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NUMERICAL ANALYSIS ***

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★ NUMERICAL ANALYSIS

⇒ **INTRODUCTION:-** So far we have used Analytic method most of the time to solve the mathematical problems. These methods gives true solutions or exact solutions of the problem. But some times it is rather difficult or impossible to use analytic method to solve a problem for example

(i) To solve the equation $x^3+2x^2e^x-7x\sin^{-1}x=5$

(ii) To evaluate the integral $\int_1^5 \frac{e^{x^2}\sin^{-1}x}{(1+x^3)^{1/2}}\,dx$

(iii) For a function $f(x)$ such that $f(0)=5$, $f(1)=9$, $f(2)=21$, $f(3)=30$, $f(5)=53$
To find $f'(1.5)$

(iv) To solve the system
$$3x+2y+xz=10$$
$$2x-y+2z=e$$
$$x+4y+5z=\pi$$

(v) For given $\frac{dy}{dx}+3x^2\sin^{-1}x\,y=e^{\tan x}$, $y(1)=2$
To find $y(2)$

In order to solve these problems we used the methods, called numerical methods, The problems with numerical methods is that, They usually do not give the exact or true solutions of the problem. However these methods gives an approximate solution instead of true solution so there occure an errors.

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SEC I

⇒ **Errors:-** An error in numerical problems, is difference between the exact value of the problem and its computed value or approximate value.

Let x be the actual value, $\bar{x}$ be the approximate (computed) value then error is

$$\text{ERROR} = E = x - \bar{x}$$

* **Sources of Errors:-** The errors that effect the mathematical computations may be divided into 5-types.

1) **Input Error:-** The input error can be the result of measurements with limited accuracy or real numbers which must be represented with fix number of digits for example
   (i) Instead of using 3.152834 we use 3.1528
   (ii) Instead of using e we use 2.7183
   (iii) Instead of using π we use 3.14 etc

2) **Modelling Error:-** The discrepency between the model and physical system is also a source of error which can be termed as modeling error. For example error caused due to friction, gravity etc.

3) **Rounding Error:-** The error which arises when we used fixed number of digits in calculation.

If the calculating device can not be handled of to 4-digits,

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numbers can not be used in subsequent calculation. Then the resulting value must be rounding off.

4) Trancation Error:- The error arises when an infinite process is replaced by a finite one. For example
$e^x = 1 + x + x^2/2! + x^3/3! + \cdots$ we use only $1 + x + x^2/2! + x^3/3!$ to solve problem

5) Initial Error:- It is also called human error. These errors can be expected in program itself by operator. Most of these errors can be minimized by proper planning.

* Types of Errors:-

1) Absolute Error:- If x is the actual value and $\bar{x}$ is approximate value then absolute error is defined by

$$E_a = A.E = |x - \bar{x}| = |\Delta x|$$

where $\Delta x = x - \bar{x}$

e.g $x = 4.83$, $\bar{x} = 4.832$

$$E_a = |x - \bar{x}| = |4.83 - 4.832| = 0.002$$

Note:- Generally if a number is correct upto n-decimal places it has a rounding error $A.E \leq \frac{1}{2} \times 10^{-n}$

2) Relative Error:- Let x be a true value of a quantity and $\bar{x}$ be approximate (computed) value then relative

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error is denoted and defined as

$$E_r = \frac{|x - \bar{x}|}{x} \quad s.t \quad x \neq 0$$

$$= \frac{|\Delta x|}{x}$$

As $x \simeq \bar{x}$ then $E_r = \frac{|\Delta x|}{\bar{x}}$

As the measure of accuracy, Relative error is more precise than absolute error. The size of absolute error depends on the unit, where as the relative is dimensionless quantity

Note:- A decimal number correct to n-significant digit has Relative Error

$$E_r \leq 5 \times 10^{-n}$$

3) Percentage Error:- Relative error is expressed in percentage is called percentage error. It is denoted and defined as

Percentage Error $= E_p = R.E \times 100$

$$E_p = \frac{|x - \bar{x}|}{x} \times 100$$

Example:- $x = 4.83$, $\bar{x} = 4.832$

$E_a = 0.002$

$$R.E = \frac{|x - \bar{x}|}{x} = \frac{0.002}{4.83} = 0.0041$$

$$P.E = R.E \times 100 = 0.0041 \times 100$$

$$= 0.41\ \%$$

It is also called probable error

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⇒ Effect of Error in Arithematic Operation :-

1) Error in Addition :- Let x_1 and x_2 be the exact value and $\bar{x}_1$, $\bar{x}_2$ be the approximation to these values respectively. Then if we denote the corresponding errors by E_1 and E_2 then

$E_1 = x_1 - \bar{x}_1$, $E_2 = x_2 - \bar{x}_2$.

Now consider the sum $A = x_1 + x_2$ & $\bar{A} = \bar{x}_1 + \bar{x}_2$ and Error in the sum $= E = A - \bar{A}$. Then

$$E = A - \bar{A} = (x_1 + x_2) - (\bar{x}_1 + \bar{x}_2)$$
$$= (x_1 - \bar{x}_1) + (x_2 - \bar{x}_2) = E_1 + E_2$$

⇒ Error in the sum is equal to the sum of errors. Further

$$E_a = |E| = |E_1 + E_2| \leq |E_1| + |E_2|$$

⇒ Absolute error in the sum is less than or equal to sum of the absolute errors. In general for n-numbers

$$|E| \leq |E_1| + |E_2| + \cdots + |E_n|$$

Hence the absolute error of the sum of n-numbers does not exceed the sum of absolute errors of numbers.

2) Error in Subtraction :- Let x_1 and x_2 be the exact values and $\bar{x}_1$ and $\bar{x}_2$ be the approximation to these values respectively. Then if we denote the corresponding errors by E_1 and E_2 then $E_1 = x_1 - \bar{x}_1$, $E_2 = x_2 - \bar{x}_2$

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Let E be the error in "$x_1 - x_2$"

$$E = (x_1 - x_2) - (\bar{x}_1 - \bar{x}_2)$$
$$= x_1 - x_2 - \bar{x}_1 + \bar{x}_2$$
$$= E_1 + \bar{x}_1 - E_2 - \bar{x}_2 - \bar{x}_1 + \bar{x}_2 \quad \because E_1 = x_1 - \bar{x}_1, E_2 = x_2 - \bar{x}_2$$
$$= E_1 - E_2 = E_1 + (-E_2)$$
$$|E| = |E_1 + (-E_2)|$$
$$\leq |E_1| + |-E_2|$$

$$\Rightarrow |E| \leq |E_1| + |E_2|$$

Thus the absolute error of numbers is not greater than the sum of absolute error of the numbers.

3) Error In Product:- Let $\bar{x}_1$ and $\bar{x}_2$ be approximation value of x_1 and x_2. Let E_1 and E_2 be errors in x_1 and x_2 respec. Then $E_1 = x_1 - \bar{x}_1$ & $E_2 = x_2 - \bar{x}_2$

Let E be error in '$x_1 \cdot x_2$'

$$E = x_1 x_2 - \bar{x}_1 \bar{x}_2$$
$$= (E_1 + \bar{x}_1)(E_2 + \bar{x}_2) - \bar{x}_1 \bar{x}_2 \quad \because E_1 = x_1 - \bar{x}_1 \Rightarrow x_1 = E_1 + \bar{x}_1$$
$$= E_1 E_2 + E_1 \bar{x}_2 + E_2 \bar{x}_1 + \bar{x}_1 \bar{x}_2 - \bar{x}_1 \bar{x}_2$$
$$= E_1(x_2 - E_2) + E_2(x_1 - E_1) + E_1 E_2 \quad \because E_2 = x_2 - \bar{x}_2 \Rightarrow \bar{x}_2 = x_2 - E_2$$
$$= E_1 x_2 - E_1 E_2 + E_2 x_1 - E_1 E_2 + E_1 E_2$$

$$E = E_1 x_2 + E_2 x_1 - E_1 E_2$$

Dividing both sides by $x_1 x_2$

$$\frac{E}{x_1 x_2} = \frac{E_1 x_2}{x_1 x_2} + \frac{E_2 x_1}{x_1 x_2} - \frac{E_1 E_2}{x_1 x_2}$$

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$$\Rightarrow \frac{E}{x_1x_2} = \frac{E_1}{x_1} + \frac{E_2}{x_2} + \frac{E_1E_2}{x_1x_2}$$

As $\frac{E_1E_2}{x_1x_2}$ is very small quantity so neglecting

$$\Rightarrow \frac{E}{x_1x_2} \simeq \frac{E_1}{x_1} + \frac{E_2}{x_2}$$

Taking absolute both sides

$$\left|\frac{E}{x_1x_2}\right| \simeq \left|\frac{E_1}{x_1} + \frac{E_2}{x_2}\right|$$

$$\leq \left|\frac{E_1}{x_1}\right| + \left|\frac{E_2}{x_2}\right| \Rightarrow E_2 \leq E_1^{(2)} + E_2^{(2)}$$

Thus the relative error of product is approximately the sum of relative error of factors.

4) Errors in Division:- Let $\bar{x}_1$ and $\bar{x}_2$ be approximate value of x_1 and x_2. Let E_1 and E_2 be errors of x_1 and x_2 resp. Let E be error in x_1/x_2

$$E = \frac{x_1}{x_2} - \frac{\bar{x}_1}{\bar{x}_2}$$

$$= \frac{x_1}{x_2} - \frac{(x_1 - E_1)}{(x_2 - E_2)} \qquad \because \bar{x}_1 = x_1 - E_1$$

$$= \frac{x_1}{x_2} - (x_1 - E_1)(x_2 - E_2)^{-1}$$

$$= \frac{x_1}{x_2} - x_1\left(1 - \frac{E_1}{x_1}\right)\left(1 - \frac{E_2}{x_2}\right)^{-1} x_2^{-1}$$

$$= \frac{x_1}{x_2} - \frac{x_1}{x_2}\left(1 - \frac{E_1}{x_1}\right)\left(1 - \frac{E_2}{x_2}\right)^{-1}$$

$$= \frac{x_1}{x_2} - \frac{x_1}{x_2}\left(1 - \frac{E_1}{x_1}\right)\left(1 + \frac{E_2}{x_2} + \dots\right)$$

$\because$ By Binomial expansion

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$$\Rightarrow E = \frac{x_1}{x_2} - \frac{x_1}{x_2}\left(1 - \frac{E_1}{x_1}\right)\left(1 + \frac{E_2}{x_2} + \cdots\right)$$

$$= \frac{x_1}{x_2} - \frac{x_1}{x_2}\left(1 - \frac{E_1}{x_1} + \frac{E_2}{x_2} + \cdots\right)$$

$$= \frac{x_1}{x_2} - \frac{x_1}{x_2} + \frac{E_1}{x_2} - \frac{E_2 x_1}{x_2^2} + \cdots$$ Neglecting Higher powers of E_1 & E_2

Dividing both sides by x_1/x_2

$$\Rightarrow \frac{E}{x_1/x_2} = \frac{E_1}{x_1}\cdot\frac{x_2}{x_1} - \frac{E_2 x_1}{x_2^2}\cdot\frac{x_2}{x_1}$$

$$\left|\frac{x_2}{x_1}E\right| = \left|\frac{E_1}{x_1} + \left(\frac{-E_2}{x_2}\right)\right|$$

$$\leq \left|\frac{E_1}{x_1}\right| + \left|\frac{E_2}{x_2}\right| \Rightarrow E^{(2)} \leq E_1^{(2)} + E_2^{(2)}$$

5) Error in Function of n-variables:- Let $y = f(x_1, x_2, \cdots, x_n)$ be a function of n-variables and let $\delta x_1, \delta x_2, \cdots, \delta x_n$ be errors in $x_1, x_2, \cdots, x_n$ and δy be error in y.

Taking increment on both sides

$$y + \delta y = f(x_1 + \delta x_1, x_2 + \delta x_2, \cdots, x_n + \delta x_n)$$

Now, using Taylor's theorem

$$y + \delta y = f(x_1, x_2, \cdots, x_n) + \left(\delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2} + \cdots + \delta x_n \frac{\partial f}{\partial x_n}\right) + \cdots$$

Neglecting square terms of $\delta x_1, \delta x_2, \cdots, \delta x_n$ we have

$$y + \delta y = y + \left(\delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2} + \cdots + \delta x_n \frac{\partial f}{\partial x_n}\right)$$

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$$\Rightarrow \delta y = \delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2} + \cdots + \delta x_n \frac{\partial f}{\partial x_n}$$

we get change in y w.r.t n- variables.
For absolute error we taking absolute on both sides. we get

$$|\delta y| = \left| \delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2} + \cdots + \delta x_n \frac{\partial f}{\partial x_n} \right|$$

$$|\delta y| \leq \left|\frac{\partial f}{\partial x_1}\right||\delta x_1| + \left|\frac{\partial f}{\partial x_2}\right||\delta x_2| + \cdots + \left|\frac{\partial f}{\partial x_n}\right||\delta x_n|$$

★★★

Question:- Find the error in product of n- numbers.

Solution:- Let $y = x_1 x_2 x_3 \cdots x_n$ ——— ①
and δy be error in y then

$$\delta y = \frac{\partial y}{\partial x_1}\delta x_1 + \frac{\partial y}{\partial x_2}\delta x_2 + \cdots + \frac{\partial y}{\partial x_n}\delta x_n \text{ ——— ②}$$

Taking log on both sides of ① we get

$$\ln y = \ln x_1 + \ln x_2 + \cdots + \ln x_n$$

Differentiate w.r.t x_1

$$\frac{1}{y}\frac{\partial y}{\partial x_1} = \frac{1}{x_1}$$

$$\Rightarrow \frac{\partial y}{\partial x_1} = \frac{y}{x_1}$$

Similarly $\frac{\partial y}{\partial x_2} = \frac{y}{x_2}, \cdots\cdots \frac{\partial y}{\partial x_n} = \frac{y}{x_n}$

Put in ②

$$\delta y = \frac{y}{x_1}\delta x_1 + \frac{y}{x_2}\delta x_2 + \cdots + \frac{y}{x_n}\delta x_n$$

For relative error we divide by 'y'

$$\frac{\delta y}{y} = \frac{\delta x_1}{x_1} + \frac{\delta x_2}{x_2} + \cdots + \frac{\delta x_n}{x_n}$$

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For absolute error

$$\left|\frac{\delta y}{y}\right| \leq \left|\frac{\delta x_1}{x_1}\right| + \left|\frac{\delta x_2}{x_2}\right| + \cdots + \left|\frac{\delta x_n}{x_n}\right|$$

$$E_2 \leq E_{12} + E_{22} + \cdots + E_{n2}$$

* * *

Question:- Find relative error of y where $y = a_1 x_1^{m_1} x_2^{m_2} x_3^{m_3} \cdots x_n^{m_n}$

Solution:-

Let δy be error in y

$$\delta y = \frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f}{\partial x_n}\delta x_n \quad \text{---} (*)$$

As $y = a_1 x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$

Taking log on both sides

$$\ln y = \ln a_1 + m_1 \ln x_1 + m_2 \ln x_2 + \cdots + m_n \ln x_n$$

Taking Partial derivative w.r.t x_1

$$\frac{1}{y}\frac{\partial y}{\partial x_1} = \frac{m_1}{x_1}$$

$$\Rightarrow \frac{\partial y}{\partial x_1} = y\frac{m_1}{x_1}$$

Similarly $\frac{\partial y}{\partial x_2} = \frac{y m_2}{x_2}, \cdots\cdots, \frac{\partial y}{\partial x_n} = \frac{y m_n}{x_n}$

$$(*) \Rightarrow \delta y = \frac{y m_1}{x_1}\delta x_1 + \frac{y m_2}{x_2}\delta x_2 + \cdots + \frac{y m_n}{x_n}\delta x_n$$

$$\left|\frac{\delta y}{y}\right| = \left|\frac{m_1}{x_1}\delta x_1 + \frac{m_2}{x_2}\delta x_2 + \cdots + \frac{m_n}{x_n}\delta x_n\right|$$

$$\leq \left|\frac{m_1}{x_1}\right|\left|\delta x_1\right| + \left|\frac{m_2}{x_2}\right|\left|\delta x_2\right| + \cdots + \left|\frac{m_n}{x_n}\right|\left|\delta x_n\right|$$

$$\Rightarrow E_2 \leq m_1 E_{11} + m_2 E_{22} + \cdots + m_n E_{n2}$$

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Thus the relative error of y can not be exceed algebric sum of there relative error

★ ★

Question:- Find relative error of $y = x^m$

Solution:- Let δy be error in y

$$\Rightarrow \delta y = \frac{\partial y}{\partial x}\delta x \quad \text{---} \quad (1)$$

As $y = x^m$

$$\ln y = m \ln x$$

$$\frac{1}{y}\frac{\partial y}{\partial x} = m\frac{1}{x}$$

$$\Rightarrow \frac{\partial y}{\partial x} = m\frac{y}{x}$$ Put in (1)

$$\delta y = m\frac{y}{x}\delta x \Rightarrow \frac{\delta y}{y} = \frac{my}{y}\cdot\frac{\delta x}{x}$$

$$\Rightarrow \left|\frac{\delta y}{y}\right| = m\left|\frac{\delta x}{x}\right|$$

$$\Rightarrow E = m E_x$$ Ans.

★ ★ ★

⇒ Significant Figures:-

Significant figures (or digits) of an approximate number is any of the digit 1, 2, 3, ..., 9 in its decimal representation, or any zero except when it is used to fix the decimal point or to fill the place of unknown or discarded digit.

Example:- (i) Number 0.00475 the significant figures are 4, 7, 5. First three zeros are used merely to fix the position

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of decimal point and indicate the place values of other digits and are therefore not significant.

