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NUMERICAL SOLUTIONS OF PDE's

INSTRUCTOR:- Prof. Dr. SHAMSUL QAMAR

→ Objectives of Course:-

* To introduce the basic concepts of numerical methods for one and two dimentional parabolic, elliptic and hyperbolic PDEs. Finite Difference Method (FDM) and Finite Element Method (FEM) will be investigated during the course. The consistency, stability and convergence of numerical methods will be thoroughly analyzed.

* It will be explained that how, why and when numerical methods will be expected to work.

* The students will be given oppurtunity to write their own computer codes through

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numerical assignments. Moreover some numerical codes will be demonstrated by the instructor during the class. The computer codes will be used for analyzing accuracy, efficiency, stability and convergence of the investigated numerical methods.

* The simulation of several real world physical and engineering problems will be performed.

* Matlab will be used as standard software for the implementation of numerical methods.

* The course will provide a firm basis for future study in numerical computation.

At the end of the course the students will gain sufficient knowledge to do numerical analysis and to write computer programms for practical problems.

Recommended Books:-

* G.D. Smith, Numerical Solutions of Partial Differential Equations: Finite Difference Method, Oxford University Press, 1986.

* J.W. Thomas, Numerical Partial Differential

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Equations, Springer-Verlag New York, Inc. 1995

* G.A. Evans, J. Blackledge, P. Yardeley, Numerical Methods for Partial Differential Equations, Springer Berlin Heidelberg. 1999.

* C. Johnson, Numerical solutions of Partial Differential Equations by the Finite Element Method, Dover, 2009.

Course Contents:-

* **Parabolic Equations:-** Explicit finite difference approximation, implicit methods, derivative boundry conditions, the local truncation error, stability analysis. Finite difference methods on rectangular grids in two space dimensions. Finite element method for parabolic equations in one and two space dimensions.

* **Hyperbolic Equations:-** Analytic solution of linear and quasi-linear equations. finite difference method on rectangular mesh for linear first order equation, stability analysis, system of linear first order equations.

* **Elliptic Equations:-** Finite difference method in rectangular co-ordinates, Finite element method for elliptic problems in one and two space dimensions.

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⇒ PDE's are frequently used to model many of the phenemona in
(i) Science (ii) Engineering (iii) Economics

⇒ PDE's are used to studying
(i) Chemical Reactions (ii) Astrophysical Phenomen
(iii) Aircraft Design (iv) Biological Pupulation
(v) Image Processing (vi) Finantial Models.

⇒ **Our Aim:-** The aim of this course is to gain both an understanding of the methedology behind numerical methods for PDE's and an appreciation of the complex issues that are unique to the field of scientific computing. Various numerical methods will be evaluated in the context of specific physical models, for example fluid flows. We will see that the first step in an accurate and robust manner. The second step requires implementing an efficient and correct numerical algorithm. The final step requires interpreting the results using intuition and mathematical reasoning.

⇒ **Conservation Laws:-** A large number of PDE's arise from the physical principle of conservation. Physicists have always been interested in describing changes in the world surrounding us. By observation, theory and experiment certain concepts have been arrived at, among which

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the concept that one can define physical quantities that remain same during some process. These quantities are said to be conserved. Typically a quantity is conserved in hypothesized isolated system, In reality no system is truely isolated and the most interesting applications come about when we study the interaction of two or more systems. This leads to the question of how one can follow the changes in physical quantities of the separate systems!!

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⇒ **Partial Differential Equation:-** A PDE is an equation involving an unknown function of two or more variables and certain of its partial derivatives.

★ **Linear First Order PDE's:-** A general linear PDE of 1st order in two space dimension has the form

$$a u_x + b u_y + c u + d = 0, \longrightarrow ①$$

where $u_x := \frac{\partial u}{\partial x}$ and $u_y := \frac{\partial u}{\partial y}$ are partial derivatives and a, b, c, d are known co-efficients which can also depend on (x, y).

★ **Linear PDE of order 2:-**

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u + g = 0 \longrightarrow ②$$

Here $u_{xy} := \frac{\partial^2 u}{\partial x \partial y}$, and the coefficients $a, b, \ldots, g$ can be functions of (x, y).

★ **Non-Linear Equation:-** If the coefficients are functions of u or derivatives of u i.e

$$a = a(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$$

$$\left.\begin{aligned} & u_t + u u_x = 0 \\ & u_t + \left(\tfrac{1}{2} u^2\right)_x = 0 \end{aligned}\right\} \text{Burger's Equation} \rightarrow \text{simplest non-linear Equation.}$$

★ **Quasilinear PDE's :-** A PDE is Quasi-linear if it is linear in the highest order derivatives with co-efficients depending on the independent variables, The unknown function and its derivatives of order

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lower than the order of the equation.

★ **Homogeneous PDE's:-** A PDE is called homogeneous if the equation does not contain a term independent of the unknow function and its derivatives. For example $u_t + f(u) = 0$. otherwise the PDE is called inhomogeneous. For example $u_t + f(u) = g(t, x)$

★ **Classification of Linear PDEs of order 2:-**
The linear PDE in ② at point (x, y) is called
Elliptic $\Leftrightarrow b^2 - 4ac < 0$ [Boundry value Problems]
Parabolic $\Leftrightarrow b^2 - 4ac = 0$ [Initial & B.V Problems]
Hyperbolic $\Leftrightarrow b^2 - 4ac > 0$ [Initial value problems]

★ First order PDEs are always Hyperbolic.

★ The type of equation tells us something about the solution and how we should go about solving it.

★ **Different PDEs & Their Applications**

1:- The laplace equation $u_{xx} + u_{yy} = 0$
$b^2 - 4ac = -4 < 0$ (Elliptic)
This equation appears in the following applications
★ Steady State Temperature.
★ Steady State Electric Field (voltage).
★ Inviscid Fluid Flow.

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* Gravitational Field.

2:- The heat conduction or diffusion equation

$$u_t = u_{xx} \quad \text{or} \quad u_t = \Delta u$$

$$b^2 - 4ac = 0 \quad \text{(Parabolic)}$$

Applications.

* Conduction of heat in bars & solids.
* Diffusion of concentration of liquid or gaseous substance in physical chemistry.
* Diffusion of neutrons in ~~viscous fluid flow~~.
* Diffusion of vorticity in viscous fluid flow
* Telegraphic transmission in cables of low inductance or capacitance.
* Equilization of charge in electromagnetic theory.
* Long wavelength electromagnetic waves in highly conducting medium.
* Slow motion in hydrodynamics.
* Evolution of probability distribution in random process.

3:- The wave equation $u_{tt} = c^{-2} u_{xx}$ or $u_{tt} = c^{-2} \Delta u$

$$b^2 - 4ac = 4c^{-2} > 0 \quad \text{(Hyperbolic)}$$

This equation appears in the following applications.

* Linearized supersonic airflow.
* Sound waves in a tube or a pipe.
* Longitudinal vibrations of a bar.
* Torsional oscillations of a rod.
* Vibration of a flexible string.
* Transmission of electricity along an insulated low-risistance cable.

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* Long water waves in a straight canal.

This classification is sencible for linear PDEs with constant coefficients, because the three types have certain characteristic properties. The solutions of parabolic equation are typically smooth for the whole time of simulation. The solution of elliptic equations are smooth for smooth initial data. The solutions of hyperbolic equations are however generate ~~of~~ ~~mixed type~~ singularities, possibly in the form of shockes.

The system of n equations is hyperbolic if n real characteristics exist. If all the characteristics are complex, the system is elliptic. If some are real and some complex, the system is of mixed type. If the system is of rank less than n, then we have a parabolic system.

⇒ **Numerical Solution of PDE's:-** Continuous space will be replaced by discrete space.

$h = \frac{b-a}{n}$, h = constant = length of interval

n = number of discretization points.

$x_j = a + jh$, $j = 1, 2, 3, \ldots, n$

$x_0 = a$, $x_1 = a + h$

$x_n = a + nh \implies x_n = b$

Demand: If $n \to \infty$ or $h \to 0$

$u_h \to u$, u_h = numerical solution and

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u = analytic or exact solution.

$$u(x+h) = u(x) + h u_x + \frac{h^2}{2!} u_{xx} + \frac{h^3}{3!} u_{xxx} + \cdots \longrightarrow ①$$

$$u(x-h) = u(x) - h u_x + \frac{h^2}{2!} u_{xx} - \frac{h^3}{3!} u_{xxx} + \cdots \longrightarrow ②$$

$$① - ② \Rightarrow u(x+h) - u(x-h) = 2h u_x + \frac{h^3}{3} u_{xxx}$$

$$\Rightarrow u_x = \frac{u(x+h) - u(x-h)}{2h} + O(h^2)$$

Difference formulae

$$① + ② \Rightarrow u_{xx} = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} + O(h^2)$$

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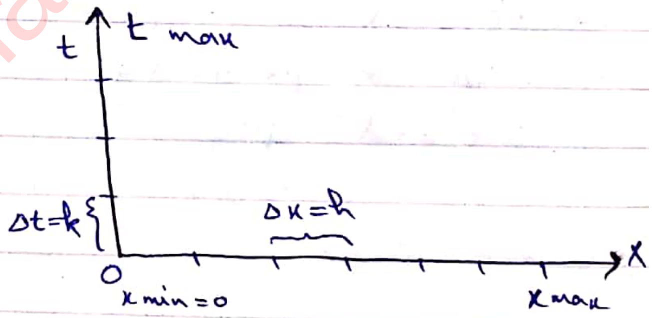
⇒ Finite Difference Method:-

(For Parabolic Equations)

* Heat Equation:-



$$u_t = a\, u_{xx}$$

$$u(0,x) = u_0(x)$$

$$u(t,0) = 0, \quad u(t,l) = 0$$

Let $l = 1 \Rightarrow 0 \le x \le 1$

$$u_t \approx \frac{u(t+k,x) - u(t,x)}{k} + O(k)$$

$$u_{xx} \approx \frac{u(t,x-h) - 2u(t,x) + u(t,x+h)}{h^2} + O(h^2)$$

$$\Rightarrow u_t = \frac{u_j^{n+1} - u_j^n}{k} \quad \& \quad u_{xx} = \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2}$$

(j for x & n for time)

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$$u_t = a\,u_{xx}$$

$$\Rightarrow \frac{u_j^{n+1} - u_j^n}{k} = a\,\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2}$$ { Explicit Sch or 2 level sch

$$\Rightarrow u_j^{n+1} - u_j^n = \frac{ak}{h^2}\left[u_{j-1}^n - 2u_j^n + u_{j+1}^n\right]$$

$$\Rightarrow u_j^{n+1} - u_j^n = a\lambda\left[u_{j-1}^n - 2u_j^n + u_{j+1}^n\right], \text{ where } \lambda = \frac{k}{h^2} \text{ const}$$

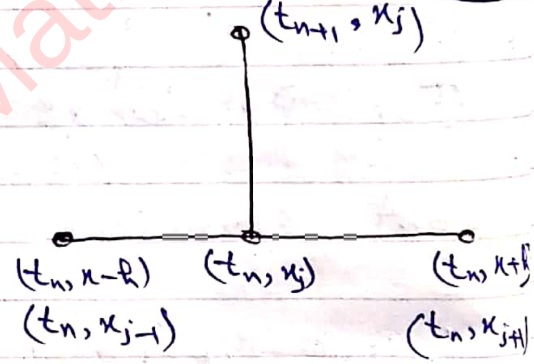
$$\Rightarrow u_j^{n+1} = u_j^n + a\lambda\left[u_{j-1}^n - 2u_j^n + u_{j+1}^n\right], \quad j = 1, 2, \cdots, N, \quad n = 0, 1, \cdots, N$$

$$\Rightarrow u_j^{n+1} = (1 - 2a\lambda)\,u_j^n + a\lambda\left(u_{j-1}^n + u_{j+1}^n\right)$$

j اس لیے 1 سے شروع کیا
چونکہ 0 پر ہمیں ویلیو دی ہوئی
ہوئی ہے.

***Explicit Scheme:-**

Each value at finite level t_{n+1} can be independently calculated from valued at time level t_n.



$\Rightarrow$ Truncation Error:-

$$u_t = a u_{xx} \quad \Rightarrow \quad u_t - a u_{xx} = 0$$

$$0 \neq L(u, t, x, k, h) = \frac{u(t+k, x) - u(t, x)}{k} - a\,\frac{u(t, x-h) - 2u(t, x) + u(t, x+h)}{h^2}$$

$L(u, t, x, k, h)$ = Truncation Error

$$u(t+k, x) = u(t, x) + k\,u_t(t, x) + \frac{k^2}{2!}u_{tt}(t, x) + O(k^3)$$

$$\frac{u(t+k, x) - u(t, x)}{k} = u_t(t, x) + \frac{k}{2}u_{tt}(t, x) + O(k^2)$$

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$$u(t, x+h) = u(t,x) + h u_x(t,x) + \frac{h^2}{2!} u_{xx}(t,x) + \frac{h^3}{3!} u_{xxx}(t,x) + \frac{h^4}{4!} u_{xxxx}(t,x) + O(h^5)$$

$$u(t, x-h) = u(t,x) - h u_x(t,x) + \frac{h^2}{2!} u_{xx}(t,x) - \frac{h^3}{3!} u_{xxx}(t,x) + \frac{h^4}{4!} u_{xxxx}(t,x) - O(h^5)$$

$$\Rightarrow \frac{u(t,x-h) - 2u(t,x) + u(t,x+h)}{h^2} = u_{xx} + \frac{h^2}{12} u_{xxxx} + O(h^4)$$

$$\Rightarrow \text{Truncation Error} = u_t(t,x) + \frac{k}{2} u_{tt} + O(k^2) - a\left[u_{xx} + \frac{h^2}{12} u_{xxxx} + O(h^4)\right]$$

$$= (u_t - a u_{xx}) + O(k, h^2)$$

$$= 0 + O(k, h^2)$$ $\left\{\right.$ Because of the given PDE $u_t - a u_{xx} = 0$

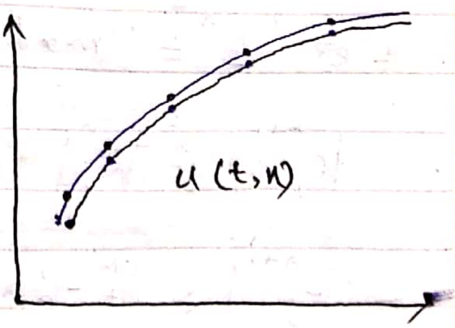
** **

★ If $h, k \to 0$ the truncation error become to zero and numerical solution approaches to analytic solution.

★ $O(k, h^2) \Rightarrow$ Error is reducing linearly if order is 1 and quadratically if order is 2 and so on.

★ $e_j^n = e(t_n, x_j) = |u(t_n, x_j) - u_j^n|$

Exact value — Numerical value



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★ L_1-Norm $= \int_0^\infty |f(x)|\,dx$

L_1 Error $= \sum_{j=1}^{N} |u(t_n, x_j) - u_j^n| \cdot h$ → height of diff → interval

★ L_2-Norm $= \sqrt{\int_0^\infty |f(x)|^2\,dx}$

L_2-Error $= \sqrt{\sum_{j=1}^{N} |u(t_n, x_j) - u_j^n|^2 \cdot h}$

⇒ **Convergence** :- The scheme is convergent if as $h \to 0$, $k \to 0$ $\forall (t^*, x^*)$, when $x_j \to x^*$, $t_n \to t^*$

$$u_j^n \to u(t^*, x^*)$$

Let $\underset{\text{Num}}{e_j^n = u_j^n} - \underset{\text{exact}}{u(t_n, x_j)}$

Numerical Scheme:-

$$u_j^{n+1} = u_j^n + a\lambda[u_{j-1}^n - 2u_j^n + u_{j+1}^n] \longrightarrow (1)$$

$$u(t_{n+1}, x_j) = u(t_n, x_j) + a\lambda[u(t_n, x_{j-1}) - 2u(t_n, x_j) + u(t_n, x_{j+1})] + L_1^n k \longrightarrow (2)$$

$(1) - (2) \Rightarrow e_j^{n+1} = e_j^n + a\lambda(e_{j-1}^n - 2e_j^n + e_{j+1}^n) + kL_j^n \to (*)$

Let $E^n = \max_j \{|e_j^n|\}$, $L = \max_j |L_j^n|$

$$|e_j^{n+1}| \leq |e_j^n| + |a\lambda|\{|e_{j-1}^n| - 2|e_j^n| + |e_{j+1}^n|\} + kL_j^n$$

$$\Rightarrow |e_j^{n+1}| \leq (1 - 2a\lambda)|e_j^n| + |a\lambda|[|e_{j-1}^n| + |e_{j+1}^n|] + k|L_j^n|$$

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taking maximum on both side

$$E^{n+1} \leq |1-2a\lambda| E^n + 2|a\lambda| E^n + k\hat{L}$$

$\because$ n time Max Error $= E^n$

assume $2a\lambda < 1 \quad \because \frac{k}{h^2} < 1$

$$\Rightarrow E^{n+1} \leq (1-2a\lambda)E^n + 2a\lambda E^n + k\hat{L}$$

$$\Rightarrow E^{n+1} \leq E^n + k\hat{L}$$

& $E^n \leq E^{n-1} + k\hat{L}$

$\leq E^{n-2} + 2k\hat{L}$

.... so on

$\leq E^0 + nk\hat{L}$

$\Rightarrow E^n \leq E^0 + t_f \hat{L}$ $\quad \because t_f$ final time

$\Rightarrow E^n \leq t_f \hat{L}$ $\quad \because$ no error in the initial data

If $\left.\begin{matrix} h \to 0, \\ k \to 0 \end{matrix}\right\} \hat{L} \to 0 \Rightarrow E^n \to 0$

** ——————————— **

★ Classical Problem of Numerical Analysis:-

★ i) **Existance & Uniqueness:-** The u_k exists and is unique.

ii) **Consistency:-** A small residuum (remaining part of Tayler series) is obtained after subtituting the exact solution u of the differential equation in the

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discretization formation (The numerical method of the formulation). Moreover, this residuum tends to zero when $h \to 0$.

iii) **Stability:-** The solution u_h remain bounded for $h \to 0$ (in a clear sense)

iv) **Convergence:-** for $h \to 0$, the discrete solution u_h tends (converges) to continuous solution u (again in a clear sense)

⇒ **Stability Analysis:-** {Fourier series is used as a basic tool}

Initial Value Problem

$$u_t = u_{xx}, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(0,x) = f(x), \quad u(t,-\infty) = 0 = u(t,\infty)$$

★ Fourier Transform:-

$$\hat{v}(t,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} v(t,x)\, dx$$

$$\hat{v}_t(t,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} v_t(t,x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} v_{xx}(t,x)\, dx \qquad \because u_t = u_{xx}$$

$$= \frac{-\omega^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} v(t,x)\, dx$$

{∵ Integrating by Part, Terms at zero & infinity are zero

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$$\Rightarrow \hat{v}_t(t,\omega) = -\omega^2 \hat{v}(t,\omega)$$

$$\Rightarrow \hat{v}_t + \omega^2 \hat{v} = 0 \quad \text{is ODE}$$

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اپلائی کرتے ہیں تو وہ ODE بن جاتی ہے

★ Inverse Fourier Transform:-

$$v(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{v}(t,\omega)\, d\omega$$

$$\|v\|_2 = \|\hat{v}\|_2 \quad \text{Parsvil's Identity}$$

where $\|\cdot\|_2 \longrightarrow L_2(\mathbb{R})$

★ In Discrete space:-

Fourier Transform $\Rightarrow \hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-ij\xi} u_j \,,$

$$\xi \in [-\pi, \pi]$$

Parsvil's Identity $\Rightarrow \|\hat{u}\|_2 = \|u\|$

Both fourier transform and Parsvil's identity are the basic tools for stability analysis.

$\Rightarrow$ **Definition:-** The two level scheme

$$u^{n+1} = Q u^n$$

is said to be stable with respect to norm $(\|\cdot\|)$ if there exist positive constants h and k and non-negative constants β s.t

$$\|u\| \leq k e^{\beta t} \|u_0\| \qquad \begin{matrix} 0 \leq t \leq nk \\ 0 \leq x \leq Nh \end{matrix}$$

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similarly $\|\hat{u}\| \leq k\, e^{\beta t} \|\hat{u}_0\|$
$\left\{ \begin{array}{l} \text{Final solution is} \\ \text{bounded by the} \\ \text{initial solution} \end{array} \right.$

$\Rightarrow \|\hat{u}\| = \|u\|$

★ Explicit Method for $u_t = a u_{xx}$

$$u_j^{n+1} = u_j^n + a\lambda \left[u_{j-1}^n - 2u_j^n + u_{j+1}^n \right], \quad j = 1, 2, \dots N$$

$$\frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-ij\xi} u_j^{n+1} = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-ij\xi} \left[u_j^n + a\lambda (u_{j-1}^n - 2u_j^n + u_{j+1}^n) \right]$$

$$\Rightarrow \hat{u}^{n+1}(\xi) = (1-2a\lambda)\hat{u}(\xi) + \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-ij\xi} a\lambda (u_{j-1}^n + u_{j+1}^n)$$

Let $m = j \mp 1 \Rightarrow j = m \pm 1$, $m \in (-\infty, \infty)$

$$\Rightarrow \hat{u}^{n+1}(\xi) = (1-2a\lambda)\hat{u}^n(\xi) + \frac{a\lambda}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-i(m+1)\xi} u_m^n + \frac{a\lambda}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-i(m-1)\xi} u_m^n$$

$\boxed{\text{change of variable}}$

$$= (1-2a\lambda)\hat{u}^n(\xi) + a\lambda \left[e^{-i\xi} + e^{i\xi} \right] \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi} u_m^n$$

$$= \left[1 - 2a\lambda + (a\lambda)(e^{-i\xi} + e^{i\xi}) \right] \hat{u}^n(\xi)$$

$$= \left[1 - 2a\lambda + 2a\lambda \cos\xi \right] \hat{u}^n(\xi)$$

$\because \cos\theta = \dfrac{e^{-i\theta} + e^{i\theta}}{2}$

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$$\Rightarrow \hat{u}^{n+1}(\xi) = \left[1-2a\lambda(1-\cos\xi)\right]\hat{u}^n(\xi)$$

$$= \left[1-4a\lambda\sin^2\frac{\xi}{2}\right]\hat{u}^n(\xi) \qquad \boxed{\because \sin^2\theta = \frac{1-\cos 2\theta}{2}}$$

Induction

$$\hat{u}^n(\xi) = \left[1-4a\lambda\sin^2\frac{\xi}{2}\right]\hat{u}^{n-1}(\xi)$$

$$\Rightarrow \hat{u}^n(\xi) = \underbrace{\left[1-4a\lambda\sin^2\frac{\xi}{2}\right]}^n\hat{u}^0(\xi)$$

Damping factor representing by D

$$\|\hat{u}^n(\xi)\| = |D|^n\|\hat{u}^0(\xi)\|$$

for stability $|D| < 1$

$\Rightarrow$ stability condition is

$$|D| = \left|1-4a\lambda\sin^2\frac{\xi}{2}\right| < 1$$

$$\Rightarrow -1 < 1-4a\lambda\sin^2\frac{\xi}{2} < 1$$

R.H.S:- $1-4a\lambda\sin^2\frac{\xi}{2} < 1$

$$\Rightarrow 4a\lambda\sin^2\frac{\xi}{2} > 1$$

$$\Rightarrow a\lambda > 0$$

L.H.S:- $-1 < 1-4a\lambda\sin^2\frac{\xi}{2}$

$$\Rightarrow 4a\lambda\sin^2\frac{\xi}{2} < 2$$

$$\Rightarrow a\lambda\sin^2\frac{\xi}{2} < \frac{1}{2}$$

$$\Rightarrow a\lambda < \frac{1}{2\sin^2\frac{\xi}{2}}$$

$$\Rightarrow 0 < a\lambda < \frac{1}{2}$$

$\because$ min value of $\frac{1}{2\sin^2\xi/2} = \frac{1}{2}$ because max value of $\sin^2\xi/2 = 1$

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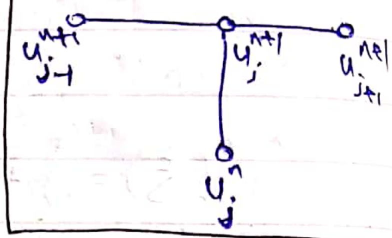
★ Implicit Scheme :- (for $u_t = a u_{xx}$)

$$\frac{u_j^{n+1} - u_j^{n}}{k} = a \, \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{h^2}$$

(Computational Molecule)

Stencil



$$\Rightarrow u_j^{n+1} - u_j^{n} = a\lambda \left[u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1} \right]$$

$$\Rightarrow -a\lambda \, u_{j-1}^{n+1} + (1+2a\lambda) u_j^{n+1} - a\lambda u_{j+1}^{n+1} = u_j^{n} \quad , j = 1, 2, 3, \dots, N$$

for $j=1$

$$-a\lambda u_0^{n+1} + (1+2a\lambda) u_1^{n+1} - a\lambda u_2^{n+1} = u_1^{n}$$

$$\Rightarrow (1+2a\lambda) u_1^{n+1} - a\lambda u_2^{n+1} = \underbrace{u_1^{n} + a\lambda u_0^{n+1}}_{\text{known}}$$

for $j=2$

$$-a\lambda u_1^{n+1} + (1+2a\lambda) u_2^{n+1} - a\lambda u_3^{n+1} = u_2^{n} \longrightarrow \text{known}$$

for $j=3$

$$-a\lambda u_2^{n+1} + (1+2a\lambda) u_3^{n+1} - a\lambda u_4^{n+1} = u_3^{n} \longrightarrow \text{known}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

for $j=N-1$

$$-a\lambda u_{N-2}^{n+1} + (1+2a\lambda) u_{N-1}^{n+1} - a\lambda u_N^{n+1} = u_{N-1}^{n} \longrightarrow \text{known}$$

for $j=N$

$$-a\lambda u_{N-1}^{n+1} + (1+2a\lambda) u_N^{n+1} - a\lambda u_{N+1}^{n+1} = u_N^{n}$$

$$\Rightarrow -a\lambda u_{N-1}^{n+1} + (1+2a\lambda) u_N^{n+1} = \underbrace{u_N^{n} + a\lambda u_{N+1}^{n+1}}_{\text{known}}$$

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$$\Rightarrow \begin{bmatrix} (1+2a\lambda) & -a\lambda & 0 & 0 & 0 & 0 \\ -a\lambda & (1+2a\lambda) & -a\lambda & 0 & 0 & 0 \\ & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & -a\lambda & 1+2a\lambda & -a\lambda & \\ 0 & 0 & 0 & -a\lambda & 1+2a\lambda \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ \vdots \\ u_{N-1}^{n+1} \\ u_N^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} u_1^n + a\lambda u_0^{n+1} \\ u_2^n \\ u_3^n \\ \vdots \\ u_{N-1}^n \\ u_N^n + a\lambda u_{N+1}^{n+1} \end{bmatrix}$$

A = known coeffici–

b^n = known values

$$\Rightarrow A u^{n+1} = b^n \quad \Rightarrow \underbrace{u^{n+1} = A^{-1} b^n}_{\text{Solution}}$$

★ Explicit scheme is conditionally stable

★ Implicit scheme is unconditionally stable.

