CH # 0
Partial Differential Equation of First order

⇒ First order P.D.E :-

A First order Partial differential equation (usually denoted by P.D.E) in Two independent variable x, y and one unknown z, also called dependent variable is an equation of the form.

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \quad \text{——— (1)}$$

Thus

$$P = \frac{\partial z}{\partial x} \qquad q = \frac{\partial z}{\partial y}$$

(1) can be written as

$$F(x, y, z, P, q) = 0$$

⇒ Classification:-

There are Four Classification

of P.D.E of First order
Linear P.D.E
Semi Linear P.D.E
Quasi Linear P.D.E
Non_Linear P.D.E

Linear P.D.E:–
if the co_efficients P, Q are Function of Independent Variables only a equation of the form

$$P(x,y)p + Q(x,y)q = R(x,y) + S(x,y)$$

is called a Linear P.D.E of First order.

Example:–

$xp - yq = xyz + xy$

$x^2p - xyq = xy^2z$ ∵ order of p, q and z equal to one

$yu_x + xu_y = xy$

$xp + yq = nz$

∵ For $n = 0, 1, 2, 3, \ldots$

ii) Semi-Linear P.D.E.:-

If the derivative p and q that appear in the Function F are Linear, while the co-efficient P, Q and R depend on the independent variable x, y and also on the dependent variable z.
Similarly an equation of the Form.

$$P(x,y)p + Q(x,y)q = R(x,y,z)$$

is called Semi-Linear P.D.E of First order.
it is also Known as almost Linear P.D.E of First order.

Example:-

i) $e^{x^2}p + xyq = xyz^2$ ∵ ⋆ order of z greater then equal to one.

or

$e^{x^2}p + xyq = xyz$

ii) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z^2$

ii) Quasi Linear:-

An equation of the Form $P(x,y,z)p+Q(x,y,z)q=R(x,y,z)$ is called a Quasi-Linear P.D.E of First order. Then $P(x,y,z)$, $Q(x,y,z)$ and $R(x,y,z)$ are independent of x, y and z.

Example:-

$$P(z)\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0$$

$$(x^2+y^2+z^2)p-xyzq=z^5x+y^2z$$

Non-Linear:-

The order of p and q are greater then one. Thus p and q are not Linear is called a non-Linear P.D.E of First order.

Example:-

ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1$ ——— ①

Thus

$p = \frac{\partial z}{\partial x}$ $\quad q = \frac{\partial z}{\partial y}$

$\Rightarrow$ ① $p^2 + q^2 = 1$

is non-Linear P.D.E of First order.

Note

* Every Linear P.D.E is a Semi Linear but converse may or may not be True.

* Every Semi-Linear P.D.E is a Quasi Linear But converse may or may not be True.

⇒ Formation of P.D.E:-

Partial differential equation is

$F(x, y, z, p, q) = 0$

i) First order of constant co-efficient.
ii) First order of function co-efficient.

Example:

$$Z = (x-a)^2 + (y-b)^2$$

Solution:- Given equation is

$$Z = (x-a)^2 + (y-b)^2 \quad \text{—— ①}$$

Partial derivative w.r.t 'x' by ①

$$\frac{\partial Z}{\partial x} = 2(x-a) + 0$$

$$\frac{\partial Z}{\partial x} = 2(x-a) \quad \text{—— (i)}$$

Partial derivative w.r.t 'y' by ①

$$\frac{\partial Z}{\partial y} = 0 + 2(y-b)(1)$$

$$\frac{\partial Z}{\partial y} = 2(y-b) \quad \text{—— (ii)}$$

if $\frac{\partial Z}{\partial x} = P$ and $\frac{\partial Z}{\partial y} = q$

By (i) and (ii)

$P = 2(x-a)$ $\qquad$ $q = 2(y-b)$

$(x-a) = \frac{P}{2}$ $\qquad$ $(y-b) = \frac{q}{2}$

Becomes eq (1)

$$Z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$Z = \frac{p^2}{2^2} + \frac{q^2}{2^2}$$

$$Z = \frac{p^2}{4} + \frac{q^2}{4}$$

$$\boxed{4Z = p^2 + q^2}$$

Which is Required Sol.

Example: 4:- Pg. No → 9

i) $Z = xy + f(x^2 + y^2)$

Solution:- Given equation is

$$Z = xy + f(x^2 + y^2) \quad \text{———} \quad (1)$$

Partial Derivative w.r.t 'x' by (1)

$$\frac{\partial Z}{\partial x} = y + f'(x^2 + y^2) \cdot 2x$$

Partial Derivative w.r.t 'y' by (1)

$$\frac{\partial Z}{\partial y} = x + f'(x^2 + y^2) \cdot 2y$$

So, $p = y + f'(x^2 + y^2)\, 2x$ ——— (i)

$q = x + f'(x^2 + y^2)\, 2y$ ——— (ii)

x Mul by (ii) y Mul by (i)

$$yp = y^2 + f'(x^2+y^2)\, 2xy \quad \text{———— (iii)}$$

and

$$xq = x^2 + f'(x^2+y^2)\, 2xy \quad \text{———— iv}$$

Subtracting eq (iii) and (iv)

$$\boxed{yp - xq = y^2 - x^2}$$

which is required Sol.

(ii) $Z = f\left(\frac{xy}{z}\right)$

Solution:- Given equation is

$$z = f\left(\frac{xy}{z}\right) \quad \text{———— (1)}$$

Partial Derivative w.r.t 'x' by ①

$$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right) \cdot \frac{y}{z}$$

Partial Derivative w.r.t 'y' by ①

$$\frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right) \cdot \frac{x}{z}$$

So,

$$P = f'\left(\frac{xy}{z}\right) \cdot \frac{y}{z} \quad \text{———— (i)}$$

and

$$q = f'\left(\frac{xy}{z}\right) \cdot \frac{x}{z} \quad \text{———— (ii)}$$

x Mul by (i) and y Mul by (ii)

$$xp = f'\left(\frac{xy}{z}\right)\frac{xy}{z} \quad \text{——— (iii)}$$

$$yq = f'\left(\frac{xy}{z}\right)\frac{xy}{z} \quad \text{——— (iv)}$$

(iii) — (iv)

$$\boxed{xp - yq = 0}$$

∴ which is required Sol.

Example : 5 :- Pg no. ⟶ 10

$z = ax + by + ab$

Solution:- Given eq is

$$z = ax + by + ab \quad \text{——— (1)}$$

Partial Derivative w.r.t 'x' by ①

$$\frac{\partial z}{\partial x} = a \quad \text{——— (i)}$$

Partial Derivative w.r.t 'y' by ①

$$\frac{\partial z}{\partial y} = b \quad \text{——— (ii)}$$

$$\frac{\partial z}{\partial x} = p = a$$

$$\frac{\partial z}{\partial y} = q = b$$

From eq ① we get

$$\boxed{Z = Px + qy + Pq}$$

Which is required Sol.

Example. 6:-

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$\because a + b + c = 1$

Solution:- Given equation is :

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{---} ①$$

and also given is $a + b + c = 1$

$\Rightarrow \quad c = 1 - a - b$

$\Rightarrow$ ①

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{1-a-b} = 1 \quad \text{---} ②$$

Partial Derivative w.r.t 'x' by ②

$$\frac{1}{a} + \frac{\partial z/\partial x}{1-a-b} = 0 \quad \text{---} (i)$$

Partial Derivative w.r.t 'y' by ②

$$\frac{1}{b} + \frac{\partial z/\partial y}{1-a-b} = 0 \quad \text{---} (ii)$$

By equation (i) and (ii)

$$\frac{p}{1-a-b} = -\frac{1}{a} \quad \text{———— (iii)}$$

$$\frac{q}{1-a-b} = -\frac{1}{b} \quad \text{———— (iv)}$$

By (iii) By (iv)

$$p = \frac{a+b-1}{a} \qquad q = \frac{a+b-1}{b}$$

$$\text{So} \Rightarrow \frac{p}{q} = \frac{a+b-1/a}{a+b-1/b} = \frac{b}{a}$$

$$\Rightarrow \frac{p}{q} = \frac{b}{a}$$

$$\Rightarrow \boxed{p = \frac{b}{a}q} \,,\quad \boxed{q = p \cdot \frac{a}{b}}$$

AS

$$p = \frac{a+b-1}{a} \,,\quad q = \frac{a+b-1}{b}$$

$$p = 1 + \frac{b}{a} - \frac{1}{a} \,,\quad q = \frac{a}{b} + 1 - \frac{1}{b}$$

$$\frac{1}{a} = 1 + \frac{p}{q} - p \,,\quad \frac{1}{b} = \frac{q}{p} + 1 - q$$

$$\frac{1}{a} = \frac{q + p - pq}{q} \,,\quad \frac{1}{b} = \frac{q + p - pq}{p}$$

$$a = \frac{q}{q + p - pq} \,,\quad b = \frac{p}{q + p - pq}$$

Put in eq ②

$$\frac{x(p+q-pq)}{q} + \frac{y(p+q-pq)}{p} + \frac{z(p+q-pq)}{-pq} = 1$$

$$\because 1-a-b = 1 - \frac{q}{p+q-pq} - \frac{p}{p+q-pq}$$

$$= \frac{p+q-pq-q-p}{p+q-pq}$$

$$c = 1-a-b = \frac{-pq}{p+q-pq}$$

So,

$$\left(\frac{x}{q} + \frac{y}{p} - \frac{z}{pq}\right)(p+q-pq) = 1$$

$$\left(\frac{px+qy-z}{pq}\right)(p+q-pq) = 1$$

$$p(x)+qy-z = \frac{pq}{p+q-pq}$$

This is Partial differential equation Form of (1).

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→ Elminate the arbitary Function in the Following and the Corresponding P.D.E.

i) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

Solution:- Given

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right) \quad \text{——— (1)}$$

Partial derivative w.r.t 'x' by (1)

$$p = \frac{\partial z}{\partial x} = 0 + 2f'\left(\frac{1}{x} + \log y\right) \cdot \left(-\frac{1}{x^2}\right)$$

$$p = \frac{-2}{x^2} f'\left(\frac{1}{x} + \log y\right)$$

Partial derivative w.r.t 'y' by (2)

$$q = \frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right) \cdot \frac{1}{y}$$

$$q = 2y + \frac{2}{y} f'\left(\frac{1}{x} + \log y\right)$$

Mul p by $\left(\frac{1}{y}\right)$ and q by $\left(\frac{1}{x^2}\right)$

$$\frac{p}{y} = \frac{-2}{x^2 y} f'\left(\frac{1}{x} + \log y\right) \quad \text{——— (i)}$$

$$\frac{q}{x^2} = \frac{2y}{x^2} + \frac{2}{x^2 y} f'\left(\frac{1}{x} + \log y\right) \quad \text{——— (ii)}$$

Adding (i) + (ii)

$$\frac{p}{y} + \frac{q}{x^2} = \frac{2y}{x^2} + \frac{2}{yx^2} f'\left(\frac{1}{x} + \log y\right) - \frac{2}{yx^2} f'\left(\frac{1}{x} + \log y\right)$$

$$\frac{p}{y} + \frac{q}{x^2} = \frac{2y}{x^2}$$

$$\frac{x^2 p + yq}{x^2 y} = \frac{2y}{x^2}$$

$$\boxed{x^2 p + yq = 2y^2}$$

It is Partial differential Equation From (1).

) $a(x+y) + b(x-y) + z^2 = c^2$

Solution:- Given

$$a(x+y) + b(x-y) + z^2 = c^2$$

$$z^2 = -a(x+y) - b(x-y) + c^2 \quad \text{——} \; (1)$$

rtial Diff Derivative w.r.t 'x' by (1)

$$2z \frac{\partial z}{\partial x} = -a - b$$

$$2zp = -(a+b)$$

$$\boxed{a+b = -2zp}$$

Partial derivative w.r.t 'y' by ①

$$2z\frac{\partial z}{\partial y} = a-b$$

$$2zq = -(a-b)$$

$$\boxed{a-b = -2zq}$$

by ①

$$a(x+y) + b(x-y) + z^2 = c^2$$

$$ax + ay + bx - by + z^2 = c^2$$

$$(a+b)x + (a-b)y + z^2 = c^2$$

Put value $(a+b)$ and $(a-b)$

$$\boxed{-2zpx - 2zqy + z^2 = c^2}$$

which is required P.D.E.

(iii) $z = x+y+f(xy)$ **E.X QNo:1**

Solution:- Given equation is

$$z = x+y+f(xy) \quad \text{———} ①$$

Partial derivative w.r.t 'x' by ①

$$\frac{\partial z}{\partial x} = 1 + f'(xy) \cdot y$$

$$p = 1 + f'(xy) \cdot y$$

Partial derivative w.r.t 'y' by (1)

$$\frac{\partial z}{\partial y} = 1 + f'(xy) \cdot x$$

$$q = 1 + f'(xy) \cdot x$$

Mul 'p' by 'x' and 'q' by 'y'

$$xp = x + xy f'(xy) \quad \text{—— (i)}$$
$$yq = y + xy f'(xy) \quad \text{—— (ii)}$$

$\Rightarrow$ (i) − (ii)

$$xp - yq = x + xyf'(xy) - y - xyf'(xy)$$

$$\boxed{xp - yq = x - y}$$

it is the P.D.E of (1)

(iv) $z = (x^2+a)(y^2+b)$ **E.x Q No. 2**

Solution:- Given eq. is

$$z = (x^2+a)(y^2+b) \quad \text{—— (1)}$$

Partial derivative w.r.t 'x' by (1)

$$p = \frac{\partial z}{\partial x} = 2x(y^2+b)$$

$$p = 2x(y^2+b) \Rightarrow (y^2+b) = \frac{p}{2x}$$

Partial derivative w.r.t 'y' by (1)

$$\frac{\partial z}{\partial y} = 2y(x^2+a)$$

$$q = 2y(x^2+a) \Rightarrow (x^2+a) = \frac{q}{2y}$$

Putting values (y^2+b) and (x^2+a) in (1)

$$z = \left(\frac{p}{2x}\right)\left(\frac{q}{2y}\right)$$

$$z = \frac{pq}{4xy}$$

$$\boxed{4xyz = pq}$$

it is P.D.E of (1)
Which is required equation.