(ii) In Number 0.0006070, The first four zeros are not significant. All remaining digits including the other two zeroes, are significant.

⇒ Rounding off Rule :-

To round off or simply round a number to n significant figures or digits, drop all digits to right of n^{th} place, or replace them by zeros(0) if the zeros are needed as place holder. In this rounding off rule, note following.

a) If the first of discarded digits is less than 5, leave remaining digits unchanged.

b) If first discarded digit is exactly 5 & there are non-zero digits among those discarded, add 1 to last retained digit.

c) If first discarded digit exceeds 5, add 1 to the last retained digit.

d) If first discarded digit, however is exactly 5 and all other discarded digits are zeros, The last retained digit is left unchanged or is increased by 1 according as it is even or odd (The even digit rule)

* * *

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⇒ The Solution of Equations In one variable:-

(Linear, Non linear, Algebraic, Transcedental)

The expression of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$$

where $a_0 \neq 0$, a_i is constant and n is positive integer is called a polynomial of degree n in variable x

$f(x) = 0$ is called an algebraic equation

Example:- $2x^3 - x + 10 = 0$ is algebraic equation.

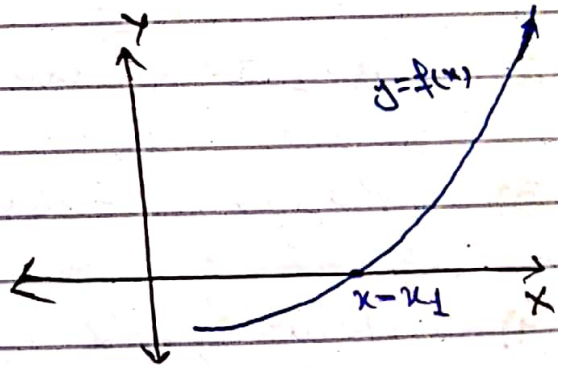
An equation which contain some other function such as trignometric, logarithmic, exponential are called Transcedental equation

Example:- $xe^x - \cos 2x = 0$

$x + \ln x = 0$

The value of x which satisfies the equation $f(x) = 0$ is called root or solution of equation.

Geometrically, a root of $f(x) = 0$ is that value of x where the graph of $y = f(x)$ cross x-axis is a root of equation $f(x) = 0$



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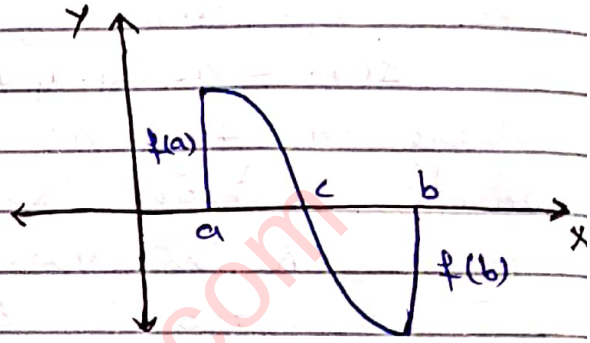
* Intermediate Value Property :-

To solve $f(x) = 0$

If f is defined on $[a, b]$ and is continuous on $[a, b]$.

If $f(a) \cdot f(b) < 0$ i.e $f(a)$ and $f(b)$ have opposite sign then there exist a point $c \in [a, b]$ s.t $f(c) = 0$.

In this case $x = c$ is called solution of the equation $f(x) = 0$



* Methods For Solving Equation in One Variable.

We shall discuss the following methods for solving equations in one variable.

1) Bisection Method (OR)
   Internal Halving Method
2) Iteration Method (OR)
   Fixed point Iteration
3) Regula Falsi Method (OR)
   Method of False position
4) Newton - Raphson Method.
5) Secant Method.

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* Interval Halving or Bisection Method.

In order to solve $f(x)=0$ consider $f(x)$ is continuous on $[a,b]$ and having opposite sign at $f(a)$ & $f(b)$ i.e $f(a).f(b)<0$

Then by intermediate value property there is a real root α of $f(x)=0$ in $]a,b[$

Let $c_1 = \frac{a+b}{2}$ be first approximation to the root.

If $f(c_1)=0$, Then c_1 is required root.

If $f(c_1)\neq 0$ &

if $f(a).f(c_1)<0$, Then root lies in $]a,c_1[$

if $f(c_1).f(b)<0$, Then root lies in $]c_1,b[$

Suppose root lies in $]c_1,b[$ then we find the mid point c_2 of $]c_1,b[$ i.e $c_2 = \frac{c_1+b}{2}$ and find $f(c_2)$

If $f(c_2)=0$. Then c_2 is required root

If $f(c_2)\neq 0$ &

if $f(c_1).f(c_2)<0$, then root lies in $]c_1,c_2[$

if $f(c_2).f(b)<0$, Then root lies in $]c_2,b[$

By halving the intervals repeatedly we can find an approximation to the exact root of $f(x)=0$

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Question:- Solve equation x^3+x-1 using Bisection method.

Solution:-

Let $f(x) = x^3+x-1$

x	0	0.2	0.4	0.6	0.8
$f(x)$	-1	-7.92	-0.536	-0.184	0.312

As sign change between 0.6 & 0.8

i.e $f(0.6) = -0.184 < 0$

& $f(0.8) = 0.312 > 0$

As $f(0.6)\cdot f(0.8) < 0$, so root lies b/w $]0.6, 0.8[$

1st Iteration:- Let $a_1 = 0.6$, $b_1 = 0.8$

$$c_1 = \frac{a_1+b_1}{2}$$

$$\Rightarrow c_1 = \frac{0.6+0.8}{2} = 0.7$$

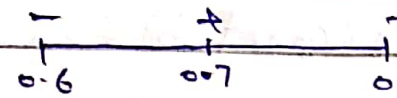
$$f(c_1) = f(0.700)$$

$$= (0.700)^3 + (0.700) + 1$$

$$= 0.0430$$

$f(0.6)\cdot f(0.7) < 0$

So root lies between $]0.6, 0.7[$

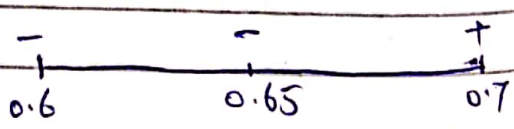


2nd Iteration:- $a_2 = 0.6$, $b_2 = 0.7$, $c_2 = \frac{a_1+b_2}{2}$

$$\Rightarrow c_2 = \frac{0.6+0.7}{2} = 0.650$$

$$f(c_2) = (0.650)^3 + (0.650) - 1$$

$$= -0.0754$$



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$f(0.6500) \cdot f(0.7000) < 0$

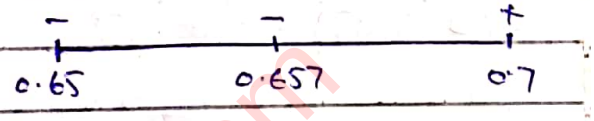
So root lies b/w $]0.6500, 0.7000[$

3rd Iteration:- Let $a_3 = 0.6500$, $b_3 = 0.7000$

$$\Rightarrow c_3 = \frac{0.6500 + 0.7000}{2} = 0.6570$$

$$f(c_3) = (0.6570)^3 + (0.6570) - 1$$

$$f(c_3) = -0.0175$$



$f(0.6570) \cdot f(0.7) < 0$

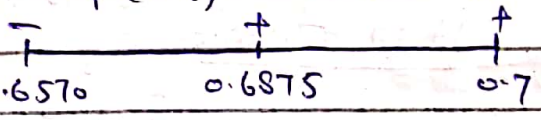
So root lies b/w $]0.6570, 0.7[$

4rth Iteration:- $a_4 = 0.6570$, $b_4 = 0.7000$

$$\Rightarrow c_4 = \frac{0.6570 + 0.7000}{2} = 0.6835$$

$$f(c_4) = (0.6835)^3 + (0.6835) - 1$$

~~$f(c_4) = -0.0091$~~ $f(c_4) = 0.0125$



$f(0.6570) \cdot f(0.6875) < 0$

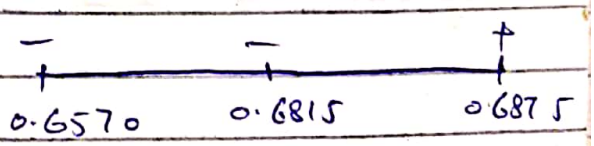
So root lies b/w $]0.6570, 0.6875[$

5th Iteration:- Let $a_5 = 0.6570$, $b_5 = 0.6875$

$$\Rightarrow c_5 = \frac{0.6570 + 0.6875}{2} = 0.6825$$

$$f(c_5) = (.6813)^3 + (.6813) - 1$$

$$= -0.0025$$



$f(0.6815) \cdot f(.6875) < 0$

So root lies b/w $]0.6815, 0.6875[$

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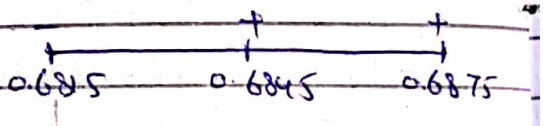
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6th Iteration:- $a_6 = 0.6815$, $b_6 = 0.6875$

$$\Rightarrow c_6 = \frac{0.6815 + 0.6875}{2} = 0.6845$$

$$f(c_6) = (0.6845)^3 + (0.6845) - 1$$

$$= 0.0052$$



$f(0.6815) \cdot f(0.6845) < 0$

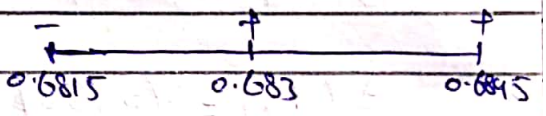
So root lies b/w $]0.6815, 0.6845[$

7th Iteration:- $a_7 = 0.6815$, $b_7 = 0.6845$

$$\Rightarrow c_7 = \frac{0.6815 + 0.6845}{2} = 0.683$$

$$f(c_7) = (0.683)^3 + (0.683) - 1$$

$$= 0.0016$$



$f(0.6815) \cdot f(0.683) < 0$

So root lies b/w $]0.6815, 0.683[$

8th Iteration:- $a_8 = 0.6815$, $b_8 = 0.683$

$$\Rightarrow c_8 = \frac{0.6815 + 0.683}{2} = 0.6823$$

$$f(c_8) = (0.6823)^3 + (0.6823) - 1$$

$$= -0.0000$$

So approximation root is 0.6823

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Question:- Solve $f(x) = \sin x - 5x + 2$ by Bisection method.

Solution:-

Let $f(x) = \sin x - 5x + 2$

x	0	0.1	0.2	0.3	0.4	0.5
$f(x)$	2	1.5995	1.1986	0.7955	0.3894	-0.0206

As $f(x)$ change sign b/w $(0.4, 0.5)$

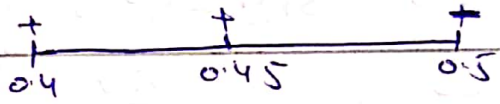
i.e $f(0.4) \cdot f(0.5) < 0$

So root lies b/w $]0.4, 0.5[$

1st Iteration:- $a_1 = 0.4$, $b_1 = 0.5$

$$\Rightarrow c_1 = \frac{0.4 + 0.5}{2} = 0.45$$

$f(c_1) = 0.1850$



$f(0.45) \cdot f(0.5) < 0$

So root lies b/w $]0.45, 0.5[$

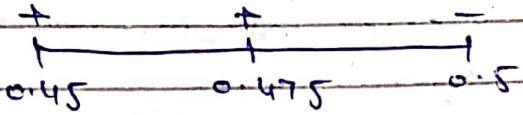
2nd Iteration:-

Let $a_2 = 0.45$, $b_2 = 0.5$, $c_2 = \frac{a_2 + b_2}{2}$

$$\Rightarrow c_2 = \frac{0.45 + 0.5}{2} = 0.475$$

$f(c_2) = \sin(0.475) - 5(0.475) + 2$

$f(c_2) = 0.0823$



$\Rightarrow f(0.475) \cdot f(0.5) < 0$

$\Rightarrow$ Root lies b/w $]0.475, 0.5[$

3rd Iteration:-

Let $a_3 = 0.475$, $b_3 = 0.5$

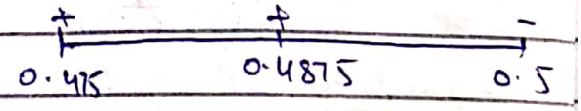
$$\Rightarrow c_3 = \frac{0.475 + 0.5}{2} = 0.4875$$

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$f(c_3) = \sin(0.4875) - 5(0.4875) + 2$

$= 0.0309$



$f(0.4875) \cdot f(0.5) < 0$

So root lies b/w $]0.4875, 0.5[$

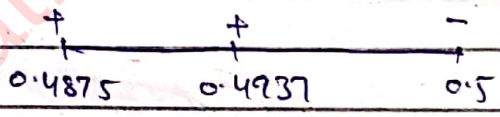
4xth Iteration:-

Let $a_4 = 0.4875$, $b_5 = 0.5$

$\Rightarrow c_4 = \frac{0.4875 + 0.5}{2} = 0.4937$

$f(c_4) = \sin(0.4937) - 5(0.4937) + 2$

$= 0.0054$



$f(0.4937) \cdot f(0.5) < 0$

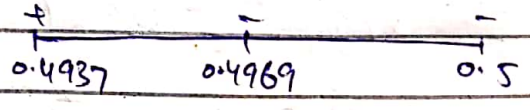
So root lies b/w $]0.4937, 0.5[$

5th Iteration:- Let $a_5 = 0.4937$, $b_5 = 0.5$

$\Rightarrow c_5 = \frac{0.4937 + 0.5}{2} = 0.4969$

$f(c_5) = \sin(0.4969) - 5(0.4969) + 2$

$= -0.0080$



$f(0.4937) \cdot f(0.4969) < 0$

So root lies b/w $]0.4937, 0.4969[$

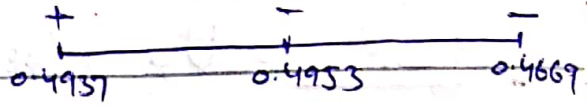
6th Iteration:- $a_6 = 0.4937$, $b_6 = 0.4969$

$\Rightarrow c_6 = \frac{0.4937 + 0.4969}{2} = 0.4953$

$f(c_6) = \sin(0.4953) - 5(0.4953) + 2$

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$f(c_6) = -0.0012$



$\Rightarrow f(0.4937) \cdot f(0.4953) < 0$

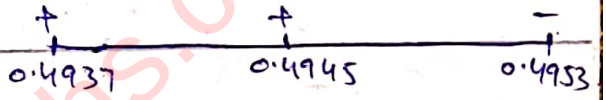
So root lies b/w $]0.4937, 0.4953[$

7th Iteration:- $a_7 = 0.4937$, $b_7 = 0.4953$

$$\Rightarrow c_7 = \frac{0.4937 + 0.4953}{2} = 0.4945$$

$f(c_7) = \sin(0.4945) - 5(0.4945) + 2$

$= 0.0020$



$f(0.4945) \cdot f(0.4953) < 0$

So root lies b/w $]0.4945, 0.4953[$

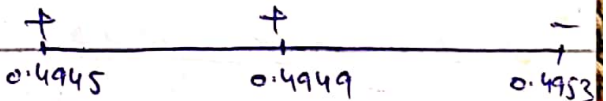
8th Iteration:-

Let $a_8 = 0.4945$, $b_8 = 0.4953$

$$\Rightarrow c_8 = \frac{0.4945 + 0.4953}{2} = 0.4949$$

$f(c_8) = \sin(0.4949) - 5(0.4949) + 2$

$= 0.0004$



$f(0.4949) \cdot f(0.4953) < 0$

So root lies b/w $]0.4949, 0.4953[$

9th Iteration:- $a_9 = 0.4949$, $b_9 = 0.4953$

$$\Rightarrow c_9 = \frac{0.4949 + 0.4953}{2} = 0.4951$$

$f(c_9) = \sin(0.4951) - 5(0.4951) + 2$

$= 0.0003$

$f(c_{10}) = \sin(0.4950) - 5(0.4950) + 2$

$= 0.0000$

Ans is 0.4950

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Question:- Use Bisection Method to find the root of $2^x - 5x + 2 = 0$ correct upto two decimal places.

