_{\epsilon}^{1}$$

$$= - \lim_{\epsilon \to 0} \left[\frac{1}{1} - \frac{1}{\epsilon} \right] = -[1 - \infty]$$

$$\int_0^1 \frac{1}{x^2} \, dx = \infty$$

i.e integral is divergent.

2) $$\int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = \lim_{\epsilon \to 0} \int_0^{1-\epsilon} \frac{1}{\sqrt{1-x^2}} \, dx$$

$$= \lim_{\epsilon \to 0} \left| \sin^{-1} x \right|_0^{1-\epsilon}$$

$$= \lim_{\epsilon \to 0} \left| \sin^{-1}(1-\epsilon) - \sin^{-1}(0) \right|$$

$$= \sin^{-1}(1) - \sin^{-1}(0)$$

$$= \sin^{-1}(1) - 0$$

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = \pi/2$$

i.e Integral is convergent.

3) $$\int_{-1}^{1} \frac{dx}{x^2}$$

Here $f(x) = \frac{1}{x^2}$

$f(x)$ has infinite discontinuity at $x = 0 \in [-1, 1]$

$$\therefore \int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2}$$

$$= \lim_{\epsilon \to 0} \int_{-1}^{0-\epsilon} \frac{dx}{x^2} + \lim_{\epsilon \to 0} \int_{0+\epsilon}^{1} \frac{dx}{x^2}$$

$$= \frac{1}{2} \lim_{\epsilon \to 0} \Big[\log x - \log(x-2)\Big]_{\epsilon}^{2-\epsilon}$$

$$= \frac{1}{2} \lim_{\epsilon \to 0} \Big[\log \frac{x}{x-2}\Big]_{\epsilon}^{2-\epsilon}$$

$$= \frac{1}{2} \lim_{\epsilon \to 0} \Big[\log\Big(\frac{2-\epsilon}{-\epsilon}\Big) - \log\Big(\frac{\epsilon}{\epsilon-2}\Big)\Big]$$

$$= \frac{1}{2} \Big[-\log\Big(\frac{2-\epsilon}{+\epsilon}\Big) - \log(0)\Big]$$

$$= \frac{1}{2} \Big[-\log\Big(\frac{2-\epsilon}{+\epsilon}\Big) - \infty\Big]$$

$$= \infty$$

$\Rightarrow$ Integral is divergent.

***

* Infinite Discontinuity :-

$f(x)$ is continuous at $x = a$ if

(i) $\lim_{x \to a} f(x)$ exist (ii) $f(a)$ is defined

(iii) $\lim_{x \to a} f(x) = f(a)$

If (ii) & (iii) conditions does not satisfied we can do some alternations OR redefined the function and removed the discontinuity.

But if condition (1) does not satisfied i.e $f(x) = \frac{1}{x^2} \Rightarrow \lim_{x \to 0} f(x) = \frac{1}{0} = \infty$

i.e height does not exist at the point. We can not removed this discontinuity it is parmanent. This is called infinite discontinuity.

* Test for convergent at "a" of $\int_a^b f(x)\,dx$.

Let "a" be the only point of infinite discontinuity of f in $[a, b]$ when the ~~interval~~

integral f is +ve in a certain nbhd $]a, c[$ of a then $\int_{a+\epsilon_1}^{b} f(x)\,dx = \int_{a+\epsilon_1}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$

It follows that $\int_a^c f(x)\,dx$, $\int_a^b f(x)\,dx$ are either both convergent at "a" or divergent.

***

Theorem:- The necessary and sufficient condition for the convergence of improper integral $\int_a^b f(x)\,dx$ at 'a' where $f(x) > 0, \forall\, x \in [a, b]$ is that $\int_{a+\epsilon}^{b} f(x)\,dx < K$, where K is a +ve real.

Proof: Suppose $\int_a^b f(x)\,dx$ is convergence

$$\Rightarrow \int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^{b} f(x)\,dx \text{ exist}$$

Since $f(x) > 0$

$$\Rightarrow \int_{a+\epsilon}^{b} f(x)\,dx > 0$$

$$\text{Let } \phi(\epsilon) = \int_{a+\epsilon}^{b} f(x)\,dx$$

then $\phi(\epsilon)$ is monotonically increasing and tends to a finite limit because $\int_a^b f(x)\,dx$ is convergent

$\Rightarrow \phi(\epsilon_1)$ is bounded above

$\Rightarrow \exists$ a +ve real K s.t $\phi(\epsilon) < K$

$$\Rightarrow \int_{a+\epsilon}^{b} f(x)\,dx < K$$

working backward we obtain the converse result.

***

Theorem (B.C.T for 2nd kind of Improper Integrals) :- If f and g be two +ve functions s.t $f(x) \leq g(x), \forall\ x \in [a,b]$ then

(i) $\int_a^b g(x)\,dx$ convergence $\Rightarrow \int_a^b f(x)\,dx$ is convergence

(ii) $\int_a^b f(x)\,dx$ divergence $\Rightarrow \int_a^b g(x)\,dx$ is divergence

Here we assume f and g have infinite discontinuity at left end.

Proof Given $f(x) \leq g(x)$

$\Rightarrow$ for $\epsilon > 0$

$$\int_{a+\epsilon}^{b} f(x)\,dx \leq \int_{a+\epsilon}^{b} g(x)\,dx \quad \text{———} \ (1)$$

(i) Suppose $\int_a^b g(x)\,dx$ convergence

$\Rightarrow \exists$ a +ve real number K s.t

$$\int_{a+\epsilon}^{b} g(x)\,dx < K$$

(1) implies $\int_{a+\epsilon}^{b} f(x)\,dx \leq \int_{a+\epsilon}^{b} g(x)\,dx < K$

$$\Rightarrow \int_{a+\epsilon}^{b} f(x)\,dx < K$$

$\Rightarrow \int_a^b f(x)\,dx$ convergence.

(ii) Given $\int_a^b f(x)\,dx$ divergent.

$\Rightarrow \int_{a+\epsilon}^b f(x)\,dx$ is unbounded for some $\epsilon > 0$

Again ① implies $\int_{a+\epsilon}^b g(x)\,dx$ is unbounded for some $\epsilon > 0$

$\Rightarrow \int_a^b g(x)\,dx$ is divergent.

* * *

Theorem (L.C.T for 2nd kind of Improper Integrals)
If f and g are +ve functions in $[a,b]$ s.t

$$\lim_{x\to a+0} \frac{f(x)}{g(x)} = l, \text{ where } l \neq 0, \; l \neq \infty$$

then $\int_a^b f(x)\,dx$ & $\int_a^b g(x)\,dx$ behaves alike.

Proof) Since $f(x) > 0$ & $g(x) > 0$ $\forall\, x \in [a,b]$
$\Rightarrow l > 0$

Then for $\epsilon > 0$ $\exists$ a nbhd $]a,c[$ s.t

$$\left|\frac{f(x)}{g(x)} - l\right| < \epsilon \quad , \forall\, x \in]a,c[$$

Consider

$$\left|\frac{f(x)}{g(x)} - l\right| < \epsilon$$

$$\Rightarrow l - \epsilon < \frac{f(x)}{g(x)} < l + \epsilon$$

$$(l-\epsilon)\,g(x) < f(x) < (l+\epsilon)\,g(x) \quad \text{——— ①}$$

Suppose $\int_a^b f(x)\,dx$ is convergent

Then by B.C.T (Previous theorem)

$\int_a^b (l-\epsilon)\, g(x)\, dx$ is convergent (by ①)

$\Rightarrow \int_a^b g(x)\, dx$ is convergent

Next suppose that $\int_a^b f(x)\, dx$ is divergent

Then from R.H inequality of ① we have
$\int_a^b g(x)\, dx$ is divergent.

So both integrals behave alike.

* * *

v.v.I

Question:- Test the convergence of the integral

$$\int_0^1 \frac{1}{\sqrt{1-x^3}}\, dx$$

Solution Here $f(x) = \dfrac{1}{\sqrt{1-x^3}}$

Clearly $f(x)$ has an infinite discontinuity at $x=1$

Consider

$$f(x) = \frac{1}{\sqrt{1-x^3}} = \frac{1}{\sqrt{(1-x)(1+x+x^2)}}$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{\sqrt{(1+x+x^2)}} \quad \text{——— ①}$$

Now $\dfrac{1}{\sqrt{1+x+x^2}}$ is bounded on $[0,1]$

$\Rightarrow \dfrac{1}{\sqrt{1+x+x^2}} \leq M$ for some +ve M

Put in ①

$$f(x) \leq \frac{M}{\sqrt{1-x}}$$

$$\Rightarrow f(x) \leq M\, g(x) \quad \text{——— ②}$$

where $g(x) = \dfrac{1}{\sqrt{1-x}}$

Consider $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{\sqrt{1-x}}\,dx$

$$= \lim_{\epsilon \to 0} \int_0^{1-\epsilon} (1-x)^{-1/2}\,dx$$

$$= -\lim_{\epsilon \to 0} \int_0^{1-\epsilon} (1-x)^{-1/2}\,(-dx)$$

$$= -\lim_{\epsilon \to 0} \left[\frac{(1-x)^{1/2}}{1/2}\right]_0^{1-\epsilon}$$

$$= -2 \lim_{\epsilon \to 0} \left[(1-x)^{1/2}\right]_0^{1-\epsilon}$$

$$= -2 \lim_{\epsilon \to 0} \left[\sqrt{1-1+\epsilon} - \sqrt{1}\right]$$

$$= -2\left[0 - \sqrt{1}\right] = -2(-1)$$

$$= 2$$

$\Rightarrow \int_0^1 \frac{1}{\sqrt{1-x}}\,dx$ is convergent then

$\int_0^1 M \frac{1}{\sqrt{1-x}}\,dx$ is convergent then from ①

$\int_0^1 f(x)\,dx$ is also convergent (By B.C.T)

* * *

Theorem:- The improper integral $\int_a^b \frac{dx}{(x-a)^n}$ convergent if $n < 1$

Proof) If $n \leq 0$ then $\int_a^b \frac{dx}{(x-a)^n}$ is R.I

If $n > 0$ then $\int_a^b \frac{dx}{(x-a)^n}$ has infinite discontinuity at $x = a$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b \frac{dx}{(x-a)^n}$$

$$= \lim_{\epsilon \to 0} \int_{a+\epsilon}^b (x-a)^{-n}\, dx$$

$$= \lim_{\epsilon \to 0} \left| \frac{(x-a)^{-n+1}}{-n+1} \right|_{a+\epsilon}^b , \quad n \neq 1$$

$$= \frac{1}{-n+1} \lim_{\epsilon \to 0} \left| \frac{1}{(b-a)^{n-1}} - \frac{1}{(\epsilon)^{n-1}} \right|$$

$$= \frac{1}{1-n} \begin{cases} -\infty & \text{if } n > 1 \\ \frac{1}{(b-a)^{n-1}} & \text{if } n < 1 \end{cases}$$

For $n = 1$

$$\int_a^b \frac{dx}{x-a} = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b \frac{dx}{x-a}$$

$$= \lim_{\epsilon \to 0} \left| \ln(x-a) \right|_{a+\epsilon}^b$$

$$= \lim_{\epsilon \to 0} \left| \ln(b-a) - \ln(\epsilon) \right|$$

$$= -\infty \qquad \because \ln 0 = \infty$$

So we conclude

$$\int_a^b \frac{dx}{(x-a)^n} = \begin{cases} -\infty & \text{if } n \geq 1 \\ \text{finite} & \text{if } n < 1 \end{cases}$$

So integral is convergent if $n < 1$

Rough:- $\frac{1}{1-n} \lim_{\epsilon \to 0} \left[\frac{1}{(b-a)^{n-1}} - \frac{1}{(\epsilon)^{n-1}} \right]$

If $n-1 > 0$

$\Rightarrow \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} = \frac{1}{0} = \infty$

$\Rightarrow \left\{ \frac{1}{(b-a)^{n-1}} - \infty \right\} = -\infty$

If $n-1 < 0 \Rightarrow \frac{1}{\epsilon^{-2}} = \epsilon_1^2 \Rightarrow \lim_{\epsilon_1 \to 0} \epsilon^2 = 0$

$\Rightarrow \left\{ \frac{1}{(b-a)^{n-1}} - 0 \right\} = \frac{1}{(b-a)^{n-1}}$

$\therefore$ Integral in convergent if $n-1 < 0 \Rightarrow n < 1$

* * *

Question:- Using L.C.T check the convergence of integral $\int_0^{\pi/2} \frac{\sin x}{x^p} dx$.

Solution Clearly this integral is improper if $p > 0$

Here $f(x) \frac{\sin x}{x^p}$ has an infinite discontinuity at $x=0$

$$f(x) = \frac{\sin x}{x} \cdot \frac{1}{x^{p-1}}$$

Let $g(x) = \frac{1}{x^{p-1}}$

Now $\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \frac{\sin x}{x}$

$= 1$

Now $\int_0^{\pi/2} g(x)\, dx = \int_0^{\pi/2} \frac{1}{x^{p-1}} dx$

Then by previous Theorem this integral

is convergence if $P-1<1 \Rightarrow P<2$

So by L.C.T $\int_0^{\pi/2} \frac{\sin x}{x^p}$ cgs if $P<2$

$\star\star\star$

Question:- Test the convergence of $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$

Solution:- Here $f(x) = \frac{1}{x^{1/3}(1+x^2)}$

0 is only singularity of $f(x)$

Let $g(x) = \frac{1}{x^{1/3}}$

Now $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{1}{\cancel{x^{1/3}}(1+x^2)} \cdot \cancel{x^{1/3}}$

$= \lim_{x\to 0} \frac{1}{1+x^2} = 1 \neq 0$

Both the integrals behave alike.

Also $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x^{1/3}}\,dx$

By P-test $\int_0^1 \frac{1}{x^{1/3}}\,dx$ convergence as $P=\frac{1}{3}<1$

So by L.C.T given integral is convergent.

$\star\star\star$

Question:- Test the convergence of the integral

$$\int_0^1 \frac{dx}{x^2(1+x^2)}$$

Solution:- Here $f(x) = \frac{1}{x^2(1+x^2)}$

0 is the only singularity of $f(x)$

Let $g(x) = 1/x^2$

Now $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x^2}{x^2(1+x^2)}$

$= \lim_{x\to 0} \frac{1}{1+x^2} = 1 \neq 0$

So both the integrals behave alike.

Now $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x^2}\,dx$

By P- test $\int_0^1 \frac{1}{x^2}dx$ divergence as $P=2>1$

So by L.C.T $\int_0^1 \frac{1}{x^2(1+x^2)}\,dx$ divergence.

***

Question:- Test the convergence of $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$

Solution:- $f(x) = \frac{1}{x^{1/2}(1-x)^{1/3}}$ is not defined at $x=0,\ x=1$

Tak $\frac{1}{2} \in]0,1[$

$$\therefore \int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}} = \int_0^{1/2} \frac{dx}{x^{1/2}(1-x)^{1/3}} + \int_{1/2}^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$$

$$= I_1 + I_2 \quad \text{——— (1)}$$

First we solve I_1

$$I_1 = \int_0^{1/2} \frac{dx}{x^{1/2}(1-x)^{1/3}}$$

Let $g(x) = \frac{1}{x^{1/2}(1-x)^{1/3}}$, $h(x) = \frac{1}{x^{1/2}}$

$$\lim_{x\to 0} \frac{g(x)}{h(x)} = \lim_{x\to 0} \frac{\cancel{x^{1/2}}}{\cancel{x^{1/2}}(1-x)^{1/3}} = \lim_{x\to 0} \frac{1}{(1-x)^{1/3}}$$

$$= 1 \neq 0$$

Therefore by Limit Comparison Test

I_1 and $\int_0^{1/2} h(x)$ behave alike.

Now $\int_0^{1/2} \frac{1}{x^{1/2}}\,dx$ is convergence by P-test $\because P<1$

Beta function $= \beta(l,m) = \int_0^1 x^{l-1} (1-x)^{m-1} \, dx$

177

$\Rightarrow I_1$ is convergent. [If it will be divergent then $I_1 + I_2$ will be divergent but as I_1 is convergent so now we have to check I_2 also)

Now consider

$$I_2 = \int_{1/2}^{1} \frac{dx}{x^{1/2}(1-x)^{1/3}}$$

Let $f_1(x) = \dfrac{1}{x^{1/2}(1-x)^{1/3}}$ & $g_1(x) = \dfrac{1}{(1-x)^{1/3}}$

$$\lim_{x\to 1} \frac{f_1(x)}{g_1(x)} = \lim_{x\to 1} \frac{\cancel{(1-x)^{1/3}}}{x^{1/2}\cancel{(1-x)^{1/3}}} = \lim_{x\to 1} \frac{1}{x^{1/2}} = 1 \neq 0$$

Both $g_1(x)$ & $f_1(x)$ behaves alike

As $\displaystyle\int_{1/2}^{1} g_1(x)\,dx = \int_{1/2}^{1} \frac{1}{(1-x)^{1/3}}$ is convergent by p-test. So I_2 is also convergent.

Hence the given integral is also convergent since I_1 & I_2 are convergent.

* * *

Question:- Show that $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ convergence if $m, n > 0$.

Solution If $m-1 \geq 0$ & $n-1 \geq 0$ then given integral is R.I

For $m-1 < 0$ & $n-1 < 0$ given integral is Improper

clearly $f(x) = x^{m-1}(1-x)^{n-1}$ has points of infinite discontinuity at $x=0$, $x=1$

Note that integral

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^1 \frac{dx}{x^{1-m}(1-x)^{1-n}}$$

Scanned by CamScanner

$$= \int_0^{1/2} \frac{dx}{x^{1-m}(1-x)^{1-n}} + \int_{1/2}^{1} \frac{dx}{x^{1-m}(1-x)^{1-n}}$$

$$\int_0^1 x^{m-1}(1-x)^{n-1} = I_1 + I_2 \longleftarrow (1)$$

Consider $I_1 = \int_0^{1/2} \frac{dx}{x^{1-m}(1-x)^{1-n}}$

Let $f_1(x) = \frac{1}{x^{1-m}(1-x)^{1-n}}$ & $g_1(x) = \frac{1}{(x)^{1-m}}$

Now $\lim_{x \to 0^+} \frac{f_1(x)}{g_1(x)} = \lim_{x \to 0^+} \frac{1}{(1-x)^{1-n}}$

$= 1$

So by L.C.T $\int_0^{1/2} f_1(x)\,dx$ & $\int_0^{1/2} g_1(x)\,dx$ behaves alike

Now $\int_0^{1/2} g_1(x) = \int_0^{1/2} \frac{1}{x^{1-m}}\,dx$

Then by well known theorem the integral convergence if $1-m < 1 \Rightarrow -m < 0 \Rightarrow m > 0$

So I_1 is convergence if $m > 0$

Now consider

$$I_2 = \int_{1/2}^{1} \frac{dx}{x^{1-m}(1-x)^{1-n}}\,dx$$

Let $f_2(x) = \frac{1}{x^{1-m}(1-x)^{1-n}}$ & $g_2(x) = \frac{1}{(1-x)^{1-n}}$

Now $\lim_{x \to 1^-} \frac{f_2(x)}{g_2(x)} = \lim_{x \to 1^-} \frac{1}{x^{1-m}}$

$= 1$

So $\int_{1/2}^{1} f_2(x)\,dx$ & $\int_{1/2}^{1} g_2(x)\,dx$ behaves alike

$$\int_{1/2}^{1} g_2(x)\,dx = \int_{1/2}^{1} \frac{1}{(1-x)^{1-n}}\,dx$$

convergence if $1-n < 1 \Rightarrow n > 0$

So I_2 is convergence if $n > 0$

So from ① give integral is convergence if $m > 0$ and $n > 0$.

* * *

Question:- Show that $\int_0^{\pi/2} \frac{x^m}{\sin^n x}$ convergence if $n < (m+1)$.

Solution:- Here $f(x) = \frac{x^m}{\sin^n x} = \frac{x^m}{\sin^n x} \cdot \frac{x^n}{x^n}$

$$= \left(\frac{x}{\sin x}\right)^n x^{m-n}$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \left(\frac{x}{\sin x}\right)^n x^{m-n} = \lim_{x\to 0} (x)^{m-n}$$

$$= \begin{cases} 0 & \text{if } m-n > 0 \\ 1 & \text{if } m-n = 0 \\ \infty & \text{if } m-n < 0 \end{cases}$$

Hence given integral is improper if $m-n < 0$

As $f(x) = \left(\frac{x}{\sin x}\right)^n x^{m-n} = \left(\frac{x}{\sin x}\right)^n x^{-(n-m)}$

$$f(x) = \left(\frac{x}{\sin x}\right)^n \frac{1}{x^{n-m}}$$

Let $g(x) = \frac{1}{x^{n-m}}$

Now $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^n \frac{1}{x^{n-m}} \cdot x^{(n-m)}$

$= \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^n = 1 \neq 0$

Therefore both integrals $\int_0^{\pi/2} f(x)\,dx$ & $\int_0^{\pi/2} g(x)\,dx$ behaves alike.

Now $\int_0^{\pi/2} g(x)\,dx = \int_0^{\pi/2} \frac{1}{x^{n-m}}\,dx$ is convergent

if $n-m < 1 \Rightarrow n < 1+m$

Hence given integral is convergent for $n < m+1$

* * *

⇒ Improper Integrals of 1st kind:-

Definition:- (i) $\int_{-\infty}^{b} f(x)\,dx = \lim_{\lambda \to \infty} \int_{\lambda}^{b} f(x)\,dx$

(ii) $\int_a^{\infty} f(x)\,dx = \lim_{\lambda \to \infty} \int_a^{\lambda} f(x)\,dx$

(iii) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$

$= \lim_{\lambda \to -\infty} \int_{\lambda}^{0} f(x)\,dx + \lim_{\lambda \to \infty} \int_0^{\lambda} f(x)\,dx$

Remark:- $\int_a^{\infty} f(x)\,dx$ exists iff for $\epsilon > 0 \ \exists$

$\left| \int_a^{\infty} f(x)\,dx - \int_a^{\lambda} f(x)\,dx \right| < \epsilon, \ \forall \lambda > K$

⇒ **Cauchy Sequence:-** A sequence $\{S_n\}$ is said to be cauchy sequence if for every $\epsilon > 0$ there exist a +ve integer n_0 s.t

$$|S_n - S_m| < \epsilon \quad \forall\, n, m \geq n_0$$

* * *

Question:- Solve $\int_1^{\infty} \frac{dx}{x(1+x)}$.

Solution:-
$$\int_1^{\infty} \frac{dx}{x(1+x)} = \lim_{\lambda \to \infty} \int_1^{\lambda} \frac{dx}{x(1+x)} \quad \text{——— (1)}$$

1st we solve $\int_1^{\lambda} \frac{dx}{x(1+x)}$

Resolve $\frac{1}{x(1+x)} = \frac{A}{x} + \frac{B}{1+x}$

$\Rightarrow \; A = 1 \;\; \& \;\; B = -1$

$$\int_1^{\lambda} \frac{dx}{x(1+x)} = \int_1^{\lambda} \left[\frac{1}{x} - \frac{1}{x+1}\right] dx$$

$$= \left[\ln x - \ln(x+1)\right]_1^{\lambda}$$

$$= \left[\ln\left(\frac{x}{x+1}\right)\right]_1^{\lambda}$$

$$= \ln \frac{\lambda}{\lambda+1} - \ln \frac{1}{2}$$

equ (1) $\Rightarrow$
$$\int_1^{\infty} \frac{dx}{x(1+x)} = \lim_{\lambda \to \infty} \left[\ln\left(\frac{\lambda}{\lambda+1}\right) - \ln \frac{1}{2}\right]$$

$$= \lim_{\lambda \to \infty} \left[\ln \frac{1}{1+1/\lambda} - \ln \frac{1}{2}\right]$$

$$= \ln 1 - \ln \frac{1}{2} \Rightarrow 0 - \ln \frac{1}{2}$$

$$= -\ln \frac{1}{2}$$

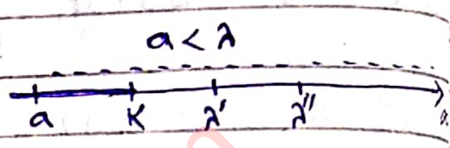
∴ Integral is convergent.

⇒ Cauchy's General Theorem of Convergence:-

Suppose f is bounded on $[a, \lambda]$, $\forall \lambda > a$.
$[a, \infty]$

Then $\int_a^\infty f(x)\,dx$ exists (converges) iff for every $\epsilon > 0$ $\exists$ real number $K > a$ (However large)

s.t $\left| \int_{\lambda'}^{\lambda''} f(x)\,dx \right| < \epsilon \quad \forall \lambda'' > \lambda' > K$ ——— ①



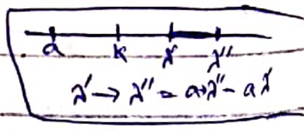
Proof:- Suppose $\int_a^\infty f(x)\,dx$ exist

⇒ for $\epsilon > 0$ there exist a real number K s.t

$$\left| \int_a^\infty f(x)\,dx - \int_a^\lambda f(x)\,dx \right| < \epsilon/2 \quad \forall \lambda > K \quad \text{——— ②}$$

To prove inequality ① hold

for this let λ'' & λ' are two real numbers s.t $\lambda'' > \lambda' > K$



Consider $\int_{\lambda'}^{\lambda''} f(x)\,dx = \int_{\lambda'}^{a} f(x)\,dx + \int_a^{\lambda''} f(x)\,dx$

$$= -\int_a^{\lambda'} f(x)\,dx + \int_a^{\lambda''} f(x)\,dx$$

$$\left| \int_{\lambda'}^{\lambda''} f(x)\,dx \right| = \left| \int_a^\infty f(x)\,dx - \int_a^{\lambda'} f(x)\,dx + \int_a^{\lambda''} f(x)\,dx - \int_a^\infty f(x)\,dx \right|$$

$$< \left| \int_a^\infty f(x)\,dx - \int_a^{\lambda'} f(x)\,dx \right| + \left| \int_a^{\lambda''} f(x)\,dx - \int_a^\infty f(x)\,dx \right|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{by ②}$$

$$\Rightarrow \left| \int_{\lambda'}^{\lambda''} f(x)\,dx \right| < \epsilon \quad \forall\ \lambda'' > \lambda' > K$$

Conversely given inequality ① is true

To prove $\int_a^{\infty} f(x)\,dx$ is convergent.

Consider the integral $S_n = \int_a^{\lambda+n} f(x)\,dx$

First we show that the sequence S_n is a cauchy sequence.

Consider $S_n - S_m = \int_a^{\lambda+n} f(x)dx - \int_a^{\lambda+m} f(x)\,dx$

$$= \int_{\lambda+m}^{\lambda+n} f(x)\,dx$$

$$\Rightarrow |S_n - S_m| = \left| \int_{\lambda+m}^{\lambda+n} f(x)\,dx \right|$$

$$< \epsilon/2 \quad \forall\ n \geq m > K \quad \text{using ①}$$

$\Rightarrow S_n$ is a cauchy sequence in $\mathbb{R}$.

As $\mathbb{R}$ is complete so S_n is convergence. (only in complete field every convergent is cauchy sequence)

Let $\lim_{n \to \infty} S_n = \ell$

Then by the definition of convergence, for $\epsilon > 0$ there exist a natural number n_0 s.t

$$|S_n - \ell| < \epsilon/2 \quad \forall\ n > n_0$$

$\lim_{\lambda \to a} S_n = \int_a^{\lambda} f(x)dx = \int_a^{\infty} f(x)dx$

Now $\left| \int_a^{\lambda} f(x)\,dx - \ell \right| = \left| \int_a^{\lambda+n} f(x)\,dx + \int_{\lambda+n}^{\lambda} f(x)dx - \ell \right|$

$$= \left| \left(\int_a^{\lambda+n} f(x)dx - \ell \right) + \int_{\lambda+n}^{\lambda} f(x)\,dx \right|$$

$$\leq \left|\int_a^{\lambda+n} f(x)\,dx - \ell\right| + \left|\int_\lambda^{\lambda+n} f(x)\,dx\right|$$

$$= |S_n - \ell| + \left|\int_\lambda^{\lambda+n} f(x)\,dx\right|$$

$$< \frac{\epsilon}{2} + \epsilon/2$$

$$\Rightarrow \left|\int_a^{\lambda} f(x)\,dx - \ell\right| < \epsilon$$

$$\Rightarrow \left|\int_a^{\lambda} f(x)\,dx - \int_a^{\infty} f(x)\,dx\right| < \epsilon$$

$$\Rightarrow \left|\int_a^{\infty} f(x)\,dx - \int_a^{\lambda} f(x)\,dx\right| < \epsilon$$

$\Rightarrow \int_a^{\infty} f(x)\,dx$ exists by previous theorem

Imp ★★★

Question:- Show that $\int_0^{\infty} \frac{\sin x}{x}\,dx$ is convergent.

Solution:- Consider $0 < \lambda' < \lambda < \lambda''$

Then $$\int_{\lambda'}^{\lambda''} \frac{\sin x}{x}\,dx = \int_{\lambda'}^{\lambda} \frac{\sin x}{x}\,dx + \int_{\lambda}^{\lambda''} \frac{\sin x}{x}\,dx$$

Then by well known theorem there are c_1 & c_2 such that,

$$0 < \lambda' < c_1 < \lambda < c_2 < \lambda''$$

So $$\int_{\lambda'}^{\lambda''} \frac{\sin x}{x}\,dx = \frac{1}{c_1}\int_{\lambda'}^{\lambda} \sin x\,dx + \frac{1}{c_2}\int_{\lambda}^{\lambda''} \sin x\,dx$$

By Mean Value theorem:-

If f is continuous on $[a,b]$ and If

$g > 0$ is integrable on $[a,b]$ then there is number $c \in [a,b]$ s.t

$$\int_a^b fg = f(c) \int_a^b g$$

$\sin x \cdot \frac{1}{x} \, dx \Rightarrow \frac{1}{c_1} \int \sin x \, dx$
($\sin x$: $g(x)$, $\frac{1}{x}$: $f(x)$)

$$\int_{\lambda'}^{\lambda''} \frac{\sin x}{x} \, dx = \frac{1}{c_1} \left[-\cos x\right]_{\lambda'}^{\lambda} + \frac{1}{c_2} \left[-\cos x\right]_{\lambda}^{\lambda''}$$

$$= -\frac{1}{c_1} \left[\cos\lambda - \cos\lambda'\right] - \frac{1}{c_2} \left[\cos\lambda'' - \cos\lambda\right]$$

$$\left|\int_{\lambda'}^{\lambda''} \frac{\sin x}{x} \, dx\right| = \left|-\frac{1}{c_1}(\cos\lambda - \cos\lambda') - \frac{1}{c_2}(\cos\lambda'' - \cos\lambda)\right|$$

$$\leq \left|-\frac{1}{c_1}(\cos\lambda - \cos\lambda')\right| + \left|-\frac{1}{c_2}(\cos\lambda'' - \cos\lambda)\right|$$

$$= \frac{1}{c_1}\left|\cos\lambda - \cos\lambda'\right| + \frac{1}{c_2}\left|\cos\lambda'' - \cos\lambda\right|$$

$$\leq \frac{1}{c_1}(2) + \frac{1}{c_2}(2) \qquad \text{max value of } \cos\lambda = 1$$

$$\left|\int_{\lambda'}^{\lambda''} \frac{\sin x}{x} \, dx\right| \leq 2\left(\frac{1}{c_1} + \frac{1}{c_2}\right) < \frac{4}{\lambda'}$$

Then for $\epsilon > 0$ we can find $k = 4/\lambda'$ s.t

$$\left|\int_{\lambda'}^{\lambda''} \frac{\sin x}{x} \, dx\right| < \epsilon \quad \text{for } \lambda'' > \lambda' > k$$

$\Rightarrow$ Given integral is convergent by cauchy critaria.

