Monday

Lecture #1 19-09-16

Numerical Analysis

Num. Sol. of ODEs
Num. Sol of PDEs

Norm:-

$$\|f(x)\|_1 = \int_0^\infty |f(x)|\,dx$$

$$\|f(x)\|_2 = \sqrt{\int_0^\infty |f(x)|^2\,dx}$$

$$\|f(x)\|_\infty = \max_{(\ )} |f(x)|$$

$$\int_a^b f(x)\,dx = \sum_{i=1}^{N} a_i f(x_i)$$

$$\frac{du}{dt} = f(t)$$

(u dependent, t independent)

$$\frac{du_1}{dt} = f_1(t, u_1, u_2)$$

$$\frac{du_2}{dt} = f_2(t, u_1, u_2)$$

$$\frac{d^2y}{dx^2} = f(x, y)$$

$y(a) = \alpha,\ y(b) = \beta,\ a \neq b$ BVP

otherwise IVP

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

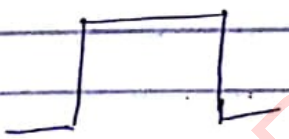
$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}$$

Tylor Expension not used for polynomial interpolation.

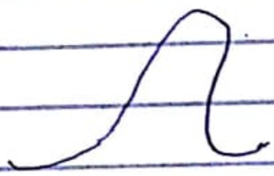
Mathematical Modeling and Simulation:-

(1) Define the physical problem.
(2) Creat a mathematical model.
(i) system of PDEs, ODEs and algebraic Equations.
(ii) Define initial and boundary conditions to get a well posed problem.
(3) Creat a discrete (numerical) model
(i) discretize the domain ⇒ generate the grid.
(ii) Solve the discrete system.
(4) Analyze errors in discrete system.
(i) consistency, stability and convergence.

Not well posed



well posed



PDEs:-

Mathematical models of continuous physical phenomena in which the dependant variable, say u is, a function of more than one independent variables, say t (time) and x (e.g spatial position)

Well Posed Problem:-

(1) Solution exists
(2) Solution unique.

(3) solution depends continuously on data.
i.e small changes in the data produces small changes in the solution (i.e the solution in continuously differentiable).

Tuesday

Lecture #02 20-09-16

Our Aim:-

To develop efficient algorithms for computing numerical solutions of PDEs, especially practical problems.
Beyond learning basic techniques, it is crucial to understand that how algorithms are constructed, why they work and what are their limitations.

Source of Errors in Mathematical model:-

(1) Model Error:-

Inexatness of the mathematical model for underline physical problem

(2) Data Error:-

Error in measurement of parameters entering the model.

(3) Round off error:-

Errors in computer arithematics

(4) Truncation Error:-

Errors in numerical method used to solve the full mathematical system.

Errors Estimates:-

Mathematical information about the accuracy of calculated numerical solution.

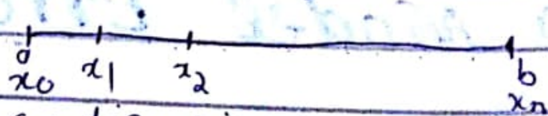
u_n, u_e

error $= \|u_e - u_n\|$

Numerical Solutions of PDEs:-

n and h are discretization parameters

main $\quad u_n \quad n \to \infty \Big\}$ numerical
$\quad\quad u_h \quad h \to \infty \Big\}$ approximation



$n =$ number of discretization points.
$h =$ step size

Classical Problems of numerical Analysis:-

(1) Existance and uniqueness:- The solution U_h exists and unique.
(2) **Stability:-** The solution U_h remains bounded for $h \to 0$ (In clear sense).
(3) **Consistancy:-** A small residium (remaining part of tylor expansion) is obtained after substituting the exact solution u of the differential equation in the discretization formulation (the numerical method formulation). Moreover this residium tends to zero $h \to 0$
(4) **Convergence:-** For $h \to 0$, the discrete solution U_h tends (converges) to continuous solution x.

PDEs:-

Linear General PDE:-

$$au_x + bu_y + cu + d = 0 \quad (\text{1st order})$$

a, b, c, d are either constant or function of independent variables x and y

$$a = a(U, U_x, u_y)$$

Hyperbolic PDE.

Linear PDE of order 2:-

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu + g = 0$$

Classification:-

(1) Hyperbolic PDEs:-

$$b^2 - 4ac > 0$$

$$u_{tt} = cu_{xx} \quad (\text{wave Equation})$$

(NTS)

$$u_t + a u_x = 0$$

$$\frac{du}{dt} = \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{dx}{dt}$$

$$= 0$$

(2) Parabolic PDE:-

$$b^2 - 4ac = 0$$

$$u_t = a u_{xx} \text{ (Heat Equation)}$$

(3) elliptic PDE:-

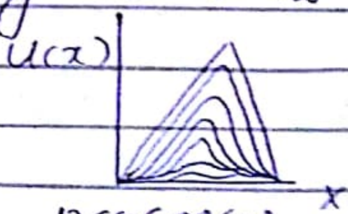
$$b^2 - 4ac < 0$$

$$\nabla u = 0 \quad \text{Laplace Equation}$$

$$u_{xx} + u_{yy} = 0$$

(1) Hyperbolic PDEs describe time-dependent conservative physical process, such as convection, that are not evolving towards steady state.

(2) Parabolic PDEs describe time-dependent dissipative physical processes, such as diffusion, that are evolving towards a steady state.



(3) Elliptic PDEs describe processes that have already reached steady state and hence are time-independent

(1) $u_t + a u_x = 0$
(2) $u_t = a u_{xx}$ $\Big\}$ IVP & BVP
(3) $u_{xx} + u_{yy} = 0 \rightarrow$ BVP

Mixed Equations:-

$$\frac{\partial^2 u}{\partial x^2} = x \frac{\partial^2 u}{\partial y^2}$$

* This Equation is elliptic when

$\chi < 0$

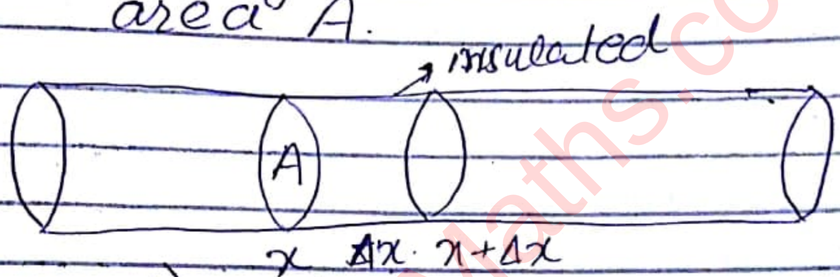
* parabolic if $\chi = 0$ * hyperbolic if $\chi > 0$.

Monday

Lecture #03 26-09-16

Heat conduction in a rod:-

consider a rod of constant length L and cross section area A.



Let $e(x,t)$ = thermal energy density
= amount of energy per unit volume.

Rod surface is insulated ⇒ No energy loss across the surface.

Then the total energy in the slice = $e(t,x)A\Delta x$.

The rate of change of heat energy is given by $\frac{\partial}{\partial t}(e(x,t))A\Delta x$. because of conservation law of heat energy

Rate of change of total energy per unit time = Heat generated by per unit time + Heat energy flowing across the boundries per unit time

Let $\phi \rightarrow$ Heat flux. (amount of thermal energy per unit time flowing to the right per unit surface

$S(x,t)$ denote heat energy generated per unit volume/ per unit time

$$\frac{\partial}{\partial t}(e(t,x)A\Delta x) = -[\phi(x+\Delta x,t) - \phi(x,t)]A + S(x,t)A\Delta x$$

$$\Rightarrow \frac{\partial}{\partial t}(e(x,t)) = -\frac{\phi(x+\Delta x,t) - \phi(x,t)}{\Delta x} + S(x,t)$$

$$\lim_{x\to 0}\frac{\partial}{\partial t}(e(x,t)) = -\lim_{x\to 0}\frac{\phi(x+\Delta x,t) - \phi(x,t)}{\Delta x}$$

$$+\lim_{x\to 0} S(x,t).$$

$$\frac{\partial}{\partial t}(e(x,t)) = -\frac{\partial \phi(x,t)}{\partial t} + S(x,t).$$

Moreover

$$e(x,t) = C(x,t)\underset{\text{density}}{\rho(x)}\,\underset{\text{temp}}{u(x,t)}$$

$c(x,t) \rightarrow$ specific heat (Heat energy to be supplied to a unit mass to raise its temprature by one degree ($J/k\cdot kg$)

Fourier law:

$$\phi(x,t) \propto -\frac{\partial u}{\partial x}$$

$$= -k_T\frac{\partial u}{\partial x}.$$

$$\partial/\partial t\left[c(x)\rho(x)u(x,t)\right] = \partial/\partial x\left(k_T\frac{\partial u}{\partial x}\right) + S(x,t)$$

In a special case c, ρ and k_T are constants.

$$\partial/\partial t\,(u(x,t)) = a\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + Q.$$

parabolic Equation

$$a = \frac{k_T}{\rho c}$$

$$Q = \frac{S(x,t)}{\rho c}$$

initial condition $u(x,0) = f(x)$

Boundary conditions

(i) Prescribed temprature (Dirichlet BCs)

$$u(0,t) = p(t),\ u(L,t) = q(t)$$

(ii) Insulate boundary conditions

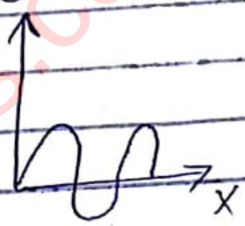
$$\frac{\partial u}{\partial n}(0,t) = 0$$

$$\partial/\partial n = -\partial/\partial x \text{ at } x = 0$$

$$\partial/\partial n = \partial/\partial x \text{ at } x = L$$

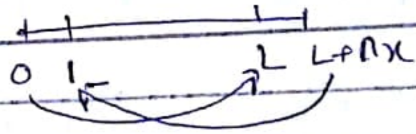
(iii) periodic boundary condition

$$u(0,t) = u(L,t)$$



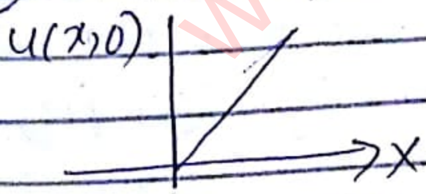
Problem:-

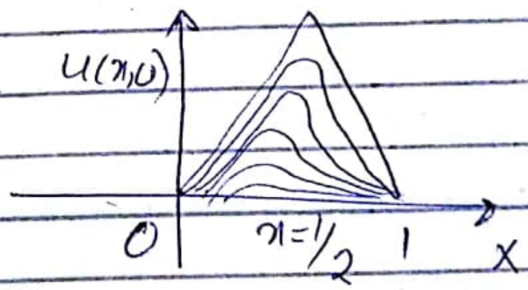
$$u(x,0) = \begin{cases} 2x, & 0 < x < \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x < 1 \end{cases}$$



$$u(0,t) = 0 = u(L,t)$$

if $u(0,t),\ u(L,t) = 1$





Finite Difference Method for parabolic Equation:-

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2},\quad 0 \le x \le 1$$

$$u(x,0) = u_0(x),\quad u(0,t) = 0,\quad u(1,t) = 0$$

$\Delta x = h \rightarrow$ space step
$\Delta t = k$ - timestep

$$u(x \pm h, t) = u(x,t) \pm h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} \pm \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots$$

$$u(x+h, t) = u(x,t) + h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots$$

$$\frac{u(x+h,t) - u(x,t)}{h} = \frac{\partial u}{\partial x} + \frac{h}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{h^2}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots$$

$$= \frac{\partial u}{\partial x} + O(h)$$

(forward difference formula) (Ist order accurate)

$$\frac{u(x,t) - u(x-h,t)}{h} = \frac{\partial u}{\partial x} + O(h)$$

(backward diff formula) (Ist order accurate)

$$u(x+h,t) + u(x-h,t) = 2u(x,t) + h^2\frac{\partial^2 u}{\partial x^2} + O(h^4)$$

$$\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} = \frac{\partial^2 u}{\partial x^2} + O(h^2)$$

(Central difference Method) (2nd order accurate)

$$u(x+h,t) - u(x-h,t) = 2h\frac{\partial u}{\partial x} + O(h^3)$$

$$\frac{u(x+h,t) - u(x-h,t)}{2h} = \frac{\partial u}{\partial x} + O(h^2)$$

central difference formula.
Ist order accurate

Example:- $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}$ $0 \le x \le 1$

$u(0,t) = 0, \quad u(1,t) = 0$

$$\frac{u(t+k,x) - u(t,x)}{k} = (a)\frac{u(t,x-h) - 2u(t,x) + u(t,x+h)}{h^2}$$

$$\frac{U_j^{n+1} - U_j^n}{k} = (a)\frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}$$

$$U_j^{n+1} = U_j^n + \frac{ak}{h^2}\left[U_{j-1}^n - 2U_j^n + U_{j+1}^n\right]$$

let $\lambda = k/h^2$

$$= U_j^n + a\lambda\left[U_{j-1}^n - 2U_j^n + U_{j+1}^n\right]$$

$$= (1-2a\lambda)U_j^n + a\lambda\left[U_{j-1}^n + U_{j+1}^n\right]$$

N = number of discretization points

$h = \frac{b-a}{N}$

$x_j = jh$

$j = 0, 1, 2, \dots N$

$t_n = nk$

$n = 0, 1, \dots M$

$t \to t_n$

$t+k \to t_{n+1}$

$x \to x_j$

$x+h \to x_{j+1}$

$x-h \to x_{j-1}$

Explicit Scheme:- Each value at time level t_{n+1} can be independently calculated from values at time level t_n.

Tuesday (12)

Lecture #04 27-09-16

Finite Difference Method for parabolic Equations:-

(Heat Equation) $\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2}$

$u(0,t) = 0 = u(1,t) \rightarrow BCs$ $\quad 0 < x < 1$

$u(x,0) = u_0(x) \rightarrow I.C$

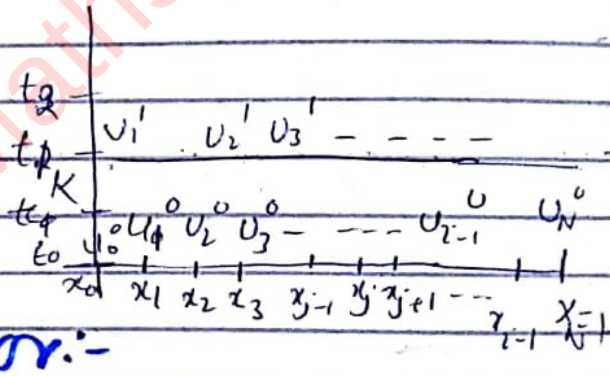
$$u_j^{n+1} = u_j^n + a\lambda \left[u_{j-1}^n - 2u_j^n + u_{j+1}^n \right]$$

$$= (1 - 2a\lambda) u_j^n + a\lambda \left[u_{j-1}^n + u_{j+1}^n \right]$$

(Discrete solution) expeicit Method

$O(k, h^2)$

First order in time and 2nd order in space.



Truncation Error:-

$L(U, x, t, k, h)$ = Truncation error produced by difference method

$$\frac{\partial u}{\partial t} - a \frac{\partial^2 u}{\partial x^2} = 0$$

$$T.E = \frac{u(x,t+k) - u(x,t)}{k} - a \left[u(x-h,t) - 2u(x-h,t) + u(x+h,t) \right.$$

$$u(x,t+k) = u(x,t) + k u_t + \frac{k^2}{2!} u_{tt} + \frac{k^3}{3!} u_{ttt} + \frac{k^4}{4!} u_{tttt} + O(k^5)$$

$$u(x \pm h, t) = u(x,t) \pm h u_x + \frac{h^2}{2!} u_{xx} + \frac{h^3}{3!} u_{xxx} + \frac{h^4}{4!} u_{xxxx} + O(h^5)$$

$$\frac{u(x,t+k)-u(t,k)}{k} = u_t + \frac{k}{2}u_{tt} + \frac{k^2}{3!}u_{ttt} + \frac{k^3}{4!}u_{tttt} + O(k^4)$$

$$u(x-h,t)+u(x+h,t) = 2u + h^2 u_{xx} + \frac{2h^4}{4!}u_{xxxx} + O(h^5)$$

$$\frac{u(x-h,t)-2u(x,t)+u(x+h,t)}{h^2} = u_{xx} + \frac{2}{4!}h^2 u_{xxxx} + O(h^3)$$

$$T.E = u_t + k/_2\, u_{tt} + k^2/_6\, u_{ttt} + \frac{k^3}{4!}u_{tttt} + O(k^4)$$

$$-a\left[u_{xx} + \frac{h^2}{12}u_{xxxx} + O(h^4)\right)$$

$$= \underbrace{(u_t - a u_{xx})}_{=0} + k/_2\, u_{tt} + O(k^3) - a\frac{h^2}{12}u_{xxxx} + O(h^3)$$

$$= O(k,h^2)$$

$$= k/_2\, u_{tt} - ah^2/_{12}\, u_{xxxx} + O(k^2, h^4) = O(k,h^2)$$

$$u_t = a u_{xx} \Rightarrow u_{tt} = a u_{xxt} = a(u_t)_{xx}$$

$$= a^2 u_{xxxx}$$

$$L(T.E) = k/_2 \cdot a^2 u_{xxxx} - a\frac{h^2}{12}u_{xxxx} + O(k^2, h^4)$$

$$= ah^2/_2\left(\frac{ak}{h^2} - 1/_6\right)u_{xxxx} + O(k^2, h^4)$$

$$= ah^2/_2\,(a\lambda - 1/_6)\,u_{xxxx} + O(k^2, h^4)$$

$$\text{if } \lambda = \frac{1}{6a} \Rightarrow T.E \sim O(k^2, h^4)$$

$$U_j^{n+1} = (1-2a\lambda)U_j^n + a\lambda\left[U_{j-1}^n + U_{j+1}^n\right]$$

$$1-2a\lambda > 0$$

$$2a\lambda < 1$$

$$\lambda < 1/2a$$

$$k/h^2 < 1/2a.$$

$$k < h^2/2a \rightarrow \text{stability condition (Courant-Friedrich-Levey condition)}$$

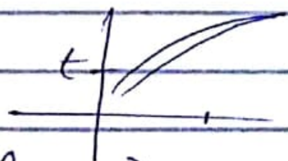
$$T.E \sim O(k^2, h^4)$$

Convergence:-

The scheme is convergent if as $k \to 0, h \to 0$

$\forall (x^*, t^*)$, when $x_j \to x^*$, $t_n \to t^*$

numerical solution $\leftarrow U_j^n \longrightarrow u(x^*, t^*) \rightarrow$ exact analytical.