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* Formation of Partial Differential Equation:-

Suppose u and v are any Two given Functions of x, y and z.

Let F be an arbitrary Function of u and v of the Form.

$$F(u,v) = 0 \quad \text{---} \quad (1)$$

We can Form a diff equation by eliminating the aribitrary Function F.

For This

We diff eq (1) Partially w.r.t x and y.

(x,z)
$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right] + \frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right] = 0 \quad \text{---} \quad (2)$$

Where $p = \frac{\partial z}{\partial x}$

and

(y,z)
$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right] + \frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right] = 0 \quad \text{---} \quad (3)$$

Now, elimination $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ From eq (2) and (3), We obtain

$$\begin{vmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p & \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q \end{vmatrix} = 0$$

(y, z) $$P = \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} \end{vmatrix}$$

(x, z) $$Q = \frac{\partial(u,v)}{\partial(x,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} \end{vmatrix}$$

(x, y) $$R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Which Simplifies to

$$p \frac{\partial(u,v)}{\partial(y,z)} + q \frac{\partial(u,v)}{\partial(x,z)} = \frac{\partial(u,v)}{\partial(x,y)}$$

This is a Linear P.D.E of the Type $Pp + Qq = R$ —— (4)

Where

$$P = \frac{\partial(u,v)}{\partial(y,z)}$$

$Q = \frac{\partial(u,v)}{\partial(z,x)}$

$R = \frac{\partial(u,v)}{\partial(x,y)}$

eqn (4) is called Lagrange's P.D.E of First order.

Example: 3:- Pg.no→7

i) $z = f(x+it) + g(x-it)$

Solution:- Given

$z = f(x+it) + g(x-it)$ —— ①

Partial Derivative w.r.t 'x' by ①

$\frac{\partial z}{\partial x} = f'(x+it) + g'(x+it)$

again

$\frac{\partial^2 z}{\partial x^2} = f''(x+it) + g''(x+it)$ —— ②

Partial Derivative w.r.t 't' by ①

$\frac{\partial z}{\partial t} = if'(x+it) - ig'(x-it)$

again

$\frac{\partial^2 z}{\partial t^2} = -f''(x+it) - g''(x-it)$ —— ③

Adding ② and ③

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$$

which is the required

ii) $f(x+y+z, x^2+y^2+z^2) = 0$

Solution:-

The Given relation is of the Form

$$F(u,v) = 0$$

where $u = x+y+z$ and $v = x^2+y^2+z^2$

Hence, The required P.D.E is of the Form

$$Pp + Qq = R \quad \text{——— (1)}$$

$$P = \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2y \\ 1 & 2z \end{vmatrix}$$

$$= 2z - 2y = 2(z-y)$$

$$Q = \frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 1 & 2z \\ 1 & 2x \end{vmatrix}$$

$$= 2x - 2z = 2(x-z)$$

$$R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & \partial x \\ 1 & \partial y \end{vmatrix}$$

$$R = \partial y - \partial x = \partial(y-x)$$

Putting P, Q, R in ①, we obtain

$$\partial(z-y)p + \partial(x-z)q = \partial(y-x)$$

$$\boxed{(z-y)p + (x-z)q = (y-x)}$$

⟹ Solution of Partial Diff. eq. of First order:–

The relation of the Form

$$F(x,y,z,a,b) = 0 \quad \text{——— ①}$$

gives P.D.E of First order of the Form

$$f(x,y,z,p,q) = 0 \quad \text{——— ②}$$

Thus The relation of the Form ①

Containing Two arbitrary constants a, b is a Solution of the Form ② and is called a complete Solution or complete integral.

* The General Solution of the Linear P.D.E

$$Pp + Qq = R \quad \text{——— ①}$$

can be written in the Form $F(u, v) = 0$ where F is an arbitrary Function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ Form a Solution of the eq.

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}$$

$$\Rightarrow \boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$$

Example: 7:- [Pg no → 14]

Find the general integral of The Following Linear P.D.E.

1) $y^2 p - xyq = x(z-2y)$

Solution:- Given

$$y^2 p - xyq = x(z-2y) \quad \text{—} ①$$

General eq

$$P(x)p + Q(x)q = R(x)$$

$P = y^2$

$Q = -xy \qquad R = x(z-2y)$

Aux. eq.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

1st two members of the eq gives us 2nd two members.

$$\frac{dx}{y^2} = -\frac{dy}{xy} \quad , \quad \frac{-dy}{xy} = \frac{dz}{x(z-2y)}$$

$$\frac{dx}{y} = \frac{-dy}{x} \quad , \quad \frac{-dy}{y} = \frac{dz}{z-2y}$$

$$x\,dx = -y\,dy \quad , \quad z\,dy - 2y\,dy = -y\,dz$$

integrating ,

$$\frac{x^2}{2} = -\frac{y}{2} + c \quad , \quad \frac{2y^2}{2} = yz + c_2$$

$$x^2 + y^2 = c_1 \text{ --- (2)} \quad , \quad y^2 = yz + c_2$$

$$y^2 - yz = c_2 \text{ --- (3)}$$

Hence the curves given by (2) and (3) generate the required integral surface. $F(c_1, c_2) = 0$

$$\Rightarrow \boxed{F(x^2+y^2,\ y^2 - yz) = 0}$$

(ii) $(y+zx)p - (x+yz)q = x^2 - y^2$

Solution:- Given equation is

$$(y+zx)p - (x+yz)q = x^2 - y^2 \text{ --- (1)}$$

Aux eq.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$P = y + zx \qquad Q = -(x+yz)$

$R = x^2 - y^2$

$$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2 - y^2}$$

To get the First integral curve,
Let us consider the First combination

$$\frac{x\,dx + y\,dy}{xy + zx^2 - xy - y^2 z} = \frac{dz}{x^2 - y^2}$$

$$\frac{x\,dx + y\,dy}{z(x^2 - y^2)} = \frac{dz}{x^2 - y^2}$$

$$x\,dx + y\,dy = z\,dz$$

on integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{z^2}{2} + C_1$$

$$\Rightarrow x^2 + y^2 - z^2 = C_1 \quad \text{--- (2)}$$

Similarly, For getting the Second integral curve, consider the combination as

$$\frac{y\,dx + x\,dy}{y^2 + xyz - x^2 - xyz} = \frac{dz}{x^2 - y^2}$$

$$\frac{y\,dx + x\,dy}{y^2 - x^2} = \frac{dz}{x^2 - y^2}$$

$$\frac{y\,dx + x\,dy}{-(x^2-y^2)} = \frac{dz}{x^2-y^2}$$

$$y\,dx + x\,dy = -dz$$

on integrating

$$\int d(xy) + \int dz = 0$$

$$xy + z = C_2 \quad \text{——— (3)}$$

Thus the curve given by eq (2) and (3) generate the required integral surface as $F(C_1, C_2)$

$$\boxed{F(x^2+y^2-z^2,\ xy+z)}$$

Example:- Pg → 16

use Lagrange's method to solve the equation.

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & -1 \end{vmatrix} = 0, \text{ where } z = z(x,y)$$

Solution:-

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & -1 \end{vmatrix} = 0$$

$$\Rightarrow x\left[-\beta-\gamma\frac{\partial z}{\partial y}\right]-y\left[-\alpha-\gamma\frac{\partial z}{\partial x}\right]+z\left[\alpha\frac{\partial z}{\partial y}-\beta\frac{\partial z}{\partial x}\right]=0$$

$$\Rightarrow -x\beta-x\gamma\frac{\partial z}{\partial y}+\alpha y+\gamma y\frac{\partial z}{\partial x}+\alpha z\frac{\partial z}{\partial y}-\beta z\frac{\partial z}{\partial x}=0$$

$$\Rightarrow (\gamma y-\beta z)\frac{\partial z}{\partial x}+(\alpha z-x\gamma)\frac{\partial z}{\partial y}=\beta x-\alpha y \quad (1)$$

The corresponding auxilliary eq. are

$$\frac{dx}{\gamma y-\beta z}=\frac{dy}{\alpha z-x\gamma}=\frac{dz}{\beta x-\alpha y} \quad (2)$$

using Multipliers x, y and z each Function is

Ruff:-

$P=\gamma y-\beta z$ $\quad Q=\alpha z-\gamma x$

$R=\beta x-\alpha y$

Mul P by x, Q by y and R by z and adding

$x\gamma y-x\beta z+\alpha yz-\gamma xy+x\beta z-\alpha yz=0$

We Find the each Fraction is

$=x\,dx+y\,dy+z\,dz$

Therfore

$$xdx + ydy + zdz = 0$$

integrating

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_0$$

$$\because 2C_0 = C_1$$

$$x^2 + y^2 + z^2 = 2C_0$$

$$x^2 + y^2 + z^2 = C_1 \quad \text{——— (3)}$$

Similarly, using Mul α, β and γ we find from eq (2) that each fraction is equal to

Ruff.

Mul P by α, Q by β, R by γ

$$\alpha\gamma y - \alpha\beta z + \alpha\beta z - \gamma\beta x + \beta\gamma x - \alpha\gamma y = 0$$

$$0 = 0$$

$$\alpha dx + \beta dy + \gamma dz = 0$$

on integrating

$$\alpha x + \beta y + \gamma z = C_2 \quad \text{——— (4)}$$

Thus General Solution of the Given eq is

$$F(C_1, C_2)$$

$$\boxed{F(x^2+y^2+z^2, \alpha x + \beta y, \gamma z) = 0}$$

Example: 09:- Pg → 17

Find the general integral of the following

i) $pz - qz = z^2 + (x+y)^2$

Solution:-

$$pz - qz = z^2 + (x+y)^2$$

The auxilliary eq. $\left[P = z \quad Q = -z \quad R = z^2+(x+y)^2 \right.$

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2} \quad (1)$$

The First two members of eq (1) gives

$$\frac{dx}{z} = \frac{dy}{-z}$$

$\Rightarrow$ $dx = -dy$

$\Rightarrow$ integrating o.b.s

$$x = -y + c_1$$

$$\boxed{x + y = c_1} \quad (2)$$

Now Ist and Last members of eq (1)

$$\frac{dx}{z} = \frac{dz}{z^2+(x+y)^2}$$

$$z^2+(x+y)^2\,dx = z\,dz$$

$$(z^2+c_1^2)\,dx = z\,dz \qquad \text{From eq ②}$$

$$dx = \frac{z}{z^2+c_1^2}\,dz$$

$$dx = \frac{1}{2}\,\frac{2z}{z^2+c_1^2}\,dz$$

$\Rightarrow$ integrating o.b.s

$$x+c = \frac{1}{2}\ln|z^2+c_1^2|$$

$$2x+2c = \ln|z^2+c_1^2|$$

$$c_2 = \ln|z^2+c_1^2| - 2x \quad —③ \quad \because 2c=c_2$$

Thus The curve given by eq ② and ③ generates the integral Surface For the given P.D.E

$$F(c_1, c_2) = 0$$

$$\boxed{F(x+y,\ \ln|z^2+c_1^2|-2x) = 0}$$

(iii) $(x^2-yz)p+(y^2-zx)q = z^2-xy$

Solution:-

$$(x^2-yz)p+(y^2-zx)q = z^2-xy$$

Auxilliary equation

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} \quad (1)$$

Eq (1) can be written as

$$\Rightarrow \frac{dx-dy}{(x^2-yz-y^2+zx)} = \frac{dy-dz}{y^2-zx-z^2+xy} = \frac{dz-dx}{z^2-xy-x^2+yz}$$

$$\Rightarrow \frac{dx-dy}{x^2-y^2+z(x-y)} = \frac{dy-dz}{y^2-z^2+x(y-z)} = \frac{dz-dx}{z^2-x^2+y(z-x)}$$

$$\Rightarrow \frac{dx-dy}{(x-y)(x+y)+z(x-y)} = \frac{dy-dz}{(y-z)(y+z)+x(y-z)} = \frac{dz-dx}{(z-x)(z+x)+y(z-x)}$$

$$\Rightarrow \frac{dx-dy}{(x-y)[x+y+z]} = \frac{dy-dz}{(y-z)[y+z+x]} = \frac{dz-dx}{(z-x)[z+x+y]} \quad (2)$$

Consider the First two terms of eq (2)

$$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(y+z+x)}$$

(with $(x+y+z)$ and $(y+z+x)$ struck out)

$$\frac{dx-dy}{(x-y)} = \frac{dy-dz}{(y-z)}$$

$\Rightarrow$ Integrating o.b.s

$$\ln(x-y) = \ln(y-z) + \ln c_1$$

$$\ln(x-y) = \ln(y-z)(c_1)$$

$$\boxed{\frac{x-y}{y-z} = c_1} \longrightarrow (3)$$

Similarly, considering last two terms of eq (2)

$$\Rightarrow \frac{dy - dz}{(y-z)(y+z+x)} = \frac{dz - dx}{(z-x)(x+y+z)}$$

$$\Rightarrow \frac{dy - dz}{y-z} = \frac{dz - dx}{z-x}$$

$$\ln(y-z) = \ln(z-x) + \ln c_2$$

$$\ln\frac{(y-z)}{z-x} = \ln c_2$$

$$\boxed{\frac{(y-z)}{z-x} = c_2} \longrightarrow (4)$$

Thus the integral curves given by eq (3) and (4) generate the integral surface $F(c_1, c_2)$

$$F\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$$

Example 10:- Pg → 19

Find the integral Surface of Linear P.D.E

$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$

containg the Straight Line

$x+y=0, \quad z=1.$

Solution:- $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$

The auxilliary equation For the given P.D.E are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z} \quad \text{--- (1)}$$