***

Rough:- $0 < \lambda' < c_1 < c_2 < \lambda''$

$c_1 > \lambda'$, $c_2 > \lambda'$

$\Rightarrow \lambda' < c_1$, $\frac{1}{c_2} < \frac{1}{\lambda'}$ — ②

$\frac{1}{\lambda'} > \frac{1}{c_1} \Rightarrow \frac{1}{c_1} < \frac{1}{\lambda'}$ — ①

$\Rightarrow \frac{1}{c_1} + \frac{1}{c_2} < 2/\lambda'$

$\lambda' > k$

$\frac{1}{\lambda'} < 1/k$

$4/\lambda' < \frac{4}{k} = \epsilon$

$\frac{4}{\lambda'} < \epsilon$

$|\cos\lambda - \cos\lambda'|$

$\leq |\cos\lambda| + |\cos\lambda'|$

$< 1 + 1 = 2$

⇒ **Comparison Test for Improper Integrals of 1st kind:-** (Basic Comparison Test).

Suppose f and g are R.I on $[a, \lambda]$ $\forall\ \lambda > a$ s.t $0 \leq f(x) \leq g(x)$ $\forall\ x \geq a$ then

(i) If $\int_a^\infty g(x)\,dx$ is cgs then $\int_a^\infty f(x)\,dx$ is cgs

(ii) If $\int_a^\infty f(x)\,dx$ is dgt then $\int_a^\infty g(x)\,dx$ is dgt

Proof) given $f(x) \leq g(x)$

$$\Rightarrow \int_a^\lambda f(x)\,dx \leq \int_a^\lambda g(x)\,dx \quad \text{——— (1)}$$

1) Given $\int_a^\infty g(x)\,dx$ is convergent

$\Rightarrow \int_a^\lambda g(x)\,dx$ is bounded $\forall\ \lambda > a$

$\Rightarrow \int_a^\lambda g(x)\,dx < M$, where M is a +ve constant

$\Rightarrow \int_a^\lambda f(x)\,dx < M$ $\forall\ \lambda > a$

$\Rightarrow \int_a^\infty f(x)\,dx$ is convergent.

2) Given $\int_a^\infty f(x)\,dx$ is divergent

$\Rightarrow \int_a^\lambda f(x)\,dx$ is unbounded for som $\lambda > a$

So from (1) $\int_a^\lambda g(x)\,dx$ is unbounded for some $\lambda > a$

$\Rightarrow \int_a^\infty g(x)\,dx$ is divergent.

* * *

⇒ **Limit Comparison Test:-** Suppose f and g are Reimann Integrable on $[a, \lambda]$ $\forall\ \lambda > a$ such that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = l > 0$$

then both the integrals $\int_a^\infty f(x)\,dx$ & $\int_a^\infty g(x)\,dx$ behaves alike.

Proof: Since $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$

So for $\epsilon > 0$ there exist a number $K > 0$ s.t

$$\left| \frac{f(x)}{g(x)} - l \right| < \epsilon \quad \forall\ x > K$$

$$\Rightarrow l - \epsilon < \frac{f(x)}{g(x)} < l + \epsilon$$

$$\Rightarrow g(x)(l-\epsilon) < f(x) < g(x)(l+\epsilon) \quad \text{——} \ (1)$$

Then by Basic comparison test from (1) we can see that

(i) If $\int_a^\infty f(x)\,dx$ converges then $\int_a^\infty g(x)\,dx$ also cgs

(ii) If $\int_a^\infty f(x)\,dx$ diverges then $\int_a^\infty g(x)\,dx$ also dgs.

So both the integrals behave alike.

* * *

Theorem:- (α-Test) Let $f(x) > 0$ on $[a, \infty[$, $a > 0$ & let $\lim_{x\to\infty} x^\alpha f(x) = l$ then

$\int_a^\infty f(x)\,dx$ is cgt if $\alpha > 1$ & dgt if $\alpha \le 1$.

Proof: Befor proving the α-test we prove another test called p-test & stated as

$\int_a^\infty \frac{1}{x^p}\,dx$ is dgt if $p \le 1$ & cgt if $p > 1$

Consider $\int_a^\infty \frac{1}{x^p}\,dx = \lim_{\lambda \to \infty} \int_a^\lambda x^{-p}\,dx$

$$= \lim_{\lambda \to \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_a^\lambda, \quad p \ne 1$$

$$= \lim_{\lambda \to \infty} \frac{1}{1-p} \left[x^{1-p} \right]_a^\lambda$$

$$= \lim_{\lambda \to \infty} \frac{1}{1-p} \left[\lambda^{1-p} - a^{1-p} \right]$$

$$= \frac{1}{1-p} \begin{cases} -a^{1-p} & \text{if } p > 1 \\ \infty & \text{if } p < 1 \end{cases}$$

for $p = 1$

Consider $\int_a^\infty \frac{1}{x}\,dx = \lim_{\lambda \to \infty} \left[\ln x \right]_a^\lambda$

$$= \lim_{\lambda \to \infty} \left[\ln \lambda - \ln a \right]$$

$$= \infty$$

So $\int_a^\infty \frac{1}{x^p}\,dx$ is dgt if $p \le 1$ & cgt

if $p > 1$

Given $\lim_{x \to \infty} x^\alpha f(x) = l$

$\Rightarrow \lim_{x \to \infty} \frac{f(x)}{x^{-\alpha}} = l$

So by L.C.T $\int_a^\infty f(x)\,dx$ & $\int_a^\infty x^{-\alpha}$ cgs or dgs together

But from P-test it is clear that $\int_a^\infty x^{-\alpha}\,dx$ is cgt if $\alpha > 1$ & dgt if $\alpha \leq 1$

So $\int_a^\infty f(x)dx$ is cgt if $\alpha > 1$ & dgt if $\alpha \leq 1$

***

Question:- Test the convergence of the Integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)\,dx$$

if f is even

$$\int_{-a}^{a} f(x)dx = 0$$

if f is odd

$f(x) = \frac{1}{1+x^2} = 1/1+x^2$

$f(-x) = \frac{1}{1+(-x)^2} = \frac{1}{1+x^2}$

$\Rightarrow f(-x) = f(x)$

so $f(x)$ is even

Solution:- $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\int_0^{\infty} \frac{dx}{1+x^2}$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\lim_{\lambda\to\infty} \int_0^{\lambda} \frac{dx}{1+x^2}$$

$$= 2\lim_{\lambda\to\infty} \left[\tan^{-1} x\right]_0^{\lambda}$$

$$= 2\lim_{\lambda\to\infty} \left[\tan^{-1}\lambda - \tan^{-1} 0\right]$$

$$= 2\left[\tan^{-1}\infty - 0\right] = 2(\pi/2)$$

$$= \pi$$

So integral is finite. So given integral is convergent.

***

Question:- Investigate the convergence of

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$$

Solution $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\int_0^{\infty} \frac{dx}{(1+x^2)^2}$

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2 \lim_{\lambda \to \infty} \int_0^{\lambda} \frac{dx}{(1+x^2)^2}$$

Put $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2 \lim_{\lambda \to \infty} \int_0^{\lambda} \frac{\sec^2\theta \, d\theta}{(1+\tan^2\theta)^2}$$

$$= 2 \lim_{\lambda \to \infty} \int_0^{\lambda} \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^2}$$

$$= 2 \lim_{\lambda \to \infty} \int_0^{\lambda} \frac{1}{\sec^2\theta} \, d\theta$$

$$= 2 \lim_{\lambda \to \infty} \int_0^{\lambda} \cos^2\theta \, d\theta$$

$$= 2 \lim_{\lambda \to \infty} \int_0^{\lambda} \frac{2\cos^2\theta}{2} \, d\theta$$

$$= 2 \lim_{\lambda \to \infty} \frac{1}{2} \int_0^{\lambda} (1+\cos 2\theta) \, d\theta$$

$$= \lim_{\lambda \to \infty} \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\lambda}$$

$$= \lim_{\lambda \to \infty} \left[\tan^{-1}x + \frac{\tan\theta}{1+\tan^2\theta}\right]_0^{\lambda}$$

$$= \lim_{\lambda \to \infty} \left[\tan^{-1}x + \frac{x}{1+x^2}\right]_0^{\lambda}$$

$$= \lim_{\lambda \to \infty} \left[\tan^{-1}\lambda + \frac{\lambda}{1+\lambda^2} - 0 - 0\right]$$

$$= \left[\tan^{-1}\infty + 0\right]$$

$$= \pi/2 \Rightarrow \text{convergent.}$$

⇒ **Drichlet Test:-** Let g be a bounded and monotonic on $[a,\infty[$ s.t $\lim_{x\to\infty} g(x)=0$. If $\int_a^{\lambda} f(x)\,dx$ is bounded $\forall \lambda > a$ then

$$\int_a^{\infty} f(x)\,g(x)\,dx \text{ is convergent.}$$

Proof) Given $\int_a^{\lambda} f(x)\,dx$ is bounded

$$\Rightarrow \left|\int_a^{\lambda} f(x)\,dx\right| < u \quad \forall \lambda > a$$

—— ⊛, where u is any +ve constant

Also $\lim_{x\to\infty} g(x) = 0$

So for $\epsilon > 0$ there exist a number K s.t

$$|g(x)-0| < \frac{\epsilon}{4u} \quad \forall x > K \quad \text{—— } \star'$$

Again by M.V.T of integration we have

$$\left|\int_{\lambda'}^{\lambda''} f(x)g(x)\,dx\right| = \left|g(\lambda')\int_{\lambda'}^{c} f(x)\,dx + g(\lambda'')\int_{c}^{\lambda''} f(x)\,dx\right| \quad \text{2nd M.V.T}$$

$$\leq |g(\lambda')|\left|\int_{\lambda'}^{c} f(x)\,dx\right| + |g(\lambda'')|\left|\int_{c}^{\lambda''} f(x)\,dx\right|$$

using $\star$ & $\star'$ we have

$$< \frac{\epsilon}{4u}\times u + \frac{\epsilon}{4u}\times u \quad \forall \lambda'' > \lambda' > K$$

$$= \epsilon/2 < \epsilon$$

$$\Rightarrow \left|\int_{\lambda'}^{\lambda''} f(x)\,g(x)\,dx\right| < \epsilon \quad \forall \lambda'' > \lambda' > K$$

So by Cauchy general principal of cgs

$\int_a^{\infty} f(x)\,g(x)\,dx$ is cgt

Question:- Using Drichlet Test show that the improper integral $\int_0^\infty \frac{\sin x}{x}$ is convergent.

Solution:- Let $f(x) = \sin x$
& $g(x) = 1/x$

Clearly $g(x)$ is monotonic & $\lim_{x\to\infty} g(x) = 0$
Also note that g is bounded
Let $\lambda > 0$
Consider $\int_0^\lambda f(x)\,dx = \int_0^\lambda \sin x\,dx$

$$= -\left[\cos x\right]_0^\lambda$$

$$= -\left[\cos\lambda - \cos 0\right]$$

$$= 1 - \cos\lambda$$

$$\left|\int_0^\lambda f(x)\,dx\right| = |1-\cos\lambda|$$

$$\leq 1 + |\cos\lambda|$$

$$\leq 2 \qquad \because |\cos\lambda| \leq 1$$

$\Rightarrow \int_0^\lambda \sin x\,dx$ is bounded $\forall\, \lambda > 0$

So by drichlet Test $\int_0^\infty \frac{\sin x\,dx}{x}$ is cgt

* * *

Definition:- $\int_a^\infty f(x)\,dx$ is said to be absolutely convergent if $\int_a^\infty |f(x)|$ is convergent.

* * *

Definition:- If $\int_0^{\infty} |f(x)|\,dx$ is divergent but $\int_0^{\infty} f(x)\,dx$ is convergent then $\int_a^{\infty} f(x)\,dx$ is said to be conditionaly convergent.

***

Example:- Show that $\int_0^{\infty} \frac{\sin x}{x}\,dx$ is conditionaly convergent.

Solution:- As we know that $\int_0^{\infty} \frac{\sin x}{x}\,dx$ is convergent (Already Proved)

Next we have to show that $\int_0^{\infty} |f(x)|\,dx$ is dgt.

Consider $\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx = \int_0^{n\pi} \frac{|\sin x|}{|x|}\,dx$

$$\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx = \int_0^{\pi} \frac{|\sin x|}{|x|}\,dx + \int_{\pi}^{2\pi} \frac{|\sin x|}{|x|}\,dx + \cdots + \int_{(n-1)\pi}^{n\pi} \frac{|\sin x|}{|x|}\,dx$$

$$= \sum_{K=1}^{n} \int_{(K-1)\pi}^{K\pi} \frac{|\sin x|}{|x|}\,dx \quad \text{——— ①}$$

Put $x = (K-1)\pi + u$, $0 < u < \pi$

when $x = (K-1)\pi$, $u = 0$

when $x = K\pi$, $u = \pi$

$$\therefore ① \Rightarrow \int_0^{n\pi} \frac{|\sin x|}{|x|}\,dx = \sum_{K=1}^{n} \int_0^{\pi} \frac{\sin (K-1)\pi + u}{K\pi - (\pi - u)}\,du$$

$$> \sum_{K=1}^{n} \int_0^{\pi} \frac{\sin u}{K\pi}\,du$$

$$= \sum_{K=1}^{n} \int_0^{\pi} \frac{\sin u \, du}{K\pi} \, du$$

$$\int_0^{n\pi} \frac{\sin|x|}{|x|} \, dx > \sum_{K=1}^{n} \frac{1}{K\pi} \left[-\cos u\right]_0^{\pi}$$

$$= \frac{2}{\pi} \sum_{K=1}^{n} \frac{1}{K}$$

So the series on R.H.S is divergent

$$\Rightarrow \int_0^{n\pi} \frac{\sin|x|}{|x|} \, dx \text{ is unbounded}$$

$$\Rightarrow \int_0^{\infty} \left|\frac{\sin x}{x}\right| \text{ is dgt}$$

So $\int_0^{\infty} \frac{\sin x}{x}$ is conditionaly cgt.

* * *

Question:- Show that $\int_0^{\infty} \frac{\sin x}{(1+x)^{\alpha}} \, dx$ is convergent for $\alpha > 0$

Solution:- Let $f(x) = \sin x$.

& $g(x) = \frac{1}{(1+x)^{\alpha}}$

Clearly, $g(x)$ is bounded if $\alpha > 0$ & also monotonic

Also $\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{1}{(1+x)^{\alpha}} = 0$ as $\alpha > 0$

$$\int_0^{\lambda} f(x) \, dx = \int_0^{\lambda} \sin x \, dx, \quad \lambda > 0$$

$$\int_0^{\lambda} f(x) \, dx = -\left[\cos x\right]_0^{\lambda}$$

$$= -\left[\cos \lambda - \cos 0\right]$$

$$= -\left[\cos \lambda - 1\right]$$

$$\int_0^{\lambda} f(x)\,dx = -\cos\lambda + 1$$

$$\left|\int_0^{\lambda} f(x)\,dx\right| \le |-\cos\lambda| + |1| \le 1+1 = 2$$

$$\Rightarrow \left|\int_0^{\lambda} f(x)\,dx\right| \le 2$$

$\Rightarrow \int_0^{\lambda} f(x)\,dx$ is bounded.

Hence by Drichlet Test

$\int_0^{\infty} f(x)\,g(x)\,dx$ is convergent i.e

$\int_0^{\infty} \frac{\sin x}{(1+x)^{\alpha}}$ is convergent

* * *

Question:- Show that $\int_0^{\infty} \frac{\cos 2x}{(100+x)^{\alpha-1}}$ is convergent for $\alpha > 1$

Solution:- Let $f(x) = \cos 2x$ & $g(x) = \frac{1}{(100+x)^{\alpha-1}}$

Clearly $g(x)$ is bounded if $\alpha > 1$ & also monotonic (decreasing)

Also $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{1}{(100+x)^{\alpha-1}} = 0$ as $\alpha > 1$

$$\int_0^{\lambda} f(x)\,dx = \int_0^{\lambda} \cos 2x\,dx \quad , \lambda > 2$$

$$= \left[\frac{\sin 2x}{2}\right]_0^{\lambda} = \left[\frac{\sin 2\lambda}{2} - 0\right]$$

$$\left|\int_0^{\lambda} f(x)\,dx\right| = \left|\frac{\sin 2\lambda}{2}\right| = \frac{|\sin 2\lambda|}{|2|} \le \frac{1}{2}$$

$\Rightarrow \int_0^{\lambda} f(x)\,dx$ is bounded

$\Rightarrow$ All the conditions of Drichlet Test are satisfied. Hence by Drichlet Test

$$\int_0^\infty \frac{\cos 2x}{(100+x)^{\alpha-1}}\,dx \quad \text{is convergent.}$$

* * *

$\Rightarrow$ **Integral Test:-** Let f be +ve & decreasing on $[1, \infty[$ Then $\int_1^\infty f(x)dx$ & $\sum_{k=1}^\infty f(k)$ cgs or dgs.

Proof) Since f is decreasing on $[1, \infty[$ so for $k-1 < x < k$

$$\Rightarrow f(k) \leq f(x) \leq f(k-1)$$

Integrate w.r.t x from $(k-1)$ to k

$$\int_{k-1}^{k} f(k)dx \leq \int_{k-1}^{k} f(x)\,dx \leq \int_{k-1}^{k} f(k-1)\,dx$$

$$f(k)\,[x]_{k-1}^{k} \leq \int_{k-1}^{k} f(x)\,dx \leq f(k-1)\,[x]_{k-1}^{k}$$

$$\Rightarrow f(k) \leq \int_{k-1}^{k} f(x)\,dx \leq f(k-1)$$

$$\sum_{k=2}^{n} f(k) \leq \sum_{k=2}^{n} \int_{k-1}^{k} f(x)dx \leq \sum_{k=2}^{n} f(k-1) \quad \text{———} \circledast$$

from $\circledast$ it is clear that

$$\sum_{k=2}^{n} f(k) \leq \int_1^n f(x)\,dx \leq \sum_{k=2}^{n} f(k-1) \quad \text{———} ①$$

from ① by B.C.T we conclude that both, The integral $\int_1^\infty f(x)\,dx$ & the series

$\sum_{k=1}^{\infty} f(k)$ behaves alike.

★ $\sum_{k=2}^{n} \int_{k-1}^{k} f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \dots + \int_{n-1}^{n} f(x)\,dx$

$$= \int_1^n f(x)\,dx$$

★ ★ ★

Question:- Show that $\int_0^\infty e^{-x^2/2}\,dx = \sqrt{\pi/2}$

Solution:- Let us consider

$$\int_0^\infty e^{-x^2/2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx \quad \text{——— (1)}$$

$\because \int_{-a}^{a} f = 2\int_0^a f$ if f is even

Let $I = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx$

$$\Rightarrow I \cdot I = \left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right)$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy \quad \text{——— (2)}$$

We Transform the above integral into polar co-ordinate system

$\therefore$ Put $x = r\cos\theta$, $y = r\sin\theta$, $0 < r < \infty$
$0 \leq \theta < 2\pi$

& $dx\,dy = r\,dr\,d\theta$

& $x^2 + y^2 = r^2$

$\therefore$ equ (2) implies $I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}(r^2)}\,r\,dr\,d\theta$

$$= \left(\int_0^\infty e^{-\frac{1}{2}r^2} r\,dr\right)\left(\int_0^{2\pi} d\theta\right)$$

$$= 2\pi \int_0^\infty e^{-\frac{1}{2}r^2} r\,dr$$

$$I^2 = 2\pi \int_0^\infty e^{-\frac{1}{2}r^2} r\,dr$$

Put $\frac{r^2}{2} = z \Rightarrow r\,dr = dz$

when $r = 0 \Rightarrow z = 0$, when $r \to \infty, z \to \infty$

$$\Rightarrow I^2 = 2\pi \int_0^\infty e^{-z}\,dz$$

$$= 2\pi \left[\frac{e^{-z}}{-1}\right]_0^\infty = -2\pi\left[e^{-z}\right]_0^\infty$$

$$= -2\pi\left[e^{-\infty} - e^{0}\right] = -2\pi[0-1] \quad \because e^{-\infty} = 0$$

$$I^2 = 2\pi$$

$I = \sqrt{2\pi}$ Put in ①

$$\int_0^\infty e^{-x^2/2}\,dx = \frac{1}{2}\sqrt{2\pi} = \sqrt{\frac{\pi}{2}}$$

***

* Improper Integral of 3rd Kind:-

Question:- Show that $\int_0^\infty e^{-x} x^{\alpha-1}\,dx$ cgs if $\alpha > 1$

Solution:- Here $f(x) = e^{-x} x^{\alpha-1}$

has infinite discontinuity at $x = 0$ if $\alpha - 1 < 0$
$\Rightarrow \alpha < 1$

Now choose any point b/w 0 and ∞ say 1 st

$$\int_0^\infty e^{-x} x^{\alpha-1}\,dx = \underbrace{\int_0^1 e^{-x} x^{\alpha-1}}_{2^{nd}\text{ kind}} + \underbrace{\int_1^\infty e^{-x} x^{\alpha-1}}_{1^{st}\text{ kind}} \quad \text{——— ①}$$

1 st we find the condition under which $\int_0^1 e^{-x} x^{\alpha-1}\,dx$ converge. Let $g(x) = \frac{1}{x^{1-\alpha}}$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{e^{-x}}{x^{1-\alpha}} \times x^{1-\alpha}$$

$$= \lim_{x\to 0} e^{-x} = 1 \neq 0$$

Hence by limit comparison Test both the integrals $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ behaves alike

But $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x^{1-\alpha}}$ is convergent for $1-\alpha < 1$
$\Rightarrow -\alpha < 0 \Rightarrow \alpha > 0$

Hence $\int_0^1 f(x)\,dx$ converges if $\alpha > 0$

Now consider the integral $\int_1^\infty e^{-x} x^{\alpha-1}\,dx$

For given α we choose sufficiently large value of x s.t $e^{-x} x^{\alpha-1} < \frac{1}{x^2}$

$(\text{i.e } e^{x} > x^{\alpha+1})$

$e^x > x^{\alpha+1}$
$x^{\alpha+1} < e^x \Rightarrow \frac{x^{\alpha+1}}{e^x} < 1$
$\Rightarrow x^{\alpha+1} e^{-x} < 1$
$\Rightarrow x^{\alpha+1+1-1} e^{-x} < 1$
$\Rightarrow x^{\alpha-1} x^2 e^{-x} < 1$
$\Rightarrow x^{\alpha-1} e^{-x} < \frac{1}{x^2}$

Hence by Basic Comparison Test if $\int_1^\infty \frac{1}{x^2}\,dx$ converges then $\int_1^\infty e^{-x} x^{\alpha-1}$ also converges

as $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent since $p = 2 > 1$

Hence $\int_1^\infty f(x)\,dx$ is convergent.

Hence R.H.S of ① is convergent if $\alpha > 0$

$\Rightarrow \int_0^\infty x^{\alpha-1} e^{-x}\,dx$ convergent if $\alpha > 0$

★ ★ ★

Question:- Use Drichlet Test to Show that $\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx$ is convergent.

Solution:- Let $f(x) = \sin x$ & $g(x) \frac{1}{\sqrt{x}}$

Note that "g" is bounded & monotonic

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

Now $\int_0^\lambda f(x)\,dx = \int_0^\lambda \sin x\,dx \qquad \lambda > 0$

$$= \left[-\cos x\right]_0^\lambda = -\cos\lambda + \cos 0$$

$$= -\cos\lambda + 1$$

$$\left|\int_0^\lambda f(x)\,dx\right| = \left|-\cos\lambda + 1\right| \leq |\cos\lambda| + |1|$$

$$= 1 + 1 = 2$$

$\Rightarrow \left|\int_0^\lambda f(x)\,dx\right| \leq 2$

$\Rightarrow \int_0^\lambda f(x)\,dx$ is bounded $\forall\ \lambda > 0$

$\Rightarrow$ All the conditions of Drichlet Test are satisfied. Hence by Drichlet Test

$\int_0^\infty f(x)g(x)\,dx$ is convegent i.e

$\int_0^\infty \frac{\sin x}{\sqrt{x}}$ is convergent.

* * *

Question:- Show that $\int_0^\infty \frac{\sin x}{x^p}$ is absolutely convergent if $p>1$

Solution:- $f(x) = \frac{\sin x}{x^p}$, $g(x) = \frac{1}{x^p}$

As $\left|\frac{\sin x}{x^p}\right| \leq \frac{1}{x^p} \quad \forall x \geq 1$

$\int_1^{\infty} \frac{1}{x^p}$ is convergent for $p > 1$. So by B.C.T

$\int_1^{\infty} \left|\frac{\sin x}{x^p}\right|$ is also convergent for $p > 1$

$\Rightarrow \int_1^{\infty} \frac{\sin x}{x^p}$ is absolutely convergent for $p > 1$

[It is not absolutely convergent for $p = 1$]

* * *

Question:- $\int_0^{\pi} \frac{dx}{1+\cos x}$ check the convergent?

Solution:- $f(x)$ has infinite discontinuity at $x = \pi/2$

$$\int_0^{\pi} \frac{dx}{1+\cos x} = \int_0^{\pi/2} \frac{dx}{1+\cos x} + \int_{\pi/2}^{\pi} \frac{dx}{1+\cos x}$$

$$= \lim_{t\to 0} \int_0^{\pi/2 - t} \frac{dx}{2\cos^2 x/2} + \lim_{t'\to 0} \int_{\pi/2+t'}^{\pi} \frac{dx}{2\cos^2 x/2}$$

$$= \lim_{t\to 0} \frac{1}{2} \int_0^{\pi/2 - t} \sec^2\frac{x}{2}\,dx + \lim_{t'\to 0} \frac{1}{2} \int_{\pi/2+t'}^{\pi} \sec^2\frac{x}{2}\,dx$$

$$= \lim_{t\to 0} 2\cdot\frac{1}{2}\left[\tan\frac{x}{2}\right]_0^{\pi/2 - t} + \lim_{t'\to 0} 2\cdot\frac{1}{2}\left[\tan\frac{x}{2}\right]_{\pi/2+t'}^{\pi}$$

$$= \lim_{t\to 0}\left[\tan\frac{1}{2}\left(\frac{\pi}{2}-t\right) - \tan 0\right] + \lim_{t'\to 0}\left[\tan\frac{\pi}{2} - \tan\frac{1}{2}\left(\frac{\pi}{2}+t'\right)\right]$$

$$= \tan\frac{\pi}{4} - 0 + \tan\frac{\pi}{2} - \tan\frac{\pi}{4}$$

$$= 1 - 0 + \infty - 1 = \infty$$

Hence The given integral is divergent.

* * *

Question:- Check the convergence of $\int_0^1 x^{m-1}(1-x)^{n-1}\log\left(\frac{1}{x}\right)dx$

Solution:- The given integral is

$$\int_0^1 x^{m-1}(1-x)^{n-1}\log\left(\frac{1}{x}\right)dx$$

Since 0 & 1 are only two points of infinite discontinuity when $m<1$ & $n<1$

Choose a point b/w 0 & 1 say $\frac{1}{2}$ s.t

$$\int_0^1 x^{m-1}(1-x)^{n-1}\log\left(\frac{1}{x}\right)dx$$

$$= \int_0^{1/2} x^{m-1}(1-x)^{n-1}\log\left(\frac{1}{x}\right)dx + \int_{1/2}^1 x^{m-1}(1-x)^{n-1}\log\left(\frac{1}{x}\right)dx$$

$$= I_1 + I_2$$

We check the convergence of integrals at 0 & 1 respectively.

* Convergence of I_1 at 0 when $m<1$

We have $f(x) = \frac{1}{x^{1-m}}(1-x)^{n-1}\log\left(\frac{1}{x}\right)$

and let $g(x) = \frac{1}{x^p}$

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0} x^p \cdot \frac{1}{x^{1-m}}(1-x)^{n-1}\log\frac{1}{x}$$

$$= \lim_{x\to 0} x^{p+m-1}(1-x)^{n-1}\log\frac{1}{x} = 0$$

If $m+p-1>0 \Rightarrow m>1-p$

Also $\int_0^{1/2}\frac{1}{x^p}dx$ convergence iff $p<1 \Rightarrow 0<1-p$

$\therefore \int_0^{1/2} f(x)\,dx$ convergence iff $m>1-p>0$

$\Rightarrow m>0$

* Convergence of I_2 at 1

$f(x) = x^{m-1}(1-x)^{n-1}\log\left(\frac{1}{x}\right)$ is proper integral if $n \geq 0$ & improper for $n < 0$. So we have

$$f(x) = \frac{x^{m-1}\log(1/x)}{(1-x)^{1-n}} \qquad g(x) = \frac{1}{(1-x)^p}$$

$$\int_{1/2}^{1} g(x)\,dx = \int_{1/2}^{1} \frac{dx}{(1-x)^p}$$ is convergent for $p<1$

Also $$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{(1-x)^p x^{m-1}\log(1/x)}{(1-x)^{1-n}}$$

$$= \frac{x^{m-1}\log\left(\frac{1}{x}\right)}{(1-x)^{1-n-p}}$$

$$= \lim_{x\to 1} \frac{x^{m-1}\log\left(\frac{1}{x}\right)}{(1-x)^{1-n-p}}$$

exists only if $1-n-p \leq 1 \Rightarrow n \geq -p > -1$ thus

$\int_{1/2}^{1} f(x)\,dx$ converges for $n > -1$

Hence the given integral converges for $m>0$, $n>-1$

FUNCTIONS OF BOUNDED VARIATION

Definition:- Let f be a real valued function defined on $[a,b]$, $-\infty < a < b < \infty$

Also let

$P = \{a = x_0, x_1, x_2, \ldots, x_{k-1}, x_k, \ldots, x_n = b\}$ be

a partition of $[a,b]$ and

$$\Delta f_k = f(x_k) - f(x_{k-1})$$

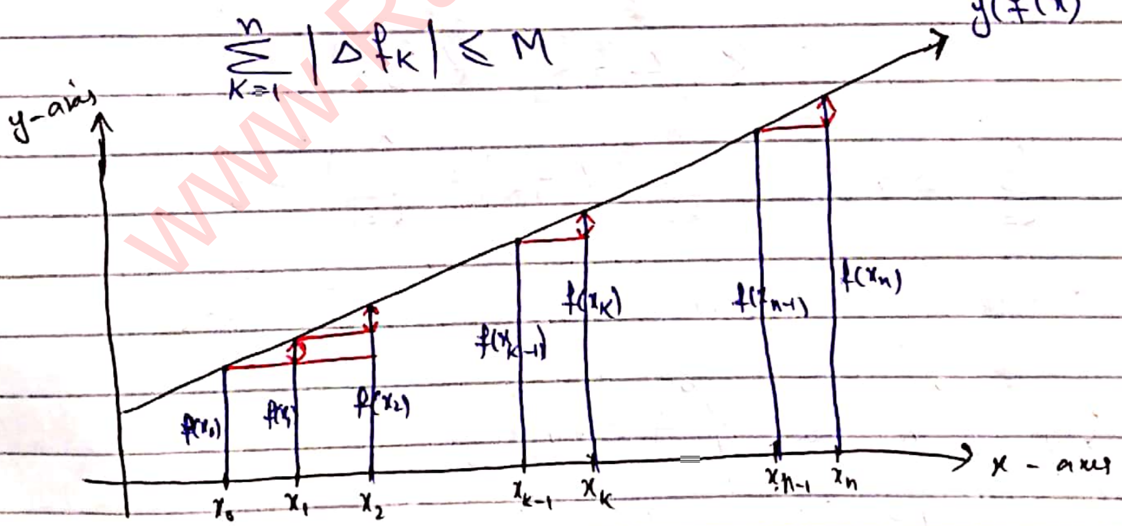
Then we define

$$V_a^b = \text{Sup}\left\{\sum_{k=1}^{n} |\Delta f_k| : P \in P(a,b)\right\}$$

In this case $V_a^b(f)$ is said to be total variation of f over $[a,b]$. If $V_a^b(f)$ is finite then f is called a function of bounded variation. OR

f is said to be function of bounded variation if $\exists$ a +ve real number M s.t

$$\sum_{k=1}^{n} |\Delta f_k| \leq M$$



$\Delta f_1 = f(x_1) - f(x_0)$

$\Delta f_2 = f(x_2) - f(x_1)$

$\vdots$

$\Delta f_n = f(x_n) - f(x_{n-1})$

When we change partitions we get another number. In this way by changing partitions we get a set of numbers. Among these Sup is $V_a^b f$

Now $|\Delta f_1| + |\Delta f_2| + \cdots + |\Delta f_n| = \sum_{k=1}^{n} |\Delta f_k|$ is a single number

Theorem:- If f and g are functions of bounded variations then so $f+g$.