① let $e_j^n = u(x_j, t_n) - U_j^n$

= (Analytical sol. − numerical so.)

$$L_j^n = \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{k} - a\,\frac{u(x_{j-1}, t_n) - 2u(x_j, t_n) + u(x_{j+1}, t_n)}{h^2}.$$

② $\rightarrow u(x_j, t_{n+1}) = (1-2a\lambda)u(x_j, t_n) + a\lambda\left[u(x_{j-1}, t_n) + u(x_{j+1}, t_n)\right] + kL_j^n$

③ $U_j^{n+1} = (1-2a\lambda)U_j^n + a\lambda\left[U_{j-1}^n + U_{j+1}^n\right] + kL_j^n$

② − ③ by using ①

$$|e_j^{n+1}| = \left|(1-2a\lambda)e_j^n + a\lambda\left[e_{j-1}^n + e_{j+1}^n\right]\right|$$

$$\leq |1-2a\lambda||e_j^n| + |a\lambda|\left(|e_{j-1}^n| + |e_{j+1}^n|\right) + k|L_j^n|$$

$$\max_{1\le j\le N} |e_j^{n+1}| \le |1-2a\lambda| \max_{1\le j\le N} |e_j^n| + |a\lambda| \left(\max_{1\le j\le N} |e_{j-1}^n| + \max_{1\le j\le N} |e_{j+1}^n| \right) + k \max_{1\le j\le N} |L_j^n|$$

$$E^n = \max_j |e_j^n| \quad \forall j$$

$$E^{n+1} \le |1-2a\lambda| E^n + a\lambda (E^n + E^n) + k\tilde{L}$$

$$2a\lambda < 1 \Rightarrow \lambda < \frac{1}{2a} \Rightarrow k < \frac{h^2}{2a}$$

(stability condition)

then

$$E^{n+1} \le (1-\cancel{2a\lambda})E^n + \cancel{2a\lambda E^n} + k\tilde{L}$$

$$= E^n + k\tilde{L}$$

$$|x| = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$$

$$E^{n+1} \le E^n + k\tilde{L}$$

$$E^n \le E^{n-1} + k\tilde{L}$$

$$\le E^{n-2} + 2k\tilde{L}$$

$$\le E^0 + nk\tilde{L}$$

if $E^0 = 0$ then $E^n \le nk\tilde{L} = t_{max}\tilde{L}$

if $k \to 0 \Rightarrow E^n \to 0$

$h \to 0 \quad \tilde{L} \to 0$

Monday (16)

Lecture # 05 03-10-16

Stability Analysis:-

Fourier series:-

$$v_t = v_{xx} \,, \; x \in \mathbb{R}, t > 0 \rightarrow (1)$$

$$v(x,0) = f(x), \quad v(0,t) = 0 = v(\infty, t)$$

$$\hat{v}_t(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} v_t(x,t)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} v_{xx}(x,t)\,dx$$

Integrating by parts and terms at $x = 0$ & ∞ are zero.

$$= \frac{-\omega^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} v(x,t)\,dx$$

$$= -\omega^2 \hat{v}(\omega,t) \quad \text{(PDE tranform into ODE)}.$$

$$\hat{v}_t + \omega^2 \hat{v}(\omega,t) = 0 \rightarrow \text{(ODE)}$$

$$v(x,t) = \int_{-\infty}^{\infty} e^{i\omega x} \hat{v}(\omega,t)\,d\omega \leftarrow \text{inverse fourier transform}$$

Porsival's identity $\|\hat{v}\|_2 = \|v\|_2$

$\| \, \|_2 \Rightarrow L_2(\mathbb{R})$

Continuous case $\rightarrow$ $\hat{u}(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} u(x,t)\,dx$

Discrete case $\rightarrow$ $\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi} u_m, \; \xi \in [-\pi, \pi]$

Definition:-

The two level scheme $u^{n+1} = Q u^n, (n \geq 0)$ is said to be stable with respect to a norm ($\|\ \|$) if there exist positive constant K and h, and non-negative constant β s.t

$$\|u^n\| \le K e^{\beta t} \|u_0\|$$

also $\|\hat{u}^n\| \le K e^{\beta t} \|\hat{u}_0\|$

Thus, both Fourier transform and parsival's Identity are the two basic tools for stability Analysis.

Explicit Finite Difference Method for heat Equation:-

$$u_t = a u_{xx} \quad u(x,0) = u_0(x),\ u(0,t) = 0 = u(L,t)$$

Explicit FDM $\leftarrow$ $u_j^{n+1} = u_j^n + a\lambda\left[u_{j-1}^n - 2u_j^n + u_{j+1}^n\right]$

$$a\lambda < 1/2 \Rightarrow k < \frac{h^2}{2a} \quad \text{CFL-condition}$$

$$\hat{u}^{n+1}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-ij\xi} u_j^{n+1} \qquad \xi \in [-\pi, \pi]$$

$$\frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} u_j^{n+1} e^{-ij\xi} = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{-ij\xi} \left\{ u_j^n + a\lambda\left[u_{j-1}^n - 2u_j^n + u_{j+1}^n\right]\right\}$$

$$\hat{u}^{n+1}(\xi) = (1 - 2a\lambda)\hat{u}^n(\xi) + \frac{a\lambda}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-ij\xi}\left[u_{j-1}^n + u_{j+1}^n\right]$$

Substitue $m = j \mp 1 \Rightarrow j = m \pm 1$

$$\hat{u}^{n+1}(\xi) = (1 - 2a\lambda)\hat{u}^n + a\lambda \frac{1}{\sqrt{2\pi}}\left[\sum_{m=-\infty}^{\infty} e^{-i(m+1)\xi} u_m^n \right.$$

$$\left. + \sum_{m=-\infty}^{\infty} u_m^n e^{-i(m-1)\xi}\right]$$

$$= (1-2a\lambda)\hat{u}^n(\xi) + a\lambda\left[e^{-i\xi}\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty} + e^{i\xi}\frac{1}{\sqrt{2\pi}} u_m^n e^{-im\xi}\right]$$

$$\hat{u}^{n+1}(\xi) = (1-2a\lambda)\hat{u}^n(\xi) + a\lambda(e^{-i\xi}+e^{i\xi})\hat{u}^n(\xi)$$

$$\boxed{\begin{aligned}\cos\xi &= 1-2\sin^2\xi/2\\ 2\sin^2\xi/2 &= 1-\cos\xi\end{aligned}}$$

$$= (1-2a\lambda)\hat{u}^n(\xi) + 2a\lambda\cos\xi\,\hat{u}^n(\xi)$$

$$= 1-2a\lambda(1-\cos\xi)\,\hat{u}^n(\xi)$$

$$\hat{u}^{n+1}(\xi) = \left[1-4a\lambda\sin^2\xi/2\right]\hat{u}^n(\xi)$$

$$\hat{u}^{n+1}(\xi) = \left[1-4a\lambda\sin^2\xi/2\right]\hat{u}^n(\xi).$$

$$\hat{u}^{n}(\xi) = \left[1-4a\lambda\sin^2\xi/2\right]\hat{u}^{n-1}(\xi).$$

$$= \underbrace{\left[1-4a\lambda\sin^2\xi/2\right]}_{\text{Damping factor } D}{}^n\,\hat{u}^0(\xi)$$

Stability $|D| < 1$

This means

~~L.H.S~~ $\left|1-4a\lambda\sin^2\xi/2\right| < 1$ Stability condition

$$-1 < 1-4a\lambda\sin^2\xi/2 < 1$$

L.H.S $\Rightarrow$ $4a\lambda\sin^2\xi/2 < 2$

$$\therefore a\lambda < \frac{1}{2\sin^2\xi/2}, \quad \xi(-\pi,\pi)$$

for $\xi = \pm\pi$ $\quad a\lambda < 1/2$

R.H.S

$$4a\lambda \sin^2 \xi/2 > 0 \qquad \text{for } \xi = \pm\pi$$

$$a\lambda > 0$$

Thus $0 < a\lambda < \frac{1}{2}$

$$\|U^n\| \leq \|U^0\|$$

Exercise:-

$$u_t = a u_{xx}$$

$$u(x,0) = u_0(x), \quad u(0,t) = 0 = u(1,t)$$

implicit scheme

$$\frac{U_j^{n+1} - U_j^n}{k} = a\left[\frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{h^2}\right]$$

$$U_j^{n+1} - U_j^n = a\lambda\left[U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}\right].$$

$$-a\lambda U_{j-1}^{n+1} + (1+2a\lambda)U_j^{n+1} - a\lambda U_{j+1}^{n+1} = U_j^n, \quad j = 1, \dots, N-1$$

$$\begin{bmatrix} 1+2a\lambda & -a\lambda & 0 & \cdots & 0 \\ -a\lambda & 1+2a\lambda & -a\lambda & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ & & & -a\lambda & 1+2a\lambda \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ \vdots \\ U_{N-1}^{n+1} \end{bmatrix} = \begin{bmatrix} U_1^n + a\lambda U_0^n \\ U_2^n \\ \vdots \\ \vdots \\ U_{N-1}^n + a\lambda U_N^n \end{bmatrix}$$

put $j = 1$

$$-a\lambda U_0^{n+1} + (1+2a\lambda)U_1^{n+1} - a\lambda U_2^{n+1} = U_1^n$$

$j = 2$,

$$-a\lambda U_1^{n+1} + (1+2a\lambda)U_2^{n+1} - a\lambda U_3^{n+1} = U_2^n$$

$$\vdots$$

$j = N-1$

$$-a\lambda U_{N-2}^{n+1} + (1+2a\lambda)U_{N-1}^{n+1} - a\lambda U_N^{n+1} = U_{N-1}^n$$

(20)

$$e^{-i\xi} = \cos\xi \mp i\sin\xi$$

Matrix becomes

$$AU^{n+1} = B^n$$

$$\underline{U}^{n+1} = \underline{A}^{-1}\underline{B}^n$$

$$-a\lambda U_{j-1}^{n+1} + (1+2a\lambda)U_j^{n+1} - a\lambda U_{j+1}^{n+1} = U_j^n$$

fourier transform

$$-a\lambda e^{-i\xi}\hat{u}^{n+1}(\xi) + (1+2a\lambda)\hat{u}^{n+1}(\xi) - a\lambda e^{i\xi}\hat{u}^{n+1}(\xi) = \hat{u}^n(\xi)$$

$$-2a\lambda\cos\xi\,\hat{u}^{n+1}(\xi) + (1+2a\lambda)\hat{u}^{n+1}(\xi) = \hat{u}^n(\xi)$$

$$[2a\lambda(1-\cos\xi)+1]\hat{u}^{n+1}(\xi) = \hat{u}^n(\xi)$$

$$[1+4a\lambda\sin^2\xi/2]\hat{u}^{n+1}(\xi) = \hat{u}^n(\xi)$$

$$\hat{u}^{n+1}(\xi) = \left|\frac{1}{1+4a\lambda\sin^2\xi/2}\right|\hat{u}^n(\xi)$$

$$\|\hat{u}^{n+1}(\xi)\|_2 = \left|\frac{1}{1+4a\lambda\sin^2\xi/2}\right|\|\hat{u}^n(\xi)\|$$

$$|D| = \frac{1}{|1+4a\lambda\sin^2\xi/2|} \le 1 \quad \forall\xi \in [-\pi,\pi]$$

$\Rightarrow$ unconditionally stable

Tuesday (21)

Lecture #06 04-10-16

Example:-

$u_t = u_{xx}$

$$u(x,0) = f(x) = \begin{cases} 2x & 0 \le x \le 1/2 \\ 2-2x & 1/2 \le x \le 1 \end{cases}$$

$u(0,t) = 0 = u(1,t)$

Exact sol:- (By separation of variables)

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

$$C_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$= \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right),\ n = 1, 2, \dots, \infty$$

Remaining

$$U_j^{n+1} = U_j^n + r\left[U_{j-1}^n - 2U_j^n + U_{j+1}^n\right]$$

$r = .48$ $\qquad r = \frac{k}{h^2},\ r < \frac{1}{2}$

$N = 20$

$h = \frac{1-0}{20} = 0.05$

$k = rh^2 = 0.48 \times (0.05)^2$

$= 0.0012$

boundary

$x_0 = 0,\ x_1 = 0.05,\ x_2 = 0.1,\ x_3 = 0.15 \dots x_{20} = 1.0$ boundary

$t=0$: $0 \quad 0.05 \quad .1 \quad .15 \quad \dots \quad 1$

$U_0^0 \quad U_1^0 \quad U_2^0 \quad \dots \quad U_{20}^0$

$$u(x,0) = \begin{cases} 2x & 0 \le x \le 0.5 \\ 2(1-x) & 0.5 \le x \le 1 \end{cases}$$

$U_0^n = 0$

$U_{20}^n = 0$

$n = 0, 1, \dots$

$u_0^0 = 0,\ U_1^0 = 0.1,\ U_2^0 = 0.2,\ U_3^0 = 0.3 \dots$

$\dots U_{10}^0 = 1,\ U_{11}^0 = 0.9, \dots U_{20}^0 =$

for $j=1$

$$u_1^1 = u_1^0 + v\left[u_0^0 - 2u_1^0 + u_2^0\right]$$

$$= 0.1 + .48\left[0 - 2(0.1) + 0.2\right]$$

$$= 0.1$$

for $j=2$

$$u_2^1 = u_2^0 + v(u_1^0 - 2u_2^0 + u_3^0) = 0.2$$

for $j=3$

$$u_3^1 = u_3^0 + v(u_2^0 - 2u_3^0 + u_4^0)$$

$$= 0.3 + .48(.2 - 2(.3) + u_4^0)$$

$$= 0.3$$

$$\|u^e - u^N\|_2 = \sqrt{\sum_{i=1}^{N-1} |u_i^e - u_i^N|^2}$$

MATLAB.

$$u_t = a u_{xx} \qquad 0 \le x \le 1$$

$$u(x,0) = \sin(\pi x)$$

$$u(0,t) = 0 = u(1,t)$$

$a = 1$

$$h = 0.02$$

$$v = r < 1/2$$

$$k = r \cdot h^2/a$$

$$x = (0:h:) \text{ ; space variable.}$$

$$u_j^{n+1} = u_j^n + r\left[u_{j-1}^n - 2u_j^n + u_{j+1}^n\right]$$

$$= (1 - 2r)u_j^n + r\left[u_{j-1}^n + u_{j+1}^n\right]$$

$$= \gamma_2 U_j^n + r\left[U_{j-1}^n + U_{j+1}^n\right]$$

Exact solution

$$= \sin(\pi x)\, e^{-a\pi^2 t}$$

MONDAY.

Lecture # 07 17-10-16

$u_t = a u_{xx}$

(1) Explicit FDM Method:- conditionally stable (i.e CFL condition needed) time step is usually small ($O(h^2)$)

(2) Implicit FDM Method:- unconditionally stable (no time-step restriction) $T.E \sim (k, h^2)$

θ-Method:-

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{a}{h^2}\left[(1-\theta)\left(U_{j-1}^n - 2U_j^n + U_{j+1}^n\right) + \theta\left(U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}\right)\right] \quad \theta \in [0,1]$$

$\theta = 0$, explicit method

$\theta = 1$, implicit method

$\theta = \frac{1}{2}$, Cranck-Nicholson method.

Cranck-Nicalson Method:-

$$U_j^{n+1} - U_j^n = \frac{a\lambda}{2}\left[U_{j-1}^n - 2U_j^n + U_{j+1}^n + U_{j-1}^{n+1} - 2U_j^{n+1} + u_{j+1}^{n+1}\right]$$

$$\lambda = \frac{k}{h^2}$$

$$-\frac{\gamma}{2} U_{j-1}^{n+1} + (1+\gamma) U_j^{n+1} - \frac{\gamma}{2} U_{j+1}^{n+1} = \frac{\gamma}{2} U_{j-1}^{n} + (1-\gamma) U_j^{n} + \frac{\gamma}{2} U_{j+1}^{n}$$

$j = 1, \dots, N$

$$\begin{bmatrix} 1+\gamma & -\gamma/2 & & 0 \\ -\gamma/2 & 1+\gamma & -\gamma/2 & \\ & & \ddots & \\ 0 & & -\gamma/2 & 1+\gamma \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ \vdots \\ U_N^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1-\gamma & \gamma/2 & 0 \\ \gamma/2 & 1-\gamma & \gamma/2 \quad 0 \\ 0 & \ddots & \gamma/2 \quad 1-\gamma \end{bmatrix} \begin{bmatrix} U_1^{n} \\ \vdots \\ U_j^{n} \end{bmatrix}$$

$$\underline{\underline{A}}\, \underline{\underline{U}}^{n+1} = \underline{\underline{B}}\, \underline{\underline{U}}^{n}$$

$$\Rightarrow \underline{U}^{n+1} = \underline{\underline{A}}^{-1} \underline{\underline{B}}\, \underline{\underline{U}}^{n}.$$