Solution:-

Let $f(x) = 2^x - 5x + 2$

x	0	0.5	1
$f(x)$	3	0.92	−1

As $f(x)$ changes sign b/w 0.5, 1

i.e $f(0.5) \cdot f(1) < 0$

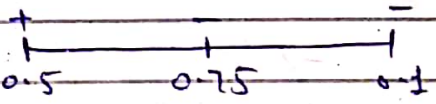
$\Rightarrow$ Root lies between $]0.5, 1[$

1st Iteration:- Let $a_1 = 0.5$, $b_1 = 1$

$$\Rightarrow c_1 = \frac{0.5 + 1}{2} = 0.75$$

$$f(c_1) = 2^{0.75} - 5(0.75) + 2$$

$$= -0.07$$



As $f(0.5) \cdot f(0.75) < 0$

So root lies b/w $]0.5, 0.75[$

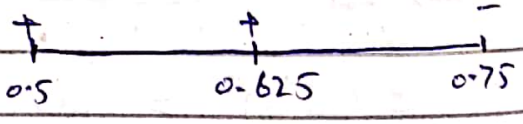
2nd Iteration:-

Let $a_2 = 0.5$, $b_2 = 0.75$

$$\Rightarrow c_2 = \frac{0.5 + 0.75}{2} = 0.625$$

$$f(c_2) = 2^{0.625} - 5(0.625) + 2$$

$$= 0.42$$



As $f(0.625) \cdot f(0.75) < 0$

$\Rightarrow$ Root lies b/w $]0.625, 0.75[$

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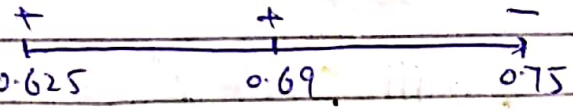
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3rd Iteration:- $a_3 = 0.625$, $b_3 = 0.75$

$\Rightarrow c_3 = \frac{0.625 + 0.75}{2} = 0.69$

$f(c_3) = 2^{0.69} - 5(0.69) + 2$

$= 0.16$



$\Rightarrow f(0.69) \cdot f(0.75) < 0$

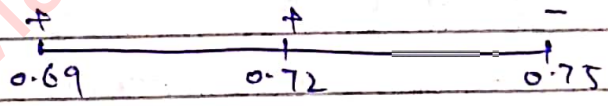
$\Rightarrow$ Root lies b/w $]0.69, 0.75[$

4rth Iteration:- Let $a_4 = 0.69$, $b_4 = 0.75$

$\Rightarrow c_4 = \frac{0.69 + 0.75}{2} = 0.72$

$f(c_4) = 2^{0.72} - 5(0.72) + 2$

$= 0.05$



$\Rightarrow f(0.72) \cdot f(0.75) < 0$

So Root lies b/w $]0.72, 0.75[$

5th Iteration:- $a_5 = 0.72$, $b_5 = 0.75$

$\Rightarrow c_5 = \frac{0.72 + 0.75}{2} = 0.73$

$f(c_5) = 2^{0.73} - 5(0.73) + 2$

$= 0.00$

$\Rightarrow$ Approximation root is 0.73

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Assignment.

Q1:- Use Bisection Method to Solve

(i) $2e^{-x} - \sin x = 0$

Ans:- 0.9210

(ii) $2x^3 + x - 2 = 0$

Ans:- 0.4836

(iii) $x + x^x = 100$

Ans:- 3.5812

Q2:- Given that $x = 1.45$ is the exact Solution of $x^2 - 5.57x + 5.974 = 0$. Apply 3 Iterations of Bisection Method for solving equation.

Q3:- Find the root of equation $2x - \cos x - 3 = 0$ upto three decimal places by bisection method.

Solutions

Q1:- (i)

Let $f(x) = 2e^{-x} - \sin x = 0$

x	0.7	0.8	0.9	1.0
$f(x)$	0.3489	0.1813	0.0298	-0.1057

As $f(x)$ changes sign between $(0.9, 1.0)$ i.e $f(0.9) \cdot f(1.0) < 0$

$\Rightarrow$ Root lies b/w $]0.9, 1.0[$

1st Iteration:-

Let $a_1 = 0.9$, $b_1 = 1.0$

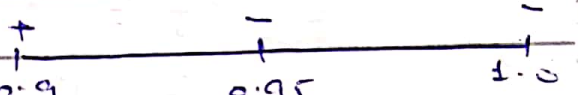
$$\Rightarrow c_1 = \frac{0.9 + 1.0}{2} = 0.95$$

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$\Rightarrow f(c_1) = 2e^{-0.95} - \sin(+0.95)$

$= -0.0399$



$f(0.9) \cdot f(0.95) < 0$

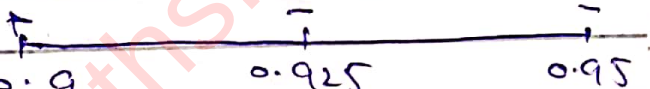
$\Rightarrow$ Root lies between $]0.9, 0.95[$

2nd Iteration:-

Let $a_2 = 0.9$, $b_2 = 0.95$

$\Rightarrow c_2 = \frac{0.9 + 0.95}{2} = 0.925$

$f(c_2) = 2e^{-0.925} - \sin(0.925) = -0.0055$



$f(0.9) \cdot f(0.925) < 0$

$\Rightarrow$ Root lies b/w $]0.9, 0.925[$

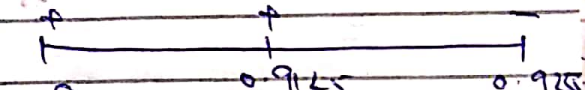
3rd Iteration:-

Let $a_3 = 0.9$, $b_3 = 0.925$

$\Rightarrow c_3 = \frac{0.9 + 0.925}{2} = 0.9125$

$f(c_3) = 2e^{-0.9125} - \sin(0.9125)$

$= 0.0120$



$f(0.9125) \cdot f(0.925) < 0$

$\Rightarrow$ Root lies b/w $]0.9125, 0.925[$

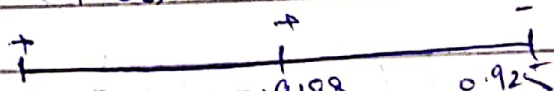
4rth Iteration:-

Let $a_4 = 0.9125$, $b_4 = 0.925$

$\Rightarrow c_4 = \frac{0.9125 + 0.925}{2} = 0.9188$

$f(c_4) = 2e^{-0.9188} - \sin(0.9188)$

$= 0.0031$



$f(0.9188) \cdot f(0.925) < 0 \Rightarrow$ Root $\in]0.9188, 0.925[$

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5th Iteration:- Let $a_5 = 0.9188$, $b_5 = 0.925$

$$\Rightarrow c_5 = \frac{0.9188 + 0.925}{2} = 0.9219$$

$$f(c_5) = 2e^{-0.9219} - \sin(0.9219)$$

$$= -0.0012$$



$f(0.9188) \cdot f(0.9219) < 0$

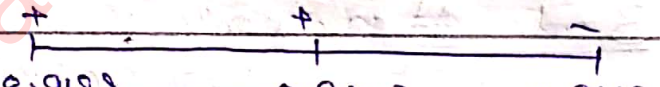
$\Rightarrow$ Root lies b/w $]0.9188, 0.9219[$

6th Iteration:- Let $a_6 = 0.9188$, $b_6 = 0.9219$

$$\Rightarrow c_6 = \frac{0.9188 + 0.9219}{2} = 0.9203$$

$$f(c_6) = 2e^{-0.9203} - \sin(0.9203)$$

$$= 0.0010$$



$\Rightarrow f(0.9203) \cdot f(9219) < 0$

$\Rightarrow$ Root lies b/w $]0.9203, 0.9219[$

7th Iteration:- Let $a_7 = 0.9203$, $b_7 = 0.9219$

$$\Rightarrow c_7 = \frac{0.9203 + 0.9219}{2} = 0.9211$$

$$f(c_7) = 2e^{-0.9211} - \sin(0.9211)$$

$$= -0.0001$$

$\Rightarrow$ Correct Answer up to four decimal places is 0.9210

* * *

(ii) Let $f(x) = 2x^3 + x - 2$

x	0.6	0.7	0.8	0.9	1
$f(x)$	−0.968	−0.614	0.176	0.358	1

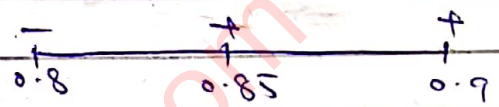
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As $f(x)$ changes sign b/w $(0.8, 0.9)$
i.e $f(0.8) \cdot f(0.9) < 0$
$\Rightarrow$ Root lies b/w $]0.8, 0.9[$

1st Iteration:- Let $a_1 = 0.8$, $b_1 = 0.9$

$$\Rightarrow c_1 = \frac{0.8+0.9}{2} = 0.85$$

$$f(c_1) = 2(0.85)^3 + 0.85 - 2$$
$$= 0.0782$$

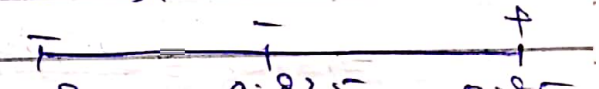


$\Rightarrow f(0.8) \cdot f(0.85) < 0$
$\Rightarrow$ Root lies b/w $]0.8, 0.85[$

2nd Iteration:- Let $a_2 = 0.8$, $b_2 = 0.85$

$$\Rightarrow c_2 = \frac{0.8+0.85}{2} = 0.825$$

$$f(c_2) = 2(0.825)^3 + 0.825 - 2$$
$$= -0.0520$$

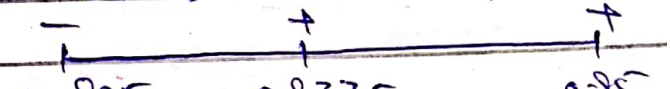


$f(0.825) \cdot f(0.85) < 0$
$\Rightarrow$ Root lies b/w $]0.825, 0.85[$

3rd Iteration:- $a_3 = 0.825$, $b_3 = 0.85$

$$\Rightarrow c_3 = \frac{0.825+0.85}{2} = 0.8375$$

$$f(c_3) = 2(0.8375)^3 + 0.8375 - 2$$
$$= 0.0123$$



$\Rightarrow f(0.825) \cdot f(0.8375) < 0$
$\Rightarrow$ Root lies b/w $]0.825, 0.8375[$

4rth Iteration:- $a_4 = 0.825$, $b_4 = 0.8375$

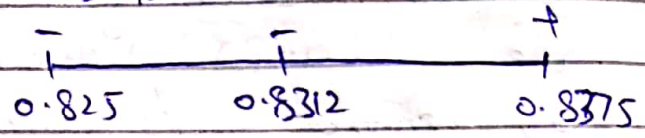
$$\Rightarrow c_4 = \frac{0.825+0.8375}{2} = 0.8312$$

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$f(c_4) = 2(0.8312)^3 + 0.8312 - 2$

$= -0.0202$



$\Rightarrow f(0.8312) \cdot f(0.8375) < 0$

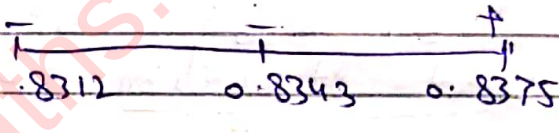
$\Rightarrow$ Root lies b/w $]0.8312, 0.8375[$

5th Iteration - Let $a_5 = 0.832$, $b_5 = 0.8375$

$\Rightarrow c_5 = \frac{0.8312 + 0.8375}{2} = 0.8343$

$f(c_5) = 2(0.8343)^3 + 0.8343 - 2$

$= -0.0045$



$f(0.8343) \cdot f(0.8375) < 0$

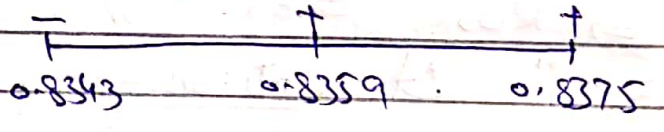
$\Rightarrow$ Root lies b/w $]0.8343, 0.8375[$

6th Iteration :- Let $a_6 = 0.8343$, $b_6 = 0.8375$

$\Rightarrow c_6 = \frac{0.8343 + 0.8375}{2} = 0.8359$

$f(c_6) = 2(0.8359)^3 + 0.8359 - 2$

$= 0.0040$



$\Rightarrow f(0.8343) \cdot f(0.8359) < 0$

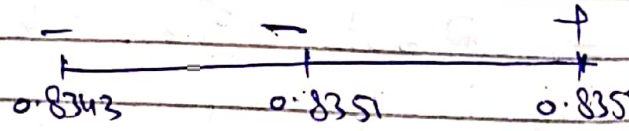
$\Rightarrow$ Root lies b/w $]0.8343, 0.8359[$

7th Iteration :- $a_7 = 0.8343$, $b_7 = 0.8359$

$\Rightarrow c_7 = \frac{0.8343 + 0.8359}{2} = 0.8351$

$f(c_7) = 2(0.8351)^3 + 0.8351 - 2$

$= -0.0011$



$f(0.8351) \cdot f(0.8359) < 0$

$\Rightarrow$ Root lies b/w $]0.8351, 0.8359[$

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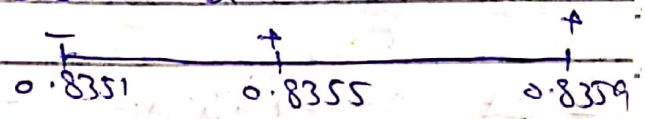
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8th Iteration:- $a_8 = 0.8351$, $b_8 = 0.8359$

$\Rightarrow c_8 = \frac{0.8351 + 0.8359}{2} = 0.8355$

$f(c_8) = 2(0.8355)^3 + 0.8355 - 2$

$= 0.0020$



$f(0.8351) \cdot f(0.8355)$

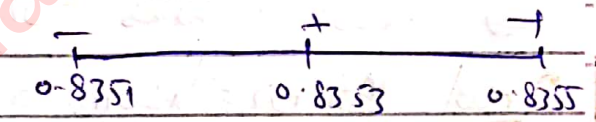
$\Rightarrow$ Root lies b/w $]0.8351, 0.8355[$

9th Iteration:- $a_9 = 0.8351$, $b_9 = 0.8355$

$\Rightarrow c_9 = \frac{0.8351 + 0.8355}{2} = 0.8353$

$f(c_9) = 2(0.8353)^3 + 0.8353 - 2$

$= 0.0009$



$f(0.8351) \cdot f(0.8353) < 0$

$\Rightarrow$ Root lies b/w $]0.8351, 0.8353[$

10th Iteration:- Let $a_{10} = 0.8351$, $b_{10} = 0.8353$

$\Rightarrow c_{10} = \frac{0.8351 + 0.8353}{2} = 0.8352$

$f(c_{10}) = 2(0.8352)^3 + 0.8352 - 2$

$= 0.0004$

$\Rightarrow$ Approximate Root correct upto four decimal places is 0.8352

* * *

(iii) $x + x^x = 100 \Rightarrow x + x^x - 100 = 0$

Let $f(x) = x + x^x - 100$

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x	3.56	3.57	3.58	3.59	3.60
$f(x)$	-4.5724	-2.4521	-0.2806	1.9433	4.2211

As $f(x)$ changes sign between 3.58, 3.59.

i.e $f(3.58) \cdot f(3.59) < 0$

$\Rightarrow$ Root lies b/w $]3.58, 3.59[$

1st Iteration:- Let $a_1 = 3.58$, $b_1 = 3.59$

$$\Rightarrow c_1 = \frac{3.58 + 3.59}{2} = 3.585$$

$$f(c_1) = 3.585 + (3.585)^{3.585} - 100$$

$$= 0.8247$$

$f(3.58) \cdot f(3.585) < 0$

$\Rightarrow$ Root lies b/w $]3.58, 3.585[$

2nd Iteration:- $a_2 = 3.58$ $b_2 = 3.585$

$$\Rightarrow c_2 = \frac{3.58 + 3.585}{2} = 3.5825$$

$$f(c_2) = 3.5825 + (3.5825)^{3.5825} - 100$$

$$= 0.2704$$

$f(3.58) \cdot f(3.5825) < 0$

$\Rightarrow$ Root lies b/w $]3.58, 3.5825[$

3rd Iteration:- $a_3 = 3.58$, $b_3 = 3.5825$

$$\Rightarrow c_3 = \frac{3.58 + 3.5825}{2} = 3.5812$$

$$f(c_3) = 3.5812 + (3.5812)^{3.5812} - 100$$

$$= -0.0165$$

$f(3.5812) \cdot f(3.5825) < 0$

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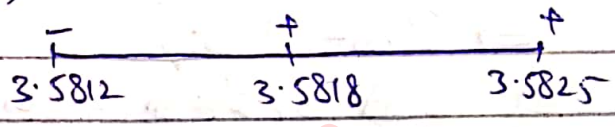
$\Rightarrow$ Root lies b/w $]3.5812, 3.5825[$

4th Iteration:- $a_4 = 3.5812$, $b_4 = 3.5825$

$\Rightarrow c_4 = \frac{3.5812 + 3.5825}{2} = 3.5818$

$f(c_4) = 3.5818 + (3.5818)^{3.588} - 100$

$= 0.1158$



$f(3.5812) \cdot f(3.5818) < 0$

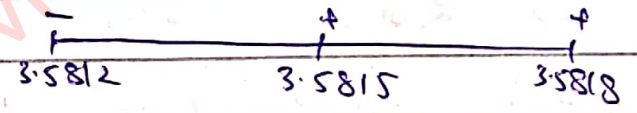
$\Rightarrow$ Root lies b/w $]3.5812, 3.5818[$

5th Iteration:- $a_5 = 3.5812$, $b_5 = 3.5818$

$\Rightarrow c_5 = \frac{3.5812 + 3.5818}{2} = 3.5815$

$f(c_5) = 3.5815 + (3.5815)^{3.5815} - 100$

$= 0.0496$



$f(3.5812) \cdot f(3.5815) < 0$

$\Rightarrow$ Root lies b/w $]3.5812, 3.5815[$

6th Iteration:- $a_6 = 3.5812$, $a_7 = 3.5815$

$\Rightarrow c_6 = \frac{3.5812 + 3.5815}{2} = 3.5813$

$f(c_6) = 3.5813 + (3.5813)^{3.5813} - 100$

$= 0.0054$

$\Rightarrow$ So Approximate Root is 3.5813

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Q2:- Let $f(x) = x^2 - 5.57x + 5.974$

x	1.38	1.46
$f(x)$	2.2484	−0.8586

As $f(1.38) \cdot f(1.46) < 0$

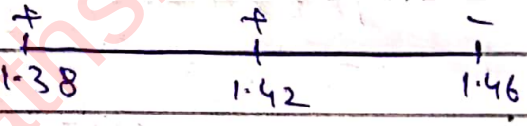
$\Rightarrow$ Root lies b/w $]1.38, 1.46[$

1st Iteration:- $a_1 = 1.38$, $b_1 = 1.46$

$\Rightarrow c_1 = \dfrac{1.38 + 1.46}{2} = 1.42$

$f(c_1) = (1.42)^2 - 5.57(1.42) + 5.974$

$= 0.081$



$f(1.42, 1.46) < 0$

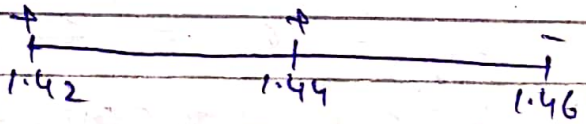
$\Rightarrow$ Root lies b/w $]1.42, 1.46[$

2nd Iteration:- $a_2 = 1.42$, $b_2 = 1.46$

$\Rightarrow c_2 = \dfrac{1.42 + 1.46}{2} = 1.44$

$f(c_2) = (1.44)^2 - 5.57(1.44) + 5.974$

$= 0.0268$



$f(1.44) \cdot f(1.46) < 0$

$\Rightarrow$ Root lies b/w $]1.44, 1.46[$

3rd Iteration:- $a_3 = 1.44$, $b_3 = 1.46$

$\Rightarrow c_3 = \dfrac{1.44 + 1.46}{2} = 1.45$

$f(c_3) = (1.45)^2 + 5.974 - 5.57(1.45)$

$= 0$

So the Solution.

