PERTURBATION METHODS-I

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Books:-

(i) Perturbation Methods by Ali H. Nayfeh.

(ii) Perturbation Methods for Scientists and Engineers by Alan W. Bush

Topics:-

Example:- Solve by perturbation method
$$x^2-(3+2\varepsilon)x+2+\varepsilon=0 \longrightarrow (1)$$

Solution

If ε is small, the root can be approximated in term of expansion of the form
$$x = x_0 + x_1\varepsilon + x_2\varepsilon^2 + \cdots \longrightarrow (2)$$

substitute equ (2) in (1)
$$\Rightarrow (x_0+x_1\varepsilon+x_2\varepsilon^2+\cdots)^2-(3+2\varepsilon)(x_0+x_1\varepsilon+x_2\varepsilon^2+\cdots)+2+\varepsilon=0$$

$$\Rightarrow x_0^2+x_1^2\varepsilon^2+2x_0x_1\varepsilon+2x_0x_2\varepsilon^2+\cdots-3x_0-3x_1\varepsilon-3x_2\varepsilon^2-2x_0\varepsilon-2x_1\varepsilon^2+\cdots+2+\varepsilon=0$$

neglecting ε^3 & higher terms

$$\Rightarrow (x_0^2-3x_0+2)+(2x_0x_1-3x_1-2x_0+1)\varepsilon+(x_1^2+2x_0x_2-3x_2-2x_1)\varepsilon^2+\cdots=0$$

Equating coefficients of powers of ε to zero

ε^0 :- $x_0^2-3x_0+2=0 \longrightarrow (3)$

equ (3) is corresponding unperturb equation i.e $\varepsilon=0$ in equ (1)

from (3)
$$\boxed{x_0 = 1, 2}$$

ε^1 :- $2x_0x_1-3x_1-2x_0+1=0 \longrightarrow (4)$

ε^2 :- $x_1^2+2x_0x_2-3x_2-2x_1=0 \longrightarrow (5)$

Put $x_0 = 1$ in equ (4) & (5) implies

from ④ : $2x_1 - 3x_1 - 2 + 1 = 0$

$\Rightarrow -x_1 - 1 = 0 \Rightarrow \boxed{x_1 = -1}$

from ⑤ $(-1)^2 + 2(1)x_2 - 3x_2 - 2(-1) = 0$

$\Rightarrow 1 + 2x_2 - 3x_2 + 2 = 0$

$\Rightarrow -x_2 + 3 = 0 \Rightarrow \boxed{x_2 = 3}$

Therefor equation ② i.e solution becomes

$$\boxed{x = 1 - \varepsilon + 3\varepsilon^2 + \cdots \longrightarrow A}$$

Now putting $x_0 = 2$ in equ ④ & ⑤ implies

from ④ $2(2)x_1 - 3x_1 - 2(2) + 1 = 0$

$\Rightarrow 4x_1 - 3x_1 - 4 + 1 = 0$

$\Rightarrow x_1 - 3 = 0 \Rightarrow \boxed{x_1 = 3}$

from ⑤ $(3)^2 + 2(2)x_2 - 3x_2 - 2(3) = 0$

$\Rightarrow 9 + 4x_2 - 3x_2 - 6 = 0$

$\Rightarrow x_2 + 3 = 0 \Rightarrow \boxed{x_2 = -3}$

So for This solution becomes

$$\boxed{x = 2 + 3\varepsilon - 3\varepsilon^2 + \cdots \longrightarrow B}$$

Exact Solution:-

$$x = \frac{(3+2\varepsilon) \pm \sqrt{(3+2\varepsilon)^2 - 4(2+\varepsilon)}}{2}$$

$$\Rightarrow x = \frac{1}{2}\left[3 + 2\varepsilon \pm \sqrt{1 + 8\varepsilon + 4\varepsilon^2}\right] \longrightarrow ⑥$$

Using binomial expan of $(1+8\varepsilon+4\varepsilon^2)^{1/2}$

$$\Rightarrow (1+8\varepsilon+4\varepsilon^2)^{\frac{1}{2}} = 1+\frac{1}{2}(8\varepsilon+4\varepsilon^2)+\frac{\frac{1}{2}(-\frac{1}{2})}{2}(8\varepsilon+4\varepsilon^2)^2+\cdots$$

$$= 1+4\varepsilon+2\varepsilon^2-\frac{1}{8}(64\varepsilon^2+\cdots)+\cdots$$

$$= 1+4\varepsilon-6\varepsilon^2$$

Put in equ ② implies

$$x = \frac{1}{2}\begin{cases} 3+2\varepsilon+1+4\varepsilon-6\varepsilon^2+\cdots \\ 3+2\varepsilon-1-4\varepsilon+6\varepsilon^2+\cdots \end{cases}$$

$$\Rightarrow x = \begin{cases} 2+3\varepsilon-3\varepsilon^2+\cdots \\ 1-\varepsilon+3\varepsilon^2+\cdots \end{cases}$$

Example:- Solve $u = 1+\varepsilon u^3$ (A.H. Nafeh)

Solution

Given $u = 1+\varepsilon u^3 \longrightarrow ①$

For small $\varepsilon(\neq 0)$ we let

$$u = u_0+\varepsilon u_1+\varepsilon^2 u_2+\cdots \longrightarrow ②$$

using equ ② in ① we get

$$u_0+u_1\varepsilon+u_2\varepsilon^2+u_3\varepsilon^3+\cdots = 1+\varepsilon(u_0+u_1\varepsilon+u_2\varepsilon^2+u_3\varepsilon^3)^3$$

$$\Rightarrow u_0+u_1\varepsilon+u_2\varepsilon^2+u_3\varepsilon^3+\cdots = 1+\varepsilon(u_0^3+u_1^3\varepsilon^3+3u_0^2u_1\varepsilon+3u_0u_1^2\varepsilon^2+3u_0^2u_2\varepsilon^2+3u_1u_2\varepsilon^3+3u_0u_2^2\varepsilon^3+3u_0^2u_2\varepsilon^2+\cdots)$$

$$\Rightarrow u_0-1+\varepsilon(u_1-u_0^3)+\varepsilon^2(u_2-u_0^2u_1)+\varepsilon^3(u_3-3u_0u_1^2-3u_0^2u_2)=0$$

Now compairing coefficients of ε

ε^0: $U_0 - 1 = 0 \longrightarrow$ (a)

ε^1: $U_1 - U_0^3 = 0 \longrightarrow$ (b)

ε^2: $U_2 - 3U_0^2 U_1 = 0 \longrightarrow$ (c)

ε^3: $U_3 - 3U_0 U_1^2 - 3U_0^2 U_2 = 0 \longrightarrow$ (d)

by (a) $\boxed{U_0 = 1}$

(b) $\Rightarrow$ $\boxed{U_1 = 1}$

(c) $\Rightarrow$ $U_2 - 3 = 0 \Rightarrow \boxed{U_2 = 3}$

(d) $\Rightarrow$ $U_3 - 3 - 9 = 0 \Rightarrow \boxed{U_3 = 12}$

Therefore equation (2) implies

$$U = 1 + \varepsilon + 3\varepsilon^2 + 12\varepsilon^3 + \cdots \longrightarrow (2)$$

If $\varepsilon = 0$ then equation (1) & (3) gives same solution i.e $U = 1$ this is solution of unperturbed equation

Example: $x^5 + x = 1$ $\quad x \approx 0.755$

Solution

Insert ε

$\Rightarrow x^5 + \varepsilon x = 1 \longrightarrow (1)$

Here we suppose $x = 1 + x_1\varepsilon + x_2\varepsilon^2 + \cdots$ b/c power of x^5 is not easy to open

If $\varepsilon = 0$

$x^5 = 1 \Rightarrow \boxed{x = 1}$

Solution of unperturb equation

For small $\varepsilon (\neq 0)$ we let

$$x = x_0 + x_1\varepsilon + x_2\varepsilon^2 + x_3\varepsilon^3 + \cdots \longrightarrow (2)$$

put (2) in (1)

$$\Rightarrow (1 + x_1\varepsilon + x_2\varepsilon^2 + x_3\varepsilon^3 + \cdots)^5 + (1 + x_1\varepsilon + x_2\varepsilon^2 + x_3\varepsilon^3 + \cdots) - 1 = 0$$

$$\Rightarrow \cancel{1} + 5(x_1\varepsilon + x_2\varepsilon^2 + x_3\varepsilon^3 + \cdots) + \frac{5(4)}{2}(x_1\varepsilon + x_2\varepsilon^2 + x_3\varepsilon^3 + \cdots)^2 + \frac{5(4)(3)}{2}(x_1\varepsilon + x_2\varepsilon^2 + x_3\varepsilon^3 + \cdots)^3 + \cdots + (1 + x_1\varepsilon + x_2\varepsilon^2 + x_3\varepsilon^3 + \cdots) - \cancel{1} = 0$$

Compairing coefficients of like powers of ε to zero

ε^1: $\quad 5x_1 + 1 = 0 \longrightarrow$ (a)

ε^2: $\quad 5x_2 + 10x_1^2 + x_1 = 0 \longrightarrow$ (b)

ε^3: $\quad 5x_3 + 10x_1x_2 + 30x_1^3 + x_2 = 0 \longrightarrow$ (c)

by (a) $\quad \boxed{x_1 = -1/5}$

(b) $\Rightarrow \quad 5x_2 + 10\left(-\frac{1}{5}\right)^2 - \frac{1}{5} = 0$

$$\Rightarrow 5x_2 + \frac{10}{25} - \frac{1}{5} = 0$$

$$\Rightarrow 5x_2 + \frac{10-5}{25} = 0 \Rightarrow 5x_2 + \frac{5}{25} = 0$$

$$\Rightarrow \boxed{x_2 = \frac{-1}{25}}$$

(c) $\Rightarrow \quad 5x_3 + 10\left(-\frac{1}{5}\right)\left(\frac{-1}{25}\right) + 30\left(\frac{-1}{5}\right)^2 - \frac{1}{25} = 0$

$$\Rightarrow 5x_3 + \frac{10}{125} + \frac{30}{25} - \frac{1}{25} = 0$$

$$\Rightarrow 5x_3 + \frac{10 + 150 - 5}{125} = 0$$

$$\Rightarrow 5x_3 + \frac{155}{125} = 0 \Rightarrow 5x_3 = -\frac{31}{25}$$

$$\Rightarrow \boxed{x_3 = \frac{-31}{125}}$$

Therefore equation ② implies

$$x = 1 - \frac{1}{5}\varepsilon - \frac{1}{25}\varepsilon^2 - \frac{31}{125}\varepsilon^3 + \cdots$$

If $\varepsilon = 0$ Then

$x = 1$ sol of unper. equ.

Put $\varepsilon = 1$

$$x = 1 - \frac{1}{5} - \frac{1}{25} - \frac{1}{25}$$

$$\Rightarrow \boxed{x \simeq 0.752}$$

Parameter Perturbation:-

Example:- Vander Pol Oscillator:-

$$\frac{d^2u}{dt^2} + u = \varepsilon(1-u^2)\frac{du}{dt} \longrightarrow (1)$$

Solution

For $\varepsilon = 0$ "Unperturbed"

$$\frac{d^2u}{dt^2} + u = 0$$

$\because u = c_1 \cos t + c_2 \sin t$
Put $c_2 = -a\sin\phi$ &
$c_1 = a\cos\phi$
$\Rightarrow u = a\cos(t+\phi)$

$\Rightarrow u = a\cos(t+\phi)$; a & ϕ are constants

To find approximate solution for (1) for ε small, we seek the solution of the form

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots ; \quad u(t,\varepsilon) \longrightarrow (2)$$

Substitute (2) in (1) implies

$$\frac{d^2u_0}{dt^2} + \varepsilon\frac{d^2u_1}{dt^2} + \varepsilon^2\frac{d^2u_2}{dt^2} + \cdots + u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots = \varepsilon\left[1-(u_0+\varepsilon u_1+\varepsilon^2 u_2+\cdots)^2\right]\left[\frac{du_0}{dt} + \varepsilon\frac{du_1}{dt} + \varepsilon^2\frac{du_2}{dt} + \cdots\right]$$

$$\Rightarrow \frac{d^2u_0}{dt^2} + u_0 + \varepsilon\left(\frac{d^2u_1}{dt^2} + u_1\right) + \varepsilon^2\left(\frac{d^2u_2}{dt^2} + u_2\right) + \cdots = \varepsilon\left[1-u_0^2-\varepsilon 2u_0u_1-\varepsilon^2(u_1^2+2u_0u_2)+\cdots\right]\left[\frac{du_0}{dt} + \varepsilon\frac{du_1}{dt} + \varepsilon^2\frac{du_2}{dt} + \cdots\right]$$

$$\Rightarrow \text{\_\_\_\_\_\_} = \varepsilon\left[(1-u_0^2)\frac{du_0}{dt} + \varepsilon\left\{(1-u_0^2)\frac{du_1}{dt} - 2u_0u_1\frac{du_0}{dt}\right\} + \cdots\right]$$

Comparing coefficients of like powers of ε

$$\varepsilon^0 :- \frac{d^2u_0}{dt^2} + u_0 = 0 \longrightarrow (3)$$

$$\varepsilon^1 :- \frac{d^2u_1}{dt^2} + u_1 = (1-u_0^2)\frac{du_0}{dt^2} \longrightarrow (4)$$

$$\varepsilon^2 :- \frac{d^2u_2}{dt^2} + u_2 = (1-u_0^2)\frac{du_1}{dt} - 2u_0u_1\frac{du_0}{dt} \longrightarrow (5)$$

from (3)

$$u_0 = a\cos(t+\phi) \; ; \; a, \phi \text{ are constants}$$

Now using u_0 in (4) we get

$$\frac{d^2u_1}{dt^2} + u_1 = (1-a^2\cos^2(t+\phi))(-a\sin(t+\phi))$$

$$= -a\sin(t+\phi) + a^3\cos^2(t+\phi)\sin(t+\phi)$$

$$= \ldots$$

$$= \frac{a^3-4a}{4}\sin(t+\phi) + \frac{a^3}{4}\sin^3(t+\phi)$$

Now this is a linear equation in u_1 because the right hand side is a known function. We can use any method to solve this linear non-homogeneous equation like method of undetermined coefficients or method of variation of parameter. After solving this equation we have following

$$U_1 = -\frac{a^3-4a}{8}t\cos(t+\phi) - \frac{1}{32}a^3\sin(t+\phi)$$

similarly using U_0 and U_1 in ⑤ we get U_2.

* The main benifit of perturbation method is that we get the easy problems i.e the difficulty of the original problem decreases however, the number of problems increases. e.g in above problem we have a non-linear equation in u variable but when we solve it by perturbation method we get three ~~no~~ linear problems.

Coordinate Perturbation:-

1:- Bessel Equation (Zill book)

Example 2:- $\frac{dy}{dx} + y = \frac{1}{x}$ ⟶ ①

Solution

for large x

Let $y = \sum_{m=1}^{\infty} a_m x^{-m}$ ⟶ ②

replace ② in ①

$$\Rightarrow \sum_{m=1}^{\infty} -m a_m x^{-m-1} + \sum_{m=2}^{\infty} a_m x^{-m} + (a_1 - 1)x^{-1} = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} -n a_n x^{-n-1} + \sum_{n=1}^{\infty} a_{n+1} x^{-n-1} + (a_1 - 1) x^{-1} = 0$$

$$\Rightarrow (a_1 - 1) x^{-1} + \sum_{n=1}^{\infty} (a_{n+1} - n a_n) x^{-n-1} = 0$$

$$\Rightarrow a_1 - 1 = 0 \Rightarrow \boxed{a_1 = 1}$$

Now

$$a_{n+1} - n a_n = 0$$

$$\Rightarrow a_{n+1} = n a_n \,;\, n = 1, 2, 3, \cdots$$

$\underline{n=1} \Rightarrow \boxed{a_2} = a_1 = \boxed{1}$

$\underline{n=2} \Rightarrow a_3 = 2a_2 \Rightarrow \boxed{a_3 = 2}$

$\therefore$ ② becomes

$$y = \frac{1}{x} + \frac{1}{x^2} + \frac{2!}{x^3} + \frac{3!}{x^4} + \cdots + \frac{n!}{x^{n+1}} + \cdots$$

⇒ The Bessel Equation of Zeroth Order:-

We consider the solution of

$$x^2 \frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0 \longrightarrow ①$$

This equation has a regular singular point at $x = 0$, which suggests that a power series solution for y can be obtained using the method of Frobenius. Thus we let.

$$y = \sum_{m=0}^{\infty} a_m x^{u+m} \longrightarrow (2)$$

where the number u and coefficients a_m must be determined so that (2) is sol of (1)
substitute (2) in (1) gives

$$\sum_{m=0}^{\infty} (u+m)(u+m-1) a_m x^{u+m-1} + \sum_{m=0}^{\infty} (u+m) a_m x^{u+m-1}$$
$$+ \sum_{m=0}^{\infty} a_m x^{u+m+1} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} [(u+m)(u+m-1)+(u+m)] a_m x^{u+m-1}$$
$$+ \sum_{m=0}^{\infty} a_m x^{u+m+1} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (u+m)(u+m-1+1) a_m x^{u+m-1} + \sum_{m=0}^{\infty} a_m x^{u+m+1} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (u+m)^2 a_m x^{u+m-1} + \sum_{m=0}^{\infty} a_m x^{u+m+1} = 0$$

which can be written as $\longrightarrow (3)$

$$u_0^2 a_0 x^{u-1} + (u+1)^2 a_1 x^{u} + \sum_{m=2}^{\infty} (u+m)^2 a_m x^{u+m-1}$$
$$+ \sum_{m=0}^{\infty} a_m x^{u+m+1} = 0$$

Replacing m by $m+2$ in the first summation of this equation, we can rewrite it as

$$u_0^2 a_0 x^{u-1} + (u+1)^2 a_1 x^{u} + \sum_{m=0}^{\infty} [(u+m+2)^2 a_{m+2} + a_m]$$
$$(x^{u+m+1}) = 0 \longrightarrow (4)$$

Since (4) is an identity in x, the coefficient of each power of x must vanish independently; that is

$$\mu_0 a_0 = 0 \longrightarrow (5)$$

$$(\mu+1)^2 a_1 = 0 \longrightarrow (6)$$

$$(\mu+m+2)^2 a_{m+2} + a_m = 0 \longrightarrow (7)$$

$m = 0, 1, 2, \cdots$

The first equation demands that $\mu_0 = 0$ if $a_0 \neq 0$; then (6) gives $a_1 = 0$ & (7) implies

$$a_{m+2} = \frac{-a_m}{(\mu+m+2)^2} ; \quad m = 0, 1, 2, \cdots \longrightarrow (8)$$

Therefore

$$a_{2m+1} = 0 , \quad m = 1, 2, 3, \cdots$$

$$a_2 = \frac{-a_0}{2^2} , \quad a_4 = \frac{a_0}{2^2 4^2} ,$$

$$a_6 = \frac{-a_0}{2^2 \cdot 4^2 \cdot 6^2} \quad \text{\& so on}$$

$$a_{2n} = (-1)^n \frac{a_0}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2} \longrightarrow (9)$$

The solution does obtained if $a_0 = 1$ is a Bessel function of zeroth order, and it is often denoted by J_0, Thus

$$J_0 = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

$$+ (-1)^n \frac{x^{2n}}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2} + \cdots \longrightarrow (10)$$

since the ration of the nth term and $(n-1)$th term is $-x^2/(2n)^2$ and tends to zero as $n \to \infty$ irrespective of the value and sign of x, the series (10) for J_0 converges uniformly and absolutely for all values of x.

⇒ Order Symbols and Gauge Functions

Suppose we are interested in a function of the single real parameter ε, denoted by $f(\varepsilon)$. In carrying out our approximations, we are interested in the limit of $f(\varepsilon)$ as ε tends to zero, denoted by $\varepsilon \to 0$. This limit might depend on whether ε tends to zero from below, denoted by $\varepsilon \uparrow 0$, or from above, denoted by $\varepsilon \downarrow 0$. If the limit of $f(\varepsilon)$ exists (i.e) it does not have an essential singularity at $\varepsilon = 0$ such as $\sin \varepsilon^{-1}$), Then there are three possibilities

$$\left.\begin{aligned} f(\varepsilon) &\to 0 \\ f(\varepsilon) &\to A \\ f(\varepsilon) &\to \infty \end{aligned}\right\} \text{ as } \varepsilon \to 0, \quad 0 < A < \infty \qquad (1)$$

In the first and last cases, the rate at which $f(\varepsilon) \to 0$ and $f(\varepsilon) \to \infty$ is expressed by comparing $f(\varepsilon)$ with known functions called gauge functions.

The simplest and most useful of these are $\ldots \varepsilon^{-n}, \ldots, \varepsilon^{-2}, \varepsilon^{-1}, 1, \varepsilon^{1}, \varepsilon^{2}, \ldots, \varepsilon^{n}, \ldots$
In some cases these must be supplemented by
$\log \varepsilon^{-1}, \log(\log \varepsilon^{-1}), e^{\varepsilon^{-1}}, e^{-\varepsilon^{-1}}$ & so on

other gauge functions are $\sin\varepsilon$, $\cos\varepsilon$, $\tan\varepsilon$, $\sinh\varepsilon$, $\cosh\varepsilon$, $\tanh\varepsilon$, and so on

The behaviour of a function $f(\varepsilon)$ is compared with a gauge function $g(\varepsilon)$ as $\varepsilon \to 0$, employing either of the Landau symbols O or o.

⇒ **The Symbol O:-** We write

$$f(\varepsilon) = O[g(\varepsilon)] \quad \text{as } \varepsilon \to 0 \longrightarrow (2)$$

If there exists a positive number A independent of ε and an $\varepsilon_0 > 0$ s.t

$$|f(\varepsilon)| \leq A|g(\varepsilon)| \quad \text{for all } |\varepsilon| \leq \varepsilon_0 \longrightarrow (3)$$

This condition can be replaced by

$$\lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| < \infty \longrightarrow (4)$$

For examples as $\varepsilon \to 0$

$\sin \varepsilon = O(\varepsilon)$, $\quad \sin \varepsilon^2 = O(\varepsilon^2)$

$\sin 7\varepsilon = O(\varepsilon)$, $\quad \sin 2\varepsilon - 2\varepsilon = O(\varepsilon^3)$

$\cos \varepsilon = O(1)$, $\quad 1 - \cos \varepsilon = O(\varepsilon^2)$

$J_0(\varepsilon) = O(1)$, $\quad J_0(\varepsilon) - 1 = O(\varepsilon^2)$

$\sinh(\varepsilon) = O(\varepsilon)$, $\cosh(\varepsilon) = O(1)$
$\tanh(\varepsilon) = O(\varepsilon)$, $\tan(\varepsilon) = O(\varepsilon)$
$\coth(\varepsilon) = O(\varepsilon^{-1})$, $\cot\varepsilon = O(\varepsilon^{-1})$

If f is a function of another variable x in addition to ε, and $g(x,\varepsilon)$ is a gauge function, we also write

$$f(x,\varepsilon) = O[g(x,\varepsilon)] \text{ as } \varepsilon \to 0 \longrightarrow (5)$$

If there exist a positive number A independent of ε and an $\varepsilon_0 > 0$ such that

$$|f(x,\varepsilon)| \leq A|g(x,\varepsilon)| \;\forall\; |\varepsilon| \leq \varepsilon_0$$

If A and ε_0 are independent of x, the relationship (1.3.5) is said to hold uniformly. For example

$\sin(x+\varepsilon) = O(1) = O[\sin(x)]$ uniformly as $\varepsilon \to 0$

while $e^{-\varepsilon t} - 1 = O(\varepsilon)$ ununiformly as $\varepsilon \to 0$

$\sqrt{x+\varepsilon} - \sqrt{x} = O(\varepsilon)$ ununiformly as $\varepsilon \to 0$

⇒ **The Symbol o:-** We write

$$f(\varepsilon) = o[g(\varepsilon)] \text{ as } \varepsilon \to 0 \longrightarrow (7)$$

If for every +ve number δ, independent of ε, there exist an ε_0 such that

$$|f(\varepsilon)| \leq \delta|g(\varepsilon)| \text{ for } |\varepsilon| \leq \varepsilon_0 \longrightarrow (8)$$

This condition can be replaced by

$$\lim_{\varepsilon \to 0} \left|\frac{f(\varepsilon)}{g(\varepsilon)}\right| = 0 \longrightarrow (9)$$

Thus as $\varepsilon \to 0$

$\sin \varepsilon = O(1)$, , $\sin \varepsilon^2 = O(\varepsilon)$

$\cos \varepsilon = o(\varepsilon^{-1/2})$, $J_0(\varepsilon) = o(\varepsilon^{-1})$

$\coth \varepsilon = o(\varepsilon^{-3/2})$, $\cot \varepsilon = o[\varepsilon^{-(n+1)/n}]$ for $n>1$

$1-\cos 3\varepsilon = O(\varepsilon)$, $\exp(-\varepsilon^{-1}) = o(\varepsilon^n)\ \forall n$

If $f = f(x,\varepsilon)$ and $g = g(x,\varepsilon)$ then (7) is said to hold uniformly if δ and ε_0 are independent of x. e.g.

$\sin(x+\varepsilon) = o(\varepsilon^{-1/3})$ uniformly as $\varepsilon \to 0$

while $e^{-\varepsilon t} - 1 = o(\varepsilon^{1/2})$ nonuniformly as $\varepsilon \to 0$

$\sqrt{x+\varepsilon} - \sqrt{x} = o(\varepsilon^{3/4})$ ——— " ———

⇒ In many instances the information about a given function may be incomplete to determine the rate at which it tends to its limit sufficient to determine whether the rate is faster or slower than that of a gauge function. In such cases we use the order symbol 'o' (little 'o') defined as follows

$$f(\varepsilon) = o(g(\varepsilon)) \quad \text{as } \varepsilon \to 0$$

⇒ Asymptotic Series:-

We found earlier that a particular solution of

$$\frac{dy}{dx} + y = \frac{1}{x} \longrightarrow (1)$$

is $$y = \frac{1}{x} + \frac{1!}{x^2} + \frac{2!}{x^3} + \frac{3!}{x^4} + \dots + \frac{(n-1)!}{x^n} + \dots \longrightarrow (2)$$

Which diverges for all values of x. To investigate whether this series is of any value for computing a particular solution of our equation, we determine the remainder if we truncate the series after n terms. To do this we note that a particular integral of our differential equation is given by

$$y = e^{-x}\int_{-\infty}^{x} x^{-1} e^{x}\, dx$$

$$\Rightarrow y = e^{-x}\left[x^{-1}e^{x}\Big|_{-\infty}^{x} - \int_{-\infty}^{x} -x^{-2} e^{x}\, dx\right]$$

$$= \frac{1}{x} + e^{-x}\int_{-\infty}^{x} x^{-2} e^{x}\, dx$$

$$= \frac{1}{x} + \frac{1}{x^2} + 2e^{-x}\int_{-\infty}^{x} x^{-3} e^{x}\, dx$$

$$= \frac{1}{x} + \frac{1}{x^2} + \frac{2!}{x^3} + \dots + \frac{(n-1)!}{x^n} + \underbrace{n!\, e^{-x}\int_{-\infty}^{x} x^{-n-1} e^{x}\, dx}_{\text{Remainder}} \longrightarrow (4)$$

Therefore if we truncate the series after n terms, the remainder is

$$R_n = n!\, e^{-x} \int_{-\infty}^{x} x^{-n-1} e^{x}\, dx \longrightarrow (5)$$

which is a function of n and x. For the series to converge, $\lim_{n \to \infty} R_n$ must be zero. This is not true in our example; In fact $R_n \to \infty$ as $n \to \infty$ so that the series diverges for all x in agreement with what we found in section 1.2.2 unig the ratio test.

For negative x

$$|R_n| \leq n!\, |x^{-n-1}|\, e^{-x} \int_{-\infty}^{\infty} e^{x}\, dx = \frac{n!}{|x^{n+1}|} \longrightarrow (6)$$

$$R_n(x) \to 0 \text{ as } |x| \to \infty$$

$$\Rightarrow y \sim \sum_{n=1}^{\infty} \frac{(n-1)!}{x^n} \text{ as } |x| \to \infty \longrightarrow (7)$$

⇒Definition:- A series $\sum_{n=0}^{\infty} a_n / x^n$, where a_n are independent of x is an Asymptotic series and $f(x) \sim \sum_{n=0}^{\infty} \frac{a_n}{x^n}$ as $|x| \to \infty$ iff

$$f(x) = \sum_{n=0}^{N} \frac{a_n}{x^n} + o\left(\frac{1}{|x|^N}\right) \text{ as } |x| \to \infty$$

$$= \sum_{n=0}^{N-1} \frac{a_n}{x^n} + \frac{a_n}{x^N} + o\left(\frac{1}{|x|^N}\right) \text{ as } |x| \to \infty$$

$$= \sum_{n=0}^{N-1} \frac{a_n}{x^n} + O\left(\frac{1}{|x|^N}\right) \text{ as } |x| \to \infty$$

Definition:- The power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to be asymptotic to a function $f(x)$ as $x \to x_0$
i.e
$$f(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{as} \quad x \longrightarrow x_0$$
if $f(x) - \sum_{n=0}^{N} a_n(x-x_0)^n << |x-x_0|^N$ as $|x| \to \infty$ for every N
Thus a power series is asymptotic to a function if the remainder after N terms is much smaller than the last retained term as $x \to x_0$.
By this definition, a series may be asymptotic with out being convergent.
We also encounter asymptotic series in non-integral power of $(x-x_0)$.
The series $\sum a_n (x-x_0)^{\alpha_n}$ is asymptotic to a function $f(x)$ if
$$y(x) - \sum_{n=0}^{N} a_n (x-x_0)^{\alpha_n} << |x-x_0|^{\alpha_n} \text{ as } x \to x_0$$

Convergent VS Asymptotic Series:-
Suppose $f(x) = \sum_{n=0}^{N} \frac{a_n}{x^n} + R_N(x)$
This series converges if and only if
$\lim_{N \to \infty} R_n(x) = 0$; x is fixed
This series is asymptotic if & only if
$$R_n(x) = o(|x|^{-N}) \quad \text{as} \quad |x| \longrightarrow \infty$$
Clearly, a convergent series is an asymptotic series, However an asymptotic

series need not convergent.

If we define asymptotic series as

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{as } x \to x_0$$

Which means that for every N the remainder $R_N(x)$ of the series is much smaller than the last retained term as $x \to x_0$,

$$R_N(x) = f(x) - \sum_{n=0}^{N} a_n (x - x_0)^N << |x - x_0|^N \text{ as } x \to x_0$$

For Convergence:- $R_N(x) = \sum_{n=N+1}^{\infty} a_n (x - x_0)^N \to 0$ as $N \to \infty$, x is fixed

For Asymptotic:- $R_N(x) << |x - x_0|^N$; as $x \to x_0$; N is fixed

⇒ Singular Perturbation:-

Consider

$$\epsilon x^2 + x + 1 = 0 \longrightarrow (1)$$

In which the small parameter ϵ is multiplied by highest degree term.

Since equation (1) is Q.E and has two roots no matter how small ϵ is. However, if $\epsilon = 0$ there is just one root $x = -1$. This is one example of singular perturbation in which problem becomes differently in the limit as $\epsilon \to 0$ then it does when $\epsilon = 0$. The expansion associated with the solution $x = -1$ of the unperturbed case is called a regular perturbation expansion

The Regular perturbation expansion continuously approaches to unperturbed solution as $\epsilon \to 0$

Example (In Bush Book)

The new solution for which there is no counter part in the unperturbed problem has what is called a singular perturbation expansion.