$\gamma \leftarrow$

Truncation Error:-

$$u_t = a u_{xx}$$

Cranck Nicolson Method:-

$$U_j^{n+1} - U_j^{n} = \frac{\gamma}{2}\left[U_{j-1}^{n} - 2U_j^{n} + U_{j+1}^{n} + U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1} \right]$$

$$T.E = u(t+k, x) - u(t, x)$$

$$-\frac{\gamma}{2}\Big[u(t, x-h) - 2(t, x) + u(t, x+h)$$

$$+ u(t+k, x-h) \cancel{-2(t+k, x-h)}$$

$$-2u(t+k, x) + u(t+k, x+h)\Big]$$

$\frac{a\theta}{h}$

$$u(t\pm k, x) = u \pm k u_t + \frac{k^2}{2!}u_{tt} \pm \frac{k^3}{3!}u_{ttt}$$

$$+ \frac{k^4}{4!}u_{tttt} + O(k^5)$$

$$u(t, x\pm h) = u \pm h u_x + \frac{h^2}{2!}u_{xx} \pm \frac{h^3}{3!}u_{xxx}$$

$$+ \frac{h^4}{4!}u_{xxxx} + O(h^5)$$

$$u(t+k, x) - u(t-k, x) = 2k u_t + \frac{2k^3}{3!}u_{ttt} + 0 + O(k^5)$$

$$\frac{u(t+k, x) - u(t-k, x)}{2k} = u_t + \frac{k^2}{3!} + O(k^4)$$

$$u(t+k, x) + u(t-k, x) = 2u + k^2 u_{tt} + \frac{2k^4}{4!}u_{tttt} + O(k^5)$$

$$u(t+k, x) + u(t-k, x) - 2u$$

$$= k^2 u_{tt} + \frac{2k^4}{4!}u_{tttt} + O(k^6)$$

$$u(t+k, x+h) - 2u(t+k, x) + u(t+k, x-h)$$
$$= h^2 u_{xx}(t+k, x) + \frac{2h^4}{4!} u_{xxxx}(t+k, x) + O(h^6)$$

$$u(t+k, x+h) - 2u(t+k, x) + u(t+k, x+h)$$
$$= h^2 u_{xx}(t+k, x) + \frac{2h^4}{4!} u_{xxxx}(t+k, x) + O(h^6)$$
$$= h^2\left[u_{xx}(t, x) + k\, u_{xxt}(t, x) + O(k^2)\right]$$
$$+ \frac{2h^4}{4!}\left[u_{xxxx}(t, x) + k\, u_{xxxxt}(t, x) + O(k^2)\right]$$
$$= h^2 u_{xx}(t, x) + O(h^4)$$

$$T.E = u_t + O(k^2) - a u_{xx} + O(h^2)$$
$$= \underbrace{(u_t - a u_{xx})}_{0} + O(k^2, h^2)$$
$$= O(k^2, h^2).$$

$$u_j^{n+1} - u_j^n = \frac{r}{2}\left[u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1} + u_{j-1}^n - 2u_j^n + u_{j+1}^n\right]$$

von - Neuman Stability Analysis -

$$\hat{U}^{n+1}(\xi) - \hat{U}^{n}(\xi) = \frac{r}{2}\Big[e^{-i\xi}\hat{U}^{n+1}(\xi) - 2\hat{U}^{n+1}(\xi) + e^{i\xi}\hat{U}^{n+1}(\xi) + e^{-i\xi}\hat{U}^{n} - 2\hat{U}^{n} + e^{i\xi}\hat{U}(\xi)\Big]$$

$$= \frac{r}{2}\Big[(2\cos\xi - 2)\hat{U}^{n+1} + (2\cos\xi - 2)\hat{U}^{n}(\xi)\Big]$$

$$= \frac{r}{2}\Big[-4\sin^2\tfrac{\xi}{2}\,\hat{U}^{n+1} - 4\sin^2\tfrac{\xi}{2}\,\hat{U}^{n}(\xi)\Big]$$

$$\Rightarrow \left(1 + 2r\sin^2\tfrac{\xi}{2}\right)\hat{U}^{n+1}$$

$$= \left(1 - 2r\sin^2\tfrac{\xi}{2}\right)\hat{U}^{n}(\xi)$$

$$\hat{U}^{n+1}(\xi) = \left(\frac{1 - 2r\sin^2\xi/2}{1 + 2r\sin^2\xi/2}\right)$$

$$\hat{U}^{n}(\xi) = \left(\frac{1 - 2r\sin^2\xi/2}{1 + 2r\sin^2(\xi/2)}\right)^{n}\hat{U}^{0}(\xi)$$

$$|D| = \left|\frac{1 - 2r\sin^2(\xi/2)}{1 + 2r\sin^2\xi/2}\right| \leq 1$$

unconditionally stable

By θ-method (28)

$$u_j^{n+1} - u_j^n = r\left[\theta\left(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}\right) + (1-\theta)\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right)\right].$$

$$U^{n+1}(\xi) = \frac{1 - 4r(1-\theta)\sin^2 \xi/2}{1 + 4r\theta \sin^2 \xi/2}$$

if $1/2 \leq \theta \leq 1$ Implicit method

$(\theta \leq [1/2, 1])$

$0 \leq \theta < 1/2$ explicit method $(\theta \in [0, 1/2))$

for $0 \leq \theta \leq 1/2$

$$|D| \leq 1$$

$$-1 < \frac{1 - 4r(1-2\theta)\sin^2 \xi/2}{1 + 4r\theta \sin^2 \xi/2} < 1$$

R.H.S $1 - 4r\sin^2 \xi/2 < 1$

$-4r\sin^2 \xi/2 < 0$ always hold.

L.H.S

$0 < 2 - 4r(1-2\theta)\sin^2 \xi/2$

if $0 \leq \theta < 1/2$

$\frac{1}{2}(1-2\theta) > 0$

$$r \leq \frac{1}{2(1-2\theta)\sin^2(\xi/2)}$$

for $\xi = \pm\pi$

$$r \le \frac{1}{2(1-2\theta)}$$

(1) Richardson Method:-

$$u_t = a u_{xx}$$

$$\frac{u_j^{n+1} - u_j^{n-1}}{2k} = \frac{a}{h^2}\left[u_{j-1}^n - 2u_j^n + u_{j+1}^n\right]$$

(2) Dufort - Method:-

$$\frac{u_j^{n+1} - u_j^{n-1}}{2k} = \frac{a}{h^2}\left[u_{j-1}^n - u_j^{n-1} - u_j^{n+1} + u_{j+1}^n\right]$$

(1) Truncation - Error
(2) von-Neuman Stability (Assignment)

Tuesday

Lecture # 08 — 18-10-16.

DuFort--Frankel Method for heat Equation:-

$$u_t = a u_{xx}.$$

$$T.E = \frac{u_j^{n+1} - u_j^{n-1}}{2k} - \frac{a}{h^2}\left[u_{j-1}^n - u_j^{n-1} - u_j^{n+1} + u_{j+1}^n\right] \neq 0$$

$$= a^2 k^2/h^2\, u_{tt} + O(k^2, h^2)$$

$$= a^2 \lambda u_{tt} + O(k^2, h^2)$$

The method is only consistent if $k \to 0$ faster than h^2.
otherwise the method is inconsistant because the method solves the following PDE

$$u_t - a u_{xx} + a\eta^2 u_{tt} = 0 \quad \eta = k/h \to 0$$

if $k \to 0$ faster than h

Stability Analysis:-

$$\frac{v_j^{n+1} - v_j^{n-1}}{2k} = \frac{a}{h^2}\left[v_{j-1}^n - v_j^{n+1} - v_j^{n-1} + v_{j+1}^n \right] \quad f=0$$

$$v_j^{n+1} - v_j^{n-1} = 2r\left[v_{j-1}^n - v_j^{n-1} - v_j^{n+1} + v_{j+1}^n \right]$$

$$v_j^{n+1} = \frac{1}{1+2r}\left[(1-2r) v_j^{n-1} + 2r\left(v_{j+1}^n + v_{j-1}^n \right) \right].$$

$$\hat{U}^{n+1}(\xi) = \frac{1}{1+2r}\left[(1-2r)\hat{U}^{n-1}(\xi) + 4r\cos\xi\, \hat{U}^n(\xi) \right]$$

$$\hat{U}^n(\xi) = \hat{U}^n(\xi)$$

$$\begin{pmatrix} \hat{U}^{n+1}(\xi) \\ \hat{U}^n(\xi) \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{4r\cos\xi}{1+2r} & \frac{1-2r}{1+2r} \\ 1 & 0 \end{pmatrix}}_{M} \begin{pmatrix} \hat{U}^n(\xi) \\ \hat{U}^{n-1}(\xi) \end{pmatrix}$$

$$|M - \lambda I| = 0$$

(31) $X^{k+1} = M X^{(k)}$

M iteration Matrix.

$$\lambda^2 - \frac{4r\cos\xi}{1+2r}\lambda - \frac{1-2r}{1+2r} = 0.$$

$$\lambda_\pm = \frac{\frac{4r\cos\xi}{1+2r} \pm \sqrt{\frac{16r^2\cos^2\xi}{(1+2r)^2} + \frac{4(1-2r)}{1+2r}}}{2}$$

$$= \frac{2r\cos\xi \pm \sqrt{1-4r^2\sin^2\xi}}{1+2r}$$

if $1-4r^2\sin^2\xi < 0$

$$\lambda_\pm = \frac{1}{1+2r}\left(2r\cos\xi \pm i\sqrt{4r^2\sin^2\xi - 1}\right)$$

$$|\lambda_\pm| = \sqrt{\frac{1}{1+2r}(2r+1)}$$

$$|\lambda_\pm| = \sqrt{\frac{2r+1}{1+2r}} = 1$$

if $1-4r^2\sin^2\xi < 0$

$$r^2 > \frac{1}{4\sin^2\xi}$$

$$r > \frac{1}{2} \Rightarrow 2r > 1$$

$$|\lambda_\pm|^2 = \frac{4r^2}{(1+2r)^2}\cos^2\xi + \frac{4r^2\sin^2\xi - 1}{(1+2r)^2}$$

$$\Rightarrow \frac{4r^2-1}{(1+2r)^2} = \frac{2r-1}{1+2r}$$

if $1-4r^2\sin^2\xi \geq 0 \Rightarrow \sqrt{1-4r^2\sin^2\xi} \leq 1$

$$|\lambda\pm| \leq \frac{1}{1+2r}\left(2r|\cos\xi| + \sqrt{1-4r^2\sin^2\xi}\right)$$

$$\leq \frac{1}{1+2r}\left(2r\cdot 1+1\right) < 1$$

This is unconditionally stable.

Monday

Lecture #09 24-10-16.

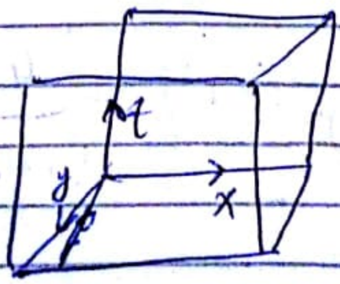
Two-Dimensional Heat Equation:-

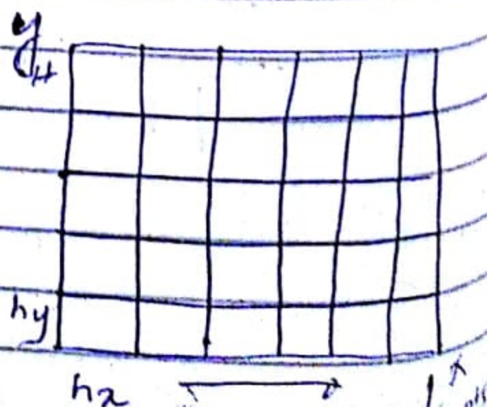
$$u_t = a(u_{xx}+u_{yy}), \quad 0\leq x\leq L, \; 0\leq y\leq H$$

B.C $u(x,y,t) = g(x,y,t)$ on boundary

I.C $u(x,y,0) = f(x,y), \; 0\leq x\leq L, \; 0\leq y\leq H$

$$h_x = \frac{L-0}{N}, \quad h_y = \frac{H-0}{M}$$





$$u_t = a(u_{xx} + u_{yy})$$

Explicit method

$$u(x_i, y_j, t_n) = U_{i,j}^{n}$$

time, x-coord., y-coord.

$$x_i = ih_x, \quad y_j = jh_y$$

$$t_n = nk \qquad i, j, n = 0, 1, 2, \ldots$$

$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{k} = a\left[\frac{U_{i-1,j}^{n} - 2U_{i,j}^{n} + U_{i+1,j}^{n}}{h_x^2} + \frac{U_{i,j-1}^{n} - 2U_{i,j}^{n} + U_{i,j+1}^{n}}{h_y^2}\right]$$

$$U_{i,j}^{n+1} - U_{i,j}^{n} = r_x\left(U_{i-1,j}^{n} - 2U_{i,j}^{n} + U_{i+1,j}^{n}\right) + r_y\left(U_{i,j-1}^{n} - 2U_{i,j}^{n} + U_{i,j+1}^{n}\right)$$

$$U_{i,j}^{n+1} = r_x U_{i-1,j}^{n} + (1 - 2r_x - 2r_y)U_{i,j}^{n} + r_x U_{i+1,j}^{n} + r_y U_{i,j-1}^{n} + r_y U_{i,j+1}^{n}$$

$i = 1, 2, \ldots N-1$

$j = 1, 2, \ldots M-1$ $\qquad T.E \sim (k, h_x^2, h_y^2)$

Stability Analysis:-

$\hat{u}(\xi, \eta)$, $\xi, \eta \in [-\pi, \pi]$

$$\hat{u}^{n+1}(\xi,\eta) = r_x e^{-i\xi}\hat{u}^{n}(\xi,\eta) + (1 - 2r_x - 2r_y)\hat{u}^{n}(\xi,\eta) + r_x e^{i\xi}\hat{u}^{n}(\xi,\eta) + r_y e^{-i\eta}\hat{u}(\xi,\eta) + r_y e^{i\eta}\hat{u}^{n}$$

$$2r_x\cos\xi + (1 - r_x - 2r_y + 2r_y\cos\eta)\hat{u}^n_{(\xi,\eta)}$$

$$= (-r_x 4\sin^2\xi/2 - r_y 4\sin^2\eta/2 + 1)\hat{u}^n_{(\xi,\eta)}$$

By induction

$$\hat{u}^n(\xi,\eta) = (1 - 4r_x\sin^2\xi/2 - 4r_y\sin^2\eta/2)^n \, \hat{u}^0(\xi,\eta)$$

$$|D| < 1$$

$$|1 - 4r_x\sin^2\xi/2 - 4r_y\sin^2\eta/2| < 1$$

$$-1 < 1 - 4r_x\sin^2\xi/2 + r_y\sin^2\eta/2 < 1$$

L.H.S

$$4r_x\sin^2\xi/2 + 4r_y\sin^2\eta/2 < 2.$$

$$\xi = \eta = \pm\pi$$

$$4r_x + 4r_y < 2.$$

$$r_x + r_y < 1/2.$$

$$ak/h^2 < 1/4$$

$$k < \frac{h^2}{4a}.$$

$$r_x = \frac{ak}{h_x^2}$$

$$r_y = \frac{ak}{h_y^2}.$$

$$r = ak/h^2.$$

$$h_x = h_y = h$$

$$r < 1/2,$$

$$r < \frac{1}{2^2}$$

$$r < \frac{1}{2d}.$$

Implicit Method:- Cranck-Nicolson method

$$u_t = a(u_{xx} + u_{yy})$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{k} = a\left[\frac{\delta x^2 u_{i,j}^{n} + \delta x^2 u_{i,j}^{n+1}}{2h_x^2} + \frac{\delta y^2 u_{i,j}^{n} + \delta y^2 u_{i,j}^{n+1}}{2h_y^2}\right]$$

$$\delta x^2 u_{i,j} = u_{i-1,j} - 2u_{i,j} + u_{i+1,j}$$

$$\delta y^2 u_{i,j} = u_{i,j-1} - 2u_{i,j} + u_{i,j+1}$$

This method is
(1) unconditionally stable
(2) 2nd order accurate in space & time
(3) Large bounded Matrix (Band is larger)
(4) If a is constant then the coefficient matrix is larger and we need to factor the matrix once

$$A U^{n+1} = B U^{n}$$

To overcome these problems we can use Alternating direction method (ADI)

ADI - Method:-

The idea is to alternate direction and thus solve one dimensional problem at each time step
The first step is to keep y-fixed.

$$\frac{U_{i,j}^{n+1/2} - U_{i,j}^{n}}{k/2} = a\left(\frac{\delta x^2 U_{i,j}^{n+1/2}}{h_x^2} + \frac{\delta y^2 U_{i,j}^{n}}{h_y^2}\right)$$

On the second step we keep x fixed

$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n+1/2}}{k/2} = a\left(\frac{\delta x^2 U_{i,j}^{n+1/2}}{h_x^2} + \frac{\delta y^2 U_{i,j}^{n+1}}{h_y^2}\right)$$

$$\Rightarrow U_{i,j}^{n+1/2} = U_{i,j}^{n} + \frac{r_x}{2}\left(U_{i-1,j}^{n+1/2} + 2U_{i,j}^{n+1/2}\right)$$

$$+ \frac{r_y}{2}\left(U_{i,j-1}^{n} - 2U_{i,j}^{n} + U_{i,j+1}^{n}\right)$$

$$-\frac{1}{2}r_x U_{i-1,j}^{n+1/2} + (1+r_x)U_{i,j}^{n+1/2} - \frac{1}{2}r_x U_{i+1,j}^{n+1/2}$$

$$= \frac{1}{2}r_y U_{i,j-1}^{n} + (1-r_y)U_{i,j}^{n}$$

$$+ \frac{1}{2}r_y U_{i,j+1}^{n}$$

$$\underline{A}\,\underline{U}^{n+1/2} = \underline{B}\,\underline{U}^{n}$$

So we have a tri-diagonal system ~~ad~~ every time step. We have to order the unknowns differentially at every timestep. The method is 2nd order in time and space and it is uncondionally stable, since the ~~denominator~~ denominater is always larger then numirater

High order Difference Formulas:-

$$\tau_{\pm h} u = u(x \pm h)$$
$$\Delta_+ u = u(x+h) - u(x)$$
$$\Delta_- u = u(x) - u(x-h)$$
$$\Delta_c u = u(x+h) - u(x-h)$$

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots$$

$$\ln(1+x) = x - x^2/2 + x^3/3 - \dots$$

$$e^x = 1 + x + x^2/2! + x^3/3! + \dots$$

$$\tau_h u(x) = u(x+h)$$
$$= u_x + h u_x + \frac{h^2}{2!} u_{xx} \dots$$
$$= \left(1 + h \,\partial/\partial x + \frac{h^2}{2} \frac{\partial^2}{\partial x^2}\right) u_x$$
$$= e^{h\partial x} u(x).$$

$$\boxed{\tau_h = e^{h\partial x}}$$

$$\tau_h u(x) = u(x+h) - u(x) + u(x)$$
$$= \Delta_h u_x + u(x)$$
$$= (\Delta_h + 1) u(x)$$

$$e^{h\partial x} = (1 + \Delta_h)$$

$$h\partial x = \ln(1 + \Delta_h).$$
$$= \Delta_h - \frac{\Delta_h^2}{2} + \frac{\Delta^3 h}{3} + \dots$$

$$\Delta_h^2 u = \Delta_h u(x+h) - u(x)$$
$$= u(x+2h) - u(x+h) - u(x+h) + u(x).$$

$$\partial_x = \frac{\Delta h}{h} - \frac{\Delta^2 h}{2h} + \frac{\Delta^3 h}{3h} - \dots$$

$$\frac{\partial y}{\partial x} = \frac{1}{h} \Delta_h u - \frac{1}{2h} \Delta_h^2 u + \frac{1}{3h} \Delta_h^3 u -$$

Tuesday (38)

Lecture #10 25-10-16.