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Q3:- Let $f(x) = 2x - \cos x - 3$

x	1.4	1.5	1.6	1.7
$f(x)$	−0.370	−0.0707	0.2292	0.5288

As $f(x)$ changes sign b/w 1.5, 1.6

i.e $f(1.5) \cdot f(1.6) < 0$

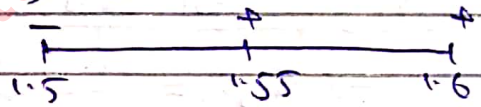
$\Rightarrow$ Root lies b/w $]1.5, 1.6[$

1st Iteration:- $a_1 = 1.5$, $b_1 = 1.6$

$$\Rightarrow c_1 = \frac{1.5+1.6}{2} = 1.55$$

$f(c_1) = 2(1.55) - \cos(1.55) - 3$

$= 0.0792$



$f(1.5) \cdot f(1.55) < 0$

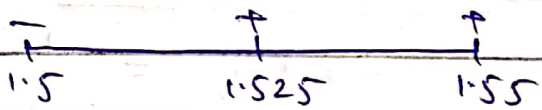
$\Rightarrow$ Root lies b/w $]1.5, 1.55[$

2nd Iteration:- $a_2 = 1.5$, $b_2 = 1.55$

$$\Rightarrow c_2 = \frac{1.5+1.55}{2} = 1.525$$

$f(c_2) = 2(1.525) - \cos(1.525) - 3$

$= 0.0042$



$f(1.5) \cdot f(1.525) < 0$

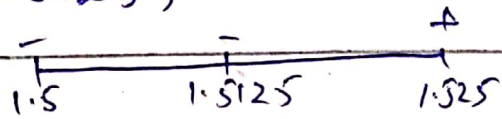
$\Rightarrow$ Root lies b/w $]1.5, 1.525[$

3rd Iteration:- $a_3 = 1.5$, $b_3 = 1.525$

$$\Rightarrow c_3 = \frac{1.5+1.525}{2} = 1.5125$$

$f(c_3) = 2(1.5125) - \cos(1.5125) - 3$

$= -0.0333$



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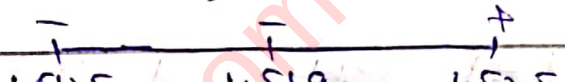
$f(1.5125) \cdot f(1.525) < 0$

$\Rightarrow$ Root lies b/w $]1.5125, 1.525[$

4th Iteration:- $a_4 = 1.5125$, $b_4 = 1.525$

$$\Rightarrow c_4 = \frac{1.5125 + 1.525}{2} = 1.519$$

$$f(c_4) = 2(1.519) - \cos(1.519) - 3$$
$$= -0.014$$



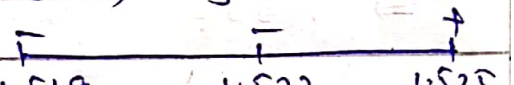
$f(1.519) \cdot f(1.525) < 0$

$\Rightarrow$ Root lies b/w $]1.519, 1.525[$

5th Iteration:- $a_5 = 1.519$, $b_5 = 1.525$

$$\Rightarrow c_5 = \frac{1.519 + 1.525}{2} = 1.522$$

$$f(c_5) = 2(1.522) - \cos(1.522) - 3$$
$$= -0.005$$



$f(1.522) \cdot f(1.525) < 0$

$\Rightarrow$ Root lies b/w $]1.522, 1.525[$

6th Iteration:- $a_6 = 1.522$, $b_6 = 1.525$

$$\Rightarrow c_6 = \frac{1.522 + 1.525}{2} = 1.5235$$

$$f(c_6) = 2(1.5235) - \cos(1.5235) - 3$$
$$= -0.000$$

$\Rightarrow$ Approximate root is

$$1.5235$$

***

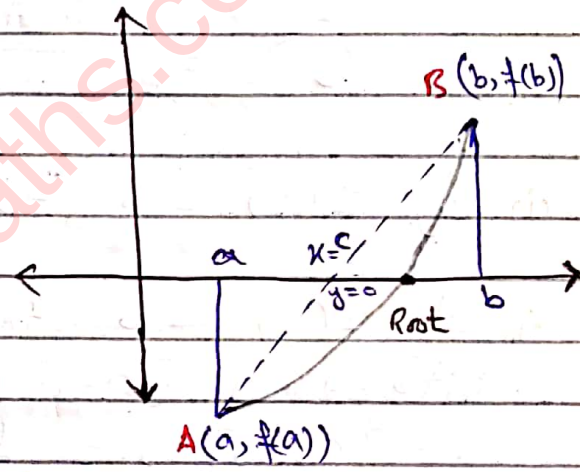
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2:- Regula Falsi Method (OR) Method of False Position.

If f is defined and continuous on $[a,b]$ with $f(a)\cdot f(b)<0$ then an approximation to the root of $f(x)=0$ is given by $c=\dfrac{af(b)-bf(a)}{f(b)-f(a)}$

Proof:- Let us consider the point $A=(a,f(a))$ & $B=(b,f(b))$. Now by using two point formula consider the equation of the line through the point A and B which is given by



$$y-f(b)=\frac{f(b)-f(a)}{b-a}(x-b) \quad \text{---} \circledast$$

Now obviously $x=c$ on x-axis is a better approximation to the root then at this point $y=0$ & $\circledast$ becomes

$$0-f(b)=\frac{f(b)-f(a)}{b-a}(c-b)$$

$$\Rightarrow c-b=-f(b)\left[\frac{b-a}{f(b)-f(a)}\right]$$

$$\Rightarrow c-b=\frac{f(b)[a-b]}{f(b)-f(a)}$$

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$$\Rightarrow c = \frac{f(b)[a-b]}{f(b)-f(a)} + b$$

$$= \frac{af(b) - bf(b)}{f(b) - f(a)} + b$$

$$= \frac{af(b) - bf(b) + bf(b) - b(f(a))}{f(b) - f(a)}$$

$$\Rightarrow c = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

(i) If $f(c) = 0$, then we have exact root

(ii) If $f(c) \neq 0$, then we consider the intervals $[a_0, c[$ s.t $f(a_0) f(c) < 0$ OR $]c, b_0]$ s.t $f(c) f(b_0) < 0$

We continuous this process and make new intervals in general for $[a_n, b_n]$ untill we have exact root for equation.

$$\Rightarrow c = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}$$

where $n = 0, 1, 2, \ldots$

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Example:- Solve $x \log_{10} x - 1.2 = 0$ correct upto four decimal places.

Solution:-

Let $f(x) = x \log_{10} x - 1.2$

Consider the following table

x	1.1	2.1	3.1
$f(x)$	−1.15	−0.523	0.323

As $f(2.1) \cdot f(3.1) < 0$. So Root lies in $[2.1, 3.1]$

1st Iteration:- Let $a_1 = 2.1$, $b_1 = 3.1$

$f(a_1) = -0.523$, $f(b_1) = 0.323$

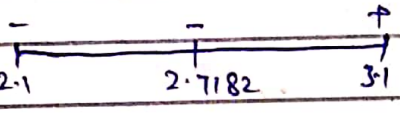
$$\Rightarrow c_1 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)}$$

$$\Rightarrow c_1 = \frac{2.1(0.323) - 3.1(-0.523)}{0.323 - (-0.523)}$$

$$c_1 = 2.7182$$

Now $f(c_1) = 2.7182 \log_{10}(2.7182) - 1.2$

$= -0.0195$



As $f(2.7182) \cdot f(3.1) < 0$

So Root lies between $]2.7182, 3.1[$

2nd Iteration:- Let $a_2 = 2.7182$, $b_2 = 3.1$

$f(a_2) = -0.0195$, $f(b_2) = 0.323$

$$\Rightarrow c_2 = \frac{a_2 f(b_2) - b_2 f(a_2)}{f(b_2) - f(a_2)} = \frac{2.7182(0.323) - 3.1(-0.0195)}{0.323 - (-0.0195)}$$

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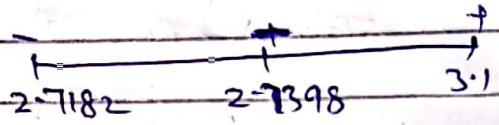
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$$\Rightarrow c_2 = \frac{0.8779 + 0.0604}{0.3425}$$

$$= 2.7398$$

$$f(c_2) = 2.7398 \log_{10}(2.7398) - 1.2$$

$$= -0.0007$$



As $f(2.7398) \cdot f(3.1) < 0$

So Root lies in $]2.7398, 3.1[$

3rd Iteration:- $a_3 = 2.7398$, $b_3 = 3.1$

$$f(a_3) = -0.0007, \quad f(b_3) = 0.323$$

$$\Rightarrow c_3 = \frac{a_3 f(b_3) - b_3 f(a_3)}{f(b_3) - f(a_3)}$$

$$= \frac{2.7398(0.323) - 3.1(-0.0007)}{0.323 - (-0.0007)}$$

$$= 2.7406$$

Now $f(c_3) = 2.7406 \log_{10} 2.7406 - 1.2$

$$= -0.0000$$

$\Rightarrow$ Required Root is 2.7406

★

Examples- Solve the equation $x - e^{-x} = 0$ by Regula falsi method, Given Root lies between $(0.5, 0.6)$

Solution:

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Let $f(x) = x - e^{-x}$

As given Root lies between $]0.5, 0.6[$

$\Rightarrow f(0.5) = -0.1065$, $f(0.6) = 0.0512$

1st Iteration:- Let

$a_1 = 0.5$, $b_1 = 0.6$, $f(a_1) = -0.1065$

$f(b_1) = 0.0512$

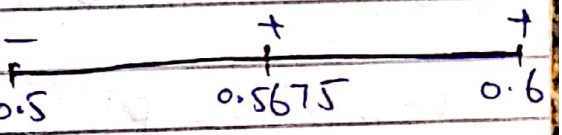
$$\Rightarrow c_1 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)}$$

$$c_1 = \frac{0.5(0.0512) - 0.6(-0.1065)}{0.0512 - (-0.1065)}$$

$$= \frac{0.0895}{0.1577} \Rightarrow c_1 = 0.5675$$

$\Rightarrow f(c_1) = 0.5675 - e^{-0.5675}$

$= .0006$



As $f(0.5) \cdot f(0.5675) < 0$

So Root lies between $]0.5, 0.5675[$

2nd Iteration:- Let

$a_2 = 0.5$, $b_2 = 0.5675$, $f(a_2) = -0.1065$

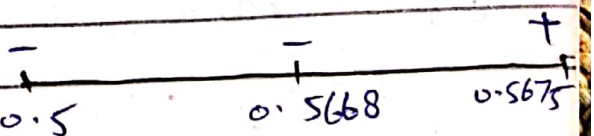
$f(b_2) = 0.0006$

$$\Rightarrow c_2 = \frac{0.5(0.0006) - 0.5675(-0.1065)}{0.0006 - (-0.1065)}$$

$$c_2 = \frac{0.0607}{0.1071} = 0.5668$$

$\Rightarrow f(c_2) = 0.5668 - e^{-0.5668}$

$= -0.0005$



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$\Rightarrow f(0.5668) \cdot f(0.5675) < 0$

$\Rightarrow$ Root lies b/w $]0.5668, 0.5675[$

3rd Iteration:- Let

$a_3 = 0.5668$, $b_3 = 0.5675$

$f(a_3) = -0.0005$, $f(b_3) = 0.0006$

$$\Rightarrow c_3 = \frac{0.5668(0.0006) - 0.5675(-0.0005)}{.0006 - (-0.0005)}$$

$$c_3 = \frac{0.00062383}{0.0011} = 0.5671$$

$$f(c_3) = 0.5671 - e^{-0.5671}$$

$$= -0.0000$$

So required root is 0.5671

* *

Examples:- Find cube root of 5 using method of false position upto 3 decimal places.

Solution:-

As $x = (5)^{1/3}$

$\Rightarrow x^3 = 5 \Rightarrow x^3 - 5 = 0$

$\Rightarrow f(x) = x^3 - 5$

x	1.6	1.7	1.8	1.9
$f(x)$	-0.904	-0.087	0.832	1.859

As $f(1.7) \cdot f(1.8) < 0$

So root lies b/w $]1.7, 1.8[$

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1st Iteration:- Let $a_1 = 1.7$, $b_1 = 1.8$

$f(a_1) = -0.087$, $f(b_1) = 0.832$

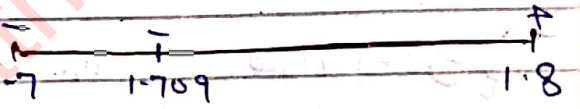
$$\Rightarrow c_1 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)}$$

$$\Rightarrow c_1 = \frac{1.7(0.832) - 1.8(-0.087)}{0.832 - (-0.087)}$$

$$c_1 = \frac{1.571}{0.919} \Rightarrow c_1 = 1.709$$

& $f(c_1) = (1.709)^3 - 5 = -0.009$

& $f(1.7) \cdot f(1.800) < 0$



$\Rightarrow$ Root lies b/w $]1.709, 1.800[$

2nd Iteration:- Now Let $a_2 = 1.709$, $b_2 = 1.800$

$\Rightarrow f(a_2) = -0.009$, $f(b_2) = 0.832$

$$\Rightarrow c_2 = \frac{1.7(-0.009) - 1.709(0.832)}{-0.009 - (-0.087)}$$

$$c_2 = \frac{1.709(0.832) - 1.8(-0.009)}{0.832 - (-0.009)}$$

$$c_2 = \frac{1.438}{0.841} \Rightarrow c_2 = 1.71$$

& $f(c_2) = (1.71)^3 - 5 = 0.000$

So Required root is 1.71

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Assignment.

Q1:- Solve By Regula falsi method

(i) $3x - \cos x - 1 = 0$

(ii) $e^{-x} - \sin\left(\frac{\pi x}{2}\right) = 0$

Q2:- Apply Regula falsi method to find approximation root of $\cos x - x = 0$ correct upto four decimal places on interval $(0.5, \pi/4)$

Q3:- Given that $x = 1.45$ is exact root of equation $x^2 - 5.57x + 5.974 = 0$ Apply three iteration of Regula falsi method for solving equation.

Q4:- Approximate the ~~soll~~ solution of followings.

(i) $3x^2 - e^x = 0$

(ii) $x^2 + 10 \cos x = 0$

Solutions.