To solve ① try the expansion

$$x = x_0 + x_1\epsilon + x_2\epsilon^2 + \cdots \longrightarrow ②$$

For one of the roots if we plug ② in ① we obtain

$$\epsilon(x_0 + x_1\epsilon + x_2\epsilon^2 + \cdots)^2 + (x_0 + x_1\epsilon + x_2\epsilon^2 + \cdots) + 1 = 0$$

$$\Rightarrow x_0 + 1 + \epsilon(x_1 + x_0^2) + \epsilon^2(x_2 + 2x_0x_1) + \cdots = 0$$

Equating like power of ϵ

ϵ^0 :- $\quad x_0 + 1 = 0 \quad \Rightarrow x_0 = -1$

ϵ^1 :- $\quad x_1 + x_0^2 = 0 \quad \Rightarrow x_1 = -1$

ϵ^2 :- $\quad x_2 + 2x_0x_1 = 0 \quad \Rightarrow x_2 = -2$

Therefore three term expansion is

$$\boxed{x = -1 - \epsilon - 2\epsilon^2 + \cdots}$$

So as expected, the above procedure yields just one solution

Lets investigate the exact solution

$$x = \frac{-1 \pm \sqrt{1-4\epsilon}}{2\epsilon} \longrightarrow ③ \quad \text{By Quadratic Formula}$$

$$\sqrt{1-4\epsilon} = (1-4\epsilon)^{1/2} = 1 - \frac{1}{2}(4\epsilon) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(4\epsilon)^2 + \cdots$$

$$= 1 - 2\epsilon - 2\epsilon^2 + \cdots$$

So ③ implies

$$x = \frac{1}{2\epsilon}\begin{cases} -1+1-2\epsilon-2\epsilon^2+\cdots \\ -1-1+2\epsilon+2\epsilon^2+\cdots \end{cases}$$

$$\Rightarrow x = \frac{1}{2\epsilon}\begin{cases} -2\epsilon-2\epsilon^2+\cdots \\ -2+2\epsilon+2\epsilon^2+\cdots \end{cases}$$

$$\Rightarrow \quad x = \begin{cases} -1-\epsilon-\cdots \\ -1/\epsilon+1+\epsilon+\cdots \end{cases}$$

* we can write x_0 here

This suggest to have

$$x = \frac{y^{*}}{\epsilon^{\nu}} + x_0 + x_1\epsilon^{\nu} + x_2\epsilon^{2\nu}$$

For 2nd root, try an expansion of the form

$$x = \frac{y}{\epsilon^{\nu}} + x_0 + \epsilon^{\nu}x_1 + \cdots \longrightarrow (*)$$

Substituting $(*)$ in ①

$$\epsilon\left(\frac{y^2}{\epsilon^{\nu}} + x_0 + \epsilon^{\nu}x_1+\cdots\right)^2 + \left(\frac{y}{\epsilon^{\nu}} + x_0 + \epsilon^{\nu}x_1+\cdots\right)+1=0$$

$$\Rightarrow \epsilon\left(\frac{y^2}{\epsilon^{2\nu}} + \frac{2yx_0}{\epsilon^{\nu}}+\cdots\right) + \left(\frac{y}{\epsilon^{\nu}} + x_0+\cdots\right)+1=0$$

$$\Rightarrow \underbrace{\epsilon^{1-2\nu}y^2 + \epsilon^{-\nu}y + 1}_{\text{dominant terms}} + \cdots = 0$$

Need to equate $1-2\nu = -\nu \Rightarrow \nu = 1$

If $\nu=1$ the leading order terms

$$y^2+y=0$$

$$\Rightarrow y(y+1)=0 \qquad \Rightarrow y = 0, -1$$

$y=0$ corresponds to the 1st root (discard)
$y=-1$ yields

$$x = \frac{-1}{\epsilon} + \dots$$

To find more terms, sub $x = \frac{-1}{\epsilon} + x_0 + x_1\epsilon + \dots$ in ①

$$\Rightarrow \epsilon\left(\frac{1}{\epsilon^2} - \frac{2x_0}{\epsilon} + x_0^2 + \dots\right) - \frac{1}{\epsilon} + x_0 + \dots + 1 = 0$$

$$\Rightarrow -2x_0 + x_0 + 1 = 0 \quad \Rightarrow x_0 = 1$$

$$\Rightarrow \boxed{x = \frac{-1}{\epsilon} + 1 + \dots}$$

Example:- For small ϵ, determine two term expansion for the roots of equation $\epsilon(x^3+x^2)+4x^2-3x-1=0 \longrightarrow ①$

Assuming $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots \longrightarrow ②$

Solution

Substitute equ ② in ① implies

$$\epsilon\left[(x_0+\epsilon x_1+\dots)^3 + (x_0+\epsilon x_1+\dots)^2\right] + 4\left[(x_0+\epsilon x_1+\dots)^2\right]$$
$$-3(x_0+\epsilon x_1+\dots) - 1 = 0$$

$$\Rightarrow \epsilon\left[x_0^3 + \epsilon^3 x_1^3 + 3(\epsilon x_0 x_1) + \dots + x_0^2 + \epsilon^2 x_1^2 + 2\epsilon x_0 x_1 + \dots\right] + 4(x_0^2 + \epsilon^2 x_1^2 + 2\epsilon x_0 x_1 + \dots) - 3(x_0 + \epsilon x_1 + \dots) - 1 = 0$$

$$\Rightarrow 4x_0^2 - 3x_0 - 1 = 0 \longrightarrow ③$$

$$(8x_0 - 3)x_1 = (-x_0^3 - x_0^2) \longrightarrow ④$$

③ $\Rightarrow x_0 = \frac{-1}{4}, 1$

The equation ④ gives $x_1 = \frac{3}{320}$ for $x_0 = \frac{-1}{4}$

and $x_1 = \frac{-2}{5}$ for $x_0 = 1$

Therefore, two of the roots of (1) have expansions

$$\boxed{\begin{aligned} x &= -\frac{1}{4} + \frac{3}{320}\epsilon + \cdots \\ \text{\&} \quad x &= 1 - \frac{2}{5}\epsilon + \cdots \end{aligned}}$$

We seek the third root in the form

$$x = \frac{y}{\epsilon^{\nu}} + x_0 + \epsilon^{\nu} x_1 + \cdots \longrightarrow (5)$$

Substituting (5) in (1) we have

$$\epsilon\left(\frac{y^3}{\epsilon^{3\nu}} + \frac{3y^2 x_0}{\epsilon^{2\nu}} + \frac{y^2}{\epsilon^{2\nu}} + \cdots\right) + 4\left(\frac{y^2}{\epsilon^{2\nu}} + \frac{2y x_0}{\epsilon^{\nu}} + \cdots\right)$$

$$-\frac{3y}{\epsilon^{\nu}} - 3x_0 + \cdots - 1 = 0 \longrightarrow (6)$$

Extracting the dominant terms in (6), we have $\epsilon^{1-3\nu} y^3 + 4\epsilon^{-2\nu} y^2 + \cdots = 0$

Setting $1 - 3\nu = -2\nu \Rightarrow \nu = 1$

For $\nu = 1$ $\quad y^3 + 4y^2 = 0 \Rightarrow y^2(y+4) = 0$

$$\Rightarrow \boxed{y = -4}$$

Therefore the third root has the expansion of the form

$$x = \frac{-4}{\epsilon} + x_0 + \epsilon x_1 + \cdots \longrightarrow (7)$$

Substitute (7) in (1) and equating the co-efficients of like powers of ϵ, we have

$$3y^2 x_0 + y^2 + 8y x_0 - 3y = 0 \Rightarrow x_0 = -7/4$$

Hence the two term expansion for the third root is

$$\boxed{x = \frac{-4}{\epsilon} - \frac{7}{4} + \cdots}$$

Example:- Solve for small ϵ

$$\epsilon x^3 + x + 2 + \epsilon = 0 \longrightarrow (1)$$

Solution

An expansion of the form

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \longrightarrow (2)$$

Substitute equ (2) in (1) implies

$$\epsilon\left[x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right]^3 + x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$
$$+ 2 + \epsilon = 0$$

$$\Rightarrow \epsilon\left(x_0^3 + \epsilon^3 x_1^3 + 3(x_0 x_1 \epsilon) + \cdots\right) + x_0 + \epsilon x_1 + \cdots + 2 + \epsilon = 0$$

$$\Rightarrow 2 + x_0 + \epsilon + \epsilon x_1 + \epsilon x_0^3 + \cdots \quad 0$$

Equating like powers of ϵ

ϵ^0 :- $\quad 2 + x_0 = 0 \quad \Rightarrow \quad \boxed{x_0 = -2}$

ϵ^1 :- $\quad 1 + x_1 + x_0^3 = 0 \quad \Rightarrow \quad \boxed{x_1 = 7}$

$$\Rightarrow \quad \boxed{x = -2 + 7\epsilon + \cdots}$$

For other roots try expansion of the form

$$x = \frac{y}{\epsilon^{\nu}} + x_0 + x_1 \epsilon^{\nu} + \cdots \longrightarrow (3)$$

Putting this in equ (1) implies

$$\epsilon\left[\frac{y}{\epsilon^{\nu}} + x_0 + x_1\epsilon^{\nu} + \cdots\right]^3 + \frac{y}{\epsilon^{\nu}} + x_0 + x_1\epsilon^{\nu} + \cdots + 2 + \epsilon = 0$$

$$\epsilon\left[\frac{y^3}{\epsilon^{3\nu}} + x_0^3 + 3\left(\frac{y}{\epsilon^{\nu}} x_0\right) + \cdots\right] + \frac{y}{\epsilon^{\nu}} + x_0 + x_1\epsilon^{\nu} +$$
$$\cdots + 2 + \epsilon = 0 \longrightarrow (4)$$

Extracting dominant terms from ④ gives $\epsilon^{1-3\nu} + \epsilon^{-\nu} y + \cdots = 0$

Equating $1-3\nu = -\nu \Rightarrow \nu = 1/2$

and $y^3 + y = 0 \Rightarrow y(y^2+1) = 0$

$\Rightarrow y = 0, \pm i$

$y=0$ corresponds to the 1st root, can be discard.

So $$x = \frac{y}{\epsilon^{1/2}} + x_0 + \epsilon^{1/2} x_1 + \cdots \longrightarrow ⑤$$

with $y = \pm i$

Substitute ⑤ in ① gives

$$\epsilon^{-\frac{1}{2}}(y^3+y) + 3y^2 x_0 + x_0 + 2 = 0$$

$$\Rightarrow 3y^2 x_0 + x_0 + 2 = 0$$

$y = \pm i \Rightarrow -3x_0 + x_0 + 2 = 0$

$$\Rightarrow \boxed{x_0 = 1}$$

Hence the two term expansion for the remaining roots are

$$\boxed{x = \frac{\pm i}{\epsilon^{1/2}} + 1 + \cdots}$$

Asymptotic Approximation of Integrals:–

There are many differential and difference equations whose solutions can not be expressed in terms of functions but can be expressed in the form of integral. There are many methods for obtaining asymptotic expansion of functions defind by definite integrals. They include expansion of integrals, Integration by parts, Laplace method, the method of stationary phase and the method of steepest descent.

1. Expansion of Integrals:– If the integrand contains a small parameter or if the limits of integration are small, we may be able to obtain an asymptotic expansion of the integral by expanding the integrand and integrating term by term.

<u>Step I</u> $\quad I(\epsilon) = \int_0^1 \sin \epsilon x^2 \, dx;$ where ϵ is small

we know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}$$

$$\Rightarrow \sin \epsilon x^2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (\epsilon x^2)^{2n-1}}{(2n-1)!}$$

$$\Rightarrow \sin \epsilon x^2 = \epsilon x^2 - \frac{1}{6}\epsilon^3 x^6 + \frac{1}{120}\epsilon^5 x^{10} + O(\epsilon^7)$$

Ratio Test

$$\lim_{n\to\infty} \frac{n^{th} \text{ term}}{(n-1)^{th} \text{ term}}$$

$$= \lim_{n\to\infty} \frac{(-1)^{n+1}(\epsilon x^2)^{2n-1}}{(2n-1)!} \cdot \frac{(2n-3)!}{(-1)^{n-1}(\epsilon x^2)^{2n-3}}$$

$$= \lim_{n\to\infty} \frac{(-1)(\epsilon x^2)^2}{(2n-1)(2n-2)} \longrightarrow 0 \quad \forall x$$

Hence the above series converges for all values of ϵx^2

$$\therefore I(\epsilon) = \int_0^1 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(\epsilon x^2)^{2n-1}}{(2n-1)!} dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} \int_0^1 (\epsilon x^2)^{2n-1} dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\epsilon^{2n-1}}{(2n-1)!} \int_0^1 x^{4n-2} dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\epsilon^{2n-1}}{(2n-1)!} \left. \frac{x^{4n-1}}{4n-1} \right|_0^1$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\epsilon^{2n-1}}{(2n-1)!} \cdot \frac{1}{4n-1}$$

$$= \frac{\epsilon}{3} - \frac{\epsilon^3}{42} + \frac{\epsilon^5}{1320} + O(\epsilon^7)$$

Example:- $I(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-m\sin^2\theta}}$, m is small $\longrightarrow (1)$

Solution

Integrand $\frac{1}{\sqrt{1-m\sin^2\theta}} = (1-m\sin^2\theta)^{-\frac{1}{2}}$

$$\Rightarrow (1-m\sin^2\theta)^{-\frac{1}{2}} = 1+\left(-\tfrac{1}{2}\right)(-m\sin^2\theta) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-m\sin^2\theta)^2$$

$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-m\sin^2\theta)^3 + \cdots$$

$$= 1 + \frac{m}{2}\sin^2\theta + \frac{3m^2}{8}\sin^4\theta + \frac{15}{48}m^3\sin^6\theta$$

$$+ O(m^4) \longrightarrow (2)$$

Using Ratio Test:-

$$\lim_{n\to\infty} \frac{n^{th}\text{ term}}{(n-1)^{th}\text{ term}} = \lim_{n\to\infty} \frac{\dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{(2n-3)}{2}\right)(-m\sin^2\theta)^{n-1}}{(n-1)!}}{\dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{(2n-5)}{2}\right)(-m\sin^2\theta)^{n-2}}{(n-2)!}}$$

$$= \lim_{n\to\infty} \frac{(2n-3)\,m\sin^2\theta}{2(n-1)}$$

$$= m\sin^2\theta \lim_{n\to\infty} \frac{2n-3}{2n-2}$$

$$= m\sin^2\theta$$

Series (2) converges if $m\sin^2\theta < 1$, which is true for small m

Substitute (2) in (1) and integrate term by term

$$I(m) = \int_0^{\pi/2} \left[1 + \frac{m}{2}\sin^2\theta + \frac{3m^2}{8}\sin^4\theta + \frac{5m^3}{16}\sin^6\theta + O(m^4)\right] d\theta$$

$$= \int_0^{\pi/2} d\theta + \frac{m}{2}\int_0^{\pi/2}\sin^2\theta\, d\theta + \frac{3m^2}{8}\int_0^{\pi/2}\sin^4\theta\, d\theta$$

$$+ \frac{5m^3}{16}\int_0^{\pi/2}\sin^6\theta\, d\theta + O(m^4)$$

$$\Rightarrow I(m) = \frac{\pi}{2}\left[1 + \frac{m}{4} + \frac{9}{64}m^2 + \frac{25}{256}m^3 + \frac{1225}{16384}m^4 + O(m^5)\right]$$

$$\star \int_0^{\pi/2} \sin^{2n}\theta \, d\theta = \frac{(2n!)\pi}{(n!)^2 \, 2^{2n+1}}$$

⇒ Home Work:-

Q 1:- $I(x) = \int_0^x t^{-\frac{3}{4}} e^{-t} dt$; for small x

Q2:- $I(x) = \int_0^\infty e^{-t^2} dt$; for small x

2:- Integration:-

$$d(uv) = u\,dv + v\,du$$

$$\Rightarrow u\,dv = d(uv) - v\,du$$

Integrate ⇒ $\boxed{\int u\,dv = uv - \int v\,du}$

$$I(x) = \int_x^\infty \frac{e^{-t}}{t^2} dt \quad \text{for large } x$$

$u = e^{-t}$ and $dv = \frac{dt}{t^2}$

$\Rightarrow du = -e^{-t} dt$ & $v = \frac{-1}{t}$

$$\Rightarrow \int_x^\infty \frac{e^{-t}}{t^2} dt = \int_x^\infty u\,dv = uv\Big|_x^\infty - \int_x^\infty v\,du$$

$$= e^{-t}\left(\frac{-1}{t}\right)\Big|_x^{t\to\infty} - \int_x^\infty \frac{1}{t} e^{-t} dt$$

$$0 + \frac{e^{-x}}{x} - \int_x^\infty e^{-t} \frac{1}{t} dt$$

$$u = e^{-t}, \quad dv = \frac{dt}{t}$$

$$\Rightarrow du = -e^{-t}, \quad v = \ln t$$

$$\Rightarrow \int_x^\infty \frac{e^{-t}}{t^2}\,dt = \frac{e^{-x}}{x} - uv\Big|_x^\infty - \int_x^\infty v\,du$$

$$= \frac{e^{-x}}{x} - e^{-t}\ln t\Big|_x^\infty + \int_x^\infty \ln t\, e^{-t}\,dt$$

$$= \frac{e^{-x}}{x} + e^{-x}\ln x + \int_x^\infty e^{-t}\ln t\,dt$$

Not that the 2nd term on the R.H.S is much bigger than the 1st term. For large x (i.e when $x \to \infty$) the above choices for u and dv dont yields an asymptotic expansion instead if we choose

$u = 1/t^2$ & $dv = e^{-t}\,dt$ we will get

$$\int_x^\infty \frac{e^{-t}}{t^2}\,dt = \frac{e^{-x}}{x^2} - 2\int_x^\infty \frac{e^{-t}}{t^3}\,dt$$

Again choosing $u = 1/t^3$ & $dv = e^{-t}dt$ we will get

$$\int_x^\infty \frac{e^{-t}}{t^2}\,dt = \frac{e^{-x}}{x^2} - \frac{2e^{-x}}{x^3} + 3!\int_x^\infty \frac{e^{-t}}{t^4}\,dt$$

Continuing with integration by parts

$$\int_x^\infty \frac{e^{-t}}{t^2}\,dt = \frac{e^{-x}}{x^2} - \frac{2e^{-x}}{x^3} + \frac{3!\,e^{-x}}{x^4} - \frac{4!\,e^{-x}}{x^5} + \cdots + \frac{(-1)^{n+1}\,n!\,e^{-x}}{x^{n+1}} + (-1)^n (n+1)!\int_x^\infty \frac{e^{-t}}{t^{n+2}}\,dt$$

Since $x \leq t \leq \infty$

$$\Rightarrow x^{n+2} \leq t^{n+2} \Rightarrow \frac{1}{t^{n+2}} \leq \frac{1}{x^{n+2}}$$

and
$$\int_x^\infty \frac{e^{-t}}{t^{n+2}} dt < \frac{1}{x^{n+2}} \int e^{-t} dt = \frac{e^{-x}}{x^{n+2}}$$

So our last expansion can be written as

$$I(x) = e^{-x} \sum_{n=1}^{N} \frac{(-1)^{n+1} n!}{x^{n+1}} + O\left(\frac{1}{|x|^{N+2}}\right)$$

By Ratio Test:-

$$\lim_{n\to\infty} \frac{n\text{th term}}{(n-1)\text{th term}}$$

$$= \lim_{n\to\infty} \frac{\frac{(-1)^{n+1} n!\, \cancel{e^{-x}}}{x^{n+1}}}{\frac{(-1)^n (n-1)!\, \cancel{e^{-x}}}{x^n}} = \lim_{n\to\infty} \frac{(-1)n}{x} \to \infty$$

This series diverges
However, for a fixed N, the remainder can be made arbitrarily small by increasing x (as x is large)

Example:- $I(x) = \int_0^x e^{-xt} f(t)\, dt$ (Laplace Integral)

For large x and $f(t)$ is analytic.

Solution

Take $u = f(t)$, $dv = e^{-xt} dt$

$$\Rightarrow du = f'(t) dt \ \& \ v = \frac{e^{-xt}}{-x}$$

Then,

$$I(x) = \int_0^x u\,dv = uv\Big|_0^x - \int_0^x v\,du$$

$$= f(t)\cdot \frac{e^{-xt}}{-x}\Big|_0^x - \int_0^x \frac{e^{-xt}}{-x} f'(t)\,dt$$

$$= \frac{f(0)}{x} + \int_0^x \frac{e^{-xt}}{x} f'(t)\,dt$$

Now setting $u = f'(t)$ & $dv = \frac{e^{-xt}}{x}dt$

$\Rightarrow du = f''(t)dt$ & $v = \frac{e^{-xt}}{-x^2}$

$$\Rightarrow I(x) = \frac{f(0)}{x} + \int_0^x u\,dv = \frac{f(0)}{x} + uv\Big|_0^x - \int_0^x v\,du$$

$$= \frac{f(0)}{x} + f'(t)\cdot\frac{e^{-xt}}{-x^2}\Big|_0^x - \int_0^x \frac{e^{-xt}}{-x^2} f''(t)\,dt$$

$$= \frac{f(0)}{x} + \frac{f'(0)}{x^2} + \int_0^x \frac{e^{-xt}}{x^2} f''(t)\,dt$$

Continuing this process we have

$$I(x) = \frac{f(0)}{x} + \frac{f'(0)}{x^2} + \frac{f''(0)}{x^3} + \dots + \frac{f^{(n)}(0)}{x^{n+1}} + \dots$$

$$= \sum_{n=0}^{N} \frac{f^{(n)}(0)}{x^{n+1}} + O\left(\frac{1}{x^{N+2}}\right)$$

Example:- $I(\alpha) = \int_0^\infty e^{i\alpha t} f(t)\,dt$ "Fourier Integ"

For large $+\alpha$, $f(t)$ is analytic.

Solution

Take $u = f(t)$, $dv = e^{i\alpha t}dt$

$du = f'(t)\,dt$ $\Rightarrow v = \frac{e^{i\alpha t}}{i\alpha}$

Then $I(\alpha) = \int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$

$$= f(t)\, e^{i\alpha t}/i\alpha \Big|_0^\infty - \int_0^\infty \frac{e^{i\alpha t}}{i\alpha} f'(t)\,dt$$

$$= \frac{f(0)}{-i\alpha} - \int_0^\infty \frac{e^{i\alpha t}}{i\alpha} f'(t)\,dt$$

Again Taking $u = f'(t)$ & $dv = \frac{e^{i\alpha t}}{i\alpha}\,dt$

$\Rightarrow du = f''(t)\,dt$ & $v = \frac{e^{i\alpha t}}{(i\alpha)^2}$

$$\Rightarrow I(\alpha) = \frac{f(0)}{-i\alpha} - \int_0^\infty u\,dv = \frac{f(0)}{-i\alpha} - \left\{ uv\Big|_0^\infty - \int_0^\infty v\,du \right\}$$

$$= \frac{f(0)}{-i\alpha} - f'(t)\frac{e^{i\alpha t}}{(i\alpha)^2}\Big|_0^\infty + \int_0^\infty \frac{e^{i\alpha t}}{(i\alpha)^2} f''(t)\,dt$$

$$= \frac{f(0)}{-i\alpha} - \left[-\frac{f'(0)}{(i\alpha)^2} \right] + \int_0^\infty \frac{e^{i\alpha t}}{(i\alpha)^2} f''(t)\,dt$$

$$= \frac{f(0)}{-i\alpha} + \frac{f'(0)}{(i\alpha)^2} + \int_0^\infty \frac{e^{i\alpha t}}{(i\alpha)^2} f''(t)\,dt$$

Continuing this process we have

$$I(\alpha) = \frac{f(0)}{-i\alpha} + \frac{f'(0)}{(i\alpha)^2} + \frac{f''(0)}{-(i\alpha)^3} + \cdots + \frac{f^{(n)}(0)}{(-i\alpha)^{n+1}} + \int_0^\infty \frac{e^{i\alpha t}}{(i\alpha)^{n+1}} f^{(n)}(t)\,dt$$

$$\Rightarrow I(\alpha) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{(-i\alpha)^{n+1}} + O\left(\frac{1}{\alpha^{N+2}}\right)$$

Example:- $I(x) = \int_a^b e^{x h(t)} f(t)\, dt \,, \quad b > a$

Generalized Laplace integral
For large x when $h(t)$ and $f(t)$ are differentiable functions.

Solution

Taking $u = f(t)$ & $dv = e^{x h(t)}$

$\Rightarrow du = f'(t)\, dt$ & $v = \int e^{x h(t)}\, dt$

As we dont know the derivative of $h(t)$, so we can not yield a solution from above choices.

Modifying the Integral:-

$$u = \frac{f(t)}{h'(t)} \quad \& \quad dv = e^{x(h(t))} h'(t)\, dt$$

$$\Rightarrow du = \left(\frac{f(t)}{h'(t)}\right)' \quad \& \quad v = \int e^{x h(t)} h'(t)\, dt$$

$$\Rightarrow v = \frac{e^{x h(t)}}{x}$$

$$\Rightarrow I(x) = \frac{f(t)}{h'(t)} \cdot \frac{e^{x h(t)}}{x}\Bigg|_a^b - \frac{1}{x}\int_a^b e^{x h(t)} \left[\frac{f(t)}{h'(t)}\right]' dt$$

$$= \frac{e^{x h(b)} f(b)}{x h'(b)} - \frac{e^{x h(a)} f(a)}{x h'(a)} - \frac{1}{x}\int_a^b e^{x h(t)} \left(\frac{f(t)}{h'(t)}\right)' dt$$

* Laplace Method :-

We continuing the asymptotic analysis of integrals of the form

$$I(x) = \int_a^b e^{x h(t)} f(t)\, dt$$

for real $h(t)$ and large +ve x. If $h'(t) = 0$ in the interval $[a,b]$, integration by parts for above integral breaks down, because the integrals in the successive integration by parts fail to exist. In this case, if $h(t)$ has a relative maximum at $t = c$, where $a \leq c \leq b$, an asymptotic expansion of the integral for large x can be obtained by expanding both $h(t)$ and $f(t)$ around $t = c$. This is Laplace's Method, which is based on the idea that the major contribution to the value of the integral arises from the neighborhoods of the points in the interval $[a,b]$ at which the integrand has its maximum value. Thus if $h(t)$ does not have a relative maximum in $[a,b]$, the major contribution to the integrand can be obtained by a succession of integration by parts. If $h(t)$ has a finite number of maxima, the interval of integration can be broken up to a finite number of intervals so that $h(t)$ has only one maximum in each interval. An asymptotic expansion of the integral can then obtained as the sum of asymptotic expansion of the resulting integrals.

Example:-

$$I(x) = \int_0^{10} \frac{e^{-xt}}{1+t}\, dt \longrightarrow (1)$$

$$\frac{1}{1+t} = 1 - t + t^2 - \cdots + (-1)^n t^n + \cdots = \sum_{n=0}^{\infty} (-1)^n t^n \longrightarrow (2)$$

convergent for $|t| < 1$.

$$\Rightarrow I(x) = \int_0^{10} e^{-xt} \sum_{n=0}^{\infty} (-1)^n t^n\, dt$$

Can break interval into two sub intervals $[0, \delta]$ & $[\delta, 10]$ where δ is small +ve number.

$$\Rightarrow I(x) = \int_0^{\delta} \frac{e^{-xt}}{1+t}\, dt + \int_{\delta}^{10} \frac{e^{-xt}}{1+t}\, dt \longrightarrow (3)$$

Next we show that the 2nd integral in (3) is exponentially small for large +ve x.

We note that $\frac{1}{1+t} < 1$ for all $t > 0$

Hence $\int_{\delta}^{10} \frac{e^{-xt}}{1+t}\, dt < \int_{\delta}^{10} e^{-xt}\, dt$

$$= \left. \frac{e^{-xt}}{-x} \right|_{\delta}^{10} = -\frac{1}{x}\left(e^{-10x} - e^{-\delta x}\right)$$

As $x \to \infty$, $e^{-10x} \to 0$ much faster than any power of x^{-1}.

For finite value of δ; $e^{-\delta x} \to 0$ much faster than any power of x^{-1}

$\therefore \int_{\delta}^{10} \frac{e^{-xt}}{1+t} dt$ tends to zero exponentially

$$\Rightarrow I(x) = \int_0^{\delta} \frac{e^{-xt}}{1+t} dt + \text{exponentially small terms as } x \to \infty$$

Thus only the immidiate nbhd of $t=0$ contributes to the asymptotic expansion of ①. Therefore

$$I(x) = \int_0^{\delta} \frac{e^{-xt}}{1+t} dt = \int_0^{\delta} e^{-xt} \sum_{n=0}^{\infty} (-1)^n t^n dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^{\delta} e^{-xt} t^n dt$$

use $\tau = xt \Rightarrow dt = d\tau/x$

$$\Rightarrow \int_0^{\delta} e^{-xt} t^n dt = \frac{1}{x^{n+1}} \int_0^{\delta x} \tau^n e^{-\tau} d\tau$$

$$= \frac{-1}{x^{n+1}} \left[\tau^n + n\tau^{n-1} + n(n-1)\tau^{n-2} + \cdots + \frac{1}{2} n! \tau^2 + n! \tau + n! \right] e^{-\tau} \Big|_0^{\delta x}$$

$$= \frac{n!}{x^{n+1}} - \underbrace{e^{-\delta x} \left[\frac{\delta^n}{x} + \frac{n\delta^{n-1}}{x^2} + \frac{n(n-1)\delta^{n-2}}{x^3} + \cdots + \frac{n!}{x^{n+1}} \right]}_{\to 0 \text{ as } x \to \infty}$$

Note:- $\int_0 t^n e^{-xt} dt = \frac{n!}{x^{n+1}}$ + exp small terms

Hence $\int_0^\delta t^n e^{-xt} dt = \frac{n!}{x^{n+1}}$ + exponentially small terms

$$\Rightarrow I(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}} \quad \text{as } x \to \infty$$

Ratio Test $\lim_{n\to\infty} \frac{n\text{th terms}}{(n-1)\text{ terms}} = \lim_{n\to\infty} \frac{(n+1)!}{x^{n+2}} \cdot \frac{x^{n+1}}{n!} \to \infty$

$\Rightarrow$ Diverges

Therefore we should use $\sim$ sign instead of $=$

$$\text{i.e}\quad I(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}} \quad \text{as } x\to\infty$$

The procedure stated above is usually reffered to as Watson's lemma, which gives the full asymp dependent of the integral of the form

$$I(x) = \int_0^b f(t) e^{-xt} dt$$

Watson's Lemma:-

$$I(x) = \int_0^b f(t) e^{-xt} dt \;;\quad b>0$$

Suppose f has expansion

$$f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{\beta n} \quad \text{as } t\to 0^+; \quad \text{where } \alpha>-1 \text{ \& } \beta>0$$

Then

$$I(x) \sim \sum_{n=0}^{\infty} a_n \int_0^\infty t^{\alpha+\beta n} e^{-xt} dt$$

or

$$I(x) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \quad \text{as } x\to\infty$$

In the above example, $h(t) = -t$ has its maximum at $t=0$. Since $h'(0) = -1$, this maximum is not a relative maximum.

As a 2nd example we consider

Example:- $$I(x) = \int_0^5 \frac{e^{-xt^2}}{1+t}\,dt \; ; \longrightarrow (1)$$

have $h(t) = -t^2$ has max at $t=0$ & $h'(0)=0$

In this example the method of integration by parts would fail as discussed before. For large x, only the immidiate ~~no~~ nbhd of $t=0$ contributes to the asymptotic development of $I(x)$ because $h(t)$ has its maximum there. By the Watson's lemma, we determine ~~the~~ any asymp. expansion for $I(x)$ by substitution the expansion of $(1+t)^{-1}$ in (1), integrating the result term by term.

Thus the result is

$$I(x) = \sum_{n=0}^{\infty} (-1)^n \int_0^5 t^n e^{-xt^2}\,dt \longrightarrow (2)$$

Let $$I_n = \int_0^{\infty} t^n e^{-xt^2}\,dt \; ; \text{ for } n = 0, 1, 2, \cdots \longrightarrow (3)$$

We start with

$$I_0 = \int_0^{\infty} e^{-xt^2}\,dt \longrightarrow (4)$$

If we let $\sqrt{x}\,t = \tau$, then $dt = d\tau/\sqrt{x}$ and ④ becomes

$$I_0 = \frac{1}{\sqrt{x}} \int_0^\infty e^{-\tau^2} d\tau \longrightarrow ⑤$$

Since

$$\int_0^\infty e^{-u^2} du = \frac{\sqrt{\pi}}{2} \longrightarrow ⑥$$

using ⑥ we write ⑤ as

$$I_0 = \frac{1}{\sqrt{x}} \int_0^\infty e^{-\tau^2} d\tau = \frac{\sqrt{\pi}}{2\sqrt{x}} \longrightarrow ⑦$$

differentiating ⑦ with respect to x, we have

$$-\int_0^\infty t^2 e^{-xt^2} dt = -\frac{\sqrt{\pi}}{4x^{3/2}}$$

Hence

$$I_2 = \int_0^\infty t^2 e^{-xt^2} dt = \frac{\sqrt{\pi}}{4x^{3/2}} \longrightarrow ⑧$$

Differentiating ⑧ w.r.t x we have

$$-\int_0^\infty t^4 e^{-xt^2} dt = -\frac{3\sqrt{\pi}}{8x^{5/2}}$$

Hence

$$I_4 = \int_0^\infty t^4 e^{-xt^2} dt = \frac{3\sqrt{\pi}}{8x^{5/2}} \longrightarrow ⑨$$

Continuing the process we find that

$$I_{2m} = \frac{1}{2}\sqrt{\pi}\,(-1)^m \frac{d^m}{dx^m}\left(x^{-1/2}\right) \longrightarrow ⑩$$

To determine the I_n for odd n, we start with

$$I_1 = \int_0^\infty t e^{-xt^2} dt \longrightarrow (11)$$

we let $xt^2 = \tau$ so that
$2xt\,dt = d\tau$ & Hence

$$I_1 = \frac{1}{2x}\int_0^\infty e^{-\tau} d\tau = \frac{-1}{2x} e^{-\tau}\Big|_0^\infty$$

$$= \frac{1}{2x} \longrightarrow (12)$$

Differentiating (12) w.r.t x yields

$$-\int_0^\infty t^3 e^{-xt^2} dt = \frac{-1}{2x^2}$$

So that $$I_3 = \int_0^\infty t^3 e^{-xt^2} dt = \frac{1}{2x^2} \longrightarrow (13)$$

Differentiating (13) w.r.t x yields

$$-\int_0^\infty t^5 e^{-xt^2} dt = -1/x^3$$

Hence $$I_5 = \int_0^\infty t^5 e^{-xt^2} dt = \frac{1}{x^3} \longrightarrow (14)$$

Continuing the process we find that

$$I_{2m+1} = \frac{1}{2}(-1)^m \frac{d^m}{dx^m}(x^{-1}) \longrightarrow (15)$$

using (10) and (15) we rewrite (2) as

$$I(x) \sim \frac{1}{2}\sqrt{\pi} \sum_{m=0}^\infty (-1)^m \frac{d^m}{dx^m}(x^{\frac{-1}{2}}) - \frac{1}{2}\sum_{m=0}^\infty (-1)^m \frac{d^m}{dx^m}(x^{-1})$$

$$\Rightarrow I(x) \sim \frac{\sqrt{\pi}}{2x^{1/2}} - \frac{1}{2x} + \frac{\sqrt{\pi}}{4x^{3/2}} - \frac{1}{2x^2} + \frac{3\sqrt{\pi}}{8x^{5/2}} - \frac{1}{x^3} + \cdots \longrightarrow (16)$$

The Gamma Function:-

Alternatively we can express (3) in terms of the gamma function defined by

$$\Gamma(z) = \int_0^\infty \tau^{z-1} e^{-\tau} \, d\tau \longrightarrow (17)$$

where z must be greater than zero, otherwise the integrand does not exist. To relate (3) to gama function, we introduce the transformation

$$\tau = x t^2 \quad , \quad d\tau = 2xt\,dt \longrightarrow (18)$$

So that (3) becomes

$$I_n = \frac{1}{2} x^{-(n+1)/2} \int_0^\infty \tau^{(n-1)/2} e^{-\tau} \, d\tau$$

$$= \frac{1}{2} x^{-(n+1)/2} \, \Gamma\left(\frac{n+1}{2}\right) \longrightarrow (19)$$

Substituting (19) into (1) yields

$$I(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \, \Gamma\left(\frac{n+1}{2}\right)}{2\, x^{(n+1)/2}} \longrightarrow (20)$$

It can be known that (20) is the same as (16)

Example:- Here we consider the following integral, which is slightly more general than the preceding example

$$I(x) = \int_a^b f(t)\, e^{x h(t)}\, dt \; ; \; b > a \longrightarrow (1)$$

$$\text{where } h'(a) = 0 \; ; \; h''(a) < 0 \longrightarrow (2)$$

$$f(t) = f_0 (t-a)^{\lambda} \text{ as } t \to a \longrightarrow (3)$$

where f_0 is a constant.

The condition (2) imply that $h(t)$ has a relative maxima at $t = a$. We assume that this maxima is also an absolute maxima as shown below. The constant λ must be greater than -1 for the integral (1) to exist. Only the immidiate nbhd of $t = a$ contributes to the asymptotic development of $I(x)$ because $h(t)$ has ~~the~~ its maximum than

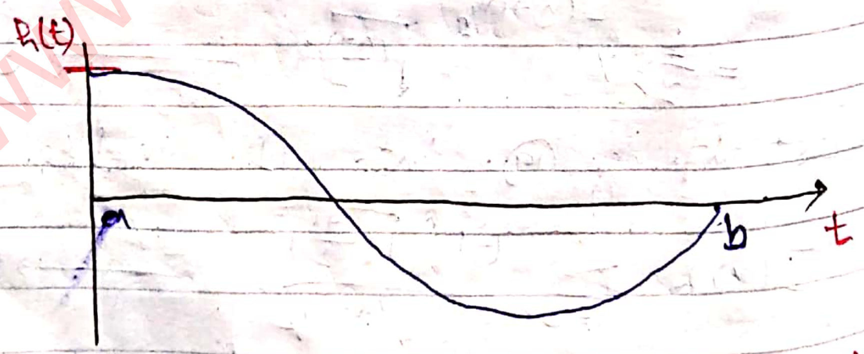


Fig A: A function with a relative and absolute maxima at $t = a$.

Thus, the upper limit can be replaced with $a + \delta$, where δ is a small

+ve number. The result is

$$I(x) \sim \int_a^{a+\delta} f(t)\, e^{x h(t)}\, dt \longrightarrow (4)$$

To determine the principal of $I(x)$, we expand $h(t)$ in a Taylor series about $t = a$, that is

$$h(t) = h(a) + h'(a)(t-a) + \frac{1}{2} h''(a)(t-a)^2 + \cdots$$

OR

$$h(t) = h(a) + \frac{1}{2} h''(a)(t-a)^2 + \cdots \longrightarrow (5)$$

because $h'(a) = 0$. Substituting (3) and (5) into (4), we have

$$I(x) \sim f_0\, e^{x h(a)} \int_a^{a+\delta} (t-a)^{\lambda}\, e^{\frac{1}{2} x h''(a)(t-a)^2}\, dt \longrightarrow (6)$$

Since $h''(a) < 0$, δ can be replaced with ∞ because only the immidiate nbhd of $t = a$ contributes to the asymptotic development of $I(x)$.
Thus we rewrite (6) as

$$I(x) \sim f_0\, e^{x h(a)} \int_a^{\infty} (t-a)^{\lambda}\, e^{\frac{1}{2} x h''(a)(t-a)^2}\, dt \longrightarrow (7)$$

The integral in (7) can be expressed in terms of the gamma function.
To this end we let

$$-\frac{1}{2} x h''(a)(t-a)^2 = \tau \longrightarrow (8)$$

and obtain

$$I(x) \sim \frac{1}{2}\left[\frac{2}{-x h''(a)}\right]^{\frac{\lambda+3}{2}} f_0 \, e^{x h(a)} \int_0^\infty \tau^{\frac{\lambda-1}{2}} e^{-\tau} \, d\tau \longrightarrow (9)$$

OR

$$I(x) \sim \frac{1}{2}\left[\frac{2}{-x h''(a)}\right]^{\frac{\lambda+3}{2}} f_0 \, e^{x h(a)} \, \Gamma\left(\frac{\lambda+1}{2}\right) ; \text{ as } x \to \infty \longrightarrow (10)$$

where Γ denote the gamma function.