Proof:- Given f & g are of bounded variations say on $[a,b]$

$\Rightarrow V_a^b f$ & $V_a^b g$ are finite

Let $h = f+g$

consider $\Delta h_k = h(x_k) - h(x_{k-1})$

$= (f+g)(x_k) - (f+g)(x_{k-1})$

$= f(x_k) + g(x_k) - f(x_{k-1}) - g(x_{k-1})$

$= f(x_k) - f(x_{k-1}) + g(x_k) - g(x_{k-1})$

$= \Delta f_k + \Delta g_k$

$|\Delta h_k| = |\Delta f_k + \Delta g_k|$

$\leq |\Delta f_k| + |\Delta g_k|$

$\Rightarrow \sum_{k=1}^{n} |\Delta h_k| \leq \sum_{k=1}^{n} |\Delta f_k| + \sum_{k=1}^{n} |\Delta g_k|$

Taking Supremum over $P \in P(a,b)$

$\Rightarrow V_a^b h \leq V_a^b f + V_a^b g$

Since $V_a^b f$ & $V_a^b g$ are finite

So $V_a^b h$ is also finite

$\Rightarrow h$ is of bounded variation.

i.e $f+g$ " " " "

***

Theorem:- If f & g are of bounded variation so is $f-g$.

Proof:- Given f & g are of bounded variation

$\Rightarrow V_a^b f$ & $V_a^b g$ are finite

Let $h = f-g$

To Prove $V_a^b(h)$ is finite

Now

$$\Delta h_K = h(x_K) - h(x_{K-1})$$

$$= (f-g)(x_K) - (f-g)(x_{K-1})$$

$$= f(x_K) - g(x_K) - f(x_{K-1}) + g(x_{K-1})$$

$$= f(x_K) - f(x_{K-1}) - [g(x_K) - g(x_{K-1})]$$

$$= \Delta f_K - \Delta g_K$$

$$|\Delta h_K| = |\Delta f_K - \Delta g_K|$$

$$\leq |\Delta f_K| + |-\Delta g_K|$$

$$|\Delta h_K| \leq |\Delta f_K| + |\Delta g_K|$$

$$\Rightarrow \sum_{K=1}^{n} |\Delta h_K| \leq \sum_{K=1}^{n} |\Delta f_K| + \sum_{K=1}^{n} |\Delta g_K|$$

Taking Supremum on both sides

$$V_a^b h \leq V_a^b f + V_a^b g \quad \text{———} \circledast$$

Since R.H.S of $*$ is finite

$\Rightarrow V_a^b h$ is finite

$\Rightarrow h$ is of bounded variation

i.e. $f-g$ " " " "

* * *

Theorem:- If f is of bounded variation on $[a,b]$ & $\lambda \in R$ then so is λf.

Proof:- Given f is of bounded variation

$\Rightarrow V_a^b f$ is finite

Let $h = \lambda f$

$$\Delta h_K = h(x_K) - h(x_{K-1})$$
$$= (\lambda f)(x_K) - (\lambda f)(x_{K-1})$$
$$= \lambda f(x_K) - \lambda f(x_{K-1})$$

$$\Delta h_K = \lambda \Delta f_K$$

$$|\Delta h_K| = |\lambda||\Delta f_K|$$

$$\sum_{K=1}^{n} |\Delta h_K| = |\lambda| \sum_{K=1}^{n} |\Delta h_K|$$

Taking Supremum on both sides

$$V_a^b h = \lambda V_a^b f$$

Since R.H.S is finite so $V_a^b h$ is finite

$\Rightarrow h$ is of bounded variation.

* * *

Remark:- If "f" is of bounded variation on $[a,b]$ then f is bounded on $[a,b]$

* * *

Theorem:- If f & g are of bounded variation on $[a,b]$ then so $f \cdot g$.

Proof: Given f & g are of bounded variation

$\Rightarrow V_a^b f$ & $V_a^b g$ are finite

Let $h = f \cdot g$

$$\Delta h_K = (fg)(x_K) - (fg)(x_{K-1})$$

$$= f(x_K)\cdot g(x_K) - f(x_{K-1})\, g(x_{K-1})$$

$$\Delta h_K = f(x_K)g(x_K) - f(x_{K-1})g(x_{K-1}) + f(x_{K-1})g(x_K) - f(x_{K-1})g(x_K)$$

$$= g(x_K)[f(x_K) - f(x_{K-1})] + f(x_{K-1})[g(x_K) - g(x_{K-1})]$$

$$= g\,x_K\,\Delta f_K + f(x_{K-1})\,\Delta g_K$$

$$|\Delta h_K| = |g(x_K)\Delta f_K + f(x_{K-1})\Delta g_K|$$

$$\leq |g(x_K)||\Delta f_K| + |f(x_{K-1})||\Delta g_K| \quad \text{--- (1)}$$

Since f & g are of bounded variation

$\Rightarrow$ f & g are bounded

$\Rightarrow$ $f(x) \leq A$ & $g(x) \leq B$

$\therefore$ 1 implies

$$|\Delta h_K| \leq B|\Delta f_K| + A|\Delta g_K|$$

$$\sum_{K=1}^{n} |\Delta h_K| \leq B\sum_{K=1}^{n}|\Delta f_K| + A\sum_{K=1}^{n}|\Delta g_K|$$

$$\operatorname{Sup}\left\{\sum_{K=1}^{n}|\Delta h_K| : P\in P(a,b)\right\} \leq \operatorname{Sup}\left\{B\sum_{K=1}^{n}|\Delta f_K| : P\in P(a,b)\right\} + \operatorname{Sup}\left\{A\sum_{K=1}^{n}|\Delta g_K| : P\in P(a,b)\right\}$$

$$= B\operatorname{Sup}\left\{\sum_{K=1}^{n}|\Delta f_K| : P\in P(a,b)\right\} + A\operatorname{Sup}\left\{\sum_{K=1}^{n}|\Delta g_K| : P\in P(a,b)\right\}$$

$$\Rightarrow V_a^b h \leq B\,V_a^b f + A\,V_a^b g$$

$\because$ R.H.S is finite so h is also finite

$\Rightarrow$ h is of bounded variation

$\Rightarrow$ $f.g$ " " " "

***

Theorem:- If f & g are of bounded variation on $[a,b]$ & $g(x)$ be such that $|g(x)| \geq \lambda > 0$ then f/g is of b.V on $[a,b]$

Proof) Let $h = f/g$ then

$$\Delta h_k = h(x_k) - h(x_{k-1})$$

$$= f/g(x_k) - f/g(x_{k-1})$$

$$= \frac{f(x_k)}{g(x_k)} - \frac{f(x_{k-1})}{g(x_{k-1})} \qquad \because \frac{f}{g}(x) = \frac{f(x)}{g(x)}$$

$$= \frac{f(x_k)\,g(x_{k-1}) - f(x_{k-1})\,g(x_k)}{g(x_k)\,g(x_{k-1})}$$

$$= \frac{f(x_k)g(x_{k-1}) - f(x_{k-1})g(x_{k-1}) + f(x_{k-1})g(x_{k-1}) - f(x_{k-1})g(x_k)}{g(x_k)\,g(x_{k-1})}$$

$$= \frac{g(x_{k-1})[f(x_k) - f(x_{k-1})] - f(x_{k-1})[g(x_k) - g(x_{k-1})]}{g(x_k)\,g(x_{k-1})}$$

$$= \frac{g(x_{k-1})\Delta f_k - f(x_{k-1})\Delta g_k}{g(x_k)\,g(x_{k-1})}$$

$$|\Delta h_k| = \left|\frac{g(x_{k-1})\Delta f_k - f(x_{k-1})\Delta g_k}{g(x_k)\,g(x_{k-1})}\right|$$

$$\leq \frac{|g(x_{k-1})||\Delta f_k| + |f(x_{k-1})||\Delta g_k|}{|g(x_k)||g(x_{k-1})|}$$

$$|\Delta h_k| \leq \frac{|g(x_{k-1})||\Delta f_k| + |f(x_{k-1})||\Delta g_k|}{\lambda^2}$$

$|g(x)| \geq \lambda$

$\frac{1}{|g(x)|} \leq \frac{1}{\lambda} \quad \forall (x)$

$\Rightarrow \frac{1}{g(x_k)} \leq \frac{1}{\lambda}$

& $\frac{1}{g(x_{k-1})} \leq \frac{1}{\lambda}$

$\frac{1}{g(x_k)\,g(x_{k-1})} \leq \frac{1}{\lambda^2}$

Let $A = |g(x_{k-1})|$ $B = |f(x_{k-1})|$

$$\Rightarrow \Delta h_k \leq \frac{A|\Delta f_k| + B|\Delta g_k|}{\lambda^2}$$

$$\sum_{k=1}^{n} \Delta h_k \leq \frac{A\sum_{k=1}^{n}|\Delta f_k| + B\sum_{k=1}^{n}|\Delta g_k|}{\lambda^2}$$

$$= \frac{A}{\lambda^2}\sum_{k=1}^{n}|\Delta f_k| + \frac{B}{\lambda^2}\sum_{k=1}^{n}|\Delta g_k|$$

Taking Supremum over all $P \in P(a,b)$

$$V_a^b h \leq \frac{A}{\lambda^2} V_a^b f + \frac{B}{\lambda^2} V_a^b g \quad \text{—— ⓐ}$$

Since f & g are of bounded variation So R.H.S of ⓐ is finite. Hence L.H.S is also finite

$\Rightarrow$ h is of bounded variation.

i.e f/g is " " "

* * *

Theorem:- If f' exists and bounded on $]a,b[$ then f is of bounded variation on $]a,b[$

Proof:- Given f' is bounded

$$\Rightarrow |f'(x)| \leq M$$

Also f' exists on $]a,b[$

$\Rightarrow$ f is differentiable on $]x_{k-1}, x_k[$

$\Rightarrow$ f is continuous on $[x_{k-1}, x_k]$

Then by M.V.T $\exists$ a point $t_k \in]x_{k-1}, x_k[$

s.t $$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(t_k)$$

$$\Rightarrow \frac{\Delta f_k}{\Delta x_k} = f'(t_k) \Rightarrow |\Delta f_k| = |f'(t_k)||\Delta x_k|.$$

$$\Rightarrow |\Delta f_k| \leq M \Delta x_k$$

$$\Rightarrow \sum_{k=1}^{n} |\Delta f_k| \leq M \sum_{k=1}^{n} \Delta x_k$$

$$= M(b-a)$$

$$\Rightarrow \sum_{k=1}^{n} |\Delta f_k| \leq M(b-a)$$

Taking supremum on both sides

$$V_a^b f \leq M(b-a)$$

Since R.H.S is finite

$\Rightarrow V_a^b f$ is finite $\Rightarrow f$ is of b.V.

***

Theorem:- Show that $f(x) = \cos x$ is of bounded variation on $[0, \pi]$

Proof

$$f(x) = \cos x$$

$$\Rightarrow f'(x) = -\sin x$$

$f'(x)$ exists on $]0, \pi[$

Also $|f'(x)| = |-\sin x|$

$$\leq 1$$

So by Previous theorem $f(x) = \cos x$ is of bounded variation.

***

Theorem:- If "f" is of bounded variation on $[a, b]$ Then f is bounded on $[a, b]$

Proof Given f is of bounded variation

To prove "f" is bounded i.e $|f(x)| \leq M, \forall x \in [a,b]$

Let $P = \{a, x, b\}$ be a partition of $[a,b]$

Now $\Delta f_1 = f(x) - f(a)$

& $\Delta f_2 = f(b) - f(x)$

$\Rightarrow |\Delta f_1| + |\Delta f_2| = |f(x) - f(a)| + |f(b) - f(x)|$

As f is of bounded variation

$\Rightarrow |\Delta f_1| + |\Delta f_2|$ is finite

$\Rightarrow |\Delta f_1| + |\Delta f_2| \leq \lambda$

$\Rightarrow |\Delta f_1| \leq \lambda \Rightarrow |f(x) - f(a)| \leq \lambda$

$\Rightarrow f(x) - f(a) \leq \lambda$

$\Rightarrow f(x) \leq \lambda + f(a) = M$

$\Rightarrow f(x) \leq M$

$\Rightarrow$ "f" is bounded

* * *

Theorem:- If f is of bounded variation on $[a,b]$ then for $a < c < b$, f is of b.V on $[a,c]$ & $[c,b]$ also $V_a^b f = V_a^c f + V_c^b f$

Proof: Let $P = \{a = x_0, x_1, x_2, \dots, x_{k-1}, x_k, \dots, x_n = b\}$

$P_1 = \{a = x_0, x_1, x_2, \dots, x_{k-1}, c\}$

& $P_2 = \{c, x_k, x_{k+1}, \dots, x_n = b\}$

be the partition of $[a,b]$, $[a,c]$ & $[c,b]$ respectively

Let $S_1 = |\Delta f_1| + |\Delta f_2| + \dots + |\Delta f_{k-1}| + |f(c) - f(k-1)|$

$$S_1 = \sum_{i=1}^{k-1} |\Delta f_i| + |f(c) - f(x_{k-1})| \quad \text{——— (1)}$$

& $S_2 = |f(x_k) - f(c)| + |\Delta f_{k+1}| + |\Delta f_{k+2}| + \dots + |\Delta f_n|$

$$S_2 = |f(x_k) - f(c)| + \sum_{i=k+1}^{n} |\Delta f_i| \quad \text{——} ②$$

From equ ① it is clear that

$$S_1 \leq \sum_{i=1}^{n} |\Delta f_i| \quad \text{——} ③$$

Similarly from ② $S_2 \leq \sum_{i=1}^{n} |\Delta f_i|$ —— ④

Taking supremum over ③ & ④ we obtain

$$V_a^c f \leq V_a^b f \quad \& \quad V_c^b f \leq V_a^b f$$

Since $V_a^b f$ is finite

$\Rightarrow V_a^c f$ & $V_c^b f$ are finite.

$\Rightarrow f$ is of bounded variation on $[a,c]$ & $[c,b]$

Next we prove that $V_a^b f = V_a^c f + V_c^b f$

Let $P^* = P_1 \cup P_2$

Consider $\sum_{P^*} |\Delta f_k| = \sum_{P_1} |\Delta f_k| + \sum_{P_2} |\Delta f_k|$

Taking supremum on both sides

$$V_a^b f = V_a^c f + V_c^b f$$

* * *

Theorem: If a function "f" is of bounded variation on $[a,b]$. Then f can be expressed as a difference of two increasing functions.

Proof: Given "f" is of b.V

Define a function $g(x) = \begin{cases} V_a^x f , & a < x \leq b \\ 0 , & x = a \end{cases}$

We claim that $g(x)$ is an increasing function

For this

Let $a < x < u \leq b$

Now $g(u) = V_a^u f$

$= V_a^x f + V_x^u f$

$\boxed{f \text{ will be increasing if for } x < y \Rightarrow f(x) < f(y) \text{. i.e we have same relation b/w heights as relation b/w the points of domain}}$

$\Rightarrow g(u) = V_a^x f + V_x^u f \geq V_a^x f \quad \because V_x^u f \geq 0$

$\Rightarrow g(u) \geq V_a^x f = g(x)$

$\Rightarrow g(u) \geq g(x) \quad \& \quad g(x) \leq g(u)$

So $x < u \Rightarrow g(x) \leq g(u)$

$\Rightarrow g$ is increasing

Now Let $h(x) = g(x) - f(x)$ ——— ⊛

We claim that $h(x)$ is an increasing function

Consider $h(u) - h(x) = [g(u) - f(u)] - [g(x) - f(x)]$

$= g(u) - g(x) - [f(u) - f(x)]$

$= V_a^u f - V_a^x f - (f(u) - f(x))$

$= V_x^u f - (f(u) - f(x))$

$\geq 0 \quad \because V_x^u f$ is supremum

$\Rightarrow h(u) - h(x) \geq 0$

$h(x) \leq h(u)$

$\Rightarrow h$ is increasing

Also note that $f(x) = g(x) - h(x)$ by ⊛

Hence f can be expressed as a difference of two increasing functions.

Question: Show that $f(x) = x\cos\left(\frac{\pi}{2x}\right)$ is not a function of b.V on $[0,1]$.

Solution Consider the partition

$$P = \left\{0, \frac{1}{2n}, \frac{1}{2n-1}, \dots, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$$

a partition of $[0,1]$

Consider $\displaystyle\sum_{K=1}^{2n} |\Delta f_K| = \sum_{K=1}^{2n} |f(x_k) - f(x_{k-1})|$

$$\sum_{K=1}^{2n} |\Delta f_K| = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \dots + |f(x_{2n}) - f(x_{2n-1})|$$

$$= |f(x_{2n}) - f(x_{2n-1})| + |f(x_{2n-1}) - f(x_{2n-2})| + \dots$$

$$+ |f(x_1) - f(x_0)|$$

$$= \left|0 - \left(-\frac{1}{2}\right)\right| + \left|\frac{1}{2} - 0\right| + \left|0 - \left(-\frac{1}{4}\right)\right| + \left|\frac{1}{4} - 0\right| + \dots + \frac{1}{n} \text{ last term}$$

$$= \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \dots + \frac{1}{n} \text{ last term}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n}$$

$$\sum_{K=1}^{2n} |\Delta f_K| = \sum_{K=1}^{n} \frac{1}{K}$$

When $n \to \infty$ the series on R.H.S is divergent series ($\sum_{K=1}^{\infty} \frac{1}{K}$ is Euler series which is divergent)

Hence $\displaystyle\sum_{K=1}^{2n} |\Delta f_K|$ is not finite

$\Rightarrow f$ is not function of b.V.

* * *

Rough:- $f(x) = x \cos\left(\frac{\pi}{2x}\right)$

$f(1) = (1)\cos\left(\frac{\pi}{2}\right) = 0$, $f\left(\frac{1}{2}\right) = \frac{1}{2}\cos(\pi) = -\frac{1}{2}$

$f\left(\frac{1}{3}\right) = \frac{1}{3}\cos\left(\frac{3\pi}{2}\right) = 0$, $f\left(\frac{1}{4}\right) = \frac{1}{4}\cos(2\pi) = 1/4$

$f\left(\frac{1}{5}\right) = \frac{1}{5}\cos\left(\frac{5\pi}{2}\right) = 0$, $f\left(\frac{1}{6}\right) = \frac{1}{6}\cos(3\pi) = -1/6$

***

Question:- Give an example of a function which is continuous on $[0,1]$ but not a function of bounded variation.

Proof:- Let $f = [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x \sin\frac{\pi}{x} & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$$

$$P = \left\{0, \frac{2}{2n+1}, \frac{2}{2n-1}, \dots, \frac{2}{5}, \frac{2}{3}, 1\right\}$$

Then $\sum_{K=1}^{n+1} |\Delta f_K| = |\Delta f_1| + |\Delta f_2| + \dots + |\Delta f_{n+1}|$

$$= |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \dots + |f(x_{2n}) - f(x_{2n-1})|$$

$$= |f(x_{2n}) - f(x_{2n-1})| + |f(x_{2n-1}) - f(x_{2n-2})| + \dots + |f(x_1) - f(x_0)|$$

$$= |f(1) - f(2/3)| + |f(2/3) - f(2/5)| + \dots + \left|f\left(\frac{2}{2n+1}\right) - f(0)\right|$$

$$= \left|0 - \left(\frac{-2}{3}\right)\right| + \left|\frac{-2}{3} - \frac{2}{5}\right| + \dots + \left|\left(\frac{2}{2n+1}\right) - 0\right|$$

$$= \frac{2}{3} + \left(\frac{2}{3} + \frac{2}{5}\right) + \left(\frac{2}{5} + \frac{2}{7}\right) + \dots + \frac{2}{2n+1}$$

$$= 2\left[\frac{1}{3} + \frac{1}{3} + \frac{1}{5} + \frac{1}{5} + \frac{1}{7} + \frac{1}{7} + \dots + \frac{1}{2n+1}\right]$$

$$= 2\left[\frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \dots + \frac{2}{2n+1}\right]$$

$$\sum_{K=1}^{n+1} |\Delta f_K| = 2(2)\left[\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2n+1}\right] \quad \text{---} (1)$$

Scanned by CamScanner

When $n \to \infty$ the series on R.H.S of ① is divergent so L.H.S is also divergent.

$$\Rightarrow \sum_{k=1}^{n+1} |\Delta f_k| \text{ is not finite}$$

$\Rightarrow f$ is not function of b.V

* * *

Theorem:- Let f be a function of bounded variation on $[a,b]$ if f is continuous at $c \in [a,b]$ then $g(x)$,

$$g(x) = \begin{cases} V_a^x f & a < x \leq b \\ 0 & x = a \end{cases}$$

is continuous at c.

Proof Given f is continuous at c.
So for $\epsilon > 0$ there exist a $\delta > 0$ s.t

$$|f(x) - f(c)| < \epsilon \quad \text{——— ①}$$

whenevere $|x - c| < \delta$

1st note that $g(x)$ is an increasing function.
Also $g(x)$ is non negative. [Already Proved]
Let us consider a partition $P = \{c, x_1, x_2, \dots, x_n = b\}$ of $[c,b]$ s.t

$$\sum_{k=1}^{n} |\Delta f_k| > V_c^b f - \epsilon/2 \quad (\text{By def } V_c^b f)$$

——— ②

$$\left\{ \begin{array}{l} V_c^b f = \text{Sup}\left\{ \sum_{k=1}^{n} |\Delta f_k| : P \in P(a,b) \right\} \\ \Rightarrow V_c^b f - \epsilon/2 < \sum_{k=1}^{n} |\Delta f_k| \end{array} \right\}$$

Suppose that $x_1 \in [a,b]$ s.t

$$|x_1 - c| < \delta$$

Then from ① we have

$$\cancel{\sum_{k=1}^{n}|\Delta f_k|} \quad |f(x_1)-f(c)| < \epsilon$$

from eqn ② we have

$$\sum_{k=1}^{n}|\Delta f_k| + \frac{\epsilon}{2} > V_c^b f$$

$$\text{or } V_c^b f < \sum_{k=1}^{n}|\Delta f_k| + \epsilon/2$$

$$= \Delta f_1 + \sum_{k=2}^{n}|\Delta f_k| + \epsilon/2$$

$$= |f(x_1)-f(x_0)| + \sum_{k=2}^{n}|\Delta f_k| + \epsilon/2$$

$$= |f(x_1)-f(c)| + \sum_{k=2}^{n}|\Delta f_k| + \epsilon/2$$

$$< \epsilon + \sum_{k=2}^{n}|\Delta f_k| + \epsilon/2 \quad \text{whenever } |x_1-c|<\delta$$

$$V_c^b f < \sum_{k=2}^{n}|\Delta f_k| + \epsilon' \quad \text{where } \epsilon' = \tfrac{3}{2}\epsilon$$

$$\leq V_{x_1}^b f + \epsilon'$$

$$\Rightarrow V_c^b f - V_{x_1}^b f < \epsilon' \quad \text{whenever } |x_1-c|<\delta$$

$$\Rightarrow V_c^{x_1} f < \epsilon' \,, \quad |x_1-c|<\delta$$

Now consider

$$|g(x_1)-g(c)| = |V_a^{x_1}f - V_a^c f|$$

$$= V_c^{x_1} f$$

$$|g(x_1)-g(c)| < \epsilon' \quad \text{whenever } |x_1-c|<\delta$$

$\Rightarrow g(x)$ is continuous at c.

* * *

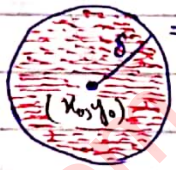
FUNCTIONS OF SEVERAL VARIABLES

dx = change in x
dy/dx = rate of change of y w.r.t x

*** Concept of Neighbourhood In Two Dimensions:-**

Let $(x_0, y_0) \in \mathbb{R}^2$. The open nbhd of (x_0, y_0) is a disc centered at (x_0, y_0) and radius δ

i.e $N_\delta(x_0, y_0) = \{(x,y) : (x-x_0)^2 + (y-y_0)^2 < \delta^2\}$



$N_\delta(x_0, y_0) = \{(x,y) : (x-x_0)^2 + (y-y_0)^2 < \delta^2\}$

⇒ Differentiability:- Let $f(x,y)$ be defined in some ϵ-nbhd of point x, y

i.e $N_\epsilon(x,y) = \{(x',y') : (x'-x)^2 + (y'-y)^2 < \epsilon^2\}$

If $(x+h), (y+k)$ be a point in this nbhd, then $f(x,y)$ is said to be differentiable at (x,y) if

$$f(x+h, y+k) - f(x,y) = Ah + Bk + h\phi(h,k) + k\psi(h,k) \quad \text{--- (1)}$$

Where $\phi, \psi \to 0$ as $(h,k) \to (0,0)$

In such a case $Ah + Bk$ is called the differential of $f(x,y)$ and it is denoted by $d\,f(x,y)$ or df i.e

$$df = Ah + Bk \quad \text{--- (2)}$$

Remark:- If we put $k=0$ in (1) then

$$f(x+h, y) - f(x,y) = Ah + h\phi(h,0)$$

$$\Rightarrow \frac{f(x+h,y) - f(x,y)}{h} = A + \phi(h,0)$$

$$\Rightarrow \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = A + 0 \qquad \because \phi(h,0) \to 0 \text{ as } h \to 0$$

$$\Rightarrow \frac{\partial f}{\partial x}(x,y) = A \quad \text{--- (3)}$$

Similarly if we put $h=0$ in ① we have

$$\frac{\partial f}{\partial y}(x,y) = B \quad \text{---} ④$$

Put in ②

$$df = \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial y} k \quad \text{---} ⑤$$

$$df = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$

Note As $h =$ change in x co-ordinate
$\therefore$ We take $h = dx$ similarly $k = dy$

Then from ⑤ $\quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

df is called total derivative of f &
$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are called partial derivative of f w.r.t its argument x and y respectively.
[Independent variable x & y are called arguments]

***

Question: If $f(x,y)$ is differentiable at (x,y) then show that $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$.

Solution: Given f is differentiable at (x,y)

$$\Rightarrow f(x+h, y+k) - f(x,y) = Ah + Bk + h\phi + k\psi \quad \text{---} ①$$

where $\phi, \psi \to 0$ as $(h,k) \to (0,0)$

From previous remark it is clear that

$$\frac{\partial f}{\partial x} = A \quad , \quad \frac{\partial f}{\partial y} = B$$

So when $(h,k) \to (0,0)$

$$① \Rightarrow \quad df = \frac{\partial f}{\partial x}h + \frac{\partial f}{\partial y}k$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad h = dx, \; k = dy$$

Theorem:- If $f(x,y)$ is differentiable at (x,y) then $f(x,y)$ is continuous at (x,y).

Proof: Given $f(x,y)$ is differentiable at (x,y)

$$\Rightarrow f(x+h, y+k) - f(x,y) = Ah + Bk + h\phi + k\psi$$

when $(h,k) \to (0,0)$ Then

$$\lim_{(h,k)\to(0,0)} [f(x+h, y+k) - f(x,y)] = 0$$

$$\Rightarrow \lim_{(h,k)\to(0,0)} f(x+h, y+k) - f(x,y) = 0$$

$$\Rightarrow \lim_{(h,k)\to(0,0)} f(x+h, y+k) = f(x,y)$$

$\Rightarrow f(x,y)$ is continuous at (x,y)

Note:- Converse of the above theorem is not true in general.