Q1(i):-

Let $f(x) = 3x - \cos x - 1$

x	0.5	0.6	0.7	0.8
$f(x)$	-0.3775	-0.0253	0.3352	0.7033

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As $f(0.6) \cdot f(0.7) < 0$. So Root lies b/w $]0.6, 0.7[$

1st Iteration:- Let

$a_1 = 0.6$, $b_1 = 0.7$

$\Rightarrow f(a_1) = -0.0253$, $f(b_1) = 0.3352$

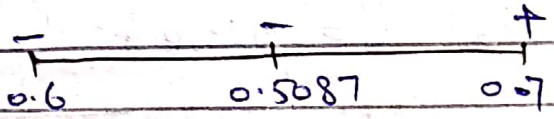
$$\Rightarrow c_1 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)}$$

$$c_1 = \frac{0.6(0.3352) - 0.7(-0.0253)}{0.3352 - (-0.0253)}$$

$$= \frac{0.1834}{0.3605} = 0.5087$$

So $f(c_1) = 3(0.5087) - \cos(0.5087) - 1$

$= -0.3473$



As $f(0.5087) \cdot f(0.7) < 0$

$\Rightarrow$ Root lies between $]0.5087, 0.7[$

2nd Iteration:-

Let $a_2 = 0.5087$, $b_2 = 0.7$

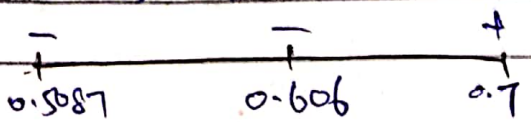
$f(a_2) = -0.3473$, $f(b_2) = 0.3352$

$$\Rightarrow c_2 = \frac{0.5087(0.3352) - 0.7(-0.3473)}{0.3352 - (-0.3473)}$$

$$= \frac{0.4136}{0.6825} = 0.6060$$

So $f(c_2) = 3(0.606) - \cos(0.606) - 1$

$= -0.004$



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As $f(0.606)\cdot f(0.7) < 0$ So Root lies b/w $]0.606, 0.7[$

3rd Iteration:-

Let $a_3 = 0.606$, $b_3 = 0.7$

$\Rightarrow f(a_3) = -0.004$, $f(b_3) = 0.3352$

$$\Rightarrow c_3 = \frac{0.606(0.3352) - 0.7(-0.004)}{0.3352 - (-0.004)}$$

$$= \frac{0.206}{0.3392} = 0.6071$$

& $f(c_3) = 3(0.6071) - e^{(0.6071)} - 1$

$= -0.0000$

So Required Root is 0.6071

Q 1(ii) Let $f(x) = e^{-x} - \sin\left(\frac{\pi}{2}x\right)$

x	0.3	0.4	0.5	0.6
$f(x)$	0.8959	0.0825	-0.1006	-0.2602

As $f(0.4)\cdot f(0.5) < 0$. So Root lies b/w $]0.4, 0.5[$

1st Iteration:- Let

$a_1 = 0.4$, $b_1 = 0.5$

$f(a_1) = 0.0825$, $f(b_1) = -0.1006$

$$\Rightarrow c_1 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)}$$

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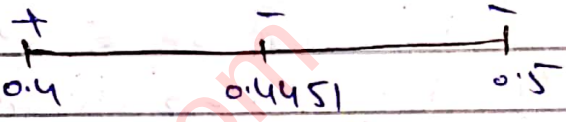
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$$\Rightarrow c_1 = \frac{0.4(-0.1006) - 0.5(0.0825)}{-0.1006 - 0.0825}$$

$$= \frac{-0.0815}{-0.1831} = 0.4451$$

Eq $f(c_1) = e^{-0.4451} - \sin\left(\frac{0.4451\,\pi}{2}\right)$

$$= -0.00281$$



As $f(0.445) \cdot f(0.4) < 0$

$\Rightarrow$ Root $\in\ ]0.400, 0.445[$

2nd Iteration:- Let $a_2 = 0.400$

$b_2 = 0.445$, $f(a_2) = 0.0825$

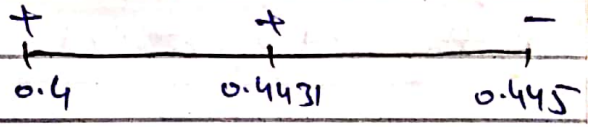
$f(b_2) = -0.0028$

$$c_2 = \frac{0.4(-0.0028) - 0.445(0.0825)}{-0.0028 - 0.0825}$$

$$= \frac{-0.0378}{-0.0853} = 0.4431$$

Eq $f(c_2) = e^{-0.4431} - \sin\left(\frac{0.4431\,\pi}{2}\right)$

$$= 0.0009$$



$\Rightarrow f(0.4431) \cdot f(0.445) < 0$

So Root $\in\ ]0.4431, 0.445[$

3rd Iteration:- Let $a_3 = 0.4431$

$b_3 = 0.445$, $f(a_3) = 0.0009$, $f(b_3) = -0.0028$

$$\Rightarrow c_3 = \frac{0.4431(-0.0028) - 0.445(0.0009)}{-0.0028 - 0.0009}$$

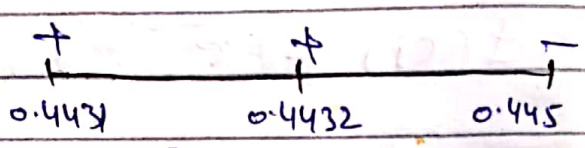
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$$\Rightarrow c_3 = \frac{-0.00164}{-0.0037} = 0.4432$$

$$f(c_3) = e^{-0.4432} - \sin\left(\frac{0.4432\pi}{2}\right)$$

$$= 0.0007$$



Clearly Root $\in]0.4432, 0.445[$

4rth Iteration:- Let $a_4 = 0.4432$

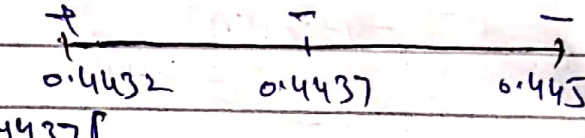
$b_4 = 0.445$, $f(a_4) = 0.0007$, $f(b_4) = -0.0028$

$$c_4 = \frac{0.4432(-0.0028) - 0.445(0.0007)}{-0.0028 - 0.0007}$$

$$= \frac{-0.001553}{-0.0035} = 0.4437$$

$$f(c_4) = e^{-0.4437} - \sin\left(\frac{0.4437\pi}{2}\right)$$

$$= -0.0002$$



Clearly Root $\in]0.4432, 0.4437[$

5 th Iteration:- Let $a_5 = 0.4432$

$b_5 = 0.4437$ $f(a_5) = 0.0007$, $f(b_5) = -0.0002$

$$\Rightarrow c_5 = \frac{0.4432(-0.0002) - 0.4437(0.0007)}{-0.0002 - 0.0007}$$

$$= \frac{-0.0004}{-0.0009} = 0.4444$$

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So Approximate Root is 0.4437 and is upto 4th Iteration.

Q2:- Let $f(x) = \cos x - 5$

& given interval is $(0.5, \frac{\pi}{4})$

So $f(0.5) = 0.3776$, $f(\frac{\pi}{4}) = -0.0783$

1st Iteration:- Let $a_1 = 0.5$, $b_1 = \frac{\pi}{4}$

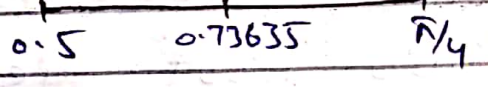
$f(a_1) = 0.3776$, $f(b_1) = -0.0783$

$$\Rightarrow c_1 = \frac{0.5(-0.0783) - \frac{\pi}{4}(0.3776)}{-0.0783 - 0.3776}$$

$$= \frac{-0.3357}{-0.4559} = 0.73635$$

$\Rightarrow f(c_2) = \cos(0.73635) - 5$

$= -0.00457$



Clearly Root $\in]0.7365, \frac{\pi}{4}[$

2nd Iteration:- Now Let $a_2 = 0.73655$

$b_2 = \frac{\pi}{4}$ $f(a_2) = 0.00457$, $f(b_2) = -0.0783$

$$\Rightarrow c_2 = \frac{0.73635(-0.0783) - \frac{\pi}{4}(0.00457)}{-0.0783 - 0.00457}$$

$$= \frac{-0.06125}{-0.08287} = 0.7391$$

$\Rightarrow f(c_2) = \cos(0.7391) - 5 = -0.0000$

So Required root is 0.7391

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Q3:- Let $f(x) = x^2 - 5.57x + 5.974$

x	1.37	1.46
$f(x)$	0.22	−0.0266

So Root lies b/w $]1.37, 1.46[$

1st Iteration:- Let $a_1 = 1.37$, $b_1 = 1.46$

$f(a_1) = 0.22$, $f(b_1) = -0.0266$

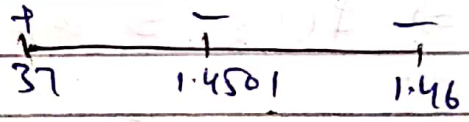
$$\Rightarrow c_1 = \frac{1.37(-0.0266) - 1.46(0.22)}{-0.0266 - 0.22}$$

$$= \frac{-0.3576}{-0.2466} = 1.4501$$

$$\Rightarrow f(c_1) = (1.4501)^2 - 5.57(1.4501) + 5.974$$

$$= -0.0003$$

$f(1.37) \cdot f(1.4501) < 0$



So Root lies b/w $(1.37, 1.4501)$.

2nd Iteration:- Now Let $a_2 = 1.37$

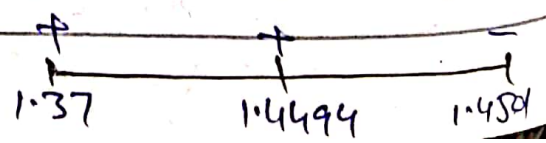
$b_2 = 1.4501$ $f(a_2) = 0.22$, $f(b_2) = -0.0003$

$$\Rightarrow c_2 = \frac{1.37(-0.0003) - 1.4501(0.22)}{-0.0003 - 0.22}$$

$$= \frac{-0.3193}{-0.2203} = 1.4494$$

$$\Rightarrow f(c_2) = (1.4494)^2 - 5.57(1.4494) + 5.974$$

$$= 0.0016$$



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As $f(1.4494) \cdot f(1.4501) < 0$ So root lies b/w $]1.4494, 1.4501[$

3rd Iteration:- $a_3 = 1.4494$, $b_3 = 1.4501$

$f(a_3) = 0.0016$, $f(b_3) = -0.0003$

$$\Rightarrow c_3 = \frac{1.4494(-0.0003) - 1.4501(0.0016)}{-0.0003 - 0.0016}$$

$$= \frac{-0.002755}{-0.0019} = 1.45$$

$$\Rightarrow f(c_3) = (1.45)^2 - 5.57(1.45) + 5.974$$

$$= 0$$

So the solution.

Q4(i) Let $f(x) = 3x^2 - e^x$

x	0.8	0.9	0.95	1
$f(x)$	-0.3055	-0.0296	0.122	0.28172

As $f(0.9) \cdot f(0.95) < 0$

So Root lies b/w $(0.9, 0.95)$

1st Iteration:- Let $a_1 = 0.9$, $b_1 = 0.95$

$f(a_1) = -0.0296$, $f(b_1) = 0.122$

$$\Rightarrow c_1 = \frac{0.9(0.122) - 0.95(-0.0296)}{0.122 - (0.0296)}$$

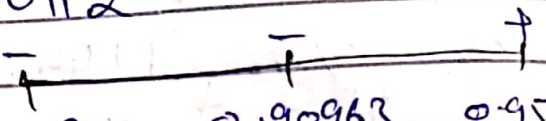
$$= \frac{0.1379}{0.1516} = 0.90963$$

$$\Rightarrow f(c_1) = 3(0.90963)^2 - e^{0.90963}$$

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$\Rightarrow f(c_1) = -0.00112$



As $f(0.90963) \cdot f(0.95) < 0$

So Root $\in (0.90963, 0.95)$

2nd Iteration:- $a_2 = 0.90963$, $b_2 = 0.95$

$f(a_2) = -0.00112$, $f(b_2) = 0.122$

$$c_2 = \frac{0.90963(0.122) - (0.95)(-0.00112)}{0.122 - (-0.00112)}$$

$$= \frac{0.11204}{0.12312} = 0.91001$$

$\Rightarrow f(c_2) = 3(0.91001)^2 - e^{0.91001}$

$= 0.00000$

So Required Root is 0.91001

Q4 (ii) Let $f(x) = x^2 + 10\cos x$

x	1.8	1.9	2
$f(x)$	0.96798	0.3771	-0.16147

As $f(1.9) \cdot f(2) < 0$

$\Rightarrow$ Root lies $\in]1.9, 2[$

1st Iteration:- $a_1 = 1.9$, $b_1 = 2$

$f(a_1) = 0.3771$, $f(b_1) = -0.1647$

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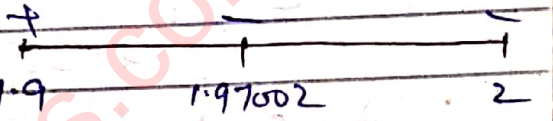
$$\Rightarrow c_1 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)}$$

$$= \frac{1.9(-0.16147) - 2(0.3771)}{-0.16147 - 0.3771}$$

$$= \frac{-1.060993}{-0.53857} = 1.97002$$

$$\Rightarrow f(c_1) = (1.97002)^2 + 10\cos(1.97002)$$

$$= -0.00605$$



$f(1.97002) \cdot f(1.9) < 0$

So Root lies b/w $(1.9, 1.97002)$

2nd Iteration:- $a_2 = 1.9$, $b_2 = 1.97002$

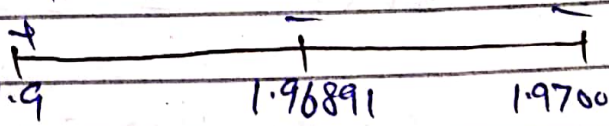
$f(a_2) = 0.3771$, $f(b_2) = -0.00605$

$$\Rightarrow c_2 = \frac{1.9(-0.00605) - 1.97002(0.3771)}{-0.00605 - 0.3771}$$

$$= \frac{0.75439}{0.38315} = 1.96891$$

$$\& \quad f(c_2) = (1.96891)^2 + 10\cos(1.96891)$$

$$= -0.00019$$



As $f(1.9) \cdot f(1.96891) < 0$

So Root lies between $(1.9, 1.96891)$

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3rd Iteration:- $a_3 = 1.9$

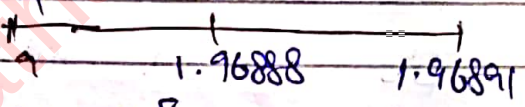
$b_3 = 1.96891$ $\quad f(a_3) = 0.3771$

$f(b_3) = -0.00019$

$$\Rightarrow c_3 = \frac{1.9(-0.00019) - 1.96891(0.3771)}{-0.00019 - 0.3771}$$

$$= \frac{-0.74283}{-0.37729} = 1.96888$$

$\Rightarrow f(c_3) = -0.00003$



Clearly Root $\in]1.90000, 1.96888[$

4rth Iteration:- $a_4 = 1.9$

$b_4 = 1.96888$ $\quad f(a_4) = 0.3771$

$f(b_4) = -0.00003$

$$\Rightarrow c_4 = \frac{1.9(-0.00003) - 1.96888(0.3771)}{-0.00003 - 0.3771}$$

$$= \frac{-0.74252}{0.37713} = 1.96887$$

$\Rightarrow f(c_4) = 0.00001$

So Approximate Root is 1.96887

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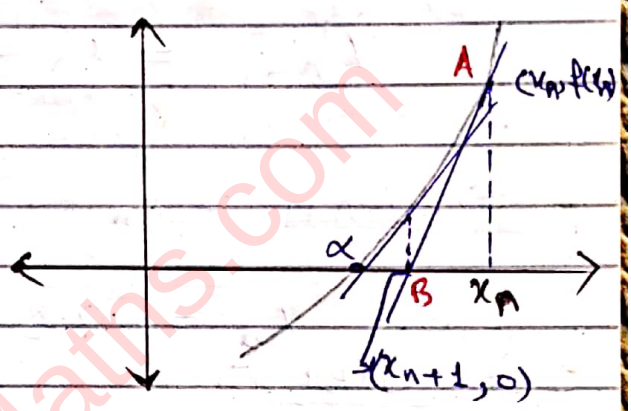
3. Newton-Raphson Method.

If x_n is an approximation to the root of $f(x) = 0$ then a better approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Proof:-

Let $f(x) = 0$ be a function cuts the x-axis at α. So α be the exaxt root of the function. Let x_n be the initial guess (approximation). Then



Slope of Tangent at A = Slope of line Through A & B

$\Rightarrow f'(x_n) = \frac{f(x_n) - 0}{x_n - x_{n+1}}$

$\Rightarrow x_n - x_{n+1} = \frac{f(x_n)}{f'(x_n)}$

$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0, 1, 2, \cdots$

***

Example:- Use Newton Raphson method to find root of equation

$$\sin x = 1 - x$$

Solution:-

$$\sin x = 1 - x$$

$$\Rightarrow x + \sin x - 1 = 0$$

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$$\Rightarrow f(x) = x + \sin x - 1$$

$$\Rightarrow f'(x) = 1 + \cos x$$

$$f'(x_n) = 1 + \cos x_n$$

Now Newton Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = x_n - \frac{x_n + \sin x_n - 1}{1 + \cos x_n} \quad \text{---} \circledast$$

Let $x_0 = 0.8$ be initial guess

1st Iteration:-

Put $n = 0$ in $\circledast$

$$x_1 = x_0 - \frac{x_0 + \sin x_0 - 1}{1 + \cos x_0}$$

$$x_1 = 0.8 - \frac{0.8 + \sin(0.8) - 1}{1 + \cos(0.8)}$$

$$= 0.8 - \frac{0.51735}{1.6967}$$

$$= 0.8 - 0.3049$$

$$= 0.4951$$

2nd Iteration:- Put $n = 1$ in $\circledast$

$$\Rightarrow x_2 = x_1 - \frac{x_1 + \sin x_1 - 1}{1 + \cos x_1}$$

$$= 0.4951 - \frac{0.4951 + \sin(0.4951) - 1}{1 + \cos(0.4951)}$$

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$$\Rightarrow x_2 = 0.4951 - \frac{(-0.02978)}{1.8799}$$

$$= 0.4951 - (-0.01584)$$

$$\Rightarrow x_2 = 0.5109$$

3rd Iteration:- Put $n=2$ in equ ⊛

$$\Rightarrow x_3 = x_2 - \frac{x_2 + \sin x_2 - 1}{1 + \cos x_2}$$

$$= 0.5109 - \frac{0.5109 + \sin(0.5109) - 1}{1 + \cos(0.5109)}$$

$$= 0.5109 - \frac{(-0.0001)}{1.8723}$$

$$= 0.5109 - (-0.0000)$$

$$\Rightarrow x_3 = 0.5109$$

As $x_2 = 0.5109$ & $x_3 = 0.5109$.