To determine the higher order terms in the development of $I(x)$, we need to keep the higher order terms in $f(t)$ and the higher order terms in $h(t)$

Example:- In all the preceeding examples $h(t)$ has its maximum at an end point, Here we consider a case in which $h(t)$ has its maxima at a point inside the interval of integration as shown below.

Specifically, we consider

$$I(x) = \int_a^b f(t) \, e^{x h(t)} \, dt \, ; \quad b > a \longrightarrow (1)$$

Where x is large

$$h'(c) = 0 \; ; \; h''(c) < 0 \; ; \; a < c < b \longrightarrow (2)$$

$$f(t) \sim f_0 (t-c)^\lambda \quad \text{as } t \to \infty \longrightarrow (3)$$

and $\lambda > -1$ for the integral to exist.

To determine the principle part of $I(x)$ for large x, we expand

$h(t)$ in a Taylor series as

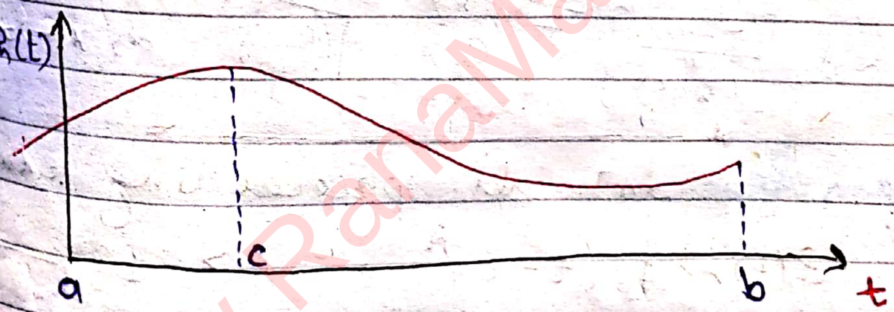
$$h(t) = h(c) + \frac{1}{2} h''(c)(t-c)^2 + \dots \longrightarrow (4)$$

because $h'(c) = 0$, Thus we have

$$I(x) \sim f_0 e^{x h(c)} \int_{c-\delta}^{c+\delta} (t-c)^{\lambda} e^{\frac{1}{2} x h''(c)(t-c)^2} dt \longrightarrow (5)$$

where δ is a positive number. Since only the immidiate nbhd of $t=c$ contributes to $I(x)$, δ can be replaced with ∞. The result is

$$I(x) \sim f_0 e^{x h(c)} \int_{-\infty}^{\infty} (t-c)^{\lambda} e^{\frac{1}{2} x h''(c)(t-c)^2} dt \longrightarrow (6)$$



A function with an absolute maxima at an interior point.

using the transformations

$$-\frac{1}{2} x h''(c)(t-c)^2 = \tau^2 \longrightarrow (7)$$

We express (6) as

$$I(x) \sim f_0 \left[\frac{-2}{x h''(c)}\right]^{\frac{\lambda+1}{2}} e^{x h(c)} \int_{-\infty}^{\infty} \tau^{\lambda} e^{-\tau^2} d\tau \quad \text{as } x \to \infty$$

If λ is an odd integer, the integral in (8) vanishes and one needs to determine the next term in the asymptotic development. If λ is an even integer

$$\int_{-\infty}^{\infty} \tau^{\lambda} e^{-\tau^2} d\tau = 2\int_0^{\infty} \tau^{\lambda} e^{-\tau^2} d\tau \longrightarrow (9)$$

using the transformation $\tau^2 = \theta$ (9) becomes

$$\int_{-\infty}^{\infty} \tau^{\lambda} e^{-\tau^2} d\tau = \int_0^{\infty} \theta^{(\lambda-1)/2} e^{-\theta} d\theta$$

$$= \Gamma\left(\frac{\lambda+1}{2}\right) \longrightarrow (10)$$

Hence (8) becomes

$$I(x) \sim f_0 \left[\frac{-2}{x h''(c)}\right]^{\frac{\lambda+1}{2}} e^{x h(c)} \Gamma\left(\frac{\lambda+1}{2}\right) \quad \text{as } x \to \infty$$

Example:- In all preceeding examples

$$f(t) \sim f_0 (t-c)^{\lambda}, \quad \lambda > -1 \longrightarrow (1)$$

Where c is the location of the maxima of $h(t)$. Here, we consider the case in which $f(t) \to 0$ as $t \to c$ faster than any power of $t-c$ so that it can not be represented as in (1)

Specifically, we consider

$$I(x) = \int_a^b e^{-1/(t-a)} e^{-x(t-a)^2} dt; \; a<b \longrightarrow (2)$$

For large +ve x.

Since $f(t) = \exp[-(t-a)^{-1}]$ tends

to zero much faster than any power of $(t-a)$, the contribution to the integral from immidiate nbhd of $t=a$ is exponentially small.
Consequently application of Watson's lemma directly to ② does not yields its asymptotic development.
Rather we should continue them and rewrite ② as

$$I(x) = \int_a^b e^{h(x,t)}\, dt \longrightarrow ③$$

where

$$h(x,t) = -(t-a)^{-1} - x(t-a)^2 \longrightarrow ④$$

The stationary value of $h(x,t)$ are located where

$$\frac{dh}{dt} = 0 = \frac{1}{(t-a)^2} - 2x(t-a) \longrightarrow ⑤$$

solving ⑤ yields

$$t = a + (2x)^{-1/3} \longrightarrow ⑥$$

For the location of maxima of $h(x,t)$. this location is a function of x, in contrast with the preceding example. Thus, to determine the asymptotic expansion of the integral, we first need to transform the variable of integration so that the maxi. of exponent is independent of x. Letting

$$t - a = x^{-\frac{1}{3}} s \longrightarrow (7)$$

rewrite (2) as

$$I(x) = \frac{1}{x^{1/3}} \int_0^{(b-a)x^{1/3}} e^{-x^{\frac{1}{3}}\left[s^2 + \frac{1}{s}\right]} ds \longrightarrow (8)$$

The maximum value of $h(s) = -(s^2 + s^{-1})$ in O at $s = 2^{-\frac{1}{3}}$. Hence, only the immidiate nbhd of this part contributes to the asymptotic expansion of $I(x)$ if b is greater than the location of the maxima of h.
Thus we expand $h(s)$ in Taylor series about this point and obtain

$$h(s) = h(2^{-1/3}) - 3(s - 2^{-\frac{1}{3}})^2 + \cdots \longrightarrow (9)$$

Then, we let

$$3x^{1/3}(s - 2^{-\frac{1}{3}})^2 = \tau^2 \longrightarrow (10)$$

in (8) and replace the upper and lower limits by $+\infty$ and $-\infty$, respectively. The result is

$$I(x) \sim \frac{e^{-\frac{3}{2}(2x)^{\frac{1}{3}}}}{\sqrt{3x}} \int_{-\infty}^{\infty} e^{-\tau^2} d\tau$$

or

$$I(x) \sim \left(\frac{\pi}{3x}\right)^{\frac{1}{2}} e^{-\left(\frac{3}{2}\right)(2x)^{\frac{1}{3}}} \longrightarrow (11)$$

Since

$$\int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}$$

Example:- Show that the as $x \to \infty$

$$\int_0^1 e^{-xt} \ln(2+t)\,dt \sim \frac{\ln 2}{x}$$

Solution For large +ve x, the major contribution to the asymptotic expansion of the integral arises from the nbhd of $t=0$. Thus using Watson's lemma, we expand $\ln(2+t)$ for small t as

$$\ln(2+t) = \ln\left[2\left(1+\frac{t}{2}\right)\right]$$
$$= \ln 2 + \ln(1+t/2)$$
$$= \ln 2 + \frac{1}{2}t + \cdots$$

Then

$$\int_0^1 e^{-xt} \ln(2+t)\,dt \sim \int_0^\infty e^{-xt} \ln 2\,dt$$
$$= \ln 2 \int_0^\infty e^{-xt}\,dt$$
$$= \ln 2 \left.\frac{e^{-xt}}{-x}\right|_0^\infty = \frac{\ln 2}{x}$$

Example:- Show that as $\omega \to \infty$

(i) $$\int_0^\infty \frac{e^{-x}}{\omega + x + x\sqrt{\omega}}\,dx \sim \frac{1}{\omega} - \frac{1}{\omega^{3/2}}$$

(ii) $$\int_0^\infty \frac{e^{-\omega x^2}}{\sqrt{x+x^2}}\,dx \sim \frac{\Gamma\left(\frac{1}{4}\right)}{2\,\omega^{1/4}}$$

Solution 1) The major contribution

to the integral arises from the nbhd of the origin. Since

$$(\omega+x+x\sqrt{\omega})^{-1}=\omega^{-1}\left(1+\frac{x+x\sqrt{\omega}}{\omega}\right)^{-1}$$

$$=\frac{1}{\omega}-\frac{x+x\sqrt{\omega}}{\omega^2}+\frac{(x+x\sqrt{\omega})^2}{\omega^3}+\cdots$$

using Watson's lemma we have

$$\int_0^\infty \frac{e^{-x}}{\omega+x+x\sqrt{\omega}}\,dx \sim \frac{1}{\omega}\int_0^\infty e^{-x}\,dx-\frac{1}{\omega^{3/2}}\int_0^\infty x\cdot e^{-x}\,dx$$

$$=\frac{1}{\omega}-\frac{\Gamma(2)}{\omega^{3/2}}=\frac{1}{\omega}-\frac{1}{\omega^{3/2}}$$

ii)

$$I(\omega)=\int_0^\infty \frac{e^{-\omega x^2}}{\sqrt{x+x^2}}\,dx$$

Since the exponent has its maximum at $x=0$, the major contribution to I arises from the nbhd of $x=0$. Hence applying Watson's lemma we have

$$I(\omega)=\int_0^\infty \frac{e^{-\omega x^2}}{x^{1/2}}(1+x)^{-\frac{1}{2}}\,dx \sim \int_0^\infty x^{-\frac{1}{2}}e^{-\omega x^2}\left(1-\frac{1}{2}x+\cdots\right)dx \longrightarrow (1)$$

We let

$$\omega x^2=\tau \text{ so that}$$

$$2\omega x\,dx=d\tau \longrightarrow (2)$$

It follows from (1) that

$$I(\omega) = \int_0^\infty x^{-\frac{1}{2}} e^{-\omega x^2} dx$$

$$= \int_0^\infty \frac{\omega^{\frac{1}{4}} \tau^{-\frac{1}{4}} e^{-\tau}}{2\omega \tau^{\frac{1}{2}} \omega^{-\frac{1}{2}}} d\tau$$

$$= \frac{1}{2\omega^{1/4}} \int_0^\infty \tau^{-\frac{3}{4}} e^{-\tau} d\tau$$

$$= \frac{\Gamma(1/4)}{2\omega^{1/4}}$$

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* The Method of Stationary Phase:-

The major contribution to the value of generalized Fourier integral

$$I(\alpha) = \int_a^b f(t)\, e^{i\alpha h(t)}\, dt \;;\; b>a$$

for large +ve α when $h(t)$ is real arises from the immidiate neighborhoods of the end points of the interval and the points at which $h(t)$ is stationary, that is, $h'(t)=0$. To the first approximation, the contribution from the nbhds of the stationary points is more important than the contribution from the nbhds of the end points of the interval.

Thus if $h'(t) \neq 0$ in $[a,b]$, an asymptotic expansion of this integral can be obtained by integration by parts. The result is

$$I(\alpha) = \frac{i}{\alpha}\left[\frac{f(a)\,e^{i\alpha h(a)}}{h'(a)} - \frac{f(b)\,e^{i\alpha h(b)}}{h'(b)}\right] + O\left(\frac{1}{\alpha^2}\right) \quad \textcircled{*}$$

as $\alpha \to \infty$

If $h'(c)=0$ where $a \leq c \leq b$, and $h'(t)$ does not vanish at any other point, The dominant term in the asymptotic expansion of this integral can be obtained by using the method of stationary ~~points~~ phase, according to which one keeps the first term in the expansions of $h(t)$ and $f(t)$ around $t=c$, that is,

$$I(\alpha) \sim f(c)e^{i\alpha h(c)} \int_{c-\delta}^{c+\delta} e^{i\alpha h''(c)(t-c)^2/2}\, dt; \text{ if } a\le c\le b$$

or

$$I(\alpha) \sim f(a)e^{i\alpha h(a)} \int_{a}^{a+\delta} e^{i\alpha h''(a)(t-a)^2/2}\, dt; \text{ if } c=a$$

or

$$I(\alpha) \sim f(b)e^{i\alpha h(b)} \int_{b-\delta}^{b} e^{i\alpha h''(b)(t-b)^2/2}\, dt; \text{ if } c=b \longrightarrow \boxed{3.163}$$

These integrals can be evaluated by using Cauchy's theorem and deforming the contour of integration so that the Fourier integral is transformed into a Laplace integral and the interval of integration is replaced by $[-\infty,\infty]$ when $a<c<b$, by $[0,\infty]$ when $a=c$ and by $[-\infty,0]$ when $c=b$

Note:- As in the case of generalized Laplace integral, the method of integration by parts fails if $h'(t)$ vanishes at any point in the interval $[a,b]$. If $h'(t)\neq 0$ in $[a,b]$, ⊛ shows that only the immediate nbhds of the end points contributes to the asymptotic development of

$I(\alpha)$. The rapid oscillations of $\exp[i\alpha h(t)]$ tend to cancel the contributions to the integral except in the nbhds of the end points as shown in Fig. Moreover both ends contribute to the asymptotic expansion in contrast with the generalized Laplace integral in which only the end with larger value of h contributes to the asymptotic expansion

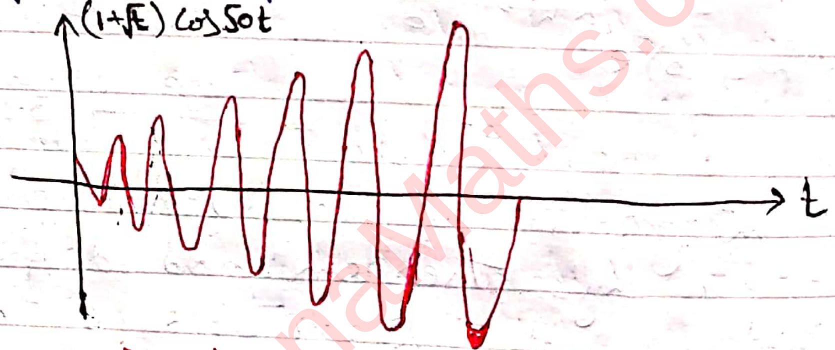

Fig :- A function with rapid oscillation

If $h'(t)$ vanishes in the interval (i.e the phase has stationary point), the contribution to the asymptotic expansion of the integral arises from the immidiate nbhds of the end and stationary points, with the major contribution arises from the nbhds of the stationary points, as evident from fig 2. The rapid osillation of $\exp[i\alpha h(t)]$ tend to cancel contributions to the integral except in the nbhds of the end and stationary points. Fig 2 shows clearly that there is less cancellation from the nbhd of a stationary point than from the nbhd of an end point. Hence, the leading terms to the asymptotic expansion of Fourier integral arise from the nbhds of stationary points.

In the absence of stationary points, the method of integration by parts yields a good approximation to the integral. As in ⊛, the principal contribution is $O(\alpha^{-1})$. In the presence of stationary points, Stokes and Kelvin developed the so-called method of stationary phase to determine the contribution of the nbhd of a stationary point $t=c$ to the asymptotic development of the integral by expanding $f(t)$ and $h(t)$ in powers of $t-c$.

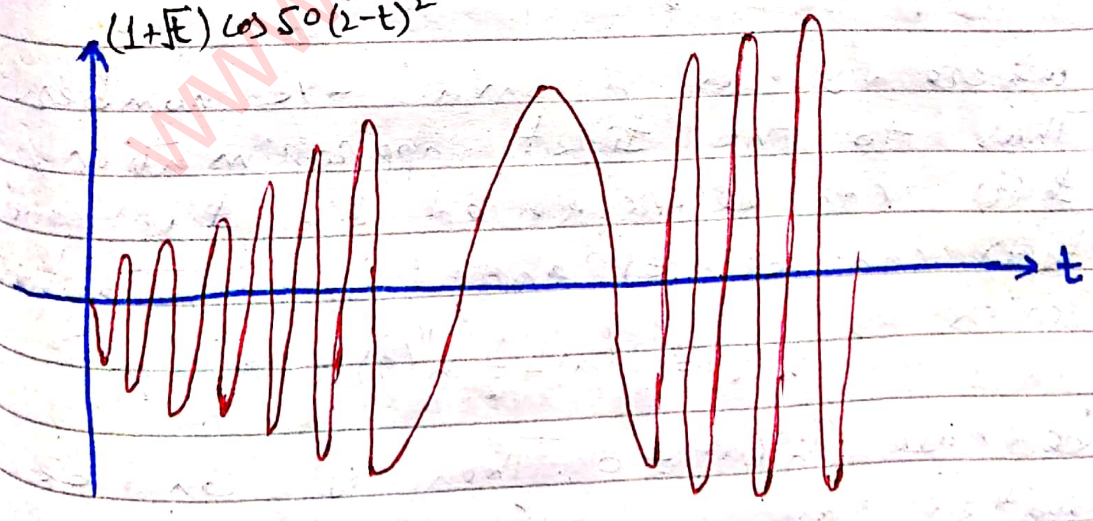

Fig 2:- A function with a stationary point

As shown the principal contribution

from the nbhd of a stationary point is $O(\alpha^{-1/2})$ and hence, only the stationary points contribute to the leading term in the asymptotic expansion of $I(\alpha)$.

Example 1:-

We begin with a case in which $h(t)$ has a stationary point at $t=a$ corresponding to a maximum or a minimum and it has no other stationary points. Moreover, we assume that $f(a)$ is finite. According to the Stoke's method of stationary phase, only the immediate nbhd of $t=a$ contributes to the leading term in the asymptotic expansion of $I(\alpha)$.

Hence

$$I(\alpha) \sim \int_a^{a+\delta} f(t)\, e^{i\alpha h(t)}\, dt \longrightarrow ①$$

where δ is a small +ve number.

Thus to the first approximation, $f(t)$ can be replaced by $f(a)$ and $h(t)$ can be expanded in a Taylor series as

$$h(t) = h(a) + \frac{1}{2} h''(a)(t-a)^2 + \cdots$$

because $h'(a) = 0$. Then ① can be written as

$$I(\alpha) \sim f(a)\, e^{i\alpha h(a)} \int_a^{a+\delta} e^{(1/2)(i\alpha h''(a)(t-a)^2)}\, dt \longrightarrow ②$$

Since only the immidiate nbhd of $t = a$ contributes to the integral, δ can be replaced with ∞. Letting $t - a = z$, we rewrite ② as

$$I(\alpha) \sim f(a) e^{i\alpha h(a)} \int_0^\infty e^{\left(\frac{1}{2}\right) i\alpha h''(a) z^2} dz \longrightarrow ③$$

To evaluate the integral in ③. we appeal to cauchy's theorem, which states that if the derivative of a function $f(z)$ of a complex variable z exists and is continuous inside and on a closed curve C (i.e F is analytic) then

$$\int_C F(z)\, dz = 0 \longrightarrow ④$$

Where the integration is carried around the closed curve C. The basic idea is to choose C in such a way that the original Fourier integral is transformed into a Laplace integral that is, the dominant part of the integrand is a real decaying exponential at ∞. In the case under consideration

$$F(z) = \exp\left[\tfrac{1}{2} i\alpha h''(a) z^2\right] \longrightarrow ⑤$$

and its derivative exists for all values of z. Hence, Cauchy's theorem applies and we have ④. To apply Cauchy's theorem, we take the closed curve

C to consist of the real axis x, a line making 45° to the real axis, and an eighth of a circle with the radius R, as shown in Fig 3. Hence

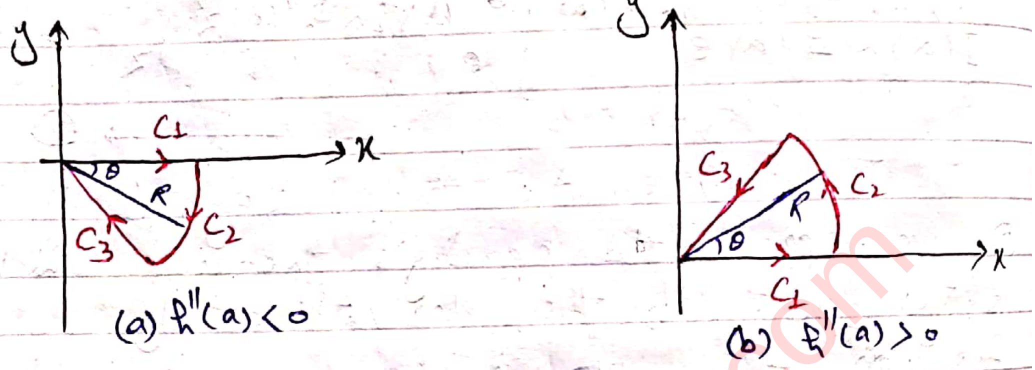

Fig 3:- Deformation of the Contour of integration.

$$\int_{C_1} F(z)\,dz + \int_{C_2} F(z)\,dz + \int_{C_3} F(z)\,dz = 0 \longrightarrow (6)$$

But on C_2

$$z = x + iy = R\cos\theta + iR\sin\theta = Re^{i\theta}$$

so that

$$z^2 = R^2 e^{2i\theta} = R^2\cos 2\theta + iR^2\sin 2\theta$$

Hence, in order that the contribution of C_2 as $R \to \infty$ vanishes, we must choose θ to be +ve or −ve depending on whether $h''(a)$ is positive or negative, respectively. Moreover, in order that the integral be converted into a Laplace integral, the angle of rotation must be $\frac{1}{4}\pi$ or $-\frac{1}{4}\pi$ depending on whether $h''(a)$ is +ve or −ve, respectively.

Hence it follows from Fig 3(b) that when $h''(a) > 0$

$$\int_{C_2} F(z)\,dz = iR\int_0^{(1/4)\pi} e^{-(\frac{1}{2})\alpha h''(a)R^2\sin 2\theta + (\frac{1}{2})i\alpha h''(a)R^2\cos 2\theta + i\theta}\,d\theta \longrightarrow (7)$$

Then

$$\left|\int_{C_2} F(z)\,dz\right| \leq R\int_0^{(1/4)\pi} e^{-(\frac{1}{2})\alpha h''(a)R^2\sin 2\theta}\,d\theta$$

$$= R\int_0^{\epsilon} e^{(-\frac{1}{2})\alpha h''(a)R^2\sin 2\theta}\,d\theta + R\int_{\epsilon}^{(\frac{1}{4})\pi} e^{-(\frac{1}{2})\alpha h''(a)R^2\sin 2\theta}\,d\theta \longrightarrow (8)$$

where ϵ is a small +ve number. Since $h''(a) > 0$ and $\sin 2\theta > 0$ in the interval $[\epsilon, \frac{1}{4}\pi]$, the integrand tends to zero uniformly, and hence, the last integral in (8) tends to zero there as $R \to \infty$. To estimate the integral over the interval $[0, \epsilon]$, we approximate $\sin 2\theta$ by 2θ and obtain

$$R\int_0^{\epsilon} e^{-(\frac{1}{2})\alpha h''(a)R^2\sin 2\theta}\,d\theta \approx R\int_0^{\epsilon} e^{-\alpha h''(a)R^2\theta}\,d\theta$$

$$= \frac{1 - e^{-\alpha h''(a)R^2\epsilon}}{\alpha R h''(a)} \to 0 \text{ as } R \to \infty$$

Therefore, the integral over the contour C_2 vanishes and it follows from (6) that

$$\int_{C_1} F(z)\,dz = -\int_{C_3} F(z)\,dz \quad \text{as } R \to \infty \longrightarrow (9)$$

Along the curve C_1, $z = x$ so that $z^2 = x^2$, where as along the curve C_3, $z = r\exp(\frac{1}{4}i\pi)$ so that $z^2 = r^2\exp(\frac{1}{2}i\pi) = ir^2$, where r is the distance from the origin to any point on C_3. Then, substituting (5) into (9) yields

$$\int_0^\infty e^{(\frac{1}{2})i\alpha h''(a)x^2}\,dx = -e^{(\frac{1}{4})i\pi}\int_\infty^0 e^{-(\frac{1}{2})\alpha h''(a)r^2}\,dr$$

$$= e^{(\frac{1}{4})i\pi}\int_0^\infty e^{-(\frac{1}{2})\alpha h''(a)r^2}\,dr \longrightarrow (10)$$

Equation (10) shows that the Fourier integral in the nbhd of a stationary point has been converted into a Laplace integral by rotating the contour of integration from the real x-axis by the angle $\frac{1}{4}\pi$. Making the substitution $\sqrt{\frac{1}{2}\alpha h''(a)}\, r = \tau$

in (10) we obtain

$$\int_0^\infty e^{(\frac{1}{2})i\alpha h''(a)x^2}\,dx = \frac{\sqrt{2}\, e^{(\frac{1}{4})i\pi}}{\sqrt{\alpha h''(a)}}\int_0^\infty e^{-\tau^2}\,d\tau = \frac{\sqrt{\pi}\, e^{(\frac{1}{4})i\pi}}{\sqrt{2\alpha h''(a)}} \longrightarrow (11)$$

using (11) in (3) yields

$$I(\alpha) \sim \frac{\sqrt{\pi}\, f(a)\, e^{i\alpha h(a) + (\frac{1}{4})i\pi}}{\sqrt{2\alpha h''(a)}} \longrightarrow (12)$$

As mentioned above, the contribution from the nbhd of a stationary point is $O(\alpha^{-1/2})$ compared with the order of $O(\alpha^{-1})$ contribution from an end point.

When $h''(a) < 0$, the contour of the integration needs to be rotated by the angle $-\frac{1}{4}\pi$ so that

$$z = r e^{-(\frac{1}{4})i\pi} \qquad z^2 = r^2 e^{-(\frac{1}{2})i\pi} = -ir^2$$

Then ③ becomes

$$I(\alpha) \sim f(a)\, e^{i\alpha h(a) - (\frac{1}{4})i\pi} \int_0^\infty e^{(\frac{1}{2})\alpha h''(a) r^2}\, dr \longrightarrow (13)$$

Which is a Laplace integral because $h''(a) < 0$. Making the substitution

$$\sqrt{-\frac{1}{2}\alpha h''(a)}\, r = \tau$$

we obtain from (13) that

$$I(\alpha) \sim \frac{\sqrt{\pi}\, f(a)\, e^{i\alpha h(a) - (\frac{1}{4})i\pi}}{\sqrt{-2\alpha h''(a)}} \quad \text{as } \alpha \to \infty \longrightarrow (14)$$

In integral, one needs to rotate the real axis by an angle θ so that $i\alpha h''(a) z^2$ in ③ becomes a $-$ve real number since

$$z = r e^{i\theta} \qquad z^2 = r^2 e^{2i\theta} = r^2\cos 2\theta + i r^2 \sin 2\theta$$

it follows that $\cos 2\theta = 0$ or $\theta = \pm\frac{1}{4}\pi$.

Then $z^2 = i r^2 \sin 2\theta$ and

$$i\alpha h''(a) z^2 = -\alpha r^2 h''(a) \sin 2\theta$$

Consequently, when $h''(a) > 0$, one

should take $\theta = \frac{1}{4}\pi$, where as when $h''(a) < 0$, one should take $\theta = -\frac{1}{4}\pi$ so that the integral in ③ becomes a Laplace integral.

Example 2:-

As a 2nd example, we consider a case in which $h(t)$ has a stationary point at $t = c$ where $a < c < b$. We assume that $h(t)$ has no other stationary points and that $f(c)$ is finite. Hence according to the Stoke's method of stationary phase, the leading term in the asymptotic expansion of $\boxed{3.163}$ arise from the immidiate neighborhood of $t = c$ and we write

$$I(\alpha) \sim \int_{c-\delta}^{c+\delta} f(t)\, e^{i\alpha h(t)}\, dt \longrightarrow (15)$$

where δ is a +ve small number. Hence $f(t)$ can be replaced with $f(c)$. Moreover, expanding $h(t)$ in a Taylor series around $t = c$, we have

$$h(t) = h(c) + \frac{1}{2} h''(c)(t-c)^2 + \cdots \longrightarrow (16)$$

because $h'(c) = 0$. substituting (16) into (15), replacing $f(t)$ with $f(c)$ and $t - c$ with z, and replacing δ with ∞, we obtain

$$I(\alpha) \sim f(c) e^{i\alpha h(c)} \int_{-\infty}^{\infty} e^{(\frac{1}{2}) i\alpha h''(c) z^2} dz \longrightarrow (17)$$

To evaluate the integral in (17) when $h''(c) > 0$, we rotate the contour of integration by the angle $\frac{1}{4}\pi$, so that $z = r \exp(\frac{1}{4} i\pi)$. The result is

$$I(\alpha) \sim f(c) e^{i\alpha h(c) + (\frac{1}{4}) i\pi} \int_{-\infty}^{\infty} e^{-(\frac{1}{2})\alpha h''(c) r^2} dr$$

We make the substitution

$$\sqrt{\tfrac{1}{2}\alpha h''(c)}\, r = \tau$$

& obtain

$$I(\alpha) \sim \frac{\sqrt{2} f(c) e^{i\alpha h(c) + (1/4) i\pi}}{\sqrt{\alpha h''(c)}} \int_{-\infty}^{\infty} e^{-\tau^2} d\tau$$

Hence

$$I(\alpha) \sim \frac{\sqrt{2\pi} f(c) e^{i\alpha h(c) + (\frac{1}{4}) i\pi}}{\sqrt{\alpha h''(c)}} \quad \text{as } \alpha \to \infty \longrightarrow (18)$$

When $h''(c) < 0$, we rotate the contour of integration by the angle $-\frac{1}{4}\pi$, make the substitution

$$\sqrt{-\tfrac{1}{2}\alpha h''(c)}\, r = \tau$$

and obtain

$$I(\alpha) \sim \frac{\sqrt{2\pi} f(c) e^{i\alpha h(c) - (\frac{1}{4}) i\pi}}{\sqrt{-\alpha h''(c)}} \quad \text{as } \alpha \to \infty \longrightarrow (19)$$

Example 3:-

As a third example, we consider a case in which $h(t)$ has a stationary point at $t=a$, no other stationary point, and

$$h'(a) = h''(a) = \dots = h^{(n-1)}(a) = 0$$

but $h^{(n)}(a) \neq 0$. We assume that $f(a)$ is finite. We expand $h(t)$ in Taylor series and obtain

$$h(t) = h(a) + \frac{1}{n!} h^{(n)}(a)(t-a)^n + \cdots \quad (20)$$

Then it follows that

$$I(\alpha) \sim f(a) e^{i\alpha h(a)} \int_0^\infty e^{i\alpha h^{(n)}(a)(t-a)^n/n!} \, dt \quad (21)$$

Letting $t-a = z$ we rewrite (21) as

$$I(\alpha) \sim f(a) e^{i\alpha h(a)} \int_0^\infty e^{i\alpha h^{(n)}(a) z^n/n!} \, dz \quad (22)$$

To evaluate the integral in (22), we convert it into a laplace integral by rotating the contour of integration by the angle $\pi/2n$ when $h^{(n)}(a) > 0$ so that $z = r \exp(i\pi/2n)$. The result is

$$I(\alpha) \sim f(a) e^{i\alpha h(a) + i\pi/2n} \int_0^\infty e^{-\alpha h^{(n)}(a) r^n/n!} \, dr \quad (23)$$

The integral (23) can be expressed in terms of the gamma function by making the substitution

$$\frac{\alpha h^{(n)}(a) r^n}{n!} = s$$

& obtain $\int_0^\infty e^{-\alpha h^{(n)}(a) r^n / n!} dr = \frac{1}{n}\left[\frac{n!}{\alpha h^{(n)}(a)}\right]^{1/n} \int_0^\infty s^{(1/n)-1} e^{-s} ds$

$$= \left[\frac{n!}{\alpha h^{(n)}(a)}\right]^{1/n} \frac{\Gamma\left(\frac{1}{n}\right)}{n} \longrightarrow (24)$$

Hence (23) can be written as

$$I(\alpha) \sim \left[\frac{n!}{\alpha h^{(n)}(a)}\right]^{1/n} \frac{f(a)\,\Gamma\left(\frac{1}{n}\right)}{n} e^{i\alpha h(a) + i\pi/2n} \quad \text{as } \alpha \to \infty \longrightarrow (25)$$

When $h^{(n)}(a) < 0$, the contour of integration needs to be rotated from the real axis by the angle $-\pi/2n$. Then following steps similar to those above, one obtains

$$I(\alpha) \sim \left[\frac{n!}{-\alpha h^{(n)}(a)}\right]^{1/n} \frac{f(a)\,\Gamma\left(\frac{1}{n}\right)}{n} e^{i\alpha h(a) - i\pi/2n} \quad \text{as } \alpha \to \infty \longrightarrow (26)$$

We note from (25) that the leading term in the contribution of a nbhd of a stationary point to the asymptotic expansion is $O(\alpha^{-1/n})$ where n corresponds to the order of lowest derivative of h that does not vanish at the stationary point. Therefore, if $h(t)$ has many stationary points in $[a, b]$, the leading term to the asymptotic expansion of $\boxed{3.163}$ arises from the nbhd of the stationary points corresponding to the largest value of n, the leading term in the asymptotic expansion of $\boxed{3.163}$ can be obtained as the summation of the leading contributions from the nbhds of these points.

Example 4:-

In all preceding examples, f(t) is finite at the stationary point. In this example, ~~we~~ we consider the preceding example when

$$f(t) \sim f_0 (t-a)^{\lambda} \quad \text{as } t \to a \longrightarrow (27)$$

Where λ must be greater than -1 for the integral $\boxed{3.163}$ to exist. Substituting for h(t) and f(t) from (20) and (27) into $\boxed{3.163}$ and replacing b with ∞ we, obtain

$$I(\alpha) \sim f_0 e^{i\alpha h(a)} \int_a^{\infty} (t-a)^{\lambda} e^{i\alpha h^{(n)}(a)(t-a)^n/n!} dt \longrightarrow (28)$$

Where λ is assumed to be less than $n-1$ for (28) to exist. Letting $t-a=z$ we rewrite (28) as

$$I(\alpha) \sim f_0 e^{i\alpha h(a)} \int_0^{\infty} z^{\lambda} e^{i\alpha h^{(n)}(a) z^n/n!} dz \longrightarrow (29)$$

As before, when $h^{(n)}(a) > 0$ we transform (29) into a Laplace integral by rotating the contour of integration from the real axis by the angle $\pi/2n$ so that $z = r\exp(i\pi/2n)$. The result is

$$I(\alpha) \sim f_0 e^{i\alpha h(a) + i(\lambda+1)\pi/2n} \int_0^\infty r^\lambda e^{-\alpha h^{(n)}(a) r^n/n!} \, dr \longrightarrow (30)$$

The integral in (30) can be expressed in terms of gamma function by making the substitution

$$\frac{\alpha h^{(n)}(a) r^n}{n!} = s$$

& obtaining

$$\int_0^\infty r^\lambda e^{-\alpha h^{(n)}(a) r^n/n!} \, dr = \frac{1}{n}\left[\frac{n!}{\alpha h^{(n)}(a)}\right]^{(\lambda+1)/n} \int_0^\infty s^{-1+(\lambda+1)/n} e^{-s} ds$$

$$= \left[\frac{n!}{\alpha h^{(n)}(a)}\right]^{(\lambda+1)/n} \frac{\Gamma\left(\frac{\lambda+1}{n}\right)}{n} \longrightarrow (31)$$

Substituting (31) into (30) yields

$$I(\alpha) \sim \left[\frac{n!}{\alpha h^{(n)}(a)}\right]^{(\lambda+1)/n} \frac{f_0 \Gamma\left(\frac{\lambda+1}{n}\right)}{n} e^{i\alpha h(a) + i(\lambda+1)\pi/2n} \quad \text{as } \alpha \to \infty \longrightarrow (32)$$

When $h^{(n)}(a) < 0$, rotating the contour of integration by the angle $-\pi/2n$ and followings steps similar to those above, we obtain,

$$I(\alpha) \sim \left[\frac{n!}{-\alpha h^{(n)}(a)}\right]^{(\lambda+1)/n} \frac{f_0\,\Gamma\!\left(\frac{\lambda+1}{n}\right)}{n} e^{i\alpha h(a) - i(\lambda+1)\pi/2n} \quad \text{as } \alpha \to \infty \longrightarrow (33)$$

We note that (32) and (33) tend to (25) and (28) as $\lambda \to 0$. We note from (32) that if $\lambda + 1 > n$, then the leading contribution to the asymptotic expancion of the integral arises from the end point $t = b$.