Suppose, we use the Multiplier yz, zx and xy the Find that each each Fraction in eq (1) is equal to

$$yz\,dx + zx\,dy + xy\,dz = 0 \quad \text{--- (2)}$$

Ruff:—

$P = x(y^2+z) = xy^2+zx \quad xy^3z + xyz^2 \quad | \; yz$

$Q = -y(x^2+z) = -yx^2 - yz \quad -x^3yz - xyz^2 \quad | \; zx$

$R = (x^2-y^2)z = x^2z - y^2z \quad x^3yz - xy^3z \quad | \; xy$

And $= 0$

$P = xy^2+zx \quad = x^2y^2 + x^2z \quad | \; x$

$Q = -yx^2 - yz \quad = -x^2y^2 - y^2z \quad | \; y$

$R = x^2z - y^2z \quad = -x^2z + y^2z \quad | \; -1$

$= 0$

Suppose we use Mul $x, y, -1$ then Find that each Fraction in eq ① is equal to

$$x\,dx + y\,dy - dz = 0 \quad \text{———} ③$$

⇒ integrating o.b.s

$$\frac{x^2}{2} + \frac{y^2}{2} - z = C_0$$

$$x^2 + y^2 - 2z = 2C_0 \qquad \because 2C_0 = C_2$$

$$\boxed{x^2 + y^2 - 2z = C_2} \quad \text{———} ④$$

⇒ integrating o.b.s in eq ②

$$xyz + xyz + xyz = C_0$$

$$3xyz = C_0$$

$$xyz = \frac{C_0}{3} \qquad \because \frac{C_0}{3} = C_1$$

$$\boxed{xyz = C_1} \quad \text{——} \quad (5)$$

we have the Equation in Parametric Form as

$x = t \quad y = -t \quad z = 1$

$x + y = 0 \quad , \quad z = 1$

Let $x = t \Rightarrow t + y = 0 \quad y = -t$

Eq (5) $\Rightarrow$ $C_1 = xyz$

$$C_1 = (t)(-t)(1)$$

$$C_1 = -t^2$$

Eq (4) $\Rightarrow$ $C_2 = (x^2 + y^2) - 2z$

$$C_2 = t^2 + (-t)^2 - 2(1)$$

$$C_2 = t^2 + t^2 - 2$$

$$C_2 = 2t^2 - 2$$

$$C_2 = 2(t^2 - 1)$$

$$C_2 = 2(C_1 - 1) \qquad \because C_1 = -t^2$$

$$C_2 = -2t^2 - 2$$
$$C_2 = -2C_1 - 2$$

Elliminating Parameter t,

$$2C_1 + C_2 + 2 = 0 \quad \text{——— (A)}$$

Hence, The required integral surface is

$$2xyz + x^2 + y^2 - 2z + 2 = 0$$
$$x^2 + y^2 - 2z + 2xyz + 2 = 0$$

$$\boxed{x^2 + y^2 - 2z + 2xyz + 2 = 0}$$

Example 11:- Pg → 19

Find the integral Surface of the Linear P.D.E

$$xp + yq = z$$

which contains the circle defined by $x^2 + y^2 + z^2 = 4$, $x + y + z = 2$

Solution:- $xp + yq = z$

The auxilliary equation

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \quad \text{——— (1)}$$

Integrating of the First two members

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\ln x = \ln y + \ln c$$

$$\ln \frac{x}{y} = \ln c$$

$$\boxed{\frac{x}{y} = c_1} \quad \text{——— (2)}$$

Similarly, Integrating of the Last two terms.

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\ln y = \ln z + \ln c$$

$$\ln \frac{y}{z} = \ln c$$

$$\boxed{\frac{y}{z} = c_2} \quad \text{——— (3)}$$

Hence, The integral Surface of the given P.D.E is

$$F\left(\frac{x}{y}, \frac{y}{z}\right) = 0 \quad \text{——— (4)}$$

if This integral surface also Contains the given circle then we Find a relation b/w x/y and y/z

The equation of two circle is

$$x^2+y^2+z^2=4 \quad \text{——} \quad (5)$$
$$x+y+z=2 \quad \text{——} \quad (6)$$

From eq (2) and (3) we have

$$(2) \Rightarrow \frac{x}{y}=c_1 \Rightarrow \boxed{y=\frac{x}{c_1}}$$

$$(3) \Rightarrow \frac{y}{z}=c_2 \Rightarrow z=\frac{y}{c_2}=\frac{x}{c_1c_2}$$

$$\boxed{z=\frac{x}{c_1c_2}}$$

Substituting the values of y and z in (5) and (6)

$$x^2+\left(\frac{x}{c_1}\right)^2+\left(\frac{x}{c_1c_2}\right)^2=4$$

$$x^2+\frac{x^2}{c_1^2}+\frac{x^2}{c_1^2+c_2^2}=4$$

$$x^2\left[1+\frac{1}{c_1^2}+\frac{1}{c_1^2c_2^2}\right]=4 \quad \text{——} \quad (7)$$

And

$$\Rightarrow \quad x + \frac{x}{c_1} + \frac{x}{c_1 c_2} = 2$$

$$x\left[1 + \frac{1}{c_1} + \frac{1}{c_1 c_2}\right] = 2$$

Squaring o.b.s

$$x^2\left[1 + \frac{1}{c_1} + \frac{1}{c_1^2 + c_2}\right]^2 = 4 \quad \text{—— (8)}$$

From (7) and (8) comparing

$$1 + \frac{1}{c_1^2} + \frac{1}{c_1^2 c_2^2} = \left[1 + \frac{1}{c_1} + \frac{1}{c_1 c_2}\right]^2$$

$$\cancel{1} + \cancel{\frac{1}{c_1^2}} + \cancel{\frac{1}{c_1^2 c_2^2}} = \cancel{1} + \cancel{\frac{1}{c_1^2}} + \cancel{\frac{1}{c_1^2 c_2^2}} + \frac{2}{c_1} + \frac{2}{c_1 c_2} + \frac{2}{c_1^2 c_2}$$

$$\frac{2}{c_1} + \frac{2}{c_1 c_2} + \frac{2}{c_1^2 c_2} = 0$$

$$\frac{2c_1 c_2 + 2c_1 + 2}{c_1^2 c_2} = 0$$

$$2c_1 c_2 + 2c_1 + 2 = 0$$

$$2[c_1 c_2 + c_1 + 1] = 0$$

$$c_1 c_2 + c_1 + 1 = 0$$

Replacing values of c_1 and c_2

$$\frac{x}{y}\cdot\frac{y}{z}+\frac{x}{y}+1=0$$

$$\Rightarrow \frac{x}{z}+\frac{x}{y}+1=0$$

$$\Rightarrow \frac{xy+xz+yz}{yz}=0$$

$$\boxed{xy+xz+yz=0}$$

Which is the required integral surface.

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Exercises

Q No 4: Find general integrals

i) $\frac{y^2 z}{x} p + xzq = y^2$

Solution:- Given Equation is

$$\frac{y^2 z}{x} p + xzq = y^2$$

The aux Equation is

$$\frac{dx}{y^2 z/x} = \frac{dy}{xz} = \frac{dz}{y^2} \quad \text{--- (1)}$$

Comparing First two members

$$\frac{dx}{y^2 z/x} = \frac{dy}{xz}$$

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz}$$

$$x^2 dx = y^2 dy$$

$\Rightarrow$ on integrating o.b.S

$$\frac{x^3}{3} = \frac{y^3}{3} + C$$

$$x^3 = y^3 + 3C$$

$$x^3 - y^3 = 3C \qquad \because 3C = C_1$$

$$\boxed{x^3 - y^3 = C_1} \quad \text{--- (2)}$$

Last and First terms

$$\frac{dx}{y^2z/x} = \frac{dz}{y^2}$$

$$\frac{x\,dx}{y^2z} = \frac{dz}{y^2}$$

$$x\,dx = z\,dz$$

$\Rightarrow$ on integrating o.b.S

$$\frac{x^2}{2} = \frac{z^2}{2} + C$$

$$x^2 = z^2 + 2C$$

$$\boxed{x^2 - z^2 = C_2} \longrightarrow (3) \quad \because 2C = C_2$$

Thus $\boxed{F(x^3 - y^3,\ x^2 - z^2) = 0}$

(ii) $(y+1)P + (x+1)Q = z$

Solution:– $(y+1)P + (x+1)Q = z$

Auxilliary equation

$P = y+1 \qquad Q = x+1 \qquad R = z$

$$\frac{dx}{y+1} = \frac{dy}{x+1} = \frac{dz}{z} \quad \text{——— (1)}$$

First two terms EV① we get

$$\frac{dx}{y+1} = \frac{dy}{x+1}$$

$$(x+1)dx = (y+1)dy$$

⇒ on integrating o.b.S

$$\frac{x^2}{2} + x = \frac{y^2}{2} + y + c$$

Mul 2

$$x^2 + 2x = 2y^2 + 2y + 2c \qquad \because 2c = C_1$$

$$\boxed{x^2 - y^2 + 2x - 2y = C_1}$$

1st and Last two terms EV① we get

$$\frac{dx}{y+1} = \frac{dz}{z}$$

$$z\,dx = (y+1)dz$$

$$zx = yz + z + C$$

$$zx - yz - z = C_2$$

$$z(x-y) = z + C_2 \qquad z + C_2 = C_2$$

$$\boxed{z(x-y) = C_2}$$

Then

$$F(c_1, c_2) = 0$$

$$\boxed{F(x^2 - y^2 + 2x - 2y,\ z(x-y)) = 0}$$

QNo5:- Find the integral Surface $xp - yq = z$ contains the circle $x^2+y^2=1$, $z=1$

Solution:- Given equation is

$$xp - yq = z$$

The Auxilliary equation is

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{z} \quad \text{---} (1)$$

Ist and 2nd terms of eq(1) we get

$$\frac{dx}{x} = \frac{dy}{-y}$$

$\Rightarrow$ integrating o.b.s

$$\ln x = -\ln y + \ln c$$

$$\ln x + \ln y = \ln c$$

$$\ln(xy) = \ln c$$

$$\boxed{xy = c_1} \quad \text{---} (2)$$

⇒ Now integrating last two members

$$\frac{dy}{-y} = \frac{dz}{z}$$

$$-\ln y = \ln z + \ln c$$

$$\ln y^{-1} = \ln z + \ln c$$

$$\ln \frac{1}{y} = \ln z + \ln c$$

$$\ln \frac{1}{y} = \ln(zc)$$

$$\boxed{\frac{1}{yz} = c_2} \quad \text{—— (3)}$$

Hence integral surface of given P.D.E is from (2) and (3)

$$c_1 = xy \quad , \quad c_2 = \frac{1}{yz}$$

$$\Rightarrow x = \frac{c_1}{y} \quad , \quad c_2 = \frac{1}{y} \quad \because z = 1$$

$$x = \frac{c_1}{1/c_2} \qquad \boxed{y = \frac{1}{c_2}}$$

$$\boxed{x = c_1 c_2}$$

Equation is circle is

$$x^2+y^2=1 \quad \text{and} \quad z=1$$

Replacing x and y

$$(c_1c_2)^2+\left(\frac{1}{c_2}\right)^2=1$$

$$c_1^2c_2^2+\frac{1}{c_2^2}=1$$

$$\frac{c_1^2c_2^4+1}{c_2^2}=1$$

$$c_1^2c_2^4+1=c_2^2$$

$$(xy)^2\left(\frac{1}{yz}\right)^4+1=\left(\frac{1}{yz}\right)^2$$

$$x^2y^2\frac{1}{y^4z^4}+1=\frac{1}{y^2z^2}$$

$$x^2\frac{1}{y^2z^4}+1=\frac{1}{y^2z^2}$$

Mul by y^2z^4 o.b.s

$$x^2+y^2z^4=z^2$$

$$\boxed{x^2+y^2z^4-z^2=0}$$

Which is required Solution.

PP

QNo3:- Find the integral Surface (general Solution) of the differential equation.