In Newton Raphson method repeated root is called root or solution of equation

$\Rightarrow$ Required root is 0.5109

* * *

Example:- Apply Newton Raphson method to find the solution of $x^3 + x - 1 = 0$

Solution:-

Let $f(x) = x^3 + x - 1$

$\Rightarrow f'(x) = 3x^2 + 1$

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$f(x_n) = x_n^3 + x_n - 1$ & $f'(x_n) = 3x_n^2 + 1$

Now N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1} \quad \text{---(A)}$$

Let $x_0 = 0.5$ be the initial guess

1st Iteration:- Put $n=0$ in (A)

$$x_1 = x_0 - \frac{x_0^3 + x_0 - 1}{3x_0^2 + 1}$$

$$\Rightarrow x_1 = 0.5 - \frac{0.5^3 + 0.5 - 1}{3(0.5)^2 + 1}$$

$$= 0.5 - \frac{(-0.375)}{1.75} = 0.5 + 0.2143$$

$$x_1 = 0.7143$$

2nd Iteration:- Put $n=1$ in (A),

$$\Rightarrow x_2 = x_1 - \frac{x_1^3 + x_1 - 1}{3x_1^2 + 1}$$

$$x_2 = 0.7143 - \frac{(0.7143)^3 + 0.7143 - 1}{3(0.7143)^2 + 1}$$

$$= 0.7143 - \frac{0.0788}{2.5308} = 0.7143 - 0.0311$$

$$x_2 = 0.6832$$

3rd Iteration:- Put $n=2$ in (A)

$$x_3 = x_2 - \frac{x_2^3 + x_2 - 1}{3x_2^2 + 1}$$

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$$\Rightarrow x_3 = 0.6832 - \frac{(0.6832)^3 + 0.6832 - 1}{3(0.6832)^2 + 1}$$

$$= 0.6832 - \frac{0.0021}{2.400} = 0.6832 - 0.0009$$

$$\Rightarrow x_3 = 0.6823$$

4 th Iteration:- Put $n=3$ in (*)

$$x_4 = 0.6823 - \frac{(0.6823)^3 + 0.6823 - 1}{3(0.6823)^2 + 1}$$

$$= 0.6823 - \frac{0.0000}{2.3966} = 0.6823 - 0$$

$$= 0.6823$$

So The Required root is 0.6823

***

Example:- Use Newton Raphson formula to find the p^{th} root of N.

Solution:-

Let x be a p^{th} root of N

$$\Rightarrow x = (N)^{1/p} \Rightarrow x^p = N$$

$$\Rightarrow x^p - N = 0$$

So $f(x) = x^p - N$

$$\Rightarrow f'(x) = p x^{p-1}$$

$$f(x_n) = x_n^p - N$$

& $f'(x_n) = p x_n^{p-1}$

Now by Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

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$$\Rightarrow x_{n+1} = x_n - \left[\frac{x_n^p - N}{p x_n^{p-1}}\right]$$

$$x_{n+1} = \frac{x_n p x_n^{p-1} - (x_n^p - N)}{p x_n^{p-1}}$$

$$= \frac{p x_n^p - x_n^p + N}{p x_n^{p-1}}$$

$$= \frac{(p-1)x_n^p + N}{p x_n^{p-1}}$$

$$\Rightarrow x_{n+1} = \frac{1}{p}\left[(p-1)x_n + \frac{N}{x_n^{p-1}}\right]$$

* * *

Examples:- Show that Newton's method of square root of N is given by $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$ and also find square root of 11 by above formula upto 4-decimal places.

Solution:-

Let $x = \sqrt{N}$

$\Rightarrow x^2 = N \quad \Rightarrow x^2 - N = 0$

So $f(x) = x^2 - N$

$\Rightarrow f'(x) = 2x$

$f(x_n) = x_n^2 - N$

& $f'(x_n) = 2x_n$

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Now by Newton Raphson's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n}$$

$$\Rightarrow x_{n+1} = \frac{2x_n^2 - (x_n^2 - N)}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$= \frac{x_n^2(2-1) + N}{2x_n} = \frac{x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right] \quad \text{——— (1)}$$

Put $N = 11$ in eqn (1):

$$\Rightarrow x_{n+1} = \frac{1}{2}\left[x_n + \frac{11}{x_n}\right] \quad \text{——— (*)}$$

Let $x_0 = 3.5$ be initial guess

1st Iteration:- Put $n = 0$ in (*)

$$\Rightarrow x_1 = \frac{1}{2}\left[x_0 + \frac{11}{x_0}\right]$$

$$= \frac{1}{2}\left[3.5 + \frac{11}{3.5}\right]$$

$$= 3.3214$$

2nd Iteration:- Put $n = 1$ in (*)

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$$x_2 = \frac{1}{2}\left[x_1 + \frac{11}{x_1}\right] = \frac{1}{2}\left[3.3214 + \frac{11}{3.3214}\right]$$

$$= \frac{1}{2}\left[3.3214 + 3.3186\right] = \frac{1}{2}\left[6.64\right]$$

$$= 3.32$$

3rd Iteration:- Put $n=2$ in (A)

$$x_3 = \frac{1}{2}\left(x_2 + \frac{11}{x_2}\right) = \frac{1}{2}\left[3.32 + \frac{11}{3.32}\right]$$

$$= \frac{1}{2}\left[3.32 + 3.3132\right] = \frac{1}{2}\left[6.6332\right]$$

$$= 3.3166$$

4th Iteration:- Put $n=3$ in (A)

$$x_4 = \frac{1}{2}\left(x_3 + \frac{11}{x_3}\right) = \frac{1}{2}\left[3.3166 + \frac{11}{3.3166}\right]$$

$$= \frac{1}{2}\left[3.3166 + 3.3166\right] = \frac{1}{2}\left[6.6332\right]$$

$$= 3.3166$$

So Approximate root is 3.3166

* * *

Assignment.

Q1:- By taking logarithm, find solution of $x^x = 10$ correct upto four-decimal places By Newton-Raphson Method

Q2:- Using N-R method, Establish

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an iterate formula for finding a cube root of real number C, Also find for $C = 30$ i.e $(30)^{1/3}$, given that initial guess $= 3$

Q3:- Find real root near to 1.5 of equation $2x - 2\sin x - 1 = 0$.

Q4:- Find the negative root of equation $e^x - \tan x = 0$ correct upto 4 decimal pla

Q5:- Using N-R method, evaluate a +ve root of equation $e^x - 3x = 0$, correct upto three decimal places.

Q6:- Use Newton-Raphson method to find the reciprocal of a number N

Solutions.

Q1:-

Given $x^x = 10$

Taking Log on both sides

$\Rightarrow x \log x = \log 10$

$\Rightarrow x \log x - \log 10 = 0$

$\Rightarrow$ So $f(x) = x \log x - \log 10$

& $f'(x) = \log x + 1$

$\Rightarrow f(x_n) = x_n \log x_n - \log 10$

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& $f'(x_n) = \log x_n + 1$

Now by N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = x_n - \frac{x_n \log x_n - \log 10}{\log x_n + 1} \quad \text{---} (*)$$

Let $x_0 = 2.4$ be initial guess

1st Iteration:- Put $n=0$ in (*)

$$x_1 = x_0 - \frac{x_0 \log x_0 - \log 10}{\log x_0 + 1}$$

$$= 2.4 - \frac{2.4 \log(2.4) - \log 10}{\log(2.4) + 1}$$

$$= 2.4 - \frac{-0.0875}{1.3802} = 2.4 + 0.0634$$

$$= 2.4634$$

2nd Iteration:- Put $n=1$ in (*)

$$\Rightarrow x_2 = x_1 - \frac{x_1 \log x_1 - \log 10}{\log x_1 + 1}$$

$$= 2.4634 - \frac{2.4634 \log(2.4634) - \log 10}{\log(2.4634) + 1}$$

$$= 2.4634 - \frac{-0.0355}{1.3915} = 2.4634 + .0255$$

$$= 2.4889$$

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3rd Iteration:- Put $n=2$ in (*)

$$\Rightarrow x_3 = x_2 - \frac{x_2 \log x_2 - \log 10}{\log x_2 + 1}$$

$$x_3 = 2.4889 - \frac{2.4889 \log(2.4889) - \log 10}{\log(2.4889) + 1}$$

$$= 2.4889 - \frac{-0.0144}{1.3960} = 2.4889 + 0.0103$$

$$= 2.4992$$

4th Iteration:- Put $n=3$ in (*)

$$x_4 = x_3 - \frac{x_3 \log x_3 - \log 10}{\log x_3 + 1}$$

$$\Rightarrow x_4 = 2.4992 - \frac{2.4992 \log 2.4992 - \log 10}{\log 2.4992 + 1}$$

$$= 2.4992 - \frac{-0.0058}{1.3978} = 2.4992 + .0041$$

$$= 2.5033$$

5th Iteration:- Put $n=4$ in (*)

$$\Rightarrow x_5 = 2.5033 - \frac{2.5033 \log(2.5033) - \log 10}{\log(2.5033) + 1}$$

$$= 2.5033 - \frac{-0.0024}{1.3985} = 2.5033 + .0017$$

$$= 2.505$$

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6th Iteration:- Put $n=5$ in (*)

$$\Rightarrow x_6 = 2.505 - \frac{2.505\log(2.505) - \log 10}{\log(2.505) + 1}$$

$$= 2.505 - \frac{-0.0001}{1.3988} = 2.505 + .0001$$

$$= 2.5057$$

7th Iteration:- Put $n=6$ in (*)

$$\Rightarrow x_7 = 2.5057 - \frac{2.5057\log(2.5057) - \log 10}{\log(2.5057) + 1}$$

$$= 2.5057 - \frac{-0.0004}{1.3989} = 2.5057 + .0003$$

$$= 2.506$$

8th Iteration:- Put $n=7$ in (*)

$$x_8 = 2.506 - \frac{2.506\log(2.506) - \log 10}{\log(2.506) + 1}$$

$$= 2.506 - \frac{-0.0001}{1.399} = 2.506 + 0.0000$$

$$= 2.506$$

So Approximate Root is 2.506

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Q 2:- Let $c^{1/3} = x$

$\Rightarrow x^3 = c \Rightarrow x^3 - c = 0$

So $f(x) = x^3 - c$

& $f'(x) = 3x^2$

$\Rightarrow f(x_n) = x_n^3 - c$

$f'(x_n) = 3x_n^2$

Now By N-R method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - c}{3x_n^2}$$

$$= \frac{3x_n^3 - x_n^3 + c}{3x_n^2}$$

$$x_{n+1} = \frac{2x_n^3 + c}{3x_n^2} \quad \text{---------- (1)}$$

When $c = 30 \Rightarrow f(x) = x^3 - 30$

(1) $\Rightarrow x_{n+1} = \frac{2x_n^3 + 30}{3x_n^2}$ ---- (*)

Let $x_0 = 3$ be initial guess

1st Iteration:- Put $n = 0$ in (*)

$$x_1 = \frac{2x_0^3 + 30}{3x_0^2} = \frac{54 + 30}{27} = \frac{84}{27}$$

$$= 3.1111$$

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2nd Iteration:- Put $n=1$ in (*)

$$\Rightarrow x_2 = \frac{2x_1^3+30}{3x_1^2} = \frac{2(3.1111)^3+30}{3(3.1111)^2}$$

$$= \frac{60.1122}{29.0368}$$

$$= 2.0702$$

3rd Iteration:- Put $n=2$ in (*)

$$\Rightarrow x_3 = \frac{2x_2^3+30}{3x_2^2} = \frac{2(2.0702)^3+30}{3(2.0702)^2}$$

$$= \frac{47.7446}{12.8572} = 3.7135$$

4th Iteration:- Put $n=3$ in (*)

$$\Rightarrow x_4 = \frac{2x_3^3+30}{3x_3^2} = \frac{2(3.7135)^3+30}{3(3.7135)^2}$$

$$= \frac{132.4189}{41.3702} = 3.2008$$

5th Iteration:- Put $n=4$ in (*)

$$\Rightarrow x_5 = \frac{2x_4^3+30}{3x_4^2} = \frac{2(3.2008)^3+30}{3(3.2008)^2}$$

$$= \frac{95.5862}{30.7354} = 3.10995$$

6th Iteration:- Put $n=5$ in (*)

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$$x_6 = \frac{2x_5^3 + 30}{3x_5^2} = \frac{2(3.10995)^3 + 30}{3(3.10995)^2}$$

$$= \frac{90.1576}{29.0154} = 3.1072$$

7th Iteration:- Put $n = 6$ in (*)

$$\Rightarrow x_7 = \frac{2x_6^3 + 30}{3x_6^2} = \frac{2(3.1072)^3 + 30}{3(3.1072)^2}$$

$$= \frac{89.9981}{28.9641}$$

$$= 3.1072$$

So Root is 3.1072

Q3:- Give $2x - 2\sin x - 1 = 0$

So $f(x) = 2x - 2\sin x - 1$

$\Rightarrow f'(x) = 2 - 2\cos x$

So $f(x_n) = 2x_n - 2\sin x_n - 1$

$f'(x_n) = 2 - 2\cos x_n$

Now By Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = x_n - \frac{2x_n - 2\sin x_n - 1}{2 - 2\cos x_n} \quad (*)$$

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Let $x_0 = 1.5$ be initial guess

1st Iteration:- Put $n=0$ in (*)

$$\Rightarrow x_1 = x_0 - \frac{2x_0 - 2\sin x_0 - 1}{2 - 2\cos x_0}$$

$$= 1.5 - \frac{2(1.5) - 2\sin(1.5) - 1}{2 - 2\cos(1.5)}$$

$$= 1.5 - \frac{0.005}{1.8585} = 1.5 - 0.0027$$

$$= 1.4973$$

2nd Iteration:- Put $n=1$ in (*)

$$\Rightarrow x_2 = x_1 - \frac{2x_1 - 2\sin x_1 - 1}{2 - 2\cos x_1}$$

$$= 1.4973 - \frac{2(1.4973) - 2\sin(1.4973) - 1}{2 - 2\cos(1.4973)}$$

$$= 1.4973 - \frac{-0.0000}{1.8531}$$

$$= 1.4973 + 0$$

$$= 1.4973$$

$\Rightarrow$ Root is 1.4973

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Q4:- To find -ve root of $e^x - \tan x = 0$

Let $f(x) = e^x - \tan x$

$\Rightarrow f'(x) = e^x - \sec^2 x$

$f(x_n) = e^{x_n} - \tan x_n$

$f'(x_n) = e^{x_n} - \sec^2 x_n$

Now By Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = x_n - \frac{e^{x_n} - \tan x_n}{e^{x_n} - \sec^2 x_n}$$

$$= x_n - \frac{e^{x_n} - \tan x_n}{e^{x_n} - \frac{1}{\cos^2 x_n}}$$

$$= x_n - \frac{e^{x_n} - \tan x_n}{\frac{e^{x_n}\cos^2 x_n - 1}{\cos^2 x_n}}$$

$$= x_n - \frac{(e^{x_n} - \tan x_n)\cos^2 x_n}{e^{x_n}\cos^2 x_n - 1}$$

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$$\Rightarrow x_{n+1} = x_n - \frac{(e^{x_n} - \tan x_n)(1+\cos 2x_n)}{e^{x_n}(1+\cos 2x_n) - 1} \quad \text{---} \quad (*)$$

Let $x_0 = -2$ be initial guess

1st Iteration:- Put $n=0$ in (*)

$$x_1 = x_0 - \frac{(e^{x_0} - \tan x_0)(1+\cos 2x_0)}{e^{x_0}(1+\cos 2x_0) - 1}$$

$$= -2 - \frac{[e^{-2} - \tan(-2)][1+\cos 2(-2)]}{e^{-2}[1+\cos 2(-2)] - 1}$$

$$= -2 - \left[\frac{-0.7099}{-0.9531}\right] = -2 - .7448$$

$$= -2.7448$$

2nd Iteration:- Put $n=1$ in (*)

$$\Rightarrow x_2 = x_1 - \frac{(e^{x_1} - \tan x_1)(1+\cos 2x_1)}{e^{x_1}(1+\cos 2x_1) - 1}$$

$$x_2 = -2.7448 - \frac{[e^{-2.7448} - \tan(-2.7448)][1+\cos 2(-2.7448)]}{e^{-2.7448}[1+\cos 2(-2.7448)] - 1}$$

$$= -2.7448 - \frac{-0.6035}{-0.8907} = -2.7448 - .6776$$

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$$\Rightarrow x_2 = -3.4224$$

3rd Iteration:- Put $n=2$ in (*)

$$\Rightarrow x_3 = x_2 - \frac{(e^{x_2} - \tan x_2)(1+\cos 2x_2)}{e^{x_2}[1+\cos 2x_2] - 1}$$

$$x_3 = -3.4224 - \frac{0.5928}{-0.9347} = -3.4224 + 0.6308$$

$$= -2.7916$$

4th Iteration:- Put $n=3$ in (*)

$$x_4 = x_3 - \frac{(e^{x_3} - \tan x_3)(1+\cos 2x_3)}{e^{x_3}[1+\cos 2x_3] - 1}$$

$$= -2.7916 - \left[\frac{-0.536}{-0.8918}\right]$$

$$= -2.7916 - 0.601 = -3.3926$$

5th Iteration:- Put $n=4$ in (*)

$$x_5 = x_4 - \frac{(e^{x_4} - \tan x_4)(1+\cos 2x_4)}{e^{x_4}[1+\cos 2x_4] - 1}$$

$$= -3.3926 - \left[\frac{0.5543}{-0.9369}\right]$$

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$$x_5 = -3.3926 + 0.5916$$

$$= -2.801$$

6th Iteration:- Put $n=5$ in (*)

$$x_6 = -2.801 - \frac{\left[e^{-2.801} - \tan(-2.801)\right]\left[1+\cos 2(-2.801)\right]}{e^{-2.801}\left[1+\cos 2(-2.801)\right] - 1}$$

$$= -2.801 - \left[\frac{-0.5218}{-0.8921}\right] = -2.801 - 0.585$$

$$= -3.386$$

7th Iteration:- Put $n=6$ in (*)

$$\Rightarrow x_7 = x_6 - \frac{\left[e^{x_6} - \tan x_6\right]\left[1+\cos 2x_6\right]}{e^{x_6}\left[1+\cos 2x_6\right] - 1}$$

$$= -3.386 - \left[\frac{0.5333}{-0.9363}\right]$$

$$= -3.386 + 0.5696 = -2.8164$$

8th Iteration:- Put $n=7$ in (*)

$$x_8 = x_7 - \frac{\left[e^{x_7} - \tan x_7\right]\left[1+\cos 2x_7\right]}{e^{x_7}\left[1+\cos 2x_7\right] - 1}$$

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$$x_8 = -2.8164 - \left|\frac{-0.4981}{-0.8926}\right| = -2.8164 + 0.5762$$

$$= -3.3926$$

As $x_4 = -3.3926 = x_8$

So Root is -3.3926

Q5:- Given $e^x - 3x = 0$

So $f(x) = e^x - 3x$

$\Rightarrow f'(x) = e^x - 3$

& $f(x_n) = e^{x_n} - 3x_n$

$f'(x_n) = e^{x_n} - 3$

Now By Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = x_n - \frac{e^{x_n} - 3x_n}{e^{x_n} - 3} \quad \text{---------} (*)$$

Let $x_0 = 1.5$ is initial guess

1st Iteration:- Put $n = 0$ in (*)

$$\Rightarrow x_1 = x_0 - \frac{e^{x_0} - 3x_0}{e^{x_0} - 3}$$

$$= 1.5 - \frac{e^{1.5} - 3(1.5)}{e^{1.5} - 3} = 1.5 - \frac{-0.0183}{1.4817}$$

$$= 1.5 + 0.0124$$

$$= 1.5124$$

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2nd Iteration:- Put $n=1$ in (*)

$$\Rightarrow x_2 = x_1 - \frac{e^{x_1} - 3(x_1)}{e^{x_1} - 3}$$

$$= 1.5124 - \frac{e^{1.5124} - 3(1.5124)}{e^{1.5124} - 3}$$

$$= 1.5124 - \frac{.0004}{1.5376} = 1.5124 - .0003$$

$$= 1.5121$$

3rd Iteration:- Put $n=2$ in (*)

$$x_3 = x_2 - \frac{e^{x_2} - 3x_2}{e^{x_2} - 3}$$

$$x_3 = 1.5121 - \frac{e^{1.5121} - 3(1.5121)}{e^{1.5121} - 3}$$

$$= 1.5121 - \frac{-0.0000}{1.5363} = 1.5121 + 0$$

$$= 1.5121$$

So Approximate root is 1.5121

Q6:- Let $x = \frac{1}{N}$

$$\Rightarrow xN = 1 \qquad \Rightarrow xN - 1 = 0$$

So $f(x) = xN - 1$

$$f'(x) = N$$

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$\Rightarrow f(x_n) = x_n N - 1$,

$f'(x_n) = N$

So By N-R method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = x_n - \frac{x_n N - 1}{N}$$ Ans.