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⇒ Stability Analysis :- (for Implicit scheme) [Von-Newmann Stability Condition]

$$u_j^{n+1} = u_j^n + a\lambda\left[u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}\right] \quad \left\{\text{Numerical Scheme}\right.$$

$$\Rightarrow u_j^{n+1} = u_j^n + a\lambda u_{j-1}^{n+1} - 2a\lambda u_j^{n+1} + a\lambda u_{j+1}^{n+1}$$

$$\Rightarrow -a\lambda u_{j-1}^{n+1} + (1+2a\lambda)u_j^{n+1} - a\lambda u_{j+1}^{n+1} = u_j^n \,; \quad j=1,2,\dots,N$$

$$\Rightarrow \frac{-a\lambda}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-ij\xi} u_{j-1}^{n+1} + \frac{(1+2a\lambda)}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-ij\xi} u_j^{n+1} - \frac{a\lambda}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-ij\xi} u_{j+1}^{n+1} = \frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-ij\xi} u_j^n \qquad \xi\in[-\pi,\pi)$$

$$\Rightarrow \frac{-a\lambda}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-ij\xi} u_{j-1}^{n+1} + (1+2a\lambda)\hat{u}^{n+1}(\xi) - \frac{a\lambda}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-ij\xi} u_{j+1}^{n+1} = \hat{u}^n(\xi)$$

Put $j\pm 1 = m \Rightarrow j = m\mp 1$

$$\Rightarrow \frac{-a\lambda}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty} e^{-i(m+1)\xi} u_m^{n+1} + (1+2a\lambda)\hat{u}^{n+1}(\xi) - \frac{a\lambda}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty} e^{-i(m-1)\xi} u_m^{n+1} = \hat{u}^n(\xi)$$

$$\Rightarrow -a\lambda\hat{u}^{(n+1)}(\xi)\left[e^{-i\xi}\right] + (1+2a\lambda)\hat{u}^{n+1}(\xi) - a\lambda\hat{u}^{n+1}(\xi)\left[e^{i\xi}\right] = \hat{u}^n(\xi)$$

$$\Rightarrow \left[-a\lambda e^{-i\xi} + (1+2a\lambda) - a\lambda e^{i\xi}\right]\hat{u}^{n+1}(\xi) = \hat{u}^n(\xi)$$

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$$\Rightarrow \left[-a\lambda(e^{-i\xi}+e^{i\xi}) + (1+2a\lambda)\right]\hat{u}^{n+1}(\xi) = \hat{u}^{n}(\xi)$$

$$\Rightarrow \left[-2a\lambda\cos(\xi) + (1+2a\lambda)\right]\hat{u}^{n+1}(\xi) = \hat{u}^{n}(\xi)$$

$$\Rightarrow \left[1+2a\lambda-2a\lambda\cos(\xi)\right]\hat{u}^{n+1}(\xi) = \hat{u}^{n}(\xi)$$

$$\Rightarrow \left[1+2a\lambda(1-\cos\xi)\right]\hat{u}^{n+1}(\xi) = \hat{u}^{n}(\xi)$$

$$\Rightarrow \left[1+2a\lambda\left(2\sin^2 \xi/2\right)\right]\hat{u}^{n+1}(\xi) = \hat{u}^{n}(\xi)$$

$\boxed{\because 1-\cos 2\theta = 2\sin^2\theta}$

$$\Rightarrow \left(1+4a\lambda\sin^2 \xi/2\right)\hat{u}^{n+1}(\xi) = \hat{u}^{n}(\xi)$$

$$\Rightarrow \hat{u}^{n+1}(\xi) = \left[\frac{\hat{u}^{n}(\xi)}{1+4a\lambda\sin^2 \xi/2}\right]$$

Induction

$$\hat{u}^{n}(\xi) = \frac{1}{1+4a\lambda\sin^2 \xi/2}\,\hat{u}^{n-1}(\xi)$$

$$\hat{u}^{n}(\xi) = \frac{1}{\left[1+4a\lambda\sin^2(\xi/2)\right]^n}\,\hat{u}^{0}(\xi)$$

$$\|\hat{u}^{n}(\xi)\| = \frac{1}{\left|1+4a\lambda\sin^2 \xi/2\right|}\,\|\hat{u}^{0}(\xi)\|$$

for stability condition

$$|D| < 1 \Rightarrow \frac{1}{\left|1+4a\lambda\sin^2 \xi/2\right|} < 1$$

$$\Rightarrow \left|1+4a\lambda\sin^2 \xi/2\right| > 1$$

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$$\Rightarrow \quad -1 > 1 + 4a\lambda \sin^2 \xi/2 > 1$$

$-1 > 1 + 4a\lambda \sin^2 \xi/2$

$\Rightarrow -2 > 4a\lambda \sin^2 \xi/2$

$\Rightarrow \frac{-2}{4} > a\lambda \sin^2 \xi/2$

$\Rightarrow \frac{-1}{2} > a\lambda \sin^2 \xi/2$

false $\because a\lambda > 0$ & $\sin^2 \xi/2 > 0$

$1 + 4a\lambda \sin^2 \xi/2 > 1$

$\Rightarrow 4a\lambda \sin^2 \xi/2 > 2$

$\Rightarrow a\lambda \sin^2 \xi/2 > \frac{1}{2}$

$\Rightarrow a\lambda > \frac{1}{2\sin^2 \xi/2}$

$\Rightarrow a\lambda > \frac{1}{2}$

$$\Rightarrow a\lambda > \frac{1}{2} \quad \text{Ans.}$$

⇒ Truncation Error:- [for Implicit Scheme]

$$u_j^{n+1} = u_j^n + a\lambda[u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}]$$

$$\Rightarrow u_j^{n+1} - u_j^n = a \cdot \frac{k}{h^2}[u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}] \qquad \because \lambda = \frac{k}{h^2}$$

$$\Rightarrow \frac{u_j^{n+1} - u_j^n}{k} = a\left[\frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{h^2}\right]$$

$$T.E = \frac{u_j^{n+1} - u_j^n}{k} - a\left[\frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{h^2}\right]$$

$$= \frac{u(t+k,x) - u(t,x)}{k} - a\left[\frac{u(t+k,x-h) - 2u(t+k,x) + u(t+k,x+h)}{h^2}\right]$$

$\longrightarrow$ ⊛

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Now

$$u(t+k,x) = u(t,x) + k u_t(t,x) + \frac{k^2}{2!} u_{tt}(t,x) + O(k^3)$$

$$\Rightarrow \frac{u(t+k,x) - u(t,x)}{k} = u_t(t,x) + \frac{k}{2} u_{tt}(t,x) + O(k^2) \longrightarrow (1)$$

$$u(t+k,x+h) = u(t+k,x) + h u_x(t+k,x) + \frac{h^2}{2!} u_{xx}(t+k,x) + \frac{h^3}{3!} u_{xxx}(t+k,x) + \frac{h^4}{4!} u_{xxxx}(t+k,x) + O(h^5) \longrightarrow (ii)$$

$$u(t+k,x-h) = u(t+k,x) - h u_x(t+k,x) + \frac{h^2}{2!} u_{xx}(t+k,x) - \frac{h^3}{3!} u_{xxx}(t+k,x) + \frac{h^4}{4!} u_{xxxx}(t+k,x) - O(h^5) \longrightarrow (iii)$$

Adding (ii) & (iii)

$$u(t+k,x-h) + u(t+k,x+h) = 2u(t+k,x) + h^2 u_{xx}(t+k,x) + \frac{h^4}{12} u_{xxxx}(t+k,x^2) + O(h^6)$$

$$\Rightarrow \frac{u(t+k,x-h) - 2u(t+k,x) + u(t+k,x+h)}{h^2} = u_{xx}(t+k,x) + \frac{h^2}{12} u_{xxxx}(t+k,x) + O(h^4) \longrightarrow (2)$$

Put (1) & (2) in (*) implies

$$T.E = u_t(t,x) + \frac{k}{2} u_{tt}(t,x) + O(k^2) - a\left[u_{xx}(t+k,x) + \frac{h^2}{12} u_{xxxx}(t+k,x) + O(h^4)\right]$$

$$= u_t(t,x) + \frac{k}{2} u_{tt}(t,x) + O(k^2) - a\left[u_{xx}(t,x) + k u_{xxt}(t,x) + \frac{k^2}{2!} u_{xxtt}(t,x) + O(k^3) + \frac{h^2}{12} u_{xxxx}(t,x) + O(h^4)\right]$$

$\because u_t = a u_{xx} \Rightarrow u_t - a u_{xx} = 0$

$\; u_{tt} = a u_{xxt}$

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★ In paper take short cuts like

Stability Analysis

$$u_j^{n+1} = u_j^n + a\lambda(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1})$$

$$\Rightarrow -a\lambda u_{j-1}^{n+1} + (1+2a\lambda)u_j^{n+1} - a\lambda u_{j+1}^{n+1} = u_j^n$$

$$\Rightarrow -a\lambda e^{-i\xi}\hat{u}^{n+1}(\xi) + (1+2a\lambda)\hat{u}^{n+1}(\xi) - a\lambda e^{i\xi}\hat{u}^{n+1}(\xi) = \hat{u}^n(\xi)$$

$$\Rightarrow -a\lambda(e^{-i\xi} + e^{i\xi})\hat{u}^{n+1}(\xi) + (1+2a\lambda)\hat{u}^{n+1}(\xi) = \hat{u}^n(\xi)$$

$$\Rightarrow [-a\lambda(2\cos\xi) + (1+2a\lambda)]\hat{u}^{n+1}(\xi) = \hat{u}^n(\xi)$$

$$\Rightarrow [-2a\lambda\cos\xi + (1+2a\lambda)]\hat{u}^{n+1}(\xi) = \hat{u}^n(\xi)$$

$$\Rightarrow [1 + 4a\lambda\sin^2\xi/2]\,\hat{u}^{n+1}(\xi) = \hat{u}^n(\xi)$$

{ So on

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★ θ-Method :- (for $u_t = a\, u_{xx}$)

$$\frac{u_j^{n+1} - u_j^n}{k} = a(1-\theta)\,\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2} + a\theta\,\frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{h^2} \quad ; \theta \in [0,1]$$

for $\theta = 0 \implies$ Explicit Method

for $\theta = 1 \implies$ Implicit Method

for $\theta = 1/2 \implies$ Crank-Nicolson Method

for $\theta \in [0, 1/2) \implies$ Explicit Method

for $\theta \in [1/2, 1] \implies$ Implicit Method

★ Crank-Nicolson Method ($\theta = 1/2$) is implicit method → Unconditionally stable.

★ $T.E \sim O(k^2, h^2)$ 2nd order in time and 2nd order in space.

⇒ **Truncation Error :-**

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{a}{2}\,\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2} + \frac{a}{2}\,\frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{h^2}$$

$$T.E = \frac{u(t+k,x) - u(t,x)}{k} - \frac{a}{2}\,\frac{u(t,x-h) - 2u(t,x) + u(t,x+h)}{h^2} - \frac{a}{2}\,\frac{u(t+k,x-h) - 2u(t+k,x) + u(t+k,x+h)}{h^2}$$

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Since for any value of θ the sum of $\theta + (1-\theta) = 1$; $\theta \in [0,1] \Rightarrow$ T-E is $O(k^2, h^2)$ for any value of θ

$$\Rightarrow T.E = u_t(t,x) + \frac{k}{2} u_{tt}(t,x) + O(k^2) - \frac{a}{2}\Big[u_{xx}(t,x) + \frac{h^2}{12} u_{xxxx}(t,x) + O(h^4)\Big] - \frac{a}{2}\Big[u_{xx}(t+k,x) + \frac{h^2}{12} u_{xxxx}(t+k,x) + O(h^4)\Big]$$

$$= u_t + \frac{k}{2} u_{tt} + O(k^2) - \frac{a}{2}\Big[u_{xx} + \frac{h^2}{12} u_{xxxx} + O(h^4)\Big] - \frac{a}{2}\Big[u_{xx} + k u_{xxt} + \frac{k^2}{2!} u_{xxtt} + O(k^3) + \frac{h^2}{12} u_{xxxx} + O(h^4)\Big]$$

$$= (u_t - a u_{xx}) + \frac{k}{2}(u_{tt} - a u_{xxt}) + O(k^2) - \frac{a}{2}\Big[\frac{h^2}{12} u_{xxxx} + O(h^4)\Big] - \frac{a}{2}\Big[\frac{k^2}{2} u_{xxtt} + O(k^3) + \frac{h^2}{12} u_{xxxx} + O(h^4)\Big]$$

$$= 0 + \frac{k}{2}(0) + O(k^2, h^2)$$

$\because u_{tt} = u_{xxt}$; $u_t - a u_x = 0$ given

$$\Rightarrow T.E = O(k^2, h^2)$$

⇒ Stability Analysis :-

$$u_j^{n+1} = u_j^n + \frac{a\lambda}{2}\Big[u_{j-1}^n - 2u_j^n + u_{j+1}^n\Big] + \frac{a\lambda}{2}\Big[u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}\Big]$$

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$$\Rightarrow u_j^{n+1} - \frac{2}{2}\left[u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}\right] = u_j^n + \frac{2}{2}\left[u_{j-1}^n - 2u_j^n + u_{j+1}^n\right] \quad \because a\lambda = 2$$

$$\Rightarrow \hat{u}^{n+1}(\xi) - \frac{2}{2}\left[e^{-i\xi}\hat{u}^{n+1}(\xi) - 2\hat{u}^{n+1}(\xi) + e^{i\xi}\hat{u}^{n+1}(\xi)\right] = \hat{u}^n(\xi) + \frac{2}{2}\left[e^{-i\xi}\hat{u}^n(\xi) - 2\hat{u}^n(\xi) + e^{i\xi}\hat{u}^n(\xi)\right]$$

$$\Rightarrow \left[1 - \frac{2}{2}\left(e^{-i\xi} + e^{i\xi} - 2\right)\right]\hat{u}^{n+1}(\xi) = \left[1 + \frac{2}{2}\left(e^{-i\xi} + e^{i\xi} - 2\right)\right]\hat{u}^n(\xi)$$

$$\Rightarrow \left[1 - \frac{2}{2}(2\cos\xi - 2)\right]\hat{u}^{n+1}(\xi) = \left[1 + \frac{2}{2}(2\cos\xi - 2)\right]\hat{u}^n(\xi)$$

$$\Rightarrow \left[1 + \frac{2}{2}\cdot 2(1 - \cos\xi)\right]\hat{u}^{n+1}(\xi) = \left[1 - \frac{2}{2}\cdot 2(1 - \cos\xi)\right]\hat{u}^n(\xi)$$

$$\Rightarrow \left[1 + 2\left(2\sin^2 \tfrac{\xi}{2}\right)\right]\hat{u}^{n+1}(\xi) = \left[1 - 2\left(2\sin^2 \tfrac{\xi}{2}\right)\right]\hat{u}^n(\xi)$$

$$\Rightarrow \left(1 + 2 2\sin^2 \tfrac{\xi}{2}\right)\hat{u}^{n+1}(\xi) = \left(1 - 2 2\sin^2 \tfrac{\xi}{2}\right)\hat{u}^n(\xi)$$

$$\Rightarrow \hat{u}^{n+1}(\xi) = \left(\frac{1 - 2 2\sin^2 \frac{\xi}{2}}{1 + 2 2\sin^2 \frac{\xi}{2}}\right)\hat{u}^n(\xi)$$

$$\Rightarrow \hat{u}^n(\xi) = \left[\frac{1 - 2 2\sin^2 \frac{\xi}{2}}{1 + 2 2\sin^2 \frac{\xi}{2}}\right]\hat{u}^{n-1}(\xi)$$

Induction

$$\hat{u}^n(\xi) = \left[\frac{1 - 2 2\sin^2 \frac{\xi}{2}}{1 + 2 2\sin^2 \frac{\xi}{2}}\right]\hat{u}^0(\xi)$$

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$$\|\hat{u}^n(\xi)\| = \left|\frac{1-2r\sin^2\xi/2}{1+2r\sin^2\xi/2}\right|^n \|\hat{u}^0(\xi)\|$$

for stability condition

$$|D|<1 \Rightarrow \left|\frac{1-2r\sin^2\xi/2}{1+2r\sin^2\xi/2}\right|<1$$

$$\frac{1-2r\sin^2\xi/2}{1+2r\sin^2\xi/2} = \left|\frac{1-0\cdots}{1+0\cdots}\right| = \frac{<1}{>1}$$

$\Rightarrow$ Scheme is unconditionally stable ($\because |D|<1$ for all value of λ)

$[\because \sin^2\xi/2 \in [0,1],\ a\lambda>0,\ r$ is +ve

$$\Rightarrow \frac{1-2r\sin^2\xi/2}{1+2r\sin^2\xi/2} = \frac{1-\text{something}}{1+\text{something}} \text{ always } <1$$

★ Du Fort-Frankel Scheme:- (for $u_t = au_{xx}$)

$$\frac{u_j^{n+1}-u_j^{n-1}}{2k} = a\,\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2}$$

$$= a\,\frac{u_{j-1}^n - 2\,\dfrac{u_j^{n-1}+u_j^{n+1}}{2} + u_{j+1}^n}{h^2}$$

$$\Rightarrow \boxed{\frac{u_j^{n+1}-u_j^{n-1}}{2k} = a\,\frac{u_{j-1}^n - u_j^{n-1} - u_j^{n+1} + u_{j+1}^n}{h^2}}$$ Du Fort Frankel Scheme

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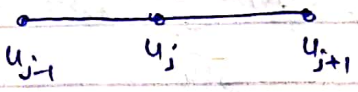
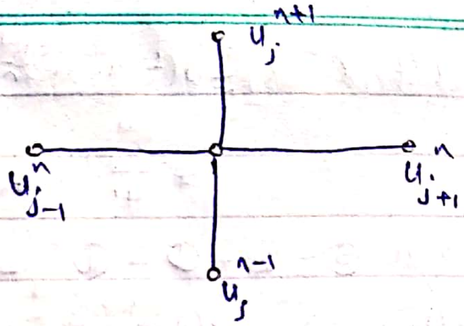
Stencil





average of $u_{j-1}, u_{j+1} = u_j$
similarly average of
$u_j^{n-1}, u_j^{n+1} = u_j^n$

$$T.E = O(h^2, k^2)$$

⇒ Truncation Error:-

$$\frac{u_j^{n+1} - u_j^{n-1}}{2k} = a\,\frac{u_{j-1}^n - 2u_j^{n-1} - u_j^{n+1} + u_{j+1}^n}{h^2}$$

$$T.E = \frac{u_j^{n+1} - u_j^{n-1}}{2k} - a\left[\frac{u_{j-1}^n - u_j^{n-1} - u_j^{n+1} + u_{j+1}^n}{h^2}\right]$$

$$= \frac{u(t+k,x) - u(t-k,x)}{2k} - a\left[\frac{u(t,x-h) - u(t-k,x) - u(t+k,x) + u(t,x+h)}{h^2}\right] \longrightarrow (A)$$

Now

$$u(t+k,x) = u(t,x) + k\,u_t(t,x) + \frac{k^2}{2!}u_{tt}(t,x) + \frac{k^3}{3!}u_{ttt}(t,x) + O(k^4) \longrightarrow ①$$

$$u(t-k,x) = u(t,x) - k\,u_t(t,x) + \frac{k^2}{2!}u_{tt}(t,x) - \frac{k^3}{3!}u_{ttt}(t,x) + O(k^4) \longrightarrow ②$$

equ ① − ② ⇒

$$u(t+k,x) - u(t-k,x) = 2k\,u_t(t,x) + \frac{k^3}{3}u_{ttt}(t,x) + O(k^5)$$

$$\Rightarrow \frac{u(t+k,x) - u(t-k,x)}{2k} = u_t(t,x) + \frac{k^2}{6}u_{ttt}(t,x) + O(k^4) \longrightarrow (*)$$

$$u(t,x-h) = u(t,x) - h\,u_x(t,x) + \frac{h^2}{2!}u_{xx}(t,x) - \frac{h^3}{3!}u_{xxx}(t,x) + \frac{h^4}{4!}u_{xxxx}(t,x) - O(h^5) \longrightarrow ③$$

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$$u(t,x+h) = u(t,x) + h u_x(t,x) + \frac{h^2}{2!} u_{xx}(t,x) + \frac{h^3}{3!} u_{xxx}(t,x) + \frac{h^4}{4!} u_{xxxx}(t,x) + O(h^5) \longrightarrow (4)$$

equ (3) − (2) − (1) + (4) $\Rightarrow$

$$u(t,x-h) - u(t-k,x) - u(t+k,x) + u(t,x+h)$$

$$= \cancel{u} - \cancel{h u_x} + \frac{h^2}{2} u_{xx} - \cancel{\frac{h^3}{6} u_{xxx}} + \frac{h^4}{24} u_{xxxx} - \cancel{O(h^5)}$$
$$+ \cancel{u} + \cancel{h u_x} + \frac{h^2}{2} u_{xx} + \cancel{\frac{h^3}{6} u_{xxx}} + \frac{h^4}{24} u_{xxxx} + \cancel{O(h^5)}$$
$$- \cancel{u} + \cancel{k u_t} - \frac{k^2}{2} u_{tt} + \cancel{\frac{k^3}{6} u_{ttt}} - O(k^4)$$
$$- \cancel{u} - \cancel{k u_t} - \frac{k^2}{2} u_{tt} - \cancel{\frac{k^3}{6} u_{ttt}} - O(k^4)$$

$$= h^2 u_{xx} + \frac{h^4}{12} u_{xxxx} + O(h^6) - k^2 u_{tt} + O(k^4)$$

$$\Rightarrow \frac{u(t,x-h) - u(t-k,x) - u(t+k,x) + u(t,x+h)}{h^2} = u_{xx} + \frac{h^2}{12} u_{xxxx} - \left(\frac{k}{h}\right)^2 u_{tt} + O(h^4) + O(k^4)$$

$$\longrightarrow (**)$$

Put equ (*) & (**) in (A) $\Rightarrow$

$$T.E = u_t + \frac{k^2}{6} u_{ttt} + O(k^4) - a u_{xx} - a \frac{h^2}{12} u_{xxxx} + a\left(\frac{k}{h}\right)^2 u_{tt} + O(h^4)$$

$$= (u_t - a u_{xx}) - \frac{a h^2}{12} u_{xxxx} + a\beta^2 u_{tt} + \frac{k^2}{6} u_{ttt} + O(h^4, k^4) \quad \boxed{\because \frac{k}{h} = \beta}$$

$$\Rightarrow T.E = -\frac{a}{12} h^2 u_{xxxx} + a\beta^2 u_{tt} + \frac{k^2}{6} u_{ttt}$$

$$= \alpha \beta^2 u_{tt} + O(h^2, k^2)$$

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when $h \to 0$, $k \to 0$, $T.E \to \alpha\beta^2 u_{tt}$

$\Rightarrow$ This is not consistence.
Will be consistence only when $k \to 0$ faster than h.

$\star$ In above case instead of study $u_t = a u_{xx}$ we are steady $u_t + \alpha\beta^2 u_{tt} = a u_{xx}$. (Then consistence

$\Rightarrow$ Stability:-

$$\frac{u_j^{n+1} - u_j^{n-1}}{2k} = a\,\frac{u_{j-1}^{n} - u_j^{n-1} - u_j^{n+1} + u_{j+1}^{n}}{h^2}$$

$$\Rightarrow u_j^{n+1} - u_j^{n-1} = \frac{2ak}{h^2}\left[u_{j-1}^{n} - u_j^{n-1} - u_j^{n+1} + u_{j+1}^{n}\right]$$

$$\Rightarrow u_j^{n+1} - u_j^{n-1} = 2r\left[u_{j-1}^{n} - u_j^{n-1} - u_j^{n+1} + u_{j+1}^{n}\right] \quad ; \; r = \frac{ak}{h^2}$$

$$\Rightarrow \hat{u}^{n+1}(\xi) - \hat{u}^{n-1}(\xi) = 2r\left[e^{-i\xi} + e^{i\xi}\right]\hat{u}^{n}(\xi) - 2r\left(\hat{u}^{n-1}(\xi) + \hat{u}^{n+1}(\xi)\right)$$

$$\Rightarrow (1+2r)\hat{u}^{n+1}(\xi) - (1-2r)\hat{u}^{n-1}(\xi) = 2r(2\cos\xi)\hat{u}^{n}(\xi)$$

$$\Rightarrow (1+2r)\hat{u}^{n+1}(\xi) - (1-2r)\hat{u}^{n-1}(\xi) = 4r\cos\xi\,\hat{u}^{n}(\xi)$$

$$\Rightarrow D(1+2r)\hat{u}^{n}(\xi) - (1-2r)D^{-1}\hat{u}^{n}(\xi) = 4r\cos\xi\,\hat{u}^{n}(\xi)$$

$$\Rightarrow \left[D^2(1+2r) - 4r\cos\xi\, D - (1-2r)\right]\hat{u}^{n}(\xi) = 0$$

characteristic equation is

$$(1+2r)D^2 - 4r\cos\xi\, D - (1-2r) = 0$$

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Hense $D = \dfrac{4r\cos\xi \pm \sqrt{16r^2\cos^2\xi + 4(1-4r^2)}}{2(1+2r)}$

$$= \frac{4r\cos\xi \pm \sqrt{4+16r^2(\cos^2\xi - 1)}}{2(1+2r)}$$

$$= \frac{4r\cos\xi \pm 2\sqrt{1+4r^2(\cos^2\xi-1)}}{2(1+2r)}$$

for maximus $\cos^2\xi = 1$

$$\Rightarrow D = \frac{4r\cos\xi \pm 2}{2(1+2r)} = \frac{2r\cos\xi \pm 1}{1+2r}$$

$$\Rightarrow D = \frac{2r\cos\xi \pm 1}{1+2r} \leq 1 \quad \forall \xi \in [-\pi, \pi]$$

$$\Rightarrow |D| \leq 1$$

$\Rightarrow$ Scheme is unconditionally stable.

* * *

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Difference Formula:-

Notation:- $\tau_{\pm} u(x) = u(x \pm h)$

$$\Delta_{\pm} u(x) = \pm[u(x \pm h) - u(x)]$$ forward & backward difference

$$\Delta_c u(x) = u(x+h) - u(x-h)$$ central difference

Now

$$u_x \approx \frac{\Delta_{\pm} u(x)}{h}, \frac{\Delta_c u(x)}{2h}$$

$$u_{xx} \approx \frac{\Delta_{\pm} u_x(x)}{h}, \frac{\Delta_c u_x(x)}{2h}$$

$$\approx \frac{\Delta_{\pm}\Delta_{\pm} u(x)}{h^2}, \frac{\Delta_{\pm}\Delta_c u(x)}{2h^2}, \frac{\Delta_c \Delta_{\pm} u(x)}{2h^2},$$

$$\frac{\Delta_c^2 u(x)}{4h^2}$$

Alternative:- Lagrange interpolation can be used to derive different order difference formulas.