Formal methods for higher order difference formula:-

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} \dots$$

$$\ln(1+x) = x - x^2/2 + x^3/3 - \frac{x^4}{4} - \dots - 1<x\leq 1$$

$$\tau_h u(x) = u(x+h), \quad \tau_h u(x) = u(x-h)$$

$$\tau_h u(x) = u(x+h) = u(x) + h u_x + \frac{h^2}{2!} u_{xx} + \dots$$

$$= \left[1 + h\,\partial x + h^2/2!\,\partial^2 x + \dots\right] u(x)$$

$$= e^{h\partial x} u(x) \quad \text{—} ②$$

$$\tau_h u(x) = u(x+h) - u(x) + u(x)$$

$$= \Delta_h u(x) + u(x)$$

$$= (1+\Delta_h)\, u(x) \quad \text{—} ②$$

comparing ① & ②

$$e^{h\partial x} = (1+\Delta_h) \Rightarrow \partial x = \frac{1}{h}\ln(1+\Delta_h)$$

$$= \frac{1}{h}\Delta_h - \frac{\Delta^2 h}{2h} + \frac{\Delta^3 h}{3h} - \frac{\Delta^4 h}{4h} + \dots$$

$$\frac{\partial}{\partial x} u(x) = \left[\frac{\Delta h}{h} - \frac{\Delta^2 h}{2h} + \frac{\Delta^3 h}{3h} - \frac{\Delta^4 h}{4h} + \dots\right] u_x$$

First order approximation of u_x.

$$u_x = \frac{\Delta_h}{h} u(x) = \frac{u(x+h) - u(x)}{h} + o(h)$$ forward diff

2nd order

$$u_x = \frac{\Delta_h}{h} u(x) - \frac{\Delta_h^2}{2h} u(x) + o(h^2)$$

$$= \frac{u(x+h) - u(x)}{h} - \frac{1}{2h}\left[u(x) - 2u(x+h) + u(x+2h)\right]$$

$$\Delta_h^2 u(x) = \Delta_h\left[u(x+h) - u(x)\right]$$

$$\left[u_x = \frac{-3u(x) + 4u(x+h) - u(x+2h)}{2h} + o(h^2)\right]$$

$$= u(x+2h) - u(x+h) - u(x+h) + u(x)$$

$$= u(x+2h) - 2u(x+h) + u(x)$$

Analogously for Δ_{-h} ? ⇒ High order backward difference formula.

$$\Delta_c u(x) = u(x+h) - u(x-h) \quad \text{(Do yourself)}$$

$$= \tau_h u(x) - \tau_{-h} u(x)$$

$$= (\tau_h - \tau_{-h}) u(x).$$

$$= \left(e^{h\partial_x} - e^{-h\partial_x}\right) u(x)$$

$$= 2\sinh(h\partial_x)\, u(x)$$

$$h\partial_x = \operatorname{arcsinh}\left(\frac{\Delta_c}{2}\right)$$

$$\operatorname{arcsinh} x = x - \frac{1}{2.3}x^3 + \frac{1.3}{2.4.5}x^5 + \ldots$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k)!\, x^{2k+1}}{2^{2k}(k!)^2 (2k+1)}$$

$$\partial x = \frac{\Delta c}{2h} - \frac{1}{6h}\left(\frac{\Delta c}{2}\right)^3 + \frac{3}{40}\left(\frac{\Delta c}{2}\right)^5 + \cdots$$

Finite difference methods for linear advection Equation:-

$$u_t + a u_x = 0 \qquad \textcircled{1} \quad u_t + a u_x = f(x)$$

$f(x)$: Source of sink

$$\frac{du}{dt} = \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial t} + \frac{\partial u}{\partial x}\cdot\frac{dx}{dt}$$

$$= \frac{\partial u}{\partial t} + \frac{dx}{dt}\cdot\frac{\partial u}{\partial x}$$

$$= u_t + a u_x = 0$$

$$u_t + a u_{xx} = f(x,t)\ \delta(x - x_0)$$

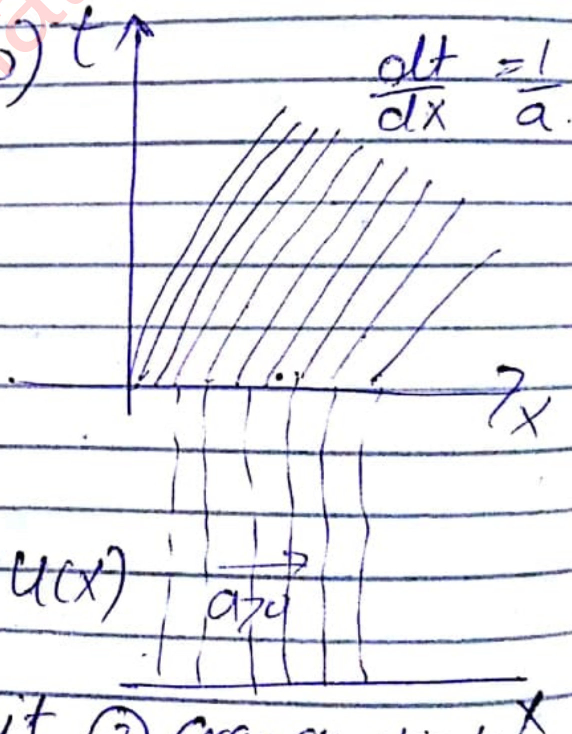
- transport eq.
- convection eq.
- Advection eq.

$$\frac{dt}{dx} = \frac{1}{a}$$

I.C $\quad u_t + a u_x = 0$

$u(a,t) = 0 = u(b,t) \quad x \in [a,b]$

$t \in (0,1)$



① Explicit ② implicit ③ Cranck Nicolson

Explicit:
- forward → unstable
- backward → stable
- central → unconditional unstable

MONDAY

Lecture # 11 31-10-16.

Finite difference methods for linear advection Equation:

$$u_t + a u_x = 0$$

$$u(x,0) = f(x)$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 , \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

FDMs:-

Explicit method:

$$\frac{U_j^{n+1} - U_j^n}{k} = -a \begin{cases} \dfrac{U_j^n - U_{j-1}^n}{h} & \text{Backward diff.} \\ \dfrac{U_{j+1}^n - U_j^n}{h} & \text{forward diff.} \end{cases}$$

$$U_j^{n+1} = U_j^n - a\lambda \begin{cases} U_j^n - U_{j-1}^n \\ U_{j+1}^n - U_j^n \end{cases}$$

$$U_j^{n+1} = \begin{cases} (1-a\lambda) U_j^n + a\lambda U_{j-1}^n \\ (1+a\lambda) U_j^n - a\lambda U_{j+1}^n \end{cases}$$

if $a > 0 \Rightarrow$ Backward diff. is stable and forward diff is unstable.

If $a < 0 \Rightarrow$ Backward diff. unstable forward is stable

$T.E \sim O(k, h)$ for both forward and backward diff formulas

Stability :-

$\cancel{T.E = u(x,t)}$

$$U_j^{n+1} = (1-a\lambda) U_j^n + a\lambda U_{j-1}^n$$

$$\hat{u}^{n+1}(\xi) = (1-a\lambda)\hat{u}^n(\xi) + a\lambda e^{-i\xi}\hat{u}^n(\xi)$$

$$= (1-a\lambda) + a\lambda\,[\cos\xi - i\sin\xi]\,\hat{u}^n(\xi)$$

$$= \{(1-a\lambda) + a\lambda\cos\xi - ia\lambda\sin\xi\}\,\hat{u}^0(\xi)$$

By induction

$$\hat{u}^n(\xi) = \{(1-a\lambda) + a\lambda\cos\xi\} - ia\lambda\sin\xi]^n\,\hat{u}^0(\xi)$$

$$|D|^2 = (1-a\lambda)^2 + a^2\lambda^2\cos^2\xi + 2a\lambda(1-a\lambda)\cos\xi + a^2\lambda^2\sin^2\xi$$

$$= (1-a\lambda)^2 + a^2\lambda^2 + 2a\lambda(1-a\lambda)\cos\xi$$

Let $a\lambda = R$.

$$|D(\xi)|^2 = (1-R)^2 + R^2 + 2R(1-R)\cos\xi$$

$$|D^2(\xi)|' = -2R(1-R)\sin\xi = 0$$

$$\Rightarrow \xi = -\pi, 0, \pi$$

for $\xi = 0$

$$|D|^2 = (1-R)^2 + R^2 + 2R(1-R)$$

$$= 1$$

for $\xi = \pm\pi$

$$|D(\xi)|^2 = (1-R)^2 + R^2 - 2R(1-R).$$

$$\therefore = (1-2R)^2$$

$$|D(\xi)| = |1-2R| < 1$$

$$-1 < 1-2R < 1$$

$R > 0$, $2R < 2$, or $R < 1$

$\Rightarrow 0 < R < 1$

$\Rightarrow ak/h < 1 \Rightarrow k \leq h/a$

$$u_t + a u_x = 0$$

Central difference (Explicit).

$$\frac{v_j^{n+1} - v_j^n}{k} = -\frac{a}{2h}\left[v_{j+1}^n - v_{j-1}^n\right] \sim O(k, h^2)$$

Stability:-

$$\hat{v}^{n+1}(\xi) - \hat{v}^n(\xi) = -\frac{a\lambda}{2}\left[e^{i\xi} - e^{-i\xi}\right]\hat{v}^n(\xi)$$

$$= -i a\lambda \sin\xi \, \hat{v}^n(\xi)$$

$$\hat{v}^n(\xi) = \left[1 - i a\lambda \sin\xi\right]^n \hat{v}^0(\xi)$$

$D = 1 - i a \sin\xi$

$|D|^2 = 1 + a^2\lambda^2 \sin^2\xi > 1 \quad \forall \xi \in [-\pi, \pi]$

unconditionally unstable.

Q:- Can we stablize this method?

Ans:- Yes, there are two ways

(1) $u(t, x)$ Modification s.t in truncation (Lax - Friedrich Method) error the coefficient of v_{xx} is less than or equal to one (≤ 1)

(2) one can also represent (formulate) the central difference method as a method with inherent diffusion (Lax, Wendroff method)

Method 1:-

$$\frac{u_j^{n+1}-u_j^n}{k} = -\frac{a}{2h}\left[u_{j+1}^n - u_{j-1}^n\right]$$

$$u_j^n = \frac{u_{j+1}^n + u_{j-1}^n}{2}$$

$$u_j^{n+1} = \frac{u_{j-1}^n + u_{j+1}^n}{2} - \frac{a\lambda}{2}\left[u_{j+1}^n - u_{j-1}^n\right]$$

Lax - Friedrich: method.

$T.E \sim O(k, h)$

and stable conditionally.

Method 2:-

$$\left[\delta_k + \delta_h^c - \frac{ka^2}{2!}\delta_h \delta_{-h}\right] u(x,t) = 0$$

$$\frac{u_j^{n+1}-u_j^n}{k} + \frac{a}{2h}\left[u_{j+1}^n - u_{j-1}^n\right] - \frac{ka^2}{2h^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right] = 0$$

$$u_j^{n+1} = u_j^n - \frac{a\lambda}{2}\left[u_{j+1}^n - u_{j-1}^n\right] + \frac{a^2\lambda^2}{2}\left[u_{j-1}^n - 2u_j^n + u_{j+1}^n\right]$$ Lax-wendroff method.

$T.E \sim O(k, h^2)$

stable (conditionally)!

(i) Implicit method:-

$a>0$

$$\frac{U_j^{n+1} - U_j^{n}}{\cancel{k}} = -a\lambda\left[U_j^{n+1} - U_{j-1}^{n+1}\right]$$

unconditionally stable

$$U_j^{n+1} - U_j^{n} = -a\lambda\left[U_{j+1}^{n+1} - U_j^{n+1}\right].$$

Implicit Backward diff:

$$U_j^{n+1} - U_j^{n} = -a\lambda\left[U_j^{n+1} - U_{j-1}^{n+1}\right]$$

$$|D|^2 = \frac{1}{1+4R(R+1)\sin^2\xi/2} \le 1 \;\forall \xi \qquad R = a\lambda = \frac{ak}{h} > 0$$

unconditionally stable.

Implicit forward diff.

$$U_j^{n+1} - U_j^{n} = -a\lambda\left[U_{j+1}^{n+1} - U_j^{n+1}\right]$$

$$|D|^2 = \frac{1}{1+4R(R-1)\sin^2\xi/2} < 1$$

for stability $R \geq 1$

The method is unstable when $0<R<1$

Implicit central difference.

$$\frac{U_j^{n+1} - U_j^{n}}{k} = \frac{-a}{2h}\left[U_{j+1}^{n+1} - U_{j-1}^{n+1}\right]$$

unconditionally stable.

General 3-Point method $(x-h, x, x+h)$

method of unknown coefficients

$$u(x,t+k) = c_1 u(x+h,t) + c_0 u(x,t) + c_{-1} u(x-h,t)$$

$$u + ku_t + \frac{k^2}{2!}u_{tt} + \frac{k^3}{3!}u_{ttt} + O(k^3)$$

$$= c_1\left\{u + hu_x + \frac{h^2}{2!}u_{xx} + O(h^3)\right\} + c_0 u$$

$$+ c_{-1}\left[u - hu_x + \frac{h^2}{2!}u_{xx} - O(h^3)\right]$$

$$= (c_0 + c_1 + c_{-1})u + h(c_1 - c_{-1})u_x + \frac{h^2}{2!}(c_1 + c_{-1})u_{xx} + O(h^3)$$

on comparing the coefficients, we obtain

$$u_t + au_x = 0$$
$$u_t = -au_x$$
$$u_{tt} = -a(u_t)_x = a^2 u_{xx}$$

$$c_{-1} + c_0 + c_1 = 1$$
$$c_1 - c_{-1} = a\lambda$$
$$c_1 + c_{-1} = a^2\lambda^2$$

order? (i) zeroth-order method

$c_{-1} + c_0 + c_1 = 1 \Rightarrow$ 2-free parameter

(ii) 1st order

u varies linearly

$$\left.\begin{aligned} c_{-1} + c_0 + c_1 &= 1 \\ c_1 - c_{-1} &= a\lambda \end{aligned}\right\} \text{one-free parameter}$$

$c_0 = 0 \Rightarrow$ Lax Friedrichs method

$c_1 = 0 \Rightarrow$ Backward difference method

$c_{-1} = 0 \Rightarrow$ forward difference method

2nd order accurate.

3 - equation for unknowns.

$$C_1 = -\frac{\lambda a}{2} + \frac{\lambda^2 a^2}{2}$$

$$C_{-1} = \lambda a/2 + \lambda^2 a^2/2$$

$$C_0 = 1 - \lambda^2 a^2$$

Remark:- one can use more points to get higher order methods

$\longleftrightarrow$

$$u(x, t+k) = C_2 U(x+2h, t) + C_1\, U(x+h, t) + C_0 U(x, t) + C_1 u(x-h, t) + C_2 (x-2h, t).$$

Lecture # 12 — 07-11-16

$$u_t + a u_x = 0, \quad a = 1$$

$$u(x,0) = \sin(2\pi x)$$

$$u(0,t) = 0 \qquad \frac{\partial}{\partial x}u(1,t) = 0$$

$$h = 0.2, \quad \lambda = \frac{ak}{h} = 0.1 \Rightarrow k = 0.02$$

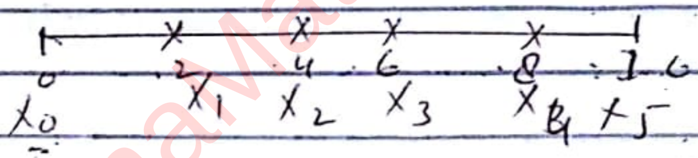
$t_{max} = 0.06$

$t_n = nk = .06 \quad | \quad N = 5$

$n = 3$

$u_0^n = 0 = u_5^n$

we need three iterations to reach t_{max}.



Explicit method
Backward

$$u_j^{n+1} = u_j^n - a\lambda\left[u_j^n - u_{j-1}^n\right].$$

$n = 0$

$$u_j^1 = u_j^0 - a\lambda\left[u_j^0 - u_{j-1}^0\right], \quad j = 1,2,3,4$$

$$u_1^1 = u_1^0 - .1\left[u_1^0 - u_0^0\right]. \qquad u_0^0 = 0$$

$$= .95 - 0.1[0.95 - 0] \qquad u_1^0 = .95$$

$$= 0.85 \qquad u_2^0 = 0.587$$

$$u_2^1 = .587 - 0.1[0.587 - .95]. \qquad u_3^0 = -.586$$

$$= 0.623 \qquad u_4^0 = -0.95$$

$$u_3^1 = -0.47 \qquad u_5^0 = 0$$

$$u_4^1 = .0914$$

$$\boxed{u_5 = u_4}$$

$n = I$

$U_1^2 = U_1^1 - 0.1\left[U_1^1 - U_0^1\right]$.

$U_2^2 = U_2^1 - 0.1\left[U_2^1 - U_1^1\right]$.

$U_3^2 = U_3^1 - 0.1\left[U_3^1 - U_2^1\right]$.

$U_4^2 = U_4^1 - 0.1\left[U_4^1 - U_3^1\right]$.