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z$$

Solution:- Given Equation

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z$$

$$P = \frac{\partial z}{\partial x} \quad , \quad q = \frac{\partial z}{\partial y}$$

$$x^2 P + y^2 q = (x+y)z \quad \text{——— (1)}$$

$P = x^2 \quad Q = y^2 \quad R = (x+y)z$

Aux. equation

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

=> Now First two terms Mul Ist 'y' and 2nd 'x'.

$$\frac{y\,dx + x\,dy}{x^2 y + x y^2} = \frac{dz}{(x+y)z}$$

$$\frac{y\,dx + x\,dy}{xy(x+y)} = \frac{dz}{(x+y)z}$$

$$\frac{y\,dx + x\,dy}{xy} = \frac{dz}{z}$$

$$\int \frac{d(xy)}{xy} = \int \frac{dz}{z}$$

$$\ln xy = \ln z + \ln c$$

$$\ln(xy) = \ln(zc)$$

$$xy = zc_1$$

$$\boxed{\frac{xy}{z} = c_1}$$

Now Second Last two terms

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$$

$$\frac{dx - dy}{(x-y)(x+y)} = \frac{dz}{(x+y)z}$$

$$\int \frac{dx - dy}{(x-y)} = \int \frac{dz}{z}$$

$$\ln(x-y) = \ln z + \ln c$$

$$\ln(x-y) = \ln zC$$

$$x-y = zC_2$$

$$\boxed{\frac{x-y}{z} = C_2}$$

So, general Sol of Surface is
$F(C_1, C_2)$

$$\boxed{F\left(\frac{xy}{z}, \frac{x-y}{z}\right) = 0}$$

which is Required

$\longleftrightarrow$

η / ita
ε / xs

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Example:-

$$xU_x - yU_y + y^2U = y^2$$

Solution:- Given Equation is

$$xU_x - yU_y + y^2U = y^2 \quad \text{---} \quad (1)$$

$$\xi(x,y) = x \quad \text{---} \quad (i)$$

$$\eta(x,y) = K$$

$$\frac{dy}{dx} = \frac{b}{a} = -\frac{y}{x}$$

$$\int \frac{dy}{dx} = \int -\frac{y}{x}$$

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\ln y = -\ln x + \ln c$$

$$\ln y + \ln x = \ln c$$

$$\ln(xy) = \ln c$$

$$xy = c$$

$$\boxed{xy = k}$$

$$\eta(x,y) = xy \quad \text{---} \quad (ii)$$

$$J = \frac{\xi(x,y)}{\eta(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}$$

$$J = \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x \neq 0$$

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial \xi_e}\frac{\partial \xi_e}{\partial x} + \frac{\partial U}{\partial M}\cdot\frac{\partial M}{\partial x} \quad \text{——— (A)}$$

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial \xi_e}\frac{\partial \xi_e}{\partial y} + \frac{\partial U}{\partial M}\frac{\partial M}{\partial y} \quad \text{——— (B)}$$

⇒ From (A)

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial \xi_e}(1) + \frac{\partial U}{\partial M}(y)$$

$$U_x = U_{\xi_e} + yU_M \quad \text{——— (C)}$$

⇒ From (B)

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial \xi_e}(0) + \frac{\partial U}{\partial M}x$$

$$U_y = xU_M$$

⇒ From (i)

$$M(x, y) = xy$$

$$M(x, y) = \xi_e y \quad \text{From (i)}$$

$$\boxed{\frac{M}{\xi_e} = y}$$

Put the value in (1) we get

$$\Rightarrow \xi_e\left(U_{\xi_e} + \frac{M}{\xi_e} U_M\right) - \frac{M}{\xi_e}(\xi_e U_M) + \frac{M^2}{\xi_e^2} U = \frac{M^2}{\xi_e^2}$$

$$\Rightarrow \xi_e U_{\xi_e} + M U_M - M U_M + \frac{M^2}{\xi_e^2} U = \frac{M^2}{\xi_e^2}$$

$$\Rightarrow \xi_e U_{\xi_e} + \frac{M^2}{\xi_e^2} U = \frac{M^2}{\xi_e^2}$$

$$\Rightarrow U_{\xi_e} + \frac{M^2}{\xi_e^3} U = \frac{M^2}{\xi_e^3} \quad \text{——— (2)}$$

$$P(\xi_e) = e^{\int \frac{M^2}{\xi_e^3} d\xi_e}$$

$$= e^{-\frac{1}{2}\frac{M^2}{\xi_e^2}}$$

$$= e^{-\frac{M^2}{2\xi_e^2}}$$

Mul in eq (2)

$$U_{\xi_e} e^{-\frac{M^2}{2\xi_e^2}} + \frac{M^2}{\xi_e^3} e^{-\frac{M^2}{2\xi_e^2}} U = \frac{M^2}{\xi_e^3} e^{-\frac{M^2}{2\xi_e^2}}$$

$$\int \frac{d}{d\xi_e} U e^{-\frac{1}{2}\frac{M^2}{\xi_e^2}} = \int \frac{M^2}{\xi_e^3} e^{-\frac{M^2}{2\xi_e^2}} d\xi_e$$

$$\Rightarrow \xi_e \left(U_{\xi_e} + \frac{M}{\xi_e} U_M \right) - \frac{M}{\xi_e} (\xi_e U_M) + \frac{M^2}{\xi_e^2} U = \frac{M^2}{\xi_e^2}$$

$$\Rightarrow \xi_e U_{\xi_e} + M U_M - M U_M + \frac{M^2}{\xi_e^2} U = \frac{M^2}{\xi_e^2}$$

$$\Rightarrow \xi_e U_{\xi_e} + \frac{M^2}{\xi_e^2} U = \frac{M^2}{\xi_e^2}$$

$$U_{\xi_e} + \frac{M^2}{\xi_e^3} U = \frac{M^2}{\xi_e^3} \quad \text{——— (2)}$$

$$P(\xi_e) = \frac{M^2}{\xi_e^3} U \qquad Q(\xi_e) = \frac{M^2}{\xi_e^3}$$

$$U = e^{-\int P(\xi_e) d\xi_e} \left\{ f(M) + \int Q(\xi_e) e^{\int P(\xi_e) d\xi_e} d\xi_e \right\}$$

$$U = e^{-\int \frac{M^2}{\xi_e^3} d\xi_e} \left\{ f(M) + e^{\int \frac{M^2}{\xi_e^3} d\xi_e} d\xi_e \right\}$$

$$U = e^{\frac{M^2}{2\xi_e^2}} \left\{ f(M) + e^{-\frac{M^2}{2\xi_e^2}} \right\}$$

$$\boxed{U = f(M) e^{M^2/2\xi_e^2} + 1}$$

General Solution

Example:-

$$xU_x + yU_y = nU$$

Solution:-

$$xU_x + yU_y - nU = 0 \quad \text{——— (A)}$$

Co-efficient are $a = x$ $y = y$ $c = -n$

A.E

$$x = \xi$$

$$\eta = K$$

Now $\frac{dy}{dx} = \frac{b}{a} = \frac{y}{x}$

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + \ln K$$

$$\ln y - \ln x = \ln K$$

$$\ln (y/x) = \ln K$$

$$\boxed{\frac{y}{x} = K}$$

$$K = \frac{y}{x} = \frac{y}{\xi}$$

$$\boxed{K = \frac{y}{\xi}}$$

Transformation

$$\xi = x$$

$$\eta = y/x \implies y = \eta \xi$$

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{-y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{1}{x} \neq 0$$

Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$\frac{\partial u}{\partial x} = u_\xi (1) + u_\eta \left(-\frac{y}{x^2}\right)$$

$$= u_\xi + u_\eta \left(\frac{-\eta \xi}{\xi^2}\right)$$

$$u_x = \frac{\partial u}{\partial x} = u_\xi - \frac{\eta}{\xi} u_\eta$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$\frac{\partial u}{\partial y} = u_\xi (0) + u_\eta \left(\frac{1}{x}\right)$$

$$u_y = \frac{\partial u}{\partial y} = \frac{1}{\xi}(u_\eta)$$

equation ④ becomes

$$\xi\left(U_\xi - \frac{\eta}{\xi}U_\eta\right) + \eta\,\xi\,\frac{1}{\xi}U_\eta - nU = 0$$

$$\xi U_\xi - \xi\frac{\eta}{\xi}U_\eta + \eta U_\eta - nU = 0$$

$$\xi U_\xi - \eta U_\eta + \eta U_\eta - nU = 0$$

$$\xi U_\xi - nU = 0$$

$$U_\xi - \frac{n}{\xi}U = 0 \quad ⑤$$

Here

$$P(\xi) = \frac{-n}{\xi}, \quad Q(\xi) = 0$$

$$U = e^{-\int P(\xi)d\xi}\left\{f(\eta) + \int Q(\xi)e^{\int P(\xi)d\xi}d\xi\right\}$$

$$U = e^{-\int \frac{-n}{\xi}d\xi}\left\{f(\eta) + \int 0\, e^{\int \frac{-n}{\xi}d\xi}d\xi\right\}$$

$$U = e^{n\ln\xi}f(\eta)$$

$$= \xi^n f(\eta)$$

$$\boxed{U = x^n f(y/x)}$$

Example:-

$$xU_x + yU_y = x^n$$

Solution:-

$$xU_x + yU_y = x^n \quad \text{——— (1)}$$

Co-efficient are

$a = x$

$b = y$

$\xi = x$

$\eta = K$

Now

$$\frac{dy}{dx} = \frac{y}{x} = \frac{b}{a}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + \ln K$$

$$\ln y - \ln x = \ln K$$

$$\ln \frac{y}{x} = \ln K$$

$$\frac{y}{x} = K \Rightarrow K = \frac{y}{x}$$

So

$$\boxed{K = \frac{y}{\xi}}$$

$x = \xi$

$$\eta = \frac{y}{\xi} \Rightarrow \boxed{y = \eta \xi}$$

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -y/x^2 & 1/x \end{vmatrix} = \frac{1}{x} \neq 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= u_\xi(1) + u_\eta\left(\frac{-y}{x^2}\right)$$

$$= u_\xi - u_\eta\left(\frac{\eta\xi}{\xi^2}\right)$$

$$u_x = \frac{\partial u}{\partial x} = u_\xi - \frac{\eta}{\xi}u_\eta$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= u_\xi(0) + u_\eta \cdot \frac{1}{\xi}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{1}{\xi}u_\eta$$

Equation (A) becomes

$$\xi\left(u_\xi - \frac{\eta}{\xi}u_\eta\right) + \eta\xi \cdot \frac{1}{\xi}u_\eta = \xi^n$$

$$\xi u_\xi - \eta u_\eta + \eta u_\eta = \xi^n$$

$$\xi_e U_{\xi_e} = \xi_e^n$$

$$U_{\xi_e} = \frac{\xi_e^n}{\xi_e} \quad \text{---} \quad (1)$$

Here

$$P(\xi_e) = 0$$

$$P(\xi) = 0 \qquad Q(\xi_e) = \xi_e^{n-1}$$

$$U = e^{-\int P(\xi_e) d\xi_e} \left\{ f(\eta) + \int Q(\xi_e) e^{\int P(\xi_e) d\xi_e} d\xi_e \right\}$$

$$= e^0 \left\{ f(\eta) + \int \xi_e^{n-1} e^0 \, d\xi_e \right\}$$

$$= f(\eta) + \int \xi_e^{n-1} d\xi_e$$

$$= f(\eta) + \frac{\xi_e^n}{n}$$

$$\boxed{U = F(y/x) + \frac{x^n}{n}} \quad \text{General Solution}$$

Example:-

$$aU_x + bU_y + CU = d \; .$$

Solution:-

$$aU_x + bU_y + CU = d \quad \text{---} \quad (A)$$

$$\left.\begin{aligned} \xi_e &= x \\ \eta &= K \end{aligned}\right] \quad \text{---} \quad (B)$$

$$\frac{dy}{dx} = \frac{b}{a} = \frac{b}{a}$$

$$\int a\,dy = \int b\,dx$$

$$ay = bx + K$$

$$ay - bx = K$$

(B) becomes

$$\xi = x$$

$$\eta = ay - bx$$

$$\Rightarrow \eta + b\xi = ay \qquad \because x = \xi$$

$$y = \frac{\eta + b\xi}{a}$$

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -b & a \end{vmatrix} = a \neq 0$$

Now,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$u_x = u_\xi(1) + u_\eta(-b)$$

$$u_x = u_\xi - b u_\eta$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$= U_{\xi}(0) + U_{\mu}(a)$$

$$U_y = a U_{\mu}$$

Equation (A) becomes

$$a(U_{\xi} - bU_{\mu}) + b(aU_{\mu}) - cU = d$$

$$aU_{\xi} - \cancel{abU_{\mu}} + \cancel{abU_{\mu}} - cU = d$$

$$aU_{\xi} - cU = d$$

$$U_{\xi} + \frac{c}{a}U = \frac{d}{a} \quad \text{——— (1)}$$

$$P(\xi) = \frac{c}{a}, \quad Q(\xi) = \frac{d}{a}$$

$$U = e^{-\int P(\xi)d\xi}\left\{f(\mu) + \int Q(\xi)e^{\int P(\xi)d\xi} d\xi\right\}$$

$$U = e^{-\int \frac{c}{a}d\xi}\left\{f(\mu) + \frac{d}{a}e^{\int \xi d\xi} \cdot d\xi\right\}$$

$$U = e^{-\frac{c}{a}\xi}\left\{f(\mu) + \frac{d}{a}\int e^{c/a\,\xi} \cdot d\xi\right\}$$

$$U = e^{-\frac{c}{a}\xi}\left\{f(\mu) + \frac{d}{a}e^{\frac{c}{a}\xi}\right\}$$

$$U = e^{-\frac{c}{a}\xi} f(\mu) + \frac{d}{a}e^{-\frac{c}{a}\xi} \cdot e^{\frac{c}{a}\xi}$$

$$U = e^{-\frac{c}{a}\xi} f(\eta) + \frac{d}{a}$$

$$\boxed{U = e^{-\frac{c}{a}\xi} f(ay - bx) + \frac{d}{a}}$$

General Solution.

Cauchy's Method of Characteristics

Example: P → 30

Find the characteristics of the equation $pq = z$ and determine the integral surface which passes through the straight line $x = 1$, $z = y$

Solution:-

Parametric Form of initial data curve is given as

$x_0(s) = 1 \quad y_0(s) = s \quad z_0(s) = s \}$ — ★

ordinarily the solution is sought in Parametric Form as

$\{x = x(t, s), \quad y = y(t, s)$
$z = z(t, s)\}$ ——— ★★

Given $x = 1 \quad y = s \quad z = s$

$pq = z$ ——— (A)

Thus using the given data, the diff. equation becomes.