* * *

Example:- Consider the function

$$f(x,y) = \begin{cases} x\sin\frac{1}{x} + y\sin\frac{1}{y}, & \text{if } x,y \neq 0 \\ 0, & \text{if } x,y = 0 \end{cases}$$

Then $f(x,y)$ is continuous at $(0,0)$

because $\left|f(x,y) - f(0,0)\right| = \left|x\sin\frac{1}{x} + y\sin\frac{1}{y}\right|$

$$\leq |x|\left|\sin\frac{1}{x}\right| + |y|\left|\sin\frac{1}{y}\right|$$

$$f(x,y) - f(0,0) \leq |x| + |y| \quad \because \left|\sin\frac{1}{x}\right| \leq 1 \;\&\; \left|\sin\frac{1}{y}\right| \leq 1 \quad \text{——(1)}$$

Now for $\epsilon > 0$ we can choose $\delta = \epsilon/2$ s.t

$f(x,y) - f(0,0) < \epsilon$ whenever $|x| < \delta$ & $|y| < \delta$

$\Rightarrow f$ is continuous at $(0,0)$

We claim that f is not differentiable at $(0,0)$
On contrary suppose that $f(x,y)$ is differentiable at $(0,0)$
Then by definition of differentiability we have

$$f(h,k) - f(0,0) = Ah + Bk + h\phi + k\psi$$

where $\phi, \psi \to 0$ as $(h,k) \to (0,0)$

$$h\sin\frac{1}{h} + k\sin\frac{1}{k} = Ah + Bk + h\phi + k\psi \quad \text{--- } ②$$

As $A = \frac{\partial f}{\partial x}(0,0) = \lim\limits_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$B = \frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0$$

Put A & B in equ ②

$$h\sin\frac{1}{h} + k\sin\frac{1}{k} = h\phi(h,k) + k\psi(h,k)$$

Put $k = h$

$$h\sin\frac{1}{h} + h\sin\frac{1}{h} = h\phi(h,h) + k\psi(h,h)$$

$$2\sin\frac{1}{h} = \phi(h,h) + \psi(h,h)$$

when $h \to 0$

$$2\sin\frac{1}{0} = 0$$

which is a contradiction. So our supposition is wrong. Therefore f is not differentiable at $(0,0)$

***

Question:- Give an example of a function which has partial derivative at (0,0) but is not differentiable at (0,0) [Same Above]

* * *

Question:- Check the continuity and differentiability of the function

$$f(x) = \sqrt{|xy|} \quad \text{at } (0,0)$$

$|x| < \delta$
$|y| < \delta$
$|x||y| < \delta^2$
$\sqrt{|xy|} < \delta$

Solution:-

Continuity at (0,0)

For $\epsilon > 0$ we choose $\delta = \epsilon > 0$ such that

$$|f(x,y) - f(0,0)| = \left|\sqrt{|xy|} - 0\right| = \sqrt{|xy|} < \delta$$

$$\Rightarrow |f(x,y) - f(0,0)| < \epsilon$$

$\Rightarrow$ f is continuous at (0,0)

Suppose f is differentiable at (0,0) then

$$f(h,k) - f(0,0) = Ah + Bk + h\phi(h,k) + k\psi(h,k)$$

$$\sqrt{|hk|} = Ah + Bk + h\phi(h,k) + k\psi(h,k) \quad \text{--- (1)}$$

where $A = f_x(0,0)$, $B = f_y(0,0)$

and $\phi, \psi \to 0$ as $(h,k) \to (0,0)$

As $A = f_x(0,0) = \lim\limits_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h}$

$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

$$B = f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{0 - 0}{k} = 0$$

Put in (1)

$$\sqrt{|hk|} = Ah + Bk + h\phi(h,k) + k\psi(h,k)$$

$$\sqrt{|hk|} = h\phi(h,k) + k\psi(h,k)$$

put $k = h$

$$\sqrt{|h^2|} = h\phi(h,h) + h\psi(h,h)$$

$$\pm h = h[\phi(h,h) + \psi(h,h)]$$

$$\pm 1 = \phi(h,h) + \psi(h,h)$$

when $h \to 0$

$$\pm 1 = 0$$

Which is a contradiction. So our supposition is wrong. Hence f is not differentiable.

***

Question:-

If $f(x) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Show that f has partial derivatives at $(0,0)$ but not differentiable at $(0,0)$

Solution,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0$$

partial derivatives exists at $(0,0)$,

Suppose f is differentiable at $(0,0)$ then

$$f(h,k) - f(0,0) = Ah + Bk + h\,\phi(h,k) + k\,\psi(h,k)$$

where $A = f_x(0,0)$ & $B = f_y(0,0)$

and $\phi, \psi \to 0$ as $(h,k) \to (0,0)$

$$\Rightarrow \frac{h^2 k}{h^2+k^2} = h\,\phi(h,k) + k\,\psi(h,k)$$

put $k = h$

$$\frac{h^3}{h^2+h^2} = h\,\phi(h,h) + h\,\psi(h,h)$$

$$\frac{\cancel{h^3}}{2\cancel{h^2}} = \cancel{h}\left[\phi(h,h) + \psi(h,h)\right]$$

$$\frac{1}{2} = \phi(h,h) + \psi(h,h)$$

when $h \to 0 \Rightarrow \frac{1}{2} = 0$

Which is a contradiction. So our supposition is wrong. Hence f is not differentiable at $(0,0)$

***

★ Sufficient Condition for Differentiability:-

Theorem:- If $f(x,y)$ be such that

(i) $f_x(a,b)$ exists

(ii) $f_y(x,y)$ is continuous at (a,b)

Then $f(x,y)$ is differentiable at (a,b)

Proof: By given condition (ii) $f_y(x,y)$ is continuous at (a,b)

$\Rightarrow f_y(x,y)$ is defined in some nbhd of (a,b)

Let $(a+h, b+k)$ be a point in the nbhd of (a,b)

M.V.T $\Rightarrow \dfrac{f(a+h)-f(a)}{h} = f'(a+\theta h)$

$\Rightarrow f(a+h)-f(a) = h f'(a+\theta h)$

Consider $f(a+h, b+k) - f(a,b) = f(a+h, b+k) - f(a+h, b) + f(a+h, b) - f(a,b)$

$f(a+h, b+k) - f(a,b) = [f(a+h, b+k) - f(a+h, b)] + [f(a+h, b) - f(a,b)]$ —— ①

Consider

$f(a+h, b+k) - f(a+h, b) = k f_y(a+h, b+\theta k)$, $0<\theta<1$

By Lagrange M.V.T

Put in ①

$f(a+h, b+k) - f(a,b) = [k f_y(a+h, b+\theta k)] + [f(a+h, b) - f(a,b)]$ —— ②

Given $f_x(a,b)$ exists

$\Rightarrow f_x(a,b) = \lim_{h\to 0} \dfrac{f(a+h, b) - f(a,b)}{h}$

$\Rightarrow \dfrac{f(a+h, b) - f(a,b)}{h} = f_x(a,b) + \phi$

$\Rightarrow f(a+h, b) - f(a,b) = h f_x(a,b) + h\phi$ put in ②

$f(a+h, b+k) - f(a,b) = k f_y(a+h, b+\theta k) + h f_x(a,b) + h\phi$

$= h f_x(a,b) + k f_y(a,b) - k f_y(a,b) + k f_y(a+h, b+\theta k) + h\phi$

$= h f_x(a,b) + k f_y(a,b) + h\phi + k\psi$ —— ③

Where $\psi = f_y(a+h, b+\theta k) - f_y(a,b)$

Clearly $\phi, \psi \to 0$ as $(h,k) \to (0,0)$

from ③ it is clear that

"f" is differentiable at (a,b)

★ ★ ★

Generalized Differentials:- A function $f(x_1, x_2, \dots, x_n)$ is said to be differentiable at $(x_1, x_2, \dots, x_n)$ if it is defined in an ϵ-nbhd and

$$f(x_1+h_1, x_2+h_2, \dots, x_n+h_n) - f(x_1, x_2, \dots, x_n)$$
$$= \frac{\partial f}{\partial x_1} h_1 + \frac{\partial f}{\partial x_2} h_2 + \dots + \frac{\partial f}{\partial x_n} h_n + h_1\phi_1 + h_2\phi_2 + \dots + h_n\phi_n$$

where $\phi_i \to 0$ as $h_i \to 0$ for $i = 1, 2, \dots, n$

and $$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots + \frac{\partial f}{\partial x_n} dx_n$$

★ ★ ★

⇒ **Higher Order partial Derivatives**:- Let a function f has the partial derivatives f_x and f_y in a nbhd of pt (x,y) Then f_x and f_y are also functions of x and y. The partial derivatives of f_x & f_y w.r.t x and y are given by (i) $\frac{\partial}{\partial x} f_x = f_{xx} = \frac{\partial^2 f}{\partial x^2} = f_{x^2}$

~~(ii) $\frac{\partial}{\partial y} f_y = f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$~~ (ii) $\frac{\partial}{\partial y} f_x = f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$

(iii) $\frac{\partial}{\partial x} f_y = f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$

(iv) $\frac{\partial}{\partial y} f_y = f_{yy} = \frac{\partial^2 f}{\partial y^2} = f_{y^2}$

These are known as 2nd order partial derivatives of f. The derivatives (ii) & (iii) are known as mixed 2nd order partial derivatives.

Similarly $\frac{\partial}{\partial x} f_{xx} = f_{xxx} = \frac{\partial^3 f}{\partial x^3} = f_{x^3}$

and so on.

Remark:- $f_{xx}(a,b) = \lim_{h\to 0} \frac{f_x(a+h,b) - f_x(a,b)}{h}$

$$g = f_x$$

$$g_x = \lim_{h\to 0} \frac{g(a+h,b) - g(a,b)}{h}$$

$$f_{yy}(a,b) = \lim_{k\to 0} \frac{f_y(a,b+k) - f_y(a,b)}{k}$$

$$f_{xy}(a,b) = \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k}$$

$$f_{yx}(a,b) = \lim_{h\to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}$$

In general $f_{xy} \neq f_{yx}$

***

Young's Theorem:- If f_x & f_y both are differentiable at (a,b) then $f_{xy}(a,b) = f_{yx}(a,b)$

Proof: Given f_x & f_y are differentiable at (a,b)

$\Rightarrow f_{xx}, f_{xy}, f_{yx}, f_{yy}$ exist

Let $(a+h, b+k)$ be a point in the nbhd of (a,b). Define a function

$$F(h) = [f(a+h,b+h) - f(a+h,b)] - [f(a,b+h) - f(a,b)] \quad \text{——— (1)}$$

& $g(x) = f(x,b+h) - f(x,b)$ ——— (2)

Then clearly (1) $\Rightarrow$

$$F(h) = g(a+h) - g(a) \quad \text{——— } (*)$$

By Applying M.V.T on g we have

$$\frac{g(a+h) - g(a)}{h} = g'(a+\theta h), \quad 0<\theta<1$$

$$\Rightarrow \quad g(a+h) - g(a) = h\, g'(a+\theta h)$$

Then (*) becomes

$$F(h) = h\, g'(a+\theta h) \quad \text{--- (3)}$$

from (2) $g'(x) = f_x(x, b+h) - f_x(x, b)$

$$\Rightarrow \quad g'(a+\theta h) = f_x(a+\theta h, b+h) - f_x(a+\theta h, b)$$

Put in (3)

$$F(h) = h[f_x(a+\theta h, b+h) - f_x(a+\theta h, b)] \quad \text{--- (4)}$$

Since f_x is differentiable at (a,b) so

$$f_x(a+\theta h, b+h) - f_x(a,b) = \theta h f_{xx}(a,b) + h f_{xy}(a,b) + \theta h \phi_1 + h\phi_2 \quad \text{--- (5)}$$

where $\phi_1, \phi_2 \to 0$ as $h \to 0$

Also $f_x(a+\theta h, b) - f_x(a,b) = \theta h f_{xx}(a,b) + \theta h \phi_3$ --- (6)

where $\phi_3 \to 0$ as $h \to 0$

Put equ (5) & (6) in equ (4)

$$F(h) = h[f_x(a,b) + \theta h f_{xx}(a,b) + h f_{xy}(a,b) + \theta h \phi_1 + h\phi_2 - f_x(a,b) - \theta h f_{xx}(a,b) - \theta h \phi_3]$$

$$\frac{F(h)}{h^2} = [f_{xy}(a,b) - \theta\phi_1 + \phi_2 - \theta\phi_3] \quad \text{--- (A)}$$

similarly if we consider

$$g(y) = f(a+h, y) - f(a,y)$$

Then as we proceed above, we have

$$\frac{F(h)}{h^2} = [f_{yx}(a,b) + \psi_1 + \theta_2\psi_2 - \theta_1\psi_3] \quad \text{--- (B)}$$

From equ A & B we have

$$f_{xy}(a,b) + \theta\phi_1 + \theta_2 - \theta\phi_3 = f_{yx}(a,b) + \psi_1 + \theta_2\psi_1 - \theta_1\psi_3$$

when $h \to 0$, ~~at~~ we have

$$f_{xy}(a,b) = f_{yx}(a,b)$$

***

Schwarz Theorem:- If for $f(x,y)$, f_x exists in a nbhd of (a,b) and f_{yx} is continuous at (a,b) then f_{xy} exists at (a,b) and

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Proof Let $(a+h, b+k)$ be a point in the nbhd of (a,b) Then define a function

$$\phi(h,k) = [f(a+h,b+k) - f(a,b+k)] - [f(a+h,b) - f(a,b)] \quad \text{--- (1)}$$

$$\phi(h,k) = g(b+k) - g(b) \quad \text{--- (2)}$$

where $g(y) = f(a+h,y) - f(a,y)$ --- (3)

Applying M.V.T on (2)

$$g(b+k) - g(b) = k\, g'(b+\theta k) \quad 0<\theta<1 \quad \text{--- (4)}$$

From (3) $g'(y) = f_y(a+h,y) - f_y(a,y)$

$$g'(b+\theta k) = f_y(a+h, b+\theta k) - f_y(a, b+\theta k)$$

Put in (4)

$$g(b+k) - g(b) = k[f_y(a+h, b+\theta k) - f_y(a, b+\theta k)] \quad \text{--- (5)}$$

$\therefore$ (2) $\Rightarrow$

$$\phi(h,k) = k[f_y(a+h, b+\theta k) - f_y(a, b+\theta k)]$$

Again Applying M.V.T

$$\phi(h,k) = k[h f_{yx}(a+\theta' h, b+\theta k)] \quad \text{--- (6)} \quad 0<\theta'<1$$

From (1) & (6)

$$[f(a+h,b+k) - f(a,b+K)] - [f(a+h,b) - f(a,b)] = k h f_{yx}(a+\theta' h, b+\theta k)$$

$$\frac{1}{k}\left[\left[\frac{f(a+h,b+k) - f(a,b+k)}{h}\right] - \left[\frac{f(a+h,b) - f(a,b)}{h}\right]\right] = f_{yx}(a+\theta' h, b+\theta k)$$

Applying limit $h \to 0$ on both sides we have

$$\lim_{h\to 0}\frac{1}{k}\left[\frac{f(a+h,b+k)-f(a,b+k)}{h}\right]-\lim_{h\to 0}\frac{1}{k}\left[\frac{f(a+h,b)-f(a,b)}{h}\right]$$

$$=\lim_{h\to 0} f_{yx}(a+\theta' h, b+\theta k)$$

$$\Rightarrow \frac{1}{k}\left[f_x(a,b+k)-f_x(a,b)\right]=\lim_{h\to 0} f_{yx}(a+\theta' h, b+\theta k)$$

$$\left\{ \because f_x(a,b+k)=\lim_{h\to 0}\frac{f(a+h,b+k)-f(a,b+k)}{h}\right\}$$

Now Applying $\lim_{k\to 0}$ on both sides

$$\lim_{k\to 0}\frac{f_x(a,b+k)-f_x(a,b)}{k}=\lim_{\substack{h\to 0\\ k\to 0}} f_{yx}(a+\theta' h, b+\theta k)$$

$$f_{xy}(a,b)=\lim_{h,k\to 0} f_{yx}(a+\theta' h, b+\theta k) \quad \text{——— Ⓐ}$$

Since f_{yx} is continuous at (a,b) and $(a+\theta' h, b+\theta k)$ lies in the nbhd of (a,b)

$$\left\{\begin{array}{l} \text{As } \theta'<1 \Rightarrow \theta' h < h \;\; \& \;\; \theta<1 \Rightarrow \theta k < k \\ a+\theta' h < a+h, \quad b+\theta k < b+k \\ \text{As } (a+h, b+k) \text{ lies in nbhd of } (a,b) \\ \text{and } (a+\theta' h, b+\theta k) < (a+h, b+k) \\ \text{so } (a+\theta' h, b+\theta k) \text{ lies in the nbhd of } (a,b) \end{array}\right\}$$

$$\therefore \lim_{\substack{h\to 0\\ k\to 0}} f_{yx}(a+\theta' h, b+\theta k)=f_{yx}(a,b)$$

As for continuity $\lim_{x\to a} f(x)=f(a)$

Hence from Ⓐ lim on R.H.S exists

$\Rightarrow f_{xy}(a,b)$ exists and

$$f_{xy}(a,b)=f_{yx}(a,b)$$

Note:- The conditions in Young's & Schwarz theorems are only sufficient i.e these conditions are not necessary.

Question:- If $f(x,y) = \begin{cases} (x^2+y^2)\log(x^2+y^2) & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x^2+y^2=0, \; x=y=0 \end{cases}$

Then $f_{xy}(0,0) = f_{yx}(0,0)$

Solution:-

$$f_x(x,y) = \begin{cases} 2x\log(x^2+y^2) + \dfrac{(x^2+y^2)}{(x^2+y^2)}(2x) & \text{if } x^2+y^2 \neq 0 \\ \lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} & \text{if } x=y=0 \end{cases}$$

$$= \begin{cases} 2x\log(x^2+y^2) + 2x & \text{if } x^2+y^2 \neq 0 \\ \lim\limits_{h\to 0} \dfrac{h^2\log h^2 - 0}{h} & \text{if } x=y=0 \end{cases}$$

$$= \begin{cases} 2x\log(x^2+y^2) + 2x & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x=y=0 \end{cases}$$

Similarly we have

$$f_y(x,y) = \begin{cases} 2y\log(x^2+y^2) + 2y & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x=y=0 \end{cases}$$

Now $f_{xy}(0,0) = \lim\limits_{k\to 0} \dfrac{f_x(0,k)-f_x(0,0)}{k}$

$$= \lim_{k\to 0} \frac{0-0}{k}$$

$$f_{xy}(0,0) = 0 \quad \text{——} (1)$$

Similarly $f_{yx}(0,0) = 0$ —— (2)

So from (1) & (2) we have

$$f_{xy}(0,0) = f_{yx}(0,0)$$

Hence in above question $f_{xy}(0,0) = f_{yx}(0,0)$ even the conditions of Young's & Schwarz's theorem are not satisfied.

$\Rightarrow$ Suppose that $f(x,y)$ satisfied the conditions of young's theorem. Then

f_x is differentiable at $(0,0)$

$$f_x(h,k) - f_x(0,0) = h f_{xx}(0,0) - k f_{xy}(0,0) + h\phi + k\psi \quad \text{—— ①}$$

where $\phi, \psi \to 0$ as $h, k \to 0$

$$\because \left(f(x+h, y+k) - f(x,y)\right) = h f_x + k f_y + \phi h + k\psi$$

Now
$$f_{xx}(0,0) = \lim_{h\to 0} \frac{f_x(h,0) - f_x(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{2h\log h^2 + 2h - 0}{h}$$

$$= \lim_{h\to 0} 2\log h^2 + 2 = \infty$$

Hence ① is not defined $\Rightarrow f_x$ is not differentiable at $(0,0)$

$\Rightarrow f$ not satisfies conditions of young's Theorem.

$$f_{yx}(x,y) = \begin{cases} \dfrac{4xy}{x^2+y^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x = y = 0 \end{cases}$$

$$\lim_{(x,y)\to(0,0)} f_{yx}(x,y) = \lim_{(x,y)\to(0,0)} \frac{4xy}{x^2+y^2}$$

put $y = mx \Rightarrow$ taking limit along $y = mx$

$$= \lim_{x\to 0} \frac{4mx^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{4m}{1+m^2}$$

which gives different values for m. So limit does not exist $\Rightarrow f_{yx}$ is not continuous. Schwarz Theorem is not satisfied.

***

Question:- Show that for the function

$$f(x,y) = \begin{cases} \dfrac{(x+y)\sqrt{x^2+y^2}+xy}{\sqrt{x^2+y^2}} & \text{If } x^2+y^2 \neq 0 \\ 0 & \text{If } x=y=0 \end{cases}$$

$$f_x(0,0) = f_y(0,0) = 1$$

Solution:-

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h\sqrt{h^2}}{h\sqrt{h^2}} = \lim_{h\to 0} 1$$

$$= 1$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{k\sqrt{k^2}}{k\sqrt{k^2}} = \lim_{k\to 0} 1$$

$$= 1$$

$$\Rightarrow f_x(0,0) = f_y(0,0) = 1$$

* * *

Question:- Show that the function

$$f(x,y) = \begin{cases} xy\,\dfrac{(x^2-y^2)}{x^2+y^2} & \text{where } x^2+y^2 \neq 0 \\ 0 & \text{when } x=y=0 \end{cases}$$

$$f_{xy}(0,0) \neq f_{yx}(0,0)$$

Proof:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0$$

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k} \quad \text{——} ①$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{hk(h^2-k^2)}{h(h^2+k^2)} = \lim_{h\to 0} \frac{k(h-k)}{h+k}$$

$$= \frac{-k^2}{k} = -k$$

Put in ①

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{-k}{k} = -1 \quad \text{——} \star$$

Again $f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h}$ —— ②

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,0+k) - f(h,0)}{k}$$

$$= \lim_{k\to 0} \frac{hk(h^2-k^2)}{k(h^2+k^2)} = h$$

Put in ②

$$f_{yx} = \lim_{h\to 0} \frac{h-0}{h} = \lim_{h\to 0} 1 = 1 \quad \text{——} \star'$$

from $\star$ & $\star'$ we have

$$f_{xy}(0,0) \neq f_{yx}(0,0)$$

* * *

⇒ **Homogeneous Function:-** $f(x,y)$ is homogeneous of degree "n" if $f(x,y) = x^n g(y/x)$

or $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

Example:-

$$f(x,y) = \frac{x^3 - y^3}{x^2+y^2}$$

$$f(\lambda x, \lambda y) = \frac{\lambda^3 x^3 - \lambda^3 y^3}{\lambda^2 x^2 + \lambda^2 y^2}$$

$$= \lambda\left[\frac{x^3-y^3}{x^2+y^2}\right]$$

$$f(\lambda x, \lambda y) = \lambda^1 [f(x,y)]$$

$$f(x,y) = \frac{x^3-y^3}{x^2+y^2}$$

$$= \frac{x^3(1-(y/x)^3)}{x^2(1+(y/x)^2)}$$

$$= x\left\{\frac{(y/x)^0 - (y/x)^3}{(y/x)^0 + (y/x)^2}\right\}$$

$$= x\, g(y/x)$$

* * *

Theorem:- If $f(\lambda x, \lambda y) = \lambda^n f(x,y)$ then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf \quad \text{[Euler's Theorem]}$$

Proof: Given $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

$\Rightarrow$ f is homogeneous of degree n.

$$\Rightarrow f(x,y) = x^n g(y/x) \quad \text{——— ①}$$

from ① $$\frac{\partial f}{\partial x} = nx^{n-1} g(y/x) + x^n g'(y/x)(-y/x^2)$$

$$\Rightarrow x\frac{\partial f}{\partial x} = nx^n g(y/x) - x^{n-1} y\, g'(y/x) \quad \text{——— ②}$$

Again differentiate ① w.r.t y

$$\frac{\partial f}{\partial y} = x^n g'(y/x)\cdot 1/x \Rightarrow \frac{\partial f}{\partial y} = x^{n-1} g'(y/x)$$

$$y\frac{\partial f}{\partial y} = x^{n-1} y\, g'(y/x) \quad \text{——— ③}$$

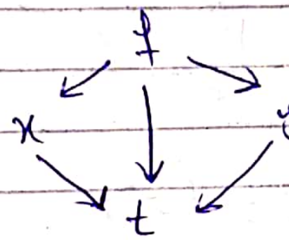
Adding ② & ③

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nx^n g(y/x)$$

$$= nf \quad \text{using ①}$$

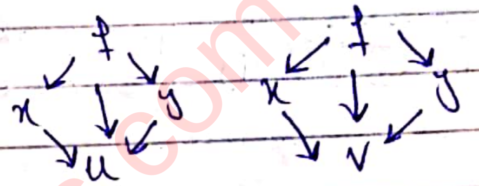
* * *

Note:- If $z = f(x,y)$, $x = g(t)$, $y = g(t)$



$$\frac{df}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$z = f(x,y)$, $x = g(u,v)$, $y = g(u,v)$



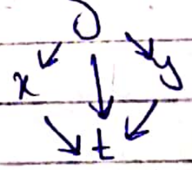
$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

* * *

Theorem:- (Chain Rule 1) If $z = f(x,y)$ & $x = g(t)$, $y = h(t)$ Then

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$



Proof

Let Δt be increment in t & $\Delta x, \Delta y, \Delta z$ be corresponding increment in x, y & z respectively.

Then

$$\Delta z=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y+\phi\,\Delta x+\psi\,\Delta y$$

Divide both sides by Δt

$$\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t}+\frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t}+\phi\frac{\Delta x}{\Delta t}+\psi\frac{\Delta y}{\Delta t}$$

when $\Delta t \to 0$ Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

***

$\Delta \Rightarrow$ Average Change $\partial \Rightarrow$ Small Change $d \Rightarrow$ Instataniour Change

***

Question: If $z = x^2 + y^2 + 2xy$ and $x = t^3 + 5$, $y = t^3 - 9$ Find $\frac{dz}{dt}$ by (i) Chain Rule 1 (ii) By direct differentiation.

Solution: (i) $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$= (2x+2y)(3t^2) + (2y+2x)(3t^2)$

$= 2(x+y)(3t^2) + 2(y+x)(3t^2)$

$= 6t^2(2t^3-4) + 6t^2(2t^3-4)$

$= 12t^5 - 24t^2 + 12t^5 - 24t^2$

$= 24t^5 - 48t^2$

(ii) $z = x^2 + y^2 + 2xy$

$= (t^3+5)^2 + (t^3-9)^2 + 2(t^3+5)(t^3-9)$

$\frac{dz}{dt} = 2(t^3+5)3t^2 + 2(t^3-9)3t^2 + 2[(3t^2)(t^3-9) + (t^3+5)(3t^2)]$

$= 6t^5 + 30t^2 + 6t^5 - 54t^2 + 2[3t^5 - 27t^2 + 3t^5 + 15t^2]$

$= 6t^5 + 30t^2 + 6t^5 - 54t^2 + 6t^5 - 54t^2 + 6t^5 + 30t^2$

$= 24t^5 - 48t^2$

***

Theorem (Chain Rule 2):- If $z = f(x,y)$

$x = g(u,v)$, $y = h(u,v)$ Then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad \&$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Proof: Let Δu & Δv be change in u & v

Then $\Delta x, \Delta y$ & Δz be corresponding changes in x, y & z respectively.

Then $\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \phi\,\Delta x + \psi\,\Delta y$

where $\phi, \psi \to 0$ as $\Delta u, \Delta v \to 0$

dividing both sides by Δu

$$\frac{\Delta z}{\Delta u} = \frac{\partial z}{\partial x}\cdot\frac{\Delta x}{\Delta u} + \frac{\partial z}{\partial y}\cdot\frac{\Delta y}{\Delta u} + \phi\frac{\Delta x}{\Delta u} + \psi\frac{\Delta y}{\Delta u}$$

taking limit $\Delta u \to 0$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

since $\phi, \psi \to 0$

Similarly $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$

* * *

Question:- Let $z = f(x,y)$, $x = u^2 - v^2$, $y = 2uv$

Prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{4(u^2+v^2)}\left\{\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right\}$$

Proof: As $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$

$$= \frac{\partial z}{\partial x}\cdot 2u + \frac{\partial z}{\partial y}\cdot 2v$$

$$\left(\frac{\partial z}{\partial u}\right)^2 = \left(\frac{\partial z}{\partial x}2u + \frac{\partial z}{\partial y}2v\right)^2$$

$$\left(\frac{\partial z}{\partial u}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 4u^2 + \left(\frac{\partial z}{\partial y}\right)^2 4v^2 + 8uv \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \quad \text{—①}$$

Now $\displaystyle \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$

$$= \frac{\partial z}{\partial x}(-2v) + \frac{\partial z}{\partial y}(2u)$$

$$\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}(-2v) + \frac{\partial z}{\partial y}(2u)\right)^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 4v^2 + \left(\frac{\partial z}{\partial y}\right)^2 4u^2 - 8uv \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \quad \text{—②}$$

Now consider

$$\frac{1}{4(u^2+v^2)} \left\{ \left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \right\}$$

$$= \frac{1}{4(u^2+v^2)} \left\{ \left(\frac{\partial z}{\partial x}\right)^2 (4u^2+4v^2) + \left(\frac{\partial z}{\partial y}\right)^2 (4u^2+4v^2) \right\}$$

$$= \frac{\cancel{(4u^2+4v^2)}}{\cancel{4(u^2+v^2)}} \left\{ \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \right\}$$

$$\frac{1}{4(u^2+v^2)} \left\{ \left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \right\} = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

* * *

Taylor's Theorem:- If $f(x,y)$ possess continuous partial derivatives of the nth order in an ϵ-nbhd of a point (a,b) and if $\theta \in (0,1)$ and $(a+h, b+k)$ be in the nbhd of (a,b) Then

$$f(a+h, b+k) = f(a,b) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(a,b) +$$

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \dots + \frac{1}{(n-1)!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^{n-1}$$

$$f(a,b) + \frac{1}{n!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^n f(a+\theta h, b+\theta k)$$

Proof:- Let $(x,y) = (a+th, b+tk)$, $0 \le t \le 1$
be a point in the nbhd of (a,b)
& $g(t) = f(a+th, b+tk)$
Then clearly g is a function of one independent variable t. So by MacLaurin's series we have

$$g(t) = g(0) + t g'(0) + \frac{t^2}{2!} g''(0) + \dots + \frac{t^{n-1}}{(n-1)!} g^{(n-1)}(0)$$
$$+ \frac{t^n}{n!} g^{(n)}(\theta t) \text{ --- } ① \,, \quad 0<\theta<1$$

As $g(t) = f(a+th, b+tk)$
$$\Rightarrow g(0) = f(a,b)$$

Note that $g(t) = f(x,y)$, where $x = a+th$, $y = b+tk$

$$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$
$$= \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial y} k$$
$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x,y)$$

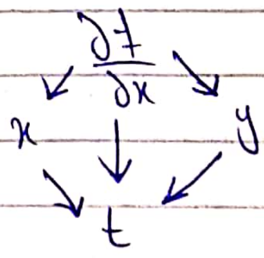
$$g'(t) = \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(a+th, b+tk)$$

$$\Rightarrow g'(0) = \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(a,b)$$

Again consider

$$g'(t) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$

$$\Rightarrow g''(t) = h\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) + k\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)$$



$$g''(t) = h\left[\frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dt}\right] + k\left[\frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}\right]$$

$$= h\left[\frac{\partial^2 f}{\partial x^2} h + \frac{\partial^2 f}{\partial y \partial x} k\right] + k\left[\frac{\partial^2 f}{\partial x \partial y} h + \frac{\partial^2 f}{\partial y^2} k\right]$$

$$= h^2 \frac{\partial^2 f}{\partial x^2} + hk\frac{\partial^2 f}{\partial y \partial x} + hk\frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}$$

$$\Rightarrow g''(t) = \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(a+th, b+tk)$$

$$g''(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b)$$

So in general

$$g^{(n)}(t) = \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^n f(a+th, b+tk)$$

$$\Rightarrow g^{(n)}(\theta t) = \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^n f(a+\theta th, b+\theta tk)$$