⇒ Formal Method:- (For high order Differences)

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots \quad (0 < x \le 2)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (-1 < x \le 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin^{-1} h x = x - \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \cdots$$

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$$\tau_h u(x) = u(x+h)$$

$$\Rightarrow \tau_h u(x) = u(x) + h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} + \cdots$$

$$= \left[1 + h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} + \cdots\right] u(x)$$

$$= e^{h\partial_x} u(x)$$

$\boxed{h \text{ mean interval length}}$

$$\Rightarrow \tau_h u(x) = e^{h\partial_x} u(x)$$

$$\Rightarrow \boxed{\tau_h = e^{h\partial_x}} \longrightarrow \circledast$$

from $\circledast$ $\ln \tau_h = h\partial_x$

Now
$$\tau_h u(x) = u(x+h)$$

$$= u(x+h) - u(x) + u(x)$$

$$\tau_h u(x) = (\Delta_h + 1)\, u(x)$$

$$\Rightarrow \boxed{\tau_h = \Delta_h + 1} \longrightarrow (1)$$

Now
$$\tau_{-h} u(x) = u(x-h)$$

$$= u(x-h) + u(x) - u(x)$$

$$= u(x) - u(x) + u(x-h)$$

$$= u(x) - [u(x) - u(x-h)]$$

$$\Rightarrow \tau_{-h} u(x) = u(x) - \Delta_{-h} u(x)$$

$$\Rightarrow \boxed{\tau_{-h} = 1 - \Delta_{-h}} \longrightarrow (2)$$

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Now from ⊛ $\ln \tau_h = h\partial_x$

$$\Rightarrow h\partial_x = \ln \tau_h = \ln(\Delta_h + 1) \quad \text{from } ①$$

$$\Rightarrow h\partial_x = \Delta_h - \frac{\Delta_h^2}{2} + \frac{\Delta_h^3}{3} - \frac{\Delta_h^4}{4} \pm \cdots$$

$$\Rightarrow \boxed{\partial_x = \frac{1}{h}\left[\Delta_h - \frac{\Delta_h^2}{2} + \frac{\Delta_h^3}{3} - \frac{\Delta_h^4}{4} \pm \cdots\right]} \rightarrow ③$$

Now

$$u_x = \partial_x u = \frac{\Delta_h u(x)}{h} - \frac{\Delta_h^2 u(x)}{2h}$$

$$\Rightarrow u_x = \frac{u(x+h) - u(x)}{h} - \frac{1}{2h}\left[\Delta_h\left[u(x+h) - u(x)\right]\right]$$

$$= \frac{u(x+h) - u(x)}{h} - \frac{1}{2h}\left[u(x+2h) - u(x+h) - u(x+h) + u(x)\right]$$

$$= \frac{2u(x+h) - 2u(x) - u(x+2h) + u(x+h) + u(x+h) - u(x)}{2h}$$

$$= \frac{-u(x+2h) + 4u(x+h) - 3u(x)}{2h} + O(h^2)$$

Analogusly we can derive 2nd order formula for Δ_{-h}

⇒ **Central Difference:-**

$$\Delta_c^h = \tau_h - \tau_{-h}$$

$$= e^{h\partial_x} - e^{-h\partial_x}$$

$$\boxed{\because e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \pm \cdots \quad \tau_{-h}^{u(x)} = u(x-h) \Rightarrow \tau_{-h} = e^{-h\partial_x}}$$

$$\Rightarrow \Delta_c^h u(x) = \left[e^{h\partial_x} - e^{-h\partial_x}\right] u(x)$$

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$$\Rightarrow 2\sinh(h\partial_x) = \Delta_c^h$$

$$\Rightarrow h\partial_x = \operatorname{arc\,sinh}\left(\frac{\Delta_c^h}{2}\right)$$

$$\Rightarrow \partial_x = \frac{1}{h}\left[\operatorname{arc\,sinh}\left(\frac{\Delta_c^h}{2}\right)\right]$$

$$\Rightarrow \partial_x = \frac{\Delta_c^h}{2h} - \frac{1}{6h}\left(\frac{\Delta_c^h}{2}\right)^3 + \frac{3}{40h}\left(\frac{\Delta_c^h}{2}\right)^5 - \cdots$$

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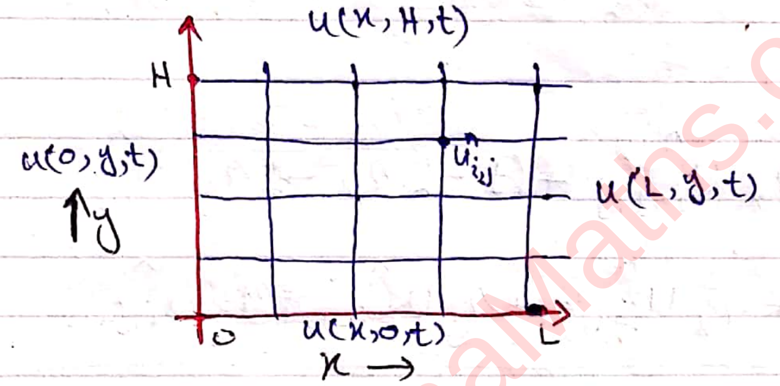
⇒ Two Dimensional Heat Equation:-

$$u_t = a(u_{xx} + u_{yy}) \qquad 0 \le x \le l, \; 0 \le y \le H$$

B.Cs:- $u(t, x, y) = g(t, x, y)$ on boundry

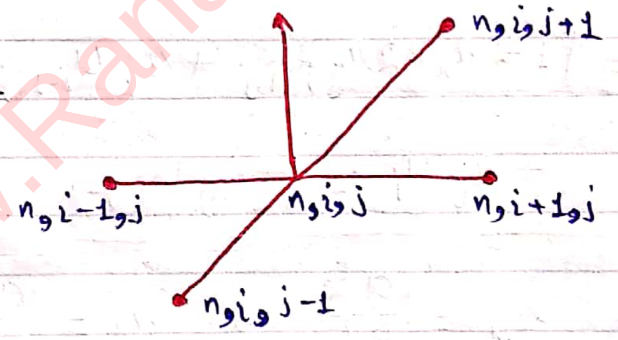
I.C:- $u(0, x, y) = f(x, y) \quad 0 < x < l, \; 0 < y < H$

$$h_x = \frac{l - 0}{N}, \qquad h_y = \frac{H - 0}{M}$$



for
$x \to i$
$y \to j$
$t \to n$

Stensil:-



1) Explicit Method:-

$$u_t = a(u_{xx} + u_{yy})$$

$$\frac{u^{n+1}_{i,j} - u^n_{i,j}}{k} = a\left[\frac{u^n_{i-1,j} - 2u^n_{i,j} + u^n_{i+1,j}}{h_x^2}\right] + a\left[\frac{u^n_{i,j-1} - 2u^n_{i,j} + u^n_{i,j+1}}{h_y^2}\right]$$

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$$\Rightarrow u_{i,j}^{n+1} - u_{i,j}^{n} = \frac{ak}{h_x^2}\left[u_{i-1,j}^{n} - 2u_{i,j}^{n} + u_{i+1,j}^{n}\right] + \frac{ak}{h_y^2}\left[u_{i,j-1}^{n} - 2u_{i,j}^{n} + u_{i,j+1}^{n}\right]$$

$$\Rightarrow u_{i,j}^{n+1} = u_{i,j}^{n} + \frac{ak}{h_x^2}\left[u_{i-1,j}^{n} - 2u_{i,j}^{n} + u_{i+1,j}^{n}\right] + \frac{ak}{h_y^2}\left[u_{i,j-1}^{n} - 2u_{i,j}^{n} + u_{i,j+1}^{n}\right]$$

Or

$$u_{i,j}^{n+1} = r_x u_{i-1,j}^{n} + (1 - 2r_x - 2r_y)u_{i,j}^{n} + r_x u_{i+1,j}^{n} + r_y u_{i,j-1}^{n} + r_y u_{i,j+1}^{n}$$

$$\boxed{\Rightarrow u_{i,j}^{n+1} = [1 - 2r_x - 2r_y]u_{i,j}^{n} + r_x u_{i-1,j}^{n} + r_x u_{i+1,j}^{n} + r_y u_{i,j-1}^{n} + r_y u_{i,j+1}^{n}}$$

where $r_x = \frac{ak}{h_x^2}$, $r_y = \frac{ak}{h_y^2}$

* Stability Analysis:-

$$u_{i,j}^{n+1} = r_x u_{i-1,j}^{n} + (1 - 2r_x - 2r_y)u_{i,j}^{n} + r_x u_{i+1,j}^{n} + r_y u_{i,j-1}^{n} + r_y u_{i,j+1}^{n}$$

$$\Rightarrow \hat{u}^{n+1}(\xi,\eta) = r_x e^{-i\xi}\hat{u}^{n}(\xi,\eta) + [1 - 2r_x - 2r_y]\hat{u}^{n}(\xi,\eta) + r_x e^{i\xi}\hat{u}^{n}(\xi,\eta) + r_y e^{-i\eta}\hat{u}^{n}(\xi,\eta) + r_y e^{i\eta}\hat{u}^{n}(\xi,\eta)$$

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$$\Rightarrow \hat{u}^{n+1}(\xi,\eta) = (e^{-i\xi}+e^{i\xi}) r_x \hat{u}^n(\xi,\eta) + (1-2r_x-2r_y)\hat{u}^n(\xi,\eta) + (e^{-i\eta}+e^{i\eta}) r_y \hat{u}^n(\xi,\eta)$$

$$= [2r_x\cos\xi + (1-2r_x-2r_y) + 2r_y\cos\eta]\,\hat{u}^n(\xi,\eta)$$

$$= [-2r_x(1-\cos\xi) - 2r_y(1-\cos\eta) + 1]\,\hat{u}^n(\xi,\eta)$$

$$= [-4r_x\sin^2\xi/2 - 4r_y\sin^2\eta/2 + 1]\,\hat{u}^n(\xi,\eta)$$

$$\hat{u}^{n+1}(\xi,\eta) = [1-4[r_x\sin^2\xi/2 + r_y\sin^2\eta/2]]\,\hat{u}^n(\xi,\eta)$$

$$\Rightarrow \hat{u}^n(\xi,\eta) = [1-4(r_x\sin^2\xi/2 + r_y\sin^2\eta/2)]\,\hat{u}^{n-1}(\xi,\eta)$$

$$\Rightarrow \hat{u}^n(\xi,\eta) = [1-4(r_x\sin^2\xi/2 + r_y\sin^2\eta/2)]^n\,\hat{u}^0(\xi,\eta)$$

$$\Rightarrow \|\hat{u}^n(\xi,\eta)\| = |D|^n \|\hat{u}^0(\xi,\eta)\|,$$

where $D = 1-4(r_x\sin^2\xi/2 + r_y\sin^2\eta/2)$

for stability $|D| < 1$

$$\Rightarrow -1 < D < 1$$

$$\Rightarrow -1 < 1-4r_x\sin^2\xi/2 + 4r_y\sin^2\eta/2 < 1$$

$\sin^2\xi/2 = \sin^2\eta/2 = 1$ $\qquad \forall\, \xi,\eta \in [-\pi,\pi]$

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$\Rightarrow -1 < 1 - 4(r_x + r_y) < 1$

R.H.S

$1 - 4(r_x + r_y) < 1$

$\Rightarrow -4(r_x + r_y) < 0 \Rightarrow r_x + r_y > 0$

L.H.S

$-1 < 1 - 4(r_x + r_y)$

$\Rightarrow 4(r_x + r_y) < 2 \Rightarrow r_x + r_y < \frac{1}{2}$

Suppose $r_x = r_y = r \Rightarrow 2r < \frac{1}{2}$

$\Rightarrow r < \frac{1}{4}$

$\Rightarrow ak/h^2 < \frac{1}{4} \Rightarrow k < h^2/4a$

$\Rightarrow$ Time step is more restricted than the 1D case

* To over come the more sever stabi restriction Crank-Nicolson method (Implicit can be used.

$\Rightarrow$ Crank - Nicolson Method:-

$u_t = a \Delta u$

$\Rightarrow u_t = a(u_{xx} + u_{yy})$

$\Delta = \partial^2/\partial x^2$ for 1D

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ for 2D

& so, on

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$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{k} = \frac{a}{2}\left[\frac{u_{i-1,j}^{n} - 2u_{i,j}^{n} + u_{i+1,j}^{n}}{h_x^2} + \frac{u_{i-1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i+1,j}^{n+1}}{h_x^2}\right]$$

$$+ \frac{a}{2}\left[\frac{u_{i,j-1}^{n} - 2u_{i,j}^{n} + u_{i,j+1}^{n}}{h_y^2} + \frac{u_{i,j-1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j+1}^{n+1}}{h_y^2}\right]$$

$$\Rightarrow \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{k} = \frac{a}{2}\left[\frac{\delta_x^2 u_{i,j}^{n} + \delta_x^2 u_{i,j}^{n+1}}{h_x^2}\right] + \frac{a}{2}\left[\frac{\delta_y^2 u_{i,j}^{n} + \delta_y^2 u_{i,j}^{n+1}}{h_y^2}\right]$$

where

$$\delta_x^2 u_{i,j} = u_{i-1,j} - 2u_{i,j} + u_{i+1,j}$$

$$\delta_y^2 u_{i,j} = u_{i,j-1} - 2u_{i,j} + u_{i,j+1} \qquad \left\{\begin{array}{l} n \text{ for } n \\ n+1 \to n+1 \end{array}\right.$$

This method is unconditionally stable. It is 2nd order accurate in the both space and time i.e $T.E \sim O(k^2, h_x^2, h_y^2)$

Problems :- * Large Banded Matrix

* It A is constant then the co-efficient matrix is larger and we need to factor the matrix area.

⇒ Alternating Direction Implicit (ADI) Method.

The idea is to alternate direction and then solve one dimensional problem at each time step

* The first step is to keep y

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fixed

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{k/2} = a\left[\frac{\delta_x^2 u_{i,j}^{n+1/2}}{h_x^2} + \frac{\delta_y^2 u_{i,j}^{n}}{h_y^2}\right]$$

$$\Rightarrow A u^{n+1/2} = B^{n}$$

* In the 2nd step we keep x-fixed

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{k/2} = a\left[\frac{\delta_x^2 u_{i,j}^{n+\frac{1}{2}}}{h_x^2} + \frac{\delta_y^2 u_{i,j}^{n+1}}{h_y^2}\right]$$

$$\Rightarrow A u^{n+1} = B^{n+1/2}$$

So we are solving tri-diagonal system at each time step.

⇒ Truncation Error (Explicit Method):-

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{k} = a\left[\frac{u_{i-1,j}^{n} - 2u_{i,j}^{n} + u_{i+1,j}^{n}}{h_x^2} + \frac{u_{i,j-1}^{n} - 2u_{i,j}^{n} + u_{i,j+1}^{n}}{h_y^2}\right]$$

$$T.E = \frac{u(t+k,x,y) - u(t,x,y)}{k} - a\left[\frac{u(t,x-h_x,y) - 2u(t,x,y) + u(t,x+h_x,y)}{h_x^2} + \frac{u(t,x,y-h_y) - 2u(t,x,y) + u(t,x,y+h_y)}{h_y^2}\right] \quad (1)$$

Now

$$u(t+k,x,y) = u(t,x,y) + k u_t(t,x,y) + \frac{k^2}{2!} u_{tt}(t,x,y) + \frac{k^3}{3!} u_{ttt}(t,x,y) + O(k^4)$$

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$$\Rightarrow \frac{u(t+k,x,y)-u(t,x,y)}{k} = u_t(t,x,y) + \frac{k}{2}u_{tt}(t,x,y) + O(k^2) \longrightarrow (*)$$

$$u(t,x-h_x,y) = u(t,x,y) - h_x u_x(t,x,y) + \frac{h_x^2}{2!}u_{xx}(t,x,y) - \frac{h_x^3}{3!}u_{xxx}(t,x,y) + \frac{h_x^4}{4!}u_{xxxx}(t,x,y) + O(h_x^5) \longrightarrow (A)$$

$$u(t,x+h_x,y) = u(t,x,y) + h_x u_x(t,x,y) + \frac{h_x^2}{2!}u_{xx}(t,x,y) + \frac{h_x^3}{3!}u_{xxx}(t,x,y) + \frac{h_x^4}{4!}u_{xxxx}(t,x,y) + O(h_x^5) \longrightarrow (B)$$

Now by adding equation (A) & (B)

$$u(t,x-h_x,y) + u(t,x+h_x,y) = 2u(t,x,y) + \frac{2h_x^2}{2}u_{xx}(t,x,y) + \frac{2h_x^4}{24}u_{xxxx}(t,x,y) + O(h_x^6)$$

$$\Rightarrow \frac{u(t,x-h_x,y) - 2u(t,x,y) + u(t,x+h_x,y)}{h_x^2} = u_{xx}(t,x,y) + \frac{h_x^2}{12}u_{xxxx}(t,x,y) + O(h_x^4) \longrightarrow (**)$$

Similarly,

$$\frac{u(t,x,y-h_y) - 2u(t,x,y) + u(t,x,y+h_y)}{h_y^2} = u_{yy}(t,x,y) + \frac{h_y^2}{12}u_{yyyy}(t,x,y) + O(h_y^4) \longrightarrow (***)$$

Put the values of (*), (**), (***) in eqn (1)

$$T.E = u_t + \frac{k}{2}u_{tt} + O(k^2) - a\left[u_{xx} + \frac{h_x^2}{12}u_{xxxx} + O(h_x^4) + u_{yy} + \frac{h_y^2}{12}u_{yyyy} + O(h_y^4)\right]$$

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$$\Rightarrow T.E = u_t - a(u_{xx}+u_{yy}) + \frac{k}{2}u_{tt} + O(k^2)$$
$$- \frac{a h_x^2}{12} u_{xxxx} + O(h_x^4) - \frac{a h_y^2}{12} u_{yyyy} + O(h$$

$$= 0 + O(k) + O(h_x^2) + O(h_y^2)$$

$$\Rightarrow T.E = O(k, h_x^2, h_y^2)$$

Stability Analysis (ADI):-

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{k/2} = a\left[\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+1/2} + u_{i-1,j}^{n+\frac{1}{2}}}{h_x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u\right.}{h_y^2}$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{k/2} = a\left[\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{h_x^2} + \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{h_y^2}\right] \quad \text{---} (2)$$

Now for (1)

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{k/2} = a\left[\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{h_x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j}^{n}}{h_y^2}\right.$$

$$\Rightarrow u_{i,j}^{n+\frac{1}{2}} = u_{i,j}^{n} + \frac{ak}{2h_x^2}\left[u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}\right]$$
$$+ \frac{ak}{2h_y^2}\left[u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}\right]$$

$$\Rightarrow u_{i,j}^{n+\frac{1}{2}} = u_{i,j}^{n} + \lambda_x\left[u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}\right]$$

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$$+ r_y\left[u^n_{i,j+1} - 2\hat{u}^n_{i,j} + u^n_{i,j-1}\right],$$

where $r_x = \frac{ak}{2h_x^2}$, $r_y = \frac{ak}{2h_y^2}$

$$\Rightarrow \hat{u}^{n+\frac{1}{2}}(\xi,\eta) = \hat{u}^n(\xi,\eta) + r_x\left[e^{i\xi}\hat{u}^{n+\frac{1}{2}}(\xi,\eta) - 2\hat{u}^{n+\frac{1}{2}}(\xi,\eta)\right.$$

$$\left. + e^{-i\xi}\hat{u}^{n+\frac{1}{2}}(\xi,\eta)\right] + r_y\left[e^{i\eta}\hat{u}^n(\xi,\eta) - 2\hat{u}^n(\xi,\eta)\right.$$

$$\left. + e^{-i\eta}\hat{u}^n(\xi,\eta)\right]$$

$$\Rightarrow \left[1 - r_x(e^{i\xi} + e^{-i\xi}) + 2r_x\right]\hat{u}^{n+\frac{1}{2}}(\xi,\eta) = \left[1 + r_y(e^{i\eta} + e^{-i\eta}) - 2r_y\right]\hat{u}^n(\xi,\eta)$$

$$\Rightarrow \left[1 + 2r_x - 2r_x\cos\xi\right]\hat{u}^{n+\frac{1}{2}}(\xi,\eta) = \left[1 - 2r_y + 2r_y\cos\eta\right]\hat{u}^n(\xi,\eta)$$

$$\Rightarrow \left[1 + 2r_x\left[1 - \cos\xi\right]\right]\hat{u}^{n+\frac{1}{2}}(\xi,\eta) = \left[1 - 2r_y(1 - \cos\eta)\right]\hat{u}^n(\xi,\eta)$$

$$\Rightarrow \left[1 + 4r_x\sin^2\xi/2\right]\hat{u}^{n+\frac{1}{2}}(\xi,\eta) = \left[1 - 4r_y\sin^2\eta/2\right]\hat{u}^n(\xi,\eta)$$

$$\Rightarrow \hat{u}^{n+\frac{1}{2}}(\xi,\eta) = \frac{\left[1 - 4r_y\sin^2\eta/2\right]}{\left[1 + 4r_x\sin^2\xi/2\right]}\hat{u}^n(\xi,\eta) \longrightarrow (A)$$

Now for (2)

$$\frac{u^{n+1}_{i,j} - u^{n+\frac{1}{2}}_{i,j}}{k/2} = a\left[\frac{u^{n+\frac{1}{2}}_{i+1,j} - 2u^{n+\frac{1}{2}}_{i,j} + u^{n+\frac{1}{2}}_{i-1,j}}{h_x^2} + \frac{u^{n+1}_{i,j+1} - 2u^{n+1}_{i,j} + u^{n+1}_{i,j-1}}{h_y^2}\right]$$

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$$\Rightarrow u_{i,j}^{n+1} = u_{i,j}^{n+\frac{1}{2}} + \frac{ak}{2h_x^2}\left[u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}\right]$$

$$+ \frac{ak}{2h_y^2}\left[u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}\right]$$

$$\Rightarrow \hat{u}^{n+1}(\xi,\eta) = \hat{u}^{n+\frac{1}{2}}(\xi,\eta) + r_x\left[e^{i\xi}\hat{u}^{n+\frac{1}{2}}(\xi,\eta) - 2\hat{u}^{n+\frac{1}{2}}(\xi,\eta)\right.$$

$$\left. + e^{-i\xi}\hat{u}^{n+\frac{1}{2}}(\xi,\eta)\right] + r_y\left[e^{i\eta}\hat{u}^{n+1}(\xi,\eta)\right.$$

$$\left. - 2\hat{u}^{n+1}(\xi,\eta) + e^{-i\eta}\hat{u}^{n}(\xi,\eta)\right]$$

$$\Rightarrow \left[1+2r_y - r_y(e^{i\eta}+e^{-i\eta})\right]\hat{u}^{n+1}(\xi,\eta) = \left[1-2r_x + r_x(e^{i\xi}+e^{-i\xi})\right]\hat{u}^{n+\frac{1}{2}}(\xi,\eta)$$

$$\Rightarrow \left[1+2r_y - 2r_y\cos\eta\right]\hat{u}^{n+1}(\xi,\eta) = \left[1-2r_x + 2r_x\cos\xi\right]\hat{u}^{n+\frac{1}{2}}(\xi,\eta)$$

$$\Rightarrow \left[1+2r_y(1-\cos\eta)\right]\hat{u}^{n+1}(\xi,\eta) = \left[1-2r_x(1-\cos\xi)\right]\hat{u}^{n+\frac{1}{2}}(\xi,\eta)$$

$$\Rightarrow \left[1+4r_y\sin^2\eta/2\right]\hat{u}^{n+1}(\xi,\eta) = \left[1-4r_x\sin^2\xi/2\right]\hat{u}^{n+\frac{1}{2}}(\xi,\eta)$$

$$\Rightarrow \hat{u}^{n+1}(\xi,\eta) = \left[\frac{1-4r_x\sin^2\xi/2}{1+4r_y\sin^2\eta/2}\right]\hat{u}^{n+\frac{1}{2}}(\xi,\eta)$$

Putting the value of $\hat{u}^{n+\frac{1}{2}}(\xi,\eta)$ from (A)

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$$\Rightarrow \hat{u}^{n+1}(\xi,\eta) = \left[\frac{1-4r_x \sin^2 \xi/2}{1+4r_y \sin^2 \eta/2}\right]\left[\frac{1-4r_y \sin^2 \eta/2}{1+4r_x \sin^2 \xi/2}\right]\hat{u}^n(\xi,\eta)$$

$$\Rightarrow \hat{u}^{n+1}(\xi,\eta) = \left[\frac{1-4r_x \sin^2 \xi/2}{1+4r_x \sin^2 \xi/2}\right]\left[\frac{1-4r_y \sin^2 \eta/2}{1+4r_y \sin^2 \eta/2}\right]\hat{u}^n(\xi,\eta)$$

$$\Rightarrow \hat{u}^{n}(\xi,\eta) = \left[\left(\frac{1-4r_x \sin^2 \xi/2}{1+4r_x \sin^2 \xi/2}\right)\left(\frac{1-4r_y \sin^2 \eta/2}{1+4r_y \sin^2 \eta/2}\right)\right]\hat{u}^{n-1}(\xi,\eta)$$

Induction implies

$$\hat{u}^{n}(\xi,\eta) = \left[\left(\frac{1-4r_x \sin^2 \xi/2}{1+4r_x \sin^2 \xi/2}\right)\left(\frac{1-4r_y \sin^2 \eta/2}{1+4r_y \sin^2 \eta/2}\right)\right]^n\hat{u}^{0}(\xi,\eta)$$

$$D = \left[\left(\frac{1-4r_x \sin^2 \xi/2}{1+4r_x \sin^2 \xi/2}\right)\left(\frac{1-4r_y \sin^2 \eta/2}{1+4r_y \sin^2 \eta/2}\right)\right]$$

$$|D| \leq 1 \quad \forall\ \xi,\eta \in [-\pi,\pi]$$

$\Rightarrow$ Scheme is unconditionally stable.

Truncation Error:- (ADI Scheme):-

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{k/2} = a\left[\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{h_x^2} + \frac{u_{i,j-1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{h_y^2}\right] \longrightarrow (1)$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{k/2} = a\left[\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{h_x^2} + \frac{u_{i,j-1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{h_y^2}\right] \longrightarrow (2)$$

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from ①

$$T.E = \frac{u(t+k/2,x,y)-u(t,x,y)}{k/2} - a\left[\frac{u(t+k/2,x+h_x,y)-2u(t+k/2,x,y)+u(t+k/2,x-h_x,y)}{h_x^2} + \frac{u(t+k,x,y-h_y)-2u(t+k,x,y)+u(t+k,x,y-h_y)}{h_y^2}\right] \longrightarrow ③$$

Now

$$u(t+\tfrac{k}{2},x,y) = u(t,x,y)+\tfrac{k}{2}u_t(t,x,y)+\left(\tfrac{k}{2}\right)^2\tfrac{1}{2!}u_{tt}(t,x,y) + \left(\tfrac{k}{2}\right)^3\tfrac{1}{3!}u_{ttt}(t,x,y)+O(k^4)$$

$$\Rightarrow \frac{u(t+\frac{k}{2},x,y)-u(t,x,y)}{k/2} = u_t(t,x,y)+\frac{k}{4}u_{tt}(t,x,y) + \frac{k^2}{24}u_{ttt}(t,x,y)+O(k^3) \longrightarrow (i)$$

Now

$$u(t+\tfrac{k}{2},x+h_x,y) = u(t+\tfrac{k}{2},x,y)+h_x u_x(t+\tfrac{k}{2},x,y) + \frac{h_x^2}{2!}u_{xx}(t+\tfrac{k}{2},x,y)+\frac{h_x^3}{3!}u_{xxx}(t+\tfrac{k}{2},x,y) + \frac{h_x^4}{4!}u_{xxxx}(t+\tfrac{k}{2},x,y)+O(h_x^5) \longrightarrow (ii)$$

&

$$u(t+\tfrac{k}{2},x-h_x,y) = u(t+\tfrac{k}{2},x,y)-h_x u_x(t+\tfrac{k}{2},x,y)+\frac{h_x^2}{2!}u_{xx}(t+\tfrac{k}{2},x,y)-\frac{h_x^3}{3!}u_{xxx}(t+\tfrac{k}{2},x,y) + \frac{h_x^4}{4!}u_{xxxx}(t+\tfrac{k}{2},x,y)-O(h_x^5) \longrightarrow (iii)$$

Adding eq (ii) & (iii) implies.