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4:- Iteration Method (OR) Fixed Point Iteration.

The process of finding successive approximation to a quantity is called iteration.

* **Working Rule:-** Let $f(x) = 0$ has a root in $[a,b]$ & $f(x) = 0$ be written in the form $x = g(x)$ such that $|g'(x)| < 1$ in $]a,b[$. Then if x_n is an approximation to the root, then a better approximation x_{n+1} is given by $x_{n+1} = g(x_n)$

***

Example:- Find by fixed point iterative method to 5 decimal point root near 0.5 of equation $\sin x = 5x - 2$

Solution:-

$$\sin x = 5x - 2$$
$$\Rightarrow 5x - \sin x - 2 = 0$$
$$\text{So } f(x) = 5x - \sin x - 2$$

Consider the table

x	0	0.5	1
$f(x)$	-2	0.0206	2.1585

$\Rightarrow f(x) = 0$ has root in $[0, 0.5]$

Now $\sin x = 5x - 2$

$$\Rightarrow x = \frac{1}{5}[2 + \sin x]$$

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Now $g'(x) = \frac{1}{5}[0 + \cos x]$

$= \frac{1}{5}\cos x$

$|g'(x)| = \frac{1}{5}|\cos x| \leq \frac{1}{5}(1) < 1 \quad \forall x \in [0, 0.5]$

So $x_n = g(x_n)$ will converge

So by Iteration Method

$x_{n+1} = g(x_n)$

$\Rightarrow x_{n+1} = \frac{1}{5}(\sin x_n + 2)$ ——— (*)

Choose $x_0 = 0.35$ as initial guess

1st Iteration:- Put $n = 0$ in (*)

$\Rightarrow x_1 = \frac{1}{5}(\sin x_0 + 2)$

$= \frac{1}{5}(\sin(0.35) + 2) = \frac{1}{5}(2.3429)$

$= 0.4686$

2nd Iteration:- Put $n = 1$ in (*)

$\Rightarrow x_2 = \frac{1}{5}(\sin x_1 + 2)$

$= \frac{1}{5}(\sin(0.4686) + 2)$

$= 0.4903$

3rd Iteration:- Put $n = 2$ in (*)

$\Rightarrow x_3 = \frac{1}{5}(\sin x_2 + 2)$

$= \frac{1}{5}(\sin(0.4903) + 2)$

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$\Rightarrow x_3 = 0.4942$

4th Iteration:- Put $n=3$ in (*)

$\Rightarrow x_4 = \frac{1}{5}(\sin x_3 + 2)$

$= \frac{1}{5}(\sin(0.4942) + 2)$

$= 0.4949$

5th Iteration:- Put $n=4$ in (*)

$\Rightarrow x_5 = \frac{1}{5}(\sin x_4 + 2)$

$= \frac{1}{5}[\sin(0.4949) + 2]$

$= 0.4950$

6th Iteration:- Put $n=5$ in (*)

$x_6 = \frac{1}{5}(\sin x_5 + 2)$

$= \frac{1}{5}[\sin(0.4950) + 2]$

$= 0.4950$

$\Rightarrow$ Approximate root is 0.4950

Example:- Find a real root correct to four decimal places of equation $x^3 + x^2 - 1 = 0$ by iteration method

Solution:-

Let $f(x) = x^3 + x - 1$

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x	0	1
$f(x)$	-1	1

$\Rightarrow$ Root lies between 0 and 1

Now $x^3 + x^2 - 1 = 0$

$\Rightarrow x^2(x+1) - 1 = 0$

$\Rightarrow x^2 = \frac{1}{1+x}$

$\Rightarrow x = \frac{1}{\sqrt{1+x}}$ ——— (1)

$g(x) = (1+x)^{-1/2}$

$g'(x) = \frac{-1}{2}(1+x)^{-3/2}$

$= \frac{-1}{2(1+x)^{3/2}}$

$|g'(x)| = \left|\frac{-1}{2(1+x)^{3/2}}\right| < 1 \quad \forall x \in [0,1]$

$\therefore$ Iteration method is applicable

Setting iteration to (1)

$x_{n+1} = \frac{1}{\sqrt{1+x_n}}$

Now guess root taken as $x_0 = 1$

1st Iteration:- Put $n=0$ in (1)

$\Rightarrow x_1 = \frac{1}{\sqrt{1+x_0}} = \frac{1}{\sqrt{1+1}}$

$= 0.7071$

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2nd Iteration:- Put $n=1$ in (*)

$$\Rightarrow x_2 = \frac{1}{\sqrt{1+x_1}} = \frac{1}{\sqrt{1+0.7071}}$$

$$= 0.7654$$

3rd Iteration:- Put $n=2$ in (*)

$$\Rightarrow x_3 = \frac{1}{\sqrt{1+x_2}} = \frac{1}{\sqrt{1+0.7654}}$$

$$x_3 = 0.7526$$

4th Iteration:- Put $n=3$ in (*)

$$\Rightarrow x_4 = \frac{1}{\sqrt{1+x_3}} = \frac{1}{\sqrt{1+0.7526}}$$

$$x_4 = 0.7554$$

5th Iteration:- Put $n=4$ in (*)

$$\Rightarrow x_5 = \frac{1}{\sqrt{1+x_4}} = \frac{1}{\sqrt{1+0.7554}}$$

$$x_5 = 0.7548$$

6th Iteration:- Put $n=5$ in (*)

$$x_6 = \frac{1}{\sqrt{1+x_5}} = \frac{1}{\sqrt{1+0.7548}}$$

$$x_6 = 0.7549$$

7th Iteration:- Put $n=6$ in (*)

$$x_7 = \frac{1}{\sqrt{1+x_6}} = \frac{1}{\sqrt{1+0.7549}}$$

$$= 0.7549$$

So Required root is 0.7549

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Assignment

Q1:- Find the root of equation $2x - \cos x - 3 = 0$ upto 3 decimal places by simple iteration method.

Q2:- Solve $x + x^x = 100$ by iteration method upto 4 d.p.

Q3:- Find the real root of equation $x - \sin x = 0.25$ using fixed point iteration.

Q4:- Set Iterate formula for solving the equation $x^{6.1} - \cos x - 10 = 0$, as starting with $x = 1.5$

Solutions.

Q1:-

$$2x - \cos x - 3 = 0$$

So $f(x) = 2x - \cos x - 3$

x	1	1.5	2
f(x)	-1.5403	-0.0707	1.4161

Root lies between 1.5 and 2

Now $2x - \cos x - 3 = 0$

$$\Rightarrow 2x = \cos x + 3$$

$$x = \frac{\cos x + 3}{2} \quad \text{---------} (1)$$

$$\Rightarrow g(x) = \frac{\cos x + 3}{2}$$

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$$\Rightarrow g'(x) = \frac{1}{2}[-\sin x]$$

$$= \frac{-1}{2}\sin x$$

$$\Rightarrow |g'(x)| = \left|\frac{-1}{2}\sin x\right|$$

$$= \frac{1}{2}\sin x \leq 1 \quad \forall x \in [1.5, 2]$$

So Iteration method is applicable.

Setting Iteration to ⓐ

$$x_{n+1} = \frac{1}{2}[\cos x_n + 3] \quad \text{——— ⓐ}$$

Let $x_0 = 1.6$ be initial guess

1st Iteration:- Put $n=0$ in ⓐ

$$\Rightarrow x_1 = \frac{1}{2}[\cos x_0 + 3]$$

$$= \frac{1}{2}[\cos(1.6) + 3] = \frac{1}{2}[2.9708]$$

$$= 1.4854$$

2nd Iteration:- Put $n=1$ in ⓐ

$$\Rightarrow x_2 = \frac{1}{2}[\cos x_1 + 3]$$

$$= \frac{1}{2}[\cos(1.4854) + 3] = \frac{1}{2}[3.0853]$$

$$= 1.5426$$

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3rd Iteration :- Put $n=2$ in (A)

$$\Rightarrow x_3 = \frac{1}{2}[\cos x_2 + 3]$$

$$= \frac{1}{2}[\cos(1.5427)+3] = \frac{1}{2}[3.0281]$$

$$= 1.5141$$

4th Iteration :- Put $n=3$ in (A)

$$\Rightarrow x_4 = \frac{1}{2}[\cos x_3 + 3]$$

$$= \frac{1}{2}[\cos(1.5141)+3] = \frac{1}{2}[3.0567]$$

$$x_4 = 1.5283$$

5th Iteration :- Put $n=4$ in (A)

$$\Rightarrow x_5 = \frac{1}{2}[\cos x_4 + 3]$$

$$= \frac{1}{2}[\cos(1.5283)+3] = \frac{1}{2}[3.0425]$$

$$= 1.5213$$

6th Iteration :- Put $n=5$ in (A)

$$\Rightarrow x_6 = \frac{1}{2}[\cos x_5 + 3]$$

$$= \frac{1}{2}[\cos(1.5213)+3] = \frac{1}{2}[3.0495]$$

$$= 1.5248$$

7th Iteration :- Put $n=6$ in (A)

$$\Rightarrow x_7 = \frac{1}{2}[\cos x_6 + 3]$$

$$= \frac{1}{2}[\cos(1.5248)+3] = \frac{1}{2}[3.0460]$$

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$x_7 = 1.523$

8th Iteration:- Put $n=7$ in (*)

$\Rightarrow x_8 = \frac{1}{2}[\cos x_7 + 3]$

$= \frac{1}{2}[\cos(1.523) + 3] = \frac{1}{2}[3.0478]$

$= 1.524$

9th Iteration:- Put $n=8$ in (*)

$x_9 = \frac{1}{2}[\cos(1.524) + 3] = \frac{1}{2}(3.04678)$

$= 1.5234$

So Approximate Root is

$1.5234 \approx 1.524$

Q 2:- $x + x^x = 100$

$\Rightarrow x + x^x - 100 = 0$

So $f(x) = x + x^x - 100$

x	3.5	3.6
$f(x)$	-16.2882	4.2211

So Root lies b/w 3.5, 3.6

Now $x + x^x = 100$

$\Rightarrow x^x = 100 - x$

$x = (100 - x)^{1/x}$

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$$\log x = \frac{1}{x}\log(100 - x)$$

$$\Rightarrow x \log x = \log(100 - x)$$

$$\Rightarrow x = \frac{\log(100 - x)}{\log x} \quad \text{——} (1)$$

$$\Rightarrow g(x) = \frac{\log(100 - x)}{\log x}$$

$$g(x) = \log(100 - x) - \log x$$

$$g'(x) = \frac{1}{100 - x}(-1) - \frac{1}{x}$$

$$= \frac{-1}{100 - x} - \frac{1}{x} = -\left[\frac{1}{100 - x} + \frac{1}{x}\right]$$

$$\Rightarrow |g'(x)| = \left|-\left[\frac{1}{100 - x} + \frac{1}{x}\right]\right|$$

$$= \frac{1}{100 - x} + \frac{1}{x} < 1 \quad \forall x \in [3.53.6]$$

So Iteration method is valid

Setting Iteration to (1)

Let $x_0 = 3.55$ be initial guess

$$\text{i.e.} \quad x_{n+1} = \frac{\log(100 - x_n)}{\log x_n} \quad \text{——} (*)$$

1st Iteration:- Put $n = 0$ in (*)

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$$x_1 = \frac{\log(100 - x_0)}{\log x_0} = \frac{\log(100 - 3.55)}{\log 3.55}$$

$$= 3.6063$$

2nd Iteration:- Put $n = 1$ in (*)

$$\Rightarrow x_2 = \frac{\log(100 - x_1)}{\log x_1} = \frac{\log(100 - 3.3063)}{\log(3.3063)}$$

$$= 3.5616$$

3rd Iteration:- Put $n = 2$ in (*)

$$\Rightarrow x_3 = \frac{\log(100 - x_2)}{\log x_2} = \frac{\log(100 - 3.5616)}{\log(3.5616)}$$

$$= 3.5970$$

4th Iteration:- Put $n = 3$ in (*)

$$\Rightarrow x_4 = \frac{\log(100 - x_3)}{\log x_3} = \frac{\log(100 - 3.5970)}{\log 3.5970}$$

$$= 3.5690$$

5th Iteration:- Put $n = 4$ in (*)

$$\Rightarrow x_5 = \frac{\log(100 - x_4)}{\log x_4} = \frac{\log(100 - 3.5690)}{\log(3.5690)}$$

$$= 3.5910$$

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6th Iteration:- Put $n=5$ in (*)

$$\Rightarrow x_6 = \frac{\log(100-x_5)}{\log x_5} = \frac{\log(100-3.5910)}{\log(3.5910)}$$

$$= 3.5736$$

7th Iteration:- Put $n=6$ in (*)

$$\Rightarrow x_7 = \frac{\log(100-x_6)}{\log x_6} = \frac{\log(100-3.5736)}{\log(3.5736)}$$

$$= 3.5874$$

8th Iteration:- Put $n=7$ in (*)

$$\Rightarrow x_8 = \frac{\log(100-x_7)}{\log x_7} = \frac{\log(100-3.5874)}{\log(3.5874)}$$

$$= 3.5764$$

9th Iteration:- Put $n=8$ in (*)

$$\Rightarrow x_9 = \frac{\log(100-x_8)}{\log x_8} = \frac{\log(100-3.5764)}{\log(3.5764)}$$

$$= 3.5851$$

10th Iteration:- Put $n=9$ in (*)

$$\Rightarrow x_{10} = \frac{\log(100-x_9)}{\log x_9} = \frac{\log(100-3.5851)}{\log(3.5851)}$$

$$= 3.5783$$

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11 th Iteration:- Put $n = 10$ in (*)

$$\Rightarrow x_{11} = \frac{\log(100 - x_{10})}{\log x_{10}} = \frac{\log(100 - 3.5783)}{\log(3.5783)}$$

$$= 3.5836$$

12th Iteration:- Put $n = 11$ in (*)

$$\Rightarrow x_{12} = \frac{\log(100 - x_{11})}{\log x_{11}} = \frac{\log(100 - 3.5836)}{\log(3.5836)}$$

$$= 3.5795$$

13 th Iteration:- Put $n = 12$ in (*)

$$\Rightarrow x_{13} = \frac{\log(100 - x_{12})}{\log x_{12}} = \frac{\log(100 - 3.5795)}{\log(3.5795)}$$

$$= 3.5826$$ & so on

$x_{14} = 3.5802$ $x_{15} = 3.5821$

$x_{16} = 3.581$ $x_{17} = 3.5815$

$x_{18} = 3.5811$ $x_{19} = 3.5814$

$x_{20} = 3.5812$ $x_{21} = 3.5813$

& $x_{22} = 3.5813$

So Root is 3.5813

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Q3:- $x - \sin x = 0.25$

$\Rightarrow x - \sin x - 0.25 = 0$

So $f(x) = x - \sin x - 0.25$

x	1	1.2
$f(x)$	-0.0915	0.018

So Root lies between 1 and 1.2

Now $x - \sin x = 0.25$

$\Rightarrow x = 0.25 + \sin x \Rightarrow g(x) = 0.25 + \sin x$ ——— (1)

$\Rightarrow g'(x) = \cos x \Rightarrow |g'(x)| = |\cos x| \leq 1$

$\Rightarrow |g'(x)| \leq 1$

So Iteration method is applicable

Setting Iteration to (1)

$\Rightarrow x_{n+1} = \sin x_n + 0.25$ ——— (*)