Put in ①

$$f(a+th, b+tk) = f(a,b) + t\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \cdots$$
$$+ \frac{t^{n-1}}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a,b)$$
$$+ \frac{t^n}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta th, b+\theta tk)$$

Put $t = 1$

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \cdots$$
$$+ \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a,b)$$
$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k)$$

* * *

Question:- The transformation from rectangular to polar co-ordination.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \text{gives}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r}\frac{\partial v}{\partial r}$$

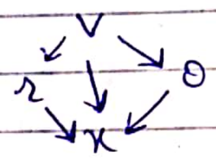
Solution:- As $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow x^2 + y^2 = r^2 \quad \text{and} \quad \theta = \tan^{-1}(y/x)$$

Let $V = V(r,\theta)$

Note that $r = g(x,y)$ & $\theta = h(x,y)$

As
$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} \quad \text{--- (1)}$$



$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial V}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} \quad \text{--- (2)}$$

As $r^2 = x^2 + y^2$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta$$

As $\theta = \tan^{-1}(y/x)$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+y^2/x^2}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} = \frac{-r\sin\theta}{r^2}$$

$$\Rightarrow \frac{\partial \theta}{\partial x} = \frac{-\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

Put in (1)

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cos\theta + \frac{\partial v}{\partial \theta}\left(\frac{-\sin\theta}{r}\right)$$

$$= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)v$$

$$\Rightarrow \frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)^2 v$$

$$= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}\right)$$

$$= \cos^2\theta\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r}\right) - \sin\theta\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial v}{\partial\theta}\right)$$

$$- \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial v}{\partial r}\right) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 v}{\partial r^2} - \sin\theta\cos\theta\left\{\frac{-1}{r^2}\frac{\partial v}{\partial\theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r\partial\theta}\right\}$$

$$- \frac{\sin\theta}{r}\left\{-\sin\theta\frac{\partial v}{\partial r} + \cos\theta\frac{\partial^2 v}{\partial\theta\partial r}\right\} + \frac{\sin\theta}{r^2}$$

$$\left\{\cos\theta\frac{\partial v}{\partial\theta} + \sin\theta\frac{\partial^2 v}{\partial\theta^2}\right\}$$

$$= \cos^2\theta\frac{\partial^2 v}{\partial r^2} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial v}{\partial\theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 v}{\partial r\partial\theta}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial v}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 v}{\partial\theta\partial r} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial v}{\partial\theta}$$

$$+ \frac{\sin^2\theta}{r^2}\frac{\partial^2 v}{\partial\theta^2}$$

$$= \cos^2\theta\frac{\partial^2 v}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial v}{\partial\theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 v}{\partial r\partial\theta}$$

$$+ \frac{\sin^2\theta}{r^2}\frac{\partial^2 v}{\partial\theta^2} \text{ ——— } (A)$$

Scanned by CamScanner

Replace θ by $\pi/2 - \theta$ we have

$$\frac{\partial^2 v}{\partial y^2} = \cos^2\left(\frac{\pi}{2}-\theta\right)\frac{\partial^2 v}{\partial r^2} + \frac{2\sin(\pi/2-\theta)\cos(\pi/2-\theta)}{r^2}\frac{\partial v}{\partial \theta}$$

$$- \frac{2\sin(\pi/2-\theta)\cos(\pi/2-\theta)}{r}\frac{\partial^2 v}{\partial r\partial\theta} + \frac{\sin^2(\pi/2-\theta)}{r^2}\frac{\partial^2 v}{\partial\theta^2}$$

$$= \sin^2\theta\frac{\partial^2 v}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial v}{\partial\theta} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 v}{\partial r\partial\theta}$$

$$+ \frac{\cos^2\theta}{r^2}\frac{\partial^2 v}{\partial\theta^2} \quad \text{———} \quad (*')$$

Adding $(*)$ & $(*)$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = (\cos^2\theta + \sin^2\theta)\frac{\partial^2 v}{\partial r^2} + \left(\frac{4\cos\theta\sin\theta}{r^2}\right)\frac{\partial v}{\partial\theta}$$

$$-\left(\frac{4\cos\theta\sin\theta}{r}\right)\frac{\partial^2 v}{\partial r\partial\theta} + \left(\frac{\sin^2\theta}{r^2} + \frac{\cos^2\theta}{r^2}\right)\frac{\partial^2 v}{\partial\theta^2}$$

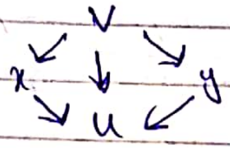
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}$$

Question:- If $x = e^u\cos v$, $y = e^u \sin v$

Then show that
$$e^{-2u}\left(\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2}\right) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

where $V = v(x,y)$ $\quad x = f(u,v)$, $y = h(u,v)$

Solution: $\dfrac{\partial v}{\partial u} = \dfrac{\partial v}{\partial x}\cdot\dfrac{\partial x}{\partial u} + \dfrac{\partial v}{\partial y}\cdot\dfrac{\partial y}{\partial u}$



$$= \frac{\partial v}{\partial x}e^u\cos v + \frac{\partial v}{\partial y}e^u\sin v$$

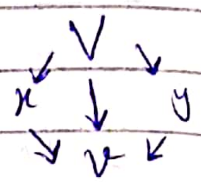
$$\frac{\partial v}{\partial u} = \left(e^u\cos v\frac{\partial}{\partial x} + e^u\sin v\frac{\partial}{\partial y}\right)V$$

$$\frac{\partial^2 V}{\partial u^2} = \left(e^u\cos v\frac{\partial}{\partial x} + e^u\sin v\frac{\partial}{\partial y}\right)^2 V$$

$$\frac{\partial^2 V}{\partial u^2} = \left(e^u \cos v \frac{\partial}{\partial x} + e^u \sin v \frac{\partial}{\partial y}\right)\left(e^u \cos v \frac{\partial V}{\partial x} + e^u \sin v \frac{\partial V}{\partial x}\right)$$

$$= e^{2u} \cos^2 v \frac{\partial^2 V}{\partial x^2} + e^{2u} \cos v \sin v \frac{\partial^2 V}{\partial x \partial y} + e^{2u} \cos v \sin v \frac{\partial^2 V}{\partial x \partial y}$$

$$+ e^{2u} \sin^2 v \frac{\partial^2 V}{\partial y^2} \quad \text{——— (1)}$$



$$\frac{\partial V}{\partial v} = \frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial V}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \frac{\partial V}{\partial x}(-e^u \sin v) + \frac{\partial V}{\partial y}(e^u \cos v)$$

$$= \left(-e^u \sin v \frac{\partial}{\partial x} + e^u \cos v \frac{\partial}{\partial y}\right) V$$

$$\frac{\partial^2 V}{\partial v^2} = \left[-e^u \sin v \frac{\partial}{\partial x} + e^u \cos v \frac{\partial}{\partial y}\right]^2 V$$

$$= \left[-e^u \sin v \frac{\partial}{\partial x} + e^2 \cos v \frac{\partial}{\partial y}\right]\left[-e^u \sin v \frac{\partial V}{\partial x} + e^u \cos v \frac{\partial V}{\partial y}\right]$$

$$= e^{2u} \sin^2 v \frac{\partial^2 V}{\partial x^2} - e^{2u} \sin v \cos v \frac{\partial^2 V}{\partial x \partial y}$$

$$- e^{2u} \cos v \sin v \frac{\partial^2 V}{\partial y \partial x} + e^{2u} \cos^2 v \frac{\partial^2 V}{\partial y^2} \quad \text{——— (2)}$$

Adding (1) & (2)

$$\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} = e^{2u} \frac{\partial^2 V}{\partial x^2} + e^{2u} \frac{\partial^2 V}{\partial y^2}$$

$$= e^{2u}\left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right]$$

$$e^{-2u}\left[\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2}\right] = \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

Question:- If $x = r\cos\theta$, $y = r\sin\theta$ and $y = f(x)$

Prove that $\dfrac{dy}{dx} = r\dfrac{d\theta}{dr}$

$$\begin{tikzcd} & y \arrow[dl] \arrow[dr] & \\ r \arrow[dr] & & \theta \arrow[dl] \\ & x & \end{tikzcd}$$

Proof) $\dfrac{dy}{dx} = \dfrac{\partial y}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial y}{\partial \theta}\dfrac{\partial \theta}{\partial x}$

$$= \sin\theta\left(\frac{x}{r}\right) + r\cos\theta\left[\frac{-y}{1+y^2/x^2}\cdot\frac{1}{x^2}\right]$$

As $r^2 = x^2 + y^2$

$\cancel{2}r\dfrac{\partial r}{\partial x} = \cancel{2}x$

$\theta = \tan^{-1}(y/x)$

$\dfrac{\partial\theta}{\partial x} = \dfrac{1}{1+y^2/x^2}\left(-y/x^2\right)$

$$= \sin\theta\left(\frac{x}{r}\right) - r\left[\frac{y}{x^2+y^2}\right]\cos\theta$$

$$= \sin\theta\,\frac{x}{r} - r\,\frac{y}{r^2}\cos\theta$$

$$= \frac{x\sin\theta}{r} - \frac{y\cos\theta}{r}$$

$$\frac{dy}{dx} = \frac{x\sin\theta - y\cos\theta}{r} \quad \text{——— (1)}$$

Now $\dfrac{d\theta}{dr} = \dfrac{\partial\theta}{\partial x}\cdot\dfrac{\partial x}{\partial r} + \dfrac{\partial\theta}{\partial y}\cdot\dfrac{\partial y}{\partial r}$

$$\begin{tikzcd} & \theta \arrow[dl] \arrow[d] \arrow[dr] & \\ x \arrow[r] & r & y \arrow[l] \end{tikzcd}$$

$$= \frac{1}{1+y^2/x^2}\left(\frac{-y}{x^2}\right)\cos\theta + \frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right)\sin\theta$$

$$= \frac{-y\cos\theta}{x^2+y^2} + \frac{x\sin\theta}{x^2+y^2} = \frac{-y\cos\theta}{r^2} + \frac{x\sin\theta}{r^2}$$

$$\frac{d\theta}{dr} = \frac{x\sin\theta - y\cos\theta}{r^2} \Rightarrow r\frac{d\theta}{dr} = \frac{x\sin\theta - y\cos\theta}{r} \quad \text{——— (2)}$$

From (1) & (2) $\dfrac{dy}{dx} = r\dfrac{d\theta}{dr}$

Question:- Prove that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = \dfrac{\partial^2 V}{\partial x'^2} + \dfrac{\partial^2 V}{\partial y'^2}$

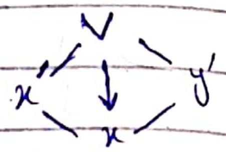
where $x' = (x-a)\cos\alpha + (y-b)\sin\alpha$

$y' = (x-a)\sin\alpha + (y-b)\cos\alpha$

} equation of rotation + Translation of axis where a, b & α are constants.

Solution:- $V = v(x', y')$ $\quad x' = f(x,y)$ $\quad y' = g(x,y)$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial v}{\partial y'} \cdot \frac{\partial y'}{\partial x}$$



$$= \frac{\partial v}{\partial x'}(\cos\alpha) + \frac{\partial v}{\partial y'}(-\sin\alpha)$$

$$\frac{\partial v}{\partial x} = \left(\cos\alpha \frac{\partial}{\partial x'} - \sin\alpha \frac{\partial}{\partial y'}\right) v$$

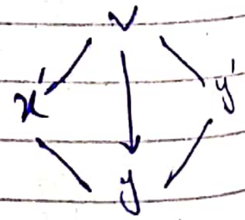
$$\frac{\partial^2 v}{\partial x^2} = \left(\cos\alpha \frac{\partial}{\partial x'} - \sin\alpha \frac{\partial}{\partial y'}\right)^2 v$$

$$= \left[\cos\alpha \frac{\partial}{\partial x'} - \sin\alpha \frac{\partial}{\partial y'}\right]\left[\cos\alpha \frac{\partial v}{\partial x'} - \sin\alpha \frac{\partial v}{\partial y'}\right]$$

$$\frac{\partial^2 v}{\partial x^2} = \cos^2\alpha \frac{\partial^2 v}{\partial x'^2} - \cos\alpha \sin\alpha \frac{\partial^2 v}{\partial x' \partial y'}$$

$$- \sin\alpha \cos\alpha \frac{\partial^2 v}{\partial y' \partial x'} + \sin^2\alpha \frac{\partial^2 v}{\partial y'^2} \quad \text{--- (1)}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial x'} \cdot \frac{\partial x'}{\partial y} + \frac{\partial v}{\partial y'} \cdot \frac{\partial y'}{\partial y}$$



$$= \frac{\partial v}{\partial x'}(\sin\alpha) + \frac{\partial v}{\partial y'}(\cos\alpha)$$

$$\frac{\partial v}{\partial y} = \left(\sin\alpha \frac{\partial}{\partial x'} + \cos\alpha \frac{\partial}{\partial y'}\right) v$$

$$\frac{\partial^2 v}{\partial y^2} = \left[\sin\alpha \frac{\partial}{\partial x'} + \cos\alpha \frac{\partial}{\partial y'}\right]^2 v$$

$$= \left[\sin\alpha \frac{\partial}{\partial x'} + \cos\alpha \frac{\partial}{\partial y'}\right]\left[\sin\alpha \frac{\partial v}{\partial x'} + \cos\alpha \frac{\partial v}{\partial y'}\right]$$

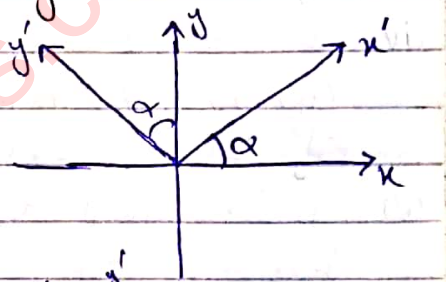
$$\frac{\partial^2 v}{\partial y^2} = \sin^2\alpha \frac{\partial^2 v}{\partial x'^2} + \sin\alpha\cos\alpha \frac{\partial^2 v}{\partial x'\partial y'} + \cos\alpha\sin\alpha \frac{\partial^2 v}{\partial y'\partial x'}$$
$$+ \cos^2\alpha \frac{\partial^2 v}{\partial y'^2} \quad \text{——— (2)}$$

By adding (1) & (2) we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial x'^2} + \frac{\partial^2 v}{\partial y'^2} \quad \text{——— (3)}$$

Question:- Show that the expression $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$ is invarient w.r.t any change in the rectangular axis

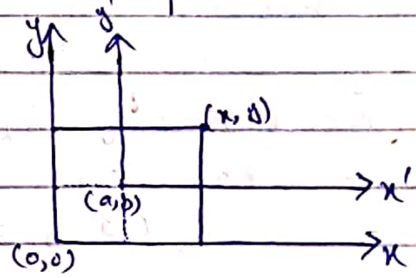
Solution:- If we rotate the axis the transformation equations are



$x' = \cos\alpha + \sin\alpha$

$y' = -\sin\alpha + \cos\alpha$

If we translate the axis The transformation equations are



$x' = (x - a)$

$y' = (y - b)$

If we shift the origin at (a,b) and rotate the axis as x' & y' Now the transformation eqns are $x' = (x-a)\cos\alpha + (y-b)\sin\alpha$

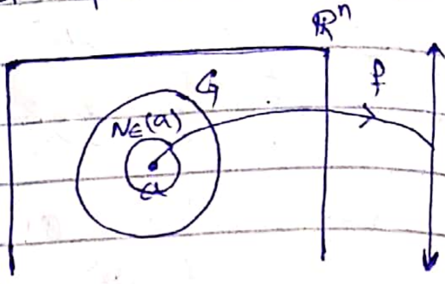
$y' = -(x-a)\sin\alpha + (y-b)\cos\alpha$

Now repeat the previous question & from eqn (3) we can see that

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$ is invariant w.r.t the change of origin + axis.

Assignment:- Calculus 9.1 → 9.6

⇒ **Maxima & Minima:-** Let $f: G \to \mathbb{R}$, G is an open set in $\mathbb{R}^n$



(i) f is said to have a local maximum at "a" if $f(x) \leq f(a)$ $\forall$ $x \in N_\epsilon(a)$

(ii) f has local minimum at "a" if $f(a) \leq f(x)$ $\forall$ $x \in N_\epsilon(a)$

(iii) f is said to have a global maximum (universal maximum, absolute maximum) at "a" if $f(x) \leq f(a)$, $\forall$ $x \in G$

(iv) f is said to have a global minimum (universal minimum, absolute minimum) at "a" if $f(a) \leq f(x)$, $\forall$ $x \in G$

(v) f is said to have local extrema if f has a local maximum or local minimum.

Theorem:- If f_x, f_y and f exist at (a,b) & f has an extrema at (a,b) Then

$$f_x(a,b) = 0 = f_y(a,b)$$

Proof: If $f(a,b)$ is an extrema value of $f(x,y)$ Then it is also extrema value of $f(x,b)$ & $f(a,y)$ at $x=a$ & $y=b$ respectively

$$\Rightarrow \frac{\partial f}{\partial x}(x,b)\Big|_{at\ x=a} = 0 \Rightarrow f_x(a,b) = 0$$

& $$\frac{\partial f}{\partial y}(a,y)\Big|_{at\ y=b} = 0 \Rightarrow f_y(a,b) = 0$$

$$\Rightarrow f_x(a,b) = 0 = f_y(a,b)$$

Scanned by CamScanner

Remark:- Converse of the above theorem is not true in general.

Theorem:- Let $f : G \to \mathbb{R}$, G is an open set in $\mathbb{R}^n$. If f has a local extrema at $a \in G$ then $\nabla f(a) = 0$ (∇ = gradiant).

Proof:- As we know that if $f = f(x_1, x_2, x_3, \cdots, x_n)$
Then $\nabla f(a) = 0$

$$\Leftrightarrow \frac{\partial f}{\partial x_i}(a) = 0 \quad \text{for } i = 1, 2, 3, \cdots, n$$

Let $(x_1, x_2, \cdots, x_{i-1}, x_{i+t}, x_{i+1}, \cdots, x_n)$ be a point in the nbhd of $a \in G$.
Then given f has a local extrema at $a \in G$.
Without any loss of generality we suppose that f has local maximum at a

$$\Rightarrow f(x) \leq f(a) \quad \forall\, x \in N_\epsilon(a)$$

$$\Rightarrow f(a_i + t) \leq f(a_i) \quad \forall\, i = 1, 2, 3, \cdots, n$$

$$\Rightarrow f(a_i + t) - f(a_i) \leq 0$$

$$\Rightarrow \frac{f(a_i + t) - f(a_i)}{t} \leq 0 \quad \text{if } t > 0$$

$$\& \quad \frac{f(a_i + t) - f(a_i)}{t} \geq 0 \quad \text{if } t < 0$$

when $t \to 0$ we have

$$\frac{\partial f}{\partial x_i}(a_i) \leq 0 \quad \& \quad \frac{\partial f}{\partial x_i}(a_i) \geq 0$$

$$\Rightarrow \frac{\partial f}{\partial x_i}(a) \leq 0 \quad \& \quad \frac{\partial f}{\partial x_i}(a) \geq 0$$

$$\Rightarrow \frac{\partial f}{\partial x_i}(a) = 0 \quad \Longrightarrow \quad \nabla f(a) = 0$$

* **Saddle point:-** There are situations when $\nabla f(a) = 0$ but f has no extremum at 'a'. Such a point are called Saddle point.

Theorem:- Let $f: V \to R$, V is a nbhd of $(a,b) \in R^2$. Suppose that all of 2nd order partial derivatives are continuous at (a,b) then

(i) (a,b) is a local minimum if $f_{xx}(a,b) > 0$ and $[f_{xx}(a,b)][f_{yy}(a,b)] - [f_{xy}(a,b)]^2 > 0$

(ii) (a,b) is a local maximum if $f_{xx}(a,b) < 0$ & $[f_{xx}(a,b)][f_{yy}(a,b)] - [f_{xy}(a,b)]^2 > 0$

(iii) (a,b) is a saddle point if $[f_{xx}(a,b)][f_{yy}(a,b)] - [f_{xy}(a,b)]^2 < 0$

Proof:- Let $(a+h, b+k)$ be a point in the nbhd of (a,b). Then by Taylor's Theorem

$$f(a+h,b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + R,$$ where R is remainder after 3 terms.

$$f(a+h,b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(a,b) + \frac{1}{2}\left(h^2\frac{\partial^2}{\partial x^2} + k^2\frac{\partial^2}{\partial y^2} + 2hk\frac{\partial^2}{\partial x \partial y}\right)f(a,b),$$

where R is neglected b/w h & k are small

$$f(a+h,b+k) - f(a,b) = h f_x(a,b) + k f_y(a,b) + \frac{1}{2}\left[h^2 f_{xx}(a,b) + k^2 f_{yy}(a,b) + 2hk f_{xy}(a,b)\right]$$

Since f has local extrema at (a,b)

$\therefore f_x(a,b) = 0 = f_y(a,b)$

Also let $A = f_{xx}(a,b)$, $B = f_{xy}(a,b)$, $C = f_{yy}(a,b)$

So ① $\Rightarrow f(a+h, b+k) - f(a,b) = \frac{1}{2}[h^2 A + 2hkB + k^2 C]$ ⊛

$\Rightarrow \Delta f = \frac{1}{2A}[h^2 A^2 + 2hkAB + k^2 AC]$

$= \frac{1}{2A}[(hA)^2 + 2(hA)(kB) + (kB)^2 - (kB)^2 + k^2 AC]$

$\Delta f = \frac{1}{2A}(hA + kB)^2 + k^2(AC - B^2)$ —— ②

(i) Given $A > 0$, $AC - B^2 > 0$

Then ② $\Rightarrow \Delta f > 0$

$\Rightarrow f(a+h, b+k) - f(a,b) > 0$

$\Rightarrow f(a,b) < f(a+h, b+k)$

$\Rightarrow f$ has a local minima at (a,b)

(ii) Given $A < 0$ & $AC - B^2 > 0$

Then ② $\Rightarrow \Delta f < 0$

$\Rightarrow f(a+h, b+k) - f(a,b) < 0$

$f(a+h, b+k) < f(a,b)$

$\Rightarrow f$ has local maxima at (a,b)

(iii) Given $AC - B^2 < 0 \Rightarrow AC < B^2$

Then we have the following three cases.

<u>Case 1</u> If $AC < 0$

$\Rightarrow A$ & C have different signs

If $h = 0$ & $k \neq 0$ then equ ② becomes

$$\Delta f = \frac{1}{2A}\left[(kB)^2 + k^2(AC-B)^2\right]$$

$$= \frac{1}{2A}(k^2 AC) = \frac{1}{2}k^2 C$$

$\Rightarrow \Delta f$ is +ve if C is +ve &

Δf is −ve if C is −ve

A similar result follows if we take $h \neq 0$ & $k = 0$

Case 2 If $AC > 0$

$\Rightarrow A$ & C have same signs

Then by discussing in the same way as in case 1 we see that Δf have different singns for different conditions.

Case 3 If $AC = 0$

$\Rightarrow$ either $A = 0$ or $C = 0$

In both cases $A = 0 = C$

(*) $\Rightarrow f(a+h, b+k) - f(a,b) = hkB$

Clearly R.H.S have different singns under different conditions. So we conclude that if $AC - B < 0$ then (a,b) is a saddle point.

Question:- Find the extrema of

$$f(x,y) = 3x^2 + y^2 - 2xy - 4y$$

Solution, $f_x = 6x - 2y$

$f_y = 2y - 2x - 4$

$f_{xx} = 6$, $f_{yy} = 2$, $f_{xy} = -2$

put $f_x = 0, f_y = 0$

$6x - 2y = 0 \Rightarrow 3x - y = 0$ ——— ①

$2y - 2x - 4 = 0 \Rightarrow y - x - 2 = 0$ ——— ②

from ① $y = 3x$ put in ②

$3x - x - 2 = 0 \Rightarrow \boxed{x = 1}$ put in ①

$3 - y = 0 \Rightarrow \boxed{y = 3}$

$\therefore$ pt $(1, 3)$

Now at $(1,3)$ $f_{xx} = 6$

$f_{yy} = 2$, $f_{xy} = -2$

$f_{xx} f_{yy} - (f_{xy})^2 = 6(2) - (-2)^2 = 12 - 4$

$= 8 > 0$

Also $f_{xx} = 6 > 0$

$\Rightarrow (1,3)$ is a local minima.

Note:- Constants, Polynomial & $\sin x$, $\cos x$ are always continuous & differentiable.

V. Imp

Question:- Find the extrema of

$f(x,y) = \sin x + \sin y + \sin(x+y)$ $\quad 0 \le x \le \pi/2$

Proof) $f_x = \cos x + \cos(x+y)$

$f_y = \cos y + \cos(x+y)$

$f_{xx} = -\sin x - \sin(x+y)$, $f_{yy} = -\sin y - \sin(x+y)$

$f_{xy} = -\sin(x+y)$

Put $f_x = 0$, $f_y = 0$

$\Rightarrow \cos x + \cos(x+y) = 0$ ——— ①

$\cos y + \cos(x+y) = 0$ ——— ②

Put $y = x$ in ①

Whenever we have trigno function and two variables we take limit along the path $x=y$

$\Rightarrow \cos x + \cos 2x = 0$

$\Rightarrow \cos x + 2\cos^2 x - 1 = 0$

$\Rightarrow 2\cos^2 x + \cos x - 1 = 0$

$\Rightarrow \cos x = \frac{-1 \pm 3}{4}$ by quadratic formula

$\Rightarrow \cos x = -1, \ ^1/_2$

$\Rightarrow x = \cos^{-1}(-1), \ \cos^{-1}(^1/_2)$

$x = \pi, \ \pi/3$

As $0 \leq x \leq \pi/2$ so we can take $x = \pi/3$

$\Rightarrow y = \pi/3 \quad \because y = x$

$\Rightarrow$ pt $(\pi/3, \pi/3)$

At $(\pi/3, \pi/3)$

$f_{xx} = -\sin \pi/3 - \sin 2\pi/3$

$= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$

$f_{yy} = -\sqrt{3}, \quad f_{xy} = \frac{-\sqrt{3}}{2}$

$f_{xx}f_{yy} - (f_{xy})^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{-\sqrt{3}}{2}\right)^2$

$= 3 - 3/4 = 9/4 > 0$

& $f_{xx} = -\sqrt{3} < 0$

$\Rightarrow f$ has local maximum at $(\pi/3, \pi/3)$

Question:- Show that $f(x,y) = \sin x \sin y \sin(x+y)$ has minimum at $(0,0)$ & maximum at $(\pi/3, \pi/3)$ in the +ve quadrant when $x^2+y^2 \leq \pi^2/2$

Solution:- $f_x = \sin y[\cos x \sin(x+y) + \sin x \cos(x+y)]$

$f_y = \sin x[\cos y \sin(x+y) + \sin y \cos(x+y)]$

$f_{xx} = \sin y\{-\sin x \sin(x+y) + \cos x \cos(x+y) +$
$\cos x \cos(x+y) - \sin x \sin(x+y)\}$

$= \sin y\{\cos(x+x+y) + \cos(x+x+y)\}$

$= 2\sin y \cos(2x+y)$

$f_{yy} = 2\sin x \cos(x+2y)$

As $f_x = \sin y\{\cos x \sin(x+y) + \sin x \cos(x+y)\}$

$= \sin y\{\sin(x+y+x)\}$

$= \sin y \sin(2x+y)$

$f_{xy} = \sin y \cos(2x+y) + \sin(2x+y)\cos y$

$= \sin(y+2x+y) = \sin(2x+2y)$

Similarly

$f_{yx} = \sin(2x+2y)$

$\Rightarrow$ 2nd order partial derivatives are continuous

Put $f_x = 0$

$\sin y \sin(2x+y) = 0$

Put $x = y \Rightarrow \sin x \sin 3x = 0$

either $\sin x = 0$ or $\sin 3x = 0 \Rightarrow 3x = 0$ or $3x = \pi$

$x = 0, \pi$ $\qquad x = 0, \pi/3$ $\qquad \Rightarrow x = \pi/3$

$\Rightarrow x = 0, \pi/3, \pi$

only $0, \pi/3$ lies in 1st quadrant

when $x = 0 \rightarrow y = 0$

$x = \pi/3 \rightarrow y = \pi/3$

So possible extreme points are $(0,0)$ & $(\pi/3, \pi/3)$

* At $(0,0)$

$$f_{xx} = 0 \ , \ f_{yy} = 0 \ , \ f_{xy} = 0 \ , \ f_{yx} = 0$$

Since all the 2nd order partial derivatives are zero at $(0,0)$

$$\Rightarrow f_{xx} f_{yy} - (f_{xy})^2 = 0$$

Therefore above theorem failed.

Now further investigation is required

Now we use basic definition to find extrema

clearly for $0 < |x| < \pi/4$

$0 < |y| < \pi/4$

f is +ve and $f(0,0) = 0$

$f(x,y) = \sin x \sin y \sin(x+y)$

$-\pi/4 < x < \pi/4 \ , \ -\pi/4 < y < \pi/4$

Put $x = y$

$$f(x,y) = (\sin x)^2 \sin 2x$$

At $x = \pi/4$ $\quad f(x,y) = \left(\frac{1}{\sqrt{2}}\right)^2 \sin\frac{\pi}{2}$

$$= \frac{1}{2}(1) = \frac{1}{2} > 0$$

Consider $|x| < \pi/4 \Rightarrow x^2 < \pi^2/16$

$y^2 < \pi^2/16$

$$x^2 + y^2 < \pi^2\left[\frac{1}{16} + \frac{1}{16}\right] = \pi^2/8 < \pi^2/2$$

$x^2+y^2 = \pi^2/2 < \left(\frac{\pi}{\sqrt{2}}\right)^2$ eqn of circle

$(x-0)^2+(y-0)^2 \leq (\pi/\sqrt{2})^2$

So f has local minimum at $(0,0)$

★ At $(\pi/3, \pi/3)$

$f_{xx} = 2\sin y \cos(2x+y) = 2\sin \pi/3 \cos\left(\frac{2\pi}{3}+\pi/3\right)$

$= 2\left(\frac{\sqrt{3}}{2}\right)(-1) = -\sqrt{3}$

$f_{yy} = \sin(2x+2y) = \sin\left(\frac{2\pi}{3}+\frac{2\pi}{3}\right)$

$= \sin\left(\frac{4\pi}{3}\right) = \frac{-\sqrt{3}}{2}$

Consider $f_{xx}f_{yy}-(f_{xy})^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{-\sqrt{3}}{2}\right)^2$

$= 3 - 3/4 = 9/4 > 0$

Also $f_{xx} = -\sqrt{3} < 0$

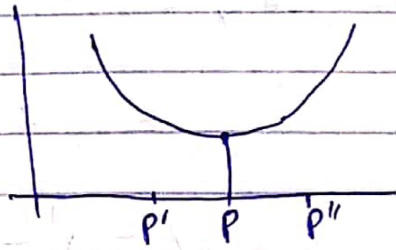
$\Rightarrow f$ has local maximum at $(\pi/3, \pi/3)$

★ First Derivative Test for Extremas.