$\boxed{U_5^2 = U_4^2}$

$U_0^1 = 0$
$U_1^1 = .85$
$U_2^1 = .623$
$U_3^1 = -0.47$
$U_4^1 = -.98$
$U_5^1 = -0.98$

Exact:

$u_0^e = \sin(2\pi(-0.06)) =$

$u_1^e = \sin(2\pi(0.2-0.06)) =$

$u_2^e = \sin(2\pi)(0.4-0.06) =$

$u_3^e = \sin 2\pi(0.6-0.06) =$

$u_4^e = \sin 2\pi(.8-0.06) =$

X	U^N	u^e	Error $\lvert u^e - u^N \rvert$
0			

$$u_t + a u_x = 0 \qquad a > 0$$

$$u(t+k, x) = c_{-1}\, u(t, x-h) + c_0\, u(t, x) + c_1\, u(t, x+h).$$

① $$u(t+k, x) = u(t, x) - \frac{\nu}{2}\left[u(t, x+h) - u(t, x-h)\right]$$

② (Alternative method) $$+ \frac{Q}{2}\left[u(t, x+h) - 2u(t, x) + u(t, x-h)\right]$$

(Coefficient of numerical diffusion) 2nd derivative (numerical diffusion)

$$c_{-1} = \frac{Q-\nu}{2}, \quad c_0 = 1 - Q, \quad c_1 = \frac{Q+\nu}{2}$$

By comparing eq ① and Eq ②

$$c_{-1} + c_0 + c_1 = 1$$

$$c_1 - c_{-1} = -\nu = -\lambda a$$

$$c_1 + c_{-1} = Q = a^2\lambda^2 = \nu^2$$

$$u(t+k, x) = \left[(1-Q) + \frac{Q-\nu}{2}\tau_h + \frac{Q+\nu}{2}\tau_{-h}\right] u(t, x)$$

$$\hat{u}(t+k, \xi) = \left[(1-Q) + \frac{Q-\nu}{2} e^{i\xi} + \frac{Q+\nu}{2} e^{-i\xi}\right] \hat{u}(t, \xi)$$

$$\tau_{\pm h}\, u = u(t, x \pm h)$$

$$D = (1-Q) + \frac{Q-\nu}{2} e^{i\xi} + \frac{Q+\nu}{2} e^{-i\xi}$$

$$= 1 - i2\sin\frac{\xi}{2}\left[\frac{\nu}{2}\cos\xi - iQ\sin\frac{\xi}{2}\right]$$

$$= 1 - 2Q\sin^2\frac{\xi}{2} - i2\nu\sin\frac{\xi}{2}\cos\frac{\xi}{2}$$

Stability $\Rightarrow |D|^2 \leq 1$

$$1 - 4Q\sin^2\frac{\xi}{2} + 4Q^2\sin^4\frac{\xi}{2} + 4\nu^2\sin^2\frac{\xi}{2}\left(1-\sin^2\frac{\xi}{2}\right) \leq 1$$

$$-Q + Q^2\sin^2\frac{\xi}{2} + \nu^2\left(1-\sin^2\frac{\xi}{2}\right) \leq 0$$

$\xi = 0, -\pi, \pi$

$\xi = 0 \quad \nu^2 - Q \leq 0 \Rightarrow Q > \nu^2 > 0$

$\xi = \pm\pi \Rightarrow \cancel{\nu^2} - Q + Q^2 - \cancel{\nu^2} \leq 0$

$\Rightarrow Q(Q-1) \leq 0$

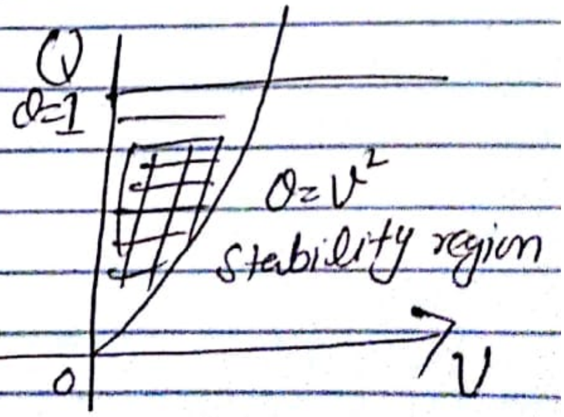
$\Rightarrow Q \leq 1$

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left[U_{j+1}^n - U_{j-1}^n\right] + \frac{Q}{2}\left[U_{j-1}^n - 2U_j^n + U_{j+1}^n\right]$$

$\frac{\nu}{2}$

if $Q = 1$

Lax-Friedrich method $\rightarrow$ stable



Backward diff $Q = V$ Stable.
Forward diff $Q = -V$ unstable.
Central diff $Q = 0$ unstable

Lax-wandroff : $Q = V^2$

Lecture 13

(Elliptic PDES) 14-11-2016

Finite Difference Method for 1D - Poison equation:-

BVP

$$-\frac{\partial^2 u}{\partial x^2} = f(x) \text{ in } \Omega = (0,1)$$

$$u(0) = 0 = u(1) \rightarrow \text{Drichlet BCs}$$

x_0 x_1 x_2 3 - - - - x_{N-1} x_N

$$u_i \approx u(x_i), \; f_i \approx f(x_i), \; x_i = ih, \; h = \frac{1-0}{N} = \frac{1}{N}$$

central difference:-

$$-\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = f_i$$

$\forall \; i = 1, 2, \ldots, N-1$, BCs. $u_0 = u_N = 0$

Result:- The original PDE is replaced by a linear system for model values

1-D Poison Eq

$$\begin{cases} i=1 & -\left[\dfrac{u_0 - 2u_1 + u_2}{h^2}\right] = f_1 \\ i=2 & -\left[\dfrac{u_1 - 2u_2 + u_3}{h^2}\right] = f_2 \\ \vdots & \\ i=N-1 & -\left[\dfrac{u_{N-2} - 2u_{N-1} + u_N}{h^2}\right] = f_{N-1} \end{cases}$$

matrix form:

$$\underline{AU=F}$$

$$A=\frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ & & 0 & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

$$U=\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{N-1} \end{bmatrix}, \quad F=\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{bmatrix}$$

Matrix A is SPD because for any $\underline{V}=\{V_1, V_2, \cdots V_{N-1}\}^T$

$$\underline{V}^T A \underline{V} = \frac{1}{h^2}\left[V_1^2 + \sum_{i=2}^{N-1} (V_i - V_{i-1})^2 + V_{N-1}^2\right]$$

$$>0 \text{ for } \underline{V} \neq \underline{0}$$

U exist and is unique

A is SPD $\Rightarrow$ A is invertible

$\Rightarrow \underline{U} = \underline{\underline{A}}^{-1}\underline{F}$ کیونکہ ہم A کا inverse لے سکتے ہیں
وہ direct طریقے کرتے ہیں

if $N \leq 100$ then Direct method is applied. otherwise iterative solvers should be used, such as Jacobi method, Gauss sidal method and SOR - method.

Iterative Solution Method:-

The iterative method proceeds by starting with a guess $u_i^{(0)}$ which in general, will not satisfy eq ①. The guess is then updates to enforce the equality at point i

$$u_i^{(n+1)} = \frac{-u_{i-1}^{(n)} - u_{i+1}^{(n)} + h^2 f_i}{2} \quad \forall\, i = 1, 2, \ldots N-1$$

where the superscript n is the iteration number. This update is repeated until a convergence criterian is reached, for example the correction drops below a specified tolerance:

$$\max_{1 \le i \le N-1} |c_i^{(n)}| = \max_{1 \le i \le N-1} \left| u_i^{(n+1)} - u_i^{(n)} \right| < \epsilon$$

Theoratical analysis guarantees that the process will eventually reach a solution, and that the number of iterations needed to reach convergence scales as.

$$K \gg \frac{\ln \epsilon}{\ln\left(1 - \sin^2 \frac{\pi}{2(N-1)}\right)} \sim \frac{-2(N-1)^2}{\pi^2} \ln \epsilon$$

The above can be used as rough estimate to bound the iteration count.

Lecture 14 15-11-2016

FDM of elliptic PDE's

Example:- $u_{xx} = f(x)$

$u(0) = 0 = u(1)$

$h = 1/5 = 0.2$

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i$$

$i=1$ $\quad -u_0 + 2u_1 - u_2 = h^2 f_1$

$i=2$ $\quad -u_1 + 2u_2 - u_3 = h^2 f_2$

$i=3$ $\quad -u_2 + 2u_3 - u_4 = h^2 f_3$

$i=4$ $\quad -u_3 + 2u_4 - u_5 = h^2 f_4$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = h^2 \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

other types of Boundary conditions:-

$-u_{xx} = f(x), \; x \in (0,1)$

$u(0) = 0, \; \frac{\partial u}{\partial x}(1) = 0$

central difference $\frac{u_{N+1} - u_{N-1}}{2h} = 0$

$$\boxed{u_{N+1} = u_{N-1}}$$

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i \quad i = 1, \ldots, N-1$$

$i=1$ $\quad \frac{-u_0 + 2u_1 - u_2}{h^2} = f_1$

$i = N$ $\quad \dfrac{-U_{N-1} + 2U_N - U_{N+1}}{h^2} = f_N$

$$\frac{-2U_{N-1} + 2U_N}{h^2} = f_N$$

$$\Rightarrow \frac{-U_{N-1} + U_N}{h^2} = \frac{1}{2} f_N$$

$$\underline{A}\,\underline{U} = \underline{F}$$

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 \\ & & \ddots & & \\ & & & 2 & 0 \\ & 0 & & -1 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ \vdots \\ \vdots \\ U_N \end{bmatrix} = \begin{bmatrix} \frac{1}{2} f_1 \\ \vdots \\ \frac{1}{2} f_N \end{bmatrix}$$

$$\frac{\partial u}{\partial x}(0) = 0 \Rightarrow \frac{U_1 - U_{-1}}{2h} = 0 \Rightarrow U_{-1} = U_N.$$

$$-U_{xx} = f(x), \quad x \in (0,1)$$

$$u(0) = g_0, \quad \frac{\partial u}{\partial x}(1) = g_1$$

$i = 1$ $\quad \dfrac{-u_0 + 2U_1 - U_2}{h^2} = f_1$

$$\frac{2U_1 - U_2}{h^2} = f_1 + g_0/h^2$$

$$\frac{U_{N+1} - U_{N-1}}{2h} = g_1$$

$$U_{N+1} = U_{N-1} + 2h g_1$$

$i=N$ $\boxed{\dfrac{-U_{N-1}+U_N}{h^2} = \dfrac{1}{2}f_N + \dfrac{g_1}{h}}$

Assignement:-

Solve $-U_{xx} = f(x)$

$f(x) = -6x + 12x^2$

(i) $u(0) = 0 = U(1)$.

(ii) $U(0) = 0, \quad \dfrac{\partial u}{\partial x}(1) = 0$.

(iii) $u(0) = 0.1 \quad \partial u/\partial x = 0.2$.

$h = 1/5 = 0.2$

Do it manually and through program.

2D-Poission Equation:-

BVP $-\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = f$ in $\Omega = [0,1] \times [0,1]$

$h_x = h_y = h$

$u_{i,j} \approx u(x_i, y_j)$, $f_{i,j} = f(x_i, x_j)$.

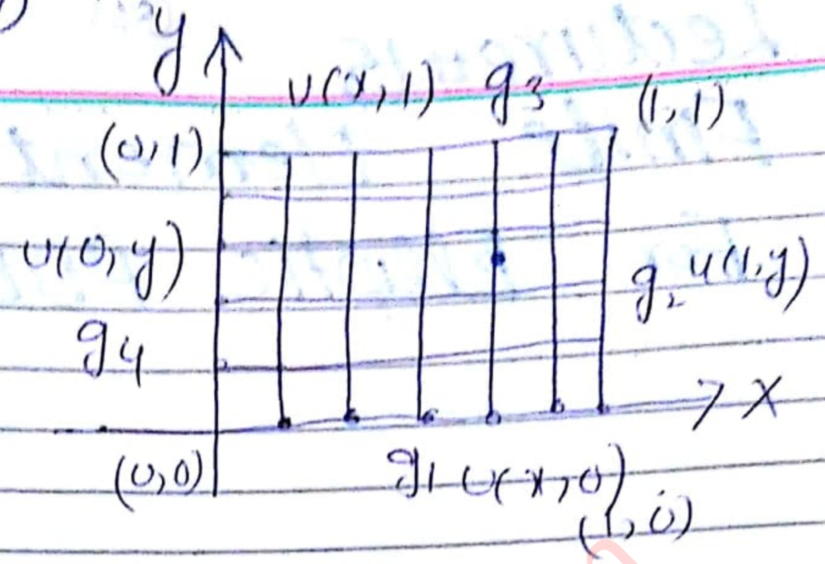
$(x_i, x_j) = (ih, jh)$, $i,j = 0, 1, \ldots, N$

$u(x,0) = g_1 = 0$

$U(1,y) = g_2 = 0$

$u(x,1) = g_3 = 0$

$u(0,y) = g_4 = 0$



$$\frac{-u_{i+1,j}+2u_{i,j}-u_{i-1,j}-u_{i,j+1}+2u_{i,j}-u_{i,j+1}}{h^2}=f_{i,j}$$

$$\frac{-u_{i+1,j}-u_{i,j-1}+4u_{i,j}-u_{i,j+1}-u_{i+1,j}}{h^2}=f_{i,j}$$

$\forall i,j = 1,2,\dots,N-1$

$A\underline{u}=\underline{F}$ $\quad A\in\mathbb{R}^{(N+1)^2\times(N-1)^2}$

$\underline{u},\underline{F}\in\mathbb{R}^{(N-1)^2}$

$\underline{u}=[u_{1,1},u_{2,1},u_{3,1}\dots u_{N+1,j}$

$u_{2,1},u_{2,2},\dots u_{N.1,j},\dots u_{N-1,j-1}]^T$

row by row.

$$A=\begin{bmatrix}\underline{B} & -I & & O\\ -I & B & -I & \\ & \ddots & \ddots & \ddots\\ & O & -\underline{\underline{I}} & \underline{\underline{B}}\end{bmatrix}$$

21-11-2016

Lecture#15
Finite Element Method For Elliptic PDE'S:-

The FEM was created to solve complicated eq. of elasticity and structured mechanic usually modeled by elliptic type PDE'S with complicated geometric. It has been developed for other application as well

our model: 1D: $-U_{xx} = f(x),\ 0 < x <$

2D: $-(U_{xx} + U_{yy}) = f(x,y)$

$x, y \in \Omega \rightarrow$ domain in (x,y) plane

$u(x,y) = U_0(x,y) \quad \forall\ x,y \in \partial\Omega$

$\boxed{\Omega \text{ interior}}$ $\quad \partial\Omega \rightarrow$ Boundary of Ω

$\frac{\partial u}{\partial t} + \frac{\partial f(x)}{\partial x} = 0 \rightarrow$ (strong formulation)

$\oint_{\partial\Omega} u\,dx - f(x)\,dV = 0$ weak formulation

↳ its solution is weak solution.

The FEM approximation the

unknown solution function u by a linear combination of basis functions each with finite (local) support, hence finite elements. The support of a function is a region when function is non-zero

$$u(x) \approx u_h(x) = \sum_{i=1}^{m+1} u_i \phi_i(x)$$

↳ unknown

The basis function $\phi_i(x)$ (finite element) are choosen (given) in advance. The residual is defined to be $\partial(x) = u''(x) + f(x)$.
The coefficient u_i are obtained by requirements on residual.

1) Galerkin: The residual is orthogonal to the basis functions.

2) collocation:- The residual is zero on n discrete points.

3) Rayleigh - Ritz:- The PDE is transformed into equivalent minimization problem
(Euler-Lagrange Equation)

Galerkin FEM:-

1) Poisson $-u_{xx} = f(x)$, $0 < x < 1$

$u(0) = 0 = u(1)$.

1) Weak formulation:-

This can be done by multiplicating a test function $\phi(x)$ on both sides of PDE

$$r(x) \perp \phi(x) \Rightarrow \int_0^1 r(x)\,\phi(x)\,dx = 0$$

$$-\int_0^1 u_{xx}\,\phi(x)\,dx = \int_0^1 f(x)\,\phi(x)\,dx$$

$\phi(0) = 0 = \phi(1)$.

Integration by parts.

$$-u_x\phi(x)\Big|_0^1 + \int_0^1 u_x\phi_x(x)\,dx = \int_0^1 f(x)\,\phi(x)\,dx$$

$$\Rightarrow \int_0^1 u'(x)\,\phi'(x)\,dx = \int_0^1 f(x)\,\phi(x)\,dx$$

$\Downarrow$

weak formulation.

2) Discrete your Domain: $(0 \le x \le 1)$

$x_i = ih$ $\quad i = 0, 1, \dots, n$.

constant $\leftarrow h = \frac{1-0}{N}$

intervals $[x_{i-1}, x_i]$ $\quad i = 1, 2, \dots N$.