* The Method of Steepest Descent :-

The Laplace's method enables us to deal with the integral of the form $\int_a^b f(t)\, e^{\alpha h(t)}\, dt$; $b>a$, where the coefficient of α in the exponent is real.

The method of stationary phase enables us to deal with the integral of the form $\int_a^b f(t)\, e^{i\alpha h(t)}\, dt$; $b>a$, where the coefficient of α in the exponent is purely imaginary.

In this section, the method of steepest Descent is used to determine approximations for large positive values of α to integral of the form

$$I(\alpha) = \int_C f(z)\, e^{\alpha h(z)}\, dz$$

when $f(z)$ and $h(z)$ are analytic functions of z and C is a contour of integration in the complex z plane. According to this method, one uses the analyticity of the integrand and appeals to Cauchy's theorem to deform the contour of integration C into a new contour C' on which the phase $\psi(x,y)$ is a constant, where $h(z) = \phi(x,y) + i\,\psi(x,y)$ and $z = x+iy$, thereby transforming the integral into a Laplace ~~method~~ integral whose asymptotic development can be obtained using Laplace's Method.

Note:- To determine the asymptotic development of $I(\alpha)$, we use the analyticity of the integrand and appeal to Cauchy's theorem to deform the contour of integration C into a new contour C' on which either ψ or ϕ is constant. Thereby transforming the integral into either a Fourier or Laplace integral. Then, the asymptotic development can be determined by using either the method of stationary phase or Laplace Method. It is preferable to transform the integral into a Laplace integral (i.e constant ψ) because the full asymptotic development of a Laplace integral arises only from the immediate nbhd of the point where the ϕ is largest on C'. In contrast, the full asymptotic development of a Fourier integral depends in general, on the end points as well as all stationary points of ψ on C'. We note that constant-phase contours are steepest descent and ascent contours.

A saddle point of $\phi(x,y)$ is also a saddle point of $\psi(x,y)$ as well as a point where $h'(z)=0$. If $z=z_0$ is a saddle point and if

$$h'(z_0) = h''(z_0) = \cdots = h^{(m)}(z_0) = 0,$$

we call $z=z_0$ a saddle point of order $m+1$.

Through a saddle point z_0, There

are two or more level curves (curves of constant ϕ), seprating the nbhd of the saddle point into sectors. Moreover, Through a saddle point, there are two or more constant-phase contours (curves of constant ψ), which are the steepest paths through the saddle point. Some of these steepest paths descend, where the others ascend.

In the simplest case, the saddle point is of order 2 so that there are two steepest descent contours and steepest ascent contours as shown in fig 1. Moreover there are four constant level curves, seprating the nbhd of the saddle point into two valleys as shown in fig 1.

For a saddle point of order 3, there are six constant-level ~~curves~~ contours, seprating the nbhd of the saddle point into three hills and three valleys. Moreover there are three steepest decent contours in the valleys and three steepent ascent contours in the hills as shown in fig 2.

The above discussion shows that an effective method of determining the asymptotic development of integral whose end points lies into two different valleys is the Method of Steepest descent developed by Reiman and Debye.

It consists of deforming the contour of integration C into a new contour C' such that:

1:- The contour of integration passes through a zero of $h'(z)$.

2:- The imaginary part ψ of $h(z)$ is constant on the contours.

3:- The contour is that of steepest Descent.

If the restrictions on the deformed contour are such that it passes through more than one saddle point, each will make its contribution to the integral with the main contribution arising from the one corresponding to the large ϕ. If $h'(z)$ does not vanish, the contour of integration chosen to satisfy the second and third conditions only.

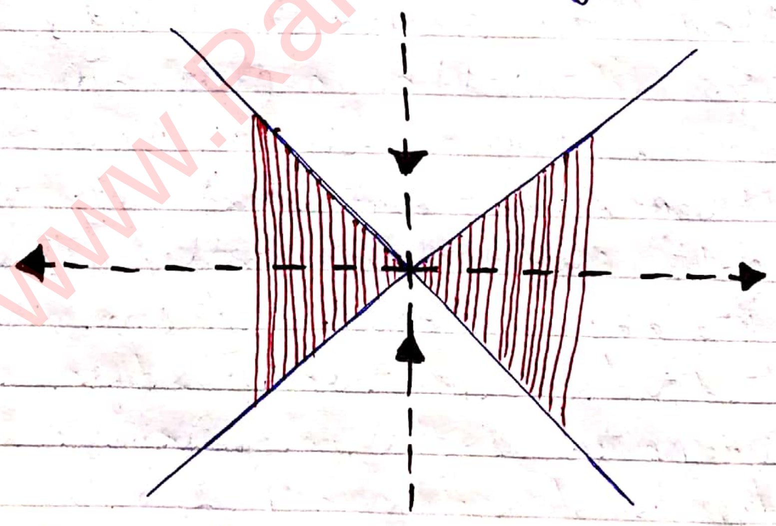

Fig 1:- Topography of the surface $\phi = Re\ h(z)$ near a saddle point of order 2. The valleys are shaded, the steepest curves doted, the solid lines are the level curves and the arrows indicate the direction in which ϕ decreases.

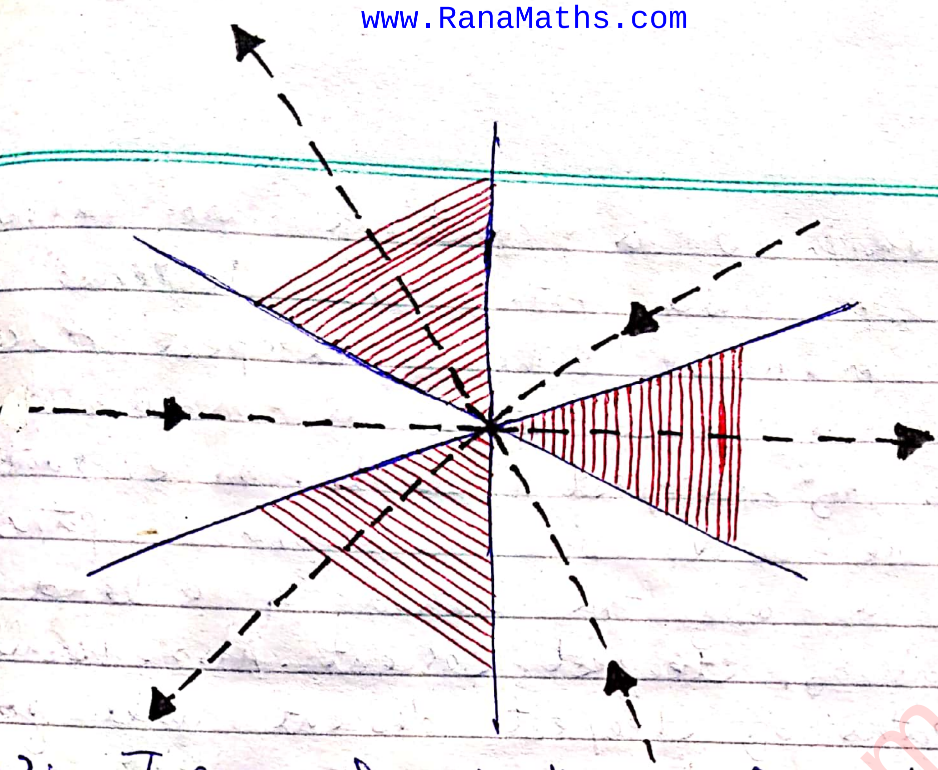

Fig 2:- Topography of the surface $\phi = \mathrm{Re}\, h(z)$ near a saddle point of order 3.

Note 2:- The basic idea of Method of Steepest Descent is to utilize the analyticity of the integrand to justify deforming the contour C to a new contour C' on which $h(z)$ has a constant imaginary point. Thus, if $h(z) = u + iv$, the integral $I(\alpha)$ becomes ($\mathrm{Im}\, h(z) = v =$ constant)

$$I(\alpha) = e^{i\alpha v} \int_{C'} f(z)\, e^{\alpha u}\, dz \longrightarrow (*)$$

Although z is complex, u is real and hence the idea used in connection with Laplace type integrals can be used to study Equ $(*)$. In this sence the Method of steepest descent is an extension of Laplaces method to integrals in the complex plane.

It turns out that paths

on which v is constant are also paths for which either the decrease of u is maximal (Paths of steepest descent) or the increase of u is maximal (Paths of steepest ascent). In evaluating $I(\alpha)$ we will use the former paths, which is why this method is called the method of steepest-Descent. Also, usually the paths of steepest descent will go through a point z_0 for which $h'(z_0)=0$. Such a point is called a saddle point and the method is alternatively referred to as the saddle point method.

We note that we could consider deforming C into a path for which u rather than v is constant (so that v varies rapidly) and then apply an extension of the method of stationary phase. However, we expect intuitively that the self cancelling of oscillations is a weaker decay of the exponential factor in the integrand. This is indeed true, which is why the asymptotic expansion of a generalized Laplace integral can be found locally.

In order to develop the method of steepest descent, we need to better understand the relationship between steepest paths and v equal costant and to integrate the direction of these paths around saddle points.

Steepest Paths :-

Let $\phi(z) = u(x,y) + iv(x,y)$, $z = x+iy$. Consider a point z_0 in the complex z plane. A direction away from z_0 in which u is decreasing is called a direction of descent. The direction on which the decrease is maximal is called direction of steepest descent. Similar considerations apply for direction of ascent.

We recall that if $f(x,y)$ is a differentiable function of two variables, then the gradient of f is the vector $\nabla f = (\partial f/\partial x, \partial f/\partial y)$. This vector points in the direction of the most rapid change of f at the point (x,y). Thus for any point $z_0 = x_0 + iy_0$ corresponding to $u(z_0) = u(x_0, y_0)$ at which $\nabla u \neq 0$, the direction of steepest ascent coincides with that of ∇u, with u increasing away from $u(z_0)$, while the direction of steepest descent coincides with that of $-\nabla u$, with u decreasing away from $u(z_0)$

Saddle Points :-

We say that the point z_0 is a saddle point of order N if the first N derivatives vanish, or alternatively, letting $n=N+1$

$$\left.\frac{d^m\phi}{dz^m}\right|_{z=z_0} = 0, \quad m=1,\cdots,n-1,$$

$$\left.\frac{d^n\phi}{dz^n}\right|_{z=z_0} = a e^{i\alpha}, \quad a>0 \longrightarrow (1)$$

When only the first derivative vanish $(N=1)$, we simply say that z_0 is a saddle point, or a "simple" saddle point, and omit the phrase "of order one".

If the 1st $n-1$ derivatives vanish, then we can show that for such points there exist n directions of steepest descent and n directions of steepest ascent.

If $z-z_0 = \rho e^{i\theta} \longrightarrow (2)$

These directions are given by

Steepest Descent direction: $\theta = -\alpha/n + (2m+1)\pi/n$,

" Ascent " : $\theta = -\alpha/n + 2m\pi/n$

$$m = 0,1,\cdots,n-1 \longrightarrow (3)$$

Where α is defined in fig 1.

For $n=2$, that is, for a simple saddle point Equ (3) imply

Descent : $\theta = -\frac{\alpha}{2} + \frac{\pi}{2}$, $\theta = -\frac{\alpha}{2} + \frac{3\pi}{2}$

Ascent : $\theta = -\frac{\alpha}{2}$, $\theta = -\frac{\alpha}{2} + \pi$ $\longrightarrow (4)$

Example:- Find the directions of steepest descent at the saddle point for the functions $\phi(z)$ given below and points z_0 indicated.

(a) $\phi(z) = z - z^3/3$. Therefore $\phi'(z) = 1 - z^2$, $\phi''(z) = -2z$. There exist two simple saddle points at $z_0 = 1$ and $z_0 = -1$

For $z_0 = 1$, $\phi''(1) = -2 = 2e^{i\pi}$, that is $\alpha = \pi$.

Hence equ ④ imply $\theta = 0, \pi$

For $z_0 = -1$, $\phi''(-1) = 2$, that is $\alpha = 0$

Hence equ ④ imply $\theta = \pi/2, 3\pi/2$.

(b) $\phi(z) = i\cosh z$

Therefore $\phi'(z) = i\sinh z$, $\phi''(z) = i\cosh z$

Consider $z_0 = 0$

$\phi''(0) = i = e^{i\pi/2}$, that is, $\alpha = \pi/2$

Hence equ ④ imply $\theta = \pi/4, 5\pi/4$

Bessel's Integrals:-

The integral representation of Bessel function, for integer values of n is:

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\tau - x\sin\tau)\,d\tau \longrightarrow ①$$

Another integral representation is

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\tau - x\sin\tau)}\,d\tau \longrightarrow ②$$

Another way to define the Bessel functions is the poisson representation formula:

$$J_k(z) = \frac{(z/2)^k}{\Gamma(k+\frac{1}{2})\,\Gamma\frac{1}{2}} \int_{-1}^{1} e^{izs} (1-s^2)^{k-\frac{1}{2}}\, ds \longrightarrow ③$$

where $k > \frac{-1}{2}$ and z is a complex number. This formula is useful especially when working with Fourier Transforms.

Example:- Find the complete asymptotic expansion of

$$I(k) = \int_0^1 \log t \; e^{ikt}\, dt, \quad \text{as } k \to \infty$$

Solution

A direct application of the method of stationary phase fails because there is no stationary points. Also, integration by parts fails because $\log t$ diverges at $t=0$. We solve this problem by the method of steepest descent, which also shows how, by deforming the contour, a fourier type integral can be mapped onto a Laplace type integral. Let us replace the real variable t by the complex variable z. Note that

$$h(z) = iz = i(x+iy) = -y+ix \longrightarrow ①$$

Clearly there does not exist any saddle point ($h'(z)=i$); we will see that the dominant contribution comes from the end points. The steepest paths $\operatorname{Im} h(z) =$ constant are given by $x =$ constant; if $y > 0$, these paths are paths of steepest descent. ~~they~~ Thus, $x = 0,\ y > 0$ and $x = 1,\ y > 0$ are the paths of steepest descent going through the end points. We note that $\operatorname{Im} h(0) \neq \operatorname{Im} h(1)$. Hence, there is no continuous contour joining $t = 0$ and $t = 1$ on which $\operatorname{Im} h(z)$ is constant. We connect the two steepest paths by the contour C_2 and use Cauchy theorem to deform the contour $[0, 1]$. (Since $t = 0$ is an integrable singularity we shall allow the contour to pass through the origin)

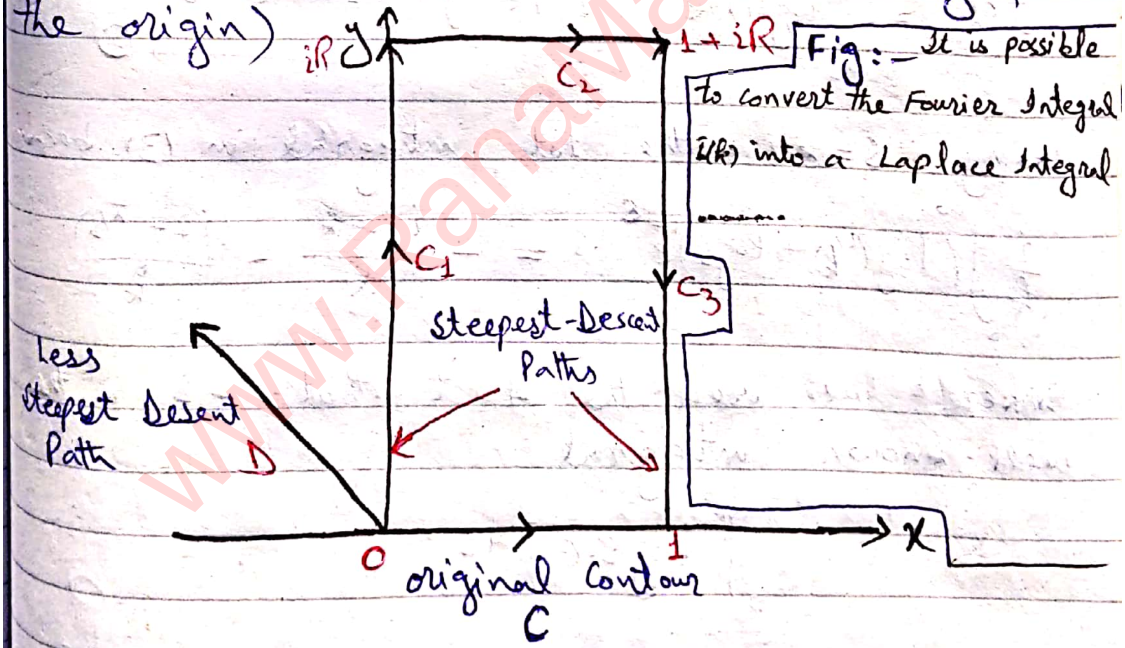

Fig: It is possible to convert the Fourier Integral $I(k)$ into a Laplace Integral ......

Note that $i = e^{i\pi/2}$

$$I(k) = \int_{C_1 + C_2 + C_3} \log z \, e^{ikz} \, dz$$

$$= i \int_0^R \log(iz)\, e^{-kz} \, dz + \int_0^1 \log(x + iR)\, e^{ikx - kR} \, dx -$$

$$i e^{ik} \int_0^R \log(1+ir) e^{-kr} \, dr \longrightarrow ②$$

It is possible to convert Fourier Integral $I(k)$ into a Laplace integral merely by deforming the original contour C into $C_1 + C_2 + C_3$ as shown above and then allowing $R \to \infty$. C_1 and C_3 are called steepest descent paths because $|\exp(\alpha h(z))|$ decreases most rapidly along these paths as t moves up from x-axis; $|\exp[\alpha h(z)]|$ also decreases along D, but less rapidly per unit length then along C_1.

Letting $R \to \infty$ we obtain

$$I(k) = i \int_0^\infty \log(ir) e^{-kr} \, dr - i e^{ik} \int_0^\infty \log(1+ir) e^{-kr} \, dr \longrightarrow ③$$

using $s = kr$, the 1st integral in (3) becomes

$$\frac{i}{k} \int_0^\infty \left(\log(i/k) + \log s\right) e^{-s} \, ds = -\frac{i \log k}{k} - \frac{(i\gamma + \pi/2)}{k}$$

Where we used the fact that $-\gamma$ is the well-known integral

$$\int_0^\infty \log s \cdot e^{-s} \, ds$$

(and the Euler constant, γ, equal $0.577216 \cdots$)

To compute the 2nd integral in ③, we use the Taylor expansion

$$\log(1+ir) = -\sum_{n=1}^\infty \frac{(-ir)^n}{n}$$

and Watson's lemma.
Thus as $k \to \infty$, the complete asymptotic expansion of the second integral is:

$$i e^{ik} \sum_{n=1}^{\infty} \frac{(-i)^n (n-1)!}{k^{n+1}}, \text{ as } k \to \infty$$

Adding these two contributions we find

$$I(k) \sim -\frac{i \log k}{k} - \frac{i\gamma + \pi/2}{k} + i e^{ik} \sum_{n=1}^{\infty} \frac{(-i)^n (n-1)!}{k^{n+1}}, k \to \infty$$

Example 2:- Consider the following integral representation of Bessel's function of the first kind and zeroth order:

$$J_0(\alpha) = \frac{1}{\pi} \int_{-1}^{1} \frac{e^{i\alpha z}}{\sqrt{1-z^2}} dz \text{ for large } \alpha \quad \text{①}$$

Integration by parts fails in this case because the choice

$$u = e^{i\alpha z} \; ; \; dv = (1-z^2)^{-1/2} dz$$

leads to a non-asymptotic expansion, where as the choice

$$u = (1-z^2)^{-1/2} \; ; \; dv = e^{i\alpha z} dz$$

leads to singular expansion at $z = \pm 1$.

To determine an approximate expansion for $J_0(\alpha)$ for large α by using the method of steepest descent, we deform the contour of integration into a constant-phase contour. We note that $h(z) = iz$ so that the phase ψ at $z = 1$

is 1, whereas the phase at $z=-1$ is -1. Thus, the contour can not be continuously deformed into a single contour along which the phase is constant. However, we can deform the contour into one that consists of three line segments: C_1, which runs up from -1 to $-1+iY$ along a straight line parallel to the y-axis; C_2, which runs parallel to x-axis from $-1+iY$ to $1+iY$; and C_3, which runs down from $1+iY$ to 1 along a straight line parallel to the y-axis, as shown below

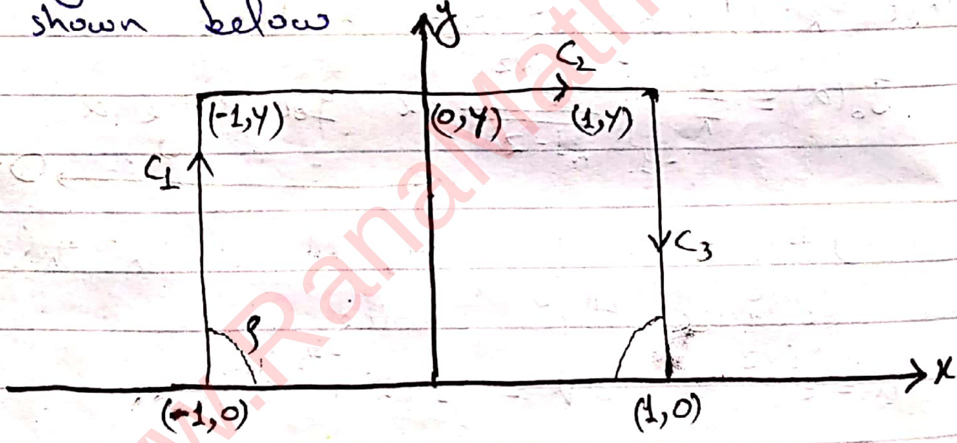

Fig:- Contour of deformation for ①.

By Cauchy theorem

$$\int_{C_1+C_2+C_3} \frac{e^{i\alpha z}}{\sqrt{1-z^2}}\,dz = \int_{-1}^{1} \frac{e^{i\alpha z}}{\sqrt{1-z^2}}\,dz \longrightarrow ②$$

Strictly speaking, the contour of integrat must be deformed as shown above to avoid the so-called branch singularities at $x=\pm 1$. However, in the limit as $\delta \to 0$ the integral tends to that given in (2). Hence, we will not worry about such border branch points in what follows. As $Y \to \infty$, the integral along C_2 vanishes because the integrand vanishes uniformly there. Then,

$$J_0(\alpha) = \frac{1}{\pi}\int_{-1}^{-1+i\infty} \frac{e^{i\alpha z}}{\sqrt{1-z^2}}\,dz + \frac{1}{\pi}\int_{1+i\infty}^{1} \frac{e^{i\alpha z}}{\sqrt{1-z^2}}\,dz \longrightarrow (3)$$

Making the substitution $z=-1+iy$ in the first integral and $z=1+iy$ in the 2nd integral, we rewrite (3) as

$$J_0(\alpha) = \frac{ie^{-i\alpha}}{\pi}\int_0^\infty \frac{e^{-\alpha y}}{\sqrt{2iy+y^2}}\,dy + \frac{ie^{i\alpha}}{\pi}\int_\infty^0 \frac{e^{-\alpha y}}{\sqrt{-2iy+y^2}}\,dy$$

or

$$J_0(\alpha) = \frac{ie^{-i\alpha-i\pi/4}}{\sqrt{2}\,\pi}\int_0^\infty e^{-\alpha y}\, y^{-1/2}\left(1-\frac{1}{2}iy\right)^{-1/2} dy$$

$$-\frac{ie^{i\alpha+i\pi/4}}{\sqrt{2}\,\pi}\int_0^\infty e^{-\alpha y}\, y^{-1/2}\left(1+\frac{1}{2}iy\right)^{-1/2} dy \longrightarrow (4)$$

The integrals in (4) are Laplace integrals and only the immidiate nbhds of $y=0$ contributes to their asymptotic developments for large α. Thus, using Watson's lemma, we expand the non-exponential parts of the integrands for small y and integrate the result term by term. The leading term in the asymptotic development of $J_0(\alpha)$ is

$$J_0(\alpha) \sim \frac{e^{-i\alpha + i\pi/4}}{\sqrt{2}\,\pi} \int_0^\infty y^{-1/2} e^{-\alpha y}\, dy + \frac{e^{i\alpha - i\pi/4}}{\sqrt{2}\,\pi} \int_0^\infty y^{-1/2} e^{-\alpha y}\, dy$$

which, with the substitution $\alpha y = t$, becomes

$$J_0(\alpha) \sim \frac{1}{\sqrt{2\alpha}} \frac{1}{\pi} \left[e^{-i\alpha + i\pi/4} + e^{i\alpha - i\pi/4} \right] \int_0^\infty t^{-1/2} e^{-t}\, dt$$

$$\sim \sqrt{\frac{2}{\alpha\pi}} \cos\left(\alpha - \frac{\pi}{4}\right) \text{ as } \alpha \to \infty$$

Singular Perturbation :-

Straight Expansions & Sources of Non-uniformity:-

It is rule rather than the exception that expansions of the poincare type (straightforward expansions), such as

$$\lim_{\epsilon \to 0} \frac{\delta_{m+1}(\epsilon)}{\delta_m(\epsilon)} = 0$$

$$f(x,\epsilon) \sim \sum_{m=0}^{\infty} \delta_m(\epsilon) f_m(x)$$

$$= f_0(x)\delta_0(\epsilon) + f_1(x)\delta_1(\epsilon) + f_2(x)\delta_2(\epsilon) + \cdots$$

Where $\delta_m(\epsilon)$ is an asymptotic sequence in terms of the parameter ϵ, are non-uniformly valid and break down in regions called regions of non-uniformity. Some of the sources of non-uniformity are:

Infinit Domain, (x becomes very large) small Parameter multiplying the highest derivative, type change of a PDE, and the presence of singularities.

In the infinite domain case, the non-uniformity manifest itself in the presence of socalled secular terms such as $x^n \cos x$ and $x^n \sin x$, which makes $f_m(x)/f_{m-1}(x)$ unbounded as x approaches infinity.

In the case of small Parameter multiplying the highest derivative, the perturbation expansion can not satisfy all the boundary and the initial conditions and the expansion thus is not valid in boundary or initial layers. Since the boundary and initial conditions required to form a well-posed problem depend on the type of the PDE under considerations, non-uniformaties might arise if the type of the perturbation equations is different from the type of the original equation. In the fourth class, singularities that are not part of the exact solution appear at some point in the expansion, generally becomes more pronounced in succeeding terms.

Example:- Consider the oscillations of a mass connected to a non-linear spring described by Duffing's equation

$$\left.\begin{aligned} &\ddot{x}+k^2x+\epsilon x^3=0, \\ \text{with } &x(0)=0,\ \dot{x}(0)=ka, \end{aligned}\right\} \longrightarrow (1)$$

Where ϵ is a small +ve number.

Let us seek an approximate solution in the form

$$x = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \longrightarrow (2)$$

using (2) in (1) and equating the co-efficients of like powers of ϵ, we have

$$\epsilon^0: \ddot{x}_0 + k^2 x_0 = 0, \quad x_0(0)=0,\ \dot{x}_0(0)=ka, \longrightarrow (3)$$

$$\epsilon^1: \ddot{x}_1 + k^2 x_1 = -x_0^3, \quad x_1(0)=0,\ \dot{x}_1(0)=0 \longrightarrow (4)$$

$\epsilon^2: \ddot{x}_2 + k^2 x_2 = -3x_0^2 x_1 ; \quad x_2(0)=0, \ \dot{x}_2(0)=0 \longrightarrow (5)$

The general solution of (3) is

$$x_0 = A_1 \cos kt + A_2 \sin kt$$

$x_0(0) = 0 \quad \Rightarrow A_1 = 0$ & so

$$x_0 = A_2 \sin kt$$

and $\quad \dot{x}_0 = A_2 k \cos kt$

$\therefore \ \dot{x}_0(0) = ka \implies A_2 = a$

Hence $\quad x_0 = a \sin kt \longrightarrow (6)$

using (6) in (4) we obtain

$$\ddot{x}_1 + k^2 x_1 = -a^3 \sin^3 kt$$

As $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

Therefore, the above becomes (by the variation of parameters)

$$\ddot{x}_1 + k^2 x_1 = -a^3\left[\frac{3}{4}\sin kt - \frac{1}{4}\sin 3kt\right]$$

with $x_1(0) = 0, \ \dot{x}_1(0) = 0 \longrightarrow (*)$

The solution of above problem is

$$x_1 = \frac{-a^3}{32k^2}\left(\sin 3kt + 9\sin kt\right) + \frac{3a^3}{8k} t\cos kt \longrightarrow (7)$$

From (2), (6) and (7) we get

$$x = a\sin kt - \frac{\epsilon a^3}{32k^2}\left(\sin 3kt + 9\sin kt\right) + \frac{3\epsilon a^3}{8k} t\cos kt$$

or

$$x = a\sin kt - \frac{\epsilon a^3}{32k^2}\left[\sin 3kt + 9\sin kt - 12kt\cos kt\right] \longrightarrow (8)$$

We note that coefficient of ϵ (i.e $x_1(t)$ is not bounded for all t. The term $t\cos kt$ is called a secular term. Moreover this secular term in $x_1(t)$ will propagate itself to higher order terms in the perturbation expansion so that none of the x_k's is bounded for all t. As a result no approximate solution can be obtained through truncation of the series i.e perturbation breaks down.

To get rid of the secular terms and obtain a regular perturbation expansion we use "straining methods". These consists (in their simplest form) of applying a perturbation expansion on both the independent and dependent variables, i.e

$$t = \tau + \epsilon f_1(\tau) + \epsilon^2 f_2(\tau) + \cdots \longrightarrow (9)$$

$$x(\tau) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \cdots \longrightarrow (10)$$

with $f_i(0) = 0$ (so that $t = 0$ will correspond to $\tau = 0$)

Eqn (9) is too general (in most cases) and is replaced by

$$t = \tau\left[1 + \epsilon b_1 + \epsilon^2 b_2 + \cdots\right] \longrightarrow (11)$$

where the b_i's are constant to be determined so that the secular terms disappear from the perturbation expansion. Thus

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} \longrightarrow (12)$$

From (11)

$$\frac{dt}{d\tau} = 1 + \epsilon b_1 + \epsilon^2 b_2 + \cdots$$

$$\frac{d\tau}{dt} = \left(1 + \epsilon b_1 + \epsilon^2 b_2 + \cdots\right)^{-1} \longrightarrow (13)$$

using (13) in (12) we get

$$\dot{x} = \left(1 + \epsilon b_1 + \epsilon^2 b_2 + \cdots\right)^{-1}\frac{dx}{d\tau}$$

$$= \left(1 - \epsilon b_1 + \cdots\right)\frac{dx}{d\tau}; \quad \boxed{x' = dx/d\tau}$$

$$\longrightarrow (14)$$

Similarly,

$$\ddot{x} = \frac{d}{dt}\dot{x} = \frac{d}{dt}\left[\frac{dx}{d\tau}\left(1 - \epsilon b_1 + \cdots\right)\right]$$

$$= \frac{d^2x}{d\tau^2}\left[\left(1 - \epsilon b_1 + \cdots\right)\right]\frac{d\tau}{dt}$$

$$= \frac{d^2x}{d\tau^2}\left[\left(1 - \epsilon b_1 + \cdots\right)\right]^2; \text{ by } (13)$$

$$= \frac{d^2x}{d\tau^2}\left(1 - 2\epsilon b_1 + \cdots\right)$$

$$= x''\left(1 - 2\epsilon b_1 + \cdots\right)$$

$$\longrightarrow (15)$$

substituting (14) and (15) in (1) we have

$$\left(x_0'' + \epsilon x_1'' + \cdots\right)\left(1 - 2\epsilon b_1 + \cdots\right) + k^2\left(x_0 + \epsilon x_1 + \cdots\right) + \epsilon\left(x_0 + \epsilon x_1 + \cdots\right)^3 = 0 \longrightarrow (16)$$

$$x_0(0) + \epsilon x_1(0) + \cdots = 0 \longrightarrow (17)$$

$$\left(x_0'(0) + \epsilon x_1'(0) + \cdots\right)\left(1 - \epsilon b_1 + \cdots\right) = ka \longrightarrow (18)$$

comparing like powers of ϵ we get

$$\epsilon^0 : x_0'' + k^2 x_0 = 0; \quad x_0(0) = 0, \; x_0'(0) = ka, \longrightarrow (19)$$

$$\epsilon^1 : x_1'' + k^2 x_1 = 2b_1 x_0'' - x_0^3 ; \quad x_1(0) = 0, \; x_1'(0) = b_1 ka \longrightarrow (20)$$

Solution of (19) is

$$x_0 = a \sin k\tau \longrightarrow (21)$$

using (21) in (20) we have

$$x_1'' + k^2 x_1 = -\left(2b_1 a k^2 \sin k\tau + a^3 \sin^3 k\tau\right)$$

using the values of $\sin^3 k\tau$ and re-arranging we get

$$x_1'' + k^2 x_1 = -\left(2b_1 k^2 + \frac{3a^2}{4}\right) a \sin k\tau + \frac{a^3}{4} \sin 3k\tau \longrightarrow (22)$$

The source of the secular term in the solution of this equation is the 1st term in the R.H.S of eqn (22). Thus, for elimination of it we choose

$$2b_1 k^2 + \frac{3a^2}{4} = 0$$

or

$$b_1 = -\frac{3a^2}{8k^2} \longrightarrow (23)$$

equation (22) thus becomes

$$x_1'' + k^2 x_1 = \frac{a^3}{4}\sin k\tau \longrightarrow (24)$$

The solution of equ (24) with the initial conditions given in equ (20) is of the following form

$$x_1 = -\frac{a^3}{32k^2}\left(9\sin k\tau + \sin 3k\tau\right) \longrightarrow (25)$$

The expression (25) is uniformly valid and gives, therefore a legitimate approximation for $x(\tau)$ which is

$$x(\tau) = x_0(\tau) + \epsilon x_1(\tau)$$

$$x(\tau) = a\sin k\tau - \frac{\epsilon a^3}{32k^2}\left(9\sin k\tau + \sin 3k\tau\right)$$

where $\tau = \dfrac{t}{\left(1-\epsilon\dfrac{3a^2}{8k^2}\right)}$

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M.S. MATH

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* The Method of Strained Co-ordinates :-

Here we describe the techniques of rendering the approximate solutions to some of the differential equations uniformly valid by introducing near-identity transformations of the independent variables. This technique goes back to the nineteenth century when astronomers, such as Lindstedt (1882), Bohlin (1889), and Gylden (1893), derived techniques to avoid the appearance of secular terms in perturbation solution of equations such as

$$\ddot{u} + \omega_0^2 u = \epsilon f(u, \dot{u}), \quad \epsilon << 1$$

This technique include Lindstedt-poincaré, and Lighthill techniques, etc.

The fundamental idea in Lindstedt's technique is based on the observation that the non-linearities alter the frequency of the system from the linear one ω_0 to $\omega(\epsilon)$. To account for this cange in frequency he introduced a new variable $\tau = \omega t$ and expand ω and u in powers of ϵ as

$$u = u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + \cdots$$

$$\omega = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \cdots$$

Then we choose the parameters $\omega_i, i \geq 1$ to prevent the appearance of secular terms. Poincare (1892) proved that the expansion obtained by Lindstedt technique are asymptotic.

Various forms of this idea have been utilized to obtain approximate solutions to problems in physics and engineering. The idea is to find a parameter in the problem (such as frequency, wave number, wave speed, eigen values or energy level) that is altered by the perturbations and then expand both the dependent variables as well as this parameter in, say, powers of the strength of these perturbations. The perturbations in the parameter are then choosen to render the expansion uniformly valid. Thus we call this technique the method of strained parameters.