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$$u(t+\tfrac{k}{2}, x+h_x, y) + u(t+\tfrac{k}{2}, x-h_x, y) = 2u(t+\tfrac{k}{2}, x, y) + h_x^2 u_{xx}(t+\tfrac{k}{2}, x, y)$$
$$+ \frac{h_x^4}{12} u_{xxxx}(t+\tfrac{k}{2}, x, y) + O(h_x^6)$$

$$\Rightarrow \frac{u(t+\frac{k}{2}, x+h_x, y) - 2u(t+\frac{k}{2}, x, y) + u(t+\frac{k}{2}, x-h_x, y)}{h_x^2}$$
$$= u_{xx}(t+\tfrac{k}{2}, x, y) + \frac{h_x^2}{12} u_{xxxx}(t+\tfrac{k}{2}, x, y) + O(h_x^4) \longrightarrow (*)$$

Similarly

$$\frac{u(t+k, x, y+h_y) - 2u(t+k, x, y) + u(t+k, x, y-h_y)}{h_y^2}$$
$$= u_{yy}(t+k, x, y) + \frac{h_y^2}{12} u_{yyyy}(t+k, x, y) + O(h_y^4) \longrightarrow (**)$$

Now Putting the values of (i) & (*) & (**) in eqn (3)

$$\Rightarrow T.E = u_t(t,x,y) + \frac{k}{4} u_{tt}(t,x,y) + \frac{k^2}{24} u_{ttt}(t,x,y) + O(k^3)$$
$$- a\Big[u_{xx}(t+\tfrac{k}{2}, x, y) + \frac{h_x^2}{12} u_{xxxx}(t+\tfrac{k}{2}, x, y) + O(h_x^4) +$$
$$u_{yy}(t+k, x, y) + \frac{h_y^2}{12} u_{yyyy}(t+k, x, y) + O(h_y^4)\Big]$$

$$= u_t + \frac{k}{4} u_{tt} + \frac{k^2}{24} u_{ttt} + O(k^3) - a\Big[u_{xx} + \frac{k}{2} u_{xxt}$$
$$+ \frac{k^2}{8} u_{xxtt} + O(k^3) + \frac{h_x^2}{12} u_{xxxx} + \frac{k h_x^2}{24} u_{xxxxt}$$
$$+ O(k^3) + O(h_x^4) + u_{yy} + k u_{yyt} + \frac{k^2}{2} u_{yytt}$$
$$+ O(k^3) + \frac{h_y^2}{12} u_{yyyy} + \frac{k h_y^2}{12} u_{yyyyt} + O(k^3) + O(h_y^4)\Big]$$

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$$\Rightarrow T.E = u_t - a(u_{xx}+u_{yy}) + \frac{k}{4}u_{tt} + \frac{k^2}{24}u_{ttt} + O(k^3)$$

$$- a\Big[\frac{k}{2}u_{xxt} + \frac{k^2}{8}u_{xxtt} + O(k^3) + \frac{h_x^2}{12}u_{xxxx}$$

$$+ \frac{k h_x^2}{12}u_{xxxxt} + O(k^3) + O(h_x^4) + k u_{yyt}$$

$$+ \frac{k^2}{2}u_{yytt} + O(k^3) + \frac{h_y^2}{12}u_{yyyy} + O(k^3) + O(h_y^4)\Big]$$

$$\because u_t - a(u_{xx}+u_{yy}) = 0$$

$$\Rightarrow T.E = 0 + \frac{k}{4}u_{tt} + O(k^2) - a\Big(\frac{k}{2}u_{xxt} + O(k^2)$$

$$+ \frac{h_x^2}{12}u_{xxxx} + O(k^2) + O(h_x^4) + k u_{yyt} + O(k^2)$$

$$+ O(h_y^2)\Big)$$

$$\Rightarrow T.E = O(k, h_x^2, h_y^2)$$

Now from ②

$$T.E = \frac{u(t+k,x,y) - u(t+\tfrac{k}{2},x,y)}{k/2} - a\left[\frac{u(t+\tfrac{k}{2},x+h_x,y) - 2u(t+\tfrac{k}{2},x,y) + u(t+\tfrac{k}{2},x-h_x,y)}{h_x^2}\right.$$

$$\left. + \frac{u(t+k,x,y+h_y) - 2u(t+k,x,y) + u(t+k,x,y-h_y)}{h_y^2}\right] \longrightarrow ④$$

Now $u(t+k,x,y) = u(t+\tfrac{k}{2},x,y) + \frac{k}{2}u_t(t+\tfrac{k}{2},x,y)$

$$+ \frac{k^2}{4}\frac{1}{2!}u_{tt}(t+\tfrac{k}{2},x,t) + O(k^3)$$

$$\Rightarrow \frac{u(t+k,x,y) - u(t+\tfrac{k}{2},x,y)}{k/2} = u_t(t+\tfrac{k}{2},x,y) + \frac{k}{4}u_{tt}(t+\tfrac{k}{2},x,y)$$

$$+ O(k^2) \longrightarrow (iv)$$

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$$\frac{u(t+\frac{k}{2}, x+h_x, y) - 2u(t+\frac{k}{2}, y, x) + u(t+\frac{k}{2}, x-h_x, y)}{h_x^2}$$

$$= u_{xx}(t+\tfrac{k}{2}, x, y) + \frac{h_x^2}{12} u_{xxxx}(t+\tfrac{k}{2}, x, y) + O(h_x^4) \longrightarrow (V)$$

&

$$\frac{u(t+k, x, y+h_y) - 2u(t+k, x, y) + u(t+k, x, y-h_y)}{h_y^2}$$

$$= u_{yy}(t+k, x, y) + \frac{h_y^2}{12} u_{yyyy}(t+k, x, y) + O(h_y^4) \longrightarrow (VI)$$

Now by putting the values of equ (iv), (V) & (VI) in equ ⓐ implies

$$T.E = u_t(t+\tfrac{k}{2}, x, y) + \frac{k}{4} u_{tt}(t+\tfrac{k}{2}, x, y) + O(k^2) - a\Big[u_{xx}(t+\tfrac{k}{2}, x, y) + \frac{h_x^2}{12} u_{xxxx}(t+\tfrac{k}{2}, x, y) + O(h_x^4) + u_{yy}(t+k, x, y)$$
$$+ \frac{h_y^2}{12} u_{yyyy}(t+k, x, y) + O(h_y^4)\Big]$$

$$= u_t + \frac{k}{2} u_{tt} + \left(\frac{k}{2}\right)^2 \frac{1}{2!} u_{ttt} + O(k^3) + \frac{k}{4} u_{tt} + \frac{k}{4} \cdot \frac{k}{2} u_{ttt}$$
$$+ O(k^3) - a\Big[u_{xx} + \frac{k}{2} u_{xxt} + \left(\frac{k}{2}\right)^2 \frac{1}{2!} u_{xxtt} + O(k^3)$$
$$+ \frac{h_x^2}{12} u_{xxxx} + \frac{k h_x^2}{24} u_{xxxxt} + O(k^3) + O(h_x^4) + u_{yy}$$
$$+ k u_{yyt} + \frac{k^2}{2} u_{yytt} + O(k^3) + \frac{h_y^2}{12} u_{yyyy} +$$
$$\frac{k h_y^2}{12} u_{yyyyt} + O(k^3) + O(h_y^4)\Big]$$

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$$\Rightarrow T.E = u_t - a(u_{xx} + u_{yy}) + \frac{k}{2} u_{tt} + O(k^2) + \frac{k}{4} u_{tt}$$
$$+ O(k^2) - a\left[\frac{k}{2} u_{xxt} + \frac{k^2}{8} u_{xxtt} + O(k^3)\right.$$
$$\left. + O(h_x^2) + k\, u_{yyt} + \frac{k^2}{2} u_{yytt} + O(h_y)^2\right.$$

$$\because u_t - a(u_{xx} + u_{yy}) = 0$$

$$\Rightarrow T.E = 0 + \frac{k}{2} u_{tt} + O(k^2) + \frac{k}{4} u_{tt} + O(k^2)$$
$$- a\left[\frac{k}{2} u_{xxt} + O(k^2) + O(h_x^2) + O(h_y^2)\right]$$

$$\Rightarrow T.E = O(k, h_x^2, h_y^2)$$

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⇒ Hyperbolic PDE :-

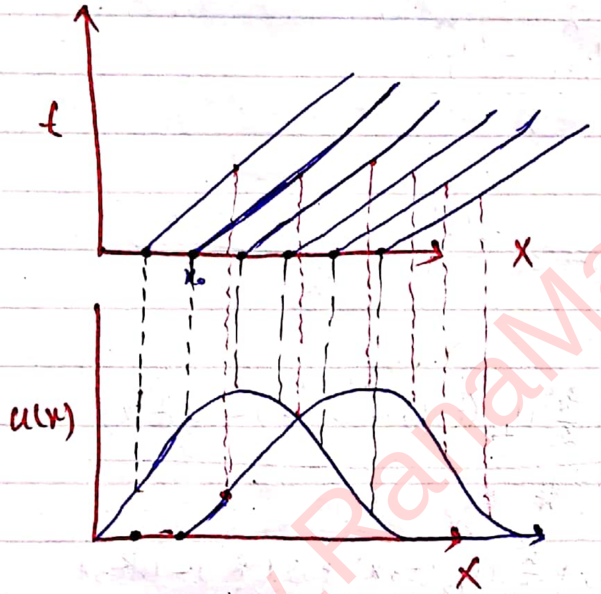
$$u_t + a u_x = 0 \longrightarrow \text{characteristic speed.}$$

$$u(0,x) = u_0(x) \quad , x \in \mathbb{R}$$

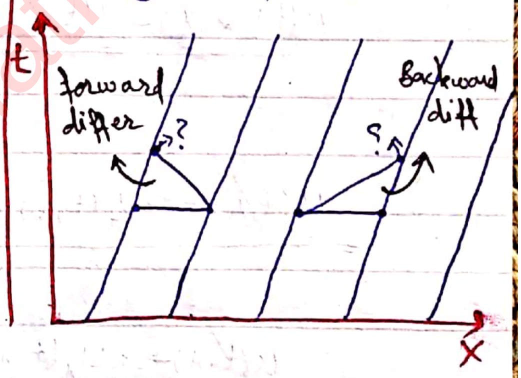
$$u(t,0) = \alpha_1 \quad , \quad u(t,l) = \alpha_2$$

If $a > 0 \longrightarrow$ flow in +ve direction

If $a < 0 \longrightarrow$ flow in −ve direction.



Characteristic :- are curves in x-t plane carry information.



Finite Difference Methods :-

$$u_t + a u_x = 0$$

★ Forward Difference :-

$$\frac{u_j^{n+1} - u_j^n}{k} + a \frac{u_{j+1}^n - u_j^n}{h} = 0$$

★ Backward Difference :-

$$\frac{u_j^{n+1} - u_j^n}{k} + a \frac{u_j^n - u_{j-1}^n}{h} = 0$$

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$$\Rightarrow u_j^{n+1} = (1+a\lambda)u_j^n - a\lambda u_{j+1}^n \rightarrow \text{forward diff}$$

$$\& \quad u_j^{n+1} = (1-a\lambda)u_j^n + a\lambda u_{j-1}^n \rightarrow \text{Backward diff}$$

where $\lambda = k/h$

Truncation Error:- (forward Difference)

$$u_t + a u_x = 0$$

$$T.E = \frac{u_j^{n+1} - u_j^n}{k} + a\frac{u_{j+1}^n - u_j^n}{h}$$

$$= \frac{u(t+k,x) - u(t,x)}{k} + a\frac{u(t,x+h) - u(t,x)}{h}$$

$$= \frac{\cancel{u(t,x)} + k u_t(t,x) + \frac{k^2}{2}u_{tt}(t,x) + O(k^3) - \cancel{u(t,x)}}{k}$$

$$+ a\frac{\cancel{u(t,x)} + h u_x(t,x) + \frac{h^2}{2}u_{xx}(t,x) + O(h^3) - \cancel{u(t,x)}}{h}$$

$$= \cancel{(u_t + a u_x)}^{0} + \frac{k}{2}u_{tt} + O(k^2) + \frac{ah}{2}u_{xx} + O(h^2)$$

$$\Rightarrow T.E = O(k, h)$$

Further $\frac{k}{2}a^2 u_{xx} + \frac{h}{2}u_{xx} + O(k^2, h^2)$

$$= \frac{h}{2}[\lambda a^2 + 1]u_{xx} + O(k^2, h^2)$$

$u_t = -a u_x$

$\Rightarrow u_{tt} = -a u_{xt}$

$= a^2 u_{xx}$

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⇒ Truncation Error:- (Backward Difference)

$$u_t + a u_x = 0$$

$$T.E = \frac{u_j^{n+1} - u_j^n}{k} + a\,\frac{u_j^n - u_{j-1}^n}{h}$$

$$= \frac{u(t+k,x) - u(t,x)}{k} + a\,\frac{u(t,x) - u(t,x-h)}{h} \longrightarrow (1)$$

Now $$u(t+k,x) = u + k u_t + \frac{k^2}{2!} u_{tt} + O(k^3)$$

$$\Rightarrow \frac{u(t+k,x) - u(t,x)}{k} = u_t + \frac{k}{2} u_{tt} + O(k^2) \longrightarrow (i)$$

$$u(t,x-h) = u - h u_x + \frac{h^2}{2!} u_{xx} - \frac{h^3}{3!} u_{xxx} + O(h^4)$$

$$\Rightarrow \frac{u(t,x-h) - u(t,x)}{h} = -u_x + \frac{h}{2} u_{xx} - \frac{h^2}{6} u_{xxx} + O(h^3)$$

$$\Rightarrow \frac{u(t,x) - u(t,x-h)}{h} = u_x - \frac{h}{2} u_{xx} + O(h^2) \longrightarrow (ii)$$

Put values of eqn (i) & (ii) in eqn (1)

$$\Rightarrow T.E = u_t + \frac{k}{2} u_{tt} + O(k^2) + a u_x - a\frac{h}{2} u_{xx} + O(h^2)$$

$$= \cancel{(u_t + a u_x)}^{\,0} + \frac{k}{2} u_{tt} + O(k^2) - \frac{ah}{2} u_{xx} + O(h^2)$$

$$= O(k, h)$$

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$\Rightarrow$ **Stability Analysis :- (Forward Difference)**

$$u_j^{n+1} = (1+a\lambda)u_j^n - a\lambda u_{j+1}^n$$

Assume $a > 0$

$$\Rightarrow \hat{u}^{n+1}(\xi) = (1+a\lambda)\hat{u}^n(\xi) - a\lambda e^{i\xi}\hat{u}^n(\xi)$$

$$= \left[1+a\lambda - a\lambda e^{i\xi}\right]\hat{u}^n(\xi)$$

$$= \left[1+a\lambda - a\lambda\cos\xi - ia\lambda\sin\xi\right]\hat{u}^n(\xi)$$

$\because e^{i\theta} = \cos\theta + i\sin\theta$

$$\hat{u}^{n+1}(\xi) = \left[1+a\lambda(1-\cos\xi) - ia\lambda\sin\xi\right]\hat{u}^n(\xi)$$

$$\hat{u}^n(\xi) = \left[1+a\lambda - a\lambda\cos\xi - ia\lambda\sin\xi\right]^n \hat{u}^0(\xi)$$

$$D = \left[1+a\lambda - a\lambda\cos\xi - ia\lambda\sin\xi\right]$$

for stability $|D| < 1 \Rightarrow |D|^2 < 1$

$$\Rightarrow |1+a\lambda - a\lambda\cos\xi - ia\lambda\sin\xi|^2 < 1$$

$\boxed{|x \pm iy| = \sqrt{x^2+y^2}}$

$$\Rightarrow (1+a\lambda)^2 + a^2\lambda^2\cos^2\xi + a^2\lambda^2\sin^2\xi - 2a\lambda(1+a\lambda)\cos\xi < 1$$

$$\Rightarrow (1+a\lambda)^2 + a^2\lambda^2 - 2a\lambda(1+a\lambda)\cos\xi < 1$$

$\cos\xi = -1$ implies

$$(1+a\lambda)^2 + a^2\lambda^2 + 2a\lambda(1+a\lambda) < 1$$

$$\Rightarrow (1+a\lambda+a\lambda)^2 < 1 \quad \Rightarrow (1+2a\lambda)^2 < 1$$

$$\Rightarrow |D^2| = (1+2a\lambda)^2 < 1$$

$$\Rightarrow |D| = |1+2a\lambda| < 1$$ which is possible

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$\Rightarrow \quad -1 < 1+2a\lambda < 1$

$\Rightarrow -1 < 1+2a\lambda$ | $1+2a\lambda < 1$

$\Rightarrow -2 < +2a\lambda$ | $+2a\lambda < 0$

$\Rightarrow -1 < a\lambda$ | $+a\lambda < 0$

$-1 < a\lambda < 0$ is false

$\Rightarrow$ for $a>0$ forward forward difference is unstable.

And if we assume $a<0$ then forward difference will be stable i.e $|1+2a\lambda|<1$ for $a<0$

$\Rightarrow$ **Stability Analysis:-** (Backward Difference)

$$u_j^{n+1} = (1-a\lambda)u_j^n + a\lambda u_{j-1}^n$$

Assume $a>0$

$$\Rightarrow \hat{u}^{n+1}(\xi) = (1-a\lambda)\hat{u}^n(\xi) + a\lambda e^{-i\xi}\hat{u}^n(\xi)$$

$$= \left[1-a\lambda+a\lambda e^{-i\xi}\right]\hat{u}^n(\xi)$$

$$\hat{u}^{n+1}(\xi) = \left[1-a\lambda+a\lambda\cos\xi + ia\lambda\sin\xi\right]\hat{u}^n(\xi)$$

$$\hat{u}^{n}(\xi) = \left[1-a\lambda+a\lambda\cos\xi + ia\lambda\sin\xi\right]\hat{u}^{n-1}(\xi)$$

$$\Rightarrow \hat{u}^{n}(\xi) = \left[1-a\lambda+a\lambda\cos\xi + ia\lambda\sin\xi\right]^n\hat{u}^{0}(\xi)$$

for stability $|D|<1 \Rightarrow |D|^2<1$,

where $D = (1-a\lambda+a\lambda\cos\xi + ia\lambda\sin\xi)$

$$|D|^2<1 \Rightarrow |1-a\lambda+a\lambda\cos\xi+ia\lambda\sin\xi|<1$$

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$\Rightarrow (1-a\lambda)^2 + a^2\lambda^2\cos^2\xi + a^2\lambda^2\sin^2\xi + 2a\lambda(1-a\lambda)\cos\xi$

$\Rightarrow (1-a\lambda)^2 + a^2\lambda^2 + 2a\lambda(1-a\lambda)\cos\xi < 1$

for $\cos\xi = 1$ $\Rightarrow 1 < 1$ not possible

$\cos\xi = -1$ implies

$(1-a\lambda)^2 + a^2\lambda^2 - 2a\lambda(1-a\lambda) < 1$

$\Rightarrow (1-a\lambda-a\lambda)^2 < 1 \Rightarrow (1-2a\lambda)^2 < 1$

$\Rightarrow |D^2| = |1-2a\lambda|^2 < 1$

$\Rightarrow |D| = |1-2a\lambda| < 1$ is true $\because a > 0$

$\Rightarrow$ Backward difference is stable when $a > 0$. and if we assume $a < 0$ then this scheme is unstable.

$|1-2a\lambda| < 1 \Rightarrow -1 < 1-2a\lambda < 1$

$\Rightarrow -1 < 1-2a\lambda$

$\Rightarrow -2 < -2a\lambda$

$\Rightarrow -1 < -a\lambda$

$\Rightarrow a\lambda < 1$

$1-2a\lambda < 1$

$\Rightarrow -2a\lambda < 0$

$-a\lambda < 0$

$\lambda < \frac{1}{a} \Rightarrow \frac{k}{h} < \frac{1}{a}$

$\Rightarrow k < \frac{h}{a}$

time step

$\Rightarrow$ Central Difference:- $u_t + au_x = 0$

$$\frac{u_j^{n+1} - u_j^n}{k} + a\frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0$$

$$\Rightarrow u_j^{n+1} = u_j^n + \frac{a\lambda}{2}(u_{j+1}^n - u_{j-1}^n)$$

Stability Analysis:-

$$u_j^{n+1} = u_j^n + \frac{a\lambda}{2}(u_{j+1}^n - u_{j-1}^n)$$

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$$\Rightarrow \hat{u}^{n+1}(\xi) = \hat{u}^n(\xi) + \frac{a\lambda}{2}(e^{i\xi} - e^{-i\xi})\hat{u}^n(\xi)$$

$$= \left[1 + \frac{a\lambda}{2}(e^{i\xi} - e^{-i\xi})\right]\hat{u}^n(\xi)$$

$$= \left[1 + \frac{a\lambda}{\cancel{2}} \cdot \cancel{2} i \sin\xi\right]\hat{u}^n(\xi)$$

$$\hat{u}^{n+1}(\xi) = \left[1 + i a\lambda \sin\lambda\right] \hat{u}^n(\xi)$$

$$\Rightarrow \hat{u}^n(\xi) = \left[1 + i a\lambda \sin\xi\right]^n \hat{u}^0(\xi)$$

$$|D| = |1 + i a\lambda \sin\xi| \quad , \xi \in [-\pi, \pi]$$

$$= \sqrt{1 + a^2\lambda^2 \sin^2\xi} \geq 1 \quad \forall \xi$$

$\Rightarrow$ This is unconditionally unstable

★ There are two ways to stable this method which we study later.

$\Rightarrow$ Truncation Error:-

$$\frac{u_j^{n+1} - u_j^n}{k} + a\frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0$$

$$T.E = \frac{u(t+k, x) - u(t,x)}{k} + a\frac{u(t, x+h) - u(t, x-h)}{2h} \longrightarrow (A)$$

$$\frac{u(t+k, x) - u(t,x)}{k} = u_t + \frac{k}{2}u_{tt} + \frac{k^2}{6}u_{ttt} + O(k^3) \longrightarrow (*)$$

Now

$$u(t, x+h) = u(t,x) + hu_x + \frac{h^2}{2}u_{xx} + \frac{h^3}{6}u_{xxx} + \frac{h^4}{24}u_{xxxx}$$

$$+ O(h^5) \longrightarrow (i)$$

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$$u(t, x-h) = u - hu_x + \frac{h^2}{2}u_{xx} - \frac{h^3}{6}u_{xxx} + \frac{h^4}{24}u_{xxxx} - O(h^5) \longrightarrow (ii)$$

Equ (i) − (ii) implies

$$u(t,x+h) - u(t,x-h) = 2hu_x + \frac{h^3}{3}u_{xxx} + O(h^5)$$

$$\Rightarrow \frac{u(t,x+h) - u(t,x-h)}{2h} = u_x + \frac{h^2}{3}u_{xxx} + O(h^4) \longrightarrow (**)$$

Putting the values of (*) & (**) in equ (1)

$$T.E = u_t + \frac{k}{2}u_{tt} + \frac{k^2}{6}u_{ttt} + O(k^3) + a\left[u_x + \frac{h^2}{3}u_{xxx} + O(h^4)\right]$$

$$= \cancel{(u_t + au_x)}^{0} + \frac{k}{2}u_{tt} + O(k^2) + \frac{ah^2}{3}u_{xxx} + O(h^4)$$

$$T.E = O(k, h^2)$$

⇒ **Richtmyer Equivalence Theorem:-** A consistent finite difference scheme for Partial Differential Equations of which the initial value problem is well posed is convergent if and only if it is stable.

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$$u_t + a u_x = 0$$

$$u_j^{n+1} = u_j^n - a\lambda[u_{j+1}^n - u_{j-1}^n], \text{ where } \lambda = \frac{k}{2h}$$

is central difference & unconditionally unstable.

★ There are two ideas for stabilizing central difference method.

1) $u(t,x)$ modified s.t in truncation the coefficient of $u_{xx} \leq 1$ [This method is called Lax-Friedrichs Method]

2) One can also represent (formulate) the central difference method as a method with inherent diffusion [Lax-Wendrof Method] OR u_{xx} is discretized s.t coefficient of u_{xx} in truncation error disappear.

★ Lax-Friedrich Method:- T.E $\sim$ $O(k, h)$

★ Lax-Wendrof Method:- T.E $\sim$ $O(k^2, h^2)$

★ **Method 1:-** $u_t + a u_x = 0$

$$u_j^{n+1} = u_j^n - a\lambda[u_{j+1}^n - u_{j-1}^n]$$

$$\boxed{\Rightarrow u_j^{n+1} = \frac{u_{j-1}^n + u_{j+1}^n}{2} - a\frac{\lambda}{2}[u_{j+1}^n - u_{j-1}^n]}$$

Stability Analysis:-

$$u_j^{n+1} = \frac{u_{j-1}^n + u_{j+1}^n}{2} - a\lambda[u_{j+1}^n - u_{j-1}^n]$$

$$\Rightarrow \hat{u}^{n+1}(\xi) = \frac{e^{-i\xi}\hat{u}^n(\xi) + e^{i\xi}\hat{u}^n(\xi)}{2} - a\lambda[e^{i\xi} - e^{-i\xi}]\hat{u}^n(\xi)$$

$$= \frac{e^{-i\xi} + e^{i\xi}}{2}\hat{u}^n(\xi) - a\lambda(i\,2\sin\xi)\hat{u}^n(\xi)$$

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$$\Rightarrow \hat{u}^{n+1}(\xi) = [\cos\xi - i2a\lambda\sin\xi]\,\hat{u}^{n}(\xi)$$

$$\Rightarrow \hat{u}^{n}(\xi) = [\cos\xi - i2a\lambda\sin\xi]\,\hat{u}^{n-1}(\xi)$$

$$\Rightarrow \hat{u}^{n}(\xi) = [\cos\xi - i2a\lambda\sin\xi]^{n}\,\hat{u}^{0}(\xi)$$

$$|D| = |\cos\xi - i2a\lambda\sin\xi| \quad ; \xi \in [-\pi, \pi)$$

$$= \sqrt{\cos^2\xi + 4a^2\lambda^2\sin^2\xi}$$

$$= 2a\lambda < 1$$

$\Rightarrow$ Scheme is unconditionally stable.

$$|D|^2 = \cos^2\xi + 4a^2\lambda^2\sin^2\xi$$
$$= 1 - \sin^2\xi + 4a^2\lambda^2\sin^2\xi$$
$$= 1 - (1 - \underbrace{4a^2\lambda^2}_{\text{small \& +ve}})\underbrace{\sin^2\xi}_{+ve \in [0,1]}$$
$$< 1 \qquad \boxed{1 - 0\ldots < 1}$$
$$\Rightarrow |D|^2 < 1 \Rightarrow |D| < 1$$

2nd Explanation:-

$$|D| = \sqrt{\cos^2\xi + 4a^2\lambda^2\sin^2\xi}$$

Let $f = |D|^2 = \cos^2\xi + 4a^2\lambda^2\sin^2\xi$

$$\frac{\partial f}{\partial \xi} = -2\cos\xi\sin\xi + 4a^2\lambda^2 2\sin\xi\cos\xi = 0$$

$$\Rightarrow [-1 + 4a^2\lambda^2]\,2\sin\xi\cos\xi = 0$$

$$\Rightarrow (-1 + 4a^2\lambda^2)\sin 2\xi = 0 \quad ; \xi \in [\pi, \pi]$$

$$\Rightarrow \sin 2\xi = 0 \Rightarrow 2\xi = n\pi, \quad n = 0, 1, 2, \ldots, -1, -2, \ldots$$

$$\Rightarrow \xi = \frac{n\pi}{2} \quad ; \xi \in [-\pi, \pi]$$

$\Rightarrow \xi = 0, \frac{\pi}{2}, -\frac{\pi}{2}$ and at these values $|D| < 1$

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* Truncation Error:-

$$u_j^{n+1} = \frac{u_{j-1}^n + u_{j+1}^n}{2} - \frac{a\lambda}{2}\left[u_{j+1}^n - u_{j-1}^n\right]$$

$$\Rightarrow u_j^{n+1} - \frac{u_{j-1}^n - u_{j+1}^n}{2} = -\frac{ak}{2h}\left[u_{j+1}^n - u_{j-1}^n\right] \quad \because \lambda = \frac{k}{h}$$

$$\Rightarrow \frac{2u_j^{n+1} - u_{j-1}^n - u_{j+1}^n}{2k} = -a\left[\frac{u_{j+1}^n - u_{j-1}^n}{2h}\right]$$

$$\Rightarrow T.E = \frac{2u_j^{n+1} - u_{j-1}^n - u_{j+1}^n}{2k} + a\left[\frac{u_{j+1}^n - u_{j-1}^n}{2h}\right] \rightarrow (A)$$

$$\therefore \quad = \frac{2u(t+k,x) - u(t,x-h) - u(t,x+h)}{2k} + a\left[\frac{u(t,x+h) - u(t,x-h)}{2h}\right]$$

Now

$$u(t+k,x) = u + ku_t + \frac{k^2}{2!}u_{tt} + \frac{k^3}{3!}u_{ttt} + O(k^4)$$

$$u(t,x-h) = u - hu_x + \frac{h^2}{2!}u_{xx} - \frac{h^3}{3!}u_{xxx} + \frac{h^4}{4!}u_{xxxx} + O(h^5)$$

$$u(t,x+h) = u + hu_x + \frac{h^2}{2!}u_{xx} + \frac{h^3}{3!}u_{xxx} + \frac{h^4}{4!}u_{xxxx} + O(h^5)$$

$$\Rightarrow 2u(t+k,x) - [u(t,x-h) + u(t,x+h)] = \cancel{2u} + 2ku_t + k^2u_{tt} + \frac{k^3}{3}u_{ttt} + O(k^4) - \cancel{2u} - h^2u_{xx} - \frac{h^4}{12}u_{xxxx} + O(h^6)$$

$$\Rightarrow \frac{2u(t+k,x) - u(t,x-h) - u(t,x+h)}{2k} = u_t + \frac{k}{2}u_{tt} + \frac{k^2}{6}u_{ttt} + O(k^3) - \frac{h^2}{2k}u_{xx} - \frac{h^4}{24k}u_{xxxx} + O(h^5)$$

&

$$\frac{u(t,x+h) - u(t,x-h)}{2h} = u_x + \frac{h^2}{6}u_{xxx} + O(h^4)$$

$$\Rightarrow T.E = u_t + \frac{k}{2}u_{tt} + \frac{k^2}{6}u_{ttt} + O(k^3) - \frac{h^2}{2k}u_{xx} - O(h^3) + a\left[u_x + \frac{h^2}{6}u_{xxx} + O(h^4)\right].$$

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$$\Rightarrow T.E = (u_t + a u_x) + \frac{k}{2} u_{tt} + O(k^2) - \frac{h^2}{2\lambda k} u_{xx}$$
$$- O(h^3) + \frac{a h^2}{6} u_{xxx} + O(h^4)$$

$$\because u_t + a u_x = 0$$

$$\Rightarrow T.E = \frac{k}{2} u_{tt} + O(k^2) - \frac{h}{2\lambda} u_{xx} + O(h^3)$$

$$\Rightarrow T.E \sim O(k, h)$$

Method 2 :-

$$u_j^{n+1} = u_j^n - \frac{a\lambda}{2}\left[u_{j+1}^n - u_{j-1}^n\right] + \frac{a^2\lambda^2}{2}\left[u_{j-1}^n - 2u_j^n + u_{j+1}^n\right]$$

Truncation Error :-

$$u_j^{n+1} - u_j^n = -\frac{ak}{2h}\left[u_{j+1}^n - u_{j-1}^n\right] + \frac{a^2k^2}{2h^2}\left[u_{j-1}^n - 2u_j^n + u_{j+1}^n\right]$$

$$\Rightarrow \frac{u_j^{n+1} - u_j^n}{k} = -a\left[\frac{u_{j+1}^n - u_{j-1}^n}{2h}\right] + \frac{a^2 k}{2}\left[\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2}\right]$$

$$\Rightarrow T.E \sim \frac{u(t+k,x) - u(t,x)}{k} + a\left[\frac{u(t,x+h) - u(t,x-h)}{2h}\right]$$
$$- \frac{a^2 k}{2}\left[\frac{u(t,x-h) - 2u(t,x) + u(t,x+h)}{h^2}\right] \quad (*)$$