$F = p_0(s)\, q_0(s) - s = 0$ ——— (B)

Strip condition

$$\frac{dz}{ds} = p_0(s)\frac{dx}{ds} + q_0(s)\frac{dy}{ds}$$

$1 = P_0(0) + q_0(1)$

$\boxed{1 = q_0}$ Put in (A)

$P_0(1) = z = s$

$\boxed{P_0 = s}$

(unique intial strip)

$\frac{dx}{ds} = \frac{d}{ds}(1) = 0$

$\frac{dy}{ds} = \frac{d}{ds}(s) = 1$

$\frac{dz}{ds} = \frac{d}{ds}(s) = 1$

Now the characteristic equation For the given P.D.E are

1) $\frac{dx}{dt} = F_p = q$ ——— (i)

2) $\frac{dy}{dt} = F_q = P$ ——— (ii)

3) $\frac{dP}{dt} = -(F_x + PF_z) = -(0 + P(1)) = P$

$\int \frac{dP}{dt} = \int P$

$\int \frac{dP}{P} = \int dt$

$\ln P = t + C_1$

$P = e^{t+C_1} = e^t \cdot e^{C_1}$

$\boxed{P = e^t C_2}$ ——— (1) $\boxed{e^{C_1} = C_2}$

$P(t,S) = C_2 e^t$

Put $t=0$ $\Rightarrow P(0,S) = C_2 e^0$

$S = C_2(1)$

$\boxed{C_2 = S}$

① $\Rightarrow \boxed{P = Se^t}$ —— (iii)

④ $\frac{dq}{dt} = -(F_y + qF_z) = -(0 + q(-1)) = q$

$\frac{dq}{dt} = q$

$\int \frac{dq}{q} = \int dt$

$\ln q = t + C_3$

$q = e^{t+C_3} = e^t \cdot e^{C_3}$

$q = C_4 e^t$ —— ② $\boxed{\because e^{C_3} = C_4}$

$q(t,S) = C_4 e^t$

Put $t=0$ $q(0,S) = C_4 e^0$

$1 = C_4$ Put in ②

$\Rightarrow \boxed{q = e^t}$ —— (iv)

⑤ $\frac{dz}{dt} = pF_p + qF_q$

$= pF_p + qF_q = pq + qp$

$$\frac{dz}{dt} = 2pq \quad \text{---} (3)$$

From eq (i)

$$\frac{dx}{dt} = q = e^t \quad \text{by eq (iii)}$$

$$dx = e^t dt$$

Integrating o.b.S

$$x = e^t + C_5 \quad \text{---} (4)$$

$$x(t,S) = e^t + C_5$$

put $t=0$ $\quad x(0,S) = e^0 + C_5$

$$1 = 1 + C_5$$

$$\boxed{C_5 = 0}$$

(4) $\Rightarrow$ $\boxed{x = e^t}$ ——— (v)

Now

From eq (ii)

$$\frac{dy}{dt} = p = Se^t \quad \text{by eq (iii)}$$

$$\frac{dy}{dt} = Se^t$$

$$dy = Se^t dt$$

Integrating o.b.S

$$y = Se^{t} + C_6 \quad \text{——} \quad (5)$$

$$y(t,S) = Se^{t} + C_6$$

Put $t=0$, $y(0,S) = Se^{0} + C_6$

$$S = S + C_6$$

$$\boxed{C_6 = 0} \quad \text{Put in (5)}$$

$$\boxed{y = Se^{t}} \quad \text{——} \quad (vi)$$

Now From eq (3)

$$\frac{dz}{dt} = 2pq$$

$$= 2(Se^{t})(e^{t}) \quad \text{by eq (iii) and (v)}$$

$$\frac{dz}{dt} = 2Se^{2t}$$

$$dz = 2Se^{2t}\,dt$$

Integrating o.b.s

$$Z = 2S\frac{e^{2t}}{2} + C_7$$

$$Z = Se^{2t} + C_7$$

$$Z = Se^{2t} + C_7 \quad \text{——} \quad (6)$$

$$Z(t,S) = Se^{2t} + C_7$$

Put $t=0$, $Z(0,S) = Se^{2(0)} + C_7$

$$S = S + C_7$$

$$\boxed{C_7 = 0}$$

(6) $\Rightarrow$ $Z = S e^{2t}$

So
$$\left\{ \begin{array}{l} P = S e^{t}, \quad q = e^{t}, \quad x = e^{t} \\ y = S e^{t}, \quad Z = S e^{2t} \end{array} \right\}$$ Ans

Now $xy = e^{t}(Se^{t})$
$= Se^{2t}$
$xy = Z$

Thus Required integral Surface is $Z = xy$.

Example: Pg $\longrightarrow$ 31

Find the characteristics of the Equation $pq = z$ and hence determine the integral surface which passes through the parabola
$x = 0, \quad y^2 = z$

Solution:- The initial data curve is

$x_0(S) = 0, \quad y_0(S) = S \quad Z_0(S) = S^2$

using this data given P.D.E becomes

$P_0(S)\, q_0(S) - S^2 = 0 = F$ — (1)

Strip Conditions

$$\frac{dz}{ds} = P_0(s)\frac{dx}{ds} + q_0(s)\frac{dy}{ds}$$

$$2s = P_0(0) + q_0(1)$$

$$2s = q_0$$

$$\boxed{q_0 = 2s}$$

$$① \Rightarrow P_0(s)(2s) = s^2$$

$$P_0(s) = \frac{s^2}{2s} = \frac{s}{2}$$

$$\boxed{P_0 = \frac{s}{2}}$$

Thus $P_0 = \frac{s}{2}$, $q_0 = 2s$ —— ②

Characteristic Equations

① $\frac{dx}{dt} = F_p = q$ —— (i)

② $\frac{dy}{dt} = F_q = P$ —— (ii)

③ $\frac{dz}{dt} = PF_p + qF_q$

$= Pq + qP$

$= 2Pq$ —— (iii)

④ $\frac{dP}{dt} = -(F_x + PF_z) = -(0 + P(-1))$

$= P$ —— (iv)

(5) $\frac{dv}{dt} = -(F_y + vF_z) = -(0 + v(-1))$

$= v$ ——— (v)

From (iv)

$$\frac{dP}{dt} = P$$

$$\int \frac{dP}{P} = \int dt$$

$$\ln P = t + C_1$$

$$P = e^{t+C_1} = e^t \cdot e^{C_1}$$

$$P = e^t C_2 \quad \text{——— (3)} \quad \because e^{C_1} = C_2$$

$$P(t,S) = e^t C_2$$

Put $t=0$, $P(0,S) = e^0 C_2$

$$\frac{S}{2} = C_2 \quad \text{Put in (3)}$$

$$\boxed{P = \frac{S}{2} e^t}$$

From (v) $\frac{dv}{dt} = v$

$$\int \frac{dv}{v} = \int dt$$

$$\ln v = t + C_3$$

$$v = e^{t+C_3} = e^t \cdot e^{C_3}$$

$$v = e^t C_4 \quad \because e^{C_3} = C_4$$

$$v = C_4 e^t \quad \text{——— (4)}$$

$$v(t,S) = C_4 e^t$$

Put $t=0$, $v(0,S) = C_4 e^0$

$$2S = C_4 \qquad \text{Put in (4)}$$

$$\boxed{v = 2Se^t}$$

From eq (i)

$$\frac{dx}{dt} = v = 2Se^t$$

$$\int dx = \int 2Se^t\, dt$$

$$x = 2Se^t + C_5 \quad \text{——— (6)}$$

$$x(t,S) = 2Se^t + C_5$$

Put $t=0$ $\quad x(0,S) = 2Se^0 + C_5$

$$0 = 2S + C_5$$

$$C_5 = -2S \qquad \text{Put in (6)}$$

$$x = 2Se^t - 2S$$

$$\boxed{x = 2S(e^t - 1)}$$

From (ii)

$$\frac{dy}{dt} = P = \frac{S}{2}e^t$$

$$dy = \frac{S}{2}e^t\, dt$$

Integrating $y = \frac{s}{2} e^{t} + c_6$ —— (7)

$$y(t,s) = \frac{s}{2} e^{t} + c_6$$

Put $t=0$ $\quad y(0,s) = \frac{s}{2} e^{0} + c_6$

$$s = \frac{s}{2} + c_6$$

$$s - \frac{s}{2} = c_6$$

$$\frac{s}{2} = c_6 \quad \text{Put in (7)}$$

$$y = \frac{s}{2} e^{t} + \frac{s}{2}$$

$$\boxed{y = \frac{s}{2}(e^{t}+1)}$$

From eq (iii)

$$\frac{dz}{dt} = 2pq = 2\left(\frac{s}{2}\right)e^{t}(2se^{t})$$

$$= 2s^2 e^{2t}$$

$$\frac{dz}{dt} = 2s^2 e^{2t}$$

$$\int dz = \int 2s^2 e^{2t}\, dt$$

$$z = 2s^2 \frac{e^{2t}}{2} + c_7$$

$$z = s^2 e^{2t} + c_7 \quad \text{—— (8)}$$

$$z(t,S) = S^2 e^{2t} + C_7$$

put $t=0$, $Z(0,S) = S^2 e^0 + C_7$

$$\cancel{S^2} = \cancel{S^2} + C_7$$

$$\boxed{C_7 = 0}$$ put in (8)

$$\boxed{Z = S^2 e^{2t}}$$

Therefore we have

$$P = \frac{S}{2} e^t, \quad q = 2Se^t$$

$$x = 2S(e^t - 1) \text{ — (a)}, \quad y = \frac{S}{2}(e^t + 1) \text{ — (b)}$$

$$Z = S^2 e^{2t} \text{ — (c)}$$

Multiply eq (b) by 4

$$4y = 2S(e^t + 1)$$

Adding in (a)

$$x + 4y = 2S(e^t + 1) + 2S(e^t - 1)$$

$$x + 4y = 2Se^t + \cancel{2S} + 2Se^t - \cancel{2S}$$

$$x + 4y = 4Se^t \text{ — (d)}$$

Squaring (d) $(x+4y)^2 = (4Se^t)^2$

$$(x+4y)^2 = 16S^2 e^{2t} \text{ — (e)}$$

Multiply eq (c) by (16)

$$16Z = 16S^2 e^{2t} \text{ — (f)}$$

By eq (e) and (f)

$$\boxed{16Z = (x+4y)^2}$$

This is the required integral surface.

Example:- Pg → 32

Find the characteristics of the PDE $p^2 + q^2 = 2$ and determine the integral surface which passes through $x=0$ $z=y$

Solution:-

The initial data curve is

$x_0(s) = 0$, $y_0(s) = s$, $z_0(s) = s$

and parametric equation

$x = x(t,s)$, $y = y(t,s)$, $z = z(t,s)$

using this data given PDE becomes

$$p_0^2(s) + q_0^2(s) - 2 = 0 = F \quad \text{——} \ (1)$$

Strip conditions $\boxed{x=0 \quad y=s \quad z=s}$

$$\frac{dz}{ds} = p(s)\frac{dx}{ds} + q(s)\frac{dy}{ds} \quad \text{——} \ (2)$$

$$1 = 0 + q_0(s)(1)$$

$q_0(s) = 1$ put in (1)

$$\boxed{q_0 = 1}$$

$p^2 + 1 = 2$

$p^2 = 2 - 1$

$p^2 = 1$

$p_0 = \pm 1$

characteristic equations

① $\frac{dx}{dt} = F_p = 2p$ —— (i)

② $\frac{dy}{dt} = F_q = 2q$ —— (ii)

③ $\frac{dz}{dt} = pF_p + qF_q$

$= p(2p) + q(2q)$

$= 2p^2 + 2q^2$

$= 2(p^2 + q^2)$

$= 2(2) = 4$

$\frac{dz}{dt} = 4$ —— (iii)

④ $\frac{dp}{dt} = -(F_x + pF_z) = 0$

$\int \frac{dp}{dt} = \int 0$

$\Rightarrow \boxed{p = c_1}$

$$p(t,x) = C_1$$

Put $t=0$, $p(0,x) = C_1$

$$C_1 = \pm 1$$

$\Rightarrow$ $\boxed{p = \pm 1}$

(5) $\frac{dq}{dt} = -(F_y + qF_z) = 0$

$$\int \frac{dq}{dt} = \int 0$$

$$q = C_2$$

$$q(t,x) = C_2$$

Put $t=0$, $q(0,x) = C_2$

$$1 = C_2$$

$\Rightarrow$ $\boxed{q = 1}$

From eq (i) $\frac{dx}{dt} = 2p = 2(\pm 1)$

$$\frac{dx}{dt} = \pm 2$$

$$\int dx = \int \pm 2\, dt$$

$$x = \pm 2t + C_3$$

$x(t,x) = \pm 2t + C_3$, Put $t=0$

$x(0,x) = \pm 2(0) + C_3$

$\boxed{0 = C_3}$

$\Rightarrow \boxed{x = \pm 2t}$

From (ii), $\frac{dy}{dt} = 2v = 2(1)$

$\frac{dy}{dt} = 2$

$\int dy = \int 2dt$

$y = 2t + C_4$

$y(t,x) = 2t + C_4$

Put $t=0$, $y(0,x) = 2(0) + C_4$

$\boxed{S = C_4}$

$\boxed{y = 2t + S}$

From (iii) $\frac{dz}{dt} = 4$

$\int dz = \int 4dt$

$$z = 4t + C_5$$

$$z(t, x) = 4t + C_5 \quad , \text{ put } t=0$$

$$z(0, x) = 4(0) + C_5$$

$$\boxed{S = C_5}$$

$$\Rightarrow \boxed{z = 4t + S}$$

Therefore we have

$$p = \pm 1 \qquad q = 1$$

$$x = \pm 2t \quad \text{—— (a)}$$

$$y = 2t + S \quad \text{—— (b)}$$

$$z = 4t + S \quad \text{—— (c)}$$

The Last three equation are Parametric equation of the desired integral Surface.