If $f'(x) < 0$ at P'

& $f'(x) > 0$ at P''

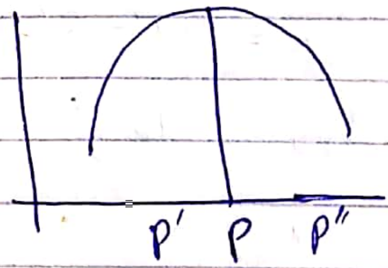
Then there is minima at P



If $f'(x) > 0$ at P'

& $f'(x) < 0$ at P''

Then there is maxima at P



where P' & P'' are nbring points of P.

Question:- Find the extremums of

(i) $-x^2+y^2$

(ii) $x^4-y^4-2(x^2-y^2)$

(iii) $y^2+3x^2y-3x^2+2$

Solution:- (i) $-x^2+y^2$

$f_x = -2x \quad , \quad f_y = 2y$

$f_{xx} = -2 \quad , \quad f_{yy} = 2$

$f_{xy} = 0 \quad , \quad f_{yx} = 0$

2nd order partial derivatives are continuous

Put $f_x = 0 \Rightarrow -2x = 0 \Rightarrow x = 0$

& $f_y = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$

The possible extreme point is $(0,0)$

At $(0,0)$

$f_{xx} = -2 \, , \, f_{yy} = 2 \, , \, f_{xy} = 0$

So $f_{xx}f_{yy} - (f_{xy})^2 = 2(-2) - 0$

$= -4 < 0$

$\Rightarrow (0,0)$ is a saddle point.

(ii) $x^4-y^4-2(x^2-y^2)$

$f_x = 4x^3 - 2(2x) = 4x^3 - 4x$

$f_{xx} = 12x^2 - 4$

$f_{xy} = 0$

$f_y = -4y^3 - 2(-2y) = -4y^3 + 4y$

$f_{yy} = -12y^2 + 4 \quad , \quad f_{yx} = 0$

Scanned by CamScanner

all 2nd order derivatives are continuous

$f_x = 0 \Rightarrow 4x^3 - 4x = 0 \Rightarrow 4x(x^2-1) = 0$

$\Rightarrow 4x = 0$ or $x^2 - 1 = 0 \Rightarrow x^2 = 1$

$\Rightarrow x = 0,$ ~~± 1~~ or $x = \pm 1$

$f_y = 0 \Rightarrow -4y^3 + 4y = 0 \Rightarrow 4y(-y^2+1) = 0$

$\Rightarrow 4y = 0$ or $1 - y^2 = 0$

$\Rightarrow y = 0$ or $y^2 = 1 \rightarrow y = \pm 1$

The possible extreme points are

$(0,0), (1,1), (1,-1), (-1,1) (-1,-1)$

★ At $(0,0)$

$f_{xx} = -4, \; f_{yy} = 4, \; f_{xy} = 0$

$\Rightarrow f_{xx} f_{yy} - (f_{xy})^2 = -16 - 0 = -16 < 0$

$\therefore (0,0)$ is saddle point

★ At $(1,1)$ $\quad f_{xx} = 8, \; f_{yy} = -8, \; f_{xy} = 0$

$\Rightarrow f_{xx} f_{yy} - (f_{xy})^2 = 8(-8) - 0 = -64 < 0$

$\therefore (1,1)$ is a saddle point

★ At $(1,-1)$

$f_{xx} = 8, \; f_{yy} = -8, \; f_{xy} = 0$

$\Rightarrow f_{xx} f_{yy} - (f_{xy})^2 = 8(-8) - 0 = -64 - 0$

$= -64 < 0$

$\therefore (1,-1)$ is saddle point

★ At $(-1,1)$ $\quad f_{xx} = 8, \; f_{yy} = -8, \; f_{xy} = 0$

$f_{xx} f_{yy} - (f_{xy})^2 = 8(-8) - 0 = -64 < 0$

$\therefore (-1,1)$ is saddle Point

★ At $(-1,-1)$ $\quad f_{xx} = 8, \; f_{yy} = -8, \; f_{xy} = 0$

$\Rightarrow f_{xx} f_{yy} - (f_{xy})^2 = -64 < 0$

$\therefore (-1,-1)$ also a saddle point

$\Rightarrow$ All the points are saddle points.

(iii) $y^3 + 3x^2y - 3x^2 - 3y^2 + 2$

$f_x = 6xy - 6x$
$f_{xx} = 6y - 6$
$f_{xy} = 6x$

$f_y = 2y + 3x^2 - 6y$
$= 3x^2 - 4y$
$f_{yy} = 2 - 6 = -4$
$f_{yx} = 6x$

$\Rightarrow$ All 2nd order partial derivatives are continuous

Put $f_x = 0 \Rightarrow 6xy - 6x = 0 \Rightarrow 6x(y-1) = 0$

$\Rightarrow 6x = 0$ or $y - 1 = 0$

$\boxed{x = 0}$ or $\boxed{y = 1}$

Now $f_y = 0 \Rightarrow 3x^2 - 4y = 0$

when $x = 0$ $3(0)^2 - 4y = 0 \Rightarrow \boxed{y = 0}$

when $y = 1$

$3x^2 - 4 = 0 \Rightarrow 3x^2 = 4 \Rightarrow x^2 = 4/3$

$\Rightarrow x = \pm 2/\sqrt{3}$

So possible extreme points are

$(0,0)$, $(2/\sqrt{3}, 1)$, $(-2/\sqrt{3}, 1)$

★ At $(0,0)$ $f_{xx} = -6$, $f_{yy} = -4$, $f_{xy} = 0$

$\Rightarrow f_{xx} f_{yy} - (f_{xy})^2 = (-6)(-4) - 0$
$= 24 > 0$

Also $f_{xx}(0,0) < 0$

$\Rightarrow$ There is maxima at $(0,0)$

★ At $(2/\sqrt{3}, 1)$

$f_{xx} = 0$, $f_{yy} = -4$, $f_{xy} = 6\left(\frac{2}{\sqrt{3}}\right)$

$\Rightarrow f_{xx} f_{yy} - (f_{xy})^2 = 0(-4) - \left(\frac{12}{\sqrt{3}}\right)^2$

$= -\frac{144}{3} = -38 < 0$

$\Rightarrow \left(\frac{2}{\sqrt{3}}, 1\right)$ is a saddle point.

★ At $\left(\frac{-2}{\sqrt{3}}, 1\right)$

$$f_{xx} = 0, \quad f_{yy} = -4, \quad f_{xy} = 6\left(\frac{-2}{\sqrt{3}}\right) = \frac{-12}{\sqrt{3}}$$

$$\Rightarrow f_{xx} f_{yy} - (f_{xy})^2 = 0 - \left(-\frac{(12)}{\sqrt{3}}\right)^2$$

$$= -\left(\frac{-12}{\sqrt{3}}\right)^2 < 0$$

$\Rightarrow \left(\frac{-2}{\sqrt{3}}, 1\right)$ is a saddle point.

Theorem:- Let $f: V \to \mathbb{R}$ & $g: V \to \mathbb{R}$, $(V \subseteq \mathbb{R}^3)$ is a nbhd of $(a,b,c) \in \mathbb{R}^3$. If $f(x,y,z)$ has local extrema at (a,b,c) subject to the condition $g(x,y,z) = 0$, where $\nabla g \neq 0$ at (a,b,c) Then there exist a real number λ s.t $\nabla f + \lambda \nabla g = 0$ at (a,b,c).
λ is called Lagrange Multiplier.

Proof: Since $\nabla g(a,b,c) \neq 0$
So by implicit function theorem there is a nbhd $V_\delta(a,b)$ and a unique function $z = \phi(x,y)$ s.t $g(x,y,\phi(x,y)) = 0, \forall (x,y) \in V_\delta$
and at (a,b)

$$\left.\phi_x = \frac{-g_x}{g_z}, \quad \phi_y = \frac{-g_y}{g_z}\right\} \quad \text{——} \ (1)$$

Since f has an extrema at (a,b,c), So by Chain Rule we have

$$f_x + f_z \phi_x = 0$$

$$\& \quad f_y + f_z \phi_y = 0$$

using (1) we have

$$f_x + f_z\left(\frac{-g_x}{g_z}\right) = 0$$

$$\& \quad f_y + f_z\left(\frac{-g_y}{g_z}\right) = 0$$

263

$$\Rightarrow f_x + \left(\frac{-f_x}{g_z}\right) g_x = 0$$

$$\& \quad f_y + \left(\frac{-f_x}{g_z}\right) g_y = 0$$

$$\left.\begin{aligned} \Rightarrow f_x + \lambda g_x &= 0 \\ \& \quad f_y + \lambda g_y &= 0 \end{aligned}\right\} \text{———} (2)$$

where $\lambda = \dfrac{-f_x}{g_z}$

$\Rightarrow$ equ (2) can be writtean as

$$\nabla f + \lambda \nabla g = 0$$

* Complicit function Theorem:-

Consider two variable function

$$f(x,y) = x^3 + y^3 = 3x$$

Then $\dfrac{dy}{dx} = \dfrac{-f_x}{f_y}$ $\dfrac{\text{Independent variable}}{\text{Dependent variable}}$

$$\phi_x = \frac{-g_x}{g_z}, \quad \phi_y = \frac{-g_y}{g_z}$$

∇(gradiant) is slope of the function

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \qquad \nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$$

$$\nabla f + \lambda \nabla g = \left(\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y}\right)$$

$$\Rightarrow f_x + \lambda g_x = 0, \quad f_y + \lambda g_y = 0$$

The above Theorem is called Lagrange Multiplier Method. λ is called Lagrange Multiplier.

Question:- Find the maxima & minima of $f(x,y) = x^2+24xy+8y^2$ where $x^2+y^2=25$

Solution:-

Given $x^2+y^2=25$

$\Rightarrow x^2+y^2-25=0$

Let $g = x^2+y^2-25$, Let $F = f+\lambda g$

$\Rightarrow F = x^2+24xy+8y^2+\lambda(x^2+y^2-25)$

Now $\frac{\partial F}{\partial x} = 2x+24y+\lambda(2x)$

$\frac{\partial F}{\partial y} = 24x+16y+\lambda(2y)$

Put $\frac{\partial F}{\partial x}=0$ & $\frac{\partial F}{\partial y}=0$

$\Rightarrow 2x+24y+\lambda(2x)=0$ ——— ①

$24x+2y+\lambda(2y)=0$ ——— ②

from eqn ① $-2\lambda = \frac{2x+24y}{x}$

& from eqn ② $-2\lambda = \frac{24x+16y}{y}$

Comparing the values

$\frac{2x+24y}{x} = \frac{24x+16y}{y} \Rightarrow 2xy+24y^2 = 24x^2+16xy$

$\Rightarrow 12x^2+7xy-12y^2=0$

$\Rightarrow (3x+4y)(4x-3y)=0$

$\Rightarrow 3x+4y=0$ ——— ③ or $4x-3y=0$ ——— ④

Also $x^2+y^2=25$ ——— ⑤

from eqn ③ $x = \frac{-4}{3}y$ ——— ⑥ Put in ⑤

$\left(-\frac{4}{3}y\right)^2+(y)^2=25 \Rightarrow \frac{25y^2}{9}=25$

$\Rightarrow 25y^2=225 \Rightarrow y^2=9 \Rightarrow \boxed{y=\pm 3}$

Put $y=3$ in equ (B) $\Rightarrow$ $\boxed{x=-4}$

$\therefore$ $\boxed{\text{pt } (-4,3)}$

Put $y=-3$ in equ (B) $\Rightarrow$ $x=4$

$\therefore$ $\boxed{\text{pt } (4,-3)}$

Next we solve of equ (4) & (5)

$$\left(\frac{3y}{4}\right)^2 = 25 \Rightarrow y = \pm 4$$

when $y=4$ $\Rightarrow$ $x=3$ $\therefore$ $\boxed{\text{pt } (3,4)}$

when $y=-4$ $\Rightarrow$ $x=-3$ $\therefore$ $\boxed{\text{pt } (-3,-4)}$

So all possible points are

$(-4,3), (4,-3), (3,4), (-3,-4)$

By considering the sign it is clear that F has maximum at $(3,4)$ & $(-3,-4)$ & minima at $(-4,3)$ & $(4,-3)$

Note:- Now consider

$$f(x,y) = x^2 + 24xy + 8y^2$$

Now the points for which $f(x,y) > 0 = +ve$ are maximas & the points for which $f(x,y) < 0$ are minimas.

Question- Find the stationary value of $f(x,y) = 7x^2 + 8xy + y^2$ subject to constraint $x^2+y^2=1$ [Stationary value mean critical point]

Solution Given $x^2+y^2=1$ $\Rightarrow$ $x^2+y^2-1=0$

Let $g = x^2+y^2-1$

Let $F = f + \lambda g$

$\Rightarrow F(x,y) = (7x^2 + 8xy + y^2) + \lambda(x^2+y^2-1)$

$$\frac{\partial F}{\partial x} = 14x + 8y + 2\lambda x$$

$$\frac{\partial F}{\partial y} = 8x + 2y + 2\lambda y$$

Put $\frac{\partial F}{\partial x} = 0$ & $\frac{\partial F}{\partial y} = 0$

$14x + 8y + 2\lambda x = 0$ —— ①

$8x + 2y + 2\lambda y = 0$ —— ②

from ① $-2\lambda = \frac{14x + 8y}{x}$ —— ③

From ② $-2\lambda = \frac{8x + 2y}{y}$ —— ④

comparing ③ & ④

$$\frac{14x + 8y}{x} = \frac{8x + 2y}{y}$$

$\Rightarrow 14x + 8y^2 = 8x^2 + 2xy \Rightarrow 8x^2 - 12xy - 8y^2 = 0$

$\Rightarrow 8x^2 - 16xy + 4xy - 8y^2 = 0$

$\Rightarrow 8x(x - 2y) + 4y(x - 2y) = 0$

$\Rightarrow (x - 2y)(8x + 4y) = 0$

$\Rightarrow$ either $x - 2y = 0$ or $8x + 4y = 0$

$\Rightarrow x = 2y$ or $x = -y/2$

Also $x^2 + y^2 = 1$ —— ⑤

★ when $x = 2y$ put in ⑤

$(2y)^2 + y^2 = 1 \Rightarrow 4y^2 + y^2 = 1$

$\Rightarrow 5y^2 = 1 \Rightarrow y^2 = 1/5 \Rightarrow y = \pm 1/5$

★ When $y = 1/5 \Rightarrow x = +2/5$

★ When $y = -1/5 \Rightarrow x = -2/5$

Stationary points are $(2/5, 1/5)$ & $(-2/5, -1/5)$

★ when $x = -y/2$ put in eqn ⑤

$\left(-\frac{y}{2}\right)^2 + y^2 = 1 \Rightarrow y^2/4 + y^2 = 1 \Rightarrow y^2\left(\frac{1}{4} + 1\right) = 1$

$\Rightarrow y^2\left(\frac{1+4}{4}\right) = 1 \Rightarrow y^2 = \frac{4}{5} \Rightarrow y = \pm \frac{2}{\sqrt{5}}$

★ When $y = \frac{2}{\sqrt{5}} \Rightarrow x = -\frac{1}{2}\left(\frac{2}{\sqrt{5}}\right) = -\frac{1}{\sqrt{5}}$

★ When $y = -\frac{2}{\sqrt{5}} \Rightarrow x = -\frac{1}{2}\left(\frac{-2}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}$

So stationary points are

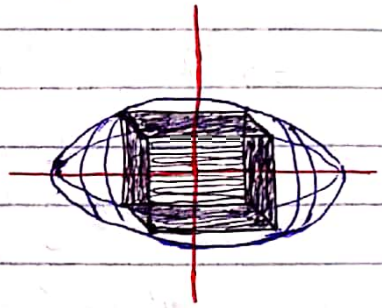
$\left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ & $\left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$

Hence the stationary points are

$$\left(\frac{2}{5}, \frac{1}{5}\right), \left(\frac{-2}{5}, \frac{1}{5}\right), \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$$

Question:- Find the greatest volume of the box contained in the ellipsoid $3x^2+2y^2+z^2 = 18$ when each of its edge is parallel to one of the co-ordinate axis

Solution centre of ellipse is at origin.



Let $2x, 2y, 2z$ be the edges of the box parallel to the co-ordinate axis. Then volume of the box is

$$f(x,y,z) = (2x)(2y)(2z) = 8xyz$$

We have to extremize the volume subject to the constraint

$$g(x,y,z) = 3x^2+2y^2+z^2-18$$

Let $F(x,y,z) = f + \lambda g$

$$= 8xyz + \lambda(3x^2+2y^2+z^2-18)$$

$$\frac{\partial F}{\partial x} = 8yz + 6\lambda x, \quad \frac{\partial F}{\partial y} = 8xz + 4\lambda y$$

$$\frac{\partial F}{\partial z} = 8xy + 2\lambda z$$

Put $\frac{\partial F}{\partial x} = 0, \; \frac{\partial F}{\partial y} = 0, \; \frac{\partial F}{\partial z} = 0$

$$\Rightarrow \frac{\partial F}{\partial x} \Rightarrow 8yz + 6\lambda x = 0 \quad \text{——— (i)}$$

$$8xz + 4\lambda y = 0 \quad \text{——— (ii)}$$

$$8xy + 2\lambda z = 0 \quad \text{——— (iii)}$$

Multiply (i), (ii) & (iii) by x, y, z respectively & taking common 2 respectively

$$4xyz + 3\lambda x^2 = 0 \quad \text{——— ①}$$

$$4xyz + 2\lambda y^2 = 0 \quad \text{——— ②}$$

$$4xyz + \lambda z^2 = 0 \quad \text{——— ③}$$

Subtract eqn ② from ①

$$3\lambda x^2 - 2\lambda y^2 = 0 \Rightarrow \lambda(3x^2 - 2y^2) = 0$$

$$\Rightarrow 3x^2 - 2y^2 = 0 \quad (\because \lambda \neq 0) \quad \text{——— ⑤}$$

[If $\lambda = 0 \Rightarrow$ constraint will be zero. So it will becomes useless. So Langrange Muliplier can not be zero]

$$⑤ \Rightarrow x^2 = \frac{2}{3}y^2 \Rightarrow x = \sqrt{\frac{2}{3}}\, y \quad \text{——— ⑥}$$

Subtract ③ from ①

$$3\lambda x^2 - \lambda z^2 = 0 \Rightarrow \lambda(3x^2 - z^2) = 0$$

$$\Rightarrow 3x^2 - z^2 = 0 \Rightarrow x^2 = \frac{1}{3}z^2$$

$$\Rightarrow x = z/\sqrt{3} \quad \text{——— ⑦}$$

x, y, z are lengthes always +ve so we are not taking $\pm$

Also $3x^2 + 2y^2 + z^2 = 18$

$$\Rightarrow 3x^2 + 2\left(\sqrt{\frac{3}{2}}\,x\right)^2 + \left(\sqrt{3}\,x\right)^2 = 18 \quad \text{by ⑥ \& ⑦}$$

$$\Rightarrow 3x^2 + 3x^2 + 3x^2 = 18 \Rightarrow 9x^2 = 18$$

$$\Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$

* when $x = \sqrt{2}$

by ⑥ $\sqrt{\frac{2}{3}}\,y = \sqrt{2} \Rightarrow y = \sqrt{3}$

by ⑦ $\frac{z}{\sqrt{3}} = \sqrt{2} \Rightarrow z = \sqrt{6}$

$\therefore$ Greatest volume

$$f = 8xyz = 8\sqrt{2}\sqrt{3}\sqrt{6} = 8\sqrt{36}$$
$$= 8(6) = 48$$

Question:- Find the extrema of $f(x,y,z) = xyz$ subject to $x^2+y^2+z^2=1$

Solution:- Let $F(x,y,z) = xyz + \lambda(x^2+y^2+z^2-1)$

$$\frac{\partial F}{\partial x} = yz + 2\lambda x$$
$$\frac{\partial F}{\partial y} = xz + 2\lambda y$$
$$\frac{\partial F}{\partial z} = xy + 2\lambda z$$

Put $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$

$\Rightarrow yz + 2\lambda x = 0$ —— ①

$xz + 2\lambda y = 0$ —— ②

$xy + 2\lambda z = 0$ —— ③

Multiply eqn ①, ② & ③ by x, y, z respectively.

$xyz + 2\lambda x^2 = 0$ —— ④

$xyz + 2\lambda y^2 = 0$ —— ⑤

$xyz + 2\lambda z^2 = 0$ —— ⑥

Subtract eqn ⑤ from ④

$2\lambda x^2 - 2\lambda y^2 = 0 \Rightarrow 2\lambda(x^2-y^2) = 0$

$\Rightarrow x^2 - y^2 = 0 \quad (\because \lambda \neq 0)$

$\Rightarrow x^2 = y^2 = x = \pm y$ —— ⑦ $\Rightarrow y = \pm x$

Subtract equ ⑥ from ④

$2\lambda x^2 - 2\lambda z^2 = 0 \Rightarrow 2\lambda(x^2 - z^2) = 0$

$\Rightarrow x^2 - z^2 = 0 \quad (\because 2\lambda \neq 0)$

$\Rightarrow x^2 = z^2 \Rightarrow x = \pm z$ — ⑧ $\Rightarrow z = \pm x$

Also we are given $x^2 + y^2 + z^2 = 1$

$\Rightarrow x^2 + x^2 + x^2 = 1$ from ⑦ & ⑧

$\Rightarrow 3x^2 = 1 \Rightarrow x^2 = 1/3$

$\Rightarrow x = \pm 1/\sqrt{3}$

★ when $x = \frac{1}{\sqrt{3}}$

$z = \pm \frac{1}{\sqrt{3}}, \quad y = \pm \frac{1}{\sqrt{3}}$

So points are $\left(\frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}\right)$

★ when $x = -\frac{1}{\sqrt{3}}$

$y = \mp \frac{1}{\sqrt{3}}, \quad z = \mp \frac{1}{\sqrt{3}}$

So points are $\left(-\frac{1}{\sqrt{3}}, \mp \frac{1}{\sqrt{3}}, \mp \frac{1}{\sqrt{3}}\right)$

Now $f(x,y,z) = xyz$

There are maximas at point $\left(\frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}\right)$

& minimas at $\left(-\frac{1}{\sqrt{3}}, \mp \frac{1}{\sqrt{3}}, \mp \frac{1}{\sqrt{3}}\right)$

Question:- Test for maxima & minima subject to the contraint $x^2 + y^2 = 1$ & $x - z = 0$ of $f(x,y,z) = xyz$

Solution:- Consider $F(x,y,z) = xyz + \lambda_1(x^2 + y^2 - 1) + \lambda_2(x - z)$

$\frac{\partial F}{\partial x} = yz + 2\lambda_1 x + \lambda_2$

$\frac{\partial F}{\partial y} = xz + 2\lambda_1 y, \quad \frac{\partial F}{\partial z} = xy - \lambda_2$

Put $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$

$yz + 2\lambda_1 x + \lambda_2 = 0$ ——— ①

$xz + 2\lambda_1 y = 0$ ——— ②

$xy - \lambda_2 = 0$ ——— ③

③ $\Rightarrow \lambda_2 = xy$ ——— ⑥

we are also given

$x^2 + y^2 = 1$ ——— ④ & $x - z = 0$

$\Rightarrow x = z$ ——— ⑤

Now from equ ①

$xy + 2\lambda_1 x + xy = 0$ by ⑤ & ⑥

$\Rightarrow 2xy + 2\lambda_1 x = 0$

$\Rightarrow -\lambda_1 = y$

From equ ② $-2\lambda_1 = \frac{xz}{y} = \frac{x^2}{y}$ $(\because x = z)$

$\Rightarrow -2(-y) = \frac{x^2}{y}$ $(\because \lambda_1 = -y)$

$\Rightarrow x^2 = 2y^2 \Rightarrow x = \pm\sqrt{2}\, y$

$\Rightarrow z = \pm\sqrt{2}\, y$ $(\because z = x)$

Also by equ ④

$x^2 + y^2 = 1 \Rightarrow 2y^2 + y^2 = 1 \Rightarrow 3y^2 = 1$

$\Rightarrow y^2 = \frac{1}{3} \Rightarrow y = \pm\frac{1}{\sqrt{3}}$

* When $y = \frac{1}{\sqrt{3}}$

$x = \pm\frac{\sqrt{2}}{\sqrt{3}}$, $z = \pm\sqrt{\frac{2}{3}}$

So the points are $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$

* when $y = \frac{-1}{\sqrt{3}}$

$$x = \mp\sqrt{\frac{2}{3}} \quad , \quad z = \mp\sqrt{\frac{2}{3}}$$

So the points are

$$\left(\mp\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}, \mp\sqrt{\frac{2}{3}}\right)$$

So by considering the signs of points its clear that f has maximum values at $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$ and minimum values at $\left(\mp\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{3}}, \mp\sqrt{\frac{2}{3}}\right)$

Question:- Find the extrema of $x+y-2z=0$ subject to $x^2+y^2+z^2=1$, $x+y+z=0$

Solution Let $F(x,y,z) = (x+y-2z) + \lambda_1(x^2+y^2+z^2-1) + \lambda_2(x+y+z)$

$$\frac{\partial F}{\partial x} = 1 + 2x\lambda_1 + \lambda_2$$

$$\frac{\partial F}{\partial y} = 1 + 2y\lambda_1 + \lambda_2$$

$$\frac{\partial F}{\partial z} = -2 + 2z\lambda_1 + \lambda_2$$

Put $F_x = F_y = F_z = 0$

$$1 + 2x\lambda_1 + \lambda_2 = 0 \quad \text{———} \quad (1)$$

$$1 + 2y\lambda_1 + \lambda_2 = 0 \quad \text{———} \quad (2)$$

$$-2 + 2z\lambda_1 + \lambda_2 = 0 \quad \text{———} \quad (3)$$

Subtract equ (2) from (1)

$$2x\lambda_1 - 2y\lambda_1 = 0$$

$$2\lambda_1(x-y) = 0 \implies x - y = 0 \quad (\because 2\lambda_1 \neq 0)$$

$$\implies \boxed{x = y}$$

Subtract equ ② from ①

$$z + 2x\lambda_1 - 2z\lambda_1 = 0$$

Als $x+y+z=0 \Rightarrow x+x+z=0 \Rightarrow 2x+z=0$

$\Rightarrow 2x = -z \Rightarrow \boxed{z = -2x}$

Now also we have

$$x^2+y^2+z^2 = 1$$

$$\Rightarrow x^2 + x^2 + (-2x)^2 = 1$$

$$\Rightarrow x^2+x^2+4x^2 = 1 \Rightarrow 6x^2 = 1 \Rightarrow x^2 = 1/6$$

$$\Rightarrow \boxed{x = \pm \frac{1}{\sqrt{6}}}$$

★ When $x = 1/\sqrt{6}$

$$y = 1/\sqrt{6}$$

& $z = -2\left(\frac{1}{\sqrt{6}}\right) = -2/\sqrt{6}$

So point is $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

★ When $x = \frac{-1}{\sqrt{6}} \Rightarrow y = -1/\sqrt{6}$

$$z = -2\left(\frac{-1}{\sqrt{6}}\right) = 2/\sqrt{6}$$

So point is $\left(\frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

Now $f(x,y,z) = x+y-2z$

There is maxima at $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$ &

minima at $\left(\frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

Question:- Test for maxima & minima for $W = x+z$ subject to $x^2+y^2+z^2 = 1$

Solution: $F(x,y,z) = x+z+\lambda(x^2+y^2+z^2-1)$

$\frac{\partial F}{\partial x} = 1 + 2\lambda x$, $\frac{\partial F}{\partial y} = 2\lambda y$, $\frac{\partial F}{\partial z} = 1 + 2\lambda z$

Put $F_x = F_y = F_z = 0$

$\Rightarrow 1 + 2\lambda x = 0$ ——— ①

$2\lambda y = 0$ ——— ②

$1 + 2\lambda z = 0$ ——— ③

From ② $\boxed{y = 0}$

Subtract equ ③ from ①

$2\lambda x - 2\lambda z = 0 \Rightarrow 2\lambda(x - z) = 0$

$\Rightarrow x - z = 0 \Rightarrow \boxed{x = z}$

Also given $x^2 + y^2 + z^2 = 1$

$\Rightarrow x^2 + 0 + x^2 = 1 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = 1/2$

$\Rightarrow x = \pm 1/\sqrt{2}$

When $x = \frac{1}{\sqrt{2}}$ when $x = \frac{-1}{\sqrt{2}}$

$z = 1/\sqrt{2}$ $z = \frac{-1}{\sqrt{2}}$

$y = 0$ $y = 0$

So the points are $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ & $\left(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$

Now $w = x + z$

w has maxima at $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ &

minima at $\left(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$

Question: Investigate the extrema of $z = 6 - 4x - 3y$ subject to $x^2 + y^2 = 1$

Solution $F(x,y) = 6 - 4x - 3y + \lambda(x^2 + y^2 - 1)$

$\frac{\partial F}{\partial x} = -4 + 2\lambda x$, $\frac{\partial F}{\partial y} = -3 + 2\lambda y$

Put $F_x = F_y = 0$

$-4+2\lambda x = 0 \Rightarrow 2\lambda = 4/x$ —— ①

$-3+2\lambda y = 0 \Rightarrow 2\lambda = 3/y$ —— ②

Comparing ① & ② $\frac{4}{x} = \frac{3}{y}$

$\Rightarrow 4y = 3x \Rightarrow x = \frac{4}{3}y$

Also given $x^2+y^2=1 \Rightarrow \left(\frac{4}{3}y\right)^2 + y^2 = 1$

$\Rightarrow \frac{16}{9}y^2 + y^2 = 1 \Rightarrow y^2\left(\frac{16}{9}+1\right) = 1 \Rightarrow y^2\left(\frac{16+9}{9}\right) = 1$

$\Rightarrow y^2 = \frac{9}{25} \Rightarrow y = \pm 3/5$

★ when $y = 3/5$

$x = \frac{4}{3}\left(\frac{3}{5}\right) = 4/5$

★ when $y = -3/5$

$x = \frac{4}{3}\left(\frac{-3}{5}\right) = \frac{-4}{5}$

So the points are $\left(\frac{4}{5}, \frac{3}{5}\right)$ & $\left(\frac{-4}{5}, \frac{-3}{5}\right)$

$z = 6-4x-3y = 6-4\left(\frac{4}{5}\right)-3\left(\frac{3}{5}\right)$

$z = 6-\frac{16}{5}-\frac{9}{5} = \frac{30-16-9}{5} = \frac{5}{5} = 1 > 0$

Also $z = 6-4\left(\frac{-4}{5}\right)-3\left(\frac{-3}{5}\right) = 6+\frac{16}{5}+\frac{9}{5} > 0$

So z has maxima at $\left(\frac{4}{5}, \frac{3}{5}\right)$ & $\left(\frac{-4}{5}, \frac{-3}{5}\right)$

Question:- Find the point (x, y, z) on the sphere $x^2+y^2+z^2=1$ which is fartherest from the point $(1, 2, 3)$

Solution

Let $P(x, y, z)$ be a point on the sphere and $A(1, 2, 3)$

distance form سے
distance جو دور ہیں
کو Max کرنا ہے

$$|AP| = \sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$$

$$|AP|^2 = (x-1)^2+(y-2)^2+(z-3)^2$$

Let $f(x, y, z) = (x-1)^2+(y-2)^2+(z-3)^2$

We have to maximize f subject to

$$x^2+y^2+z^2=1$$

Let $F(x, y, z) = (x-1)^2+(y-2)^2+(z-3)^2+\lambda(x^2+y^2+z^2-1)$

$$\frac{\partial F}{\partial x} = 2(x-1)+2\lambda x$$

$$\frac{\partial F}{\partial y} = 2(y-2)+2\lambda y$$

$$\frac{\partial F}{\partial z} = 2(z-3)+2\lambda z$$

Put $F_x = F_y = F_z = 0$

$2(x-1)+2\lambda x = 0 \implies x-1+\lambda x = 0$ ——— ①

$2(y-2)+2\lambda y = 0 \implies y-2+\lambda y = 0$ ——— ②

$2(z-3)+2\lambda z = 0 \implies z-3+\lambda z = 0$ ——— ③

From ① $x(1+\lambda)-1=0 \implies x = \dfrac{1}{1+\lambda}$

From ② $y(1+\lambda)-2=0 \implies y = \dfrac{2}{1+\lambda}$

From ③ $z(1+\lambda)-3=0 \implies z = \dfrac{3}{1+\lambda}$?