Now

$$u(t+k,x) = u + k u_t + \frac{k^2}{2!} u_{tt} + \frac{k^3}{3!} u_{ttt} + O(k^4)$$

$$\Rightarrow \frac{u(t+k,x) - u(t,x)}{k} = u_t + \frac{k}{2} u_{tt} + \frac{k^2}{6} u_{ttt} + O(k^3) \quad (1)$$

Now

$$u(t,x+h) = u + h u_x + \frac{h^2}{2!} u_{xx} + \frac{h^3}{3!} u_{xxx} + \frac{h^4}{4!} u_{xxxx} + O(h^5)$$

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$$u(t,x-h) = u - hu_x + \frac{h^2}{2!}u_{xx} - \frac{h^3}{3!}u_{xxx} + \frac{h^4}{4!}u_{xxxx} - O(h^5)$$

$$\Rightarrow \frac{u(t,x+h)-u(t,x-h)}{2h} = u_x + \frac{h^2}{6}u_{xxx} + O(h^4) \longrightarrow ②$$

&

$$\frac{u(t,x+h)-2u(t,x)+u(t,x-h)}{h^2} = u_{xx} + \frac{h^2}{12}u_{xxxx} + O(h^4) \longrightarrow ③$$

Putting equation ①, ② & ③ in equ ⊛ implies that

$$T.E = u_t + \frac{k}{2}u_{tt} + \frac{k^2}{6}u_{ttt} + O(k^3) + a\left[u_x + \frac{h^2}{6}u_{xxx}\right.$$

$$\left. + O(h^4)\right) - \frac{a^2 k}{2}\left[u_{xx} + \frac{h^2}{12}u_{xxxx} + O(h^4)\right]$$

$$= \overset{0}{(u_t + au_x)} + \frac{k}{2}u_{tt} + \frac{k^2}{6}u_{ttt} + O(k^3) + \frac{ah^2}{6}u_{xxx}$$

$$+ O(h^4) - \frac{k}{2}a^2u_{xx} - \frac{a^2kh^2}{24}u_{xxxx} + O(h^4)$$

$$= \cancel{\frac{k}{2}u_{tt}} + \frac{k^2}{6}u_{ttt} + O(k^3) + \frac{ah^2}{6}u_{xxx} + O(h^4)$$

$$- \cancel{\frac{k}{2}u_{tt}} + O(h^3)$$

$$\boxed{\because \; u_t = -au_x \;\Rightarrow u_{tt} = a^2u_{xx}}$$

$$\Rightarrow T.E \sim O(h^2, k^2)$$

Stability Analysis:-

$$u_j^{n+1} = u_j^n - \frac{a\lambda}{2}\left[u_{j+1}^n - u_{j-1}^n\right] + \frac{a^2\lambda^2}{2}\left[u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

After applying fourier series we have

$$\hat{u}^{n+1}(\xi) = \hat{u}^n(\xi) - \frac{a\lambda}{2}\left[e^{i\xi} - e^{-i\xi}\right]\hat{u}^n(\xi) + \frac{a^2\lambda^2}{2}\left[e^{i\xi} - 2 + e^{-i\xi}\right]\hat{u}^n(\xi)$$

$$= \hat{u}^n(\xi) - \frac{a\lambda}{2}(2i\sin\xi)\hat{u}^n(\xi) + \frac{a^2\lambda^2}{2}(2\cos\xi - 2)\hat{u}^n(\xi)$$

$$= \left[1 - \frac{a\lambda}{2}(2i\sin\xi) + \frac{a^2\lambda^2}{2}(2\cos\xi - 2)\right]\hat{u}^n(\xi)$$

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$$\Rightarrow \hat{u}^{n+1}(\xi) = \left[1 - ia\lambda\sin\xi - 2a^2\lambda^2\sin^2\xi/2\right]\hat{u}^n(\xi)$$

$$= \left[1 - 2a^2\lambda^2\sin^2\xi/2 - ia\lambda\sin\xi\right]\hat{u}^n(\xi)$$

$$\Rightarrow \hat{u}^n(\xi) = \left[1 - 2a^2\lambda^2\sin^2\xi/2 - ia\lambda\sin\xi\right]^n \hat{u}^0(\xi)$$

$$\Rightarrow \hat{u}^n(\xi) = |D|\,\hat{u}^0(\xi)$$

where $|D| = \left|1 - 2a^2\lambda^2\sin^2\xi/2 - ia\lambda\sin\xi\right|$

$$|D|^2 = \left(1 - 2a^2\lambda^2\sin^2\xi/2\right)^2 + (a\lambda\sin\xi)^2$$

$$= 1 + 4a^4\lambda^4\sin^4\xi/2 - 4a^2\lambda^2\sin^2\xi/2 + a^2\lambda^2\sin^2\xi$$

$$= 1 + 4a^4\lambda^4\sin^4\xi/2 - 4a^2\lambda^2\sin^2\xi/2 + 4a^2\lambda^2\sin^2\xi/2(1-$$

$$= 1 + 4a^4\lambda^4\sin^4\xi/2 - \cancel{4a^2\lambda^2\sin^2\xi/2} + \cancel{4a^2\lambda^2\sin^2\xi/2} - 4a^2\lambda^2\sin^4\xi/2$$

$$|D|^2 \leq 1 \Rightarrow \cancel{1} + 4a^4\lambda^4\sin^4\xi/2 - 4a^2\lambda^2\sin^4\xi/2 \leq \cancel{1}$$

$$\Rightarrow (a^2\lambda^2 - 1)a^2\lambda^2\sin^4\xi/2 \leq 0$$

Now $f(\xi) = \sqrt{a^2\lambda^2 - 1}\,a\lambda\sin^2\xi/2$

$$\frac{\partial f}{\partial \xi} = 0 \Rightarrow \sqrt{a^2\lambda^2-1}\,a\lambda\cdot 2\sin\xi/2\cos\xi/2\cdot\frac{1}{2} = 0$$

$$\Rightarrow \frac{\sin\xi}{2} = 0 \Rightarrow \sin\xi = 0$$

$$\Rightarrow \xi = 0, \pm\pi \quad \because \xi \in [-\pi, \pi]$$

Now $\xi = 0 \Rightarrow$ Ident

& $\xi = \pm\pi \Rightarrow a^2\lambda^2 - 1 \leq 0$

$$\Rightarrow a^2\lambda^2 \leq 1$$

$$\Rightarrow |D|^2 \leq 1 \Rightarrow |D| \leq 1$$

$\Rightarrow$ Scheme is unconditionally stable.

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⇒ General Three Point Method:-

$(x-h,\ x,\ x+h)$

[Method of Unknown Coefficients]

$$u(t+k,x) = c_{-1}u(t,x-h)+c_0u(t,x)+c_1u(t,x+h) \longrightarrow (*)$$

$$\Rightarrow u(t,x)+ku_t(t,x)+\frac{k^2}{2!}u_{tt}(t,x)+O(k^3)$$

$$= c_{-1}\left[u(t,x)-hu_x(t,x)+\frac{h^2}{2!}u_{xx}(t,x)+O(h^3)\right]$$

$$+c_0u(t,x)+c_1\left[u(t,x)+hu_x(t,x)+\frac{h^2}{2!}u_{xx}(t,x)+O(h^3)\right]$$

$$=(c_{-1}+c_0+c_1)u(t,x)+(c_1-c_{-1})hu_x$$

$$+(c_1+c_{-1})\frac{h^2}{2!}u_{xx}+O(h^3)$$

As $u_t+au_x=0$
$u_t=-au_x$
$u_{tt}=a^2u_{xx}$

$$\Rightarrow u(t,x)-kau_x(t,x)+\frac{a^2k^2}{2}u_{xx}(t,x)+O(k^3)$$

$$=(c_{-1}+c_0+c_1)u(t,x)+(c_1-c_{-1})hu_x$$

$$+(c_1+c_{-1})\frac{h^2}{2!}u_{xx}+O(h^3)$$

Now comparing the coefficients of u, u_x, u_{xx}

$$\Rightarrow \quad c_{-1}+c_0+c_1=1$$

$$(c_1-c_{-1})h=-ka \Rightarrow c_1-c_{-1}=\frac{-ka}{h}$$

$$(c_1+c_{-1})\frac{h^2}{2}=\frac{a^2k^2}{2} \Rightarrow c_1+c_{-1}=\frac{a^2k^2}{h^2}$$

As $\lambda=\frac{k}{h}$. This implies

$$c_{-1}+c_0+c_1=1$$

$$c_1-c_{-1}=-a\lambda, \quad c_1+c_{-1}=a^2\lambda^2$$

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⇒ Order of The Method:-

1) Zero oder ⇒ $u = $ constant

$\Rightarrow c_{-1} + c_0 + c_1 = 1$

we have two free parameters.

2) First order ⇒ $u = $ linear

$\Rightarrow c_{-1} + c_0 + c_1 = 1 \longrightarrow (1)$

$c_1 - c_{-1} = -a\lambda \longrightarrow (2)$

we have one free parameter

Assume $c_0 = 0 \Rightarrow$

$$\left.\begin{aligned} c_{-1} + c_1 &= 1 \\ c_1 - c_{-1} &= -a\lambda \end{aligned}\right\} \text{addition} \Rightarrow 2c_1 = 1 - a\lambda$$

$\Rightarrow \boxed{c_1 = \frac{1-a\lambda}{2}}$ Put in (2) ⇒

$\frac{1-a\lambda}{2} - c_1 = -a\lambda \Rightarrow c_1 = \frac{1-a\lambda}{2} + a\lambda$

$\Rightarrow \boxed{c_{-1} = \frac{1+a\lambda}{2}}$

$\Rightarrow u(t+k, x) = \frac{1+a\lambda}{2} u(t, x-h) + \frac{1-a\lambda}{2} u(t, x+h)$

$= \frac{u(t, x-h) + u(t, x+h)}{2} - \frac{a\lambda}{2}\left[u(t, x+h) - u\right.$

$\Rightarrow u_j^{n+1} = \frac{u_{j-1}^n + u_{j+1}^n}{2} - \frac{a\lambda}{2}\left[u_{j+1}^n - u_{j-1}^n\right]$

This is Lax - Friedrich Method.

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Assume $c_1 = 0 \Rightarrow$

$$\left.\begin{aligned} C_{-1} + C_0 &= 1 \\ -C_{-1} &= -a\lambda \end{aligned}\right\} \text{Addition} \Rightarrow \boxed{C_0 = 1 - a\lambda}$$

$$\Rightarrow C_{-1} + 1 - a\lambda = 1 \Rightarrow \boxed{C_{-1} = a\lambda}$$

$$\Rightarrow u(t+k, x) = a\lambda[u(t, x-h)] + (1-a\lambda)u(t,x)$$

$$= a\lambda[u(t,x-h) - u(t,x)] + u(t,x)$$

$$= u(t,x) - a\lambda[u(t,x) - u(t,x-h)]$$

$$\Rightarrow u_j^{n+1} = u_j^n - a\lambda[u_j^n - u_{j-1}^n]$$

This is Backward difference.

Assume $c_{-1} = 0 \Rightarrow$

$$\left.\begin{aligned} C_0 + C_1 &= 1 \\ C_1 &= -a\lambda \end{aligned}\right\} \text{subtraction} \Rightarrow \boxed{C_0 = 1 + a\lambda}$$

$$\Rightarrow C_1 + 1 + a\lambda = 1 \Rightarrow \boxed{C_1 = -a\lambda}$$

$$\Rightarrow u(t+k, x) = (1+a\lambda)u(t,x) - a\lambda\, u(t, x+h)$$

$$\Rightarrow u_j^{n+1} = (1+a\lambda)u_j^n - a\lambda u_{j+1}^n$$

This is Forward difference.

ii) 2nd order Scheme $\Rightarrow$ u is quadratic

$$\Rightarrow C_{-1} + C_0 + C_1 = 1 \longrightarrow ①$$

$$C_1 - C_{-1} = -a\lambda \longrightarrow ② \qquad C_1 + C_{-1} = a^2\lambda^2 \longrightarrow ③$$

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Solve three equations simultaneously

Adding equ ③ + ② $\Rightarrow C_1 = \frac{-a\lambda}{2} + \frac{\lambda^2 a^2}{2}$

Subtracting equ ① − ② $\Rightarrow C_{-1} = \frac{a\lambda}{2} + \frac{\lambda^2 a^2}{2}$

Putting c_1, c_{-1} in equ ① $\Rightarrow C_0 = 1 - \lambda^2 a^2$

$$\Rightarrow u(t+k, x) = \left(\frac{a\lambda}{2} + \frac{a^2\lambda^2}{2}\right) u(t, x-h) + (1 - a^2\lambda^2) u(t,x)$$
$$+ \left(\frac{-a\lambda}{2} + \frac{a^2\lambda^2}{2}\right) u(t, x+h)$$
$$= u(t,x) + \frac{a\lambda}{2}\left[u(t,x-h) - u(t,x+h)\right]$$
$$+ \frac{a^2\lambda^2}{2}\left[u(t,x-h) + u(t,x+h)\right] - a^2\lambda^2 u(t,x)$$

$$u(t+k,x) = u(t,x) - \frac{a\lambda}{2}\left[u(t,x+h) - u(t,x-h)\right]$$
$$+ \frac{a^2\lambda^2}{2}\left[u(t,x-h) - 2u(t,x) + u(t,x+h)\right]$$
$$+ O(k^2, h^2)$$

This is Lax-Wendroff Method.

Other Substitution:- $\nu = a\lambda$

$$u(t+k,x) = u(t,x) - \frac{\nu}{2}\left[u(t,x+h) - u(t,x-h)\right]$$
$$+ \frac{Q}{2}\left[u(t,x-h) - 2u(t,x) + u(t,x+h)\right]$$

$$= \frac{\nu + Q}{2} u(t,x-h) + (1-Q) u(t,x)$$
$$+ \frac{Q - \nu}{2} u(t,x+h)$$

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Compare with

$$u(t+k,x) = C_{-1}\, u(t,x-h) + C_0\, u(t,x) + C_1\, u(t,x+h)$$

$$\Rightarrow C_{-1} = \frac{Q+V}{2}, \quad C_0 = 1-Q, \quad C_1 = \frac{Q-V}{2}$$

$$\Rightarrow C_{-1} + C_0 + C_1 = 1 \longrightarrow \text{Consistency condition}$$

$$C_1 - C_{-1} = -\nu = -a\lambda \longrightarrow \text{1st order}$$

$$C_1 + C_{-1} = Q = a^2\lambda^2 \longrightarrow \text{for Consistency}$$

$$\Rightarrow \boxed{Q = a^2\lambda^2 = \nu^2}$$

⇒ Stability Analysis:-

$$u(t+k,x) = u(t,x) - \frac{\nu}{2}\left[u(t,x+h) - u(t,x-h)\right] + \frac{Q}{2}\left[u(t,x-h) - 2u(t,x) + u(t,x+h)\right]$$

$$\Rightarrow \hat{u}^{n+1}(\xi) = \hat{u}^n(\xi) - \frac{\nu}{2}\left[e^{i\xi} - e^{-i\xi}\right]\hat{u}^n(\xi) + \frac{Q}{2}\left[e^{-i\xi} - 2 + e^{i\xi}\right]\hat{u}^n(\xi)$$

$$= \left[1 - \frac{\nu}{2}(i2\sin\xi) + \frac{Q}{2}(2\cos\xi - 2)\right]\hat{u}^n(\xi)$$

$$\hat{u}^{n+1}(\xi) = \left[1 - Q(1-\cos\xi) - i\nu\sin\xi\right]\hat{u}^n(\xi)$$

$$\Rightarrow \hat{u}^n(\xi) = \left[1 - Q(1-\cos\xi) - i\nu\sin\xi\right]^n \hat{u}^0(\xi)$$

$$D = 1 - Q(1-\cos\xi) - i\nu\sin\xi$$

$$= 1 - 2Q\sin^2\xi/2 - i\nu\sin\xi$$

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$$|D|^2 = 1+4Q^2\sin^4\tfrac{\xi}{2} - 4Q\sin^2\tfrac{\xi}{2} + \nu^2\sin^2\xi$$

$$= 1+4Q^2\sin^4\tfrac{\xi}{2} - 4Q\sin^2\tfrac{\xi}{2} + 4\nu^2\sin^2\tfrac{\xi}{2}\left(1-\sin^2\tfrac{\xi}{2}\right)$$

for stability $|D| \leq 1 \Rightarrow |D|^2 \leq 1$

$$\Rightarrow \cancel{1} + 4Q^2\sin^4\tfrac{\xi}{2} - 4Q\sin^2\tfrac{\xi}{2} + 4\nu^2\sin^2\tfrac{\xi}{2}\left(1-\sin^2\tfrac{\xi}{2}\right) \leq \cancel{1}$$

$$\Rightarrow Q^2\sin^2\tfrac{\xi}{2} - Q + \nu^2\left(1-\sin^2\tfrac{\xi}{2}\right) \leq 0$$

$$\Rightarrow \nu^2 - Q + (Q^2-\nu^2)\sin^2\tfrac{\xi}{2} \leq 0$$

If $\xi = 0 \Rightarrow Q \geq \nu^2$

If $\xi = \pm\pi \Rightarrow \nu^2 - Q + (Q^2-\nu^2) \leq 0$

$$\Rightarrow Q(Q-1) \leq 0 \qquad \because Q = \nu^2$$

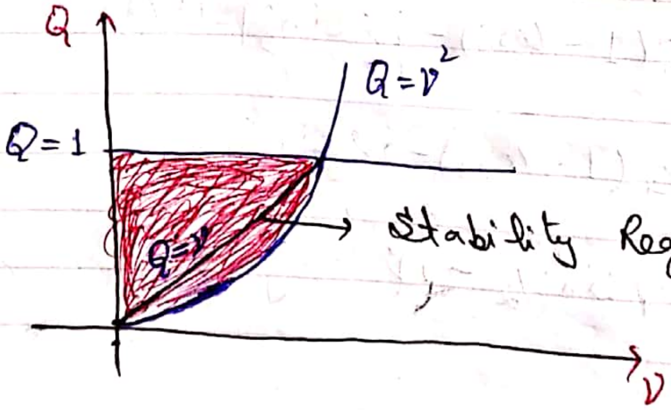
$$\Rightarrow Q \leq 1$$

$\because$ If $f(\xi) = \nu^2 - Q + (Q^2-\nu^2)\sin^2\xi$

$$\frac{df}{d\xi} = 0 \Rightarrow 2\sin\tfrac{\xi}{2}\cos\tfrac{\xi}{2} = 0$$

$$\Rightarrow \sin\xi = 0 \quad \Rightarrow \xi = 0, \pm\pi$$

$\Rightarrow 0 < \nu \leq Q \leq 1$ stability Region



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$$u(t+k,x) = u(t,x) - \frac{\nu}{2}[u(t,x+h) - u(t,x-h)] + \frac{Q}{2}[u(t,x-h) - 2u(t,x) + u(t,x+h)]$$

1) $Q = 1 \Rightarrow$ Lax - Friedrich (Stable)
2) $Q = \nu \Rightarrow$ Backward difference (Stable)
3) $Q = -\nu \Rightarrow$ Forward difference (Unstable)
4) $Q = 0 \Rightarrow$ Central differe (Unstable)

★ $u_j^{n+1} = u_j^n - \frac{a\lambda}{2}[u_{j+1}^n - u_{j-1}^n]$ (Central Difference)

This is unconditionally unstable. And if we modify it as (Explicit $\rightarrow$ Emplicit)

$$u_j^{n+1} = u_j^n - \frac{a\lambda}{2}[u_{j+1}^{n+1} - u_{j-1}^{n+1}]$$

This is unconditionally stable.

Now

$$\hat{u}^{n+1}(\xi) = \hat{u}^n(\xi) - \frac{a\lambda}{2}[e^{i\xi} - e^{-i\xi}]\hat{u}^{n+1}(\xi)$$

$$= \hat{u}^n(\xi) - \frac{a\lambda}{2}(i2\sin\xi)\hat{u}^{n+1}(\xi)$$

$$\hat{u}^{n+1}(\xi) = \hat{u}^n(\xi) - ia\lambda\sin\xi\,\hat{u}^{n+1}(\xi)$$

$$\Rightarrow (1 + ia\lambda\sin\xi)\hat{u}^{n+1}(\xi) = \hat{u}^n(\xi)$$

$$\Rightarrow \hat{u}^{n+1}(\xi) = \frac{1}{1 + ia\lambda\sin\xi}\hat{u}^n(\xi)$$

$$\Rightarrow \hat{u}^n(\xi) = \left[\frac{1}{1 + ia\lambda\sin\xi}\right]^n \hat{u}^0(\xi)$$

Here $D = \dfrac{1}{1 + ia\lambda\sin\xi}$

Now

$$|D|^2 = \frac{1}{|1 + ia\lambda\sin\xi|^2}$$

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$$\Rightarrow |D|^2 = \frac{1}{1+a^2\lambda^2\sin^2\xi} \leq 1 \quad \forall\, \xi \in [-\pi,\pi]$$

$$\Rightarrow |D| \leq 1$$

$\Rightarrow$ Scheme is unconditionally stable

★ $$\frac{u_j^{n+1}-u_j^n}{k} = \frac{-a}{2h}\left[u_{j+1}^n - u_{j-1}^n\right]$$ (Unconditionally Unstable)

If we modify it as

$$\frac{u_j^{n+1}-u_j^{n-1}}{2k} = \frac{-a}{2h}\left[u_{j+1}^n - u_{j-1}^n\right]$$ This is unconditionally stable.

$$\Rightarrow u_j^{n+1} = u_j^{n-1} - \frac{ak}{h}\left[u_{j+1}^n - u_{j-1}^n\right]$$

$$\Rightarrow \begin{bmatrix} u(t+k,x) = -a\lambda\Delta_c u(t,x) + u(t-k,x) \\ u(t,x) = u(t,x) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u(t+k,x) \\ u(t,x) \end{bmatrix} = \begin{bmatrix} -a\lambda\Delta_c & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u(t,x) \\ u(t-k,x) \end{bmatrix}$$

$$\Rightarrow \hat{u}^{n+1}(\xi) = \begin{bmatrix} -a\lambda(e^{i\xi}-e^{-i\xi}) & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \hat{u}^n(\xi) \\ \hat{u}^{n-1}(\xi) \end{bmatrix}$$

$$\Rightarrow D = \begin{bmatrix} -2ia\lambda\sin\xi & 1 \\ 1 & 0 \end{bmatrix}$$

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Eigen Values are

$$\begin{vmatrix} \lambda_1 + 2ia\lambda\sin\xi & 1 \\ 1 & \lambda_1 \end{vmatrix} = 0$$

$$\Rightarrow \lambda_1^2 + i2a\lambda\lambda_1\sin\xi - 1 = 0$$

$$\Rightarrow \lambda_1 = \pm\sqrt{1 - a^2\lambda^2\sin^2\xi} - ia\lambda\sin\xi$$

$a\lambda \leq 1$ implies that

$$|\lambda_1| = 1 - a^2\lambda^2\sin^2\xi + a^2\lambda^2\sin^2\xi$$

$$= 1$$

If $a\lambda > 1$ implies that

$$|\lambda_1| = -i\sqrt{a^2\lambda^2\sin^2\xi - 1} - ia\lambda\sin\xi$$

$$= \left[\sqrt{a^2\lambda^2\sin^2\xi - 1} + a\lambda\sin\xi\right]^2$$

$$= \sqrt{a^2\lambda^2 - 1} + a\lambda > 1 \quad \because a\lambda > 1,\ \xi = \frac{\pi}{2}$$

<u>Other Method</u> $u_j^{n+1} = u_j^{n-1} - \frac{ak}{h}\left(u_{j+1}^n - u_{j-1}^n\right)$

$$\Rightarrow \hat{u}^{n+1}(\xi) = \hat{u}^{n-1}(\xi) - a\lambda\left[e^{i\xi} - e^{-i\xi}\right]\hat{u}^n(\xi)$$

$$= \hat{u}^{n-1}(\xi) - a\lambda\left[i2\sin\xi\right]\hat{u}^n(\xi)$$

$$\hat{u}^{n+1}(\xi) = \hat{u}^{n-1}(\xi) - i2a\lambda\sin\xi\,\hat{u}^n(\xi)$$

$$\Rightarrow \hat{u}^{n+1}(\xi) - \hat{u}^{n-1}(\xi) = -i2a\lambda\sin\xi\,\hat{u}^n(\xi)$$

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$$\Rightarrow D\hat{u}^n(\xi) - D^{-1}\hat{u}^n(\xi) = -i2a\lambda \sin\xi\, \hat{u}^n(\xi)$$

$$\Rightarrow D^2\hat{u}^n(\xi) + D\, i2a\lambda\sin\xi\, \hat{u}^n(\xi) - \hat{u}^n(\xi) = 0$$

$$\Rightarrow D = \frac{-i2a\lambda\sin\xi \pm \sqrt{(-i2a\lambda\sin\xi)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{-i2a\lambda\sin\xi \pm \sqrt{4 - 4a^2\lambda^2\sin^2\xi}}{2}$$

$$= \frac{-i2a\lambda\sin\xi \pm 2\sqrt{1 - a^2\lambda^2\sin^2\xi}}{2}$$

$$D = \pm\sqrt{1 - a^2\lambda^2\sin^2\xi} - ia\lambda\sin\xi$$

$$|D|^2 = 1 - a^2\lambda^2\sin^2\xi + a^2\lambda^2\sin\xi$$

$$= 1$$

$$\Rightarrow |D| = 1$$

$\Rightarrow$ Scheme is unconditionally stable.

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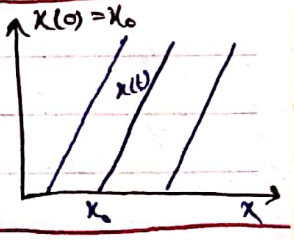
⇒ Method of Characteristics:-

The method of characteristics is a technique for solving PDEs typically, it applies to 1st order equations. Although more generally the method of characteristics is valid for any hyperbolic PDEs. The method is to reduce a PDE into an ODE along which the solution can be integrated from same initial data.

Example:- $u_t + a(u)u_x = 0 \longrightarrow ①$

As $\dfrac{du}{dt} = \dfrac{\partial u}{\partial t}\cdot\dfrac{dt}{dt} + \dfrac{\partial u}{\partial x}\dfrac{dx}{dt}$ (total differential)

$$= \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}$$



Let $\dfrac{dx}{dt} = a(u)$ implies

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0 \quad \because \text{by } ①$$

$$\Rightarrow \frac{du}{dt} = 0 \quad \Rightarrow u(t,x) = u_0 = \text{constant}$$

Now $\dfrac{dx}{dt} = a(u_0) \quad \Rightarrow x = a(u_0)t + c$

$$\Rightarrow x = a(u_0)t + x_0$$

$$\Rightarrow \boxed{x_0 = x - a(u_0)t}$$

$$\Rightarrow u(t,x) = u_0(x_0) = u_0[x - a(u_0)t]$$

$\Rightarrow u = \text{constant} \quad \Rightarrow a(u) = \text{constant}$

$\Rightarrow$ characteristics are straight lines.