Eliminating Parameters S and t,

$$y \pm x = 2t + S \pm 2t$$

$$y \pm x = 4t \pm S$$

$$z = 4t + S$$

$$\boxed{z = y \pm x}$$

This is the required integral Surface.

QNo6:- Pg → 49

Find the equation of the integral Surface of the PDE

$$2y(z-3)p + (2x-z)q = y(2x-3)$$

Solution:-

Given $2y(z-3)p + (2x-z)q = y(2x-3)$ — ①

Comparing it with $Pp + Qq = R$

$P = 2y(z-3)$

$Q = 2x - z$

$R = y(2x-3)$

Auxilliary eq i.e

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{2y(z-3)} = \frac{dy}{(2x-z)} = \frac{dz}{y(2x-3)} \quad ②$$

$$\frac{1/2\,dx}{y(z-3)} = \frac{y\,dy}{y(2x-z)} = \frac{dz}{y(2x-3)}$$

$$\frac{\frac{1}{2}dx + y\,dy - dz}{y(z-3) + y(2x-z) - y(2x-3)} = 0$$

$$\frac{\frac{1}{2}dx + y\,dy - dz}{y(z-3) + y(2x-z) - y(2x-3)} = 0$$

$$\frac{\frac{1}{2}dx + y\,dy - dz}{yz - 3y + 2xy - yz - 2xy + 3y} = 0$$

$$\frac{1}{2}dx + y\,dy - dz = 0$$

Integrating,

$$\frac{1}{2}x + \frac{y^2}{2} - z = C_1$$

$$x + y^2 - 2z = 2C_0$$

$$\Rightarrow x + y^2 - 2z = C_1 \quad \text{—— (i)}$$

Now

$$\frac{dx}{2y(z-3)} = \frac{dz}{y(2x-3)}$$

$$\frac{dx}{2(z-3)} = \frac{dz}{2x-3}$$

$$\int (2x-3)\,dx = \int (2z-6)\,dz$$

$$\frac{2x^2}{2} - 3x = \frac{2z^2}{2} - 6z + C_2$$

$$x^2 - 3x = z^2 - 6z + C_2$$

$$x^2 - 3x - z^2 + 6z = C_2 \quad \text{——— (ii)}$$

Since

$$x^2 + y^2 = 2x \quad \text{and} \quad z = 0$$

$$x^2 + y^2 - 2x = 0 \quad \text{and} \quad z = 0$$

$$x^2 + y^2 - 2x + 1 - 1 = 0 \quad \text{and} \quad z = 0$$

$$(x^2 - 2x + 1) + (y^2 - 1) = 0 \quad \text{and} \quad z = 0$$

$$(x^2 - 2x + 1) + y^2 = 1 \quad \text{and} \quad z = 0$$

$$(x - 1)^2 + y^2 = 1 \quad \text{and} \quad z = 0$$

The Parametric equations of this system are

$$x - 1 = \cos\theta \quad , \quad y = \sin\theta \; , \; z = 0$$

$$x = 1 + \cos\theta \quad , \quad y = \sin\theta \; , \; z = 0$$

Put in (i) and (iii), we have

$$(i) \Rightarrow (1 + \cos\theta) + \sin^2\theta - 2(0) = C_1$$

$$1 + \cos\theta + \sin^2\theta = C_1$$

$$1 + \cos\theta + 1 - \cos^2\theta = C_1$$

$$2 + \cos\theta - \cos^2\theta = C_1$$

$$-(\cos^2\theta - \cos\theta - 2) = C_1 \quad \text{——— (iii)}$$

Now (ii) $\Rightarrow$

$(1+\cos\theta)^2 - 3(1+\cos\theta) - 0 + 0 = C_2$

$1+\cos^2\theta + 2\cos\theta - 3 - 3\cos\theta = C_2$

$\cos^2\theta - \cos\theta - 2 = C_3$ —— (iv)

Comparing (iii) and (iv)

$-C_1 = C_2$

$C_1 + C_2 = 0$ —— (v)

$F(C_1, C_2) = 0$

Adding (i) and (ii)

$x + y^2 - 2z + x^2 - 3x - z^2 + 6z = 0$

$$\boxed{x^2 + y^2 - 2x + 4z - z^2 = 0}$$

which is required Solution.

QNo 9:- Pg → 49

Find the characteristics of the equation $pq = xy$ and determine the integral surface which passes through the curve $z = x,\ y = 0$

Solution:- The initial data curve is

$x_0(S) = S$, $y_0(S) = 0$, $z_0(S) = S$

And the Parametric equations

$x = x(t,S)$, $y = y(t,S)$

$z = z(t,S)$

using the data given PDE becomes

$$P_0(S)\, q_0(S) = 0 = F \quad \text{——— (1)}$$

Strip conditions

$$\frac{dz}{dS} = P(S)\frac{dx}{dS} + q(S)\frac{dy}{dS} \quad \text{——— (2)}$$

$$1 = P_0(1) + q_0(0)$$

$$\boxed{P_0 = 1} \quad \text{put in (1)}$$

$$(1)(q_0) = 0$$

$$\boxed{q_0 = 0}$$

Characteristic equations

1) $\dfrac{dx}{dt} = F_p = q$ ——— (i)

2) $\dfrac{dy}{dt} = F_q = P$ ——— (ii)

3) $\frac{dz}{dt} = pF_p + qF_q$

$= p(q) + q(p)$

$= 2pq$ ——— (iii)

4) $\frac{dp}{dt} = -(F_x + pF_z)$

$= -(1 + p(0))$

$= -1$ ——— (iv)

5) $\frac{dq}{dt} = -(F_y + qF_z)$

$= -(1 + q(0))$

$= -1$ ——— (v)

From eq (iii)

$\frac{dp}{dt} = -1$

$\int dp = \int -1\, dt$

$p = -t + c_1$

$p(t, s) = -t + c_1$, Put $t=0$

$p(0, s) = -0 + c_1$

$1 = c_1$

$\Rightarrow \boxed{p = 1-t}$

From eq (v)

$$\frac{dv}{dt} = -1$$

$$\int dv = \int -1\,dt$$

$$v = -t + C_2$$

$$v(t,s) = -t + C_2$$

$$v(0,s) = 0 + C_2$$

$$0 = C_2$$

$$\boxed{v = -t}$$

From (i) $\frac{dx}{dt} = v = 0$

Integrating

$$x = C_3$$

$$x(t,s) = C_3 \quad , \text{Put } t=0$$

$$x(0,s) = C_3$$

$$\boxed{s = C_3}$$

$$\boxed{x = s}$$

From (ii) $\frac{dy}{dt} = p = 1$

Integrating

$$y = t + C_4$$

$$y(t,s) = t + C_4 \quad , \text{Put } t=0$$

$$y(0,s) = 0 + C_4$$

$$\boxed{C_4 = 0}$$

$$\Rightarrow \boxed{y = t}$$

From (iii)

$$\frac{dz}{dt} = 2pq = 2(1)(0)$$
$$= 0$$

on Integrating

$$z = C_5$$

$$z(t, s) = C_5 \quad , \quad \text{Put } t=0$$
$$z(0, s) = C_5$$
$$\boxed{s = C_5}$$

$$\boxed{z = s}$$

Thus we have

$$p = 1 \quad , \quad q = 0$$

$x = s$ — (a)

$y = t$ — (b)

$z = s$ — (c)

(a) – (c)

$\Rightarrow x - z = 0$ ——— (d)

(b) × (d) $\Rightarrow$ $\boxed{y(x - z) = 0}$

This is the required integral surface.

⟹ Compatible System of First order Equations:–

Two First order PDE'S are said to be compatible if they have a common solution.

⟹ Necessary and Sufficient Condition:–

The Necessary and Sufficient condition for two partial diff. equation

$$f(x, y, z, p, q) = 0 \quad \text{——— (1)}$$

and

$$g(x, y, z, p, q) = 0 \quad \text{——— (2)}$$

to be compatible is

$$\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$$

Example:– Pg ⟶ 35

Show that the following PDE'S $xp - yq = x$ and $x^2p + q = xz$ are compatible and hence, Find their solution.

Solution:- We have

$$F = xp - yq - x = 0 \quad \text{---} ①$$

$$\Rightarrow qy + x = xp$$

$$g = x^2 p + q - xz = 0 \quad \text{---} ②$$

- $$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p-1 & x \\ 2xp-z & x^2 \end{vmatrix}$$

$$= x^2(p-1) - x(2xp - z)$$

$$= x^2 p - x^2 - 2x^2 p + xz$$

$$= xz - x^2 p - x^2$$

- $$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & x \\ -x & x^2 \end{vmatrix}$$

$$= 0 + x^2$$

$$= x^2$$

- $$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & -y \\ 0 & 1 \end{vmatrix}$$

$$= -q - 0$$

$$= -q$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -y \\ -x & 1 \end{vmatrix}$$

$$= -xy$$

Now

$$\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$$

$$= xz - x^2p - x^2 + px^2 - q - q \cdot xy$$

$$= xz - qxy - q - x^2$$

$$= xz - q - qxy - x^2$$

$$= xz - q - x(qy - x)$$

$$= xz - q - x(xp) \quad \text{by eq. (1)}$$

$$= xz - q - x^2p$$

$$= 0 \quad \text{by eq (2)}$$

Then given PDE'S are compatiable

Now solving eq (1) and (2) For p and q,

$$xp - yq = x \quad \text{——— (1)}$$

$$x^2p + q = xz \quad \text{——— (2)}$$

Multiply eq (2) by 'y' and adding in (1)

$$x^2py + qy = xyz$$
$$xp - yq = x$$

$$xp + x^2py = x + xyz$$
$$xp + x^2py = x(1+yz)$$
$$xp(1+xy) = x(1+yz)$$

$$\Rightarrow \boxed{p = \frac{(1+yz)}{(1+xy)}}$$ put in eq ①

From eq ①

$$xp - x = yq$$
$$x\left(\frac{1+yz}{1+xy}\right) - x = yq$$
$$\frac{x + xyz - x - x^2y}{1+xy} = yq$$
$$\frac{xyz - x^2y}{1+xy} = yq$$
$$\frac{xy(z-x)}{1+xy} = yq$$
$$\boxed{q = \frac{x(z-x)}{1+xy}}$$

In order to get the solution of the given system we have

$$dz = P\,dx + Q\,dy$$

$$\Rightarrow dz = \frac{(1+yz)}{(1+xy)}dx + \frac{x(z-x)}{(1+xy)}dy$$

$$\Rightarrow (1+xy)dz = (1+yz)dx + x(z-x)dy$$

$$(1+xy)dz = dx + yz\,dx + xz\,dy - x^2dy$$

$$= dx + z(y\,dx + x\,dy) - x^2dy$$

$$(1+xy)dz = dx + z\,d(xy) - x^2dy$$

$$(1+xy)dz - z\,d(xy) = dx - x^2dy$$

Dividing by $(1+xy)^2$ as both sides

$$\frac{(1+xy)dz - z\,d(xy)}{(1+xy)^2} = \frac{dx - x^2dy}{(1+xy)^2}$$

Quotient Rule

$$d\left(\frac{z}{(1+xy)}\right) = \frac{x^2\left(\frac{1}{x^2}dx - dy\right)}{(1+xy)^2}$$

$$d\left(\frac{z}{1+xy}\right) = \frac{-dy + d\left(\frac{1}{x}\right)}{\left(\frac{1}{x}+y\right)^2}$$

$$d\left(\frac{z}{1+xy}\right) = \frac{-d\left(y+\frac{1}{x}\right)}{\left(y+\frac{1}{x}\right)^2}$$

Let $t = y + \frac{1}{x} = \frac{xy+1}{x}$

③ becomes $d\left(\frac{z}{1+xy}\right) = \frac{-d(t)}{t^2} = -t^{-2}dt$

$$\int d\left(\frac{z}{1+xy}\right) = -\int t^{-2}\,dt$$

$$\frac{z}{1+xy} = -\frac{t^{-2+1}}{-2+1} + C$$

$$\frac{z}{1+xy} = -\frac{t^{-1}}{-1} + C$$

$$\frac{z}{1+xy} = t^{-1} + C$$

$$\frac{z}{1+xy} = \frac{1}{t} + C$$

$$\frac{z}{1+xy} = \frac{1}{\frac{xy+1}{x}} + C = \frac{x}{xy+1} + C$$

$$\frac{z}{(1+xy)} = \frac{x}{(1+xy)} + c$$

$$\boxed{z = x + c(1+xy)}$$

QNo11:-

Show that PDE'S

$xp = yq$ and $z(xp+yq) = 2xy$

are compatible and have Find its Solution.