If we interpret this parameter expansion as a near-identity ~~parameter~~ transformation, then Lighthill's technique of rendering approximate solutions uniformly valid is a generalization of this technique. According to Lighthill (1949), If we counter a non-uniformity in expanding a function such as $u(x_1, x_2, \cdots x_n; \varepsilon)$

in powers of ϵ, we expand not only dependent variable u but also the independent variable exibiting the non-uniformity, say x_1, in powers of ϵ in terms of a new independent variable as

$$u = \sum_{m=0}^{N-1} \epsilon^m u_m(s, x_2, x_3, \cdots, x_n) + O(\epsilon^N)$$

$$x_1 = s + \sum_{m=1}^{N} \epsilon^m \xi_m(s, x_2, x_3, \cdots, x_n) + O(\epsilon^{N+1})$$

The last expansion can be viewed as a near-identity transformation from x_1 to s. The function ξ_m are called straining functions, and they are determined such that the expansion for u is uniformly valid. In other words, $u_m/u_{m-1} < \infty$ for all values of x_1 of interest, or equivalently higher approximations are no more singular than the first. Note that if $\xi_m = \omega_m s$ with ω_m constant, Lighthill's tenchnique becomes the Lindstedt's-Poincare' technique. Since Lighthill's transformations strains a coordinate rather than a parameter, ~~this~~ this technique is called the method of strained co-ordinates.

Note that Lighthill's technique is an extension of the Lindstedt-Poincare technique.

The Lindstedt-Poincaré Method:-

The previous example (see page) shows that truncated straightforward expansions in powers of ϵ of equations of the form

$$\ddot{u} + \omega_0^2 u = \epsilon f(u, \dot{u}) \longrightarrow (1)$$

where ϵ is a small quantity and ω_0 is a constant, are valid only for short intervals of time because of the presence of secular terms. The assence of Lindstedts poincaré technique is to prevent the appearance of these secular terms by introducing a new variable

$$t = s(1 + \epsilon\omega_1 + \epsilon^2\epsilon_2 + \cdots) \longrightarrow (2)$$

in (1) to obtain

$$(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)^{-2} \frac{d^2u}{ds^2} + \omega_0^2 u$$

$$= \epsilon f\left[u, (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)^{-1} \frac{du}{ds}\right] \rightarrow (3)$$

Letting $\quad u = \sum_{n=0}^{\infty} \epsilon^n u_n(s) \longrightarrow (4)$

in (3) and equating coefficients of like powers of ϵ, we obtain equations to determine U_m in succession. The solutions for U_m contains secular terms unless the ω_m have specific values

→ Page 178

ω_1 is chosen to eliminate the coefficient of $\cos(s+\phi)$ on the RHS of eqn (11). This conditions determines ω_1 to be

$$\omega_1 = \frac{-3}{8} a^2 \longrightarrow (12)$$

Then the solution of (11) becomes

$$u_1 = \frac{1}{32} a^3 \cos 3(s+\phi) \longrightarrow (13)$$

Substituting for u_0, u_1 and ω_1 into (9), we obtain

$$\frac{d^2u_2}{ds^2} + u_2 = \left(\frac{51}{128} a^4 - 2\omega_2\right) a \cos(s+\phi) + NST \longrightarrow (14)$$

Where NST stands for terms that do not produce secular terms. secular terms eliminated if

$$\omega_2 = \frac{51}{256} a^4 \longrightarrow (15)$$

Therefore

$$u = a\cos(\omega t+\phi) + \frac{\epsilon}{32} a^3 \cos 3(\omega t+\phi) + O(\epsilon^2) \longrightarrow (16)$$

where a and ϕ are constants of integration, and

$$\omega = \left(1 - \frac{3}{8} a^2 \epsilon + \frac{51}{256} a^4 \epsilon^2 + \cdots\right)^{-1}$$

$$= 1 + \frac{3}{8} a^2 \epsilon - \frac{51}{256} a^4 \epsilon^2 + O(\epsilon^3)$$

Example:- As an example, we consider Diffing's Equation

$$\frac{d^2u}{dt^2} + u + \epsilon u^3 = 0 \longrightarrow (5)$$

Under the transformation (2) it becomes

$$\frac{d^2u}{ds^2} + (1+\epsilon\omega_1+\epsilon^2\omega_2+\cdots)^2(u+\epsilon u^3) = 0 \longrightarrow (6)$$

We substitute (4) into (6) and equating co-efficients of like powers of ϵ to obtain

$$\epsilon^0: \frac{d^2u_0}{ds^2} + u_0 = 0 \longrightarrow (7)$$

$$\epsilon^1: \frac{d^2u_1}{ds^2} + u_1 = -u_0^3 - 2\omega_1 u_0 \longrightarrow (8)$$

$$\epsilon^2: \frac{d^2u_2}{ds^2} + u_2 = -3u_0^2u_1 - 2\omega_1(u_1+u_0^3) - (\omega_1^2+2\omega_2)u_2 \longrightarrow (9)$$

The general solution of (7) is

$$u_0 = a\cos(s+\phi) \longrightarrow (10)$$

where a and ϕ are constants of integration

With (8), (10) becomes

$$\frac{d^2u_1}{ds^2} + u_1 = -\frac{1}{4}a^3\cos 3(s+\phi) - \left(\frac{3}{4}a^2+2\omega_1\right)a\cos(s+\phi) \longrightarrow (11)$$

If a straightforward perturbation expansion is used, $\omega_m \equiv 0$, and (11) reduces to the form (*) [see page 173] whose particular solution contains a secular term. In order to avoid this secular term, ω_1 is chosen,

Example: Redo previous example using the transformation $\tau = \omega t$.

Solution

Introducing the transformation

$$\tau = \omega t \longrightarrow (17)$$

where ω is a constant that depend on ϵ. Using chain rule, we transform the derivatives according to

$$\frac{d}{dt} = \frac{d\tau}{dt} \cdot \frac{d}{d\tau} = \omega \frac{d}{d\tau}$$

$$\frac{d^2}{dt^2} = \omega \frac{d^2}{dt\, d\tau} = \omega \frac{d\tau}{dt} \cdot \frac{d^2}{d\tau^2} = \omega^2 \frac{d^2}{d\tau^2}$$

Hence (5) becomes

$$\omega^2 u'' + u + \epsilon u^3 = 0 \longrightarrow (18)$$

Where prime denote the derivative w.r.t τ. Let

$$u = u_0(\tau) + \epsilon u_1(\tau) + \cdots \longrightarrow (19)$$

$$\omega = 1 + \epsilon \omega_1 + \cdots \longrightarrow (20)$$

using (19) & (20) in (18) we get

$$(1+\epsilon\omega_1 + \cdots)^2 (u_0'' + \epsilon u_1'' + \cdots) + u_0 + \epsilon u_1 + \cdots$$
$$+ \epsilon(u_0 + \epsilon u_1 + \cdots)^3 = 0 \longrightarrow (21)$$

Equating each of the coefficient of ϵ^0 and ϵ^1 to zero yields

$$u_0'' + u_0 = 0 \longrightarrow (22)$$

$$u_1'' + u_1 = -u_0^3 - 2\omega_1 u_0'' \longrightarrow (23)$$

The general solution of (22) is

$$u_0 = a\cos(\tau+\beta) \longrightarrow (24)$$

where a and β are constants. Then equ (23) becomes

$$u_1'' + u_1 = -a^3\cos^3(\tau+\beta) + 2\omega_1 a\cos(\tau+\beta)$$

or

$$u_1'' + u_1 = \left(2\omega_1 a - \frac{3}{4}a^3\right)\cos(\tau+\beta) - \frac{1}{4}a^3\cos(3\tau+3\beta) \longrightarrow (25)$$

solution of (25) is

$$u_1 = \frac{1}{2}\left(2\omega_1 a - \frac{3}{4}a^3\right)\tau\sin(\tau+\beta) + \frac{1}{32}a^3\cos(3\tau+3\beta) \longrightarrow (26)$$

To eliminate secular term, we let

$$2\omega_1 a - \frac{3}{4}a^3 = 0$$

or $$\omega_1 = \frac{3}{8}a^2 \longrightarrow (27)$$

Hence $$u_1 = \frac{1}{32}a^3\cos(3\tau+3\beta) \longrightarrow (28)$$

Thus, two term expansion is

$$u = a\cos(\tau+\beta) + \frac{1}{32}a^3\cos(3\tau+3\beta)$$

or

$$u = a\cos(\omega t+\beta) + \frac{\epsilon}{32}a^3\cos(3\omega t+3\beta) \longrightarrow (29)$$

with $$\omega = 1 + \frac{3}{8}\epsilon a^2 + \cdots$$

which is similar to solution (16)

Example:- Consider the equation

$$\ddot{u} + \omega_0^2 u = \epsilon u\dot{u}^2 \longrightarrow (1)$$

Determine a first order uniform expansion by using the Lindtedt - Poincare technique.

Solution Let $\tau = \omega t = (\omega_0 + \epsilon\omega_1 + \cdots)t \longrightarrow (2)$

Introducing the change of variable (2) into (1) we have

$$(\omega_0^2 + 2\epsilon\omega_0\omega_1 + \cdots)u'' + \omega_0^2 u = \epsilon\omega_0^2 uu'^2 + \cdots \longrightarrow (3)$$

Letting $u = u_0(\tau) + \epsilon u_1(\tau) + \cdots\cdots \longrightarrow (4)$ in (3) and equating co-efficients of the powers of ϵ, we have

$$u_0'' + u_0 = 0 \longrightarrow (5)$$

$$u_1'' + u_1 = -2\frac{\omega_1}{\omega_0}u_0'' + u_0 u_0'^2 \longrightarrow (6)$$

The solution of (5) can be expressed as

$$u_0 = a\cos(\tau + \beta) \longrightarrow (7)$$

Then (6) becomes

$$u_1'' + u_1 = \frac{2\omega_1}{\omega_0}a\cos(\tau+\beta) + a^3\sin^2(\tau+\beta)\cos(\tau+\beta)$$

$$= \left(\frac{2\omega_1}{\omega_0}a + \frac{1}{4}a^3\right)\cos(\tau+\beta) - \frac{1}{4}a^3\cos(3\tau+3\beta) \longrightarrow (8)$$

Eliminating the term that lead to secular terms in (8) yields

$$\omega_1 = \frac{-\omega_0 a^2}{8} \longrightarrow (9)$$

Hence $$u = a\cos\left[\omega_0\left(1 - \frac{\epsilon a^2}{8}\right)t + \beta\right] + \cdots$$

The Method of Renormalization:-

Instead of substituting transformation $\tau = \omega t$ into the governing equation and expanding both ω and u as in the Lindtedt-Poincaré technique, one can substitute this transformation and the expansion for ω into the straight forward expansion, expand the result for small ϵ, and choose the ω_n to eliminate the secular terms. Thus one lets

$$\omega_0 t = \frac{\omega_0 \tau}{\omega} = \frac{\omega_0 \tau}{\omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots}$$

$$= \frac{\cancel{\omega_0}\tau}{\cancel{\omega_0}\left(1 + \frac{\epsilon\omega_1}{\omega_0} + \frac{\epsilon^2\omega_2}{\omega_0} + \cdots\right)}$$

$$= \tau\left[1 + \frac{\epsilon\omega_1}{\omega_0} + \frac{\epsilon^2\omega_2}{\omega_0} + \cdots\right]^{-1}$$

$$= \tau\left(1 - \frac{\epsilon\omega_1}{\omega_0} + \frac{\epsilon^2\omega_1^2}{\omega_0^2} - \frac{\epsilon^2\omega_2}{\omega_0} + \cdots\right) \longrightarrow (1)$$

In the straight forward expansion, expand the result for small ϵ, keeping τ fixed, and choose the ω_n to eliminate the secular terms.

For example, a first order straight forward expansion of the Duffing's equation is

$$u = a\cos(t+\beta) + \epsilon\left[-\frac{3}{8}a^3 t\sin(t+\beta) + \frac{1}{32}a^3\cos(3t+3\beta)\right] + \cdots \longrightarrow (2)$$

Substituting the transformation ① with ω_0 being unity into the straightforward expansion ② of the Duffing's equation, we get

$$u = a\cos(\tau+\beta - \epsilon\omega_1\tau + \cdots) +$$
$$\epsilon a^3\left[-\frac{3}{8}(\tau - \epsilon\omega_1\tau + \cdots)\sin(\tau+\beta - \epsilon\omega_1\tau + \cdots)\right.$$
$$\left. + \frac{1}{32}\cos(3\tau + 3\beta - 3\epsilon\omega_1\tau + \cdots)\right] + \cdots \longrightarrow ③$$

using Taylor's series expansion we have

$$\cos(\tau+\beta - \epsilon\omega_1\tau + \cdots) = \cos(\tau+\beta) + \epsilon\omega_1\tau\sin(\tau+\beta) + \cdots$$
$$\sin(\tau+\beta - \epsilon\omega_1\tau + \cdots) = \sin(\tau+\beta) - \epsilon\omega_1\tau\cos(\tau+\beta) + \cdots$$
$$\longrightarrow ④$$

using these expansions we rewrite ③ as

$$u = a\cos(\tau+\beta) + \epsilon\left[\left(\omega_1 - \frac{3}{8}a^2\right)a\tau\sin(\tau+\beta) + \frac{1}{32}a^3\right.$$
$$\left.\cos(3\tau+3\beta)\right] + \cdots \longrightarrow ⑤$$

Choosing ω_1 to eliminate the secular term gives

$$\omega_1 = \frac{3}{8}a^2 \longrightarrow ⑥$$

and

$$u = a\cos(\tau+\beta) + \frac{1}{32}a^3\cos(3\tau+3\beta) + \cdots \longrightarrow ⑦$$

where

$$\tau = \left(1 + \frac{3\epsilon}{8}a^2 + \cdots\right)t$$

In agreement with the expansion obtained using the Lindstedt-Poincaré technique. It should be noted that in general the method of renormalization yields a uniform approximation with much less algebra than that involved in applying the Lindstedt-Poincaré technique.

Example :- An example of the failure of renormalization technique

Renormalization does not always succeed in rendering the expansion of oscillator problems. An example of its failure is provided by the governing equation for the Van der Pol oscillator,

$$\frac{d^2u}{dt^2}+u=\epsilon(1-u^2)\frac{du}{dt};\ u(0)=1,\ \frac{du}{dt}(0)=0 \longrightarrow (1)$$

The straight forward two term expansion is easily found to be

$$u=\cos t+\epsilon\left(\frac{3}{8}t\cos t-\frac{9}{32}\sin t-\frac{1}{32}\sin 3t\right) \longrightarrow (2)$$

Letting $t=\tau(1-\epsilon\omega_1+\cdots) \longrightarrow (3)$

using (3) into (2) we have

$$u=\cos(\tau-\epsilon\omega_1\tau+\cdots)+\epsilon\left[\frac{3}{8}(\tau-\epsilon\omega_1\tau+\cdots)\cos(\tau-\epsilon\omega_1\tau+\cdots)-\frac{9}{32}\sin(\tau-\epsilon\omega_1\tau+\cdots)-\frac{1}{32}\sin(3\tau-3\epsilon\omega_1\tau+\cdots)\right] \longrightarrow (4)$$

using Taylor series expansion we have

$$\cos(\tau-\epsilon\omega_1\tau+\cdots)=\cos\tau+\epsilon\omega_1\tau\sin\tau+\cdots$$

$$\sin(\tau-\epsilon\omega_1\tau+\cdots)=\sin\tau-\epsilon\omega_1\tau\cos\tau+\cdots \longrightarrow (5)$$

using (5) in (4) we have

$$u=(\cos\tau+\epsilon\omega_1\tau\sin\tau+\cdots)+\frac{3\epsilon}{8}\Big[(\tau-\epsilon\omega_1\tau+\cdots)$$

$$(\cos\tau+\epsilon\omega_1\tau\sin\tau+\cdots)\Big]-\frac{9\epsilon}{32}\left[\sin\tau-\epsilon\omega_1\tau\cos\tau+\cdots\right]$$

$$-\frac{1}{32}\epsilon\left[\sin 3\tau-3\epsilon\omega_1\tau\cos 3\tau+\cdots\right]$$

or

$$u=\cos\tau+\epsilon\omega_1\tau\sin\tau+\epsilon\Big[\frac{3}{8}\tau\cos\tau-\frac{9}{32}\sin\tau$$

$$-\frac{1}{32}\sin 3\tau\Big]+\cdots \longrightarrow (6)$$

To remove secular term in the $O(\epsilon)$ term we require

$$\omega_1\tau\sin\tau+\frac{3}{8}\tau\cos\tau=0$$

$$\text{or}\quad \omega_1=-\frac{3}{8}\cot\tau \longrightarrow (7)$$

Thus

$$u=\cos\tau-\epsilon\left[\frac{9}{32}\sin\tau+\frac{1}{32}\sin 3\tau\right]+O(\epsilon^2) \longrightarrow (8)$$

where

$$t=\tau\left[1+\frac{3}{8}\epsilon\cot\tau+\cdots\right]$$

This is invalid because the cotangent function is singular when

$$\tau=0,\pi,2\pi,\cdots$$

The Method of Averaging:-

We discuss this technique in connection with the general weakly non-linear (oscillators) second order equation

$$\frac{d^2u}{dt^2} + k^2 u + \epsilon f\left(u, \frac{du}{dt}\right) = 0 \longrightarrow (1)$$

$$|\epsilon| << 1$$

When $\epsilon = 0$ (The unperturbed oscillator equation) the solution of (1) can be written as

$$u = a\cos(kt + \phi) \longrightarrow (2)$$

where a and ϕ are constants whose values are determined by the initial conditions.

To determine the approximate solution to (1) for ϵ small but different from zero, Krylov and Bogoliubove (1947) assumed that the solution of (1) is still given by (2) but with the varying a and ϕ. This is

$$u = a(t)\cos(kt + \phi(t)) \longrightarrow (3)$$

Since the function of time have been introduced, any convenient constraint may be used in addition to the governing equation. The ~~stran~~ standard constraint is to impose the following

$$\frac{du}{dt} = -a(t)\,k\sin(kt + \phi(t)) \longrightarrow (4)$$

mean that $a(t)$ and $\phi(t)$ are slowly varying function of t)

$\phi(t)$ are slowly varying function of time; hence they change very little during the time $2\pi/k$ (The period of the terms on RHS)

Since equation (10) and (11) are rather complicated in general we approximate them by their time average over one period of the system i.e $[0, 2\pi/k]$. Thus

$$\frac{da}{dt} = \frac{\epsilon}{2\pi k}\int_0^{2\pi} \sin\psi\, f(a\cos\psi, -ak\sin\psi)\,d\psi \longrightarrow (12)$$

$$\frac{d\phi}{dt} = \frac{\epsilon}{2\pi k a}\int_0^{2\pi} \cos\psi\, f(a\cos\psi, -ak\sin\psi)\,d\psi \longrightarrow (13)$$

where $\psi = kt + \phi$

Thus the original 2nd order differential equation (1) for u has been replaced by the two first order differential equations (12) and (13) for the amplitude a and the phase ϕ.

The solution of these equations would give a and ϕ. Using these values of a and ϕ in (3) we get the approximate solution.

This averaging technique is usually reffered to as Van der Pol method or the Krylov-Bogoliubov method.

———————————

Example:- As an example let us consider Duffing's Equation

$$\ddot{u} + u + \epsilon u^3 = 0 \longrightarrow ①$$

Comparing ① with

$$\ddot{u} + ku + \epsilon f(u, \dot{u}) = 0$$

we have $k = 1$, $f(u, \dot{u}) = u^3$

We seek the solution of ① in the form

$$u = a(t)\cos(kt + \phi) \longrightarrow ②$$

$$= a \cos \psi \longrightarrow ②^*$$

with

$$\dot{a} = \frac{\epsilon}{2\pi k}\int_0^{2\pi} \sin\psi\, f(\underbrace{a\cos\psi}_{u^3}, \underbrace{-ak\sin\psi}_{du/dt})\, d\psi \longrightarrow ③$$

$$\dot{\phi} = \frac{\epsilon}{2\pi k a}\int_0^{2\pi} \cos\psi\, f(a\cos\psi, -ak\sin\psi)\, d\psi \longrightarrow ④$$

and $\psi = kt + \phi$

From ③ we have

$$\dot{a} = \frac{\epsilon}{2\pi}\int_0^{2\pi} \sin\psi\,(a^3\cos^3\psi)\, d\psi$$

$$= \frac{a^3\epsilon}{2\pi}\int_0^{2\pi} \cos^3\psi \sin\psi\, d\psi = 0$$

i.e $\boxed{\dot{a} = 0}$

Integrating $a = \text{constant} \longrightarrow ⑤$

Also from ④

$$\dot{\phi} = \frac{\epsilon}{2\pi a}\int_0^{2\pi} \cos\psi(a^3\cos^3\psi)\, d\psi$$

$$= \frac{a^2\epsilon}{2\pi}\left(\frac{3\pi}{4}\right) = \frac{3a^2\epsilon}{8}$$

$$\Rightarrow \quad \dot{\phi} = \frac{3a^2\epsilon}{8}$$

Integrating $\quad \phi = \frac{3a^2}{8}\epsilon t + \phi_0 \longrightarrow (6)$

where ϕ_0 is a constant of integration

Therefore, to first approximation

$$u = a\cos\left(t + \frac{3a^2}{8}\epsilon t\right) + O(\epsilon)$$

or

$$u = a\cos\left[\left(1 + \frac{3}{8}\epsilon a^2\right)t\right] + O(\epsilon)$$

Example:- Consider the equation

$$\ddot{u} + u + \epsilon(1 - \dot{u}^2)\dot{u} = 0 \longrightarrow (1)$$

comparing with

$$\ddot{u} + k^2 u + \epsilon f(u, \dot{u}) = 0$$

we have

$$k = 1, \quad f(u, \dot{u}) = (1 - \dot{u}^2)\dot{u}$$

Let

$$u = a\cos(kt + \phi)$$

$$= a\cos\psi \longrightarrow (2)$$

with

$$\dot{a} = \frac{\epsilon}{2\pi k}\int_0^{2\pi} \sin\psi f(a\cos\psi, -ak\sin\psi)\, d\psi \longrightarrow (3)$$

$$\dot{\phi} = \frac{\epsilon}{2\pi k a}\int_0^{2\pi} \cos\psi\, f(a\cos\psi, -ak\sin\psi)\, d\psi \longrightarrow (4)$$

substituting values in (3) and (4), we have

$$\dot{a} = \frac{\epsilon}{2\pi}\int_0^{2\pi} \sin\psi(1-a^2\sin^2\psi)(-a\sin\psi)\, d\psi$$

$$= \frac{\epsilon}{2\pi}\int_0^{2\pi} (-a\sin^2\psi + a^3\sin^4\psi)\, d\psi$$

$$= \frac{-\epsilon a}{2\pi}\int_0^{2\pi} \sin^2\psi\, d\psi + \frac{\epsilon a^3}{2\pi}\int_0^{2\pi} \sin^4\psi\, d\psi$$

$$= \frac{-\epsilon a}{2\pi}(\pi) + \frac{\epsilon a^3}{2\pi}\left(\frac{3}{4}\pi\right)$$

$$\Rightarrow \boxed{\dot{a} = \frac{\epsilon}{2} a\left(\frac{3}{4}a^2 - 1\right)} \longrightarrow (5)$$

$$\dot{\phi} = \frac{\epsilon}{2\pi a}\int_0^{2\pi} \cos\psi[1-a^2\sin^2\psi](-a\sin\psi)\, d\psi$$

$$= \frac{\epsilon}{2\pi a}\int_0^{2\pi} -a\cos\psi\sin\psi(1-a^2\sin^2\psi)\, d\psi = 0$$

Thus $\dot{\phi} = 0$

$$\Rightarrow \phi = \text{constant} \longrightarrow (6)$$

From (5)

$$\frac{da}{dt} + \frac{\epsilon}{2}a = \frac{3}{8}\epsilon a^3$$

or

$$a^{-3}\frac{da}{dt} + \frac{\epsilon}{2}a^{-2} = \frac{3}{8}\epsilon$$

Let $a^{-2} = z \Rightarrow a^{-3}\frac{da}{dt} = -\frac{1}{2}\frac{dz}{dt}$

So

$$-\frac{1}{2}\frac{dz}{dt} + \frac{\epsilon}{2}z = \frac{3}{8}\epsilon$$

or

$$\frac{dz}{dt} - \frac{\epsilon}{2}z = -\frac{3}{4}\epsilon$$

$$I.F = e^{-\int \epsilon dt} = e^{-\epsilon t} \text{ and thus}$$

$$\frac{d}{dt}(z.e^{-\epsilon t}) = -\frac{3}{4}\epsilon e^{-\epsilon t}$$

$$\Rightarrow ze^{-\epsilon t} = \frac{3}{4}e^{-\epsilon t} + c$$

$$\Rightarrow z = \frac{3}{4} + ce^{\epsilon t}$$

$$\Rightarrow \frac{1}{a^2} = \frac{3}{4} + ce^{\epsilon t}$$

$$\Rightarrow a^2 = \frac{1}{3/4 + ce^{\epsilon t}}$$

$$\Rightarrow a = \left[\sqrt{\frac{3}{4} + ce^{\epsilon t}}\right]^{-1} \longrightarrow (7)$$

Thus from (2), (6) and (7) we have

$$u = \left[\sqrt{\frac{3}{4} + ce^{\epsilon t}}\right]^{-1} \cos(t + \text{constant})$$

* The Method of Multiple Scales:-

The multiple scale technique will be introduced by considering the following problem, which could not be successfully dealt with by renormalization. The exact solution of

$$\frac{d^2u}{dt^2} + u = -\epsilon\frac{du}{dt}; \quad u(0)=1, \quad \frac{du}{dt}(0)=0 \quad \text{①}$$

is

$$u_{ex} = e^{-\epsilon t/2}\left[\cos\sqrt{1-\frac{\epsilon^2}{4}}\,t + \frac{\epsilon}{2\sqrt{1-\frac{\epsilon^2}{4}}}\sin\sqrt{1-\frac{\epsilon^2}{4}}\,t\right] \quad \text{②}$$

The straight-forward two term expansion is

$$u_{2T} = \cos t - \frac{\epsilon}{2}(t\cos t - \sin t) \quad \text{③}$$

This can be constructed from exact solution by expanding the exponential, square root and trignometric functions. Non uniformities are generate in forming the expansions of the exponential and trignometric functions. While it is uniformly true to state that

$$u = \cos t + O(\epsilon) \quad \text{for } t = O(1)$$

It is not uniformly valid for $t = O(1/\epsilon)$. If we are interested in values of t which are $O(1/\epsilon)$ then the combination ϵt must be preserved in the exponential function. Then it is uniformly valid to state that

$$u = e^{-\epsilon t/2}\cos t + O(\epsilon) \quad \text{for} \quad t = O(1/\epsilon) \longrightarrow (4)$$

If we are interested in time which are $O(1/\epsilon^2)$ then (4) is no longer valid. In this case terms of the form $\epsilon^2 t$ must be preserved in the cosine function appearing in (2)

$$\cos\sqrt{1-\left(\frac{\epsilon}{2}\right)^2}\,t = \cos\left[1-\frac{1}{2}\left(\frac{\epsilon}{2}\right)^2 + O(\epsilon^4)\right]t$$

$$= \cos\left(1-\frac{\epsilon^2}{8}\right)t + O(\epsilon^4 t)$$

Thus

$$u = e^{-\frac{\epsilon t}{2}}\cos\left(1-\epsilon^2/8\right)t + O(\epsilon),$$

is uniformly valid statement for $t = O(1/\epsilon^3)$.

Notice that if we are concerned only with uniformly valid leading order expansions then the 2nd member of the bracket in (2) never contribute since it is uniformly of $O(\epsilon)$ for all t

⇒ Time Scales:-

The basic idea behind the method of multiple scales is to avoid the introduction of non-uniformity associated with expansions of functions by preserving the combinations ϵt, $\epsilon^2 t$, $\epsilon^3 t$ etc. As variables on which these functions depend. these combinations are called scales and denoted as follows

$T_0 = t$, $T_1 = \epsilon t$, $T_2 = \epsilon^2 t, \ldots, T_n = \epsilon^n t$

To demonstrate the idea consider the following function

$$f = \frac{1}{1-\epsilon} \exp(t + \epsilon t + \epsilon^2 t + \epsilon^3 t + \epsilon^4 t)$$

using one scale T_0 we have

$$f \sim \left(1 + \epsilon + \frac{\epsilon^2}{2} + \cdots\right) \exp(T_0)\Big(1 + \epsilon T_0 + \epsilon^2 T_0 + \cdots$$
$$+ \frac{1}{2}(\epsilon T_0 + \epsilon^2 T_0 + \cdots)^2 + \cdots\Big)$$

$$\sim \exp T_0 + \epsilon(1+T_0)\exp T_0 + \epsilon^2\left(1 + 2T_0 + \frac{1}{2}T_0^2\right)\exp T_0$$

The region of non-uniformity is $T_0 = O(1/\epsilon)$ i.e. $t = O(1/\epsilon)$. using two scales

$$f = \frac{1}{1-\epsilon} \exp(T_0 + T_1 + \epsilon^2 T_0 + \epsilon^3 T_0 + \epsilon^4 T_0)$$

$$\sim \left(1 + \epsilon + \frac{\epsilon^2}{2} + \cdots\right) \exp(T_0 + T_1)\left[1 + \epsilon^2 T_0 + \cdots\right]$$

$$\sim \exp(T_0 + T_1)\left\{1 + \epsilon + \epsilon^2\left(\frac{1}{2} + T_0\right) + \cdots\right\}$$

If $T_0 = O(1/\epsilon)$ The 2nd the 3rd members of the above expansion are of the same order, namely $O(\epsilon)$. There is no point in preserving the 2nd member of the above expansion since other terms are of this order, but we may state that

$$f = \exp(T_0 + T_1)\left(1 + O(\epsilon, \epsilon T_0)\right),$$

where this expansion is uniformly valid for $t = O(1/\epsilon)$. using three scales we may either state

$$f = \exp(T_0+T_1+T_2)\cdot(1+\epsilon+O(\epsilon^2, \epsilon^3 T_0))$$

which is uniformly valid for $t=O(1/\epsilon)$ or we may state

$$f = \exp(T_0+T_1+T_2)\cdot(1+O(\epsilon, \epsilon^2, \epsilon^3 T_0)),$$

which is uniformly valid for $t=O(1/\epsilon^2)$

Introducing further scales allows either more terms to be included in the expansion for $t=O(1/\epsilon)$ or fewer terms may be used with one extended region of uniformity.

We will restrict our application of multiple scale technique to the introduction of two scales, T_0 and T_1, are obtain one-term uniformly valid expansions for $t=O(1/\epsilon)$. using the scales $T_0=t$ and $T_1=\epsilon t$. The time derivatives becoms

$$\frac{d}{dt} = \frac{dT_0}{dt}\cdot\frac{\partial}{\partial T_0} + \frac{dT_1}{dt}\cdot\frac{\partial}{\partial T_1} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} \longrightarrow (5a)$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\epsilon\frac{\partial^2}{\partial T_0 \partial T_1} + \epsilon^2\frac{\partial^2}{\partial T_1^2} \longrightarrow (5b)$$

Returning to the example providing by equ (1) which becomes

$$\frac{\partial^2}{\partial T_0^2} + 2\epsilon\frac{\partial^2 u}{\partial T_0 \partial T_1} + \frac{\epsilon^2 \partial^2 u}{\partial T_1^2} + u = -\epsilon\frac{\partial u}{\partial T_0} - \epsilon^2\frac{\partial u}{\partial T_1}$$

we expand u in the form

$$u \equiv u(T_0, T_1; \epsilon) \sim u_0(T_0, T_1) + \epsilon u_1(T_0, T_1) + \cdots$$

and obtain the following order equations

$$O(1): \quad \frac{\partial^2 u_0}{\partial T_0^2} + u_0 = 0 \longrightarrow (6)$$

$$O(\epsilon): \quad \frac{\partial^2 u_1}{\partial T_0^2} + u_1 = -2\frac{\partial^2 u_0}{\partial T_0 \partial T_1} - \frac{\partial u_0}{\partial T_0} \longrightarrow (7)$$

It is convenient to express the solution of the harmonic oscillator equation (6) in complex form

$$u_0 = A(T_1)e^{iT_0} + A^*(T_1)e^{-iT_0} \longrightarrow (8)$$

where the (*) symbol indicate the complex conjugate and the function A depends on T_1 only. The complimentry function associated with equ (7) has the same form as (8). The particular integral will contain secular terms if the right hand side of (7) contains $\exp(\pm iT_0)$ terms. The condition that these should be absent determines the function $A(T_1)$ in equ (8).

The R.H.S of (7) is

$$-2i\left[\frac{dA}{dT_1}e^{iT_0} - \frac{dA^*}{dT_1}e^{-iT_0}\right] - i\left[Ae^{iT_0} - A^*e^{-iT_0}\right]$$

$$= -i\left(2\frac{dA}{dT_1} + A\right)e^{iT_0} + i\left(2\frac{dA^*}{dT_1} + A^*\right)e^{-iT_0}$$

Thus to avoid secular terms we require

$$2\frac{dA}{dT_1} + A = 0 \longrightarrow (9a)$$

and

$$2\frac{dA^*}{dT_1} + A_1^* = 0 \longrightarrow (9b)$$

Imposing (9a) automatically ensures that (9b) is satisfied.

It is convenient to express the complex function, $A(T_1)$, in polar form $R(T_1)\exp[i\theta(T_1)]$, where $R(T_1)$ & $\theta(T_1)$ are real functions. Then

$$\frac{dA}{dT_1} = \left(\frac{dR}{dT_1} + iR\frac{d\theta}{dT_1}\right)e^{i\theta}$$

So that (9a) becomes

$$\left(2\frac{dR}{dT_1} + 2iR\frac{d\theta}{dT_1} + R\right)e^{i\theta} = 0$$

The bracketed term in the above equation must be zero. Equating real part to be zero yields

$$2\frac{dR}{dT_1} + R = 0 \longrightarrow (10)$$

While the imaginary part yields

$$R\frac{d\theta}{dT_1} = 0,$$

with non-trivial solution $\theta = \theta_0$ (a constant) the of ⑩ is $R = R_0 e^{-T_1/2}$

where R_0 is a +ve constant. Then

$$A(T_1) = R_0 e^{-T_1/2} e^{i\theta_0},$$

and substituting into ⑧ leads to

$$u_0 = R_0 e^{-T_1/2}\left[e^{i(T_0+\theta_0)} + e^{-i(T_0+\theta_0)}\right]$$

$$u_0 = 2R_0 e^{-T_1/2} \cos(T_0+\theta_0) \longrightarrow (11)$$

At this stage the initial conditions are imposed. In this case we have

$$u(0) = 1 \quad , \quad \frac{du}{dt}(0) = 0$$

which must be expressed in terms of the scale T_0 and T_1. When $t=0$ both T_0 and T_1 are zero, so that initial conditions becomes

$$u(T_0=0, T_1=0) = 1 \sim u_0(0,0) + \epsilon u_1(0,0) + \cdots \quad \&$$

$$\frac{\partial u}{\partial T_0}(0,0) + \epsilon \frac{\partial u}{\partial T_1}(0,0) = 0 \sim \frac{\partial u_0}{\partial T_0}(0,0)$$
$$+ \epsilon\left(\frac{\partial u_1}{\partial T_0}(0,0) + \frac{\partial u_0}{\partial T_1}(0,0)\right) + \cdots$$

Thus the conditions satisfied by u_0 are

$$u_0(0,0) = 1 \quad \& \quad \frac{\partial u_0}{\partial T_0}(0,0) = 0$$

Imposing these on the expression (11) yields $2R_0 \cos\theta_0 = 1$ and $-2R_0 \sin\theta_0 = 0$. Thus $\theta_0 = 0$ and $R_0 = 1/2$ so that

$$u_0 = e^{-T_1/2} \cos T_0$$

Just as before, we do not evaluate u_1 but merely ensure that secular terms are absent so that we may write

$$u = e^{-T_1/2} \cos T_0 + O(\epsilon)$$

where this expression is uniformly valid for $t = O(1/\epsilon)$. At this stage the original variable is used and we have

$$u = e^{-\epsilon t/2} \cos t + O(\epsilon)$$

This agrees with expansion ④ of the exact solution.