$P = \left(\dfrac{-1}{\sqrt{14}}, \dfrac{-2}{\sqrt{14}}, \dfrac{-3}{\sqrt{14}}\right)$ will be the fartheset point ~~from~~ from the point $A = (1, 2, 3)$

As P & A are in different octants but A & $\left(\frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$ lies in same octant.

Now as $x^2+y^2+z^2=1$

$$\Rightarrow \frac{1}{(1+\lambda)^2} + \frac{4}{(1+\lambda)^2} + \frac{9}{(1+\lambda)^2} = 1$$

$$\Rightarrow \frac{14}{(1+\lambda)^2} = 1 \Rightarrow (1+\lambda)^2 = 14$$

$$\Rightarrow 1+\lambda = \pm\sqrt{14}$$

So $x = \pm\frac{1}{\sqrt{14}}$, $y = \pm\frac{2}{\sqrt{14}}$, $z = \pm\frac{3}{\sqrt{14}}$

So the extrema points are

$$\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) \ \& \ \left(\frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$$

Question:- Calculate extrema of $f(x,y) = x^2+y^2$ subject to $x^3+y^3-6xy=0$

Solution:- Let $f(x,y) = x^2+y^2$

$$F(x,y) = x^2+y^2+\lambda(x^3+y^3-6xy)$$

$$\frac{\partial F}{\partial x} = 2x+3\lambda x^2-6y\lambda$$

$$\frac{\partial F}{\partial y} = 2y+3\lambda y^2-6x\lambda$$

Put $F_x = F_y = 0$

$$2x+3\lambda x^2-6y\lambda=0 \Rightarrow 2x+3\lambda(x^2-2y)=0 \quad \text{—(1)}$$

$$2y+3\lambda y^2-6x\lambda=0 \Rightarrow 2y+3\lambda(y^2-2x)=0 \quad \text{—(2)}$$

From (1) & (2)

$$-3\lambda = \frac{2x}{x^2-2y}, \quad -3\lambda = \frac{2y}{y^2-2x}$$

$$\Rightarrow \frac{2x}{x^2-2y} = \frac{2y}{y^2-2x}$$

$$2xy^2 - 4x^2 = 2x^2y - 4y^2$$

$$\Rightarrow 2xy^2 - 2x^2y - 4x^2 + 4y^2 = 0$$

$$\Rightarrow 2xy(y-x) - 4(x^2-y^2) = 0$$

$$\Rightarrow 2xy(y-x) - 4(x+y)(x-y) = 0$$

$$\Rightarrow -2xy(x-y) - 4(x+y)(x-y) = 0$$

$$\Rightarrow -2(x-y)\{xy+2x+2y\} = 0$$

$$\Rightarrow -2(x-y) = 0 \quad \text{OR} \quad y(x+2)+2x = 0$$

$$\Rightarrow x = y \quad \text{OR} \quad y = \frac{-2x}{x+2}$$

As we have $x^3+y^3-6xy = 0$

when $y = x$ $\quad x^3+x^3-6x^2 = 0 \Rightarrow 2x^3-6x^2=0$

$$\Rightarrow 2x^2(x-3) = 0 \Rightarrow x^2 = 0 \quad \text{OR} \quad x = 3$$

$$\Rightarrow x = 0 \quad \text{OR} \quad x = 3$$

So $y = 0$ OR $y = 3$

So Points are $(0,0)$, $(3,3)$

★ — ★ — ★ — ★ — ★ — ★ — ★

Prepared By

MUHAMMAD TAHIR (0344-8563284)

M.Sc :- Punjab University
M.S :- COMSATS Islamabad

⇒ Jacobian:- Suppose the system of equations is given by

$$y_1 = f_1(x_1, x_2, x_3, \ldots, x_n)$$
$$y_2 = f_2(x_1, x_2, x_3, \ldots, x_n)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$y_n = f_n(x_1, x_2, x_3, \ldots, x_n)$$

Then by definition of total differential

$$dy_1 = \frac{\partial f_1}{\partial x_1}dx_1 + \frac{\partial f_1}{\partial x_2}dx_2 + \cdots + \frac{\partial f_1}{\partial x_n}dx_n$$

$$dy_2 = \frac{\partial f_2}{\partial x_1}dx_1 + \frac{\partial f_2}{\partial x_2}dx_2 + \cdots + \frac{\partial f_2}{\partial x_n}dx_n$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$dy_n = \frac{\partial f_n}{\partial x_1}dx_1 + \frac{\partial f_n}{\partial x_2}dx_2 + \cdots + \frac{\partial f_n}{\partial x_n}dx_n$$

The Transformation matrix is given by

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

The determinant of matrix A is called

The Jacobian of the above transformation. It is denoted by J i.e

$$J = \frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

Question: If $x = r\cos\theta$, $y = r\sin\theta$
Find Jacobian of this transformation

Solution

As $J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

Question: Find the Jacobian of following Transformation

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

Solution

As $J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

$$\Rightarrow J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & -r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi[0 + r^2\sin^2\theta\cos\phi] + r\cos\theta\cos\phi[r\sin\theta\cos\phi\cos\theta]$$
$$- r\sin\theta\sin\phi[-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi]$$
$$= r^2\sin^3\theta\cos^2\phi + r^2\cos^2\theta\cos^2\phi\sin\theta + r^2\sin^3\theta\sin^2\phi$$
$$+ r^2\cos^2\theta\sin^2\phi\sin\theta$$
$$= r^2\sin^3\theta(\cos^2\theta + \sin^2\theta) + r^2\cos^2\theta\sin\theta(\cos^2\theta + \sin^2\theta)$$
$$= r^2\sin^3\theta + r^2\cos^2\theta\sin\theta$$
$$J = r^2\sin\theta(\sin^2\theta + \cos^2\theta)$$
$$J = r^2\sin\theta$$

★ **Implicit function:-** Consider the equation $f(x,y,z) = 0$ of three variables x, y, z. Where by assigning values to two variables x, y the value of third variable can be found. In this case z is said to be an implicit function of x & y. For example $\sin(x+y+z) - \cos z = 1$ defines an implicit function.

$x^3 + y^3 = 5 \Rightarrow y = (5 - x^3)^{1/3}$ is implicit function of y.

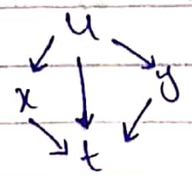
Question:- $f(x,y) = 0$ Let $u = f(x,y)$

$x = t$, $y = g(t)$

Then show that $\dfrac{dy}{dx} = \dfrac{-\partial f/\partial x}{\partial f/\partial y} = \dfrac{-f_x}{f_y}$

Solution

As $u = f(x,y)$



$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$0 = \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y}(g'(t))$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \qquad (\because y = g(t),\ t = x \Rightarrow y = g(x) \Rightarrow \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{\partial f}{\partial x}\frac{dy}{dx} = \frac{-\partial f}{\partial x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$$

$$= \frac{-f_x}{f_y}$$

Question:- If $f(x,y,u,v) = 0$, $g(x,y,u,v) = 0$

with $u = \phi(x,y)$ $v = \psi(x,y)$

Then find $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$

Solution Given $f(x,y,u,v) = 0$

$$\Rightarrow df = 0$$

$$\Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv = 0 \qquad \text{—— (1)}$$

$$g(x,y,u,v) = 0$$

$$\Rightarrow dg = 0$$

$$\Rightarrow \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial u}du + \frac{\partial g}{\partial v}dv = 0 \qquad \text{—— (2)}$$

$$① \Rightarrow \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv = -\frac{\partial f}{\partial x} dx - \frac{\partial f}{\partial y} dy \quad \text{---} ③$$

$$② \Rightarrow \frac{\partial g}{\partial u} du + \frac{\partial g}{\partial v} dv = -\frac{\partial g}{\partial x} dx - \frac{\partial g}{\partial y} dy \quad \text{---} ④$$

Note :- Cramer Rule :-

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \Delta} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \Delta}$$

$$\Delta = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix} = \frac{\partial(f,g)}{\partial(u,v)}$$

$$\Delta_1 = \begin{vmatrix} -\frac{\partial f}{\partial x} dx - \frac{\partial f}{\partial y} dy & \frac{\partial f}{\partial v} \\ -\frac{\partial g}{\partial x} dx - \frac{\partial g}{\partial y} dy & \frac{\partial g}{\partial v} \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} \frac{\partial f}{\partial u} & -\frac{\partial f}{\partial x} dx - \frac{\partial f}{\partial y} dy \\ \frac{\partial g}{\partial u} & -\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y} dy \end{vmatrix}$$

Now consider

$$\Delta_1 = \begin{vmatrix} -\frac{\partial f}{\partial x} dx - \frac{\partial f}{\partial y} dy & \frac{\partial f}{\partial v} \\ -\frac{\partial g}{\partial x} dx - \frac{\partial g}{\partial y} dy & \frac{\partial g}{\partial v} \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} -\frac{\partial f}{\partial x}dx & \frac{\partial f}{\partial v} \\ -\frac{\partial g}{\partial x}dx & \frac{\partial g}{\partial v} \end{vmatrix} + \begin{vmatrix} -\frac{\partial f}{\partial y}dy & \frac{\partial f}{\partial v} \\ -\frac{\partial g}{\partial y}dy & \frac{\partial g}{\partial v} \end{vmatrix}$$

$$= -\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial v} \end{vmatrix} dx - \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial y} \end{vmatrix} dy$$

$$= \frac{-\partial(f,g)}{\partial(x,v)}dx - \frac{\partial(f,g)}{\partial(y,v)}dy$$

by Cramer Rule

$$du = \frac{\Delta_1}{\Delta} \qquad \text{provided } \Delta \neq 0$$

$$= \frac{-\frac{\partial(f,g)}{\partial(x,v)}dx}{\frac{\partial(f,g)}{\partial(u,v)}} + \frac{-\frac{\partial(f,g)}{\partial(y,v)}dy}{\frac{\partial(f,g)}{\partial(u,v)}} \quad \text{———} \; (*)$$

Also $u = \phi(x,y)$

$$du = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy \qquad (\because u = \phi)$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad \text{———} \; (*')$$

from $*$ & $*'$

$$\frac{\partial u}{\partial x} = -\frac{\partial(f,g)/\partial(x,v)}{\partial(f,g)/\partial(u,v)}$$

$$\frac{\partial u}{\partial v} = -\frac{\partial(f,g)/\partial(y,v)}{\partial(f,g)/\partial(u,v)}$$

Now consider

$$\Delta_2 = \begin{vmatrix} \frac{\partial f}{\partial u} & -\frac{\partial f}{\partial x}dx - \frac{\partial f}{\partial y}dy \\ \frac{\partial g}{\partial u} & -\frac{\partial g}{\partial x}dx - \frac{\partial g}{\partial y}dy \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial f}{\partial u} & -\frac{\partial f}{\partial x}dx \\ \frac{\partial g}{\partial u} & -\frac{\partial g}{\partial x}dx \end{vmatrix} + \begin{vmatrix} \frac{\partial f}{\partial u} & -\frac{\partial f}{\partial y}dy \\ \frac{\partial g}{\partial u} & -\frac{\partial g}{\partial y}dy \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial x} \end{vmatrix} dx - \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial y} \end{vmatrix} dy$$

$$\Delta_2 = \frac{-\partial(f,g)}{\partial(u,x)}dx - \frac{\partial(f,g)}{\partial(u,v)}dy$$

By Cramer Rule

$$dv = \frac{\Delta_2}{\Delta} \quad \text{provided } \Delta \neq 0$$

$$dv = \frac{-\partial(f,g)/\partial(u,x)}{\partial(f,g)/\partial(u,v)}dx + \frac{-\partial(f,g)/\partial(u,v)}{\partial(f,g)/\partial(u,v)}dy \qquad \circledast$$

Also $v = \psi(x,y)$

$$dv = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \qquad (\star')$$

by $\star$ & $\star'$

$$\frac{\partial v}{\partial x} = \frac{-\partial(f,g)/\partial(u,x)}{\partial(f,g)/\partial(u,v)}, \qquad \frac{\partial v}{\partial y} = -\frac{\partial(f,g)/\partial(u,v)}{\partial(f,g)/\partial(u,v)}$$

* Directional Derivatives:-

Let $f: V \to \mathbb{R}$, $V \subseteq \mathbb{R}^n$
is defined in the nbhd of $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$
then the directional derivative of f at 'a'
in the direction of $\beta = (b_1, b_2, b_3, \dots, b_n) \in \mathbb{R}^n$
is denoted & defined as

$$D_\beta(f) = \lim_{h \to 0} \frac{f(a + h\beta) - f(a)}{h}$$

Note If $f(x,y)$ & $a = (a_1, a_2)$ $\beta = (b_1, b_2) = (1,0) \hat{i}$

then $$D_\beta(f) = \lim_{h \to 0} \frac{f(a_1 + h, a_2) - f(a_1, a_2)}{h}$$

$$= \frac{\partial f(a_1, a_2)}{\partial x}$$

$a + h\beta = a_1, a_2 + h(1,0)$
$= a_1 + h, a_2 + 0$
$= a_1 + h, a_2$

Similarly if $\beta = (0,1) = \hat{j}$

then $D_\beta(f) = \frac{\partial f}{\partial y}(a_1, a_2)$

Hence partial derivatives are the directional derivatives in the direction of co-ordinate axis.

MUHAMMAD TAHIR WATTOO

M.Sc MATH (PUNJAB UNIVERSITY)

M.S MATH (CIIT ISLAMABAD)

0344 8563284

CHAPTER NO 8.

SEQUENCE AND SERIES ★★

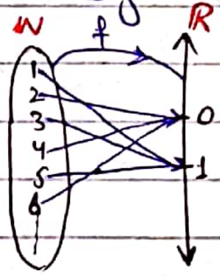
★ ★ ★
★ ★
★

⇒ **Sequence:-** A sequence is a function whose domain is the set of natural numbers. If the range of the sequence is the set $\mathbb{R}$ then the sequence is called real sequence. OR sequence of real numbers. i.e $f: \mathbb{N} \to \mathbb{R}$ defined by $f(n) = a_n$ is a real sequence. "a_n" is called the nth term of the sequence.

Sequence is denoted as $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ or $a_1, a_2, a_3, \dots, a_n, \dots$

Example:- $f: \mathbb{N} \to \mathbb{R}$ is given by

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$



★ **Limit of a Sequence:-** Let $\{a_n\}$ be a sequence of real numbers. we say that

$$\lim_{n \to \infty} a_n = l$$

if for every $\epsilon > 0$ $\exists$ a +ve integer m s.t $|a_n - l| < \epsilon$ for all $n \geq m$

Example:- $\lim_{n \to \infty} \frac{1}{n} = 0$

Suppose $\varepsilon = 0.1 > 0$ Here $a_n = \frac{1}{n}$, $l = 0$

$|a_n - l| < \varepsilon$, $\forall n \geq m$ (?)

$\Rightarrow \left|\frac{1}{n} - 0\right| < 0.1 \Rightarrow \frac{1}{n} < \frac{1}{10}$ $\forall n \geq m$

$\Rightarrow n > 10 = m$ (say)

So $\left|\frac{1}{n} - 0\right| < \varepsilon \ , \forall \ n \geq 11$

Remark:- from the definition of limit

$$|a_n - l| < \varepsilon$$

$$\Leftrightarrow \ l - \varepsilon < a_n < l + \varepsilon$$

$$\Leftrightarrow \ a_n \in \,]l - \varepsilon, l + \varepsilon[\quad , \forall \ n \geq m$$

This shows that almost all the points of the sequence belongs to the interval $(l - \varepsilon, l + \varepsilon)$

Theorem:- Let $\{a_n\}$ be a convergent sequence of real numbers if

$$\lim_{n\to\infty} a_n = l_1 \quad \& \quad \lim_{n\to\infty} a_n = l_2$$

Then $l_1 = l_2$

Proof

Given $\lim_{n\to\infty} a_n = l_1$

Then for $\varepsilon > 0$ there exist m_1 s.t

$$|a_n - l_1| < \varepsilon/2 \quad , \forall \ n \geq m_1 \quad \text{--- (1)}$$

Also $\lim_{n\to\infty} a_n = l_2 \Rightarrow$ for $\varepsilon > 0 \ \exists \ m_2$ s.t

$$|a_n - l_2| < \varepsilon/2 \quad , \forall \ n \geq m_2 \quad \text{--- (2)}$$

Let $m = \max\{m_1, m_2\}$

So (1) & (2) becomes

$$|a_n - l_1| < \varepsilon/2 \quad , \forall \ n \geq m \quad \text{--- (3)}$$

$$\& \ |a_n - l_2| < \varepsilon/2 \quad , \forall \ n \geq m \quad \text{--- (4)}$$

Consider $|l_1 - l_2| = |l_1 - a_n + a_n - l_2|$

$\leq |l_1 - a_n| + |a_n - l_2|$ — Triangle inequality

$< \varepsilon/2 + \varepsilon/2 \quad , \forall n \geq m$

$\Rightarrow |l_1 - l_2| < \varepsilon$

$\because \varepsilon$ is arbitrary $\Rightarrow |l_1 - l_2| = 0$

$\Rightarrow l_1 = l_2$

⇒ **Sub Sequence:-** If $\{a_n\}$ is a sequence and $\{a_{n_i}\}$ is taken from $\{a_n\}$ then $\{a_{n_i}\}$ is called sub sequence of $\{a_n\}$

Example:- If $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is a sequence Then $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$ is a sub sequence.

Theorem:- If the sequence $\{a_n\}$ of real numbers converges to l then every subsequence $\{a_{n_i}\}$ of $\{a_n\}$ also converges to l.

Proof: Given $\lim_{n\to\infty} a_n = l$

$\Rightarrow$ for $\varepsilon > 0$ there exist $m \in \mathbb{N}$ s.t

$|a_n - l| < \varepsilon \quad , \forall n \geq m$

Since this condition hold for almost all the points of the sequence, so in particular it hold for almost all the points of the subsequence.

i.e $|a_{n_i} - l| < \varepsilon \quad \forall n_i \geq m$

$\Rightarrow \lim_{n\to\infty} a_{n_i} = l$

⇒ **Divergent Sequence:-** A sequence $\{a_n\}$ is divergent if it not convergent.

Remark:- $\lim_{n\to\infty} a_n = \infty$ if for any real number $M>0$ there exist a +ve integer m s.t $a_n > M$ whenever $n \geq m$

for example n^2.

⇒ **Bounded Sequence:-** The sequence $\{a_n\}$ is bounded if & only if there exist $M \in \mathbb{R}^+$ s.t $|a_n| \leq M, \forall n$

Theorem If the sequence $\{a_n\}$ of real numbers is convergent then it is bounded.

Proof Suppose $\lim_{n\to\infty} a_n = l$ then for $\varepsilon > 0$ there exist $m \in \mathbb{N}$ s.t

$$|a_n - l| < \varepsilon, \forall n \geq m \quad \text{—①}$$

consider $|a_n| = |(a_n - l) + l|$

$$\leq |a_n - l| + |l|$$

$$|a_n| < \varepsilon + |l|, \forall n \geq m \quad \text{using ①}$$

$$\Rightarrow |a_n| < M', \quad M' = \varepsilon + |l|$$

Let $M'' = \max\{|a_1|, |a_2|, \dots, |a_{m-1}|\}$

$$\Rightarrow |a_i| \leq M'' \text{ for } i = 1, 2, 3, \dots, m-1$$

Let $M = M' + M''$

So $|a_n| < M, \forall n$

$\Rightarrow \{a_n\}$ is bounded.

Remark:- Converse of the above theorem is not true in general.

Example:- $a_n = (-1)^n$

$$= \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

Let $M = 2$ Then clearly $|a_n| = 1 < M, \forall n$

$\Rightarrow \{a_n\}$ is bounded

Clearly $\{a_n\}$ is not convergent.

$\Rightarrow$ **Monotonic Sequences:-** A sequence $\{a_n\}$ of real numbers is said to be

(i) Monotonically increasing if $a_n \leq a_{n+1}$ $\forall n = 1, 2, \ldots$

(ii) Monotonically decreasing if $a_{n+1} \leq a_n$ $\forall n = 1, 2, 3, \ldots$

V.V. Imp

Theorem- An increasing sequence which is bounded above is convergent.

Proof: Let $\{a_n\}$ be an increasing sequence which is bounded above.

Let $A = \{a_1, a_2, \ldots\}$ be the set of points of the sequence. Then A is bounded above.

Since A is bounded above & $A \subseteq \mathbb{R}$ with $\mathbb{R}$ is complete so A has supremum in $\mathbb{R}$.

Let $M = \operatorname{Sup} A$

We claim that $\lim_{n \to \infty} a_n = M$

As $M = \operatorname{Sup} A \Rightarrow$ for $\varepsilon > 0$ $M - \varepsilon$ is not an upper bound of A

So there exist $a_m \in A$ s.t $a_m > M - \varepsilon$

$\Rightarrow a_n > M - \varepsilon \quad \forall n \geq m$ $\because a_n$ is increas —— ①

Again $M = \text{Sup } A$

$\Rightarrow a_n \leq M, \forall n$ —— ②

from ① & ② we have

$M - \varepsilon < a_n \leq M < M + \varepsilon \quad , \forall n \geq m$

$M - \varepsilon < a_n < M + \varepsilon$

$\Rightarrow |a_n - M| < \varepsilon \quad , \forall n \geq m$

$\Rightarrow \lim\limits_{n \to \infty} a_n = M$

So every increasing bounded above sequence is convergent to least upper bound (i.e Supremum)

Theorem:- Every decreasing sequence which is bounded below is convergent & converges to Infimum.

Proof Let $\{a_n\}$ be an decreasing sequence which is bounded below.

Let $A = \{a_1, a_2, a_3, \ldots\}$ be the set of points of the sequence. Then A is bounded below Since A is bounded below & $A \subseteq \mathbb{R}$ with $\mathbb{R}$ is complete So A has infimum in $\mathbb{R}$.

Let $M = \text{Inf } A$

We claim that $\lim\limits_{n \to \infty} a_n = M$

As $M = \text{Inf } A \Rightarrow$ for $\varepsilon > 0$ $M + \varepsilon$ is not lower bound of A.

15

So there exist $a_m \in A$ s.t $a_m < M+\varepsilon$

$\Rightarrow a_n < M+\varepsilon, \ \forall n \geq m$ $\because a_n$ is decreasing

——— ①

Again $M = \text{Inf } A$

$\Rightarrow a_n \geq M, \ \forall n$

——— ②

from ① & ②

$M \leq a_n < M+\varepsilon \quad , \forall n \geq m$

$\Rightarrow M-\varepsilon < M \leq a_n < M+\varepsilon$

$M-\varepsilon < a_n < M+\varepsilon$

$\Rightarrow |a_n - M| < \varepsilon \quad \forall n \geq m$

$\Rightarrow \lim\limits_{n \to \infty} a_n = M$

$\Rightarrow$ Every decreasing bounded sequence is convergent to greatest lower bound.

Theorem:- Every bounded monotonic sequence is convergent

Proof) اوپر والے 2 Theorems میں سے ایک کا Complete proof کر کے دوسرے کے لیے similarly کر کے proof دیا ہے۔

★ This Theorem is known as bounded monotonic sequence theorem.

★ Every bounded monotonic sequence attain its bound. Same case for decreasing

Proof) Above two theorems.

Example:- Test the convergence of:

(i) $\left\{2 - \frac{1}{2^{n-1}}\right\}$ (ii) $\left\{\left(1+\frac{1}{n}\right)^n\right\}$

Solution

(i) $a_n = 2 - \frac{1}{2^{n-1}}$

$\lim\limits_{n\to\infty} a_n = 2 - \lim\limits_{n\to\infty} \frac{1}{2^{n-1}}$

$= 2 - 0$

$\lim\limits_{n\to\infty} a_n = 2$

So sequence is convergent & its limit is 2.

(ii) $a_n = \left\{\left(1+\frac{1}{n}\right)^n\right\}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

$= e$

So sequence is convergent.

★ Operation on Convergent Sequences:-

Theorem:- If $\{a_n\}$ & $\{b_n\}$ are convergent sequences of real numbers & if $\lim\limits_{n\to\infty} a_n = a$ & $\lim\limits_{n\to\infty} b_n = b$ then

(a) $\lim\limits_{n\to\infty} (a_n \pm b_n) = a \pm b$

(b) $\lim\limits_{n\to\infty} c a_n = ca$ where c is real number

(c) $\lim\limits_{n\to\infty} a_n b_n = ab$

(d) $\lim_{n\to\infty} (a_n)^2 = a^2$ (e) $\lim_{n\to\infty} \frac{1}{a^n} = \frac{1}{a}$ provided that $a_n \& a \neq 0$

(f) If $a_n \leq b_n$ then $a \leq b$.

Proof: Given $\lim_{n\to\infty} a_n = a$

& $\lim_{n\to\infty} b_n = b$

Then for $\varepsilon > 0$ $\exists$ $m_1, m_2 \in N$ s.t

$|a_n - a| < \varepsilon/2$, $\forall n \geq m_1$

& $|b_n - b| < \varepsilon/2$, $\forall n \geq m_2$

Let $m = \max\{m_1, m_2\}$

Therefore $|a_n - a| < \varepsilon/2$, $\forall n \geq m$

& $|b_n - b| < \varepsilon/2$, $\forall n \geq m$

(a) Let us consider

$$|(a_n + b_n) - (a+b)| = |(a_n - a) + (b_n - b)|$$
$$\leq |a_n - a| + |b_n - b|$$
$$< \varepsilon/2 + \varepsilon/2 \quad \forall n \geq m$$

$$\Rightarrow \lim_{n\to\infty} (a_n + b_n) = a + b$$

(b) Consider $|ca_n - ca| = |c||a_n - a|$

$$< c\,\varepsilon/2 \quad \forall n \geq m$$

$$\Rightarrow |ca_n - ca| < \varepsilon , \forall n \geq m$$

$$\Rightarrow \lim_{n\to\infty} ca_n = ca$$

(d) To prove $\lim_{n\to\infty} a_n b_n = ab$

As we know that every convergent sequence is bounded so

$|a_n| \leq m_1$ & $|b_n| \leq m_2$, $\forall n$

Let $M = \max(m_1, m_2)$

$\therefore |a_n| \leq M$ & $|b_n| \leq M$, $\forall n$.

Consider

$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab|$

$= |b_n(a_n - a) + a(b_n - b)|$

$\leq |b_n||a_n - a| + |a||b_n - b|$

$< M \cdot \varepsilon/2 + |a| \varepsilon/2$

$= \varepsilon/2 (M + |a|)$

$= \varepsilon'$, $\forall n \geq m$

$\Rightarrow \lim_{n\to\infty} a_n b_n = ab$

(e) To prove $\lim_{n\to\infty} a_n^2 = a^2$

As a_n is convergent so

$|a_n| < m$, $\forall n$

Consider $|a_n^2 - a^2| = |(a_n - a)(a_n + a)|$

$= |a_n - a||a_n + a|$

Theorem:- If $\{a_n\}$, $\{b_n\}$ & $\{c_n\}$ are sequences s.t $a_n \leq b_n \leq c_n$, $\forall\ n \in \mathbb{N}$ & if

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l$$

Then $\lim_{n\to\infty} b_n = l$.

Proof:- Given $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l$

so for $\varepsilon > 0$ there exist $m_1, m_2 \in \mathbb{N}$ s.t

$$|a_n - l| < \varepsilon, \quad \forall\ n \geq m_1$$

$$\& \quad |c_n - l| < \varepsilon, \quad \forall\ n \geq m_2$$

Let $m = \max\{m_1, m_2\}$

Therefore $|a_n - l| < \varepsilon \quad, \forall\ n \geq m$

$\& \quad |c_n - l| < \varepsilon \quad, \forall\ n \geq m$

As $|a_n - l| < \varepsilon \Rightarrow l - \varepsilon < a_n < l + \varepsilon$ —— ①

Also $l - \varepsilon < c_n < l + \varepsilon$ —— ②

Given $a_n \leq b_n \leq c_n \quad \forall\ n \in \mathbb{N}$

so for $n \geq m$ we have

$$l - \varepsilon < a_n \leq b_n \leq c_n < l + \varepsilon$$

$$\Rightarrow l - \varepsilon < b_n < l + \varepsilon$$

$$\Rightarrow |l - \varepsilon| < b_n, \forall\ n \geq m$$

$$\Rightarrow \lim_{n\to\infty} b_n = l$$

⇒ Cauchy Sequence:- A sequence $\{a_n\}$ is said to be cauchy sequence if for $\varepsilon > 0$ There exist $n_0 \in \mathbb{N}$ s.t $|a_n - a_m| < \varepsilon$, $\forall\ n, m \geq n_0$

Theorem:- Every convergent sequence is cauchy sequence

Proof: Let $\{a_n\}$ be a convergent sequence and

$$\lim_{n\to\infty} a_n = l$$

$\Rightarrow$ for $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$|a_n - l| < \varepsilon/2 \quad , \forall\ n \geq n_0 \qquad \text{——— (1)}$$

Let $m \geq n_0$ then (1) implies

$$|a_m - l| < \varepsilon/2 \qquad \text{——— (2)}$$

consider $|a_n - a_m| = |a_n - l + l - a_m|$

$$\leq |a_n - l| + |l - a_m|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall\ n, m \geq n_0$$

$\Rightarrow |a_n - a_m| < \varepsilon, \ \forall\ n, m \geq n_0$

$\Rightarrow a_n$ is cauchy's sequence.