$\Downarrow$

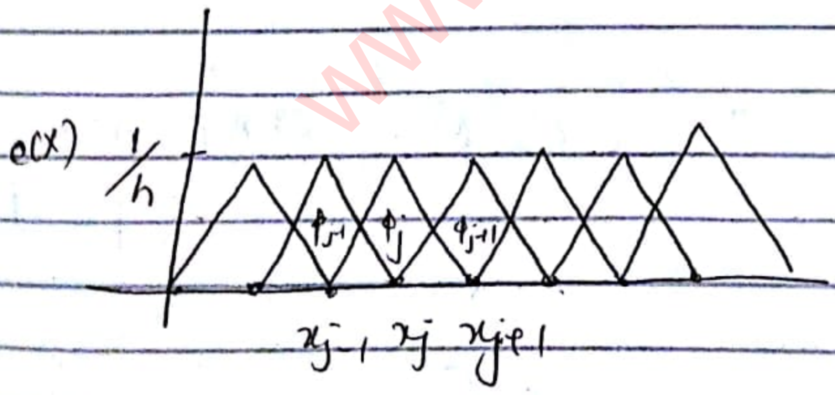
line element. Equidistant (uniform mesh)

(3) Let $\phi_j(x) = e_j \rightarrow$ (Basis function with compact support)

Hat Function:

$$e_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h} & x_{j-1} \le x \le x_j \\ \dfrac{x_{j+1} - x}{h} & \text{if } x_j \le x \le x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

$$e_j'(x) = \begin{cases} 1/h & \text{if } x_{j-1} \le x \le x_j \\ -1/h & x_j \le x \le x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$



4) Approximate the Solution as a linear combination of basis function

$$U_h(x) = \sum_{i=1}^{n-1} u_i\, e_i(x) \text{, where } u_i \text{ are unknown}$$

replace $\phi(x) = e_j(x)$,

$$\int_0^1 u_h'(x) \cdot e_j'(x)\, dx = \int_0^1 f(x)\, e_j(x)\, dx$$

$j = 1, 2, \dots, N-1$

$$\int_0^1 \sum_{i=1}^{n-1} u_i\, e_i'(x)\, e_j'(x)\, dx = \int_0^1 f(x)\, e_j(x)\, dx.$$

$$\sum_{i=1}^{n-1} u_i \int_0^1 e_i'(x)\, e_j'(x)\, dx = \int_0^1 f(x)\, e_j(x)\, dx$$

$j = 1, \dots, n-1$

$$\left(\int_0^1 e_1' e_1' dx\right) u_1 + \left(\int_0^1 e_1' \cdot e_2' dx\right) u_2 + \dots$$

$$+ \left(\int_0^1 e_1' e_{n-1}' dx\right) u_{n-1} = \int_0^1 f e_1 dx$$

$$\left(\int_0^1 e_2' e_1' dx\right) u_1 + \dots + \left(\int_0^1 e_2' e_{n-1}' dx\right) u_{n-1} = \int_0^1 f e_2 dx$$

$$\left(\int_0^1 e_{n-1}' e_1' dx\right) u_1 + \dots + \left(\int_0^1 e_{n-1}' e_{n-1}' dx\right) u_{n-1} = \int_0^1 f e_{n-1} dx$$

$$\begin{bmatrix} a(e_1,e_1) & a(e_1,e_2) & \cdots & a(e_1,e_{n-1}) \\ \vdots & & & \\ a(e_{n-1},e_1) & a(e_{n-1},e_2) & & a_n(e_{n-1},e_{n-1}) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} f, e_1 \\ \vdots \\ f, e_{n-1} \end{bmatrix}$$

$$a(e_i, e_j) = \int_0^1 e_i', e_j' \, dx, \quad (f, e_j) = \int_0^1 f, e_j \, dx$$

Galerkin FEM

$$\int_0^1 e_i' e_j' \, dx \; ?$$

$i = j$

$$\int_0^1 e_i' e_j' \, dx = \int_0^1 (e_j')^2 \, dx.$$

$$= \int_{x_{j-1}}^{x_{j+1}} (e_j')^2 \, dx = \int_{x_{j-1}}^{x_j} (e_j')^2 \, dx + \int_{x_j}^{x_{j+1}} (e_j')^2 \, dx.$$

$$= h/h^2 + h/h^2 = 2/h.$$

$i = j-1$

$$\int_0^1 e_{j-1}' e_j' \, dx = \int_{x_{j-1}}^{x_j} e_{j-1}' e_j' \, dx + \int_{x_j}^{x_{j+1}} e_{j-1}' e_j' \, dx.$$

$$= \left(-1/h^2\right) \cdot h = -\frac{1}{h}$$

$i = j+1$

$$\int_0^1 \phi_{j+1}' \phi_j' \, dx = \int_{x_{j-1}}^{x_j} \phi_{j+1}' \phi_j' \, dx + \int_{x_j}^{x_{j+1}} \phi_{j+1}' \phi_j' \, dx$$

$$= 0 + \frac{-1}{h^2}(h) = -1/h$$

22-11-16.

$$\int_0^1 e_i' e_j' dx \,?$$

for $i=j$ $\quad \int_0^1 e_j' e_j' = \int_{x_{j-1}}^{x_{j+1}} |e_j'|^2 dx$

$$= \int_{x_{j-1}}^{x_j} |e_j'|^2 dx + \int_{x_j}^{x_{j+1}} |e_j'|^2 dx$$

$$= \frac{1}{h} + \frac{1}{h} = \frac{2}{h}.$$

for $i=j-1$

$$\int_0^1 e_{j-1}' e_j' dx = \int_{x_{j-1}}^{x_j} e_{j-1}' e_j' dx + \int_{x_j}^{x_{j+1}} e_{j-1}' e_j' dx$$

$$= -\frac{1}{h^2} \cdot h = -\frac{1}{h}.$$

for $i=j+1$ $\quad \int_0^1 e_j' e_{j+1}' = \int_{x_{j-1}}^{x_j} e_j' e_{j+1}' dx + \int_{x_j}^{x_{j+1}} e_j' e_{j+1}' dx$

$$= 0 + \left(-\frac{1}{h^2} h\right) = -\frac{1}{h}$$

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix}$$

(Tri-diagonal)

$u_0 = 0$

$-u_0 + 2u_1 - u_3 = f$

$$\begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ -1 & 2 & -1 & \cdots & & 0 \\ & & & & & \\ & & & -1 & 2 & -1 \\ & & & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ \\ \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} 0 \\ f_1 \\ \vdots \\ f_{n-1} \\ 0 \end{bmatrix}$$

$$\underline{A}\,\underline{U} = \underline{F}$$

$$\int_0^1 f e_j \, dx = ?$$

$f = c$ $$c\int_0^1 e_j \, dx = c\int_{x_{j-1}}^{x_{j+1}} e_j \, dx$$

$$= c\int_{x_{j-1}}^{x_j} e_j \, dx + c\int_{x_j}^{x_{j+1}} e_j \, dx.$$

$$= c\left[\int_{x_{j-1}}^{x_j} \left(\frac{x - x_{j-1}}{h}\right) dx + \int_{x_j}^{x_{j+1}} \left(\frac{x_{j+1} - x}{h}\right) dx\right]$$

$$= c\left[\frac{(x - x_{j-1})^2}{2}\Big|_{x_{j-1}}^{x_j} + \frac{1}{h}\frac{(x_{j+1} - x)^2}{2}\Big|_{x_j}^{x_{j+1}}\right]$$

$$= c\,h/2 + ch/2 = ch$$

$$-u_{xx} = f(x), \; u_0 = 0, \; u_n = 0$$

$$-\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h}\right) = hf_i \quad \forall\, i = 1, 2, \ldots, n-1$$

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots \\ & & & \\ & & & \end{bmatrix}$$

Both FEM and FDM agree with each other because
1) Mesh is equidistant (uniform).
2) Linear basis function (Piecewise linear)

Advantages of FEM:-

1) Mesh is non uniform.
2) irregular Geometery (Multi-dimensional problems).

Summary of Galerkin method:-

1) Basis functions ϕ_j can be choosen to fit the requirement of the problem common choices are low order (Multidimensional) polynomials.
2) The necessary integrals can easily be calculated for a standard element and the whole process is easily automated.
3- Much easier to generalize for orbitrary geometery and non-uniform irregular meshes then FDM

4. Finite elements are very flexible and easily adaptable
5. This made the FEM as most popular method

Example1 -

$$-u_{xx} = x, \quad u(0) = 0 = u(1)$$

(i) $u(x) = u_1 x(1-x)$. $\phi(x) = x(1-x)$

(ii) $u(x) = u_1 x(1-x) + u_2 x^2(1-x)$

$\phi_1(x) = x(1-x), \phi_2(x) = x^2(1-x)$

Use Galerkin method to find u_1, u_2

(i) $$\int_0^1 (u_{xx} + x) \cdot \phi(x)\, dx = 0$$

$$\int_0^1 [u_1(-2) + x]\, x(1-x)\, dx = 0$$

$$\Rightarrow \int_0^1 [-2u_1 + x][x - x^2]\, dx = 0$$

$$\Rightarrow \int_0^1 \{-2u_1 x + 2u_1 x^2 + x^2 - x^3\}\, dx = 0$$

$$-2u_1 \frac{x^2}{2}\Big|_0^1 + 2u_1 \frac{x^3}{3}\Big|_0^1 + \frac{x^3}{3}\Big|_0^1 - \frac{x^4}{4}\Big|_0^1 = 0$$

$$-u_1 + \frac{2}{3}u_1 + \frac{1}{3} - \frac{1}{4} = 0$$

$$\frac{-3u_1 + 2u_1}{3} = \frac{1}{4} - \frac{1}{3}$$

$$-\frac{U_1}{3} = \frac{\cancel{3} - 4}{1\cancel{2}}$$

$$\cancel{-}\frac{U_1}{3} = \cancel{+}\frac{1}{12}$$

$$U_1 = \frac{3}{12}$$

$$\boxed{U_1 = \frac{1}{4}}$$

(ii) $\int r(x)\phi_1\, dx = 0, \int_0^1 r(x)\phi_2(x)\, dx = 0$

(a) $\int_0^1 u'\phi_1'(x)\, dx = \int f(x)\phi_1'(x)\, dx$

(b) $\int_0^1 u'\phi_2'(x)\, dx = \int f(x)\phi_2(x)\, dx$

Example:-

$$-U_{xx} = f(x).$$

$$u(0) = 0, \quad u(1) = 1$$

$$\int u'\phi'(x)\, dx = \int f(x)\phi(x)\, dx. \; ?$$

$$w(x) = u(x) - x \checkmark$$

$$w(0) = 0 - 0 = 0$$

$$w(1) = 1 - 1 = 0$$

$$-w_{xx} = f(x).$$

at the end $\quad u(x) = w(x) + x.$

$a(v,u) = \int_0^1 |v'(x)|^2 dx > 0 \quad \forall\, v' \neq 0$

$\Rightarrow A$ is +ive definit matrix

(symetric $a(u,v) = a(v,u)$ and positive eign values)

(The system of equations always give a unique solution.

Proof:-

Symmetry

$$(A_h)_{jk} = \int_0^1 \frac{\partial \phi_j(x)}{\partial x} \cdot \frac{\partial \phi_k}{\partial x}\, dx = \int_0^1 \frac{\partial \phi_k}{\partial x} \cdot \frac{\partial \phi_j}{\partial x}\, dx$$

$$= (A_h)_{kj}$$

$\Rightarrow$ Symmetric

Positive definit matrix

$(X^T A X > 0)$

For any vecter $V \in \mathbb{R}^{n-1} / \{0\}$ holds.

$$v^T A v^T = \sum_{j,k=1}^{n-1} v_j v_k \int_0^1 \frac{\partial \phi_j}{\partial x} \cdot \frac{\partial \phi_k}{\partial x}\, dx$$

$$= \int_0^1 \left(\sum_{j=1}^{n-1} v_j \frac{\partial \phi_j}{\partial x} \right) \cdot \left(\sum_{k=1}^{n-1} v_k \frac{\partial \phi_k}{\partial x} \right) dx$$

$$= \int_0^1 \left(\sum_{j=1}^{n-1} v_j \frac{\partial \phi_j}{\partial x} \right)^2 dx > 0$$

stability $\int_0^1 \left(\frac{\partial u_h}{\partial x}\right)^2 dx = \int_0^1 \left(\sum_{j=1}^{m-1} u_j^h \frac{\partial \phi_j}{\partial x}\right)^2 dx$

$$u_h(x) = \sum_{j=1}^{n-1} u_j \phi_j(x)$$

$$= U^T(AU) = U^T F_h.$$

$$= \sum_{j=1}^{n-1} \int_0^1 u_j^h \cdot \phi_j(x) f(x)\, dx.$$

$$= \int_0^1 u^h f(x)\, dx \longrightarrow (1)$$

$$\left[\int_a^b f(x) g(x)\, dx\right] \le \left[\int_a^b [f(x)]^2 dx \times \int_a^b [g(x)]^2 dx\right]^{1/2}.$$

(1) $\Rightarrow$

$$\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx = \int_0^1 u(x) f(x)\, dx.$$

$$\le \sqrt{\int_0^1 |u(x)|^2 dx} \sqrt{\int_0^1 |f(x)|^2 dx} \longrightarrow (2)$$

(Cauchy-schwarz)

Also

$$\int_0^1 |u(x)|^2 \le \frac{1}{4} \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx \text{ for}$$

function with BCs $u(0) = 0 = u(1)$

poincare inequality.

$$\int_0^1 |u(x)|^2 dx \le \frac{1}{4}\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx \le \frac{1}{4}\sqrt{\int_0^1 |u(x)|^2 dx}\sqrt{\int_0^1 |f(x)|^2 dx}$$

$$\sqrt{\int_0^1 |u(x)|^2 dx}\sqrt{\int_0^1 |u(x)|^2 dx} \le \frac{1}{4}\sqrt{\int_0^1 |u(x)|^2 dx}\sqrt{\int_0^1 |f(x)|^2 dx}$$

$$\Rightarrow \sqrt{\int_0^1 |u(x)|^2 dx} \le \frac{1}{4}\sqrt{\int_0^1 |f(x)|^2 dx}$$

$$\|u(x)\|_2 \le \frac{1}{4}\|f(x)\|$$

we get an estimate which is independent from h and L_2 norm.

28-11-2016

Exercise:-

use Galerkin method to solve steady state $au_x - bu_{xx} = 0 \rightarrow (1)$

$u(0) = 0, \quad u(1) = 1$ $\qquad a = 5$
$\qquad b = 1, 0.5, 0.1$

$$\varepsilon u_x - u_{xx} = 0 \rightarrow (2)$$

$w = u - x.$

$w(0) = 0$

$w(1) = 0$

$u = w + x.$

$② \Rightarrow \quad \epsilon(w_x + 1) - w_{xx} = 0.$

$\epsilon w_x + \epsilon - w_{xx} = 0 \quad — ③$

$w(0) = 0 = w(1)$

Galerken method :- multiply with test function and integrate over $[0,1]$.

$$\int_0^1 \epsilon w_x \phi(x) + \int_0^1 \epsilon \phi(x) - \int_0^1 w_{xx} \phi(x) = 0$$

integrate by parts

$$= \int_0^1 w_x \phi_x(x)\, dx$$

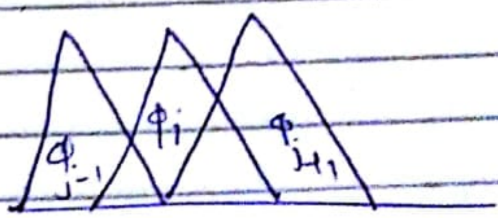
$\phi(x) = e_j$

$$w_h(x) = \sum_{i=1}^{n-1} w_i e_i(x).$$

$$\sum_{i=1}^{n-1} \epsilon w_i \underbrace{\int_0^1 e_i' e_j\, dx}_{B} + \epsilon \underbrace{\int_0^1 e_j\, dx}_{\epsilon h} + \sum_{i=1}^{n-1} w_i \underbrace{\int_0^1 e_i' e_j'\, dx}_{A} = 0$$

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \cdots & & & 2 & -1 \\ \cdots & & & -1 & 2 \end{bmatrix}$$

$$\int_0^1 e_i' e_j\, dx = ?$$



for $i = j$

$$\int_0^1 e_j' e_j\, dx = \int_{x_{j-1}}^{x_j} e_j' e_j\, dx + \int_{x_j}^{x_{j+1}} e_j' e_j\, dx.$$

$$= \int_{x_{j-1}}^{x_j} \frac{1}{h}\left(\frac{x-x_{j-1}}{h}\right) dx + \int_{x_j}^{x_{j+1}} \left(-\frac{1}{h}\right) \frac{x_{j+1}-x}{h} dx$$

$$= \frac{1}{h^2} \frac{(x-x_{j-1})^2}{2}\Bigg|_{x_{j-1}}^{x_j} + \frac{1}{h^2} \frac{(x_{j+1}-x)^2}{1}\Bigg|_{x_j}^{x_{j+1}}$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$

for $i = j$

$$= \int_{x_{j+1}}^{x_j} -\frac{1}{h}\left(\frac{x-x_{j-1}}{h}\right) dx + \int_{x_j}^{x_{j+1}} (-)\frac{x_{j+1}-x}{h} dx$$

$$= -\frac{1}{h^2} \frac{(x-x_{j-1})^2}{2}\Bigg|_{x_{j-1}}^{x_j} - \frac{1}{2}$$

for $i = j+1$

$$= \int_{x_{j-1}}^{x_j} -\frac{1}{h} \overset{0}{\nearrow}\left(\frac{x-x_{j-1}}{h}\right) dx + \int_{x_j}^{x_{j+1}} \frac{1}{h}\left(\frac{x_{j+1}-x}{h}\right) dx$$

$$= 0 + \left(\frac{1}{h^2}\right) - \frac{(x_{j+1}-x)^2}{2}\Bigg|_{x_j}^{x_{j+1}}$$

$$B = \begin{bmatrix} 0 & \epsilon/2 & \cdots & & 0 \\ -\epsilon/2 & 0 & \epsilon/2 & \cdots & 0 \\ & & & & \\ 0 & 0 & 0 & -\epsilon/2 & 0 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 2/h & -1/h+\epsilon/2 & 0 & \cdots & 0 \\ -1/h-\epsilon/2 & 2/h & -1/h+\epsilon/2 & \cdots & \\ \cdots & & & & \\ & & & & 2/h \\ & & & -\frac{1}{h}-\epsilon/2 & 2/h \end{bmatrix} = \tilde{A}$$

$$\int_0^1 e_j\,dx = \int_{x_{j-1}}^{x_j} \left(\frac{x - x_{j-1}}{h}\right)dx + \int_{x_j}^{x_{j+1}} \left(\frac{x_{j+1}-x}{h}\right)dx$$

$$= \frac{1}{h}\frac{(x - x_{j-1})^2}{2}\Bigg|_{x_{j-1}}^{x_j} + -\frac{1}{h}\frac{(x_{j+1}-x)^2}{2}\Bigg|_{x_j}^{x_{j+1}}$$

$$= \frac{h}{2} + h/2 = h$$

$AW = F$

$$\begin{bmatrix} 2/h & -\frac{1}{h}+\epsilon/2 & 0 & \cdots & 0 \\ -1/h-\epsilon/2 & 2/h & -1/h+\epsilon/2 & 0 & \\ \cdots & & & & \\ & & & & 2/h \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{n-1} \end{bmatrix} = \begin{bmatrix} -\epsilon h \\ -\epsilon h \\ \vdots \\ -\epsilon h \end{bmatrix}$$

Finally

$u = w + x$ final solution

Exact solution

$$u(x) = \frac{1-e^{\epsilon x}}{1-e^{\epsilon}}$$

The maximum principle holds if ϵh (Peclet no.) does not exceeds unity. When $\epsilon h \le 1$. If ϵh exceeds unity, the solution has boundary layer which can not resolved by standard Galerkin method. In that case some stabilization technique is needed. In this problem for $b=1$ & $b=0.5$ the maximum principle holds.