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Example 1 $u_t + a\,u_x = 0$, $a = \text{constant}$

$$u_0(x) = \begin{cases} x/2 & , 0 < x < 1 \\ 0 & \text{otherwise i.e } 0 \text{ for } x \geq 1 \end{cases}$$

Solution

$x(0) = a$

$$\frac{dx}{dt} = a \Rightarrow x = at + c$$

$$\Rightarrow x = at + x_0 \Rightarrow \boxed{x_0 = x - at}$$

$$u(t,x) = u_0(x_0) = \begin{cases} \frac{x_0}{2} & 0 < x_0 < 1 \\ 0 & x_0 \geq 1 \end{cases}$$

$$= \begin{cases} \frac{x-at}{2} , & 0 < x - at < 1 \\ 0 , & x - at \geq 1 \end{cases}$$

$$= \begin{cases} \frac{x-at}{2} , & 0 < x < 1+at \\ 0 , & x \geq 1+at \end{cases}$$

Example 2 :- $u_t - 2u_x = e^{2x}$;

$$u(0,x) = f(x)$$

Solution Here $a = -2$

$$\frac{dx}{dt} = -2 \Rightarrow x = -2t + x_0$$

$$\Rightarrow \boxed{x_0 = x + 2t}$$

$$\frac{du}{dt} = e^{2x} = e^{2(-2t+x_0)} = e^{-4t} \cdot e^{2x_0}$$

$$\Rightarrow u(t,x) = e^{2x_0} \cdot \frac{e^{-4t}}{-4} + c$$

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Now $u(0,x) = \frac{-e^{2x_0}}{4} + c$

$$\Rightarrow f(x_0) + \frac{e^{2x_0}}{4} = c$$

$$\Rightarrow u(t,x) = \frac{e^{2x_0}}{-4} \cdot e^{-4t} + f(x_0) + \frac{e^{2x_0}}{4}$$

$$= \frac{e^{2(x+2t)}}{-4} \cdot e^{-4t} + f(x+2t) + \frac{e^{2(x+2t)}}{4}$$

$$\Rightarrow u(t,x) = \frac{-e^{+2x}}{4} + f(x+2t) + \frac{e^{2(x+2t)}}{4}$$

Example If $u_t + uu_x = 0$ i.e. $u_t + (\frac{1}{2}u^2)_x = 0$

Here $a(u) = u$

$$u(t,x) = u_0[x - a(u)t]$$

$$= u_0[x - ut] \text{ is solution}$$

$\xi = x - ut$

$$\frac{\partial u}{\partial t} = \frac{\partial u_0}{\partial t} = \frac{\partial u_0}{\partial \xi} \cdot \frac{\partial \xi}{\partial t}$$

$$\Rightarrow \frac{\partial u_0}{\partial t} = u_0' \cdot \frac{\partial}{\partial t}\{x - ut\}$$

$$= u_0'(0 - u_t t - u1)$$

$$= u_0'[-tu_t - u]$$

To check the solution

$$\frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x}$$

$$= \frac{\partial u_0}{\partial \xi} \cdot \frac{\partial \xi}{\partial x}$$

$$= u_0'[1 - u_x t]$$

$$\Rightarrow u_t + uu_x = u_0'[-tu_t - u] + uu_0'[1 - u_x t]$$

$$= u_0[t\,uu_x - u] + uu_0'[1 - u_x t] \quad \because u_t = -uu_x$$

$$= 0$$

Example. $u_t + uu_x = 0$

$$u(0,x) = \begin{cases} 0 & , x < 0 \\ x & , 0 \le x \le 1 \\ 1 & , x > 1 \end{cases}$$

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Solution

$$u(t,x) = u_0(x_0) = \begin{cases} 0 & , x_0 < 0 \\ x_0 & , 0 \leq x_0 \leq 1 \\ 1 & , x_0 > 1 \end{cases}$$

Now

$$\frac{dx}{dt} = a(u_0) = u_0$$

① $\frac{dx}{dt} = u_0(x_0) = 0 \Rightarrow \boxed{x = x_0}$

② $\frac{dx}{dt} = u_0(x_0) = x_0 \Rightarrow x = x_0 t + c \Rightarrow x = x_0 t + x_0$

$$\Rightarrow \boxed{x_0 = \frac{x}{1+t}}$$

③ $\frac{dx}{dt} = 1 \Rightarrow x = t + c = t + x_0$

$$\Rightarrow \boxed{x_0 = x - t}$$

$$\Rightarrow u(t,x) = \begin{cases} 0 & , x_0 < 0 \\ \frac{x}{1+t} & 0 \leq \frac{x}{1+t} \leq 1 \\ 1 & \frac{x}{1+t} > 1 \end{cases}$$

$$= \begin{cases} 0 & , x < 0 \\ \frac{x}{1+t} & 0 \leq x \leq 1+t \\ 1 & x > 1+t \end{cases}$$

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Example 2:- $u_t + uu_x = 0$

$$u(0,x) = \begin{cases} 0 & , x < 0 \\ x & , 0 \leq x \leq 2 \\ 2 & , x > 2 \end{cases}$$

Solution

$$u(t,x) = u_0(x_0) = \begin{cases} 0 & , x_0 < 0 \\ x_0 & , 0 \leq x_0 \leq 2 \\ 2 & , x_0 > 2 \end{cases}$$

Now $\dfrac{dx}{dt} = a(u_0) = u_0$

① $\dfrac{dx}{dt} = u_0(x_0) = 0 \Rightarrow \boxed{x = x_0}$

② $\dfrac{dx}{dt} = u_0(x_0) = x_0 \Rightarrow x = x_0 t + c$

$\Rightarrow x = x_0 t + x_0$

$\Rightarrow \boxed{x_0 = \dfrac{x}{1+t}}$

③ $\dfrac{du}{dt} = u_0(x_0) = 2 \Rightarrow x = 2t + c$

$\Rightarrow x = 2t + x_0$

$\Rightarrow \boxed{x_0 = x - 2t}$

$$\Rightarrow u(t,x) = \begin{cases} 0 & , x_0 < 0 \\ \dfrac{x}{1+t} & , 0 \leq \dfrac{x}{1+t} \leq 2 \\ 2 & \dfrac{x}{1+t} > 2 \end{cases}$$

$$= \begin{cases} 0 & , x < 0 \\ \dfrac{x}{1+t} & , 0 \leq x \leq 2+2t \\ 1 & , x > 2+2t \end{cases}$$

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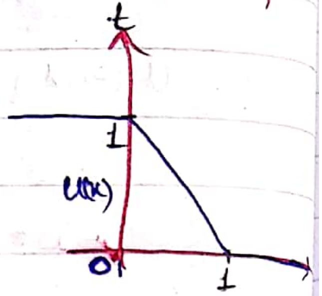
Example:- $u_t + uu_x = 0$ {Burger Equation}

$$u(0,x) = \begin{cases} 1 & x<0 \\ 1-x & 0\le x\le 1 \\ 0 & x>1 \end{cases}$$



Solution

$$\frac{dx}{dt} = u_0(x_0)$$

$$x(0) = x_0 \quad \Rightarrow u(t,x) = u_0(x_0) = \begin{cases} 1, & x_0<0 \\ 1-x_0, & 0\le x_0\le 1 \\ 0, & x_0>1 \end{cases}$$

$\Rightarrow$ ① $\frac{dx}{dt} = 1 \Rightarrow x = t + c$

$\Rightarrow x = t + x_0 \quad \Rightarrow \boxed{x_0 = x - t}$

② $\frac{dx}{dt} = 1 - x \Rightarrow x = (1-x_0)t + c$

$\Rightarrow x = (1-x_0)t + x_0 \quad \Rightarrow \boxed{x_0 = \frac{x-t}{1-t}}$

③ $\frac{dx}{dt} = 0 \Rightarrow \boxed{x = x_0}$

$$\Rightarrow u(t,x) = \begin{cases} 1, & x<t \\ \frac{1-x}{1-t}, & 0\le \frac{x-t}{1-t} \le 1 \\ 0, & x>1 \end{cases}$$

$$= \begin{cases} 1, & x<t \\ \frac{1-x}{1-t}, & t\le x\le 1 \\ 0, & x>1 \end{cases}$$

This is a solution of PDE for $0\le t<1$
{∵ At $t=1$ is discontinuity}
At $t=1$

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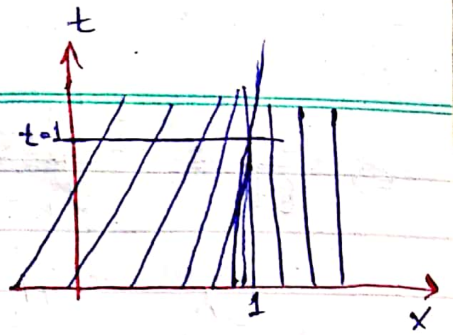
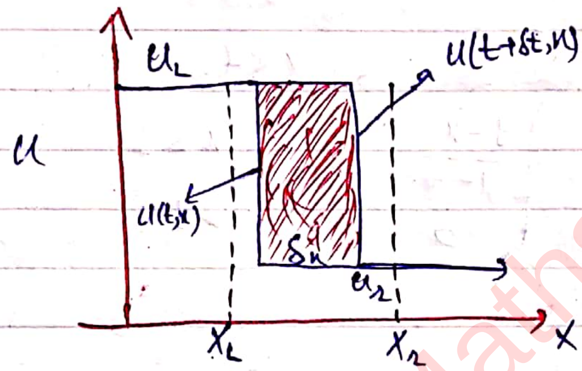
$$u(t,x) = \begin{cases} 1, & x < 1 \\ 0 & x \geq 1 \end{cases}$$



Now

$$u_t + uu_x = 0 \iff u_t + \left(\tfrac{1}{2}u^2\right)_x = 0$$

Let $f(u) = \frac{1}{2}u^2$ $\quad \Leftrightarrow u_t + f(u)_x = 0$



$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0$$

Integrating from x_L to x_2

$$\int_{x_L}^{x_2} \left(\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x}\right) dx = 0$$

$$\Rightarrow \frac{d}{dt}\int_{x_L}^{x_2} u\,dx + f(u_2) - f(u_L) = 0$$

$$-\frac{1}{\delta t}(u_2 - u_L)\cdot \delta x = -\left[f(u_2) - f(u_L)\right]$$

Shock speed $\Rightarrow \dfrac{\delta x}{\delta t} = \dfrac{f(u_2) - f(u_L)}{u_2 - u_L}$

$$= \frac{\frac{1}{2}u_2^2 - \frac{1}{2}u_L^2}{u_2 - u_L} \qquad \because f(u) = \tfrac{1}{2}u^2$$

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$$\Rightarrow \frac{\delta x}{\delta t} = \frac{\frac{1}{2}(u_2^2 - u_L^2)}{u_2 - u_2} = \frac{\frac{1}{2}(\cancel{(u_2 - u_L)})(u_2 + u_L)}{\cancel{(u_2 - u_L)}}$$

$$\Rightarrow \frac{\delta x}{\delta t} = \frac{u_L + u_2}{2}$$

$$\Rightarrow S = \frac{\delta x}{\delta t} = \frac{u_L + u_2}{2}$$

Shock speed

$$u(t,x) = \begin{cases} 1 & , x < t \\ \frac{1-x}{1-t} & , t \leq x \leq 1 \\ 0 & , x > 1 \end{cases}$$

for $0 \leq t < 1$

for $t \geq 1$

$$u_L = 1 \quad , u_2 = 0$$

$$\frac{dX_s}{dt} = \frac{f(u_L) - f(u_2)}{u_L - u_2} = \frac{\frac{1}{2} - 0}{1 + 0}$$

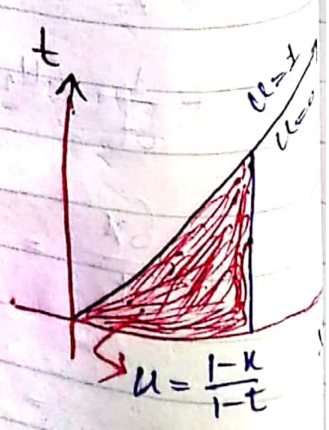
$$\Rightarrow \frac{dX_s}{dt} = \frac{1}{2}$$

$$\Rightarrow X_s = \frac{1}{2}t + c$$

$$\Rightarrow 1 = \frac{1}{2} \cdot 1 + c \Rightarrow c = \frac{1}{2} \quad \because \text{at } t=1,\ x=1$$

$$\Rightarrow X_s = \frac{1+t}{2}$$

$$\Rightarrow u(t,x) = \begin{cases} 1 & X_s < 1 \\ 0 & X_s \geq 1 \end{cases}$$



Example:- $u_t + uu_x = 0$ {Burger Equation}

Initial Condition = $\begin{cases} 0 & , \ x<0 \\ x & , \ 0 \le x \le 1 \\ 0 & , \ x>1 \end{cases}$

Solution

$$u(t,x) = u_0(x_0) = \begin{cases} 0 & , \ x_0<0 \\ x_0 & , \ 0 \le x_0 \le 1 \\ 0 & , \ x_0>1 \end{cases}$$

① $\frac{dx}{dt} = 0 \Rightarrow \boxed{x = x_0}$

② $\frac{dx}{dt} = x_0 \Rightarrow x = x_0 t + C \Rightarrow x = x_0 t + x_0$

$\Rightarrow x = x_0(t+1) \Rightarrow \boxed{x_0 = \frac{x}{1+t}}$

③ $\frac{dx}{dt} = 0 \Rightarrow \boxed{x = x_0}$

$$\Rightarrow u(t,x) = \begin{cases} 0 & , \ x<0 \\ \frac{x}{1+t} & , \ 0 \le \frac{x}{1+t} \le 1 \\ 0 & , \ x>1 \end{cases}$$

$$= \begin{cases} 0 & ; \ x<0 \\ \frac{x}{1+t} & ; \ 0 \le x \le 1+t \\ 0 & ; \ x>1 \end{cases}$$

$$\frac{dX_s}{dt} = \frac{f(u_L) - f(u_R)}{u_L - u_R} = \frac{1}{2}(u_L + u_R)$$

$$= \frac{1}{2}\left(\frac{x_0}{1+t} + 0\right) = \frac{1}{2} \frac{X_s}{1+t}$$

$$\Rightarrow \frac{dX_s}{X_s} = \frac{1}{2} \frac{dt}{1+t}$$

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$$\Rightarrow \ln X_s = \frac{1}{2}\ln(1+t) = \ln(1+t)^{\frac{1}{2}}$$

$$\Rightarrow X_s = \sqrt{1+t}$$

$$\Rightarrow u(t,x) = \begin{cases} 0 & x<0 \\ \frac{x}{1+t} & 0 \le x < \sqrt{1+t} \\ 0 & x > \sqrt{1+t} \end{cases}$$

$$u_t + uu_x = 0 \quad \Leftrightarrow \quad u_t + \left(\tfrac{1}{2}u^2\right)_x = 0$$

Conserved quantity is u
multiply with u

$$\Rightarrow uu_t + u^2 u_x = 0 \longrightarrow ①$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\tfrac{1}{2}u^2\right) + \frac{\partial}{\partial x}\left(\tfrac{1}{3}u^3\right) = 0 \longrightarrow ②$$

Let $v = \frac{1}{2}u^2$

$$\boxed{u = \sqrt{2v} \Rightarrow u^3 = (2v)^{\frac{3}{2}}}$$

Thus eqn ② implies

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\sqrt{8}}{3}v^{\frac{3}{2}}\right) = 0 \longrightarrow ③$$

v is conserved quantity $\because v = \frac{1}{2}u^2$

Characterist speed of eqn ① is $\frac{dX_s}{dt} = \frac{u_L+u_R}{2}$

characteristic speed of eqn ③ is

$$\frac{dX_s}{dt} = \frac{f(v_L)-f(v_R)}{v_L+v_R} = \frac{2}{3}\frac{u_L^3-u_R^3}{u_L^2-u_R^2} \neq \frac{1}{2}(u_L+u_R)$$

$$= \frac{\frac{\sqrt{8}}{3}v_L^{\frac{3}{2}} - \frac{\sqrt{8}}{3}v_R^{\frac{3}{2}}}{v_L - v_R}$$

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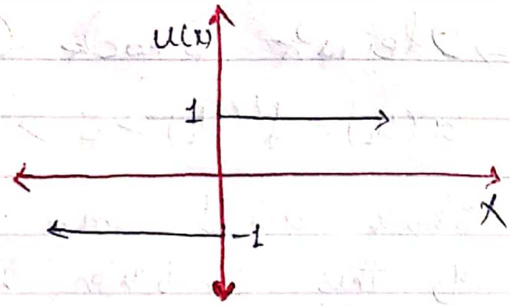
Example:- $u_t + uu_x = 0$

$$u_0(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$



Solution

$$s = \frac{f(u_L) - f(u_R)}{u_L - u_R} = \frac{1}{2}(u_L + u_R)$$

$$= \frac{1}{2}(-1+1) = 0$$

(No shock)

Possible Solution 1:-

$$u(t,x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

Possible Solution 2:-

$$u(t,x) = \begin{cases} -1 & x < -t \\ \frac{x}{t} & -t \le x < t \\ 1 & x > t \end{cases}$$

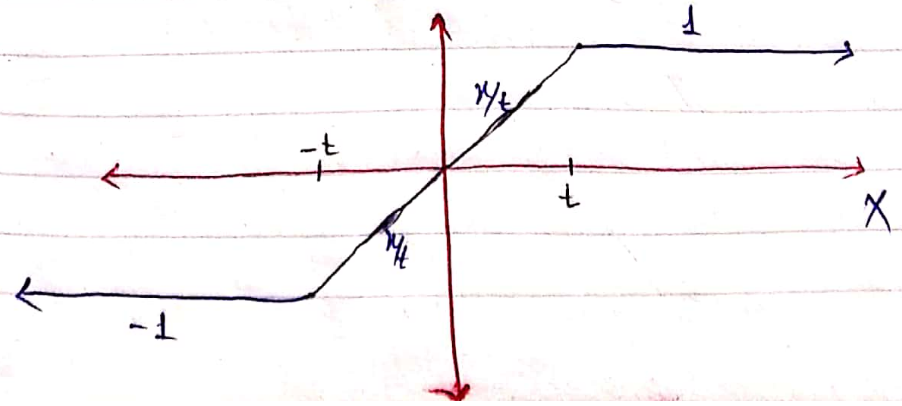
$\because \frac{dx}{dt} = u \Rightarrow$

$\frac{dx}{dt} = -1 \Rightarrow x = -t + c \Rightarrow x = -t + x_0$

$\Rightarrow \boxed{x_0 = x + t}$

$\frac{dx}{dt} = 1 \Rightarrow x = t + c \Rightarrow x = t + x_0$

$\Rightarrow \boxed{x_0 = x - t}$



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⇒ Oleinik Shock Condition:-

$$a(u_L) = f'(u_L) > S > f'(u_R) = a(u_R)$$

$u_t + f(u)_x = 0$

$u_t + \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} = 0$

$u_t + \underbrace{f'(u)}_{a(u)} u_x = 0$

Shock will form only if this condition holds.

In the current situation $a(u_L) \not> a(u_R)$

Thus there is no shock.

———————

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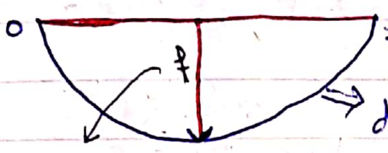
⇒ Finite Element Method for Elliptic PDE's:-

$$-\frac{\partial^2 u}{\partial x^2} = f(x) \quad , \quad x \in (0,1)$$

$$u(0) = 0 = u(1)$$



⇒ derivation from $u(x) = 0$

after force

⇒ Variational Formulation:-

$$I(u) = \int_0^1 \left[\frac{1}{2}(u'(x))^2 - f(x)\,u(x)\right] dx \longrightarrow (*)$$

goal is to minimize $I(u)$ i.e find $\min I(u)$, Here u is continuously differentiable function with $u(0) = 0 = u(1)$

$I(u)$ is function at functions space

$$X = \{ u \in C^1[0,1] \mid u(0) = 0 = u(1) \}$$

u is continuously differentiable at $(0,1)$, Continuous at $[0,1]$ and derivatives at 0 and 1 are continuously extable [no pole]

⇒ Banach Space:- (Completely vector-norm space)

with $\|u\|_x = \max\limits_{x \in [0,1]} |u(x)| + \max\limits_{x \in [0,1]} |u'(x)|$ (uniform convergence)

★ I is convex if

$$I(\lambda u + (1-\lambda)u) \leq \lambda I(u) + (1-\lambda) I(u) \; ; \; \lambda \in [0,1]$$

★ Necessary condition for extrema

$I'(x) = 0$ differentiable at Banach space like $\mathbb{R}^n$

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★ Concrete condition

$$0 = \frac{d}{d\varepsilon} I(u+\varepsilon\phi)\Big|_{\varepsilon=0} \quad ; \quad u, \phi \in X$$

$$\Rightarrow \frac{d}{d\varepsilon} \int_0^1 \left[\frac{1}{2}(u'(x)+\varepsilon\phi'(x))^2 - (u+\varepsilon\phi)f(x)\right] dx \Big|$$

$$= \int_0^1 \left[(u'(x)+\varepsilon\phi'(x))\phi'(x) - f\phi\right] dx \quad \left\{ \because \text{ differentiate w.r.t } \varepsilon \right.$$

★ for minima

$$\frac{d}{d\varepsilon}(u+\varepsilon\phi)\Big|_{\varepsilon=0} = 0$$

$$\Rightarrow \int_0^1 \left[u'(x)\phi'(x) - f(x)\phi(x)\right] dx = 0$$

using integration by parts and

$$\phi(0) = u(0) = 0, \; \phi(1) = u(1) = 0$$

$$\Rightarrow \int_0^1 \left[-u'' - f\right]\phi(x)\, dx = 0$$

$$\Rightarrow -u'' = f(x)$$

⇒ Finite Element Method:-

$$-u_{xx} = f(x)$$

$$u(t,x) = \sum_{i=1}^{n} u_i \, \phi(x) \rightarrow C_0^1$$

unknowns

are choosen in advance.

⇒ There are three finite element methods are generally used.

$$\boxed{r(x) = u^h_{xx} - f_h(x) \neq 0}$$

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① Galerkin FEM:-

$$\int_0^1 (u^h_{xx} - f)\,\phi(x)\,dx = 0$$

② Collocation Method:- The residual at certain choosen points in the given domain called collocation points.

③ Rayleigh-Ritz Method:- using the variational technique, the given PDE is re-written in an equivallent integral form (functional). The unknowns u_i are obtained from the minima of that functional.

1. Galerkin Method:-

$$\boxed{\begin{array}{l} -u'' = f \\ u(0) = 0 = u(1) \end{array}}$$

$$\int_0^1 (-u'' - f)\,\phi(x)\,dx = 0$$

$$\Rightarrow \int_0^1 -u''(x)\,\phi(x)\,dx - \int_0^1 f(x)\,\phi(x)\,dx = 0$$

$\because$ BCs $\phi(0) = 0 = \phi(1)$

$$\Rightarrow \cancel{-\phi(x)\,u'(x)\Big|_0^1}^{\,0} + \int_0^1 u'(x)\,\phi'(x)\,dx - \int_0^1 f(x)\,\phi(x)\,dx = 0$$

$$\Rightarrow \int_0^1 [u'(x)\,\phi'(x) - f(x)\,\phi(x)]\,dx = 0$$

$$\boxed{\begin{array}{l} u(t,x) = \sum_{j=1}^{n} u_j e_j(x) \\ u' = \sum_{j=1}^{n} u_j e_j' \end{array}}$$

$$= \int_0^1 [u'(t,x)\,e_i' - f e_i]\,dx \qquad i = 1,2,3,\ldots,n$$

$$\Rightarrow \int_0^1 \left[\left(\sum_{j=1}^{n} u_j e_j'\right) e_i' - f e_i\right] dx = 0$$

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$$\Rightarrow \sum_{j=1}^{n} u_j \int_0^1 e_i' e_j' \, dx = \int_0^1 f e_i \, dx \quad ; i = 1,2,3,\ldots$$

$$= \begin{bmatrix} \int_0^1 e_1' e_1' dx & \int_0^1 e_1' e_2' dx & \cdots & \int_0^1 e_1' e_n' dx \\ \int_0^1 e_2' e_1' dx & \int_0^1 e_2' d_2' dx & \cdots & \int_0^1 e_2' e_n' dx \\ \vdots & \vdots & & \vdots \\ \int_0^1 e_n' e_1' dx & \int_0^1 e_n' e_2' dx & \cdots & \int_0^1 e_n' e_n' dx \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^1 f e_1 dx \\ \int_0^1 f e_2 dx \\ \vdots \\ \int_0^1 f e_n dx \end{bmatrix} \longrightarrow \circledast$$

$$\Rightarrow \underline{\underline{A}} \, \underline{u} = \underline{F}$$

Basis Functions:-

$$e_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h} & \text{if } x \in [x_{j-1}, x_j] \\ \dfrac{x_{j+1} - x}{h} & \text{if } x \in [x_j, x_{j+1}] \\ 0 & \text{if } |x - x_j| > h \end{cases}$$

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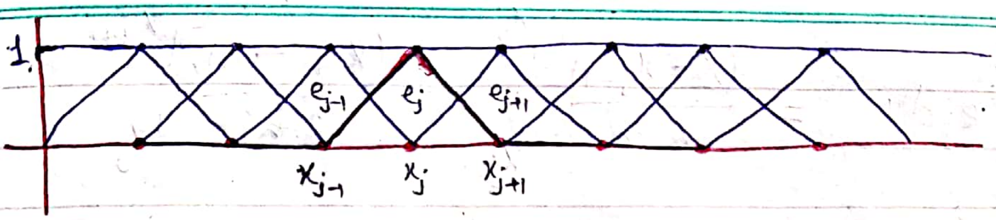

$$e_j'(x) = \begin{cases} \dfrac{1}{h} & \text{if } x \in [x_{j-1}, x_j] \\ -1/h & \text{if } x \in [x_j, x_{j+1}] \\ 0 & \text{otherwise} \end{cases}$$

Now $$\int_0^1 e_j' e_i' \, dx = \int_{x_{j-1}}^{x_{j+1}} e_j' e_i' \, dx + 0 + 0$$

$$i = j \Rightarrow \int_0^1 e_j'^2 \, dx = \int_{x_{j-1}}^{x_{j+1}} e_j'^2 \, dx = \int_{x_{j-1}}^{x_j} e_j'^2 \, dx + \int_{x_j}^{x_{j+1}} e_j'^2 \, dx$$

$$= \left(\frac{1}{h}\right)^2 \cdot h + \left(-\frac{1}{h}\right)^2 \cdot h = \frac{2}{h}$$

$$i = j-1 \Rightarrow \int_0^1 e_j' e_{j-1}' \, dx = \int_{x_{j-1}}^{x_{j+1}} e_j' e_{j-1}' \, dx = \int_{x_{j-1}}^{x_j} e_j' e_{j-1}' \, dx + \int_{x_j} e_j' e_{j-1}' \, dx$$

$$= \left(\frac{1}{h} \cdot \frac{-1}{h}\right) \cdot h + \left(-\frac{1}{h} \cdot 0\right) h = -\frac{1}{h}$$

$$i = j+1 \Rightarrow \int_0^1 e_j' e_{j+1}' \, dx = \int_{x_{j-1}}^{x_{j+1}} e_j' e_{j+1}' \, dx$$

$$= \frac{-1}{h}$$

And $$\int_0^1 e_j \, dx = \int_{x_{j-1}}^{x_j} \frac{x - x_{j-1}}{h} \, dx + \int_{x_j}^{x_{j+1}} \frac{x_{j+1} - x}{h} \, dx$$

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$$\Rightarrow \int_0^1 e_j \, dx = \frac{(x - x_{j-1})^2}{2h}\Bigg|_{x_{j-1}}^{x_j} + \frac{(x_{j+1} - x)^2}{2h}\Bigg|_{x_j}^{x_{j+1}}$$

$$= \frac{(x_j - x_{j-1})^2}{2h} - \frac{(x_{j-1} - x_{j-1})^2}{2h} + \frac{(x_{j+1} - x_{j+1})^2}{2h} - \frac{(x_{j+1} \ldots}{2 \ldots}$$

$$= \frac{h^2}{2h} - 0 + \frac{h^2}{2h} - 0 = \frac{h}{2} + \frac{h}{2}$$

$$= h$$

So ⊛ implies

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} ch \\ ch \\ \vdots \\ ch \end{bmatrix}$$

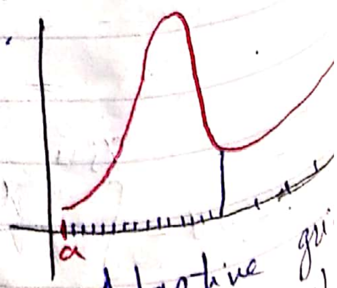
$\Rightarrow \underline{AU} = c\underline{H}$

$$-u_{xx} = c \Rightarrow \frac{-[u_{j-1} - 2u_j + u_{j+1}]}{h^2} = c$$

$$\Rightarrow -u_{j-1} + 2u_j - u_{j+1} = ch^2$$

★ Greater Advantages of FEM:-

- ★ If the mesh is not equidistent (non-uniform
- ★ Multi-dimensional problems.