Example:- For our next example, we consider the Van der Pol oscillator

$$\frac{d^2u}{dt^2} + u = \epsilon(1-u^2)\frac{du}{dt}; \quad u(0) = a, \ \frac{du}{dt}(0) = 0 \quad \text{①}$$

using the scale $T_0 = t$ & $T_1 = \epsilon t$. This becomes

$$\frac{\partial^2 u}{\partial T_0^2} + 2\epsilon \frac{\partial^2 u}{\partial T_0 \partial T_1} + \epsilon^2 \frac{\partial^2 u}{\partial T_1^2} + u = \epsilon(1-u^2)\left[\frac{\partial u}{\partial T_0} + \epsilon \frac{\partial u}{\partial T_1}\right]$$

substituting

$$u \sim u_0(T_0, T_1) + \epsilon u_1(T_0, T_1) + \cdots$$

leads to the following order equations

$$O(1): \quad \frac{\partial^2 u_0}{\partial T_0^2} + u_0 = 0$$

$O(\epsilon)$: $$\frac{\partial^2 u_1}{\partial T_0^2} + u_1 = -2\frac{\partial^2 u_0}{\partial T_0 \partial T_1} + (1-u_0^2)\frac{\partial u_0}{\partial T_0}$$

The order ① equation has solution

$$u_0 = A(T_1)e^{iT_0} + A^*(T_1)e^{-iT_0}$$

So the R.H.S of the order $O(\epsilon)$ equation becomes

$$-2i\left[\frac{dA}{dT_1}e^{iT_0} - \frac{dA^*}{dT_1}e^{-iT_0}\right] + i\left[1 - A^2e^{2iT_0} - 2AA^* - A^{*2}e^{-2iT_0}\right]\left[Ae^{iT_0} - A^*e^{-iT_0}\right]$$

The coefficient of e^{iT_0} is

$$-2i\frac{dA}{dT_1} + iA - iA^2A^*,$$

and the coefficient of e^{-iT_0} is the complex conjugate of this

To avoid secular terms appearing in u_1, the coefficient of $e^{\pm iT_0}$ must be zero. using polar form

$$A = Re^{i\theta}$$

we require

$$-2\left(\frac{dR}{dT_1} + iR\frac{d\theta}{dT_1}\right) + R - R^3 = 0$$

and real and imaginary parts of this equation lead to

$$-2\frac{dR}{dT_1} + R - R^3 = 0 \longrightarrow (2a)$$

$$\frac{d\theta}{dT_1} = 0 \longrightarrow (2b)$$

The solution of (2b) is $\theta = \theta_0$ (a constant)
Equation (2a) is of separable form

$$\frac{1}{R(R^2-1)}\frac{dR}{dT} = \frac{-1}{2}$$

using partial fraction leads to

$$\int\left(\frac{-1}{R} + \frac{1}{2}\frac{1}{R-1} + \frac{1}{2}\frac{1}{R+1}\right)dR = \int \frac{-1}{2}\, dT_1$$

and on integration we obtain

$$-\ln|R| + \frac{1}{2}\ln|R-1| + \frac{1}{2}\ln|R+1| = \frac{-T_1}{2} + c$$

or

$$\ln\sqrt{\left|\frac{R^2-1}{R^2}\right|} = \frac{-T_1}{2} + c$$

Thus $\dfrac{R^2-1}{R^2} = K e^{-T_1}$

~~wh~~ where K is arbitrary constant.
Solving for R yields

$$R = \frac{1}{\sqrt{1-Ke^{-T_1}}}$$

So we have

$$u_0 = R\left[e^{i(T_0+\theta_0)} + e^{-i(T_0+\theta_0)}\right] = 2R\cos(T_0+\theta_0)$$

The initial conditions require

$$u_0(0,0) = a \quad \& \quad \frac{\partial u_0}{\partial T_0}(0,0) = 0$$

Thus

$2R(0)\cos\theta_0 = a$ & $-2R(0)\sin\theta_0 = 0$
with solution $\theta_0 = 0$ and $R(0) = a/2$
Then since

$$R = 1/\sqrt{1-Ke^{-T_1}}, \text{ we have}$$

$$\frac{a}{2} = 1/\sqrt{1-K} \Rightarrow K = 1 - \frac{4}{a^2}$$

and finally

$$u_0 = \frac{2}{\sqrt{1+\left(\frac{4}{a^2}-1\right)e^{-T_1}}}\cos T_0$$

with $u_1 = O(1)$ for all t
Thus the uniformly valid one term expansion of the solution of the van der pol eqn is

$$u = \frac{2}{\sqrt{1+\left(\frac{4}{a^2}-1\right)e^{-\epsilon t}}}\cos t + O(\epsilon) \longrightarrow (3)$$

This shows that as $t \to \infty$ the expansion tends to the limit cycle $u = 2\cos t + O(\epsilon)$ for all initial values. Furthermore, it is only for the particular initial value $a = 2$ that the solution is periodic.

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Assignment # 1

* Modeling of Duffing's Equation

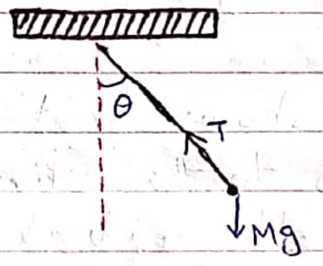
Introduction:-

The equation $\frac{d^2u}{dt^2} + k^2u + \epsilon u^3 = 0$

is known as duffing equation. This equation represents non-linear oscillators. The variable 'u' can represent a variety of quantities such as angle of oscillation, the deformation of a elastic system, a current or a voltage. The independent variable 't' is time.

We will see how Duffing's equation arises in the description of two different mechanical systems.

(i) The Pendulum:-



Consider a point mass 'M' which is connected by a rod of length 'l' to a firm support As shown in adjoing figure.

The mass swings in a vertical plane under the action of gravity. The length of rod is fixed and its mass is negligible.

Two forces acting of the mass 'm' are gravity and tension in the rod.

The component of motion tangential to the sector of the circle on which

The mass moves is driven by the force $-Mg\sin\theta$. The tangential accelleration of the partical moving on a circle of fixed radius is $L(d^2\theta/dt^2)$.

So by Newton's 2nd law we have

$$ML\frac{d^2\theta}{dt^2} = -Mg\sin\theta \Rightarrow \frac{d^2\theta}{dt^2} = \frac{-g}{L}\sin\theta$$

Suppose the pendulum is released from rest when $\theta = \theta_0$ at $t=0$

If we put $u = \theta/\theta_0$ and $g/L = \Omega^2$ then above equation becomes.

$$\frac{d^2}{dt^2}(u\theta_0) = -\Omega^2 \sin(u\theta_0)$$

$$\Rightarrow \theta_0\frac{d^2u}{dt^2} = -\Omega^2(u\theta_0)$$

$$\Rightarrow \frac{d^2u}{dt^2} = -\Omega^2\frac{\sin(u\theta_0)}{\theta_0}$$

With initial conditions that $u=1$ & $\frac{du}{dT} = 0$ at $t=0$

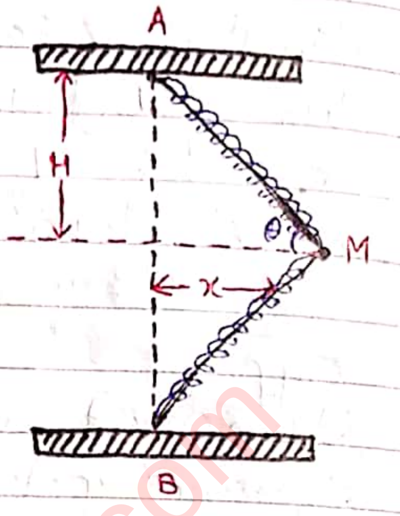
If 'θ' is small the sine term can be approximated by a truncated Maclaurin expansion. On keeping two terms in the expansion we obtained

$$\frac{d^2u}{dt^2} = -\Omega^2\left(u - \frac{\theta_0^2 u^3}{6}\right)$$

$$\Rightarrow \boxed{\frac{d^2u}{dt^2} + \Omega^2 u + \epsilon u^3 = 0} \; ; \text{where } \epsilon = \frac{-\theta_0^2}{6}$$

Which is duffing equation.

(ii) A mass and Spring Oscillator:-

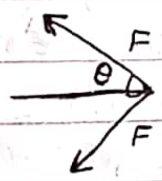


Consider a point mass 'M' which is connected to the fixed points 'A' and 'B' by identical spring of negligible mass. The natural length of rods spring is 'L' and spring constant is 'K'. The separation between 'A' and 'B' is '2H' where $H > L$. The mass is displaced by a perpendicular distance x_0 from AB and released as shown in figure. The mass oscillator on a path perpendicular to AB.

Gravity is neglected and the spring force F, vary linearly with extension.

By Hook's law we have

$$F = K\left(\frac{\sqrt{H^2+X^2} - L}{L}\right)$$



And Newton's 2nd law yields

$$M\frac{d^2X}{dt^2} = -2F\cos\theta$$

& $\cos\theta = X/\sqrt{H^2+X^2}$ so that

$$M\frac{d^2X}{dt^2} = \frac{-2K}{L}X\left(1 - \frac{L}{\sqrt{X^2+H^2}}\right)$$

When the displacement X is small as compared with H the inverse square of term may be replaced by a truncated Maclaurin expansion.

On keeping two terms in the expansion we obtain

$$\frac{d^2X}{dt^2} = \frac{-2K}{ML} X\left[1 - \frac{L}{H}\left(1 - \frac{1}{2}\frac{X^2}{H^2}\right)\right]$$

If we put $u = X/X_0$ and $\Omega = \sqrt{2K(H-L)/MLH}$ the above equation becomes

$$\frac{d^2u}{dt^2} = -\Omega^2\left(u + \frac{L}{2(H-L)}\frac{X_0^2}{H^2}u^3\right)$$

$$\Rightarrow \boxed{\frac{d^2u}{dt^2} + \Omega^2 u + \varepsilon u^3 = 0}$$

which is duffing's equation.

where $\epsilon = LX_0^2/2(H-L)H_0^2$

★ Uniform and Non-Uniform Expansions:-

An expansion of the form

$$f(x,\varepsilon) = \sum a_n(x)\,\delta_n(\varepsilon)$$

is said to be non-uniform when subsequent terms are no longer small corrections to the previous terms. This occurs when subsequent terms are of same order or of dominant order than previous terms.

★ Region of Non-Uniformity:-

Examples:-

① Consider the expansion

$$f(x;\varepsilon) = 1+\varepsilon x+\varepsilon^2 x^2+\varepsilon^3 x^3+\dots \quad \text{as } \varepsilon \to 0$$

In above expansion subsequent terms are of same order when $x = O(1/\varepsilon)$ as $\varepsilon \to 0$. and of dominant order for larger x i.e. $x = O(1/\varepsilon^2)$.

In above expansion non-uniformity occurs when $\varepsilon x = O(1)$ i.e $x = O(\frac{1}{\varepsilon})$ as $\varepsilon \to 0$

Also above expansion is valid if $x = O(1)$ so that εx decreases by a factor ε. The expansion remains valid for large 'x' provided x is not larger as $1/\varepsilon$. For example expansion is valid for $x = O(1/\sqrt{\varepsilon})$ as $\varepsilon \to 0$

② Consider the expansion

$$f(x;\varepsilon) = 1 + \varepsilon e^{x} + \varepsilon^2 e^{2x} + \cdots \quad a\ \varepsilon \to 0$$

The region of non-uniformity is ocurs when $\varepsilon e^{x} = O(1)$ i.e $e^{x} = O(1/\varepsilon)$

$$\Rightarrow x = O(-\ln \varepsilon) \quad \text{as } \varepsilon \to 0$$

③ Consider the expansion

$$f(x;\varepsilon) = 1 + \frac{\varepsilon}{x} + \frac{\varepsilon^2}{x^2} + \frac{\varepsilon^3}{x^3} + \cdots \quad \text{as } \varepsilon \to 0$$

The region of non-uniformity occurs when $\frac{\varepsilon}{x} = O(1)$ i.e $\frac{x}{\varepsilon} = O(1)$

$$\Rightarrow x = O(\varepsilon)$$

④ Consider the expansion

$$\sin(x+\varepsilon) = \sin x \cos\varepsilon + \cos x \sin\varepsilon$$

$$= \sin x\left[1 - \frac{\varepsilon^2}{2} + O(\varepsilon^4)\right] + \cos x\left[\varepsilon - \frac{\varepsilon^3}{3!} + O(\varepsilon^5)\right]$$

$$= \sin x + \varepsilon\cos x - \frac{\varepsilon^2}{2}\sin x - \frac{\varepsilon^3}{6}\cos x + O(\varepsilon^4)$$

As in above expansion, each coefficients of the ascending powers of ε is bounded by a fixed quantity [$\sin x$ and $\cos x$ are bounded between $-1 \le x \le 1$] are small corrections to the previous terms for all 'x' thus expansion is uniform for all x

⑤ Consider the expansion

$$\sin[x(1+\varepsilon)] = \sin(x + x\varepsilon)$$

$$= \sin x \cos x\varepsilon + \cos x \sin x\varepsilon$$

$$= \sin x\left[1 - \frac{x^2\varepsilon^2}{2} + O(\varepsilon^4 x^4)\right] + \cos x\left[\varepsilon x - \frac{\varepsilon^3 x^3}{3!} + O(\varepsilon^5 x^5)\right]$$

$$\Rightarrow \sin[x(1+\varepsilon)] = \sin x + \varepsilon x \cos x - \frac{x^2\varepsilon^2}{2}\sin x$$
$$- \frac{x^3\varepsilon^3}{6}\cos x + O(\varepsilon^4 x^4)$$

In above expansion coefficients of ascending powers of ε can not be bounded by a fixed quantity and consequently it is possible for subsequent term to have same order as that of previous terms. The region of non-uniformity occurs when $\varepsilon x = O(1)$ i.e $x = O(1/\varepsilon)$

Note:- Trignometric functions $\sin x$ and $\cos x$ are assigned order unity $\forall x$

Definition:- Non uniformaity occurs when subsequent terms are of same order or of dominant order than previous terms.

The critical case is that subsequent terms are of same order. This determins the region of non-uniformity.

Summary:- Non uniformities do not occur in the expansions

$$f(x;\varepsilon) \simeq \sum_{n=0}^{\infty} f_n(x)\,\delta_n(\varepsilon) \quad \text{as } \varepsilon \to 0$$

If the coefficients $f_n(x)$ are bounded.

$\Rightarrow$ For a non-uniformity we require

$$f_{n+1}(x)\,\delta_{n+1}(\varepsilon) = O[f_n(x)\,\delta_n(\varepsilon)]$$

i.e $f_{n+1}(x) = f_n(x)\, O\left[\dfrac{\delta_n(\varepsilon)}{\delta_{n+1}(\varepsilon)}\right] \longrightarrow ①$

Since the functions $\delta_n(\varepsilon)$ form as asymptotic sequence as $\varepsilon \to 0$ so that the ratio $\delta_n(\varepsilon)/\delta_{n+1}(\varepsilon)$ is singular

⇒ The non-uniformity condition ① requires not only that $f_{n+1}(x)$ is singular but also that it is more singular than $f_n(x)$.

Example:- Consider the expansion
$$x + \varepsilon x + \varepsilon^2 x + \varepsilon^n x$$
is uniform even throught the coefficient functions as $x \to \infty$. The singularity does not increase in strength with subsequent terms. So that the increasing powers of ε ensure that subsequent terms are small corrections to the previous terms regardless of how large x becomes.

Note:- ① To obtain a region of non-uniformity we equate the order of subsequent term and previous term.

② To see whether a expansion is uniformly valid or not, we have to watch $f_n(x)$. If $f_{n+1}(x)$ is more singular than $f_n(x)$ then expansion is non-uniform. If singularity of $f_n(x)$ does not increases as n increases then expansion is uniform.

⇒ Lindstend-Poincaré Technique:-

This technique is used to render the non-uniformity by straining the co-ordinate.

In this technique we apply transformation to independent coordinate (t) as follows

$$\tau = (1+\varepsilon\omega_1+\varepsilon^2\omega_2+\cdots)t$$

Then expanding the dependant coordinate in terms of τ i.e

$$u(\tau;\varepsilon) \sim u_0(\tau)+\varepsilon u_1(\tau)+\varepsilon^2 u_2(\tau)+\cdots \quad \text{as } \varepsilon \to 0$$

Idea of above technique can be explained by examples (Which we have done before)

⇒ LightHill Technique:-

The method of strained coordinate has been extended by lighthill to allow a broader class of straining transformation than that used in lindstend-Poincare' technique.

LightHill uses the straining transformation

$$x \sim s+\varepsilon f_1(s)+\varepsilon^2 f_2(s)+\cdots$$

where 'x' is original independent variable and 's' is strained coordinate.

Note that Lindstend-Poincare' tech is special case of LightHill's technique.

The procedure is analogous to the Lindstend poincare technique.

⇒ Renormalization:-

Renormalization technique is equivalent to above manipulatively less complicated. This method starts from a straight forward expansion of the original governing equation.

$$u(x;\varepsilon) = u_0(x) + \varepsilon_1 u_1(x) + \varepsilon_2 u_2(x) + \cdots \quad \text{as } \varepsilon \to 0$$

which will be non-uniform in general. Then in straight forward expansion we use the Light hill's straining transformation.

$$x \sim s + \varepsilon f_1(s) + \varepsilon^2 f_2(s) + \cdots \longrightarrow (1)$$

Then we determine $f_n(s)$ by using light hill's condition to both $u(s;\varepsilon)$ and transformation (1)

The Light hill condition is

"Subsequent coefficient functions should be no more singular than previous function"

ASSIGNMENT # 2

Question 1:- Obtain two term straight forward expansion of the solution of

$$(x+\varepsilon y)\frac{dy}{dx}+4y=1 \quad \text{where } 0<\varepsilon \ll 1,$$
$$y(1)=1/2$$

Determine region of non-uniformity. Obtain using method of renormalization, the uniformly valid one term expansion.

Solution

Given equation is

$$(x+\varepsilon y)\frac{dy}{dx}+4y=1 \quad , \quad y(1)=\frac{1}{2} \longrightarrow ①$$

we assume solution of ① of the form

$$y(x;\varepsilon)=y_0(x)+\varepsilon y_1(x)+\varepsilon^2 y_2(x)+\cdots \longrightarrow ②$$

Putting this in ① we get

$$\left[x+\varepsilon(y_0+\varepsilon y_1+\cdots)\right]\left[\frac{dy_0}{dx}+\varepsilon\frac{dy_1}{dx}+\cdots\right]$$
$$+4\left[y_0+\varepsilon y_1+\cdots\right]=1$$

Also $\quad y_0(1)+\varepsilon y_1(1)+\cdots=1/2$

Equating coefficients of like powers of ε we have.

ε^0: $$x\frac{dy_0}{dx}+4y_0=1 \longrightarrow ③$$

Also $\quad y_0(1)=1/2$

ε^1: $$y_0\frac{dy_0}{dx}+x\frac{dy_1}{dx}+4y_1=0 \ , \ y_1(1)=0 \longrightarrow ④$$

from ③ we have

$$x\frac{dy_0}{dx} + 4y_0 = 1$$

$$\Rightarrow \frac{dy_0}{dx} + \frac{4}{x}y_0 = \frac{1}{x}$$

Which is linear equation

$$\text{I.F} = e^{\int \frac{4}{x}dx} = e^{4\int \frac{1}{x}dx} = e^{4\ln x} = e^{\ln x^4} = x^4$$

$$\Rightarrow x^4\frac{dy_0}{dx} + 4x^3y_0 = x^3$$

$$\Rightarrow \frac{d}{dx}(x^4y_0) = x^3$$

By Integrating we get

$$x^4y_0 = \frac{x^4}{4} + c$$

using $y_0(1) = \frac{1}{2}$ $\Rightarrow (1)^4\left(\frac{1}{2}\right) = \frac{(1)^4}{4} + c$

$$\Rightarrow \boxed{c = 1/4}$$

So $\boxed{y_0 = 1/4 + 1/4x^4}$

From equation ④ we have

$$\left(\frac{1}{4} + \frac{1}{4x^4}\right)\left(\frac{-1}{x^5}\right) + x\frac{dy_1}{dx} + 4y_1 = 0$$

$$\Rightarrow x\frac{dy_1}{dx} + 4y_1 = \frac{1}{4x^5} + \frac{1}{4x^9}$$

$$\Rightarrow \frac{dy_1}{dx} + \frac{4}{x}y_1 = \frac{1}{4x^6} + \frac{1}{4x^{10}}$$

Which is linear equation

$$I.F = x^4$$

$$\Rightarrow x^4 \frac{dy_1}{dx} + 4x^3 y_1 = \frac{1}{4x^2} + \frac{1}{4x^6}$$

$$\Rightarrow \frac{d}{dx}(x^4 y_1) = \frac{1}{4}x^{-2} + \frac{1}{4}x^{-6}$$

Integrating we get

$$x^4 y_1 = \frac{-1}{4x} - \frac{1}{20x^5} + c$$

using $y_1(1) = 0$

$$\Rightarrow (1)^4(0) = \frac{-1}{4} - \frac{1}{20} + c \Rightarrow \boxed{c = \frac{3}{10}}$$

So

$$\boxed{y_1 = \frac{-1}{4x^5} - \frac{1}{20x^9} + \frac{3}{20x^4}}$$

So equation ① becomes

$$y(x;\varepsilon) = \frac{1}{4} + \frac{1}{4x^4} + \varepsilon\left(-\frac{1}{4x^5} - \frac{1}{20x^9} + \frac{3}{10x^4}\right) + \cdots \longrightarrow ⑤$$

Which is required two term expansion.

Region of Non-Uniformity:-

Region of non-uniformity is obtained by equating the order of y_0 and εy_1 as $\varepsilon \to 0$

i.e $\frac{1}{x^4} = O\left(\frac{\varepsilon}{x^9}\right)$

Hence the order of non-uniformity is

$$x = O(\varepsilon^{1/5})$$

To renormalize the straight forward expansion we introduce strained coordinate 's' where

$$x = s + \varepsilon f_1(s) + \cdots \longrightarrow ⑥$$

so equ ⑤ becomes

$$y(s,\varepsilon) = \frac{1}{4} + \frac{1}{4(s+\varepsilon f_1(s)+\cdots)^4} + \varepsilon\left[\frac{-1}{4(s+\cdots)^5}\right.$$

$$\left. - \frac{1}{20(s+\cdots)^9} + \frac{3}{10(s+\cdots)^4}\right] + \cdots$$

$$= \frac{1}{4} + \frac{1}{4s^4}\left(1 - \frac{4\varepsilon_1 f_1}{s} + \cdots\right) + \varepsilon\left(\frac{-1}{4s^5} - \frac{1}{20s^9}\right.$$

$$\left. + \frac{3}{10s^4}\right) + \cdots$$

$$\Rightarrow y(s,\varepsilon) = \frac{1}{4} + \frac{1}{4s^4} + \varepsilon\left(\frac{-f_1}{s^5} - \frac{1}{4s^5} - \frac{1}{20s^9} + \frac{3}{10s^4}\right) + \cdots$$

In order to make the above expansion uniform the singularities of order $O(1/s^5)$ and $O(1/s^9)$ in 2nd term must be removed by some choice of f_1 i.e.

$$\frac{f_1}{s^5} - \frac{1}{4s^5} - \frac{1}{20s^9} = 0$$

$$\Rightarrow f_1 = -\frac{1}{4} - \frac{1}{20s^4}$$

So
$$y(s;\varepsilon) = \frac{1}{4} + \frac{1}{4s^4} + O(\varepsilon)$$

Also from ⑥ we have

$$x = s - \frac{\varepsilon}{20}\left(5 + \frac{1}{s^4}\right) + O(\varepsilon^2) \quad \text{as } \varepsilon \longrightarrow 0$$

Question 2:- Use the renormalization technique to obtain a one-term uniformly valid expansion for the solution of

$$\frac{d^2u}{dt^2} + u = \varepsilon u^5 , \quad u(0) = 1 , \quad \frac{du}{dt}(0) = 0 \longrightarrow ①$$

Solution We assume solution of ① of the form

$$u(t) = u_0(t) + \varepsilon u_1(t) + \cdots \longrightarrow ②$$

So ① becomes

$$\left(\frac{d^2u_0}{dt^2} + \varepsilon \frac{d^2u_1}{dt^2} + \cdots\right) + (u_0 + \varepsilon u_1 + \cdots) = \varepsilon (u_0 + \varepsilon u_1 + \cdots)^5$$

Also $u_0(0) + \varepsilon u_1(0) + \cdots = 1$,

$$\frac{du_0}{dt} + \varepsilon \frac{du_1}{dt} + \cdots = 0$$

Equating coefficients of like powers of ε we have

ε^0: $\frac{d^2u_0}{dt^2} + u_0 = 0 ; \quad u_0(0) = 1 , \quad \frac{du_0}{dt}(0) = 0 \longrightarrow ③$

ε^1: $\frac{d^2u_1}{dt^2} + u_1 = u_0^5 ; \quad u_1(0) = 0 , \quad \frac{du_1}{dt}(0) = 0 \longrightarrow ④$

General solution of ③ is

$$u_0 = A\cos t + B\sin t$$

using $u_0(0) = 1 \Rightarrow \boxed{A = 1}$

$$\Rightarrow \frac{du_0}{dt} = -\sin t + B\cos t$$

using $\frac{du_0}{dt}(0) = 0$, we have $\boxed{B = 0}$

Hence $\boxed{u_0 = \cos t}$ $\longrightarrow$ (A)

Now from (4) we have

$$\frac{d^2u_1}{dt^2} + u_1 = u_0^5 , \quad u_1(0) = 0 , \quad \frac{du_1}{dt}(0) = 0$$

$$\Rightarrow \quad \frac{d^2u_1}{dt^2} + u_1 = \cos^5 t \longrightarrow (5)$$

Now we express $\cos^5 t$ in terms of multiples of angles, so that particular integral of above can easily be found

As $\cos^5 t = \left[\dfrac{e^{it} + e^{-it}}{2}\right]^5$

$$= \frac{1}{2^5}\left[e^{it} + e^{-it}\right]^5$$

By using binomial theorem

$$\cos^5 t = \frac{1}{32}\left[\binom{5}{0}e^{i5t} + \binom{5}{1}e^{i4t}e^{-it} + \binom{5}{2}e^{i3t}e^{-i2t} + \binom{5}{3}e^{i2t}e^{-i3t} + \binom{5}{4}e^{it}e^{-i4t} + \binom{5}{5}e^{-i5t}\right]$$

$$= \frac{1}{32}\left[e^{i5t} + 5e^{i3t} + 10e^{it} + 10e^{-it} + 5e^{-i3t} + e^{-i5t}\right]$$

$$= \frac{1}{32}\left[(e^{i5t} + e^{-i5t}) + 10(e^{it} + e^{-it}) + 5(e^{i3t} + e^{-i3t})\right]$$

$$= \frac{1}{32}\left[(2\cos 5t) + 5(2\cos 3t) + 10(2\cos t)\right]$$

$$\Rightarrow \cos^5 t = \frac{1}{16}\left[\cos 5t\right] + \frac{5}{16}\cos 3t + \frac{5}{8}\cos t$$

So ⑤ becomes

$$\frac{d^2u_1}{dt^2}+u_1=\frac{1}{16}\cos 5t+\frac{3}{16}\cos 3t+\frac{5}{8}\cos t \longrightarrow ⑥$$

$$u_1(0)=0\ ,\quad \frac{du_1}{dt}(0)=0$$

Complementry function of above equation can be easily found as

$$u_{1c}=D\cos t+C\sin t$$

For Particular solution we put u_{1p} in equ ⑥ imply

$$(D^2+1)u_{1p}=\frac{1}{16}\cos 5t+\frac{5}{16}\cos 3t+\frac{5}{8}\cos t$$

where $D^2=\frac{d^2}{dt^2}$

$$\Rightarrow u_{1p}=\frac{1}{D^2+1}\left[\frac{1}{16}\cos 5t+\frac{5}{16}\cos 3t+\frac{5}{8}\cos t\right]$$

$$=\frac{1}{16}\frac{1}{D^2+1}\cos 5t+\frac{5}{16}\frac{1}{D^2+1}\cos 3t+\frac{5}{8}\frac{1}{D^2+1}\cos t$$

$$\Rightarrow u_{1p}=\frac{1}{16}\left\{Re\frac{1}{D^2+1}e^{i5t}\right\}+\frac{5}{16}\left\{Re\frac{1}{D^2+1}e^{i3t}\right\}$$

$$+\frac{5}{8}\left\{Re\frac{1}{D^2+1}e^{it}\right\}$$

Now we use the following two results

(a) $\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}$ if $f(a)\neq 0$

(b) If $f(D)$ has a K-fold root at

'a', then we can write $f(D)=\phi(D)(D-a)^k$

Then we have $$\frac{1}{f(D)}e^{ax}=\frac{1}{\phi(D)(D-a)^k}e^{ax}=\frac{x^k e^{ax}}{k!\,\phi(a)}$$

[We use result (a) on first two terms on R.H.S and result (b) on third term on R.H.S]

So
$$u_{1p}=\frac{1}{16}\left\{Re\ \frac{e^{i5t}}{(5i)^2+1}\right\}+\frac{5}{16}\left\{Re\ \frac{e^{i3t}}{(3i)^2+1}\right\}$$
$$+\frac{5}{8}\left\{Re\ \frac{1}{(D+i)(D-i)}e^{it}\right\}$$
$$=\frac{1}{16}\left\{Re\ \frac{e^{i5t}}{-24}\right\}+\frac{5}{16}\left\{Re\ \frac{e^{i3t}}{-8}\right\}+\frac{5}{8}\left\{Re\ \frac{t\,e^{it}}{(i+i)\,1!}\right\}$$
$$=\frac{1}{16}\left\{\frac{\cos 5t}{-24}\right\}+\frac{5}{16}\left\{\frac{\cos 3t}{-8}\right\}+\frac{5}{8}\left\{Re\ \frac{ti(\cos t+i\sin t)}{2i^2}\right\}$$
$$=\frac{1}{16}\left\{\frac{\cos 5t}{-24}\right\}+\frac{5}{16}\left\{\frac{\cos 3t}{-8}\right\}+\frac{5}{8}\left\{\frac{t\sin t}{2}\right\}$$
$$\Rightarrow u_{1p}=\frac{-1}{384}\cos 5t-\frac{5}{128}\cos 3t+\frac{5}{16}t\sin t$$

Now as $u_1=u_{1c}+u_{1p}$

$$\Rightarrow u_1=D\cos t+C\sin t-\frac{1}{384}\cos 5t-\frac{5}{128}\cos 3t+\frac{5}{16}t\sin t$$

using $u_1(0)=0$ imply

$$0=D-\frac{1}{384}-\frac{5}{128}\Rightarrow 0=D-\frac{1}{24}$$
$$\Rightarrow \boxed{D=\frac{1}{24}}$$

Now $$\frac{du_1}{dt}=\frac{-1}{24}\sin t+C\sin t+\frac{5}{384}\sin 5t+\frac{15}{128}\sin 3t$$
$$+\frac{5}{16}\left[t\cos t+\sin t\right]$$

using $\frac{du_1}{dt}(0) = 0$ imply $\boxed{c = 0}$

So
$$\boxed{u_1 = \frac{1}{24}\cos t - \frac{1}{384}\cos 5t - \frac{5}{128}\cos 3t + \frac{5}{16} t \sin t} \rightarrow (B)$$

using (A) and (B) in (2) we have

$$u = \cos t + \varepsilon\left[\frac{1}{24}\cos t - \frac{1}{384}\cos 5t - \frac{5}{128}\cos 3t + \frac{5}{16} t \sin t\right] \longrightarrow (7)$$

Above solution is non-uniform due to presence of secular term $\left(\varepsilon \frac{5}{16} t \sin t\right)$

Hence we use the following transformation to render the above solution uniform

$$t = s + \varepsilon f_1(s) + \cdots \longrightarrow (8)$$

So equ (7) becomes

$$u(s) = \cos(s + \varepsilon f_1(s) + \cdots) + \varepsilon\left[\frac{1}{24}\cos(s + \varepsilon f_1(s) + \cdots) - \frac{1}{384}\cos(5s + \varepsilon 5 f_1(s) + \cdots) - \frac{5}{128}(3s + 3\varepsilon f_1(s) + \cdots) + \frac{5}{16}(s + \varepsilon f_1(s) + \cdots)\sin(s + \varepsilon f_1(s) + \cdots)\right]$$

using Taylor series

$$u(s) = \cos s - \varepsilon f_1 \sin s + \cdots + \varepsilon\left[\frac{\cos s + \cdots}{24} - \frac{1}{384}\cos 5s + \cdots - \frac{5}{128}\cos 3s + \cdots + \frac{5}{16} s \sin s + \cdots\right]$$

$$\Rightarrow u(s) = \cos s + \varepsilon\left[\frac{1}{24}\cos s - \frac{1}{384}\cos 5s - \frac{5}{128}\cos 3s\right.$$
$$\left. + \left(-f_1 + \frac{5}{16}s\right)\sin s\right] + O(\varepsilon^2)$$

In order to remove secular term, we must have

$$-f_1 + \frac{5}{16}s = 0 \quad \text{i.e.} \quad \boxed{f_1 = \frac{5}{16}s}$$

Hence one term uniformly valid expansion is

$$\boxed{\begin{array}{l} u = \cos s + O(\varepsilon)\,; \\ \text{where } t = s + \frac{5}{16}s\varepsilon + O(\varepsilon^2) \end{array}}$$

** ——— **

★ **Time Scales:-** The basic idea behind the multiple scales is to avoid the introduction of non-uniformities associated with expansions of functions by preserving the combination $\varepsilon t, \varepsilon^2 t, \varepsilon^3 t$, etc as a variable on which these functions depend. These combinations are called scales and denoted as follows

$$T_0 = t, \quad T_1 = \varepsilon t, \quad T_2 = \varepsilon^2 t, \ldots, T_n = \varepsilon^n t$$

We can demonstrate the idea by considering the following example.

$$f \sim \frac{1}{1-\varepsilon} \exp(t + \varepsilon t + \varepsilon^2 t + \varepsilon^3 t + \varepsilon^4 t + \cdots)$$

(a) Using One Scale i.e. $t = T_0$

$$\Rightarrow f \sim (1-\varepsilon)^{-1} e^{T_0 + \varepsilon T_0 + \varepsilon^2 T_0 + \cdots}$$

$$\sim \left(1 + \varepsilon + \frac{\varepsilon^2}{2} + \cdots\right) e^{T_0} \cdot e^{\varepsilon T_0 + \varepsilon^2 T_0 + \cdots}$$

Let $x = \varepsilon T_0 + \varepsilon^2 T_0 + \cdots$

$$\Rightarrow f \sim \left(1 + \varepsilon + \frac{\varepsilon^2}{2} + \cdots\right) e^{T_0} e^{x}$$

$$\sim \left(1 + \varepsilon + \frac{\varepsilon^2}{2} + \cdots\right) e^{T_0} \left[1 + x + \frac{x^2}{2} + \cdots\right]$$

$$\sim e^{T_0}\left(1 + \varepsilon + \frac{\varepsilon^2}{2} + \cdots\right)\left[1 + (\varepsilon T_0 + \varepsilon^2 T_0 + \cdots) + \frac{1}{2}(\varepsilon T_0 + \varepsilon^2 T_0 + \cdots)^2\right]$$

$$\sim e^{T_0}\left[1 + (\varepsilon T_0 + \varepsilon^2 T_0 + \cdots) + \frac{1}{2}(\varepsilon T_0 + \varepsilon^2 T_0 + \cdots)^2 + \varepsilon + \varepsilon(\varepsilon T_0 + \varepsilon^2 T_0 + \cdots) + \frac{\varepsilon^2}{2} + \cdots\right]$$

$$\Rightarrow f \sim e^{T_0}\left[1 + \varepsilon(1+T_0) + \varepsilon^2\left(2T_0 + \frac{1}{2} + \frac{1}{2}T_0^2 + \cdots\right) + O(\varepsilon^3)\right]$$