Finite Element method for Heat Equation:-

$$u_t - a u_{xx} = 0 \,, \quad t > 0 \,,$$

$$x \in [b_1, b_2],$$

$$u(b_1) = 0 = u(b_2). \qquad a > 0$$

$$u(x,0) = u_0(x)$$

$$\int_{b_1}^{b_2} u_t \, \phi(x) dx - a \int_{b_1}^{b_2} u_{xx} \phi(x) dx = 0$$

$$\Rightarrow \int_{b_1}^{b_2} u_t \phi(x) dx + a \int_{b_1}^{b_2} u_x \phi_x(x) dx = 0$$

Discretization

$$b_1 = x_0 < x_1 < x_2 < \cdots < x_n = b_2$$

$\phi_j = e_j$

$$u_h(x,t) = \sum_{i=1}^{n-1} u_i e_i(x).$$

$$\int_{b_1}^{b_2} \sum_{i=1}^{n-1} (u_i)_t e_i e_j \, dx + a \int_{b_1}^{b_2} \sum_{i=1}^{n-1} u_i e_i' e_j' \, dx = 0$$

$$\Rightarrow \sum_{i=1}^{n-1} (u_i)_t \underbrace{\int_{b_1}^{b_2} e_i e_j \, dx}_{\text{Mass matrix, } M} + a \sum_{i=1}^{n-1} u_i \underbrace{\int_{b_1}^{b_2} e_i' e_j' \, dx}_{\substack{A \\ \text{stiffness matrix.}}} = 0$$

$$M = \begin{bmatrix} \int_{b_1}^{b_2} e_1 e_1 \, dx & \int_{b_1}^{b_2} e_1 e_2 \, dx & \cdots \\ \vdots & \ddots & \vdots \end{bmatrix}$$

$$A = \begin{bmatrix} \int_{b_1}^{b_2} e_1' e_1' \, dx & \int_{b_1}^{b_2} e_1' e_2' \, dx & \cdots \\ \vdots & \ddots & \vdots \end{bmatrix}$$

$$\underline{M}\, \underline{U}_t + a \underline{A}\, \underline{U} = \underline{0}$$

$i = j$

$$\int_{b_1}^{b_2} e_j e_j \, dx = \int_{x_{j-1}}^{x_{j+1}} e_j^2 \, dx = \int_{x_{j-1}}^{x_j} e_j^2 \, dx + \int_{x_j}^{x_{j+1}} e_j^2 \, dx$$

$$= \int_{x_{j-1}}^{x_j} \frac{(x - x_{j-1})^2}{h^2} \, dx + \int_{x_j}^{x_{j+1}} \frac{(x_{j+1} - x)^2}{h^2} \, dx$$

$$= \frac{h}{3} + \frac{h}{3} = \frac{2h}{3}$$

for $i = j+1$

$$\int_{b_1}^{b_2} \varphi_{j+1}\varphi_j \, dx = \int_{x_{j-1}}^{x_{j+1}} \varphi_{j+1}\varphi_j \, dx = \int_{x_{j-1}}^{x_j} \varphi_{j+1}\varphi_j \, dx + \int_{x_j}^{x_{j+1}} \varphi_{j+1}\varphi_j \, dx$$

$$= \int_{x_j}^{x_{j+1}} \left(\frac{x - x_j}{h}\right)\left(\frac{x_{j+1} - x}{h}\right) dx$$

$$= \frac{h}{6}$$

$$i = j-1 \quad \int_{b_1}^{b_2} \varphi_{j-1}\varphi_j \, dx = \frac{h}{6}.$$

$$\underline{\underline{M}} = h \begin{bmatrix} 2/3 & 1/6 & 0 & \cdots & 0 \\ 1/6 & 2/3 & \frac{1}{6} & \cdots & \\ & 0 & & 1/6 & 2/3 \end{bmatrix}$$

$$\underline{\underline{M}}\, \underline{U}_t + a \underline{\underline{A}}\, \underline{U} = \underline{0}$$

system of ODEs.

approximate derivative by implicit Euler method

$$\underline{\underline{M}} \frac{\underline{U}^{n+1} - \underline{U}^{n}}{k} + a \underline{\underline{A}}\, U^{n+1} = 0$$

$$\left(\frac{1}{k}\underline{\underline{M}} + aA\right)\underline{U}^{n+1} = \frac{1}{k}\underline{\underline{M}}\underline{U}^n$$

$$u_t - a u_{xx} = 0$$

$$\underline{\underline{M}}\dot{U} + a\underline{A}\,\underline{U} = 0$$

$$M = h\begin{bmatrix} 2/3 & 1/6 & 0 & \cdots & 0 \\ 1/6 & \cdots & 2/3 & 1/6 \cdots & 0 \\ & & \ddots & & \\ & 0 & & 1/6 & 2/3 \end{bmatrix}$$

$$\underline{A} = 1/h\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & 0 \\ & & & & \\ & 0 & & -1 & 2 \end{bmatrix}$$

- Discretization in terms of space variable only.
- while time derivative is kept continous.

Any ODE solve can be applied to solve the resulting ODE system

Here as an example, we apply implicit Euler method

$$\underset{\text{mass matrix}}{\underline{M}}\ \frac{U^{n+1} - U^n}{h} + a \overset{\text{stiffness matrix}}{\hat{A}} U^{n+1} = 0$$

$$\underbrace{\left(\tfrac{1}{k}\underline{M} + a\underline{A}\right)}_{B} \underline{u}^{n+1} = \tfrac{1}{k}\underline{M}\,\underline{U}^n$$

$$\underline{u}^{n+1} = \underline{B}^{-1}\tfrac{1}{k}\underline{M}\,\underline{u}^n$$

sometimes one would like to solve instead of $\underline{M}U_t + a\underline{A}\underline{U} = 0$, the system of equations $\underline{U}_t = \underline{B}\underline{U}$. This could be obtained by lamping of mass matrix

Trapezoidal rule:

$$Q_h^P(g) = \int_a^b I_g(g(x))\,dx$$

We use trapezoidal rule on the left integration

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left[f(a)+f(b)\right] = \frac{h}{2}[0+1] = h/2$$

$\Rightarrow U_t\, Q_h(e_i e_j) + a\sum_{i=1}^{n-1} x_i \int_{b_1}^{b_2} e_i' e_j'\,dx = 0$

mass matrix will be approximated as

$$\tilde{M} = (Q_h(e_i e_j))$$

Calculation of $Q_B(e_i e_j)$

if $i=j$

$$Q_B(e_j^2) = \int_{b_1}^{b_2} e_j^2\,dx$$

$$\begin{cases} \dfrac{x_i - x_{j-1}}{h} & x_{j-1} \le x \le x_j \\ \dfrac{x_{j+1} - x}{h} & x_j \le x \le x_{j+1} \\ 0 \text{ otherwise} \end{cases}$$

$$= \int_{x_{j-1}}^{x_j} e_j^2\,dx + \int_{x_j}^{x_{j+1}} e_j^2\,dx$$

$$= h/2\,[0+1] + h/2\,[1+0]$$

$$= h$$

if $i=j-1$ $\int_{b_1}^{b_2} e_{j-1}e_j\,dx = \int_{x_{j-1}}^{x_{j+1}} e_{j-1}e_j\,dx$

$= 0$ Due to trapezoidal rule.

if $i=j+1 \Rightarrow Q_h(e_j e_{j+1}) = 0$

$$\tilde{M} = h\begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} = hI$$

$\tilde{M} U_t + a A U = 0$

$h I U_t + a A U = 0$

$U_t = -\frac{a}{h} A U$

$U_t = f(U)$

$U_t = B U$

$u_t - a u_{xx} = 0$

$u_t - a \left[\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} \right] = 0, \forall, j = 1, \ldots, n-1$

$U_t + \frac{a}{h} A U = 0$ FDM

B/C of Equidistant + Limping

$\Rightarrow$ FDM and FEM are equidistant

Reaction - Diffusion Equation

$u_t = a u_{xx} + f(x, u, t), \; t \geq 0,$

$x \in [b_1, b_2]$

$u(x, 0) = u_0(x), \; u(b_1) = 0 = u(b_2)$

Galerkin method $\int_{b_1}^{b_2} u_t \phi \, dx = -a \int_{b_1}^{b_2} u_x \phi_x dx + \int_{b_1}^{b_2} f(x,u,t) \phi(x) dx$

$\phi = e_j$

$$\sum_{i=1}^{n-1} (u_i)_t \underbrace{\int_{b_1}^{b_2} e_i e_j \, dx}_{I_1} = -a \sum_{i=1}^{n-1} \underbrace{\int_{b_1}^{b_2} e_i' e_j' \, dx}_{I_2} + \underbrace{\int_{b_1}^{b_2} f(x,u,t) e_j \, dx}_{I_3} \quad \forall j = 1, \dots, n-1$$

$$\int_{b_1}^{b_2} f(x,u,t) e_j \, dx \, ?$$

If f is non-linear, then I_3 is complicates.

$$I_3 \approx Q(f e_j) = \int_{b_1}^{b_2} (f e_j) \, dx$$

$$= f(x_j, u_j, t) \int_{b_1}^{b_2} e_j \, dx.$$

$$= h f(x_j, u_j, t).$$

$\underline{\underline{M}} \, \underline{U}_t + a \underline{\underline{A}} \, \underline{U} = h f_j$ Non-linear ODE.

General Boundary conditions:-

$$-(p(x)u_x)_x + qu = f \; , \; a \le x \le b$$

$$u(a) = 0 \; , \; \alpha u(b) + \beta u'(b) = \gamma \; ,$$

$$\beta \ne 0 \; , \; \alpha/\beta \ge 0$$

where α, β, γ are know constants, but $u(b)$ and $u'(b)$ are unknown's.

Galerkin method :- Multiply with test function and integrate

Integration by parts.

$$-P(b)u'(b)\phi(b) + P(a)u'(a)\phi(a) + \int_a^b (P'u'\phi' + qu\phi)\,dx = \int_a^b f\phi\,dx.$$

$$-P(b)\phi(b)\cdot\frac{\gamma - \alpha u(b)}{\beta} + \int_a^b (P'u'\phi' + qu\phi)\,dx = \int_a^b f\phi\,dx$$

$$\Rightarrow \int_a^b ((P'u'\phi' + qu\phi)\,dx + \alpha/\beta\, P(b)u(b)\phi(b)$$

$$= \int_a^b f\phi\,dx + \frac{\gamma}{\beta}P(b)\phi(b)$$

This is the weak form (variational form) of the sturm-Liouville problem with given BCs

Define $a(u,\phi) = \int_a^b ((P'u'\phi' + qu\phi)\,dx$

$$+ \alpha/\beta\, P(b)u(b)$$

$$L(\phi) = \int_a^b f\phi\, dx + \gamma/\beta\, p(b)\phi(b)$$

we can prove that

(1) $a(u,\phi) = a(\phi,u) \Rightarrow$ symmetrix.

(2) $a(u,\phi)$ is a bilinear form. i.e

$$a(\alpha u + \beta w, \phi) = \alpha a(u,\phi) + \beta a(w,\phi)$$

$$a(u, \alpha\phi + \beta w) = \alpha a(u,\phi) + \beta a(u,w)$$

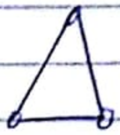
(3) $a(u,\phi)$ is a inner product. The every norm

$$\|u\|_a = \sqrt{a(u,u)} = \left\{ \int_a^b (pu'^2 + qu^2)\, dx + \alpha/\beta\, p(b)\, u^2(b) \right\}^{1/2}$$

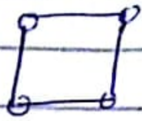
Now we can see why we need to have $\beta \neq 0$ & $\alpha/\beta > 0$

05-12-16.

Two Dimensional FEM :-



$p(x,y) = ax+by+c$

Rectangle :  Bilinear function

$$p(x,y) = (ax+b)(cy+d)$$

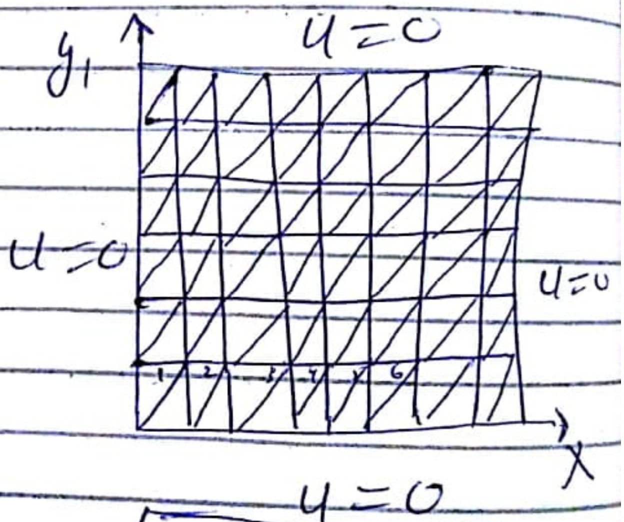
$$= acxy + adx + bcy + bd$$

$$= exy + fx + gy + h.$$

Example :-

$$-\Delta u = 1 \qquad \Omega = [0,1] \times [0,1]$$

$$u\big|_{\partial\Omega} = 0$$

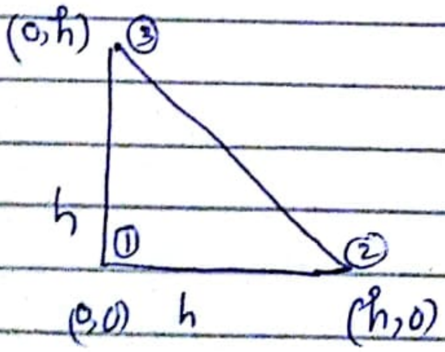


Numbering in the interior nodals points

$$x_1, x_2, \dots x_{n^2}$$

$$h = \frac{1}{n+1}$$

$$\boxed{h_x = h_y = h}$$

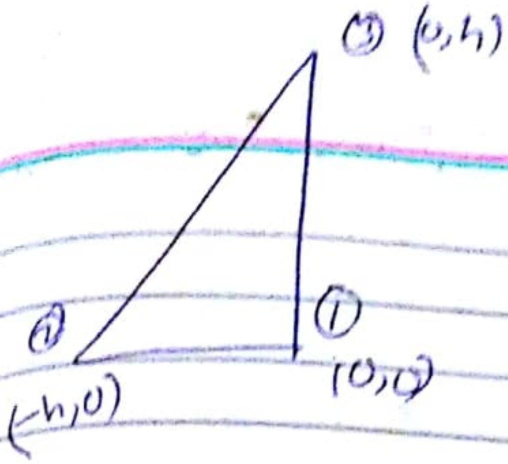


(1) $\varphi_1 = \dfrac{h-x-y}{h}$

② $\varphi_2 = x/h$

③ $\varphi_3 = y/h$

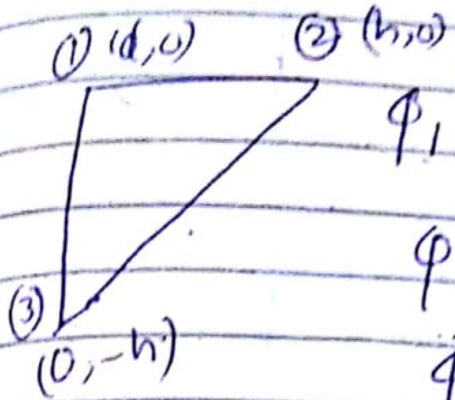
$$\Sigma \varphi_i = 1$$



(1) $\varphi_1 = \dfrac{h+x-y}{h}$

② $\varphi_2 = -x/h$

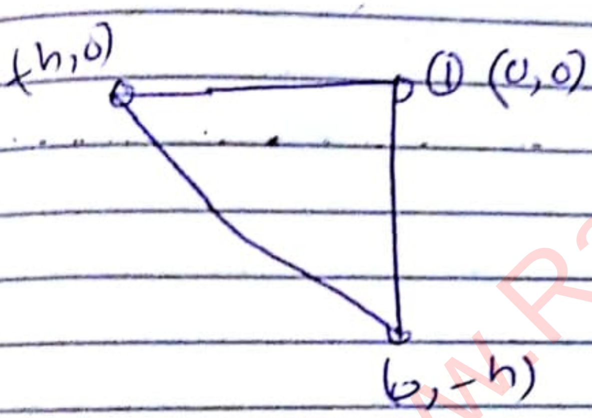
③ $= y/h$



$\varphi_1 = \dfrac{h-x+y}{h}$

$\varphi_2 = x/h$

$\varphi_3 = -y/h$



$\varphi_1 = \dfrac{h+x+y}{h}$

$\varphi_2 = -x/h$

$\varphi_3 = -y/h$

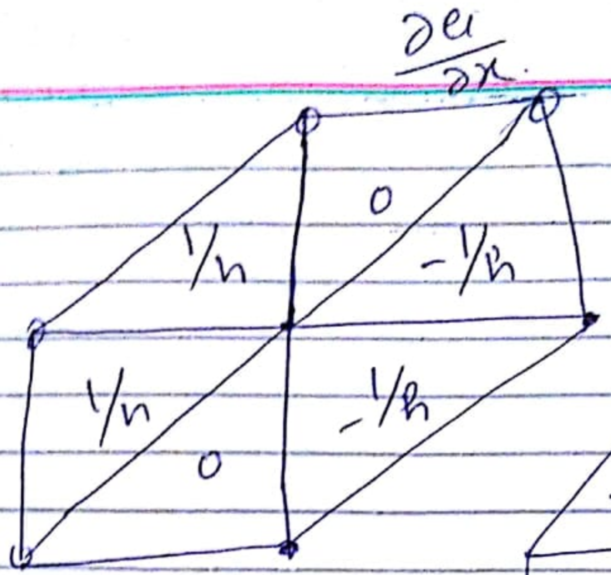
Example:-

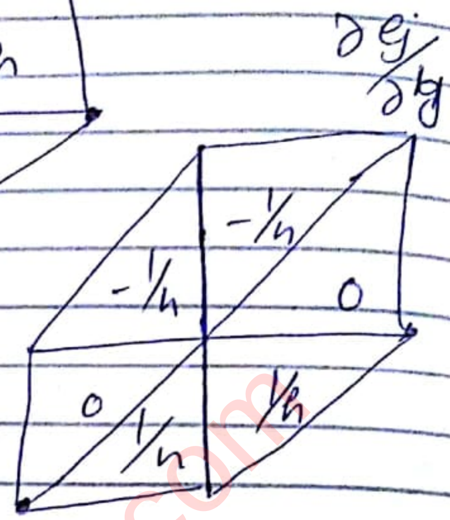
$$-\Delta u = 1$$

$$-u_{xx} - u_{yy} = 1$$

$$\sum u_i \int \nabla\varphi_i \nabla\varphi_j \, dx = \int \varphi_j \, dx$$

$$\nabla\varphi_i \nabla\varphi_j = \frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x} + \frac{\partial\varphi_i}{\partial y}\cdot\frac{\partial\varphi_j}{\partial y}$$