Solution:-

Let $f = xp - yq = 0$ — ①

$g = z(xp+yq) - 2xy = 0$

$\Rightarrow g = xpz + yzq - 2xy = 0$ — ②

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p & x \\ 2p - 2y & 2x \end{vmatrix}$$

$$= 2xp - 2xp + 2xy$$

$$= 2xy$$

- $\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} -q & -y \\ qz-2x & yz \end{vmatrix} = -qyz + yqz - 2xy$

$= -2xy$

- $\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} 0 & x \\ xp+yq & 2z \end{vmatrix} = -x^2p - xyq$

- $\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} 0 & -y \\ xp+yq & zy \end{vmatrix} = +xyp + y^2q$

we have

$\Rightarrow \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$

$= 2xy - x^2p^2 - xypq - 2xy + xypq + y^2q^2$

$= -x^2p^2 + y^2q^2$

From eq. ① $\quad xp = yq$

$x^2p^2 = y^2q^2$

$y^2q^2 - x^2p^2 = 0$

$\Rightarrow -x^2p^2 + y^2q^2 = 0$

So it has compatible solution

Now For Solution

Put eq ① in eq ②, $xP = yq$ — ①

② ⇒ $z(xP + yq) = 2xy$

$$z(yq + yq) = 2xy$$

$$2yzq = 2xy$$

$$zq = x$$

$$q = \frac{x}{z}$$

Put in ①

$$xP = \frac{xy}{z}$$

$$P = \frac{y}{z}$$

Solution as Follows

$$dz = p\,dx + q\,dy$$

$$dz = \frac{y}{z}dx + \frac{x}{z}dy$$

$$z\,dz = y\,dx + x\,dy$$

$$z\,dz = d(xy)$$

Integrating

$$\frac{z^2}{2} = xy + C$$

$$\Rightarrow z^2 = 2xy + 2c$$

$$\Rightarrow \boxed{z^2 = 2xy + C_1} \quad \because 2c = C_1$$

$\Rightarrow$ which is required

QNo12:-

Show that the equations $p^2+q^2=1$ and $(p^2+q^2)x = pz$ are compatible. And have Find its Solution.

Solution:-

Let $f = p^2+q^2 = 1$

$$f = p^2+q^2-1 = 0 \quad \text{——} \quad (1)$$

$$g = (p^2+q^2)x = pz$$

$$g = (p^2+q^2)x - pz = 0 \quad \text{——} \quad (2)$$

- $$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} 0 & 2p \\ p^2+q^2 & 2px-z \end{vmatrix} = -2p^3 - 2pq^2$$

- $$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} 0 & 2p \\ -p & 2px-z \end{vmatrix} = 2p^2$$

- $$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} 0 & 2q \\ 0 & 2qx \end{vmatrix} = 0$$

- $\frac{\partial(f,g)}{\partial(z,y)} = \begin{vmatrix} 0 & 2q \\ -p & 2qx \end{vmatrix} = 2pq$

we have

$= \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$

$= -2p^2 - 2pq^2 + 2p^2 + 0 + 2pq^2$

$= 0$

Now

$p^2 + q^2 = 1$ —— ①

$(p^2+q^2)x = pz$ —— ②

Put eq ① in ②

$2 \Rightarrow 1(x) = pz$

$\boxed{p = \frac{x}{z}}$ put in ①

$\left(\frac{x}{z}\right)^2 + q^2 = 1$

$q^2 = 1 - \left(\frac{x}{z}\right)^2$

$q = \sqrt{1 - \frac{x^2}{z^2}}$

$\boxed{q = \frac{\sqrt{z^2 - x^2}}{z}}$

Solution is: $dz = p\,dx + q\,dy$

$$dz = \frac{x}{z}dx + \frac{\sqrt{z^2-x^2}}{z}dy$$

$$z\,dz = x\,dx + \sqrt{z^2-x^2}\,dy$$

$$z\,dz - x\,dx = \sqrt{z^2-x^2}\,dy$$

$$\Rightarrow dy = \frac{z\,dz - x\,dx}{\sqrt{z^2-x^2}}$$

(2) Divi and Mul

$$dy = \frac{1}{2\sqrt{z^2-x^2}}\left(2z\,dz - 2x\,dx\right)$$

$$\int dy = \int d\left(\sqrt{z^2-x^2}\right)$$

$$y + c = \sqrt{z^2-x^2}$$

$$(z^2-x^2)^{1/2} = (y+c)$$

$$(z^2-x^2) = (y+c)^2$$

$$z^2 = (y+c)^2 + x^2$$

$$\boxed{z^2 = x^2 + (y+c)^2}$$

This is the required Solution of given PDE.

⟹ CHARPIT'S METHOD:-

Example 16:- Pg → 38

Find the complete integral of $(p^2+q^2)y = qz$

Solution:-

Let $f = (p^2+q^2)y - qz = 0$ ——— ①

$$f_x = \frac{\partial f}{\partial x} = 0$$

$$f_y = \frac{\partial f}{\partial y} = p^2+q^2$$

$$f_z = \frac{\partial f}{\partial z} = -q$$ ——— ②

$$f_p = \frac{\partial f}{\partial p} = 2py$$

$$f_q = \frac{\partial f}{\partial q} = 2qy - z$$

Charpits auxiliary equations are

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-(f_x+pf_z)} = \frac{dq}{-(f_y+qf_z)}$$

$$\Rightarrow \frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{p(2py) + q(2qy - z)}$$

$$= \frac{dp}{-(0 + p(-q))} = \frac{dq}{-(p^2 + q^2 + q(-q))}$$

$$\Rightarrow \frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{2p^2y + 2q^2y - qz}$$

$$= \frac{dp}{pq} = \frac{dq}{-p^2} \quad \text{--- (3)}$$

From the Last two members of eq. (3)

$$\frac{dp}{pq} = \frac{dq}{-p^2}$$

$$\Rightarrow \frac{dp}{q} = \frac{dq}{-p}$$

$$-p\,dp = q\,dq$$

$$+p\,dp + q\,dq = 0$$

Integrating

$$\frac{p^2}{2} + \frac{q^2}{2} = c$$

$$\because 2c = a$$

$$(p^2 + q^2) = 2c$$

$$p^2 + q^2 = a \quad (\text{constant}) \quad \text{--- (4)}$$

Put in eq ①

$(p^2+q^2)y = qz$

$ay = qz$

$$\boxed{q = \frac{ay}{z}}$$ put in eq ④

$p^2+q^2 = a$

$p^2 = a - q^2 = a - \left(\frac{ay}{z}\right)^2$

$p^2 = \frac{az^2 - a^2y^2}{z^2}$

$$\boxed{p = \frac{\sqrt{az^2 - a^2y^2}}{z}}$$

Solution as Follows

$dz = p\,dx + q\,dy$

$dz = \frac{\sqrt{az^2 - a^2y^2}}{z}dx + \frac{ay}{z}dy$

$z\,dz = \sqrt{az^2 - a^2y^2}\,dx + ay\,dy$

$z\,dz - ay\,dy = \sqrt{az^2 - a^2y^2}\,dx$

$$\frac{1}{a} \cdot a\,(z\,dz - ay\,dy) = \sqrt{az^2 - a^2y^2}\,dx$$

$$\frac{1}{a}\left(\frac{az\,dz - a^2y\,dy}{\sqrt{az^2 - a^2y^2}}\right) = dx$$

Equation can be written as

$$\frac{d\,(az^2 - a^2y^2)^{1/2}}{a} = dx$$

Integrating

$$\frac{(az^2 - a^2y^2)^{1/2}}{a} = x + b$$

$$\frac{a\left(\frac{z^2}{a} - y^2\right)^{1/2}}{a} = x + b$$

$$\Rightarrow (x+b)^2 = \left(\sqrt{\left(\frac{z^2}{a}\right) - y^2}\right)^2$$

$$\Rightarrow (x+b)^2 = \frac{z^2}{a} - y^2$$

$$\boxed{(x+b)^2 + y^2 = \frac{z^2}{a}}$$

which is the required complete integral

Example:- Pg → 39

Find the complete integral of PDE

$$z^2 = pqxy \quad \text{———} \quad (A)$$

Solution:-

Given $f = z^2 - pqxy$ ——— (1)

So we have

$$f_x = \frac{\partial f}{\partial x} = -pqy \;,\; f_y = \frac{\partial f}{\partial y} = -pqx$$

$$f_z = \frac{\partial f}{\partial z} = 2z \;,\; f_p = \frac{\partial f}{\partial p} = -xyq$$

$$f_q = \frac{\partial f}{\partial q} = -pxy$$

Charpits auxiliary Equations

$$\Rightarrow \frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + q$$

$$\Rightarrow \frac{dx}{-xyq} = \frac{dy}{-pxy} = \frac{dz}{p(-xyq) + q(-xyp)} =$$

$$\frac{dp}{-(-pqy + 2pz} = \frac{dq}{-(-pqx + 2qz)}$$

$$\Rightarrow \frac{dx}{-xyq} = \frac{dy}{-xyp} = \frac{dz}{-2pqxy}$$

$$= \frac{dp}{pqy - 2pz} = \frac{dq}{pqx - 2qz}$$

$$\rightarrow p(qy - 2z) \qquad q(px - 2z)$$

$$\Rightarrow \frac{dx/x}{-qy} = \frac{dy/y}{-px} = \frac{dz}{-2pqxy} = \frac{dp/p}{qy - 2z}$$

$$= \frac{dq/q}{px - 2z} \qquad \because pqxy = z^2$$

$$\Rightarrow \frac{dx/x}{-qy} = \frac{dy/y}{-px} = \frac{dz}{-2z^2} = \frac{dp/p}{qy - 2z}$$

$$\frac{dq/q}{px - 2z} \qquad \text{——— } (2)$$

From eq (2)

$$\Rightarrow \frac{dp/p}{qy - 2z} = \frac{dq/q}{px - 2z} = \frac{dx/x}{-qy} = \frac{dy/y}{-px}$$

$$\Rightarrow \frac{dp/p - dq/q}{qy - 2z - px + 2z} = \frac{-dx/x + dy/y}{qy - px}$$

$$\Rightarrow \frac{dp/p - dq/q}{qy - px} = \frac{-dx/x + dy/y}{qy - px}$$

$$\Rightarrow \int \frac{dp}{p} - \int \frac{dq}{q} = \int -\frac{dx}{x} + \int \frac{dy}{y}$$

$$\ln p - \ln q = -\ln x + \ln y + \ln c$$

$$\ln\left(\frac{p}{q}\right) = \ln\left(\frac{y}{x}\right) + \ln c$$

$$\ln \frac{p}{q} = \ln \frac{y}{x} c$$

$$\frac{p}{q} = \frac{y}{x} c$$

$$\frac{p}{q} \cdot \frac{x}{y} = c$$

$$\boxed{p = \frac{cqy}{x}}$$

Put in (A)

$$z^2 = pqxy = \frac{(cqy)}{x}(qxy)$$

$$z^2 = cq^2y^2$$

$$q^2 = \frac{z^2}{cy^2} \Rightarrow q = \frac{z}{\sqrt{c}\,y}$$

$$\boxed{q = \frac{az}{y}}$$

where $1/\sqrt{c} = a$

$a^2 = \frac{1}{c}$

$\therefore c = \frac{1}{a^2}$

Put in p,

$$p = \frac{cy}{x} q = \frac{cy}{x} \frac{az}{y}$$

$$p = \frac{az}{x}\left(\frac{1}{a^2}\right)$$

$$\boxed{P = \frac{z}{ax}}$$

Now, $dz = Pdx + qdy$

Substituting P and q

$$dz = \frac{z}{ax}dx + \frac{az}{y}dy$$

$$dz = z\left(\frac{1}{ax}dx + \frac{a}{y}dy\right)$$

$$\int \frac{dz}{z} = \int \frac{1}{a}\frac{dx}{x} + \int a\frac{dy}{y}$$

$$\ln z = \frac{1}{a}\ln x + a\ln y + \ln b$$

$$\ln z = \ln x^{1/a} + \ln y^{a} + \ln b$$

$$\ln z = \ln(x^{1/a}\, y^{a} . b)$$

$$\boxed{z = b x^{1/a} y^{a}}$$

which is the complete integral of the given PDE.

Example. 18:- Pg → 41

Find the complete integral of $x^2p + y^2q^2 - 4 = 0$ using

Charpits method.

Solution:-

$$x^2p^2 + y^2q^2 - 4 = 0 \quad \text{(A)}$$

Let $f = x^2p^2 + y^2q^2 - 4 \quad (1)$

we have

$$f_x = \frac{\partial f}{\partial x} = 2xp^2, \quad f_y = \frac{\partial f}{\partial y} = 2yq^2$$

$$f_z = \frac{\partial f}{\partial z} = 0, \quad f_p = \frac{\partial f}{\partial p} = 2px^2$$

$$f_q = \frac{\partial f}{\partial q} = 2qy^2$$

Charpits auxilliary equations are

$$\Rightarrow \frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)}$$

$$= \frac{dq}{-(f_y + qf_z)}$$

$$\Rightarrow \frac{dx}{2px^2} = \frac{dy}{2qy^2} = \frac{dz}{p(2px^2) + q(2qy^2)}$$

$$= \frac{dp}{-(2xp^2 + (p)(0))} = \frac{dq}{-(2yq^2 - q(0))}$$

$$\Rightarrow \frac{dx}{2px^2} = \frac{dy}{2qy^2} = \frac{dz}{2p^2x^2 + 2q^2y^2}.$$

$$= \frac{dp}{-2xp^2} = \frac{dq}{-2yq^2}$$

$$\Rightarrow \frac{dx}{2px^2} = \frac{dy}{2qy^2} = \frac{dz}{2(p^2x^2+q^2y^2)} = \frac{dz}{0}$$

$$= \frac{dp}{-2xp^2} = \frac{dq}{-2yq^2} \quad \text{——— (2)}$$

considering the terms.

$$\frac{dx}{2x^2p} = \frac{dp}{-2xp^2}$$

$$\frac{dx}{x} = \frac{-dp}{p}$$

$$\int \frac{dx}{x} + \int \frac{dp}{p} = 0$$

$$\ln x + \ln p = \ln a$$

$$\ln(xp) = \ln a$$

$$xp = a \quad \text{——— (2)}$$

$\Rightarrow x^2p^2 = a^2$ put in eq (A)

$$a^2 + y^2q^2 = 4$$

$$y^2q^2 = 4 - a^2$$

$$q^2 = \frac{4-a^2}{y^2}$$

$$\boxed{q = \sqrt{\frac{4-a^2}{y^2}}}$$

From eq (2),

$$P = \frac{a}{K}$$

Now

$$dz = P\,dx + q\,dy$$

Substituting values of P and q,

$$dz = \frac{a}{x}\,dx + \frac{\sqrt{4-a^2}}{y}\,dy$$

$$\int dz = \int a\,\frac{dx}{x} + \int (\sqrt{4-a^2})\,\frac{dy}{y}$$

$$\boxed{z = a\ln x + \sqrt{4-a^2}\cdot\ln y + b}$$

Which is the required complete integral of the given PDE.