Remark:- Converse of the above theorem is not true in general.

Examples:- (a) If $p > 0$ then $\lim_{n\to\infty} \frac{1}{n^p} = 0$

(b) If $p > 0$ then $\lim_{n\to\infty} (p)^{1/n} = 1$

(c) $\lim_{n\to\infty} (n)^{1/n} = 1$ (d) $\lim_{n\to\infty} \frac{(n)^\alpha}{(1+p)^n} = 0$

where $p > 0$ & α is a real number.

Solution:- (a) To prove $\lim_{n\to\infty} \frac{1}{n^p} = 0$

For this we have to find $m \in \mathbb{N}$ s.t

$$\left|\frac{1}{n^p} - 0\right| < \varepsilon$$

One way of finding m is to work backward

Suppose $\left|\frac{1}{n^p} - 0\right| < \varepsilon$

$\Rightarrow \frac{1}{n^p} < \varepsilon \Rightarrow n^p < \frac{1}{\varepsilon} \Rightarrow n > \left(\frac{1}{\varepsilon}\right)^{1/p}$

let m be a +ve integer s.t

$$n > \left(\frac{1}{\varepsilon}\right)^{1/p} > m$$

So $\left|\frac{1}{n^p} - 0\right| < \varepsilon \quad , \forall \; n > m$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n^p} = 0$$

(b) To prove $\lim_{n\to\infty} (p)^{1/n} = 1$

case 1

If $p > 1$.

Let $x_n = (p)^{1/n} - 1 \Rightarrow (p)^{1/n} = 1 + x_n$

$$\Rightarrow p = (1+x_n)^n \quad \text{——} ①$$

using binomial series we have

$$(1+x_n)^n = 1 + n x_n + \frac{n(n-1)}{2!} x_n^2 + \cdots$$

$$\geqslant 1 + n x_n$$

$\Rightarrow 1 + n x_n \leqslant (1+x_n)^n$ using ①

$1 + n x_n \leqslant p \Rightarrow$

$\Rightarrow x_n \leqslant \frac{p-1}{n} \Rightarrow 0 < x_n \leqslant \frac{p-1}{n}$

$0 < \lim_{n\to\infty} x_n \leqslant \lim_{n\to\infty} \frac{p-1}{n}$

$0 < \lim_{n\to\infty} x_n \leqslant 0$

$\Rightarrow \lim_{n\to\infty} x_n = 0 \Rightarrow \lim_{n\to\infty} \left[p^{\frac{1}{n}} - 1\right] = 0$

$\Rightarrow \lim_{n\to\infty} p^{1/n} = 1$

case 2 If $P = 1$ The proof is trivial

case 3 If $0 < P < 1$

Let $y = 1/p \Rightarrow y > 1$

Consider $\lim_{n\to\infty} \left(\frac{1}{y}\right)^{1/n} = 1$

$\Rightarrow \lim_{n\to\infty} p^{1/n} = 1$

So in all possible cases $\lim_{n\to\infty} p^{1/n} = 1$

(C) To prove $\lim_{n\to\infty} n^{1/n} = 1$

Let $y = n^{1/n} \Rightarrow \ln y = \frac{1}{n}\ln n$

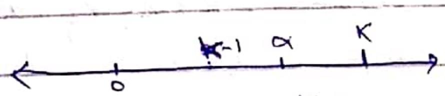
$\Rightarrow \lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln n}{n}$ (for $\frac{\infty}{\infty}$)

$= \lim_{x\to\infty} \frac{\ln x}{x}$, $x \in [1, \infty[$

$= \lim_{x\to\infty} \frac{1}{x}$

$\lim_{n\to\infty} \ln y = 0 \Rightarrow \lim_{n\to\infty} y = e^0$

$\Rightarrow \lim_{n\to\infty} n^{\frac{1}{n}} = 1$



(d) Let k be a +ve integer s.t $k > \alpha$ & $k > 0$

For $n > 2k$ we have

$(1+p)^n = 1 + np + \frac{n(n-1)}{2!}p^2 + \dots + \frac{n(n-1)\dots(n-(k-1))}{k!}p^k + \dots$

$> \frac{n(n-1)\dots(n-(k-1))}{k!}p^k$ neglecting remaining terms —— ①

As we know that $n > \frac{n}{2}$

$n - 1 > \frac{n}{2}$ ..... $n-(k-1) > \frac{n}{2}$

Multipling vertically we have

$n(n-1)\cdots(n-K+1) > \frac{n^K}{2^K}$

So inequality ① becomes

$$(1+p)^n > \frac{n^K}{2^K}\cdot\frac{p^K}{K!}$$

Taking resiprocal $\frac{1}{(1+p)^n} < \frac{2^K \cdot K!}{n^K\, p^K}$

$$\Rightarrow \frac{n^\alpha}{(1+p)^n} < \frac{2^K \cdot K!\, n^\alpha}{n^K\, p^K}$$

$$\Rightarrow 0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^K K!\, n^{\alpha-K}}{p^K}$$

Taking limit $n\to\infty$

$$0 < \lim_{n\to\infty}\frac{n^\alpha}{(1+p)^n} < \frac{2^K K!}{p^K}\lim_{n\to\infty} n^{\alpha-K}$$

$$\Rightarrow 0 < \lim_{n\to\infty}\frac{n^\alpha}{(1+p)^n} < 0 \quad \because \alpha < K \Rightarrow \alpha - K < 0$$

$$\Rightarrow \lim_{n\to\infty}\frac{n^\alpha}{(1+p)^n} = 0$$

Question:- Let $\alpha > 0$ & let $x_1 > 0$ define the rest of the sequence by $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$, $n = 1,2,3,\ldots$ show that $x_n \to \sqrt{a}$ as $n\to\infty$

Solution Clearly all the terms of given sequence is +ve. let $\lim_{n\to\infty} a_n = l$, $l > 0$

Given $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$

Taking limit $n\to\infty$ we have

$$l = \frac{1}{2}\left(l + \frac{a}{l}\right)$$

$$\Rightarrow l = \frac{l^2+a}{2l} \Rightarrow 2l^2 = l^2 + a \Rightarrow l^2 = a$$

$$\Rightarrow l = \pm\sqrt{a} \Rightarrow l = \sqrt{a} \quad \because l > 0$$

Question:- Let $Y = (y_n)$ define inductively by $y_1 = 1$, $y_{n+1} = \frac{1}{4}(2y_n + 3)$ for $n \geq 1$.

Show that $\lim\limits_{n\to\infty} Y = \frac{3}{2}$

Solution:- Let $\lim\limits_{n\to\infty} y_n = l$, $l > 0$

Given $y_{n+1} = \frac{1}{4}(2y_n + 3)$

Taking $\lim\limits_{n\to\infty}$ we have

$$l = \frac{1}{4}(2l + 3) \Rightarrow l = \frac{l}{2} + \frac{3}{4}$$

$$\Rightarrow l = \frac{2l+3}{4} \Rightarrow 4l = 2l + 3 \Rightarrow 2l = 3$$

$$\Rightarrow l = \frac{3}{2}$$

$\Rightarrow$ **Series:-** When we add the terms of a sequence we get a series.

A series of the form $a_1 + a_2 + a_3 + \dots + a_n + \dots$
$= \sum\limits_{n=1}^{\infty} a_n$ —— ① is called infinite series.

★ **Nth Partial Sum:-** $S_1 = a_1$

$S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$ ....

$S_n = a_1 + a_2 + a_3 + \dots + a_n$

S_n is called the nth partial sum.

★ $S_1, S_2, S_3, \dots, S_n, \dots$ is called sequence of partial sums.

★ The series of ① is convergent if its sequence of partial sums is convergent.

Ratio Test :- Used for (Factorial), $(\text{constant})^n$

Root Test:- $(\text{whole})^n$, L.C.T:- Polynomial/Polynomial $= \frac{n^3 + n^{1/2}}{n^5 + n^{1/2} + 1}$

B.C.T:- Trignometric function i.e sin, cos, log etc

Integral Test:- decreasing sequence.

P-Test:- $\sum \frac{1}{n^p}$

→ **Cauchy General Principle of Convergence:-** The necessary & sufficient condition for the convergence of the series $\sum a_n$ is that for every $\varepsilon > 0$ there is a +ve integer n_0 s.t $|S_n - S_m| < \varepsilon \quad \forall\ m, n > n_0$

Proof:- Suppose $\sum a_n$ is convergent

$\Rightarrow \{S_n\}$ is convergent, where $S_n = a_1 + a_2 + \cdots + a_n$

Let $\lim\limits_{n\to\infty} S_n = l$

Then for $\varepsilon > 0$ there exist a +ve integer n_0 s.t $|S_n - l| < \varepsilon/2 \quad , \forall\ n > n_0$ ——— ①

Let $m > n_0$

$\therefore \quad |S_m - l| < \varepsilon/2$

Consider $|S_n - S_m| = |(S_n - l) + (l - S_m)|$

$\leq |S_n - l| + |l - S_m|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ by ① & ②

$\Rightarrow |S_n - S_m| < \varepsilon \quad \forall\ n, m > n_0$

Conversely suppose that $|S_n - S_m| < \varepsilon, \forall\ n, m > n_0$ ——— ③

Suppose that for some fixed, large m, we have $S_m \to l$ as $m \to \infty$ i.e $S_m = l$ as $m \to \infty$

So ③ implies

$|S_n - l| < \varepsilon, \quad \forall\ n > n_0$

$\Rightarrow \lim\limits_{n\to\infty} S_n = l \Rightarrow \sum a_n$ is convergent.

Assignment:- Proof of L.C.T, B.C.T from BSC.

$\Rightarrow$ **Cauchy Root Test:-** If $a_n > 0$ and $\lim_{n\to\infty}(a_n)^{1/n} = l$ then $\sum a_n$ converges if $l < 1$ & diverges if $l > 1$.

Proof:- Case 1:- Given $\lim_{n\to\infty}(a_n)^{1/n} = l$, $l < 1$

let $(a_n)^{1/n} < r$ where $l < r < 1$

$\Rightarrow a_n < r^n \Rightarrow \sum a_n < \sum r^n$ —— ①

Then series on R.H.S of ① is converges so by B.C.T Given series is convergent.

Case 2:-

Given $\lim_{n\to\infty}(a_n)^{1/n} = l$, where $l > 1$

$\Rightarrow (a_n)^{1/n} > 1 \Rightarrow a_n > 1$

$\Rightarrow \lim_{n\to\infty} a_n \neq 0$. So by divergent Test series is divergent.

Theorem:- The necessary condition for the convergence of $\sum a_n$ is that $\lim_{n\to\infty} a_n = 0$

Proof:- Let $\sum a_n$ is convergent

$\Rightarrow \{S_n\}$ is convergent, where $S_n = a_1 + a_2 + \cdots + a_n$

let $\lim_{n\to\infty} S_n = l$

$\Rightarrow \lim_{n\to\infty} S_{n-1} = l$

As $a_n = S_n - S_{n-1} \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$

$\Rightarrow \lim_{n\to\infty} a_n = l - l \Rightarrow \lim_{n\to\infty} a_n = 0$

Converse of this theorem is not True. Because in $\sum \frac{1}{n}$, $a_n = \frac{1}{n} \to 0$ as $n \to \infty$ but series is divergent [it is p-series]

* **Ratio Test:-** If $a_n > 0$, $n = 0, 1, 2, \ldots$ and if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$ exist then the series $\sum a_n$ is convergent if $0 \le l < 1$ & divergent if $l > 1$.

Proof:- Case 1 $\quad l < 1$

Given $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$

So $\frac{a_{n+1}}{a_n} \le r$ for some integer m s.t $n \ge m$

$$\Rightarrow a_{n+1} \le r a_n \quad , \forall n \ge m$$

$$\text{i.e } a_{m+1} \le r a_m$$

Now $a_{m+2} \le r a_{m+1} \le r(r a_m) = r^2 a_m$

$$\Rightarrow a_{m+2} \le r^2 a_m$$

So in general $a_{m+k} \le r^k a_m$

$$\Rightarrow \sum a_{m+k} \le \sum r^k a_m \quad \text{——— (A)}$$

from A it is clear that $\sum a_n$ is convergent because series on R.H.S is convergent being a geometric series with $r < 1$

Case 2:-

Let $l > 1$

If m is sufficiently large we have

$\frac{a_{n+1}}{a_n} \ge r$, $r > 1$, for $n = m, m+1, \ldots$

$$\Rightarrow \lim_{n\to\infty} a_n \ne 0$$

So by divergent Test given series is divergent.

v. v. Imp

* **Cauchy's Condensation Test** :- If $a_n \geq a_{n+1} \geq 0$ for all n, & if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

Proof:- Clearly $a_1 \leq a_1$

$$a_2 + a_3 \leq a_2 + a_2 \qquad \because a_3 \leq a_2$$
$$= 2a_2$$

$$a_4 + a_5 + a_6 + a_7 \leq a_4 + a_4 + a_4 + a_4 = 4a_4$$

$$\Rightarrow a_4 + a_5 + a_6 + a_7 \leq 2^2 a_{2^2}$$

$$\Rightarrow a_{2^2} + a_{2^2+1} + a_{2^2+2} + a_{2^3-1} \leq 2^2 a_{2^2}$$

So in general for $n \in \mathbb{N}$ we have

$$a_{2^n} + a_{2^n+1} + a_{2^n+2} + \cdots + a_{2^{n+1}-1} \leq 2^n a_{2^n}$$

Then by vertically addition

$$\sum a_n \leq \sum 2^n a_{2^n}$$

Since series on R.H.S is convergent so by B.C.T given series is convergent.

Example:- Using Cauchy condensation Test show that $\sum \frac{1}{n(\log n)^p}$ is convergent if $p > 1$ & divergent if $p \leq 1$

Solution

$$a_n = \frac{1}{n(\log n)^p}$$

Consider $\sum 2^n a_{2^n} = \sum 2^n \frac{1}{2^n (\log 2^n)^p}$

$$= \sum \frac{1}{(n \log 2)^p} = \frac{1}{(\log 2)^p} \sum \frac{1}{n^p}$$

which is convergent if $p > 1$ & dgt if $p \leq 1$

So by Cauchy condensation Test given series is convergent if $P > 1$ & dgt if $P \leq 1$.

* * *

⇒ Sequence & Series of Functions:-

★ Pointwise Convergence of Sequence of Functions:-

Let $\{f_n\}$ be a sequence of real functions defined on a set E. Then we say that $\{f_n\}$ convergence to the function f on E if

$$\lim_{n\to\infty} f_n(x) = f(x), \forall x \in E \quad \text{—— ①}$$

If ① holds then we say that sequence $\{f_n\}$ converges pointwise to f on E OR for every $\varepsilon > 0$ there exist a +ve integer m s.t

$$|f_n(x) - f(x)| < \varepsilon, \forall n \geq m$$

Remark:- In general the number m depends on ε and x i.e $m = m(\varepsilon, x)$

Example:- Show that the sequence $\left\{\frac{x}{1+nx}\right\}$ Converges to $f(x) = 0$, $x \in [0, \infty)$

Solution: Let $f_n(x) = \frac{x}{1+nx}$.

we want to prove that $\lim_{n\to\infty} f_n(x) = f(x)$, $x \in [0,\infty)$

or for $\varepsilon > 0$ there exist m s.t

$$|f_n(x) - f(x)| < \varepsilon$$

or $\left|\frac{x}{1+nx} - 0\right| < \varepsilon$ i.e $\left|\frac{x}{1+nx}\right| < \varepsilon, \forall n \geq m$

consider $\left|\frac{x}{1+nx}\right| \leq \frac{1}{n}$, $\forall x \in [0, \infty[$ —— ②

Note that $\left|\frac{x}{1+nx}\right| < \varepsilon$ if $n > \frac{1}{\varepsilon} = m(\varepsilon)$ | $\frac{1}{n} < \varepsilon \Rightarrow \frac{1}{\varepsilon} < n$

So $\frac{x}{1+nx} \to 0$ as $n \to \infty$, $\forall x \in [0, \infty[$

پاور سیریز کا Sum ایک پاور فنکشن ہوتا ہے اور پاور فنکشن کی سیریز پاور سیریز ہوتی ہے

* Series of power functions is power series.

★ **Uniform Convergence of Sequence of Functions:-**

In previous definition if $m = m(\varepsilon)$ then the sequence $\{f_n(x)\}$ convergence uniformly to f on E

Example:- In previous example sequence converge uniformly.

Example:- For $n = 1, 2, 3, \ldots$ & $x > 0$ let $f_n(x) = \frac{1}{x+n}$ show that f_n converge uniformly to $f(x) = 0$

Solution:-

The given sequence converge uniformly to $f(x) = 0$ if for $\varepsilon > 0$ there exist $m = m(\varepsilon)$ s.t $|f_n(x) - f(x)| < \varepsilon$

i.e $\left|\frac{1}{x+n} - 0\right| < \varepsilon \Rightarrow \left|\frac{1}{x+n}\right| < \varepsilon$

consider $\left|\frac{1}{x+n}\right| < \varepsilon \Rightarrow \frac{1}{x+n} < \varepsilon$

$\Rightarrow \frac{1}{\varepsilon} < n + x \Rightarrow n > \frac{1}{\varepsilon} - x$

Since $x > 0$ so $\frac{1}{\varepsilon} > \frac{1}{\varepsilon} - x$, $\forall x$

So we select $m(\varepsilon) = 1/\varepsilon$

So given series converge uniformly.

★ Convergence & Uniform Convergence of Series of Functions:- Let $\{f_n\}$, $n = 1, 2, 3, \cdots$ be a sequence of functions defined on a set E. The series $\sum_{n=1}^{\infty} f_n$ is convergent if $\{S_n\}$ is convergent, where $S_n = f_1 + f_2 + f_3 + \cdots$

Remark:- Let $f_1, f_2, f_3, \cdots$ be real valued functions on a set E. we say that the series $\sum f_n$ converge uniformly to the function $S(x)$ if

$$\lim_{n\to\infty} S_n(x) = S(x) \quad \text{or for } \varepsilon > 0$$

there exist $m = m(\varepsilon)$ s.t

$$|S_n(x) - S(x)| < \varepsilon, \ \forall\, n \geq m \quad \text{---} \circledast$$

Let $R_n(x) = S_n(x) - S(x)$

so $\circledast \Rightarrow R_n(x) < \varepsilon, \ \forall\, n \geq m$

Theorem:- The necessary & sufficient condition that the sequence $\{S_n\}$ should converge uniformly in E, $E = [a, b]$ is that given $\varepsilon > 0$ there exist $m = m(\varepsilon)$ s.t

$$|S_n(x) - S_{n+p}(x)| < \varepsilon, \forall\, n \geq m$$

Proof Suppose $\{S_n\}$ converge

where $S_n = f_1 + f_2 + f_3 + \cdots + f_n$

Let $\lim_{n\to\infty} S_n(x) = l$

then for $\varepsilon > 0$ there exist $m = m(\varepsilon)$

s.t $|S_n(x) - l| < \varepsilon/2, \ \forall\, n \geq m \quad \text{---} (1)$

As $n > m$ so $n+p > m$, $p > 0$

Then by ①

$$|S_{n+p}(x) - \ell| < \varepsilon/2, \forall\ n+p \geq m \quad \text{——— ②}$$

Consider

$$|S_n(x) - S_{n+p}(x)| = |(S_n(x) - \ell) + (\ell - S_{n+p}(x))|$$

$$\leq |S_n(x) - \ell| + |\ell - S_{n+p}(x)|$$

$$< \varepsilon/2 + \varepsilon/2 \quad \text{by ① \& ②}$$

$$\Rightarrow |S_n(x) - S_{n+p}(x)| < \varepsilon \quad \forall\ n \geq m$$

Conversely suppose that

$$|S_n(x) - S_{n+p}(x)| < \varepsilon, \forall\ n \geq m \quad \text{——— ③}$$

Suppose that for some fixed, large $n+p$ we have $S_{n+p}(x) \to \ell$ as $n+p \to \infty$ i.e $S_{n+p}(x) = \ell$ as $n+p \to \infty$ so ③ $\Rightarrow$

$$|S_n(x) - \ell| < \varepsilon \quad \forall\ n \geq m$$

$$\Rightarrow \lim_{n \to \infty} S_n = \ell \Rightarrow \{S_n\} \text{ is convergent}$$

Example: Check the convergence of the series $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$, $x \in (-\infty, \infty)$

Solution

$$S_n(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots + \frac{x^2}{(1+x^2)^{n-1}}$$

$$= x^2\left[1 + \left(\frac{1}{1+x^2}\right) + \left(\frac{1}{(1+x^2)^2}\right) + \cdots + \left(\frac{1}{(1+x^2)^{n-1}}\right)\right]$$

$$S_n(x) = \frac{x^2\left[1 - \left(\frac{1}{1+x^2}\right)^n\right]}{1 - \frac{1}{1+x^2}}$$

Scanned by CamScanner

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{x^2\left[1-\left(\frac{1}{1+x^2}\right)^n\right]}{1-\frac{1}{1+x^2}}$$

$$= \frac{x^2}{1-\frac{1}{1+x^2}} = \frac{x^2}{\frac{1+x^2-1}{1+x^2}}$$

$$\lim_{n\to\infty} S_n(x) = 1+x^2 .$$

So given series is convergent & its sum is $1+x^2$.

⇒ **Weierstrass M_n-Test:-** Let $S_n(x)$ be finite & continuous in $[a,b]$ & Let $S_n(x) \to S(x)$ $\forall\ x \in [a,b]$. We know that $S(x)$ is continuous on $[a,b]$

The function $|S_n(x) - S(x)|$ for fixed n is continuous on $[a,b]$ & Hence bounded because every continuous function is bounded.

Let M_n be the upper bound of $|S_n(x) - S(x)|$ i.e $M_n = \text{Max}\,|S_n(x) - S(x)|$

Theorem:- The necessary & sufficient condition for the uniform convergence of the series $\sum f_n$ is that $M_n \to 0$ as $n \to \infty$

Proof) Let $\lim_{n\to\infty} M_n = 0$

Then for $\varepsilon > 0$ there exist $m_1(\varepsilon)$ s.t

$$|M_n - 0| < \varepsilon \ , \forall\ n \geq m$$

$$\Rightarrow |M_n| < \varepsilon \Rightarrow \text{Max}\,|S_n(x) - S(x)| < \varepsilon$$

$$\Rightarrow |S_n(x) - S(x)| < \varepsilon \ \forall\ n \geq m_1(\varepsilon)$$

$\Rightarrow S_n$ is uniformly convergence.

Working back ward we obtain the converse.

Example:- Test the uniform convergence of the series $\sum f_n$, where $f_n(x) = nx(1-x)^n$, $0 \le x \le 1$

Solution

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x)^n$$

$$= \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}}$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)}$$

$$\Rightarrow \lim_{n\to\infty} f_n(x) = 0, \quad \forall x \in [0,1]$$

So given series $\sum f_n$ converges to 0

Next we check the uniform convergence of this series using M_n-Test

Let $y = |f_n(x) - f(x)|$

$$= |nx(1-x)^n - 0|$$

$$y = nx(1-x)^n$$

$$\frac{dy}{dx} = n\{(1-x)^n + x \cdot n(1-x)^{n-1}(-1)\}$$

$$= n(1-x)^n - n^2x(1-x)^{n-1}$$

$$\frac{d^2y}{dx^2} = n \cdot n(1-x)^{n-1}(-1) - n^2\{(1-x)^{n-1} + x(n-1)(1-x)^{n-2}(-1)\}$$

$$= -n^2(1-x)^{n-1} - n^2(1-x)^{n-1} - x(n-1)(1-x)^{n-2}$$

$$= -2n^2(1-x)^{n-1} - x(n-1)(1-x)^{n-2} \quad \text{--- (1)}$$

Put $\frac{dy}{dx} = 0 \Rightarrow n(1-x)^n - n^2x(1-x)^{n-1} = 0$

$$\Rightarrow n(1-x)^{n-1}\{1 - x - nx\} = 0$$

$\Rightarrow 1 - x - nx = 0 \Rightarrow x = \frac{1}{1+n}$ Put in (1)

$$\frac{d^2y}{dx^2} = -2n^2\left(1-\frac{1}{1+n}\right)^{n-1} - \frac{1}{1+n}(n-1)\left[1-\frac{1}{1+n}\right)^{n-2}$$

$$= -2n^2\left[\frac{\cancel{1}+n\cancel{-1}}{1+n}\right]^{n-1} - \frac{n-1}{1+n}\left[\frac{1+n-1}{1+n}\right]^{n-2}$$

$$= -2n^2\left[\frac{n}{1+n}\right]^{n-1} - \left(\frac{n-1}{n+1}\right)\left[\frac{n}{1+n}\right]^{n-2}$$

$$= -\left[\frac{n}{1+n}\right]^{n-2}\left\{2n^2\left(\frac{n}{1+n}\right) - \frac{n-1}{n+1}\right\}$$

$$= -\left[\frac{n}{1+n}\right]^{n-2}\left\{\frac{2n^3-n-1}{n+1}\right\}$$

$$= -\left[\frac{n}{1+n}\right]^{n-2}\left\{\frac{2n^3-n-1}{n+1}\right\} < 0 \quad \because n \in [0,1]$$

$$\frac{d^2y}{dx^2} < 0 \quad \text{at } x = \frac{1}{1+n}$$

So y is max at $x = \frac{1}{1+n}$

$$\therefore M_n = y\left(\frac{1}{1+n}\right)$$

$$= \frac{n}{1+n}\left[1-\frac{1}{1+n}\right]^n = \frac{n}{1+n}\left(\frac{n}{1+n}\right)^n$$

$$M_n = \left(\frac{n}{1+n}\right)^{n+1}$$

Now $\lim\limits_{n\to\infty} M_n = \lim\limits_{n\to\infty}\left(\frac{n}{1+n}\right)^{1+n}$

$$= \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^{1+n}$$

$$= \lim_{n\to\infty}\frac{1}{(1+1/n)^n}\lim_{n\to\infty}\frac{1}{(1+1/n)}$$

$$= \frac{1}{e}\cdot 1 = \frac{1}{e} \neq 0$$

So given series is not uniformly convergent.

Example:- Using M_n-Test check the convergence of

$$\sum\left[\frac{nx}{1+n^2x^2}-\frac{(n+1)x}{1+(n+1)^2x^2}\right],\ x\in[0,1]$$

Solution:- $f_n(x)=\frac{nx}{1+n^2x^2}-\frac{(n+1)x}{1+(n+1)^2x^2}$

$$f_1(x)=\frac{x}{1+x^2}-\cancel{\frac{2x}{1+2^2x^2}}$$

$$f_2(x)=\cancel{\frac{2x}{1+2^2x^2}}-\cancel{\frac{3x}{1+3^2x^2}}$$

$$f_3(x)=\cancel{\frac{3x}{1+3^2x^2}}-\cancel{\frac{4x}{1+4^2x^2}}$$

$$\vdots$$

$$f_n(x)=\cancel{\frac{nx}{1+n^2x^2}}-\frac{(n+1)x}{1+(n+1)^2x^2}$$

Adding vertically

$$\sum_{i=1}^{n}f_i=\frac{x}{1+x^2}-\frac{(n+1)x}{1+(n+1)^2x^2}=S_n(x)$$

$$\lim_{n\to\infty}S_n(x)=\lim_{n\to\infty}\frac{x}{1+x^2}-\lim_{n\to\infty}\frac{(n+1)x}{1+(n+1)^2x^2}$$

$$=\frac{x}{1+x^2}=S(x)$$

So given series is convergent & its sum is $\frac{x}{1+x}$.

Next we use M_n-Test for uniform convergence.

Let $y = |S_n(x) - S(x)|$

$$= \left| \frac{x}{1+x^2} - \frac{(n+1)x}{1+(n+1)^2x^2} - \frac{x}{1+x^2} \right|$$

$$= \left| - \frac{(n+1)x}{1+(n+1)^2x^2} \right|$$

$$y = \frac{(n+1)x}{1+(n+1)^2x^2}$$

$$\frac{dy}{dx} =$$

$$\frac{d^2y}{dx^2} = 0$$

Put $\frac{dy}{dx} = 0$

$\Rightarrow x = \frac{1}{n+1}$ Put in ①

$\frac{d^2y}{dx^2} < 0$ at $x = \frac{1}{n+1}$

$$M_n = y\left(\frac{1}{n+1}\right)$$

$$y = \frac{(n+1)\frac{1}{n+1}}{1+(n+1)^2\left(\frac{1}{n+1}\right)^2} = \frac{1}{2} \neq 0$$

So given series is not uniformly convergent.

Exercise:- Using M_n - Test check the convergence of the following series.

(i) $\sum \dfrac{2n^2 x}{e^{n^2x^2}} - \dfrac{2(n-1)^2 x}{e^{(n-1)^2x^2}}$ on $[0,1]$

(ii) $\sum f_n$ where $f_n(x) = nx\,e^{-nx^2}$ on $[0,1]$

*** Weierstrass M-Test for uniform Convergence of a series:-** Let $\sum\limits_{k=1}^{\infty} f_k$ be a series of functions defined on E. if there exist +ve numbers $M_1, M_2, \ldots$ with $\sum\limits_{k=1}^{\infty} M_k$ is convergent s.t

$$|f_k(x)| \leq M_k, \forall x \in E \text{ \& for each } k = 1,2,3,\ldots$$

then $\sum\limits_{k=1}^{\infty} f_k$ converge uniformly on E

Example:- Show that the series $\sum\limits_{n=1}^{\infty} \dfrac{\sin(x+nx)}{n(n+1)}$ is uniformly convergent $\forall x \in (-\infty, \infty)$

Solution:- Let $f_n(x) = \dfrac{\sin(x+nx)}{n(n+1)}$

$$|f_n| = \left|\frac{\sin(x+nx)}{n(n+1)}\right| \leq \frac{1}{n(n+1)} = M_n$$

$$\Rightarrow |f_n(x)| \leq M_n, \forall n \text{ \& } x \in (-\infty, \infty)$$

consider $\sum\limits_{n=1}^{\infty} M_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+1)}$

$$a_n = \frac{1}{n^2+n} \quad \text{take } b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+n}$$

$$= 1 \neq 0$$

Now $\sum b_n = \sum \frac{1}{n^2}$ is convergent so by L.C.T ~~$\sum$~~ $\sum M_n$ is convergent.

So given series is ~~un~~ uniformly convergence by M-Test.

Remark:- Every Power series is uniformly convergent inside its interval of convergence.

Question:- Show that the following series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{\lfloor 2n+1}$$ uniformly converges in every interval $[a, b]$

Solution

Here $|a_n| = \frac{1}{(2n+1)!}$

$$|a_{n+1}| = \frac{1}{(2n+3)!}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(2n+1)!}{(2n+3)!}$$

$$= 0$$

So radius of convergence $= \infty$

As we know that every power series is uniformly converges inside interval of convergence so given series converge uniformly in $(-\infty, \infty)$. In particular in each interval $[a, b]$

* * * * *