Adaptive gri
(non-uniform)

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$\underline{Au} = c\,\underline{H}$

$$a(u,u) = \int_0^1 |u'(x)|^2\,dx > 0 \quad \text{for all } u \neq 0$$

$\Rightarrow$ A is +ve definit matrix

Symmetry $\left[\begin{array}{l} a(u,v) = a(v,u) \\ \text{and +ve eigen values} \end{array}\right]$

$\Rightarrow$ system of equations always give unique solution.
[Banded Tridiagonal system]

★ Symmetry :- $(A_h)_{jk} = \int_0^1 \frac{\partial \phi_j^h}{\partial x} \cdot \frac{\partial \phi_k^h}{\partial x}\,dx = \int_0^1 \frac{\partial \phi_k^h}{\partial x} \cdot \frac{\partial \phi_j^h}{\partial x}\,dx$

$$= (A_h)_{kj}$$

★ Positive Definite :- $X^T A X > 0$

★ For a +ve vector $V \in \mathbb{R}^m | \{0\}$ holds

$$\underline{V}^T \cdot (\underline{A}\underline{V}) = \sum_{j,k=1}^{m} V_j V_k \int_0^1 \frac{\partial \phi_j^h}{\partial x} \cdot \frac{\partial \phi_k^h}{\partial x}\,dx$$

$$= \int_0^1 \left(\sum_{j=1}^{m} V_j \frac{\partial \phi_j^h}{\partial x} \right) \left(\sum_{k=1}^{m} V_k \frac{\partial \phi_k^h}{\partial x} \right) dx$$

$$= \int_0^1 \left(\sum_{j=1}^{m} V_j \frac{\partial \phi_j^h}{\partial x} \right)^2 dx > 0$$

★ Cauchy-Schwarz Inequality :-

$$\left[\int_a^b f(x) g(x)\,dx \right]^2 \leq \int_a^b (f(x))^2\,dx \cdot \int_a^b (g(x))^2\,dx$$

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OR

$$\int_a^b f(x)\,g(x)\,dx \leq \left[\int_a^b f^2(x)\,dx \cdot \int_a^b g^2(x)\,dx\right]^{1/2}$$

$$= \sqrt{\int_a^b f^2(x)\,dx} \cdot \sqrt{\int_a^b g^2(x)\,dx}$$

*** Poincore - Inequality :-**

$$\int_0^1 |u(x)|^2 dx \leq \frac{1}{4}\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx$$

for function u with B.Cs $u(0) = 0 = u(1)$

*** Stability Analysis (Galerkin Method)**

$$\int_0^1 \left(\frac{\partial u_h}{\partial x}\right)^2 dx = \int_0^1 \left(\sum_{j=1}^{m} u_j^h \frac{\partial \phi_j}{\partial x}\right)^2 dx$$

$$= \underline{u}^T \cdot (\underline{A}\,\underline{V})$$

$$= \underline{u}^T \cdot F \qquad (\because \underline{A}\underline{V} = \underline{F} = c\underline{H})$$

$$= \sum_{j=1}^{m} \int_0^1 u_j^h \phi_j^h(x)\, f(x)\, dx \qquad \because u(x) = \sum_{j=1}^{m} u_j \phi_j(x)$$

$$= \int_0^1 u(x)\, f(x)\, dx \longrightarrow ①$$

Now

$$\int_0^1 |u(x)|^2 dx \leq \frac{1}{4}\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx \quad \text{by Poincore-Inequ}$$

$$= \frac{1}{4}\int_0^1 u(x)\, f(x)\, dx \quad \text{by ①}$$

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$$\leq \frac{1}{4}\sqrt{\int_0^1 (u(x))^2 dx}\ \sqrt{\int_0^1 (f(x))^2 dx} \quad \because \text{by C-S Ineq}$$

$$\Rightarrow \sqrt{\int_0^1 |u(x)|^2 dx}\ \sqrt{\int_0^1 |u(x)|^2 dx} \leq \frac{1}{4}\sqrt{\int_0^1 (u(x))^2 dx}\ \sqrt{\int_0^1 (f(x))^2 dx}$$

$$\Rightarrow \int_0^1 |u(x)|^2 dx \leq \frac{1}{16}\int_0^1 |f(x)|^2 dx \quad \because \text{by squaring}$$

Thus we got an estimate which is independent from h on the L_2-norm.

Question Use Galerkin Finite Element method to show

$$a u_x - b u_{xx} = 0$$
$$u(0) = 0 \quad , \quad u(1) = 1$$

Solution Let $\varepsilon = \frac{a}{b}$

$$\Rightarrow \varepsilon u_x - u_{xx} = 0 \longrightarrow (*)$$

Let $z(x) = u(x) - x \longrightarrow (1)$

$$\Rightarrow z(0) = u(0) - 0 = 0 - 0 = 0$$
$$z(1) = u(1) - 1 = 1 - 1 = 0$$

$$\Rightarrow u(x) = z(x) + x \longrightarrow (2)$$

$$\Rightarrow \varepsilon(z+x)_x - (z+x)_{xx} = 0$$

$$\Rightarrow \varepsilon z_x + \varepsilon - z_{xx} = 0$$

with $z(0) = 0 = z(1)$

from (2)
$u_x = z_x + 1$
$u_{xx} = z_{xx}$ $\Big\}$ Put in (*) $\Rightarrow$
$\varepsilon(z_x + 1) - z_{xx} = 0$
$\Rightarrow \varepsilon z_x + \varepsilon - z_{xx} = 0$

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Using Galerkin FEM

$$\int_0^1 (\varepsilon z_x + \varepsilon - z_{xx})\,\phi(x)\,dx = 0$$

$$\Rightarrow \varepsilon\int_0^1 z_x\,\phi(x)\,dx + \varepsilon\int_0^1 \phi(x)\,dx - \int_0^1 z_{xx}\,\phi(x)\,dx = 0$$

$$\phi = e_j$$

$$\Rightarrow \varepsilon\int_0^1 z_x\,e_j\,dx + \varepsilon\int_0^1 e_j\,dx - \int_0^1 z_{xx}\,e_j\,dx = 0$$

$z(x) = \sum$

$$\Rightarrow \varepsilon\sum_{i=1}^{N} z_i\int_0^1 e_i' e_j\,dx + \varepsilon\int_0^1 e_j\,dx + \sum_{i=1}^{n} z_i\int_0^1 e_i' e_j'\,dx = 0$$

$\because$ integrating by parts & Putting $z(0) = 0 = z(1)$

$$\Rightarrow \varepsilon\sum_{i=1}^{N} z_i\int_0^1 e_i' e_j\,dx + \varepsilon\int_0^1 e_j\,dx + \underbrace{\sum_{i=1}^{N} z_i\int_0^1 e_i' e_j'\,dx}_{A} = 0 \quad , j = 1, 2, \dots, N$$

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & & \vdots \\ 0 & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & & -1 & 2 \end{bmatrix}$$

Now

$$e_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & \text{if } x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{h} & \text{if } x \in [x_i, x_{i+1}] \\ 0 & , \text{ otherwise} \end{cases}$$

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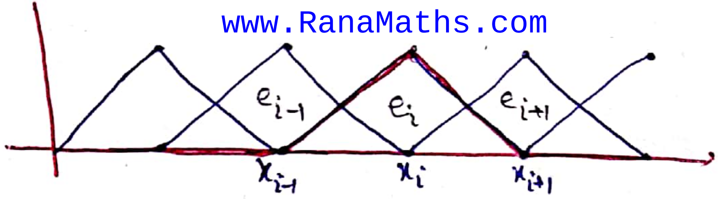

4 $e_i'(x) = \begin{cases} \frac{1}{h} & \text{if } x \in [x_{i-1}, x_i] \\ -\frac{1}{h} & \text{if } x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise}. \end{cases}$

Now $\int_0^1 e_i' e_j \, dx = ?$

Case I $i=j \Rightarrow \int_0^1 e_i' e_i \, dx = \int_{x_{i-1}}^{x_{i+1}} e_i' e_j \, dx = \int_{x_{i-1}}^{x_i} e_i' e_i \, dx + \int_{x_i}^{x_{i+1}} e_i' e_i \, dx$

$$\Rightarrow \int_0^1 e_i' e_i \, dx = \int_{x_{i-1}}^{x_i} \frac{1}{h}\left(\frac{x - x_{i-1}}{h}\right) dx + \int_{x_i}^{x_{i+1}} -\frac{1}{h}\left(\frac{x_{i+1} - x}{h}\right) dx$$

$$= \frac{1}{h^2} \left.\frac{(x - x_{i-1})^2}{2}\right|_{x_{i-1}}^{x_i} + \frac{1}{h^2} \left.\frac{(x - x_{i+1})^2}{2}\right|_{x_i}^{x_{i+1}}$$

$$= \frac{1}{h^2} \cdot \frac{h^2}{2} + \frac{1}{h^2}\left(0 - \frac{h^2}{2}\right) = \frac{h^2}{2h^2} - \frac{h^2}{2h^2}$$

$$= 0$$

Case II $j = i-1 \Rightarrow \int_0^1 e_i' e_{i-1} \, dx = \int_{x_{i-1}}^{x_{i+1}} e_i' e_{i-1} \, dx$

$$\Rightarrow \int_0^1 e_i' e_{i-1} \, dx = \int_{x_{i-1}}^{x_i} e_i' e_{i-1} \, dx + \int_{x_i}^{x_{i+1}} e_i' e_{i-1} \, dx$$

$$= \int_{x_{i-1}}^{x_i} \frac{1}{h}\left(\frac{x_i - x}{h}\right) dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right) \cdot 0 \, dx$$

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$$\Rightarrow \int_0^1 e_i' e_{i-1} = \frac{-1}{h^2}\left(\frac{(x_i - x_j)^2}{2}\right)\Bigg|_{x_{i-1}}^{x_i} + 0$$

$$= \frac{-1}{h^2}\left(\frac{-h^2}{2}\right) = +\frac{1}{2}$$

Cas 3 $j = i+1 \Rightarrow$

$$\int_0^1 e_i' e_{i+1} dx = \int_{x_{i-1}}^{x_{i+1}} e_i' e_{i+1} dx = \int_{x_{i-1}}^{x_i} e_i' e_{i+1} dx + \int_{x_i}^{x_{i+1}} e_i' e_{i+1} dx$$

$$= \int_{x_{i-1}}^{x_i} +\frac{1}{h}(0)\, dx + \int_{x_i}^{x_{i+1}} -\frac{1}{h}\left(\frac{x - x_i}{h}\right) dx$$

$$= -\frac{1}{h^2}\left(\frac{(x - x_i)^2}{2}\right)\Bigg|_{x_i}^{x_{i+1}} = \frac{-1}{h^2}\left[\frac{h^2}{2} - 0\right]$$

$$= \frac{-1}{2}$$

&

$$\int_0^1 e_i\, dx = \int_{x_{i-1}}^{x_{i+1}} e_i\, dx = \int_{x_{i-1}}^{x_i} e_i\, dx + \int_{x_i}^{x_{i+1}} e_i\, dx$$

$$= \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{h}\right) dx + \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1} - x}{h}\right) dx$$

$$= \frac{1}{h}\, \frac{(x - x_{i-1})^2}{2}\Bigg|_{x_{i-1}}^{x_i} - \frac{1}{h}\, \frac{(x - x_{i+1})^2}{2}\Bigg|_{x_i}^{x_{i+1}}$$

$$= \frac{1}{h}\left[\frac{h^2}{2} - 0\right] - \frac{1}{h}\left[0 - \frac{h^2}{2}\right]$$

$$= \frac{1}{h}\left[\frac{h^2}{2} + \frac{h^2}{2}\right] = h$$

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$$\Rightarrow \varepsilon \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \cdots & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & 0 \\ & 0 & & \ddots & 0 & \frac{1}{2} \\ & & & & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_N \end{bmatrix} + \varepsilon \begin{bmatrix} h \\ h \\ \vdots \\ h \end{bmatrix}$$

$$+ \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & 0 \\ & 0 & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{2}{h} & -\frac{1}{h}+\frac{\varepsilon}{2} & 0 & 0 & \cdots & 0 \\ -\frac{1}{h}-\frac{\varepsilon}{2} & \frac{2}{h} & -\frac{1}{h}+\frac{\varepsilon}{2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & 0 \\ & 0 & & \ddots & & -\frac{1}{h}+\frac{\varepsilon}{2} \\ & & & & -\frac{1}{h}-\frac{\varepsilon}{2} & \frac{2}{h} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_N \end{bmatrix} = \begin{bmatrix} -\varepsilon h \\ -\varepsilon h \\ \vdots \\ -\varepsilon h \end{bmatrix}$$

$$\Rightarrow \tilde{A}\underline{Z} = \underline{H} \Rightarrow \underline{Z} = A^{-1}\underline{H}$$

As $\underline{Z} = \underline{U} - \underline{X}$

$\Rightarrow \boxed{\underline{U} = \underline{Z} + \underline{X}}$ is solution at the end.

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★ Finite Element Method For Parabolic PDE's:-

★ Heat Equation:- (Galarkin Method)

$$u_t - a u_{xx} = 0 \qquad x \in [b_1, b_2]$$

B.Cs:- $u(b_1) = u(b_2) = 0$

I.Cs:- $u(0,x) = u_0(x)$

I.B.P

$$\Rightarrow \int_{b_1}^{b_2} u_t \cdot \phi \, dx + a \int_{b_1}^{b_2} u_x \phi_x \, dx = 0 \qquad \text{(Ref Page 52)}$$

$$\phi = e_j \quad , \quad u_h(t,x) = \sum_{i=1}^{m} u_i(t) e_i(x)$$

$$\Rightarrow \int_{b_1}^{b_2} \sum_{i=1}^{m} (u_i)_t e_i e_j \, dx + a \int_{b_1}^{b_2} \sum_{i=1}^{m} u_i (e_i)_x (e_j)_x \, dx = 0$$

$$\Rightarrow \sum_{i=1}^{m} (u_i)_t \underbrace{\int_{b_1}^{b_2} e_i e_j \, dx}_{M = \text{Mass matrix}} + a \sum_{i=1}^{m} u_i \underbrace{\int_{b_1}^{b_2} (e_i)_x (e_j)_x \, dx}_{A = \text{Stifnic matrix}} = 0 \; ; \; j = 1,2,\cdots,m \quad \longrightarrow (*)$$

Now $\displaystyle\int_{b_1}^{b_2} e_i e_j \, dx = \quad ?$

$\underline{j = i} \Rightarrow$

$$\int_{b_1}^{b_2} e_i e_j \, dx = \int_{b_1}^{b_2} e_i^2 \, dx = \int_{x_{i-1}}^{x_{i+1}} e_i^2 \, dx = \int_{x_{i-1}}^{x_i} e_i^2 \, dx + \int_{x_i}^{x_{i+1}} e_i^2 \, dx$$

$$= \int_{x_{i-1}}^{x_i} \frac{(x - x_{i-1})^2}{h^2} \, dx + \int_{x_i}^{x_{i+1}} \frac{(x_{i+1} - x)^2}{h^2} \, dx$$

$$= \left. \frac{(x - x_{i-1})^3}{3h^2} \right|_{x_{i-1}}^{x_i} + \left. \frac{(x_{i+1} - x)^3}{3h^2} \right|_{x_i}^{x_{i+1}}$$

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$$\Rightarrow \int_{b_1}^{b_2} e_i e_j \, dx = \left[\frac{(x_i - x_{i-1})^3}{3h^2} - \frac{(x_{i-1} - x_{i-1})^3}{3h^2}\right] + \left[\frac{(x_{i+1} - x_{i+1})^3}{3h^2} - \frac{(x_{i+1} - x_i)^3}{3h^2}\right]$$

$$= \left(\frac{h^3}{3h^2} - 0\right) + \left(0 - \frac{-h^3}{3h^2}\right) = \frac{1}{3}h + \frac{1}{3}h$$

$$= \frac{2}{3}h$$

$j = i+1 \Rightarrow$

$$\int_{b_1}^{b_2} e_i e_{i+1} \, dx = \int_{x_{i-1}}^{x_{i+1}} e_i e_{i+1} \, dx = \int_{x_{i-1}}^{x_i} e_i e_{i+1} \, dx + \int_{x_i}^{x_{i+1}} e_i e_{i+1} \, dx$$

$$= \int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{h} \cdot 0 + \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1} - x}{h}\right) \cdot \left(\frac{x - x_i}{h}\right) dx$$

$$= 0 + \left(\frac{x_{i+1} - x}{h}\right) \cdot \frac{(x - x_i)^2}{2h}\Bigg|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} \frac{(x - x_i)^2}{2h}\left(-\frac{1}{h}\right) dx$$

$$= 0 + \frac{1}{2h^2} \int_{x_i}^{x_{i+1}} (x - x_i)^2$$

$$= \frac{1}{2h^2} \cdot \frac{(x - x_i)^3}{3}\Bigg|_{x_i}^{x_{i+1}}$$

$$= \frac{1}{6h^2} \left[(x_{i+1} - x_i)^3 - (x_i - x_i)^3\right]$$

$$= \frac{1}{6h^2} \left[h^3 - 0\right] = \frac{h^3}{6h^2}$$

$$= \frac{h}{6}$$

$j = i-1 \Rightarrow$

$$\int_{b_1}^{b_2} e_i e_{i-1} \, dx = \int_{x_{i-1}}^{x_{i+1}} e_i e_{i-1} \, dx$$

Similarly $= \dfrac{h}{6}$

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from ⊛

$$\underline{\underline{M}}U_t + a\,\underline{\underline{A}}\,U = 0$$

where

$$\underline{\underline{A}} = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \\ \vdots & & & 2 & & -1 \\ 0 & 0 & \cdots & -1 & & 2 \end{bmatrix}$$

$$\underline{\underline{M}} = \begin{bmatrix} \frac{2}{3}h & \frac{h}{6} & 0 & 0 & \cdots & 0 \\ \frac{h}{6} & \frac{2}{3}h & \frac{h}{6} & 0 & \cdots & 0 \\ 0 & \frac{h}{6} & \frac{2}{3}h & \frac{h}{6} & & \\ \vdots & & \ddots & \ddots & \ddots & \frac{h}{6} \\ & & & & \frac{h}{6} & \frac{2}{3}h \end{bmatrix}$$

$$\underline{\underline{M}}U_t + a\,\underline{\underline{A}}\,\underline{U} = 0 \quad \text{(Semi-discrete Scheme)}$$

$$\Rightarrow \underline{\underline{M}}\underbrace{\left[\frac{\underline{U}^{n+1} - \underline{U}^n}{k}\right]}_{\text{forward difference}} + a\,\underline{\underline{A}}\,\underline{U}^{n+1} = 0 \quad \text{(emplicit Scheme)}$$

$$\Rightarrow \left[\frac{1}{k}\underline{\underline{M}} + a\,\underline{\underline{A}}\right]\underline{U}^{n+1} = \frac{1}{k}\underline{\underline{M}}\,\underline{U}^n$$

Let $\frac{1}{k}\underline{\underline{M}} + a\,\underline{\underline{A}} = \underline{\underline{B}}$

$$\Rightarrow \underline{\underline{B}}\,\underline{U}^{n+1} = 1/k\,\underline{\underline{M}}\,\underline{U}^n$$

$$\Rightarrow \underline{U}^{n+1} = \frac{1}{k}\underline{\underline{B}}^{-1}\underline{\underline{M}}\,\underline{U}^n$$

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⇒ Lumping of the Mass Matrix:-

Sometime one would like to solve instead of $\underline{M}U_t + a\underline{A}U = 0$, the system of equations of the form $U_t = \underline{B}U$. This can be obtained by Lumping the mass matrix.

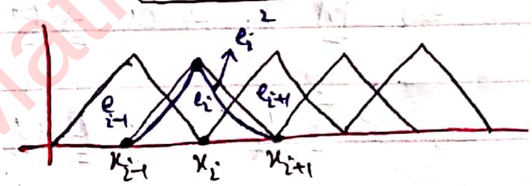
(*) is

$$\sum_{i=1}^{m}(U_i)_t \underbrace{\int_{b_1}^{b_2} e_i e_j \, dx}_{\underline{M}} + a\sum_{i=1}^{m} U_i \int_{b_1}^{b_2} (e_i)_x (e_j)_x \, dx = 0$$

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left(f(a)+f(b)\right)$$

↳ Approximate this integral by Trapezoidal rule

$$i=j \Rightarrow \int_{b_1}^{b_2} e_i e_j \, dx$$



$$= \int_{x_{i-1}}^{x_{i+1}} e_i^2 \, dx = \int_{x_{i-1}}^{x_i} e_i^2 \, dx + \int_{x_i}^{x_{i+1}} e_i^2 \, dx$$

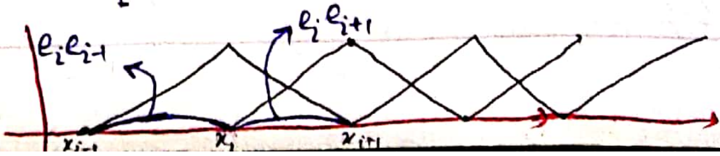
$$= \frac{(x_i - x_{i-1})}{2}\left[f(x_{i-1}) + f(x_i)\right] + \frac{(x_{i+1}-x_i)}{2}\left[f(x_i) + f(x_{i+1})\right]$$

$$= \frac{(x_i - x_{i-1})}{2}\left[0+1\right] + \frac{(x_{i+1}-x_i)}{2}\left[1+0\right]$$

$$= \frac{h}{2} + \frac{h}{2} = h$$

$$j = i-1 \Rightarrow \int_{b_1}^{b_2} e_i e_{i-1} \, dx = \int_{x_{i-1}}^{x_{i+1}} e_i e_{i-1} \, dx = \int_{x_{i-1}}^{x_i} e_i e_{i-1} \, dx + \int_{x_i}^{x_{i+1}} e_i e_{i-1} \, dx$$

$$\Rightarrow \int_{b_1}^{b_2} e_i e_{i-1} \, dx = \frac{(x_i - x_{i-1})}{2}\left[f(x_{i-1}) + f(x_i)\right] + \frac{(x_{i+1}-x_i)}{2}\cancel{\left[f(x_{i+1}) + f(x_i)\right]}^{0}$$



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$$\Rightarrow \int_{b_1}^{b_2} e_i\, e_{i-1}\, dx = \frac{(x_i - x_{i-1})}{2}\left[0+0\right]$$

$$= 0$$

Similarly for $j = i+1$ $= 0$

$$\Rightarrow \underline{\underline{M}} = k \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & \ddots & & \vdots \\ & 0 & & & & 0 \\ & & & & 0 & 1 \end{bmatrix} = k\underline{\underline{I}}$$

$$\Rightarrow \underline{\underline{M}}\, \underline{U}_t + a\, \underline{\underline{A}}\, \underline{U} = 0$$

$$\Rightarrow k\, \underline{\underline{I}}\, \underline{U}_t + a\, \underline{\underline{A}}\, \underline{U} = 0$$

$$\Rightarrow \underline{U}_t = -\frac{a}{k}\, \underline{\underline{A}}\, \underline{U}$$

$$\Rightarrow \underline{U}_t = \underline{\underline{B}}\, \underline{U} \; ; \text{ where } \underline{\underline{B}} = -\frac{a}{k}\underline{\underline{A}}$$

$$\Rightarrow \frac{\underline{U}^{n+1} - \underline{U}^n}{k} = \underline{\underline{B}}\, \underline{U}^n$$

$$\Rightarrow \underline{U}^{n+1} = k\underline{\underline{B}}\,\underline{U}^n + \underline{U}^n = (k\underline{\underline{B}} + \underline{\underline{I}})\underline{U}^n$$

$$\Rightarrow \underline{U}^{n+1} = \underline{\underline{C}}\, \underline{U}^n \; ; \text{ where } \underline{\underline{C}} = k\underline{\underline{B}} + \underline{\underline{I}}$$

★ Equidistent Mesh + Lumping $\Rightarrow$ Same equations as obtained by FDM.

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⇒ Reaction Diffusion Equation:-

$$u_t = a\,u_{xx} + f(t,x) \quad ; x \in [b_1, b_2]$$

u_t: mol/sec, a: cm²/sec, u_{xx}: mol/cm², $f(t,x) \to -\lambda u$, λ: 1/sec, u: mol

⇒ mol/sec = mol/sec

a → diffusion coefficient

λ → Reaction rate constant [or decay rate constant]

$$u_t = a\,u_{xx} + f(t,x,u)$$

$$\Rightarrow \int_{b_1}^{b_2} u_t \cdot \phi\, dx = \int_{b_1}^{b_2} a\,u_{xx} \cdot \phi\, dx + \int_{b_1}^{b_2} f \cdot \phi\, dx$$

$$\phi = e_j \quad , \quad u = \sum_{i=1}^{m} u\, e_i$$

$$\Rightarrow \sum_{i=1}^{m} (u_i)_t \underbrace{\int_{b_1}^{b_2} e_i e_j\, dx}_{I_1 = \text{Mass Matrix}} = -a \sum_{i=1}^{m} \underbrace{\int_{b_1}^{b_2} (e_i)_x (e_j)_x\, dx}_{I_2 = A} + \underbrace{\int_{b_1}^{b_2} f(t,x,u)\, e_j\, dx}_{I_3}$$

If f is non-linear then I_3 is complicated

Simplification:- Use Trapezoidal rule

$$\sum_{i=1}^{m} (u_i)_t \int_{b_1}^{b_2} e_i e_j\, dx = -a \sum_{i=1}^{m} \int (e_i)_x (e_j)_x\, dx + f(t, x_j, u_j) \int_{x_{j-1}}^{x_{j+1}} e_j\, dx$$

$$\Rightarrow \underline{M}\,\underline{u}_t = -a\,\underline{A}\,\underline{u} + \underline{F} \; ,$$

$$\text{where } F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix} \cdot h \quad , \text{ where } \int_{x_{j-1}}^{x_{j+1}} e_j\, dx = h$$

$$* \int_a^b f(x)\,dx = f\left(\frac{a+b}{2}\right) \int_a^b dx = (b-a)\, f\left(\frac{a+b}{2}\right) \Rightarrow \text{Mid point rule}$$

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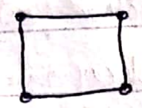
★ Two Dimensional Finite Element Method:-

Triangle

$$p(x) = ax + by + c$$

Rectangle



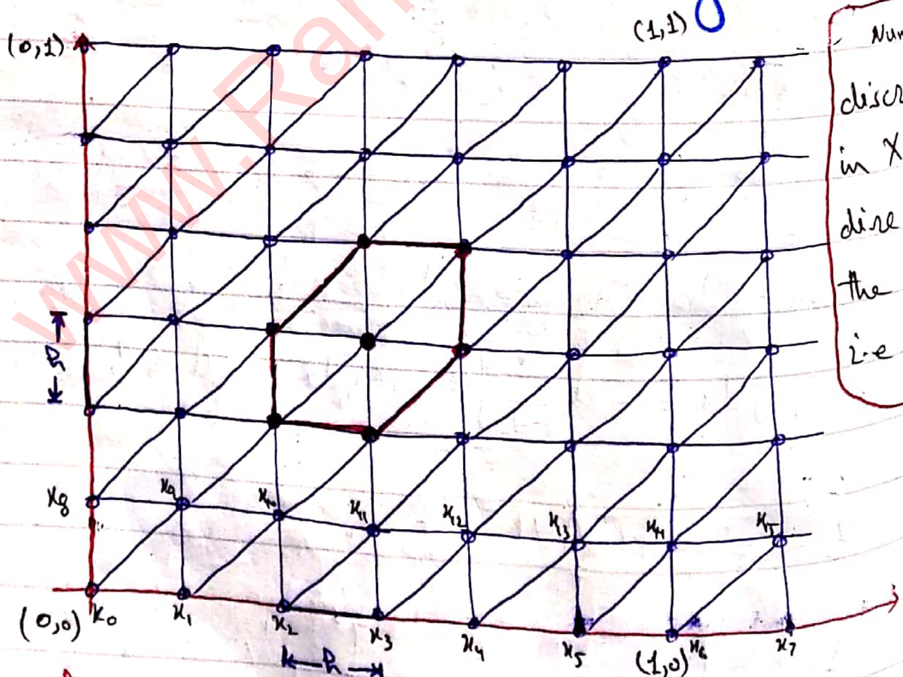
Bilinear functions

$$= (ax+b)(cy+d)$$
$$= acxy + adx + bcy + bd$$
$$= exy + fx + gy + h$$

They have compact support
Globally continuous.