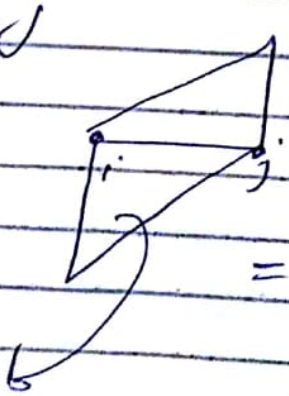
$$\int_\Omega \frac{\partial e_j}{\partial x}\cdot\frac{\partial e_i}{\partial x}\,dx = \frac{4}{h^2}\cdot\frac{h^2}{2} = 2$$

area of triangle

$$\int_\Omega \frac{\partial e_j}{\partial y}\cdot\frac{\partial e_j}{\partial y} = 2$$

if $i=j$ $\displaystyle\int_\Omega \nabla e_j \nabla e_j \, dx = 4$

if $i \neq j$



$$\nabla e_j \nabla e_i = \frac{\partial e_i}{\partial x}\frac{\partial e_j}{\partial x} + \frac{\partial e_i}{\partial y}\frac{\partial e_j}{\partial y}$$

$$= \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) + 0\left(-\frac{1}{h}\right)$$

$$= -\frac{1}{h^2}$$

$$\left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) + 0\left(\frac{1}{h}\right)$$

$$= -\frac{1}{h^2}$$

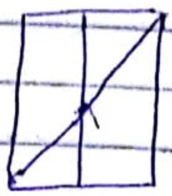
$$\cancel{\nabla e_i \nabla e_j = \frac{\partial e_i}{\partial x}\frac{\partial e_j}{\partial x} + \partial o}$$

$$\int \nabla e_i \nabla e_j \, dx = \frac{-1}{h^2} + \frac{-1}{h^2} \times \frac{h^2}{2}$$

$$= \frac{-2}{h^2} \cdot \frac{h^2}{2}$$

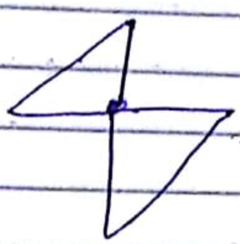
$$= -1$$

(2)



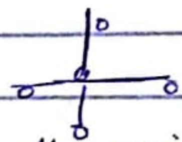
$= 0$

(3)



$= -1$

$$\int_{\Omega} \nabla e_i \nabla e_j \, dx = \begin{cases} 4 & i = j \\ -1 & \\ 0 & \text{otherwise} \end{cases}$$



$$A = \begin{matrix} \tilde{A} & I & \\ I & \tilde{A} & \\ 0 & & \tilde{A} \end{matrix}$$

$$\tilde{A} = \begin{bmatrix} 4 & -1 & 0 & 0 & \cdots \\ -1 & 4 & -1 & 0 & \cdots \\ & & & & \\ & 0 & & -1 & 4 \end{bmatrix}$$

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① $\int_0^h \int_0^x e_i \, dy \, dx$

$\doteq \int_0^h \int_0^x 1 \; e_i \, dy \, dx = \int_0^h \int_0^x -x/h \, dy \, dx$

$= -h^2/3$

② $\int_0^h \int_0^x -y/h \, dy \, dx = -h^2/6$

③ $\int_0^h \int_0^x \frac{h+x-y}{h} \, dy \, dx = 2/3 \, h^2$.

④ $\int_0^h \int_0^x x/h \, dy \, dx = h^2/3$.

⑤ $\int_0^h \int_0^x y/h \, dy \, dx = h^2/6$.

⑥ $\int_0^h \int_0^x \frac{h-x+y}{h} \, dy \, dx = h^2/3$.

At the end

$\int_\Omega e_i \, dx = -h^2/3 + h^2/3 - h^2/6 + \frac{h^2}{6} + \frac{h^2}{3} + \frac{2h^2}{3}$.

$= \frac{h^2 + 2h^2}{3}$

$= h^2$

Do yourself

13-12-2016

Raylaigh-Ritz method for elliptic PDE,s:-

$$-u_{xx} = f \quad , 0 < x < 1$$

19-12-2016.

Method of characteristics:-

It is a technique for solving PDE,s, especially first order PDE,s.
This method can be applied to hyperbolic PDE,s.
This method reduces given PDE to an ODE, which can be then integrated easily

Example:-

$u_t + a(u)u_x = 0$, quasi_linear

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} = 0$$

$$\frac{du}{dt} = \frac{\partial u}{\partial t}\cdot\frac{dt}{dt} + \frac{\partial u}{\partial x}\cdot\frac{dx}{dt}$$

$$= \frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{du}{dt} = 0 \Rightarrow u(x,t) = u_0(x_0) \text{ constant}$$

$x(0) = x_0$

$$\frac{dx}{dt} = a(u) = a(u_0)$$

$$x = a(u_0)t + c$$

$$x_0 = c$$

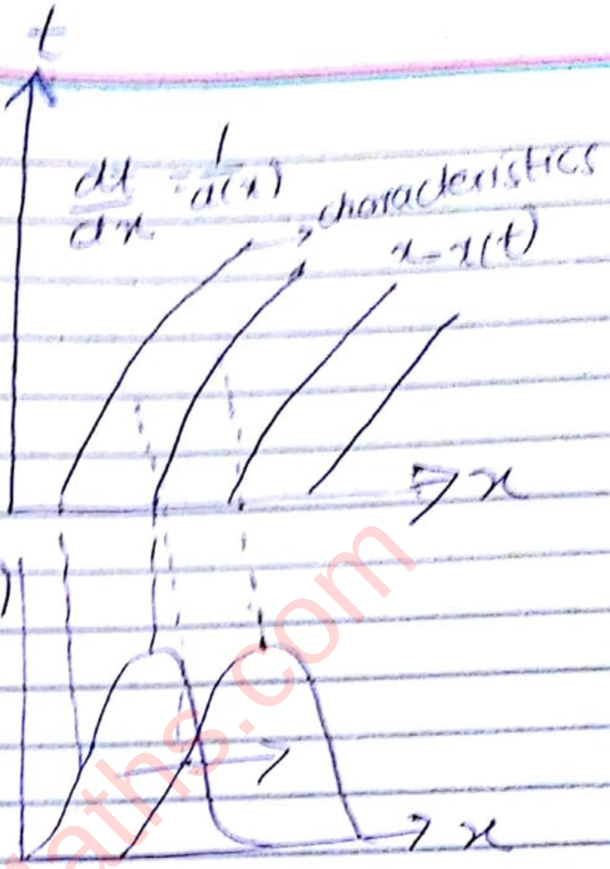
$$x = a(u_0)t + x_0$$

$$\Rightarrow x_0 = x - a(u_0)t$$

$$\Rightarrow u(x,t) = u_0(x - a(u_0)t)$$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \; a>0$$

$$u_0(x) = \begin{cases} \frac{1}{2}x, & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$



$$\frac{dx}{dt} = a$$

$$\Rightarrow x = at + c \Rightarrow$$

$$x_0 = x - at$$

$$x(0) = x_0$$

$$u(x,t) = u_0(x_0) = \begin{cases} \frac{1}{2}x_0, & 0<x_0<1 \\ 0, & \text{otherwise} \end{cases}$$

$$u(x,t) = \begin{cases} \frac{1}{2}(x-at), & 0<x-at<1 \\ 0, & \text{otherwise} \end{cases}$$

final solution.

Example:-

$$u_t - 2u_x = e^{2x}, \; u(x,0) = f(x)$$

Solve by M.C.

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial t} - 2\frac{\partial u}{\partial x} = e^{2x}$$

$$\Rightarrow \boxed{\frac{du}{dt} = e^{2x}}$$

$$\frac{du}{dt} = e^{2(x_0 - 2t)}$$

$$= e^{2x_0 - 4t}$$

$$u(x,t) = \frac{e^{2x_0 - 4t}}{4} + c$$

$$u(x,0) = \frac{1}{4} e^{2x_0} + c$$

$$u(x_0,0) = -\frac{1}{4} e^{2x_0} + c$$

$$\frac{dx}{dt} = -2, \ x(0) = x_0$$

$$x = -2t + C$$

$$x_0 = +C$$

$$\boxed{C = x_0}$$

$$x = 2t + x_0$$

$$x_0 = x + 2t$$

or

$$\boxed{x = x_0 - 2t}$$

$$f(x) = C - \frac{1}{4} e^{2x_0}$$

$$C = f(x_0) + \frac{1}{4} e^{2x_0}$$

$$u(x,t) = -\frac{1}{4} e^{2x_0 - 4t} + f(x_0) + \frac{1}{4} e^{2x_0}$$

$$u(x,t) = -\frac{1}{4} e^{2x} + f(x+2t) + \frac{1}{4} e^{2x+4t}$$

Example:-

$$u_t + 2u_x = \cos(x)$$

$$u(x,0) = e^x$$

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t}$$

$$= \frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = \cos(x)$$

$$\frac{du}{dt} = \cos(x)$$

$$\frac{dx}{dt} = 2.$$

$$x = 2t + c.$$

$$x(0) = 0 + c$$

$$\boxed{c = x_0}$$

$$x = 2t + x_0.$$

$$\boxed{x_0 = x - 2t}$$

$$\frac{du}{dt} = \cos(x_0 + 2t)$$

$$\cancel{u(x,t) = -\sin(x) + c.}$$

$$\cancel{u(x,0) = -\sin(x) + c}$$

$$\cancel{e^x = -\sin(x) + c.}$$

$$\cancel{c = e^x + \sin(x).}$$

$$\cancel{u(x,t) = -\sin(x) + e^x + \sin(x)}$$

$$u = \frac{1}{2}(-\sin(x_0 + 2t)). + c.$$

$$u(x_0, t) = -\frac{1}{2}\sin(x_0 + 2t) + c$$

(98)

$$u(x_0,0) = -\frac{1}{2}\sin(x_0+2(0)) + c$$

$$e^{x_0} = -\frac{1}{2}\sin(x_0) + c$$

$$c = e^{x_0} + \frac{1}{2}\sin(x_0)$$

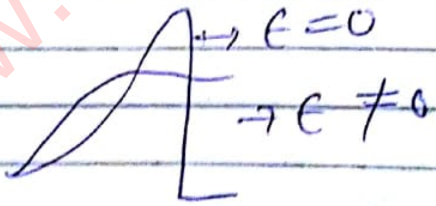
$$u(x_0,t) = +\frac{1}{2}(\sin(x_0+2t) + e^{x_0} + \frac{1}{-2}\sin(x_0)$$

$$u(x,t) = +\frac{1}{2}\sin(x-2t+2t) + e^{x-2t} + \frac{1}{-2}(\sin(x-2t))$$

$$= -\frac{1}{2}\sin(x-2t) + e^{x-2t} + \frac{1}{2}\sin(x)$$

Simple non linear PDE :-

Non linear hyperbolic $\leftarrow u_t + uu_x = 0$ Burger Equation



$$a(u) = u$$

$$u(x,t) = u_0(x-ut) \quad \text{implicit Sol.}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} \qquad u_t = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial t}$$

$$u_x = u'\cdot(1-u_x t) \qquad u_t = u'(-u_t t - u)$$

$$u_x = \frac{u'}{1+u't} \qquad u_t = \frac{-uu'}{1+u't}$$

20-12-16 (99)

$u_t + u u_x = 0$

$$-\frac{uu'}{1+u_t} + \frac{uu'}{1+u't} = 0$$

Example:-

$u_t + u u_x = 0 \Rightarrow u(x,0) = u_0(x) = \begin{cases} 0 & x<0 \\ x & 0<x<1 \\ 1 & x>1 \end{cases}$

$$u(x,0) = u_0(x_0) = \begin{cases} 0 & x_0 < 0 \\ x_0 & 0 < x_0 < 1 \\ 1 & x_0 > 1 \end{cases}$$

$\cancel{u_t}$; $\frac{dx}{dt} = u_0(x_0)$

$x(0) = x_0$ | $\frac{dx}{dt} = 0$, $x = \cancel{\;} x_0 + c$, $\boxed{x_0 = x - \cancel{\;}}$

$$\frac{dx}{dt} = x_0.$$

$$x = x_0 t + x_0$$

$$= x_0(1+t).$$

$$x_0 = \frac{x}{1+t}.$$

$$\frac{dx}{dt} = 1 \quad , x_0 = x - t$$

$$u(x,t) = \begin{cases} 0 & , x < t \\ \frac{x}{1+t} & 0 \le x \le 1+t \\ 1 & x > t+1 \end{cases}$$

$x(0)=x_0$

20-12-16

Example:-

$u_t + uu_x \rightarrow$ Burger Eq (inviscid)

$$u(x,0) = u_0(x) = \begin{cases} 1 & x<0 \\ 1-x & 0<x<1 \\ 0 & x>1 \end{cases}$$

$$\frac{dx}{dt} = u_0(x_0) = \begin{cases} 1 & x_0<0 \\ 1-x_0 & 0\le x\le 1 \\ 0 & x_0>1 \end{cases}$$

$\frac{dx}{dt}=1 \Rightarrow x = t+C \Rightarrow x_0 = x-t$

$\frac{dx}{dt} = 1-x_0 \Rightarrow x = (1-x_0)t + C \Rightarrow x = (1-x_0)t + x_0$

$$\boxed{x_0 = \frac{x-t}{1-t}}$$

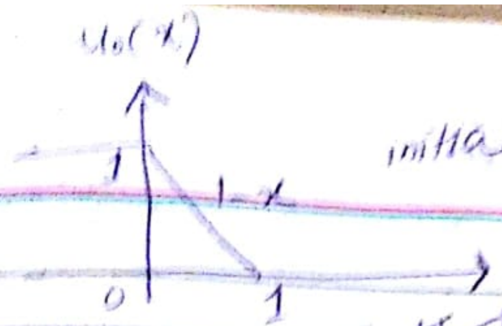
$\frac{dx}{dt} = 0 \Rightarrow x = x_0$

$$\Rightarrow u(t,x) = u_0(x_0) = \begin{cases} 1 & x<t \\ \frac{1-x}{1-t} & 0<\frac{x-t}{1-t}<1 \quad (t\le x\le 1) \\ 0 & x>1 \end{cases}$$

The solution has a pole (singularity) at $t=1$

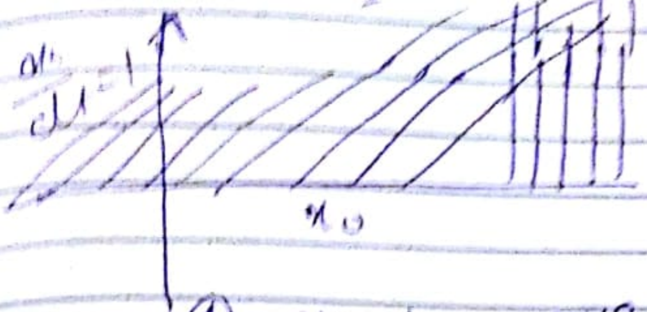
Thus above solution is only valid for $0\le t\le 1$

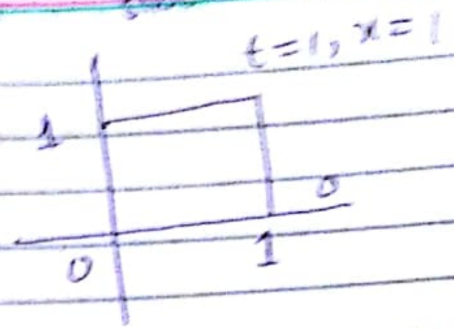
For $t=1$: The characteristics intersect each other, the slope of solution is infinite.



initial data

shock discontinuity





① shock wave
② refraction wave
③ contact wave.

$$u_t + uu_x = 0$$

$$\Rightarrow u_t + \left(\frac{1}{2}u^2\right)_x = 0$$

$$u_t + f(u)_x = 0 \Rightarrow \text{Non-linear hyperbolic Equation}$$

$$\int_{x_L}^{x_R} u_t\, dx + \int_{x_L}^{x_R} f(u)_x\, dx = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\int_{x_L}^{x_R} u\, dx = -\left(f(u_R) - f(u_L)\right)$$

$$\frac{1}{\delta t}(u_L - u_R)\cdot \delta x = f(u_L) - f(u_R)$$

$$\dot{x}_s = \frac{\delta x}{\delta t} = \frac{f(u_L) - f(u_R)}{u_L - u_R} = \frac{1}{2}(u_L + u_R)$$

Shock speed for Burger Eq

$$\dot{X}_s = \frac{1}{2}(U_L + U_R)$$

$$= \frac{1}{2}(1+0)$$

$$X_s = \frac{1}{2}t + C$$

$$X_s(1) = 1$$

$$1 = \frac{1}{2} + C \Rightarrow \boxed{C = \frac{1}{2}}$$

$$X_s = \frac{t+1}{2} \quad \text{for } t \geq 1$$

$$u(t,x) = \begin{cases} 01 & x < x_s \\ 0 & x > x_s \end{cases}$$

Example:-

$$u_t + 2uu_x = 0$$

$$u(x,0) = U_0(x) = \begin{cases} \frac{1}{2} & x < 0 \\ \frac{1-x}{2} & 0 \leq x < 1 \\ 0 & x > 1 \end{cases}$$

$$\frac{dx}{dt} = 2U_0(x_0) = \begin{cases} 1 & x_0 < 0 \\ 1 - x_0 & 0 \leq x_0 \leq 1 \\ 0 & x_0 > 1 \end{cases}$$

$$\frac{dx}{dt} = 1 \quad , \quad x = t + C \Rightarrow x_0 = x - t$$

$$\frac{dx}{dt} = 1 - x, \quad x = (1-x)t + C$$

~~$x_0 = (1-x_0)0 + c$~~

~~$x_0 = -t + x_0 t$~~

~~$x_0 = t(x_0 - 1)$~~

~~$x_0 = (1-x)t + (x_0-1)t$~~

$x_0 = c$

$x = (1-x)t + x_0$

~~$x_0 = x$~~

$x_0 = x - (1-x)t$

$= x - t + tx$

$x_0 = x - t + tx$

$\frac{dx}{dt} = 0$

$x = c$ $\boxed{x = x_0}$

Q $\dot{x}_s = \frac{f(u_L) - f(u_R)}{u_L - u_R}$

$= \frac{1}{2} + 0$

$x_s = \frac{1}{2}$ same as above

Example

$$u_t + u u_x = 0$$

$$\frac{dx}{dt} = u(x,0) = \begin{cases} 2 & x_0 < 0 \\ 2 - x_0 & 0 \le x_0 < 2 \\ 0 & x_0 \geq 2 \end{cases}$$

$$u(t,x) = \begin{cases} 2 & x < 2t \\ \dfrac{2-x}{1-t} & 2t \le x \le 2 \\ 0 & x \geq 2 \end{cases}$$