Region of Non-Uniformity:-

$$\frac{\varepsilon^2 T_0^2}{\varepsilon T_0} = O(1) \Rightarrow T_0 = O\left(\frac{1}{\varepsilon}\right)$$

$$\Rightarrow t = O\left(\frac{1}{\varepsilon}\right)$$

(b) Using Two Scales ie $T_0 = t$, $T_1 = \varepsilon t$

$$f \sim \frac{1}{1-\varepsilon} e^{t + \varepsilon t + \varepsilon^2 t + \varepsilon^3 t + \cdots}$$

$$\sim \left(1 + \varepsilon + \frac{\varepsilon^2}{2} + \cdots\right) e^{T_0 + T_1 + \varepsilon^2 T_0 + \varepsilon^3 T_0}$$

$$\sim \left(1 + \varepsilon + \frac{\varepsilon^2}{2} + \cdots\right) e^{T_0 + T_1} \cdot e^{\varepsilon^2 T_0 + \varepsilon^3 T_0 + \cdots}$$

Let $x = \varepsilon^2 T_0 + \varepsilon^3 T_0 + \cdots$

$$\Rightarrow f \sim e^{T_0 + T_1}\left(1 + \varepsilon + \frac{\varepsilon^2}{2} + \cdots\right) e^{x}$$

$$\sim e^{T_0+T_1}\left[1 + \varepsilon + \frac{\varepsilon^2}{2} + \cdots\right]\left[1 + (\varepsilon^2 T_0 + \cdots) + \frac{1}{2}(\varepsilon^2 T_0 + \cdots)^2 + \cdots\right]$$

$$\sim e^{T_0+T_1}\left[1 + (\varepsilon^2 T_0 + \cdots) + \frac{1}{2}(\varepsilon^2 T_0 + \cdots)^2 + \varepsilon + \frac{\varepsilon^2}{2} + \cdots\right]$$

$$\sim e^{T_0+T_1}\left[1 + \varepsilon + \varepsilon^2\left(\frac{1}{2} + T_0\right) + \cdots\right] \longrightarrow (*)$$

Region of Non Uniformity:-

$$\frac{\varepsilon^2 T_0}{\varepsilon} = O(1) \Rightarrow T_0 = O\left(\frac{1}{\varepsilon}\right)$$

$$\Rightarrow t = O\left(\frac{1}{\varepsilon}\right)$$

In order to keep one term

uniform expansion equ (*) can be written as

$$f = \exp(T_0, T_1)\left[1 + O(\varepsilon, \varepsilon^2 T_0)\right]$$

where this expansion is uniformly valid for

$$t = O(1/\varepsilon)$$

We can also expand our function for three or more scales, but here we will restrict our application of the multiple scale technique to the introduction of two scales "T_0" and "T_1" and obtain one term uniformly valid expansion for $t = O\left(\frac{1}{\varepsilon}\right)$ using the scales $T_0 = t$, $T_1 = \varepsilon t$

the time derivative becomes.

$$\frac{d}{dt} = \frac{dT_0}{dt}\frac{\partial}{\partial T_0} + \frac{dT_1}{dt}\frac{\partial}{\partial T_1}$$

$$\Rightarrow \frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1} \longrightarrow (a_1)$$

again differentiate w.r.t time

$$\Rightarrow \frac{d^2}{dt^2} = \frac{d}{dt}\left[\frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1}\right]$$

$$= \frac{\partial}{\partial T_0}\left(\frac{d}{dt}\right) + \varepsilon\frac{\partial}{\partial T_1}\left(\frac{d}{dt}\right)$$

$$= \frac{\partial}{\partial T_0}\left(\frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1}\right) + \varepsilon\frac{\partial}{\partial T_1}\left(\frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1}\right)$$

$$= \frac{\partial^2}{\partial T_0^2} + \varepsilon\frac{\partial^2}{\partial T_0 \partial T_1} + \varepsilon\frac{\partial^2}{\partial T_0 \partial T_1} + \varepsilon^2\frac{\partial^2}{\partial T_1^2}$$

$$\Rightarrow \frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon\frac{\partial^2}{\partial T_0 \partial T_1} + \varepsilon^2\frac{\partial^2}{\partial T_1^2} \longrightarrow (a_2)$$

Example:- Consider

$$\frac{d^2u}{dt^2} + u = -\varepsilon \frac{du}{dt} \; ; \; u(0)=1, \; \frac{du}{dt}(0)=0$$

Solution

using equation (a_1) and (a_2) in above differential equation

$$\frac{\partial^2 u}{\partial T_0^2} + 2\varepsilon \frac{\partial^2 u}{\partial T_0 \partial T_1} + \varepsilon^2 \frac{\partial^2 u}{\partial T_1^2} + u = -\varepsilon \frac{\partial u}{\partial T_0} - \varepsilon^2 \frac{\partial u}{\partial T_1} \longrightarrow (A)$$

We expand u in the form

$$u(T_0, T_1, \varepsilon) \sim u_0(T_0, T_1) + \varepsilon u_1(T_0, T_1) + \cdots \longrightarrow (1)$$

using equ (1) in (A) implies

$$\frac{\partial^2 u_0}{\partial T_0^2} + \varepsilon \frac{\partial^2 u_1}{\partial T_0^2} + 2\varepsilon \frac{\partial^2 u_0}{\partial T_0 \partial T_1} + u_0 + \varepsilon u_1$$

$$= -\varepsilon \frac{\partial u_0}{\partial T_0} - \varepsilon^2 \frac{\partial u_1}{\partial T_0}$$

Equating coefficients of $O(\varepsilon^0)$, $O(\varepsilon^1)$ implies

$O(\varepsilon^0)$:- $$\frac{\partial^2 u_0}{\partial T_0^2} + u_0 = 0 \longrightarrow (2)$$

$O(\varepsilon^1)$:- $$\frac{\partial^2 u_1}{\partial T_0^2} + u_1 = -2\frac{\partial^2 u_0}{\partial T_0 \partial T_1} - \frac{\partial u_0}{\partial T_0} \longrightarrow (3)$$

And initial conditions becomes

at $t=0 \Rightarrow T_0 = 0, \; T_1 = 0$

So $u(0)=1$ imply

$$u_0(T_0=0, T_1=0) + \varepsilon u_1(T_0=0, T_1=0) + \cdots = 1$$

$$\Rightarrow u_0(0,0) = 1 \;\; \& \;\; u_1(0,0) = 0 \longrightarrow (4)$$

And $\frac{du}{dt}(0) = 0$ imply

$$\frac{\partial u_0}{\partial T_0}(0,0) + \varepsilon \frac{\partial u_0(0,0)}{\partial T_1} + \varepsilon \frac{\partial u_1}{\partial T_1}(0,0) = 0$$

$$\Rightarrow \quad \frac{\partial u_0}{\partial T_0}(0,0) = 0 \quad \& \quad \frac{\partial u_0}{\partial T_0}(0,0) + \frac{\partial u_0}{\partial T_1}(0,0) = 0 \longrightarrow (5)$$

Now from equation (2)

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

Solution of (2) in exponential form for non repeated roots is

$$u_0 = c(T_1)e^{iT_0} + c^*(T_1)e^{-iT_0} \longrightarrow (B)$$

where $c^*(T_1)$ is conjugate of $c(T_1)$ and c is function of T_1 only.

As the particular integral of equ (3) contain secular term due to the term $\exp(\pm iT_0)$. So the R.H.S of equ (3) is

$$-2\frac{\partial^2 u_0}{\partial T_1 \partial T_0} - \frac{\partial u_0}{\partial T_0} = -2\frac{\partial}{\partial T_1}\left[\frac{\partial}{\partial T_0}\left\{c(T_1)e^{iT_0} + c^*(T_1)e^{-iT_0}\right\}\right]$$

$$- \frac{\partial}{\partial T_0}\left[c(T_1)e^{iT_0} + c^*(T_1)e^{-iT_0}\right]$$

$$= -2\frac{\partial}{\partial T_1}\left[ice^{iT_0} - ic^*e^{-iT_0}\right] - \left[ice^{iT_0} - ic^*e^{-iT_0}\right]$$

$$= -2ie^{iT_0}\frac{\partial c}{\partial T_1} + 2ie^{-iT_0}\frac{\partial c^*}{\partial T_1} - ice^{iT_0}$$

$$+ ic^* e^{-iT_0}$$

$$= -ie^{iT_0}\left[2\frac{\partial c(T_1)}{\partial T_1} + c(T_1)\right] + ie^{-iT_0}\left[2\frac{\partial c^*}{\partial T_1}\right.$$

$$\left. + c^*(T_1)\right]$$

So in order to avoid secular terms we require

$$2\frac{\partial c}{\partial T_1}+c=0 \longrightarrow (6a) \quad \& \quad 2\frac{\partial c^*}{\partial T_1}+c^*=0 \longrightarrow (6b)$$

We know that conjugate of a derivative of a complex number and derivative of a conjugate of a complex number are same.

$\Rightarrow$ Solution of (6a) and (6b) are same.

We can write $C(T_1)$ in polar form also

$$\Rightarrow C(T_1)=R(T_1)\,e^{i\theta(T_1)} \longrightarrow (7)$$

$\because z=x+iy$
$\Rightarrow z=r(\cos\theta+i\sin\theta)$
$\Rightarrow z=re^{i\theta}$

where $R(T_1)$ and $\theta(T_1)$ are real functions. We have to find R and θ.

For this differentiate (7) w.r.t T_1

$$\Rightarrow \frac{\partial c}{\partial T_1}=e^{i\theta}\frac{dR}{dT_1}+Ri\frac{d\theta}{dT_1}e^{i\theta}$$

$$=e^{i\theta}\left(\frac{dR}{dT_1}+iR\frac{d\theta}{dT_1}\right)$$

Equation (6a) imply

$$2\left(\frac{dR}{dT_1}+iR\frac{d\theta}{dT_1}\right)e^{i\theta}+Re^{i\theta}=0$$

$$\Rightarrow \left(2\frac{dR}{dT_1}+2iR\frac{d\theta}{dT_1}+R\right)e^{i\theta}=0 \quad ; \quad e^{i\theta}\neq 0$$

$$\Rightarrow R+2\frac{dR}{dT_1}+2iR\frac{d\theta}{dT_1}=0$$

equating real and imaginary parts

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$$R + 2\frac{dR}{dT_1} = 0$$

$$\Rightarrow \frac{dR}{dT_1} = \frac{-R}{2}$$

$$\Rightarrow \frac{dR}{R} = -\frac{1}{2}dT_1$$

$$\Rightarrow \ln R = -\frac{1}{2}T_1$$

$$\Rightarrow \ln R = \left(-\frac{1}{2}T_1\right)\cdot 1 + \ln c$$

$$= \left(-\frac{1}{2}T_1\right)\cdot \ln e + \ln R_0$$

$$= \ln e^{-\frac{1}{2}T_1} + \ln R_0$$

$$\Rightarrow \ln R = \ln\left(R_0 e^{-\frac{1}{2}T_1}\right)$$

$$\Rightarrow R = R_0 e^{-\frac{1}{2}T_1}$$

&

$$2R\frac{d\theta}{dT_1} = 0$$

$$\Rightarrow \frac{d\theta}{dT_1} = 0$$

$$\Rightarrow \theta = \theta_0 \quad \text{constant}$$

$$\Rightarrow C(T_1) = R_0 e^{-\frac{T_1}{2}} e^{i\theta_0}$$

$$\Rightarrow \bar{C}(T_1) = R_0 e^{-\frac{T_1}{2}} e^{-i\theta_0}$$

Put values of $C(T_1)$ and $\bar{C}(T_1)$ in (B) imply

$$u_0 = R_0 e^{-\frac{T_1}{2}} e^{i\theta_0} \cdot e^{iT_0} + R_0 e^{-\frac{T_1}{2}} e^{-i\theta_0} \cdot e^{-iT_0}$$

$$\Rightarrow u_0(T_0, T_1) = R_0 e^{-\frac{T_1}{2}}\left[e^{i(\theta_0+T_0)} + e^{-i(T_0+\theta_0)}\right]$$

$$= R_0 e^{-\frac{T_1}{2}}\left[\cos(\theta_0+T_0) + i\sin(\theta_0+T_0) + \cos(T_0+\theta_0) - i\sin(\theta_0+T_0)\right]$$

$$= R_0 e^{-\frac{T_1}{2}} 2\cos(\theta_0+T_0)$$

$$\Rightarrow u_0(T_0,T_1) = 2R_0 e^{-\frac{T_1}{2}} \cos(\theta_0 + T_0)$$

$\frac{du_0}{dT_0}(0,0) = 0$

$\Rightarrow -2R_0 e^{-\frac{T_1}{2}} \sin(\theta_0+T_0) = 0$

$\Rightarrow -2R_0 \neq 0$

$\Rightarrow \sin\theta_0 = 0$

$\Rightarrow \theta_0 = 0$

using $u_0(0,0) = 1$

$\Rightarrow 1 = 2R_0 \cdot 1 \cdot \cos 0$

$$\Rightarrow \boxed{R_0 = 1/2}$$

$$\Rightarrow \boxed{u_0 = e^{-\frac{T_1}{2}} \cos T_0} \longrightarrow (8)$$

We did not evaluate u_1, but merely ensure that secular terms are absent from u_1, so we can write one term uniformly valid expansion as

$$u(T_0,T_1) = e^{-T_1/2} \cos T_0 + O(\varepsilon) \longrightarrow (9)$$

And this expansion is also valid for $t = O(1/\varepsilon)$

$$\Rightarrow \boxed{u(t,\varepsilon) = e^{-\frac{\varepsilon t}{2}} \cos t + O(\varepsilon)}$$

Example 2: We consider the Van der Pol oscillator i.e

$$\frac{d^2u}{dt^2} + u = \varepsilon(1-u^2)\frac{du}{dt}$$

$$u(0) = a, \quad \frac{du}{dt}(0) = 0$$

Solution

using $T_0 = t$ and $T_1 = \varepsilon t$ and using equations "a_1" and "a_2" in given differential equation:

$$\frac{\partial^2 u}{\partial T_0^2} + 2\varepsilon \frac{\partial^2 u}{\partial T_0 \partial T_1} + \varepsilon^2 \frac{\partial^2 u}{\partial T_1^2} + u = \varepsilon(1-u^2)\left[\frac{\partial u}{\partial T_0} + \varepsilon \frac{\partial u}{\partial T_1}\right] \longrightarrow (1)$$

Let $u(T_0, T_1, \varepsilon) \sim u_0(T_0, T_1) + \varepsilon u_1(T_0, T_1) + \cdots$ $\longrightarrow$ ②

$O(\varepsilon^0)$: $\dfrac{\partial^2 u_0}{\partial T_0^2} + u_0 = 0 \longrightarrow$ ③

with $u_0(0,0) = a$, $\dfrac{\partial u_0}{\partial T_0}(0,0) = 0$

$O(\varepsilon^1)$: $\dfrac{\partial^2 u_1}{\partial T_1^2} + 2\dfrac{\partial^2 u_0}{\partial T_0 \partial T_1} + u_1 = \dfrac{\partial u_0}{\partial T_0} - u_0^2 \dfrac{\partial u_0}{\partial T_0}$

$$\Rightarrow \frac{\partial^2 u_1}{\partial T_0^2} + u_1 = -2\frac{\partial^2 u_0}{\partial T_0 \partial T_1} + (1 - u_0^2)\frac{\partial u_0}{\partial T_0} \longrightarrow ④$$

with $u_1(0,0) = 0$, $\dfrac{\partial u_0}{\partial T_1}(0,0) + \dfrac{\partial u_1}{\partial T_0}(0,0) = 0$

Initial condition $\Rightarrow$ at $t = 0$, $T_0 = 0$, $T_1 = 0$

$u(0) = a$

$\Rightarrow u_0(0,0) + \varepsilon u_1(0,0) + \cdots = a$

$\Rightarrow u_0(0,0) = a$, $u_1(0,0) = 0$

$\dfrac{\partial u}{\partial t}(0) = 0 \Rightarrow \dfrac{\partial u_0}{\partial T_0}(0,0) + \varepsilon \dfrac{\partial u_0}{\partial T_1} + \varepsilon \dfrac{\partial u_1}{\partial T_0}(0,0) = 0$

$\Rightarrow \dfrac{\partial u_0(0,0)}{\partial T_0} = 0$, $\dfrac{\partial u_0}{\partial T_1}(0,0) + \dfrac{\partial u_1(0,0)}{\partial T_0} = 0$

Solution of ③ in exponential form is

$$u_0 = B(T_1)e^{iT_0} + B^*(T_1)e^{-iT_0} \longrightarrow ⑤$$

where $B(T_1)$ and $B^*(T_1)$ are conjugate complex numbers.

As the particular integral of eqn ④ contains the secular terms due to the term $\exp(\pm iT_0)$. To find R.H.S of equ ④ i.e.

$$= -2\frac{\partial}{\partial T_1}\left[\frac{\partial}{\partial T_0}\left\{B(T_1)e^{iT_0} + B^*(T_1)e^{-iT_0}\right\}\right]$$
$$+\left[1-\left\{Be^{iT_0}+B^*e^{-iT_0}\right\}^2\right]\cdot\frac{\partial}{\partial T_0}\left\{Be^{iT_0}+B^*e^{-iT_0}\right\}$$

$$= -2\frac{\partial}{\partial T_1}\left[Bie^{iT_0} - B^* i e^{-iT_0}\right] + \left[1 - B^2e^{2iT_0} - 2BB^*\right.$$
$$\left. - B^{*2}e^{-2iT_0}\right]\left[Bie^{iT_0} - B^* i e^{-iT_0}\right]$$

$$= -2ie^{iT_0}\frac{\partial B(T_1)}{\partial T_1} + 2ie^{-iT_0}\frac{\partial B^*(T_1)}{\partial T_1} + Bie^{iT_0}$$
$$- B^* i e^{-iT_0} - B^{*2}Bie^{-iT_0} + B^{*3}ie^{-3iT_0} - B^3 i e^{3iT_0}$$
$$+ B^2B^* i e^{iT_0} - 2iB^2B^*e^{iT_0} + 2BB^{*2} i e^{-iT_0}$$

$$= ie^{iT_0}\left[-2\frac{\partial B}{\partial T_1} + B + B^2B^* - 2B^2B^*\right] + ie^{-iT_0}\left[2\frac{\partial B^*}{\partial T_1}\right.$$
$$\left. - B^* - B^{*2}B + 2BB^{*2}\right]$$

$$= e^{iT_0}\left[-2i\frac{\partial B}{\partial T_1} + Bi - B^2B^* i\right] + e^{-iT_0}\left[2i\frac{\partial B^*}{\partial T_1}\right.$$
$$\left. + B^{*2}Bi - B^* i\right]$$

As coefficient of e^{-iT_0} is complex conjugate of e^{iT_0}, so their solution becomes same.

To avoid secular terms appearing in u_1, the coefficients of $e^{\pm iT_0}$ must be zero. Using polar form of $B(T_1)$ i.e

$$B(T_1) = R(T)\, e^{i\theta(T_1)} \longrightarrow (6)$$

$$\Rightarrow -2i\left[Ri\, e^{i\theta}\frac{d\theta}{dT_1} + e^{i\theta}\frac{dR}{dT_1}\right] + iRe^{i\theta} - iR^2 e^{2i\theta}.Re^{-i\theta} = 0$$

$$\Rightarrow \left\{2R\frac{d\theta}{dT_1} + i\left[-2\frac{dR}{dT_1} + R - R^3\right]\right\}e^{i\theta} = 0 \;;\; e^{i\theta} \neq 0$$

Equating real and imaginary parts

$$2R\frac{d\theta}{dT_1} = 0$$

$$\Rightarrow \frac{d\theta}{dT_1} = 0$$

$$\Rightarrow \theta = \theta_0 \text{ constant}$$

$$-2\frac{dR}{dT_1} + R - R^3 = 0$$

$$\Rightarrow 2\frac{dR}{dT_1} = R(1-R^2)$$

$$\Rightarrow \frac{dR}{dT_1} = \frac{R(1-R)(1+R)}{2}$$

$$\Rightarrow \frac{dR}{R(1-R)(1+R)} = \frac{1}{2}dT_1$$

$$\Rightarrow \frac{dR}{R(R-1)(R+1)} = -\frac{1}{2}dT_1$$

$$\Rightarrow \frac{1}{R(R-1)(R+1)} = \frac{A_1}{R} + \frac{B_1}{R-1} + \frac{C_3}{R+1}$$

$$\Rightarrow A_1 = \frac{1}{(R-1)(R+1)} = -1$$

$$B_1 = \frac{1}{R(R+1)} = \frac{1}{2}, \quad C_3 = \frac{1}{R(R-1)} = \frac{1}{2}$$

$$\Rightarrow \int\left[-\frac{1}{R} + \frac{1}{2(R-1)} + \frac{1}{2(R+1)}\right]dR = -\frac{1}{2}dT_1$$

$$\Rightarrow -\ln R + \frac{1}{2}\left[\ln(R-1) + \ln(R+1)\right] = -\frac{1}{2}T_1 + X$$

$$\Rightarrow -\ln R + \frac{1}{2}\ln(R^2-1) = \left(-\frac{1}{2}T_1 + X\right)\cdot 1$$

$$\ln R^{-1} + \ln(R^2-1)^{\frac{1}{2}} = \left(-\frac{1}{2}T_1 + X\right)\cdot \ln e$$

$$\Rightarrow \ln \frac{(R^2-1)^{\frac{1}{2}}}{R} = \ln e^{-\frac{1}{2}T_1 + X} = e^{-\frac{1}{2}T_1}\cdot K$$

$$\Rightarrow \frac{R^2-1}{R^2} = K_1 e^{-T_1}$$

$$\Rightarrow 1 - \frac{1}{R^2} = K_1 e^{-T_1} \Rightarrow 1 - K_1 e^{-T_1} = \frac{1}{R^2}$$

$$\Rightarrow R^2 = \frac{1}{1-K_1 e^{-T_1}} \Rightarrow R = \sqrt{\frac{1}{1-K_1 e^{-T_1}}}$$

equ ⑥ in p.by

$$B(T_1) = \sqrt{\frac{1}{1-K_1 e^{-T_1}}}\cdot e^{i\theta_0}$$

equ ⑤ $\Rightarrow$

$$U_0(T_0,T_1) = \sqrt{\frac{1}{1-K_1 e^{-T_1}}}\, e^{i\theta_0}\cdot e^{iT_0} + \sqrt{\frac{1}{1-K_1 e^{-T_1}}}\, e^{-i\theta_0}\cdot e^{-iT_0}$$

$$\Rightarrow U_0(T_0,T_1) = \sqrt{\frac{1}{1-K_1 e^{-T_1}}}\left[e^{i(\theta_0+T_0)} + e^{-i(\theta_0+T_0)}\right]$$

using $U_0(0,0) = a$ and $\frac{dU_0(0,0)}{dT_0} = 0$

$$\Rightarrow \frac{dU_0(T_0,T_1)}{dT_0} = \sqrt{\frac{1}{1-K_1 e^{-T_1}}}\left[e^{i(\theta_0+T_0)} - i e^{-i(\theta_0+T_0)}\right]$$

$$\frac{dU_0(0,0)}{dT_0} = i\sqrt{\frac{1}{1-K_1}}\left[e^{i\theta_0} - e^{-i\theta_0}\right]$$

$$\Rightarrow \quad 0 = i\sqrt{\frac{1}{1-K_1}}\left[2i\sin\theta_0\right]$$

$$\Rightarrow \quad -2\sqrt{\frac{1}{1-K_1}}\sin\theta_0 = 0 \Rightarrow -2\sqrt{\frac{1}{1-K_1}} \neq 0$$

$$\Rightarrow \quad \sin\theta_0 = 0 \quad \Rightarrow \boxed{\theta_0 = 0}$$

$$u_0(0,0) = \sqrt{\frac{1}{1-K_1}}\left[1+1\right]$$

$$\Rightarrow \quad \frac{a}{2} = \sqrt{\frac{1}{1-K_1}}, \quad a^2(1-K_1) = 4$$

$$1-K_1 = 4/a^2$$

$$\Rightarrow \quad \boxed{K_1 = 1 - 4/a^2}$$

As

$$u_0(T_0,T_1) = \sqrt{\frac{1}{1-K_1e^{-T_1}}}\cdot 2\cos(\theta_0+T_0)$$

$$\Rightarrow \quad u_0(T_0,T_1) = \sqrt{\frac{1}{1-\left(1-\frac{4}{a^2}\right)e^{-T_1}}}\cdot 2\cos T_0$$

Thus the uniformly valid one term expansion tends to the limit cycle $u = 2\cos t + O(\varepsilon)$ for all initial values.

Question:- Use the multiple scale with scales $T_0 = t$ and $T_1 = \epsilon t$ to obtain a uniformly valid one term expansion of th

$$\frac{d^2u}{dt^2} + 9u = \epsilon\left[(1-u^3)\frac{du}{dt} + u^3\right]; \quad u(0)=1, \quad \frac{du}{dt}(0) = 0$$

Solution

using equation (a_1) and (a_2) in given equ

$$\Rightarrow \frac{\partial^2 u}{\partial T_0^2} + 2\epsilon\frac{\partial^2 u}{\partial T_0 \partial T_1} + \epsilon^2\frac{\partial^2 u}{\partial T_1^2} + 9u = \epsilon\left[(1-u^3)\left[\frac{\partial u}{\partial T_0} + \epsilon\frac{\partial u}{\partial T_1}\right] + u^3\right] \longrightarrow (1)$$

Let $u(T_0, T_1, \epsilon) \sim u_0(T_0, T_1) + \epsilon u_1(T_0, T_1) + \cdots \longrightarrow (2)$

using (2) in (1) we obtain

$$O(\epsilon^0): \quad \frac{\partial^2 u_0}{\partial T_0^2} + 9u_0 = 0 \longrightarrow (3)$$

$$O(\epsilon^1): \quad \frac{\partial^2 u_1}{\partial T_0^2} + 2\frac{\partial^2 u_0}{\partial T_0 \partial T_1} + 9u_1 = (1-u_0^3)\frac{\partial u_0}{\partial T_0} + u_0^3$$

$$\Rightarrow \frac{\partial^2 u_1}{\partial T_0^2} + 9u_1 = -2\frac{\partial^2 u_0}{\partial T_0 \partial T_1} + (1-u_0^3)\frac{\partial u_0}{\partial T_0} + u_0^3 \longrightarrow (4)$$

solution of (3) in exponential form is

$$u_0 = A(T_1)e^{3iT_0} + A^*(T_1)e^{-3iT_0} \longrightarrow (5)$$

where $A(T_1)$ is function of T_1 only and $A^*(T_1)$ is complex conjugate of $A(T_1)$

The complementary function of equ (4) has the same form as equ (5). The particular integral contain secular term if the R.H.S of equ ④ contains $\exp(\pm 3iT_0)$

R.H.S of equ ④ is

$$\frac{\partial}{\partial T_1}\left[\frac{\partial}{\partial T_0}\left\{Ae^{3iT_0}+A^*e^{-3iT_0}\right\}\right]+\left[1-\left\{Ae^{3iT_0}+A^*e^{-3iT_0}\right\}^3\right]$$

$$\frac{\partial}{\partial T_0}\left[Ae^{3iT_0}+A^*e^{-3iT_0}\right]+\left\{Ae^{3iT_0}+A^*e^{-3iT_0}\right\}^3$$

$$\Rightarrow -2\frac{\partial}{\partial T_1}\left[3Ai\,e^{3iT_0}-3iA^*e^{-3iT_0}\right]+\left[1-A^3e^{9iT_0}-A^{*3}e^{-9iT_0}\right.$$

$$\left.-3A^2A^*e^{3iT_0}-3AA^{*2}e^{-3iT_0}\right]\left[3Ai\,e^{3iT_0}-3A^*i\,e^{-3iT_0}\right]$$

$$+\left[A^3e^{9iT_0}+A^{*3}e^{-9iT_0}+3A^2A^*e^{3iT_0}+3AA^{*2}e^{-3iT_0}\right]$$

$$\Rightarrow -6ie^{3iT_0}\frac{\partial A}{\partial T_1}+6ie^{-3iT_0}\frac{\partial A^*}{\partial T_1}+3Ai\,e^{3iT_0}-3A^*ie^{-3iT_0}$$

$$+3A^2A^*e^{3iT_0}+3AA^{*2}e^{-3iT_0}$$

$$=e^{3iT_0}\left[-6i\frac{\partial A}{\partial T_1}+3Ai+3A^2A^*\right]+e^{-3iT_0}\left[6i\frac{\partial A^*}{\partial T_1}\right.$$

$$\left.-3A^*i+3A^2A^*\right]$$

We know that terms in bracket are complex conjugate each other and their solution is same.

So, in order to avoid secular term

$$-6i\frac{\partial A}{\partial T_1}+3Ai+3A^2A^*=0\longrightarrow ⑥$$

Polar form of $A(T_1)$ is

$$A(T_1) = R(T_1)\, e^{i\theta(T_1)} \longrightarrow (*)$$

$$\frac{dA}{dT_1} = e^{i\theta}\frac{dR}{dT_1} + iR\frac{d\theta}{dT_1}e^{i\theta} \longrightarrow (7)$$

using (7) in (6) imply

$$6i\left[\frac{dR}{dT_1} + iR\frac{d\theta}{dT_1}\right]e^{i\theta} + 3iRe^{i\theta} + 3R^2e^{2i\theta}\cdot Re^{-i\theta} = 0$$

$$\Rightarrow \left[-6i\frac{dR}{dT_1} + 6R\frac{d\theta}{dT_1} + 3iR + 3R^3\right]e^{i\theta} = 0$$

$$\Rightarrow \left(6R\frac{d\theta}{dT_1} + 3R^3\right) + i\left(-6\frac{dR}{dT_1} + 3R\right) = 0 \quad \because e^{i\theta} \neq 0$$

$$\Rightarrow 6R\frac{d\theta}{dT_1} + 3R^3 = 0$$

$$\Rightarrow \frac{d\theta}{dT_1} = -\frac{1}{2}R^2$$

$$\Rightarrow \frac{d\theta}{dT_1} = -\frac{1}{2}K^2e^{T_1}$$

$$\Rightarrow d\theta = -\frac{1}{2}K^2e^{T_1}\,dT_1$$

$$\theta = -\frac{K^2}{2}e^{T_1} + K_1$$

$$\Rightarrow \theta(T_1) = -\frac{K^2}{2}e^{T_1} + K_1 \longrightarrow (8)$$

$$-6\frac{dR}{dT_1} + 3R = 0$$

$$\Rightarrow \frac{dR}{R} = \frac{dT_1}{2}$$

$$\Rightarrow \ln R = \left(\frac{1}{2}T_1 + C_1\right)\cdot 1$$

$$= \left(\frac{1}{2}T_1 + C_1\right)\cdot \ln e$$

$$\Rightarrow \ln R = \ln e^{\frac{1}{2}T_1 + C_1}$$

$$\Rightarrow R = Ke^{\frac{1}{2}T_1}$$

$$\Rightarrow R(T_1) = Ke^{\frac{1}{2}T_1} \longrightarrow (9)$$

using (8) and (9) eqn (*) imply

$$A(T_1) = Ke^{\frac{1}{2}T_1}\cdot e^{i\left[-\frac{K^2}{2}e^{T_1} + K_1\right]}$$

$$A(0) = Ke^{i\left[-\frac{K^2}{2} + K_1\right]}$$

$$A(0) = K_2 e^{-i\frac{K^2}{2}} \longrightarrow (10)$$

Initial Conditions are

$u(0) = 1$ at $t = 0$, $T_0 = 0$, $T_1 = 0$

$\Rightarrow$ $u_0(0,0) = 1$, $u_1(0,0) = 0$

And $\frac{du}{dt}(0) = 0 \Rightarrow \left[\frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1}\right]\left[u_0 + \varepsilon u_1 + \cdots\right] = 0$

$$\Rightarrow \frac{\partial u_0}{\partial T_0}(0,0) + \varepsilon \frac{\partial u_0}{\partial T_0}(0,0) + \varepsilon \frac{\partial u_0}{\partial T_1}(0,0) = 0$$

$$\Rightarrow \frac{\partial u_0(0,0)}{\partial T_0} = 0$$

we have equ (5)

$$u_0(T_0, T_1) = R(T_1) e^{i\theta(T_1)} e^{3iT_0} + R(T_1) e^{-i\theta(T_1)} e^{-3iT_0}$$

$$= R\left[e^{i\theta + 3iT_0} + e^{-i(\theta + 3T_0)}\right]$$

$$= R\left[\cos(\theta + 3T_0) + \cancel{i\sin(\theta + 3T_0)} + \cos(\theta + 3T_0) \cancel{- i\sin(\theta + 3T_0)}\right]$$

$$\Rightarrow u(T_0, T_1) = 2R(T_1)\cos\left[\theta + 3T_0\right]$$

using $u_0(0,0) = 1$

$$u_0(0,0) = 2R(0)\cos\left[\theta(0) + 3\times 0\right]$$

$$\Rightarrow 1 = 2K\cos\left[-\frac{K^2}{2} + K_1\right] \longrightarrow (11)$$

$$\frac{\partial u_0(T_0, T_1)}{\partial T_0} = -6R(T_1)\sin(\theta(T_1) + 3T_0)$$

using $\frac{\partial u_0}{\partial T_0}(0,0) = 0$

$$\left[\begin{array}{l} R(T_1) = K e^{\frac{1}{2}T_1} \Rightarrow R(0) = K \\ \theta(T_1) = -\frac{K^2}{2} e^{T_1} + K_1 \\ \theta(0) = -\frac{K^2}{2} + K_1 \end{array}\right]$$

$$\Rightarrow 0 = -6R(0)\sin\theta(0), \quad 0 = -6K\sin\theta(0)$$

$$\Rightarrow -6K \sin\left[-\frac{K^2}{2} + K_1\right] = 0 \quad ; -6K \neq 0$$

$$\Rightarrow \sin\left[-\frac{K^2}{2} + K_1\right] = 0$$

$$\Rightarrow K_1 = \frac{K^2}{2}$$

eqn (10) $\Rightarrow$ $1 = 2K\cos\left[-\frac{K^2}{2} + \frac{K^2}{2}\right]$, $1 = 2K\cos 0$

$$\boxed{K = \frac{1}{2}} \Rightarrow K_1 = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8}$$

$$\Rightarrow R(T_1) = \frac{1}{2} e^{\frac{T_1}{2}}, \quad \theta(T_1) = -\frac{1}{8} e^{T_1} + \frac{1}{8}$$

$$= -\frac{1}{8}\left[e^{T_1} - 1\right]$$

$$\Rightarrow u_0(T_0, T_1) = 2 \cdot \frac{1}{2} e^{\frac{T_1}{2}} \cdot \cos\left\{-\frac{1}{8}(e^{T_1} - 1) + 3T_0\right\}$$

uniformly One term valid expansion is

$$u_0(t) = e^{\frac{\varepsilon t}{2}} \cos\left[3t - \frac{e^{\varepsilon t}}{8} + \frac{1}{8}\right] + O(\varepsilon)$$

⇒ **Boundary Layer:-** Boundary layers are region in which a rapid change occurs in the value of variable.