★ $\Delta = \dfrac{\partial}{\partial x^2} + \dfrac{\partial}{\partial y^2}$ (Laplace operator)

★ Friedrich Keller Triangulation:-



Number discretization in X- and Y-directions the same i.e. $h = \frac{1}{\ldots}$

Example:-

$$-\Delta u = 1 \qquad \Omega \in [0,1]^2$$

$$u\Big|_{\partial\Omega} = 0$$

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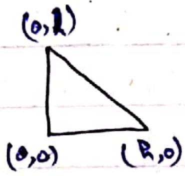
* Refrence Triangles & Corresponding Basis Functions:-

(a)

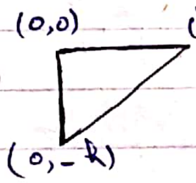


Refrence Triangle (a) $\sum \phi_i = 1$

corresponding Basis functions:- $\sum \phi_i = 1$

$$\phi_1 = \frac{h-x-y}{h}, \quad \phi_2 = \frac{x}{h}, \quad \phi_3 = \frac{y}{h}$$

(b)



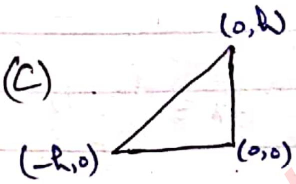
$\sum \phi_i = 1$

implies

$$\phi_1 = \frac{h-x+y}{h} \text{ (corresponding to } (0,0)\text{)}, \quad \phi_2 = \frac{x}{h} \text{ (corresponding to } (h,0)\text{)}$$

$$\phi_3 = -y/h \text{ (corresponding to } (0,-h)\text{)}$$

(C)



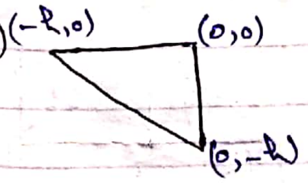
$\sum \phi_i = 1$

implies

$$\phi_1 = \frac{h+x-y}{h} \text{ (corresponding to } (0,0)\text{)}, \quad \phi_2 = \frac{-x}{h} \text{ (corresponding to } (-h,0)\text{)}$$

$$\phi_3 = y/h \text{ (corresponding to } (0,h)\text{)}$$

(d)



implies $\sum \phi_i = 1$

$$\phi_1 = \frac{h+x+y}{h}, \quad \phi_2 = \frac{-x}{h}, \quad \phi_3 = -y/h$$

Basis Function:- $\{e_1, e_2, e_3, \dots, e_{n^2}\}$

$$e_k(x_l) = \begin{cases} 1 & , k = l \\ 0 & , \text{otherwise} \end{cases}$$

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Now $\quad -\Delta u = 1$

$$\Rightarrow -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 1$$

Implying finite Element method

$$\Rightarrow -\int\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\phi \, dx = \int 1\cdot\phi \, dx$$

$$\Rightarrow \int_{\partial\Omega}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial \phi}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial \phi}{\partial y}\right)dx = \int 1\cdot\phi \, dx \quad \longrightarrow (*)$$

$$\phi = e_j \,, \quad u = \sum_{i=1}^{N^2} u_i e_i$$

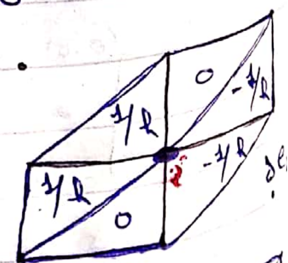
$$\Rightarrow \frac{\partial u}{\partial x} = \sum_{i=1}^{N^2} u_i \frac{\partial e_i}{\partial x}$$

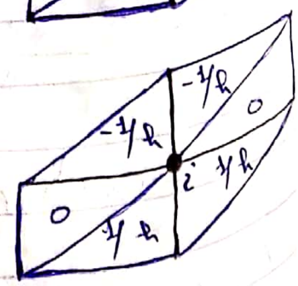
$$(*) \Rightarrow \sum_{i=1}^{N^2} u_i \underbrace{\int_{\partial\Omega}\left(\frac{\partial e_i}{\partial x}\cdot\frac{\partial e_j}{\partial x} + \frac{\partial e_i}{\partial y}\cdot\frac{\partial e_j}{\partial y}\right)dx}_{A} = \int e_j \, dx \quad \longrightarrow (A)$$

Now $\quad \nabla e_i \, \nabla e_j$

Now let $\displaystyle\int\left(\frac{\partial e_i}{\partial x}\cdot\frac{\partial e_j}{\partial x} + \frac{\partial e_i}{\partial y}\cdot\frac{\partial e_j}{\partial y}\right)dx$

$$i=j \Rightarrow \int_{\partial\Omega}\left[\left(\frac{\partial e_i}{\partial x}\right)^2 + \left(\frac{\partial e_i}{\partial y}\right)^2\right]dx$$





$$= \left[\left(-\frac{1}{h}\right)^2 + (0)^2 + \left(\frac{1}{h}\right)^2 + \left(\frac{1}{h}\right)^2 + (0)^2 + \left(\frac{-1}{h}\right)^2\right]\cdot\frac{h^2}{2}$$

$$+\left[(0)^2 + \left(\frac{-1}{h}\right)^2 + \left(-\frac{1}{h}\right)^2 + (0)^2 + \left(\frac{1}{h}\right)^2 + \left(\frac{1}{h}\right)^2\right]\cdot\frac{h^2}{2}$$

where $\frac{h^2}{2}$ is the area of the triangle.

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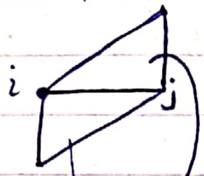
$$\Rightarrow \iint\left[\left(\frac{\partial e_i}{\partial x}\right)^2 + \left(\frac{\partial e_i}{\partial y}\right)^2\right] dx = \left[\frac{4}{h^2} + \frac{4}{h^2}\right]\cdot\frac{h^2}{2} = \frac{8}{h^2}\cdot\frac{h^2}{2}$$

$$= 4$$

$i \neq j \Rightarrow$



$$\Rightarrow \int\left(\frac{\partial e_i}{\partial x}\cdot\frac{\partial e_j}{\partial x} + \frac{\partial e_i}{\partial y}\cdot\frac{\partial e_j}{\partial y}\right) dx$$

$$= \left[-\frac{1}{h}\left(\frac{1}{h}\right) + 0\left(-\frac{1}{h}\right)\right]\cdot\frac{h^2}{2} = \frac{-1}{h^2}\cdot\frac{h^2}{2}$$

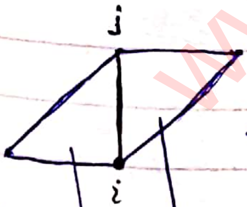
$$= \frac{-1}{2}$$

$$= \left[\left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) + \left(\frac{1}{h}\right)(0)\right]\cdot\frac{h^2}{2} = \frac{-1}{h^2}\cdot\frac{h^2}{2}$$

$$= \frac{-1}{2}$$

$$\Rightarrow \iint\left[\frac{\partial e_i}{\partial x}\cdot\frac{\partial e_j}{\partial x} + \frac{\partial e_i}{\partial y}\cdot\frac{\partial e_j}{\partial y}\right] dx = \frac{-1}{2} + \frac{-1}{2}$$

$$= -1$$



$$\Rightarrow \iint\left[\frac{\partial e_i}{\partial x}\cdot\frac{\partial e_j}{\partial x} + \frac{\partial e_i}{\partial y}\cdot\frac{\partial e_j}{\partial y}\right] dx$$

$$= \left[0\left(-\frac{1}{h}\right) + \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right)\right]\frac{h^2}{2} = \frac{-1}{h^2}\cdot\frac{h^2}{2} = \frac{-1}{2}$$

$$= \left[\frac{1}{h}(0) + \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right)\right]\cdot\frac{h^2}{2} = \frac{-1}{h^2}\cdot\frac{h^2}{2} = \frac{-1}{2}$$

$$\Rightarrow \iint\left[\frac{\partial e_i}{\partial x}\cdot\frac{\partial e_j}{\partial x} + \frac{\partial e_i}{\partial y}\cdot\frac{\partial e_j}{\partial y}\right] dx = \frac{-1}{2} + \frac{-1}{2}$$

$$= -1$$

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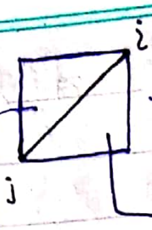


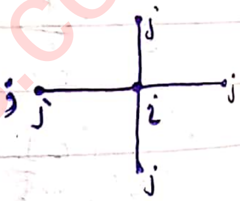
$$\Rightarrow \iint \left[\frac{\partial e_i}{\partial x}\cdot\frac{\partial e_j}{\partial x} + \frac{\partial e_i}{\partial y}\cdot\frac{\partial e_j}{\partial y}\right] dx$$

$$= \left[0\left(-\frac{1}{h}\right) + \frac{1}{h}(0)\right]\cdot\frac{h^2}{2} = 0$$

$$= \left[\frac{1}{h}(0) + 0\left(-\frac{1}{h}\right)\right]\frac{h^2}{2} = 0$$

$$\Rightarrow \iint \left[\frac{\partial e_i}{\partial x}\cdot\frac{\partial e_j}{\partial x} + \frac{\partial e_i}{\partial y}\cdot\frac{\partial e_j}{\partial y}\right] dx = 0$$

$$\Rightarrow \int_\Omega \nabla e_i \nabla e_j \, dx = \begin{cases} 4 & ; \; i = j \\ -1 & ; \\ 0 & ; \; \text{otherwise} \end{cases}$$



And $\int_\Omega e_j \, dx = h^2$

Now from (A)

$$\underline{A}\;\underline{u} = F$$

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where

$$A = \left[\begin{array}{ccccc|ccccc|c}
4 & -1 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & \\
-1 & 4 & -1 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 & \\
0 & -1 & 4 & \ddots & 0 & & & \ddots & & & \\
 & & \ddots & \ddots & -1 & & & & & & \\
 & \cdots & 0 & -1 & 4 & & & & & -1 & \\
\hline
-1 & 0 & 0 & \cdots & 0 & 4 & -1 & 0 & \cdots & 0 & -1 \quad 0 \quad 0 \cdots 0 \\
0 & -1 & 0 & \cdots & 0 & -1 & 4 & -1 & \cdots & 0 & 0 \quad -1 \quad 0 \cdots 0 \\
 & & \ddots & & & & \ddots & \ddots & \ddots & -1 & \ddots \\
 & & & & -1 & & & & -1 & 4 & -1 \\
\hline
0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 4 \quad -1 \quad 0 \cdots 0 \\
0 & 0 & \cdots & & 0 & 0 & -1 & 0 & \cdots & 0 & -1 \quad 4 \quad -1 \quad 0 \cdots 0 \\
 & & 0 & & & & & \ddots & & & \ddots \quad \ddots \quad -1 \\
 & & & 0 & 0 & & & & & -1 & -1 \quad 4 \\
\hline
- & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & -
\end{array}\right]$$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \\ \vdots \\ u_{n^2} \end{bmatrix} = \begin{bmatrix} h^2 \\ h^2 \\ h^2 \\ \vdots \\ h^2 \end{bmatrix}$$

Jordon Matrix due to two dimensinal space

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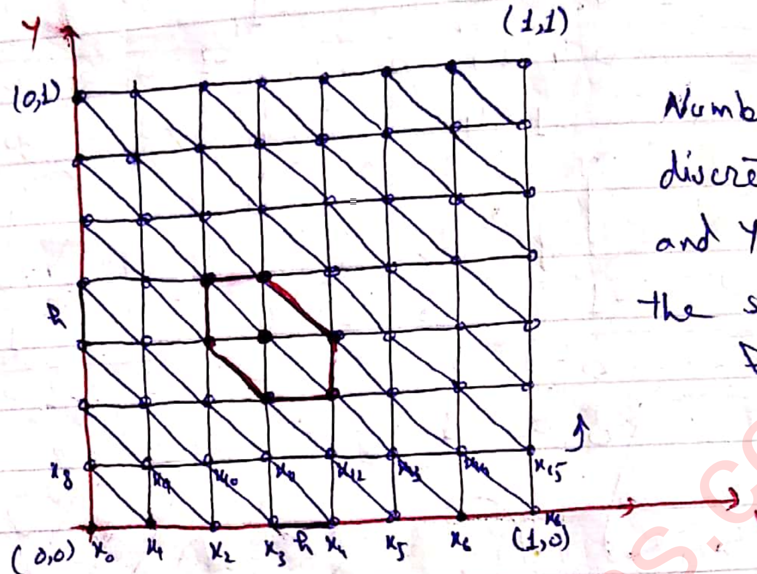
★ If we alter the triangulation, then the final result will remain same. (As Below)



Number of discretization in X and Y- direction are the same i.e $h = \frac{1-0}{N}$

Refrence Triangles & corresponding Basis functions:-

(a) (0,h), (0,0), (h,0) $\Rightarrow \phi_1 = \frac{h-x-y}{h}$, $\phi_2 = \frac{x}{h}$

$\phi_3 = y/h$; s.t $\sum \phi_i = 1$

(b) −h,0 , (0,0), (0,−h) $\Rightarrow \phi_1 = \frac{h+x+y}{h}$; $\phi_2 = -\frac{x}{h}$

$\phi_3 = -y/h$; s.t $\sum \phi_i = 1$

Basis functions:- $\{e_1, e_2, e_3, \dots, e_{N^2}\}$

$$\Rightarrow \quad e_k(x_l) = \begin{cases} 1 & ; k=l \\ 0 & ; \text{otherwise} \end{cases}$$

Now $\quad -\Delta u = 1$

$u|_{\partial\Omega} = 0 \quad ; \Omega \in [0,1]^2$

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$$\Rightarrow \quad -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 1$$

$$\Rightarrow -\int\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\cdot \phi \, dx = \int 1\cdot \phi \, dx$$

$$\Rightarrow \int\left(\frac{\partial u}{\partial x}\cdot\frac{\partial \phi}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial \phi}{\partial y}\right) dx = \int 1\cdot\phi \, dx \longrightarrow (*)$$

where $\phi = e_j$, $u = \sum_{i=1}^{N^2} u_i e_i$

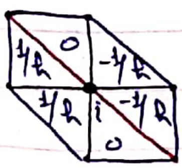
$$\Rightarrow \frac{\partial u}{\partial x} = \sum_{i=1}^{N^2} u_i \frac{\partial e_i}{\partial x}$$

So $(*) \Rightarrow \sum_{i=1}^{N^2} u_i \int\left(\frac{\partial e_i}{\partial x}\cdot\frac{\partial e_j}{\partial x} + \frac{\partial e_i}{\partial y}\cdot\frac{\partial e_j}{\partial y}\right) dx = \int e_j \, dx$

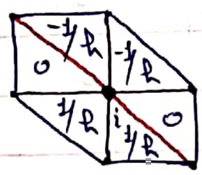
$$\Rightarrow \sum_{i=1}^{N^2} u_i \int \nabla e_i \nabla e_j \, dx = \int e_j \, dx \longrightarrow (1)$$

Now Let $A = \int \nabla e_i \nabla e_j \, dx$

$i=j \Rightarrow \iint\left[\left(\frac{\partial e_i}{\partial x}\right)^2 + \left(\frac{\partial e_i}{\partial y}\right)^2\right] dx$



$\frac{\partial e_i}{\partial x}$



$\frac{\partial e_i}{\partial y}$

$$\Rightarrow \iint\left[\left(\frac{\partial e_i}{\partial x}\right)^2 + \left(\frac{\partial e_i}{\partial y}\right)^2\right] dx = \left(\frac{4}{h^2} + \frac{4}{h^2}\right)\cdot\frac{h^2}{2}$$

$$= \frac{8}{h^2}\cdot h^2/2$$

$$= 4$$

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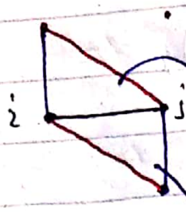
$$i \neq j \Rightarrow \int \left(\frac{\partial e_i}{\partial x} \cdot \frac{\partial e_j}{\partial x} + \frac{\partial e_i}{\partial y} \cdot \frac{\partial e_j}{\partial y} \right) dx$$

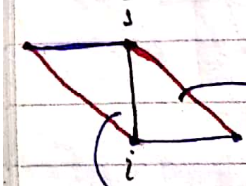


$$= \left[\left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) + \left(-\frac{1}{h}\right)(0) \right] \cdot \frac{h^2}{2} = -\frac{1}{2}$$

$$= \left[\left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) + (0)\left(\frac{1}{h}\right) \right] \cdot \frac{h^2}{2} = -\frac{1}{2}$$

$$\Rightarrow \int \left(\frac{\partial e_i}{\partial x} \cdot \frac{\partial e_j}{\partial x} + \frac{\partial e_i}{\partial y} \cdot \frac{\partial e_j}{\partial y} \right) dx = -\frac{1}{2} + -\frac{1}{2}$$

$$= -1$$

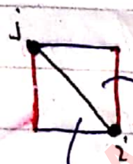


$$= \left[\left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) + \left(-\frac{1}{h}\right)(0) \right] \cdot \frac{h^2}{2} = -\frac{1}{2}$$

$$= \left[0\left(\frac{1}{h}\right) + \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) \right] \cdot \frac{h^2}{2} = -\frac{1}{2}$$

$$\Rightarrow \int \left(\frac{\partial e_i}{\partial x} \cdot \frac{\partial e_j}{\partial x} + \frac{\partial e_i}{\partial y} \cdot \frac{\partial e_j}{\partial y} \right) dx = -\frac{1}{2} + -\frac{1}{2}$$

$$= -1$$



$$= \left[0\left(-\frac{1}{h}\right) + \left(-\frac{1}{h}\right)(0) \right] \cdot \frac{h^2}{2} = 0$$

$$= \left[\left(\frac{1}{h}\right)(0) + 0\left(\frac{1}{h}\right) \right] \cdot h^2/2 = 0$$

$$\Rightarrow \int \left(\frac{\partial e_i}{\partial x} \cdot \frac{\partial e_j}{\partial x} + \frac{\partial e_i}{\partial y} \cdot \frac{\partial e_j}{\partial y} \right) dx = 0$$

$$\Rightarrow \int_\Omega \nabla e_i \nabla e_j \, dx = \begin{cases} 4 & ; i = j \\ -1 & ; j \end{cases}$$

$$\qquad 0 \quad ; \text{otherwise}$$

Now from ① $\underline{A}\,\underline{U} = \underline{F}$ implies

(Same result as on Page 63)

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* System of Hyperbolic PDE's:-

A system of the form $u_t + A u_x = 0$ is hyperbolic if matrix A is diagonalizable with real eigen values. The matrix is diagonalizable if there exist a non-singular matrix K s.t $K^{-1}AK = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)$ Λ is a diagonal matrix. The eigen values $\lambda_1, \lambda_2, \dots, \lambda_m$ are the characteristics speeds of the system of m equations.

(i) $\underline{u}_t + A\underline{u}_x = 0$, $A \in \mathbb{R}^m \times \mathbb{R}^m$ (a_{ij} constant)

(ii) λ_i are real eigen values with non-linear independent eigen vectors $K^{(i)}$; $i = 1, 2, \dots, m$

(iii) A is diagonalizable if $A = K^{-1}AK$ or KAK^{-1}

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}, \quad \underline{\underline{K}} = [K^{(1)}, K^{(2)}, \dots K^{(m)}]$$

$$AK^{(i)} = \lambda_i K^{(i)}$$

* Characteristic Variables:-

K^{-1} exists

$\Rightarrow$ New variables $\underline{\omega} = [\omega_1, \omega_2, \dots, \omega_m]^T$

s.t $\underline{\omega} = K^{-1}\underline{u}$ or $\underline{u} = K\underline{\omega}$

Since K is constant

$\Rightarrow \underline{u}_t = K\underline{\omega}_t$ and $\underline{u}_x = K\underline{\omega}_x$

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$$\underline{u}_t + A\underline{u}_x = 0$$

$$\Rightarrow K^{-1}\underline{u}_t + \underbrace{K^{-1}AK}K^{-1}\underline{u}_x = 0$$

$$\Rightarrow \underline{\omega}_t + \Lambda \underline{\omega}_x = 0$$

$$\Rightarrow \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix}_t + \begin{bmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_m \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_m \end{bmatrix}_x = 0 \quad \left\{ \begin{array}{l} \text{Decoupled} \\ \text{system} \end{array} \right.$$

$$\Rightarrow (\omega_i)_t + \lambda_i(\omega_i)_x = 0 \quad ; i = 1, 2, 3, \cdots, m$$

$$\omega_i(t,x) = \omega_i^{(0)}(x - \lambda_i t)$$

$\because \frac{dx}{dt} = \lambda_i$

$\Rightarrow x = \lambda_i t + c$

$\Rightarrow x = \lambda_i t + x_0 \Rightarrow x_0 = (x-\lambda_i t)$

Now $\omega_i(t,x) = \omega_i^{(0)}(x_0)$

$\Rightarrow \omega_i(t,x) = \omega_i^{(0)}(x - \lambda_i t)$

At the end

$$\underline{u} = K\underline{\omega}$$

Example:-

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u(0,x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \Big| \qquad v(0,x) = 0$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad ; \quad \lambda_{1,2} = \{3, 1\}$$

$$K = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad , \quad K^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

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$$\underline{w} = K^{-1}\underline{u} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \because u = u_1,\ v = u_2$$

$$\Rightarrow \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_t + \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (\omega_1)_t + 3(\omega_1)_x = 0$$
$$(\omega_2)_t + (\omega_2)_x = 0$$

$$\Rightarrow \left.\begin{aligned} \omega_1(t,x) &= \omega_1^{(0)}(x-3t) \\ \omega_2(t,x) &= \omega_2^{(0)}(x-t) \end{aligned}\right] \longrightarrow (*)$$

Now $$W^{(0)} = \begin{bmatrix} \omega_1^{(0)} \\ \omega_2^{(0)} \end{bmatrix} = K^{-1}\begin{bmatrix} u_1^{(0)} \\ u_2^{(0)} \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1^{(0)} \\ u_2^{(0)} \end{bmatrix}$$

$$\Rightarrow \omega_1^{(0)}(x) = \frac{1}{2}\left[u_1^{(0)}(x) + u_2^{(0)}(x)\right]$$
$$\omega_2^{(0)}(x) = \frac{1}{2}\left[u_1^{(0)}(x) - u_2^{(0)}(x)\right]$$

$(*)$ implies

$$\Rightarrow \omega_1(t,x) = \frac{1}{2}\left[u_1^{(0)}(x-3t) + u_2^{(0)}(x-3t)\right]$$
$$\omega_2(t,x) = \frac{1}{2}\left[u_1^{(0)}(x-t) - u_2^{(0)}(x-t)\right]$$

Now $\underline{u} = K\underline{\omega}$

$$\Rightarrow u = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1^{(0)}(x-3t) + u_2^{(0)}(x-3t) \\ u_1^{(0)}(x-t) - u_2^{(0)}(x-t) \end{bmatrix}$$

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Example:- Linearized Gas Dynamics

$$\begin{bmatrix} \rho \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & \rho_0 \\ a^2/\rho_0 & 0 \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rho(0,x) = \rho^{(0)}(x) \quad , \quad u(0,x) = u^{(0)}(x)$$

Let $u_1 = \rho$ & $u_2 = u$

$\Rightarrow u_1(0,x) = u_1^{(0)}(x) \quad , \quad u_2(0,x) = u_2^{(0)}(x)$

characteristic variables $[w_1, w_2]^t = K^{-1}\underline{u}$

Now

$$A = \begin{bmatrix} 0 & \rho_0 \\ a^2/\rho_0 & 0 \end{bmatrix} \Rightarrow \lambda_1 = -a, \ \lambda_2 = a$$

$$K = \begin{bmatrix} \rho_0 & \rho_0 \\ -a & a \end{bmatrix} , \quad K^{-1} = \frac{1}{2a\rho_0}\begin{bmatrix} a & -\rho_0 \\ a & \rho_0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_t + \begin{bmatrix} -a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_x = 0$$

$$\Rightarrow \left.\begin{aligned} \frac{\partial w_1}{\partial t} - a\frac{\partial w_1}{\partial x} &= 0 \\ \frac{\partial w_2}{\partial t} + a\frac{\partial w_2}{\partial x} &= 0 \end{aligned}\right\} \longrightarrow (*)$$

I.e. $w_1^{(0)} = \dfrac{1}{2a\rho_0}\left[a u_1^{(0)}(x) - \rho_0 u_2^{(0)}(x)\right]$

$w_2^{(0)} = \dfrac{1}{2a\rho_0}\left[a u_1^{(0)}(x) + \rho_0 u_2^{(0)}(x)\right]$

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$$\Rightarrow \omega_1(t,x) = \omega_1^{(0)}(x-\lambda_1 t) = \omega_1^{(0)}(x+at)$$

$$\omega_2(t,x) = \omega_2^{(0)}(x-\lambda_2 t) = \omega_2^{(0)}(x-at)$$

or

$$\omega_1(t,x) = \frac{1}{2a\rho_0}\left[a u_1^{(0)}(x+at) - \rho_0 u_2^{(0)}(x+at)\right]$$

$$\omega_2(t,x) = \frac{1}{2a\rho_0}\left[a u_1^{(0)}(x-at) + \rho_0 u_2^{(0)}(x-at)\right]$$

At the end

$$u_1(t,x) = \frac{1}{2a}\left[a u_1^{(0)}(x+at) - \rho_0 u_2^{(0)}(x+at)\right] + \frac{1}{2a}\left[a u_1^{(0)}(x-at) + \rho_0 u_2^{(0)}(x-at)\right]$$

$$u_2(t,x) = \frac{-1}{2\rho_0}\left[a u_1^{(0)}(x+at) - \rho_0 u_2^{(0)}(x+at)\right] + \frac{1}{2\rho_0}\left[a u_1^{(0)}(x-at) + \rho_0 u_2^{(0)}(x-at)\right]$$

Example:- $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t + \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$u_1(0,x) = \sin x \;,\; u_2(0,x) = \cos x$$

$$u_1(0,x) = u_1^{(0)}(x) \;,\; u_2(0,x) = u_2^{(0)}(x)$$

Characteristic variables $[\omega_1, \omega_2]^t = K^{-1}\underline{u}$

Now

$$A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$$

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Now for eigen values

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 - 16 = 0$$

$$\Rightarrow 1+\lambda^2 - 2\lambda - 16 = 0 \Rightarrow \lambda^2 - 2\lambda - 15 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 3\lambda - 15 = 0 \Rightarrow \lambda(\lambda-5)+3(\lambda-5) =$$

$$\Rightarrow (\lambda-5)(\lambda+3) = 0$$

$$\Rightarrow \boxed{\lambda_1 = 5}, \quad \boxed{\lambda_2 = -3}$$

Now for eigen vector

$$K_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad K_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow K = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow K^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad K^{-1}U_t + K^{-1}AKK^{-1}U_x = 0$$

$$\Rightarrow \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_t + \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_x = 0$$

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$$\Rightarrow (\omega_1)_t + 5(\omega_1)_x = 0$$

$$(\omega_2)_t - 3(\omega_2)_x = 0$$

$$\Rightarrow \left.\begin{array}{l} \omega_1(t,x) = \omega_1^{(0)}(x-5t) \\ \omega_2(t,x) = \omega_2^{(0)}(x+3t) \end{array}\right\} \longrightarrow \textcircled{A}$$

Now $W^{(0)} = \begin{bmatrix} \omega_1^{(0)} \\ \omega_2^{(0)} \end{bmatrix} = K^{-1}\begin{bmatrix} U_1^{(0)} \\ U_2^{(0)} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \omega_1^{(0)} \\ \omega_2^{(0)} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} U_1^{(0)} \\ U_2^{(0)} \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -U_1^{(0)} - U_2^{(0)} \\ -U_1^{(0)} + U_2^{(0)} \end{bmatrix}$$

$$\Rightarrow \omega_1^{(0)}(x) = +\frac{1}{2}\left[U_1^{(0)}(x) + U_2^{(0)}(x)\right]$$

$$\omega_2^{(0)}(x) = +\frac{1}{2}\left[U_1^{(0)}(x) - U_2^{(0)}(x)\right]$$

So $\textcircled{A} \Rightarrow$

$$\omega_1(t,x) = \frac{1}{2}\left[U_1^{(0)}(x-5t) + U_2^{(0)}(x-5t)\right]$$

$$\omega_2(t,x) = \frac{1}{2}\left[U_1^{(0)}(x+3t) - U_2^{(0)}(x+3t)\right]$$

Now at the end

$$\underline{U} = K\underline{\omega}$$

$$\Rightarrow U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} U_1^{(0)}(x-5t) + U_2^{(0)}(x-5t) \\ U_1^{(0)}(x+3t) - U_2^{(0)}(x+3t) \end{bmatrix}\cdot\frac{1}{2}$$

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$$\Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} u_1^{(0)}(x-5t) + u_2^{(0)}(x-5t) + u_1^{(0)}(x+3t) - u_2^{(0)}(x+3t) \\ u_1^{(0)}(x-5t) + u_2^{(0)}(x-5t) - u_1^{(0)}(x+3t) + u_2^{(0)}(x+3t) \end{bmatrix}$$

$$\Rightarrow u_1 = \frac{1}{2}\left[u_1^{(0)}(x-5t) + u_2^{(0)}(x-5t)\right] + \frac{1}{2}\left[u_1^{(0)}(x+3t) - u_2^{(0)}(x+3t)\right]$$

$$u_2 = \frac{1}{2}\left[u_1^{(0)}(x-5t) + u_2^{(0)}(x-5t)\right] - \frac{1}{2}\left[u_1^{(0)}(x+3t) - u_2^{(0)}(x+3t)\right]$$

$$\Rightarrow u_1(t,x) = \frac{1}{2}\left[\sin(x-5t) + \cos(x-5t)\right] + \frac{1}{2}\left[\sin(x+3t) - \cos(x+3t)\right]$$

$$u_2(t,x) = \frac{1}{2}\left[\sin(x-5t) + \cos(x-5t)\right] - \frac{1}{2}\left[\sin(x+3t) - \cos(x+3t)\right]$$

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