Q.13:- Find the complete integral of the equation

$$(p^2+q^2)x = pz \quad \text{where}$$

$$p = \frac{\partial z}{\partial x} \quad \text{and} \quad q = \frac{\partial z}{\partial y}$$

Solution:- Given

$$(p^2+q^2)x = pz \quad \text{——— (A)}$$

Let $F = (p^2+q^2)x - pz = 0$ ——— (1)

Now

$$F_x = \frac{\partial f}{\partial x} = p^2+q^2, \quad f_y = \frac{\partial f}{\partial y} = 0$$

$$F_z = -p, \quad F_p = 2px - z,$$

$$F_q = 2qx$$

Charpit's auxiliary equations

$$\Rightarrow \frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-(f_x+pf_z)}$$

$$= \frac{dq}{-(f_y+qf_z)}$$

$$\Rightarrow \frac{dx}{2px-z} = \frac{dy}{2qx} = \frac{dp}{-(p^2+q^2+p(-p))}$$

$$= \frac{dq}{-(0+q(-p))} = \frac{dz}{p(2px-z)+q(2qx)}$$

$-(p^2+q^2-p^2)$
$= -q^2$

$$\Rightarrow \frac{dx}{2px-z} = \frac{dy}{2qx} = \frac{dz}{pz} = \frac{dp}{-q^2}$$

$$= \frac{dq}{pq} \quad \text{——— (2)}$$

Considering last two terms

$$\frac{dp}{-q^2} = \frac{dq}{pq}$$

$$\Rightarrow \frac{dp}{-q} = \frac{dq}{p}$$

$$\Rightarrow p\,dp = -q\,dq$$

$$\Rightarrow \int p\,dp + \int q\,dq = 0$$

$$\frac{p^2}{2} + \frac{q^2}{2} = C$$

$$p^2 + q^2 = 2C \qquad \because 2C = a$$

$$\boxed{p^2 + q^2 = a}$$

Put in Ⓐ.

$$a(x) = pz$$

$$\Rightarrow \boxed{p = \frac{ax}{z}} \quad \text{put in ③}$$

$$\left(\frac{ax}{z}\right)^2 + q^2 = a$$

$$q^2 = a - \left(\frac{ax}{z}\right)^2$$

$$q^2 = a - \frac{a^2x^2}{z^2}$$

$$q^2 = \frac{az^2 - a^2x^2}{z^2}$$

$$\boxed{q = \frac{\sqrt{az^2 - a^2x^2}}{z}}$$

Now, Solution as Follows.

$$dz = p\,dx + q\,dy$$

Substituting p and q,

$$dz = \frac{ax}{z}\,dx + \frac{\sqrt{az^2 - a^2x^2}}{z}\,dy$$

$$z\,dz = ax\,dx + \left(\sqrt{az^2 - a^2x^2}\right)dy$$

$$z\,dz - ax\,dx = \sqrt{az^2 - a^2x^2}\,dy$$

$$\frac{1}{a}\cdot a(z\,dz - ax\,dx) = \sqrt{az^2 - a^2x^2}\,dy$$

$$\frac{(az\,dz - a^2x\,dx)}{a} = \sqrt{az^2 - a^2x^2}\,dy$$

$$\Rightarrow dy = \frac{(az\,dz - a^2x\,dx)}{a\sqrt{az^2 - a^2x^2}\,dy}$$

$$\Rightarrow \int dy = \int \frac{1}{a}\,d(az^2 - a^2x^2)$$

$$y = \frac{1}{a}(az^2 - a^2x^2)^{1/2} + c$$

$$y + b = \frac{1}{a}(z^2 - x^2)^{1/2}$$

Squaring o.b.s

$$(y+b)^2 = \frac{1}{a^2}(az^2 - a^2x^2)$$

$$(y+b)^2 = \frac{1}{a^2}a^2(z^2/a - x^2)$$

$$(y+b)^2 = \frac{z^2}{a} - x^2$$

$$\boxed{\frac{z^2}{a} = (y+b)^2 + x^2}$$

which is the required

→ Special Types of First order Equations:-

Type - I

Equations involving p and q only. i.e Equation of the Type

$$f(p,q)=0 \quad \text{—— (1)}$$

Let $z = ax+by+c=0$ —— (A) is a Solution of the given PDE, described by $F(p,q)=0$ then

$$p = \frac{\partial z}{\partial x} = a \quad \text{and}$$

$$q = \frac{\partial z}{\partial y} = b$$

Substituting these values of p and q in (1)

$$\Rightarrow \quad f(a,b)=0 \quad \text{—— (2)}$$

Solving For b, we get

$$b = \phi(a) \quad , \text{Say}$$

Then

$$z = ax + \phi a y + c$$

is the complete integral of the Equation.

Example: 19:- Pg → 42

Find complete integral of the Equation $\sqrt{p} + \sqrt{q} = 1$

Solution:-

Given PDE is of the Form $F(p, q) = 0$

Let us Suppose Solution is of the Form $z = ax + by + c$ — ①

where

$$\sqrt{a} + \sqrt{b} = 1$$

$$\sqrt{b} = 1 - \sqrt{a}$$ ∵ $p = a$, $q = b$

$$b = (1 - \sqrt{a})^2$$ Squaring o.b.s

Thus, ① becomes

~~$z = ax + \sqrt{1-(\sqrt{a})^2}$~~

$$z = ax + (1 + \sqrt{a})^2 y + c$$

⇒ Which is the required complete integral

Example 20_

Find the complete integral of the PDE $pq = 1$

Solution:_

Given PDE is of the form

$f(p, q) = 0$

Let $z = ax + by + c$ be the solution of the given PDE.

$\because p = a, q = b$

$pq = ab = 1$

$\Rightarrow ab = 1 \Rightarrow$ $\boxed{b = 1/a}$

Hence, the complete integral is

$$\boxed{z = ax + \frac{1}{a}y + c}$$

Q15:_ Find the complete integral of the PDE $p + q = pq$

Solution:_

Given PDE is of the form

$f(p, q) = 0$

Let us suppose $z = ax + by + c$ — (1) be the solution of given PDE.

$$p+q = pq$$

$$a+b = ab \qquad \because p=a,\ q=b$$

$$a = ab - b$$

$$a = b(a-1)$$

$$\boxed{b = \frac{a}{a-1}}$$

Hence,
The complete integral is

$$\boxed{z = ax + \left(\frac{a}{a-1}\right)y + c}$$

Type II

Equation not involving the Independent variable (x, y) i.e equation is of the Type $F(z, p, q) = 0$ —— ①

Let use assume that z is a Function of $u = x + ay$, where a is any arbitrary constant.

$$Z = f(u) = f(x+ay) \quad \text{--- (2)}$$

$$P = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \quad \because \frac{dz}{du} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}$$

$$P = \frac{dz}{du}(1) = \frac{dz}{du} \Rightarrow \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x}$$

and $$q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$$

$$q = \frac{dz}{du}(a) = a\frac{dz}{du}$$

Substituting the values of P and q in (1) we get

$$f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0 \quad \text{--- (3)}$$

which is an ordinary diff. equation of First order.

Solving eq (3) we obtain

$$\frac{dz}{du} = \phi(z, a) \text{ say}$$

$$\Rightarrow \frac{dz}{\phi(z,a)} = du$$

Integrating $\quad \because U = x+ay$

$$f(z,a) = U + C$$

$$f(z,a) = x+ay+C$$

which is the complete integral of the given PDE.

Example:- F Pg → 43

Find the complete Integral of $P(1+q) = qz$

Solution:-

Given $P(1+q) = qz$ —— ①

Let us assume the Solution is in the Form

$$Z = f(u) = x + ay$$

$$P = \frac{dz}{du}, \quad q = a\frac{dz}{du}$$

Substituting these values in the given PDE

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du}(z)$$

$$1 + a\frac{dz}{du} = az$$

$$a\frac{dz}{du} = az - 1$$

$$a\frac{dz}{az-1} = du$$

Integrating $\int \frac{a\,dz}{az-1} = \int du$

$$\ln(az-1) = u+c$$

$$\Rightarrow \boxed{\ln(az-1) = x+ay+c}$$

which is the required complete integral

Example:- Pg → 44

Find the complete integral of the PDE.

$$p^2z^2 + q^2 = 1$$

Solution:- Let us assume that the $z = F(u) = x + ay$ is solution of the given PDE, Then

$$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}.$$

Substituting the values in the given PDE.

$$\left(\frac{dz}{du}\right)^2 z^2 + \left(a\frac{dz}{du}\right)^2 = 1$$

$$\left(\frac{dz}{du}\right)^2 (z^2 + a^2) = 1$$

$$\left(\frac{dz}{du}\right)^2 = \frac{1}{z^2+a^2}$$

$$\frac{dz}{du} = \frac{1}{\sqrt{z^2+a^2}}$$ (Taking square root a.b.s)

$$\int \sqrt{z^2+a^2}\, dz = \int du$$

$$\frac{z}{2}\sqrt{z^2+a^2} + \frac{a^2}{2}\ln\left[\frac{z+\sqrt{z^2+a^2}}{a}\right] = u+c$$

$$\because \int \sqrt{a^2+x^2}\, dx = \frac{x}{a}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|$$

Hence the required complete integral is

$$\frac{z\sqrt{z^2+a^2}}{2} + \frac{a^2}{2}\ln\left[\frac{z+\sqrt{z^2+a^2}}{a}\right] = x+ay+c$$

$\because u = x+ay$

Q16 (i) Find the complete integral of the PDE.

$$zpq = p+q$$

Solution:- $zpq = p+q$ ——— (1)

Let us assume that $z = f(u)$, $u = x + ay$ is a solution of the given PDE.

$$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}.$$

Substituting these values in (1)

$$z\left(\frac{dz}{du}\right)\left(a\frac{dz}{du}\right) = \frac{dz}{du} + a\frac{dz}{du}$$

$$z\left(\frac{dz}{du}\right)\left(a\frac{dz}{du}\right) = \frac{dz}{du}(1+a)$$

$$za\frac{dz}{du} = 1 + a$$

$$\int\left(\frac{a}{1+a}\right)z\,dz = \int du$$

$$\left(\frac{a}{1+a}\right)\frac{z^2}{2} = u + c_1$$

$$az^2 = 2(1+a)(u+c_1)$$

$$az^2 = 2(1+a)(x+ay+c_1)$$

$$\boxed{z^2 = 2(1+a)\left(\frac{x}{a} + y + c\right)}$$

which is the required complete integral of the given PDE.

Example:0.6:- Pg → 10

Find the Partial differential Equation of the Family of Planes the Sum of whose x, y, z intercepts is equal to unity.

Solution:-

Let $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ be The eq of plane in intercept Form So That

$a+b+c=1$, Thus

$1-a-b=c$

So, $\frac{x}{a}+\frac{y}{b}+\frac{z}{1-a-b}=1$ ——— (i)

Diff Partially w.r.t x

$$\frac{1}{a}+0+\frac{\partial z/\partial x}{1-a-b}=0$$

$$\frac{\partial z}{\partial x}=-\frac{(1-a-b)}{a}$$

$$\Rightarrow \boxed{p=\frac{a+b-1}{a}}$$

Diff. Partially w.r.t y,

$$0+\frac{1}{b}+\frac{\partial z/\partial y}{1-a-b}=0$$

$$\frac{\partial z}{\partial y} = \frac{-(1-a-b)}{b}$$

$$\boxed{q = \frac{a+b-1}{b}}$$

So,

$$\frac{p}{q} = \frac{a+b-1/a}{a+b-1/b} = \frac{b}{a}$$

$$\frac{p}{q} = \frac{b}{a}$$

$$\Rightarrow \boxed{p = \frac{b}{a}q} \quad \Rightarrow \boxed{q = \frac{a}{b}p}$$

AS

$$p = \frac{a+b-1}{a}, \quad q = \frac{a+b-1}{b}$$

$$p = 1 + \frac{b}{a} - \frac{1}{a}, \quad q = \frac{a}{b} + 1 - \frac{1}{b}$$

$$\frac{1}{a} = 1 + \frac{b}{a} - p, \quad \frac{1}{b} = \frac{a}{b} + 1 - q$$

$$\because \frac{b}{a} = \frac{p}{q}$$

$$\frac{1}{a} = 1 + \frac{p}{q} - p, \quad \frac{1}{b} = \frac{q}{p} + 1 - q$$

$$\frac{1}{a} = \frac{q+p-pq}{q}, \quad \frac{1}{b} = \frac{q+p-pq}{p}$$

$$a = \frac{q}{p+q-pq}, \quad b = \frac{p}{p+q-pq}$$

Put in eq (i)

$$\frac{x(p+q-pq)}{q}+\frac{y(p+q-pq)}{p}+\frac{z(p+q-pq)}{-pq}=1$$

$$\because 1-a-b = 1-\frac{q}{p+q-pq}-\frac{p}{p+q-pq}$$

$$=\frac{p+q-pq-q-p}{p+q-pq}$$

$$=\frac{-pq}{p+q-pq}$$

$$\Rightarrow \left(\frac{x}{q}+\frac{y}{p}-\frac{z}{pq}\right)(p+q-pq)=1$$

$$\boxed{(px+qy-z)=\frac{pq}{p+q-pq}}$$

This is required PDE.