Mathematically the occurrence of boundary layer is associated with the presence of a small parameter multiplying the highest derivative in the governing equation of a process. In such cases straight forward expansion can not be satisfied by all initial or boundary conditions. The straight forward expansion is reffered to as the "outer expansion". The "inner expansion" associated the boundary layer region is expressed in terms of stretched variable. The inner and outer expansions are matched over a region located at the edge of the Boundary layer. The technique is called the method of "matched asymptotic expansions". The composite expansion can be written as

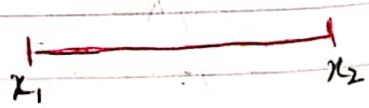
$$f_0^{comp} = f_0^{in} + f_0^{out} - f_0^{match}$$

Criteria for the location of Boundary Layer:-

Consider general linear equation

$$\varepsilon \frac{d^2 f}{dx^2} + a(x)\frac{df}{dx} + b(x) f = c(x)$$

$$x_1 < x < x_2$$



① If $a(x) > 0$, within $x_1 < x < x_2$, then boundary layer occur at $x = x_1$ and streching transformation will be

$$\xi = \frac{x - x_1}{\varepsilon}$$

② If $a(x) < 0$, within $x_1 < x < x_2$, then boundary layer occur at $x = x_2$ and stretching transformation will be $\xi = \dfrac{x_2 - x}{\varepsilon}$

Question (Exercise P-156) Obtain a one term composite expansion for the solution of

$$\varepsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} + \frac{f}{x+1} = 2 \; ; \; 0 < x < 1$$

$$f(0) = 0 \; \& \; f(1) = 3$$

Solution Here coefficient of $\frac{df}{dx}$ i.e $a(x) = 1$ which is +ve within $0 < x < 1$, so boundary layer occur at $(x = 0)$ and stretching variable will be

$$\xi = \frac{x - 0}{\varepsilon} = \frac{x}{\varepsilon}$$

The outer Expansion:-

Let $f^{out} = f_0^{out} + \varepsilon f_1^{out} + \cdots$

Put in given differential equ.

$$\Rightarrow \frac{d f_0^{out}}{dx} + \frac{f_0^{out}}{x+1} = 2 \longrightarrow ②$$

Since for one term ε and higer neglect

With $f(1) = 3 \Rightarrow f_0^{out}(1) + \varepsilon f_1^{out}(1) + \cdots = 3$

$$\Rightarrow f_0^{out}(1) = 3$$

$$I.F = e^{\int \frac{1}{x+1}dx} = e^{\ln(x+1)} = x+1$$

$$\Rightarrow \frac{d}{dx}\left[f_0^{out}(x+1)\right] = 2(x+1)$$

$$\Rightarrow f_0^{out}(x+1) = (x+1)^2 + c$$

$$f_0^{out} = (x+1) + \frac{c}{x+1}$$

Now using $f_0^{out}(1) = 3$

$$\Rightarrow 3 = 2 + \frac{c}{2} \Rightarrow \boxed{c=2}$$

$$\Rightarrow f_0^{out} = (x+1) + \frac{2}{x+1} = \frac{x^2+2x+1+2}{x+1}$$

$$\Rightarrow f_0^{out} = \frac{x^2+2x+3}{x+1}$$

The inner expansion:-

Introducing the stretched variable

$$s = \frac{x}{\varepsilon}$$

$$\Rightarrow \frac{df}{dx} = \frac{df}{ds}\cdot\frac{ds}{dx} = \frac{1}{\varepsilon}\frac{df}{ds}$$

$$\frac{d^2f}{dx^2} = \frac{1}{\varepsilon}\frac{d}{dx}\left[\frac{df}{ds}\right] = \frac{1}{\varepsilon}\frac{d^2f}{ds^2}\frac{ds}{dx}$$

$$\Rightarrow \frac{d^2f}{dx^2} = \frac{1}{\varepsilon^2}\frac{d^2f}{ds^2}$$

Putting values of $\frac{df}{ds}$ and $\frac{d^2f}{ds^2}$ in given differential eqn implies

$$\varepsilon \cdot \frac{1}{\varepsilon^2}\frac{d^2 f}{ds^2} + \frac{1}{\varepsilon}\frac{df}{ds} + \frac{f}{s\varepsilon+1} = 2$$

with $f(0) = 0$

$$\Rightarrow \frac{d^2 f}{ds^2} + \frac{df}{ds} + \frac{\varepsilon f}{s\varepsilon+1} = 2\varepsilon$$

$f(x=0) = 0$

$\Rightarrow f(s=0) = 0$

$\Rightarrow f_0^{in}(0) + \varepsilon f_1^{in}(0) + \cdots = 0$

$\Rightarrow f_0^{in}(0) = 0$

Let $f_0^{in} \sim f_0^{in}(s) + \varepsilon f_1^{in}(s) + \cdots$

$$\Rightarrow \frac{d^2 f_0^{in}}{ds^2} + \frac{df_0^{in}}{ds} = 0$$

characteristic equation is

$$m^2 + m = 0 \Rightarrow m = 0, m = -1$$

$$\Rightarrow f_0^{in} = A + Be^{-s}$$

Now using $f_0^{in}(0) = 0$

$$\Rightarrow 0 = A + B \Rightarrow \boxed{B = -A}$$

$$\Rightarrow f_0^{in} = A - Ae^{-s}$$

Using Prandtl's matching Condition:-

$$\lim_{x\to 0} f_0^{out} = \lim_{s\to\infty} f_0^{in}$$

$$\Rightarrow \lim_{x\to 0} \frac{x^2+2x+3}{x+1} = \lim_{s\to\infty}(A - Ae^{-s})$$

$$\Rightarrow 3 = A - Ae^{-\infty} \Rightarrow A - 0 = 3$$

$$\Rightarrow \boxed{A = 3}$$

$$\Rightarrow f_0^{in} = 3 - 3e^{-s}$$

So,

$$f_0^{comp} \sim f_0^{out} + f_0^{in} - f_0^{match}$$

$$\sim \frac{x^2+2x+3}{x+1} + 3 - 3e^{-\xi} - 3$$

$$\Rightarrow \boxed{f_0^{comp} \sim \frac{x^2+2x+3}{x+1} - 3e^{-\xi}}$$

OR

$$\boxed{f_0^{comp} \sim \frac{x^2+2x+3}{x+1} - 3e^{-x/\varepsilon}}$$

Question:- $\varepsilon \frac{d^2 f}{dx^2} - \frac{df}{dx} + \frac{f}{x+1} = 2 \; ; \; 0 < x < 1 \longrightarrow (A)$

$0 < \varepsilon \leq 1 \;,\; f(0) = 0 \;,\; f(1) = 2$

Solution

Here $a(x) = -1 < 0$ with $0 < x < 1$

So B.L is at $x = 1$ and stretched variable is

$$\xi = \frac{x_2 - x}{\varepsilon} = \frac{1-x}{\varepsilon}$$

<u>Outer expansion:-</u>

Let $f^{out} \sim f_0^{out} + \varepsilon f_1^{out} + \cdots \longrightarrow (1)$

Put in given differential equation.

$$\Rightarrow -\frac{d f_0^{out}}{dx} + \frac{1}{x+1} f_0^{out} = 2$$

with $f_0^{out}(0) = 0$

$$\Rightarrow \frac{d f_0^{out}}{dx} - \frac{1}{x+1} f_0^{out} = -2 \longrightarrow (2)$$

$$I.F = e^{\int -\frac{1}{x+1}dx} = e^{-\ln(x+1)} = e^{\ln(x+1)^{-1}}$$

$$\Rightarrow I.F = (x+1)^{-1} = \frac{1}{x+1}$$

$$\Rightarrow \frac{d}{dx}\left[f_0^{out}\left(\frac{1}{x+1}\right)\right] = \frac{-2}{x+1}$$

$$\Rightarrow f_0^{out}\left(\frac{1}{x+1}\right) = -2\ln(x+1) + c$$

Now using $f_0^{out}(0) = 0$

$$\Rightarrow 0 = -2\ln(1) + c \Rightarrow \boxed{c = 0}$$

$$\Rightarrow f_0^{out} = -2(1+x)\ln(x+1)$$

Inner expansion:-

We have $s = \frac{1-x}{\varepsilon}$

$$\Rightarrow s\varepsilon = 1 - x \Rightarrow x = 1 - s\varepsilon$$

As $\frac{df}{dx} = \frac{df}{ds}\frac{ds}{dx} = -\frac{1}{\varepsilon}\frac{df}{ds}$

& $\frac{d^2f}{dx^2} = \frac{1}{\varepsilon^2}\frac{d^2f}{ds^2}$

Put these values in ⊛ imply

$$\frac{\varepsilon}{\varepsilon^2}\frac{d^2f}{ds^2} + \frac{1}{\varepsilon}\frac{df}{ds} + \frac{f}{+1-s\varepsilon+1} = 2$$

$$\Rightarrow \frac{d^2f}{ds^2} + \frac{df}{ds} - \frac{\varepsilon f}{2+s\varepsilon} = 2\varepsilon \longrightarrow (3)$$

With $f(1) = 3 \Rightarrow$ at $x = 1$, $f = 3$

at $x = 1$, $s = 0 \Rightarrow f(s=0) = 3$

$$\Rightarrow f_0^{in}(0) = 3$$

Let $f^{in} = f_0^{in} + \varepsilon f_1^{in} + \cdots$

$$\Rightarrow \frac{d_2 f_0^{in}}{ds^2} + \frac{d f_0}{ds} = 0$$

Characteristic equ is $m^2 + m = 0$

$\Rightarrow m = 0, -1$

$\Rightarrow f_0^{in} = A + Be^{-s}$

Now using $f_0^{in}(0) = 3$

$\Rightarrow 3 = A + B \quad \Rightarrow B = 3 - A$

$\Rightarrow f_0^{in} = A + (3 - A)e^{-s}$

Matching Condition:

$$\lim_{x \to 0} f_0^{out} = \lim_{s \to \infty} f_0^{in}$$

$$\Rightarrow \lim_{x \to 0} -2(1+x)\ln(1+x) = \lim_{s \to \infty} A + 3 - Ae^{-s}$$

$\Rightarrow -2(1+0)\ln(1+0) = A + 3 - 0$

$\Rightarrow -2\ln 2 = A + 3$

$\Rightarrow A = -3 + \ln 2^{-2}$

$\Rightarrow f_0^{in} = \ln 1 - \ln 4 + [3 - \ln 1 - \ln 4]e^{-s}$

$\Rightarrow f_0^{in} = -\ln 4 + (3 - \ln 4)e^{-s}$

$$\Rightarrow f_0^{comp} = f_0^{out} + f_0^{in} - f_0^{match}$$

$$\sim -2(1+x)\ln(1+x) - \ln 4 + (3-\ln 4)e^{-s}$$

* Thickness of Boundary Layer:-

If boundary layer has a thickness of $O(\varepsilon^{P})$ then by this we mean that a variation of $O(\varepsilon^{P})$ in the independent variable will hold with in region of rapid change in the dependent variable.

Location of Boundary Layer:-

Consider general linear differential equation

$$\varepsilon \frac{d^2f}{dx^2} + a(x)\frac{df}{dx} + b(x) f = c(x)$$

where $x_1 < x < x_2$

1) If $a(x) > 0$ throughout $x_1 < x < x_2$, Then the boundary layer will occur at left end point i.e at $x = x_1$. and the stretching transformation will be

$$\xi = \frac{x - x_1}{\varepsilon^{P}}$$

, where boundary layer is assumed to have a thickness of $O(\varepsilon^{P})$

2) If $a(x) < 0$ throughout $x_1 < x < x_2$ then the boundary layer will occur at right end point i.e at $x = x_2$ and the stretching transformation will be

$$\xi = \frac{x_2 - x}{\varepsilon^{P}}$$

, where boundary layer is assumed to be have a thickness of $O(\varepsilon^{P})$

(iii) If $a(x)$ changes sign in the interval $x_1 < x < x_2$ then the boundary layer occurs at an interior point x_0 where $a(x_0)=0$ and the boundary layers may occur at both ends x_1 and x_2.

Question (Exercise P.156) Obtain one term composite expansion for the solution of

$$\varepsilon \frac{d^2 f}{dx^2} + x\frac{df}{dx} + f = 0 \ ; \quad 2 < x < 4$$

where $0 << \varepsilon << 1$; $f(2)=0$, $f(4)=1$

Solution

Here $a(x)=x$ which is +ve for $2<x<4$. So boundary layer occur at left end point i.e at $x=2$ and stretching variable will be $\xi = \frac{x-2}{\varepsilon^p}$

where p is to be determined

Outer expansion:-

$$\text{Let } f^{out} \sim f_0^{out} + \varepsilon f_1^{out} + \cdots$$

Put this in eqn (1) we get

$$\varepsilon\left(\frac{d^2 f_0^{out}}{dx^2} + \varepsilon \frac{d^2 f_1^{out}}{dx^2} + \cdots\right) + x\left(\frac{df_0^{out}}{dx} + \varepsilon \frac{df_1^{out}}{dx} + \cdots\right)$$

$$+\left(f_0^{out} + \varepsilon f_1^{out} + \cdots\right) = 0$$

Also $f_0^{out}(4) + \varepsilon f_1^{out}(4) + \cdots = 1$

Comparing coefficient of ε^0:-

$$x\frac{df_0^{out}}{dx} + f_0^{out} = 0 \ ; \text{ with } f_0^{out}(4) = 1$$

$$\Rightarrow \frac{df_0^{out}}{f_0^{out}} = -\frac{1}{x}\,dx$$

Integrating

$$\ln f_0^{out} = -\ln x + \ln c$$

$$\Rightarrow \ln f_0^{out} = \ln\frac{c}{x} \Rightarrow f_0^{out} = \frac{c}{x}$$

using $f_0^{out}(4) = 1$

$$\Rightarrow 1 = \frac{c}{4} \Rightarrow \boxed{c=4}$$

$$\Rightarrow \boxed{f_0^{out} = 4/x}$$

Inner Expansion:- the stretched variable is

$$s = \frac{x-2}{\varepsilon^p} \Rightarrow \varepsilon^p s + 2 = x$$

where p is to be determined

$$\frac{df}{dx} = \frac{df}{ds}\frac{ds}{dx} = \frac{1}{\varepsilon^p}\frac{df}{ds}$$

And $\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(\frac{1}{\varepsilon^p}\frac{df}{ds}\right) = \frac{1}{\varepsilon^p}\frac{d}{ds}\left(\frac{df}{ds}\right)\frac{ds}{dx}$

$$\Rightarrow \frac{d^2f}{dx^2} = \frac{1}{\varepsilon^{2p}}\frac{d^2f}{ds^2}$$

So equ ① becomes

$$\varepsilon^{1-2p}\frac{d^2f}{ds^2} + (2+\varepsilon^p s)\frac{1}{\varepsilon^p}\frac{df}{ds} + f = 0$$

$$\Rightarrow \varepsilon^{1-2p}\frac{d^2f}{ds^2} + \frac{2}{\varepsilon^p}\frac{df}{ds} + s\frac{df}{ds} + f = 0$$

Equating order of dominant terms

i.e $1-2P = -P \Rightarrow \boxed{P=1}$

So $\quad s = \dfrac{x-2}{\varepsilon}$

$f(x=0)=0$

$\Rightarrow f\left(s = \frac{2-2}{\varepsilon}\right) = 0$

$\Rightarrow f(s=0) = 0$

and equation ① becomes

$$\frac{1}{\varepsilon}\frac{d^2f}{ds^2} + \frac{2}{\varepsilon}\frac{df}{ds} + s\frac{df}{ds} + f = 0$$

$$\Rightarrow \frac{d^2f}{ds^2} + 2\frac{df}{ds} + s\varepsilon\frac{df}{ds} + \varepsilon f = 0 \text{, with } f(0)=0 \longrightarrow ②$$

Now let $f^{in} = f_0^{in} + \varepsilon f_1^{in} + \cdots$

Put this in equ ②, equating coefficient of ε^0, we get

$$\frac{d^2 f_0^{in}}{ds^2} + 2\frac{df_0^{in}}{ds} = 0 \qquad \text{Also } f_0^{in}(0) = 0$$

Characteristic equ of above is

$$m^2 + 2m = 0 \Rightarrow m = 0, -2$$

So $f_0^{in} = Ae^{0s} + Be^{-2s}$

$\Rightarrow f_0^{in} = A + Be^{-2s}$

using $f_0^{in}(0) = 0 \Rightarrow 0 = A+B$

$\Rightarrow \boxed{B = -A}$

So $\boxed{f_0^{in} = A - Ae^{-2s}}$

Prandtle Matching Condition:-

$$\lim_{x\to 2} f_0^{out} = \lim_{\xi\to\infty} f_0^{in}$$

i.e $$\lim_{x\to 2} \frac{4}{x} = \lim_{\xi\to\infty} A - Ae^{-2\xi}$$

$$\Rightarrow \frac{4}{2} = A - A(0) \Rightarrow \boxed{A = 2} \to f_0^{match}$$

So $$\boxed{f_0^{in} = 2 - 2e^{-2\xi}}$$

So
$$f_0^{comp} = f_0^{in} + f_0^{out} - f_0^{match}$$
$$= \frac{4}{x} + \cancel{2} - 2e^{-2\xi} - \cancel{2}$$

$$\Rightarrow f_0^{comp} = \frac{4}{x} - 2\exp\left[-2\left(\frac{x-2}{\varepsilon}\right)\right] \quad \because \xi = \frac{x-2}{\varepsilon}$$

(ii) Obtain one term composite expansion for the solution of

$$\varepsilon\frac{d^2f}{dx^2} - \frac{df}{dx} + \frac{f}{1+x} = 2 \; ; \; 0<x<1 \longrightarrow (1)$$

where $0<\varepsilon<<1$; $f(0)=0$, $f(1)=3$

Solution

Here $a(x) = -1$ which is negative within $0<x<1$. So boundary layer occurs at right end point i.e at $x=1$ and the stretching variable will be

$$\xi = \frac{1-x}{\varepsilon^p} \text{ , where } p \text{ is to be determined.}$$

outer expansion:- Let $f^{out} \sim f_0^{out} + \varepsilon f_1^{out} + \cdots$

Putting this in equ ① and equating coefficients of ε^0, we get

$$-\frac{df_0^{out}}{dx} + \frac{f_0^{out}}{1+x} = 2 \quad \text{Also } f_0^{out}(0) = 0$$

$$\Rightarrow \frac{df_0^{out}}{dx} - \frac{f_0^{out}}{1+x} = -2$$

which is linear equation

$$I.F = \exp\left[-\int \frac{1}{1+x}dx\right] = \exp\left[-\ln(1+x)\right] = \frac{1}{1+x}$$

$$\Rightarrow \frac{1}{1+x}\frac{df_0^{out}}{dx} - \frac{1}{(1+x)^2} f_0^{out} = \frac{-2}{1+x}$$

$$\Rightarrow \frac{d}{dx}\left[\frac{f_0^{out}}{x+1}\right] = \frac{-2}{1+x}$$

Integrating $\frac{1}{1+x} f_0^{out} = -2\ln(1+x) + c$

using $f_0^{out}(0) = 0$

$$\Rightarrow 0 = -2\ln(1) + c \quad \Rightarrow \boxed{c = 0}$$

So $\boxed{f_0^{out} = -2(1+x)\ln(1+x)}$

Inner Expansion:- The stretched variable is

$$\xi = \frac{1-x}{\varepsilon^p} \quad \Rightarrow x = 1 - \varepsilon^p \xi$$

where p is to be determined

Now
$$\frac{df}{dx} = \frac{d\xi}{dx}\frac{df}{d\xi} = -\frac{1}{\varepsilon^p}\frac{df}{d\xi}$$

$$\therefore \frac{d^2 f}{dx^2} = \frac{1}{\varepsilon^{2p}}\frac{d^2 f}{d\xi^2}$$

So equ (1) becomes.

$$\varepsilon^{1-2P}\frac{d^2f}{ds^2}+\frac{1}{\varepsilon^P}\frac{df}{ds}+\frac{f}{1-\varepsilon^P s+1}=2$$

$$\Rightarrow \varepsilon^{1-2P}\frac{d^2f}{ds^2}+\frac{1}{\varepsilon^P}\frac{df}{ds}+\frac{f}{2-\varepsilon^P s}=2$$

Equating the order of dominant terms

i.e $1-2P=-P \Rightarrow \boxed{P=1}$

So

$$\frac{1}{\varepsilon}\frac{d^2f}{ds^2}+\frac{1}{\varepsilon}\frac{df}{ds}+\frac{f}{2-\varepsilon s}=2$$

$$\Rightarrow \frac{d^2f}{ds^2}+\frac{df}{ds}+\frac{\varepsilon f}{2-\varepsilon s}=2\varepsilon \longrightarrow (2)$$

Also $f(x=1)=3$

$$\Rightarrow f\left(s=\frac{1-1}{\varepsilon}\right)=3 \Rightarrow f(s=0)=3$$

Let $f^{in}\sim f_0^{in}+\varepsilon f_1^{in}+\cdots$

Put this in (2) and equating leading order terms we get

$$\frac{d^2f_0^{in}}{ds^2}+\frac{df_0^{in}}{ds}=0 \qquad \text{Also } f_0^{in}(0)=3$$

Characteristic equ is $m^2+m=0$

$$\Rightarrow m=0,\ m=-1$$

So $f_0^{in}=Ae^{0s}+Be^{-s} \Rightarrow f_0^{in}=A+Be^{-s}$

using $f_0^{in}(0)=3 \Rightarrow 3=A+B \Rightarrow \boxed{B=3-A}$

Hence $\boxed{f_0^{in}=A+(3-A)e^{-s}}$

Prandtle matching Condition:-

$$\lim_{x\to 1} f_0^{out} = \lim_{\xi\to\infty} f_0^{in}$$

i.e $\lim_{x\to 1} -2(1+x)\ln(1+x) = \lim_{\xi\to\infty} A+(3-A)e^{-\xi}$

$$\Rightarrow -2(2)\ln(2) = A$$

$$\Rightarrow \boxed{A = -4\ln 2} \longrightarrow f_0^{match}$$

So $\boxed{f_0^{in} = -4\ln 2 + (3+4\ln 2)e^{-\xi}}$

Now as $f_0^{comp} = f_0^{out} + f_0^{in} - f_0^{match}$

$$= -2(1+x)\ln(1+x) - 4\cancel{\ln 2} + (3+4\ln 2)e^{-\xi} + 4\cancel{\ln 2}$$

$$\Rightarrow f_0^{comp} = -2(1+x)\ln(1+x) + (3+4\ln 2)\exp\left[-\left(\frac{1-x}{\varepsilon}\right)\right]$$

Example:- (Page 160 Bush Book) Obtain one term composite expansion for the solution of

$$\varepsilon\frac{d^2f}{dx^2} + x^2\frac{df}{dx} - f = 0 \;;\quad 0<x<1 \longrightarrow (1)$$

where $0<\varepsilon<<1$; $f(0)=1$, $f(1)=2$

Solution

Here $a(x)=x^2$ which is +ve in $0<x<1$ so boundary layer occurs at left end point i.e at $x=0$

And the streching variable will be

$$\xi = \frac{x-0}{\varepsilon^P} = \frac{x}{\varepsilon^P}$$

where P is to be determined

Outer expansion:-

Let $f^{out} \sim f_0^{out} + \varepsilon f_1^{out} + \dots$

Put this in ① and equating coefficients of ε^0; we get

$$\frac{x^2 d f_0^{out}}{dx^2} - f_0^{out} = 0 \quad \text{Also } f_0^{out}(1) = 2$$

$$\Rightarrow \frac{d f_0^{out}}{f_0^{out}} = \frac{1}{x^2} dx$$

Integrating $\ln|f_0^{out}| = -\frac{1}{x} + c$

using $f_0^{out}(1) = 2$

$$\Rightarrow \ln|2| = -1 + c \Rightarrow c = 1 + \ln 2$$

$$\Rightarrow \ln f_0^{out} - \ln 2 = 1 - 1/x$$

$$\Rightarrow \ln \frac{f_0^{out}}{2} = 1 - \frac{1}{x} \Rightarrow \frac{f_0^{out}}{2} = e^{1-\frac{1}{x}}$$

$$\Rightarrow \boxed{f_0^{out} = 2e^{1-1/x}}$$

Inner Expansion:- The stretched variable is $\xi = \frac{x-0}{\varepsilon^P} = \frac{x}{\varepsilon^P} \Rightarrow x = \xi \varepsilon^P$

where P is to be determined.

Now $\dfrac{df}{dx} = \dfrac{df}{ds}\dfrac{ds}{dx} = \dfrac{1}{\varepsilon^{p}}\dfrac{df}{ds}$

& $\dfrac{d^2f}{dx^2} = \dfrac{1}{\varepsilon^{2p}}\dfrac{d^2f}{ds^2}$

So equ (1) becomes

$$\varepsilon^{1-2p}\frac{d^2f}{ds^2} + \frac{1\cdot s^2\varepsilon^{2p}}{\varepsilon^{p}}\frac{df}{ds} - f = 0$$

$$\Rightarrow \varepsilon^{1-2p}\frac{d^2f}{ds^2} + \varepsilon^{p}s^2\frac{df}{ds} - f = 0$$

Equating order of dominant terms

i.e $1-2p = 0 \Rightarrow \boxed{p = 1/2}$

~~Also f(x f(s= f(s=0~~

$$\text{So}\quad \frac{d^2f}{ds^2} + \varepsilon^{\frac{1}{2}}s^2\frac{df}{ds} - f = 0 \longrightarrow (2)$$

Also $f(x=0)=1 \Rightarrow f\left(s = \dfrac{0}{\varepsilon^{1/2}}\right) = 1$

$\Rightarrow f(s=0) = 1$

Let $f^{in} \sim f_0^{in} + \varepsilon f_1^{in} + \cdots$

Putting this in equ (2) and equating leading order terms we get

$\dfrac{d^2 f_0^{in}}{ds^2} - f_0^{in} = 0$ Also $f_0^{in}(0) = 1$

ch. equ of above is

$m^2 - 1 = 0 \Rightarrow m = \pm 1$

So $f_0^{in} = Ae^{s} + Be^{-s}$

using $f_0^{in}(0) = 1 \Rightarrow 1 = A + B$

$$\Rightarrow \boxed{B = 1 - A}$$

So $$\boxed{f_0^{in} = A e^{\xi} + (1-A) e^{-\xi}}$$

Prandtl matching condition:-

$$\lim_{x \to 0} f_0^{out} = \lim_{\xi \to \infty} f_0^{in}$$

i.e $$\lim_{x \to 0} 2 e^{1-\frac{1}{x}} = \lim_{\xi \to \infty} A e^{\xi} + (1-A) e^{-\xi}$$

Above equation satisfied only if

$$\boxed{A = 0} \rightarrow f_0^{match}$$

So $$f_0^{in} = e^{-\xi}$$

Hence $$f_0^{comp} = f_0^{out} + f_0^{in} - f_0^{match}$$

$$\Rightarrow f_0^{comp} = 2 e^{1-\frac{1}{x}} + \exp[-\xi]$$

$$\Rightarrow \boxed{f_0^{comp} = 2 e^{1-\frac{1}{x}} + \exp\left[\frac{-x}{\varepsilon^{1/2}}\right]}$$

Note:- Do practice remaining questions of Exercise P-156

Error Function:- Just like gamma function there is an error function and defined by

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-t^2}\, dt$$

OR

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2}\, dt$$

Properties:-

(i) $\operatorname{erf}(\infty) = 1$

(ii) $\operatorname{erf}(-\infty) = -1$

An Interior Boundary Layer:-

Example:- (P-161) Obtain the one term composite expansion for the solution of

$$\varepsilon \frac{d^2 f}{dx^2} + x \frac{df}{dx} + x f = 0 \; ; \; -1 < x < 1 \longrightarrow (1)$$

where $0 < \varepsilon << 1$; $f(-1) = e$, $f(1) = 2e^{-1}$

Solution

Here $a(x) = x$ which changes the sign in $-1 < x < 1$. which is the case of interior boundary layer. We see that $a(x) > 0$ in $0 < x < 1$, so boundary layer occurs at left end point which is $x = 0$, and $a(x) < 0$ in

$-1<x<0$

~~$x<0$~~, so boundary layer occurs in right end point which is again $x=0$.

So there will be two outer expansions for +ve and negative x and an inner expansion in the boundary layer located at $x=0$.

We denote outer expansion for +ve x by f_0^+, and outer expansion for −ve x by f_0^-

Outer Expansion:–

Let $f^+ = f_0^+ + \varepsilon f_1^+ + \cdots$

Putting this in eqn ① and equating coefficients of ε^0 we get

$$x\frac{df_0^+}{dx} + x f_0^+ = 0 \quad \text{also } f_0^+(1) = 2e^{-1}$$

As $x \neq 0$ for outer expansion.

$$\Rightarrow \frac{df_0^+}{dx} + f_0^+ = 0 \quad , \quad f_0^+(1) = 2e^{-1}$$

$$\Rightarrow \frac{df_0^+}{f_0^+} = -dx$$

Integrating

$\ln|f_0^+| = -dx$

using $f_0^+(1) = 2e^{-1}$

$\Rightarrow \ln|2e^{-1}| = -1 + c$

$\ln 2 + \ln e^{-1} = -1 + c$

$\ln 2 - 1 = -1 + c$

$\Rightarrow \boxed{c = \ln 2}$

So $\ln|f_0^+| = -x + \ln 2$

$\Rightarrow \ln f_0^+ - \ln 2 = -x$

$\Rightarrow \ln \dfrac{f_0^+}{2} = -x$

$\Rightarrow \dfrac{f_0^+}{2} = e^{-x}$

$\Rightarrow \boxed{f_0^+ = 2e^{-x}}$

Now let

$f^- \sim f_0^- + \varepsilon f_1^- + \cdots$

Putting this in ① and equating coefficients of ε^0 we get

$x \dfrac{df_0^-}{dx} + x f_0^- = 0$

Also $f_0^-(0) = e$

As $x \neq 0$ for outer expansion.

So $\dfrac{df_0^-}{dx} + f_0^- = 0$

$\Rightarrow \dfrac{df_0^-}{f_0^-} = -dx$

Integrating

$\ln|f_0^-| = -x + d$

using $f_0^-(-1) = e$

$\Rightarrow \ln e = -(-1) + d$

$\Rightarrow 1 = 1 + d$

$\Rightarrow \boxed{d = 0}$

So $\ln|f_0^-| = -x$

$\Rightarrow \boxed{f_0^- = e^{-x}}$

P.T.O

Inner Expansion:-

The stretching variable will be

$\xi = \frac{x}{\varepsilon^P}$, where P is to be determined $\Rightarrow x = \varepsilon^P \xi$

$$\frac{df}{dx} = \frac{1}{\varepsilon^P} \frac{df}{d\xi}$$

& $$\frac{d^2 f}{dx^2} = \frac{1}{\varepsilon^{2P}} \frac{d^2 f}{d\xi^2}$$

So ① becomes

$$\varepsilon^{1-2P} \frac{d^2 f}{d\xi^2} + \frac{\varepsilon^P}{\varepsilon^P} \frac{df}{d\xi} + \varepsilon^P \xi f = 0$$

$$\Rightarrow \varepsilon^{1-2P} \frac{d^2 f}{d\xi^2} + \xi \frac{df}{d\xi} + \varepsilon^P \xi f = 0$$

Equating order of dominant term

i.e $1 - 2P = 0 \Rightarrow \boxed{P = 1/2}$

So $$\frac{d^2 f}{d\xi^2} + \xi \frac{df}{d\xi} + \varepsilon^{\frac{1}{2}} \xi f = 0 \longrightarrow ②$$

Now let

$$f_0^{in} = f_0^{in} + \varepsilon f_1^{in} + \cdots$$

Putting this in equ ② and equating coefficient of ε^0 we get

$$\frac{d^2 f_0^{in}}{d\xi^2} + \xi \frac{df_0^{in}}{d\xi} = 0$$

Let $w = \frac{d f_0^{in}}{ds}$

$$\Rightarrow \frac{dw}{ds} + s w = 0$$

$$\Rightarrow \frac{dw}{w} = -s\, ds$$

Integrating

$$\ln w = -\frac{s^2}{2} + a$$

$$\Rightarrow w = e^{-s^2/2 + a}$$

$$\Rightarrow w = e^{-s^2/2} e^{a}$$

$$\Rightarrow w = A e^{-s^2/2} \quad \text{where } A = e^{a}$$

$$\Rightarrow \frac{d f_0^{in}}{d\xi} = A e^{-\xi^2/2}$$

Integrating above expansion from 0 to ξ w.r.t t

$$\int_0^{\xi} \frac{d f_0^{in}}{dt}\, dt = A \int_0^{\xi} e^{-t^2/2}\, dt$$

$$\Rightarrow f_0^{in}(\xi) - f_0^{in}(0) = A \int_0^{\xi} e^{-t^2/2}\, dt$$

$$\Rightarrow f_0^{in}(\xi) = A \int_0^{\xi} e^{-t^2/2}\, dt + f_0^{in}(0) \longrightarrow (3)$$

As we know that

If we consider $-1 < x < 0$

Then $s = \frac{0 - x}{\varepsilon^p} = \frac{-x}{\varepsilon^p}$

If we consider $0 < x < 1$

$\xi = \frac{x - 0}{\varepsilon^p} \Rightarrow s = \frac{x}{\varepsilon^p}$

In both cases ξ is +ve so we consider

$\xi = x/\varepsilon^p$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

$$\Rightarrow \int_0^s e^{-t^2/2}\, dt = \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(s/\sqrt{2}\right)$$

So ③ becomes

$$\boxed{f_0^{in} = A\frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{s}{\sqrt{2}}\right) + f_0^{in}(0)}$$

Prandtle matching condition.

When

$$\lim_{x \to 0^+} f_0^+ = \lim_{s \to \infty} f_0^{in}$$

i.e $$\lim_{x \to 0^+} 2e^{-x} = \lim_{s \to \infty} \left[A\frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{s}{\sqrt{2}}\right) + f_0^{in}(0)\right]$$

$$\Rightarrow 2 = A\frac{\sqrt{\pi}}{2} \operatorname{erf}(\infty) + f_0^{in}(0)$$

$$\Rightarrow 2 = A\frac{\sqrt{\pi}}{2} + f_0^{in}(0) \qquad \because \operatorname{erf}(\infty) = 1$$

$$\Rightarrow 2 = \frac{A\sqrt{\pi}}{2} + f_0^{in}(0) \longrightarrow ④$$

When $$\lim_{x \to 0^-} f_0^- = \lim_{s \to \infty} f_0^{in}$$

i.e $$\lim_{x \to 0^-} e^{-x} = \lim_{s \to \infty} \left[A\frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{s}{\sqrt{2}}\right) + f_0^{in}(0)\right]$$

$$\Rightarrow 1 = A\frac{\sqrt{\pi}}{2} \operatorname{erf}(-\infty) + f_0^{in}(0)$$

$$\Rightarrow 1 = A\frac{\sqrt{\pi}}{2}(-1) + f_0^{in}(0) \qquad \because erf(-\infty) = -1$$

$$\Rightarrow 1 = -\frac{A\sqrt{\pi}}{2} + f_0^{in}(0) \longrightarrow (5)$$

Now (4) + (5) $\Rightarrow$

$$3 = 2f_0^{in}(0) \Rightarrow \boxed{f_0^{in}(0) = 1.5}$$

(4) − (5) $\Rightarrow$ $1 = \frac{2A\sqrt{\pi}}{2} \Rightarrow \boxed{A = \frac{1}{\sqrt{\pi}}}$

So $f_0^{in} = \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} erf\left(\frac{s}{\sqrt{2}}\right) + 1.5$

$$\Rightarrow \boxed{f_0^{in} = \frac{1}{2} erf\left(\frac{s}{\sqrt{2}}\right) + 1.5}$$

Composite expansion:- The leading order terms over the region are

$$f_0^{+} = 2e^{-x} \qquad x > O(\sqrt{\varepsilon})$$

$$f_0^{in} = 0.5\, erf\left(\frac{x}{\sqrt{2\varepsilon}}\right) + 1.5 \qquad x = O(\sqrt{\varepsilon})$$

$$f_0^{-} = e^{-x} \qquad x < O(\sqrt{\varepsilon})$$

Also we can see that

$$f_0^{in}[x > O\sqrt{\varepsilon})] = 0.5(1) + 1.5 + T.S.T = 2 + T.S.T$$

and

$$f_0^{in}[x < O\sqrt{\varepsilon}] = 0.5(-1) + 1.5 + T.S.T = 1 + T.S.T$$

where T.S.T stands for Transiently Small terms

So we can write

$$\boxed{f_0^{comp} = \left[0.5\, erf\left(\frac{x}{\sqrt{2\varepsilon}}\right) + 1.5\right] e^{-x}}$$

which yield correct coefficient of e^{-x} outside the boundary layer and correct leading order behaviour with in the boundary layer.