Let $x_0 = 1$ be initial guess

1st Iteration:- Put $n=0$ in (*)

$\Rightarrow x_1 = \sin x_0 + 0.25 = \sin(1) + 0.25$

$= 1.0915$

2nd Iteration:- Put $n=1$ in (*)

$\Rightarrow x_2 = \sin x_1 + 0.25 = \sin(1.0915) + 0.25$

$= 1.1373$

3rd Iteration:- Put $n=2$ in (*)

$\Rightarrow x_3 = \sin x_2 + 0.25 = \sin(1.1373) + 0.25$

$= 1.1578$

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4th Iteration:- Put $n = 3$ in (*)

$\Rightarrow x_4 = \sin x_3 + 0.25 = \sin(1.1575) + 0.25$

$= 1.1658$

5th Iteration:- Put $n = 4$ in (*)

$\Rightarrow x_5 = \sin x_4 + 0.25 = \sin(1.1658) + 0.25$

$= 1.1691$

6th Iteration:- Put $n = 5$ in (*)

$\Rightarrow x_6 = \sin x_5 + 0.25 = \sin(1.1691) + 0.25$

$= 1.1704$

7th Iteration:- Put $n = 6$ in (*)

$\Rightarrow x_7 = \sin x_6 + 0.25 = \sin(1.1704) + 0.25$

1.1709

8th Iteration:- Put $n = 7$ in (*)

$\Rightarrow x_8 = \sin x_7 + 0.25 = \sin(1.1709) + 0.25$

1.1711

9th Iteration:- Put $n = 8$ in (*)

$\Rightarrow x_9 = \sin x_8 + 0.25 = \sin(1.1711) + 0.25$

$= 1.1712$

10th Iteration:- Put $n = 9$ in (*)

$\Rightarrow x_{10} = \sin x_9 + 0.25 = \sin(1.1712) + 0.25$

$= 1.1712$

So Required root is 1.1712

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Q4:- $x^{6.1} - \cos x - 10 = 0$

So $f(x) = x^{6.1} - \cos x - 10$

x	1.4	1.5
$f(x)$	-2.3828	1.7912

So Root lies b/w 1.4 & 1.5

Now $x^{6.1} - \cos x - 10 = 0$

$\Rightarrow x^{6.1} = \cos x + 10$

$\Rightarrow x = [\cos x + 10]^{1/6.1}$ ——— (1)

$\Rightarrow g(x) = [\cos x + 10]^{1/6.1}$

$g'(x) = \frac{1}{6.1}[\cos x + 10]^{\frac{1}{6.1}-1}[-\sin x]$

$= \frac{1}{6.1}[\cos x + 10]^{\frac{-51}{61}}[-\sin x]$

$= \frac{-1}{6.1}[\cos x + 10]^{\frac{-51}{61}}[\sin x]$

$|g'(x)| = \left|\frac{-1}{6.1}[\cos x + 10]^{\frac{-51}{61}}[\sin x]\right|$

$= \frac{1}{6.1}[\cos x + 10]^{\frac{-51}{61}}[\sin x] < 1$

$\Rightarrow |g'(x)| < 1 \quad \forall\ x \in [1.4, 1.5]$

So Iteration method is applicable

Setting Iteration on (1)

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$$\Rightarrow x_{n+1} = [\cos x_n + 10]^{1/6.1} \quad \text{---------- (*)}$$

Let $x_0 = 1.5$ be initial guess

1st Iteration:- Put $n=0$ in (*)

$$\Rightarrow x_1 = [\cos x_0 + 10]^{1/6.1}$$

$$= [\cos(1.5) + 10]^{1/6.1}$$

$$= 1.4603$$

2nd Iteration:- Put $n=1$ in (*)

$$\Rightarrow x_2 = [\cos x_2 + 10]^{1/6.1} = [\cos(1.4603) + 10]^{1/6.1}$$

$$= 1.4612$$

3rd Iteration:- Put $n=2$ in (*)

$$\Rightarrow x_3 = [\cos x_2 + 10]^{1/6.1}$$

$$= [\cos(1.4612) + 10]^{1/6.1}$$

$$= 1.4612$$

So the required root is

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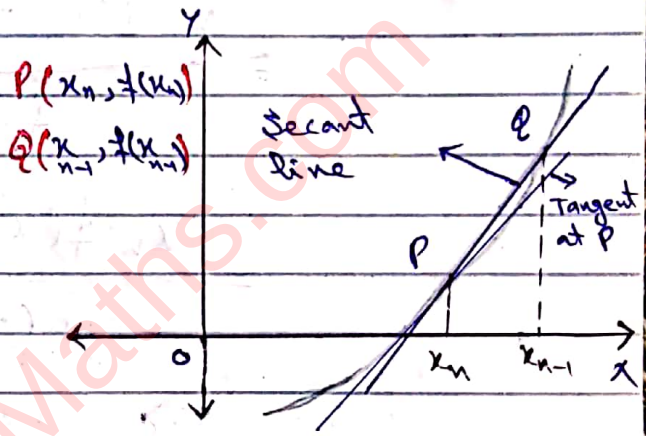
5:- Secant Method:-

If x_{n-1} & x_n are two approximations to the root of $f(x) = 0$ then a better approximation x_{n+1} is given by

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Proof:-

Slop of Tangent line at P ≈ Slope of Secant through P and Q



$$\Rightarrow f'(x_n) \approx \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n} \quad \text{---------- (1)}$$

2 ، 3 ، 4 ، 5 ، 6 ، 7 ، 8 ، 9 ، 10 Secant ، 11 ، 12 ، 13 ، 14 ، 15 Points

Now

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{(by N-R)}$$

$$\Rightarrow \frac{f(x_n)}{f'(x_n)} = x_n - x_{n+1}$$

$$\Rightarrow f(x_n) = f'(x_n)\left[x_n - x_{n+1}\right]$$

$$\Rightarrow f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}} \quad \text{Put in (1)}$$

$$\frac{f(x_n)}{x_n - x_{n+1}} \approx \frac{f(x_{n-1}) - f(x_n)}{\cancel{f(x_n)}\; x_{n-1} - x_n}$$

$$\Rightarrow \frac{1}{x_n - x_{n+1}} \approx \frac{f(x_{n-1}) - f(x_n)}{f(x_n)\left[x_{n-1} - x_n\right]}$$

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$$\Rightarrow x_n - x_{n+1} = \frac{f(x_n)[x_{n-1} - x_n]}{f(x_{n-1}) - f(x_n)}$$

$$x_{n+1} = x_n - \frac{x_{n-1} f(x_n) - x_n f(x_n)}{f(x_{n-1}) - f(x_n)}$$

$$= \frac{x_n f(x_{n-1}) - x_n f(x_n) - x_{n-1} f(x_n) + x_n f(x_n)}{f(x_{n-1}) - f(x_n)}$$

$$= \frac{x_n f(x_{n-1}) - x_{n-1} f(x_n)}{f(x_{n-1}) - f(x_n)}$$

$$\Rightarrow x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

* * *

Examples- Find the true root of equation $x^3 - x - 10 = 0$ by secant method.

Solution:- Let $f(x) = x^3 - x - 10$

secant formula is,

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{---(A)}$$

Choose $x_0 = 1$ and $x_1 = 2$

$$\Rightarrow f(x_0) = (1)^3 - 1 - 10 = 0 - 10 = -10$$

$$f(x_1) = (2)^3 - 2 - 10 = -4$$

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1st Iteration:- Put $n=1$ in (*)

$$\Rightarrow x_{n+1} = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{1(-4) - 2(-10)}{-4-(-10)} = \frac{-4+20}{-4+10}$$

$$= \frac{16}{6} = 2.6667 \Rightarrow f(x_2) = 6.3642$$

2nd Iteration:- Put $n=2$ in (*)

$$\Rightarrow x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{2(6.3642) - 2.6667(-4)}{6.3642-(-4)}$$

$$= \frac{23.3952}{10.3642} = 2.2573$$

$$\& f(x_3) = (2.2573)^3 - 2.2573 - 10$$

$$= -0.7554$$

3rd Iteration:- Put $n=3$ in (*)

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{2.6667(-0.7554) - 2.2573(6.3642)}{-0.7554 - 6.3642}$$

$$= \frac{-16.3803}{-7.1196} \Rightarrow x_4 = 2.3007$$

$$\& f(x_4) = (2.3007)^3 - 2.3007 - 10$$

$$= -0.1221$$

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4th Iteration:- Put $n=4$ in (*)

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$$

$$= \frac{2.2573(-0.1221) - 2.3007(-0.7554)}{-0.1221 - (-0.7554)}$$

$$= \frac{1.4623}{0.6333} \Rightarrow x_5 = 2.3090$$

& $f(x_5) = (2.3090)^3 - 2.3090 - 10$

$$= 0.0016$$

5th Iteration:- Put $n=5$ in (*)

$$\Rightarrow x_6 = \frac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)}$$

$$= \frac{2.3007(0.0016) - 2.3090(-0.1221)}{0.0016 - (-0.1221)}$$

$$= \frac{0.2856}{0.1237} = 2.3088$$

$f(x_6) = (2.3088)^3 - 2.3088 - 10$

$$= -0.0014$$

6th Iteration- Put $n=6$ in (*)

$$\Rightarrow x_7 = \frac{x_5 f(x_6) - x_6 f(x_5)}{f(x_6) - f(x_5)}$$

$$= \frac{2.3090(-0.0014) - 2.3088(0.0016)}{-0.0014 - 0.0016}$$

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$$\Rightarrow x_7 = \frac{-0.0069}{-0.0030} \Rightarrow x_7 = 2.3$$

& $f(x_7) = (2.3)^3 - 2.3 - 10$

$= -0.1330$

7th Iteration:- Put $n=7$ in (*)

$$\Rightarrow x_8 = \frac{x_6 f(x_7) - x_7 f(x_6)}{f(x_7) - f(x_6)}$$

$$= \frac{2.3088(-0.1330) - 2.3(-0.0014)}{-0.1330 - (-0.0014)}$$

$$= \frac{-0.3039}{-0.1316} \Rightarrow x_8 = 2.3093$$

$\Rightarrow f(x_8) = (2.3093)^3 - 2.3093 - 10$

$= 0.0054$

8th Iteration:- Put $n=8$ in (*)

$$x_9 = \frac{x_7 f(x_8) - x_8 f(x_7)}{f(x_8) - f(x_7)}$$

$$= \frac{2.3(0.0054) - 2.3093(-0.1330)}{0.0054 - (-0.1330)}$$

$$= \frac{0.3196}{0.1384} \Rightarrow x_9 = 2.3092$$

& $f(x_9) = (2.3092)^3 - 2.3092 - 10$

$= 0.0051$

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9th Iteration:- Put $n=9$ in (*)

$$\Rightarrow x_{10} = \frac{x_8 f(x_9) - x_9 f(x_8)}{f(x_9) - f(x_8)}$$

$$= \frac{2.3093(0.0051) - 2.3092(0.0054)}{0.0051 - 0.0054}$$

$$= \frac{-0.0007}{-0.0003} \Rightarrow x_{10} = 2.3333$$

& $f(x_{10}) = (2.3333)^3 - 2.3333 - 10$

$$= 0.3704$$

10 th Iteration:- Put $n=10$ in (*)

$$\Rightarrow x_{11} = \frac{x_9 f(x_{10}) - x_{10} f(x_9)}{f(x_{10}) - f(x_9)}$$

$$= \frac{2.3092(0.3704) - 2.3333(0.0051)}{0.3704 - 0.0051}$$

$$= \frac{0.8434}{0.3653} \Rightarrow x_{11} = 2.3088$$

& $f(x_{11}) = -0.0014$

Approximate Root is 2.3089

Examples- Find root of $x \log_{10} x - 1.2 = 0$ upto four decimal places by Secant Method.

Solution:-

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Let $f(x) = x \log_{10} x - 1.2$

Secant formula is

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{---------- (*)}$$

Let $x_0 = 2$ & $x_1 = 3$

1st Iteration:- Put $n = 1$ in (*)

$$\Rightarrow x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \text{---------- (1)}$$

Now $f(x_0) = x_0 \log_{10} x_0 - 1.2$

$= -0.5979$

& $f(x_1) = x_1 \log_{10} x_1 - 1.2$

$= 0.2314$ put values in (1)

$$\Rightarrow x_2 = \frac{2(0.2314) - 3(-0.5979)}{0.2314 - (-0.5979)}$$

$= 2.7210$

& $f(x_2) = (2.710) \log_{10} (2.7210) - 1.2$

$= -0.0171$

2nd Iteration:- Put $n = 2$ in (*)

$$\Rightarrow x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

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$$\Rightarrow x_3 = \frac{3(-0.0171) - (2.7210)(0.2314)}{-0.0171 - 0.2314}$$

$$= 2.7404$$

$$f(x_3) = (2.7404)\log_{10}(2.7404) - 1.2$$

$$= -0.0002$$

3rd Iteration:- Put $n=3$ in (A)

$$\Rightarrow x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{2.7210(-0.0002) - (2.7404)(-0.0171)}{-0.0002 - (-0.0171)}$$

$$= \frac{+0.0403}{0.0169}$$

$$= 2.7406$$

$$f(x_4) = (2.7406)\log_{10}(2.7406) - 1.2$$

$$= 0.0000$$

So Required root is 2.7406

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Assignment*

Q1:- By Taking Logrithm, find the solution of $x^x = 10$ correct upto 4 significant figure by secant Method

Q2:- Using secant method calculate a root of $x^3 + 2x^2 + 10x - 20 = 0$

Q3:- Solve equation $\sin x - 5x + 2 = 0$ by secant method correct upto 3 dp given root lies b/w 0.6 & 0.4

Q4:- Use secant formula to compute a real root of equation $x^4 - x - 10 = 0$

Q5:- Use secant Method to find the root of $\sin x = \frac{x}{2}$.

Q6:- Find smallest +ve root of equation $f(x) = \cos x \cosh x - 1$ by secant method

Solutions

Q1:- Given $x^x = 10 \Rightarrow x^x - 10 = 0$

$f(x) = x^x - 10$

Now $x^x = 10$

Taking Log on both sides

$\log x^x = \log 10$

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$\Rightarrow x\log x = \log 10 \Rightarrow x\log x - \log 10 = 0$

$\Rightarrow f(x) = x\log x - \log 10$

Secant formula

$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad (*)$$

Let $x_0 = 2.5$ & $x_1 = 2.6$

$\Rightarrow f(x_0) = -0.00515$ & $f(x_1) = 0.0789$

1st Iteration:- Put $n = 1$ in (*)

$$\Rightarrow x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{2.5(0.0789) - 2.6(-0.00515)}{0.0789 - (-0.00515)}$$

$$= \frac{0.2106}{0.0738} = 2.8537$$

$f(x_2) = (2.8537)\log(2.8537) - \log 10$

$= 0.2996$

2nd Iteration:- Put $n = 2$ in (*)

$$\Rightarrow x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{2.6(0.2996) - 2.8537(0.0789)}{0.2996 - 0.0789}$$

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$\Rightarrow x_3 = \frac{0.5538}{0.2207} \Rightarrow x_3 = 2.5093$

$f(x_3) = (2.5093)\log(2.5093) - \log 10$

$= 0.0026$

3rd Iteration:- Put $n = 3$ in (*)

$\Rightarrow x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$

$= \frac{2.8537(0.0026) - 2.5093(0.2996)}{0.0026 - 0.2996}$

$= \frac{-0.7443}{-0.2970} \Rightarrow x_4 = 2.5061$

$f(x_4) = (2.5061)\log(2.5061) - \log 10$

$= -0.0000$

So Approximate root is 2.5061

Q2:- Given $x^3 + 2x^2 + 10x - 20 = 0$

$\Rightarrow f(x) = x^3 + 2x^2 + 10x - 20$

Secant formula is

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{---} (*)$$

Let $x_0 = 1.3$, $f(x_0) = -1.480$

& $x_1 = 1.4 \Rightarrow f(x_1) = 0.6640$

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1st Iteration:- Put $n=1$ in (*)

$$\Rightarrow x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{1.3(0.6640) - 1.4(-1.4230)}{0.6640 - (-1.4230)}$$

$$= \frac{2.8554}{2.0870} \Rightarrow x_2 = 1.3682$$

$$\Rightarrow f(x_2) = (1.3682)^3 + 2(1.3682)^2 + 10(1.3682) - 20$$

$$= -0.0132$$

2nd Iteration:- Put $n=2$ in (*)

$$\Rightarrow x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{1.4(-0.0132) - 1.3682(0.6640)}{-0.0132 - (0.6640)}$$

$$= \frac{-0.9270}{0.6772} \Rightarrow x_3 = 1.3689$$

$$f(x_3) = (1.3689)^3 + 2(1.3689)^2 + 10(1.3689) - 20$$

$$= 0.0019$$

3rd Iteration:- Put $n=3$ in (*)

$$\Rightarrow x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

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$$x_4 = \frac{1.3682(0.0019) - 1.3689(-0.0132)}{0.0019 - (-0.0132)}$$

$$= \frac{0.0207}{0.0151} \Rightarrow x_4 = 1.3709$$

$$f(x_4) = (1.3709)^3 + 2(1.3709)^2 + 10(1.3709) - 20$$

$$= 0.0442$$

4th Iteration:- Put $n = 4$ in (A)

$$\Rightarrow x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$$

$$= \frac{1.3689(0.0442) - 1.3709(0.0019)}{0.0442 - 0.0019}$$

$$= \frac{0.0579}{0.0423} \Rightarrow x_5 = 1.3688$$

$$f(x_5) = (1.3688)^3 + 2(1.3688)^2 + 10(1.3688) - 20$$

$$= -0.0002$$

So Approximate root is 1.3688

Q3:- Given $\sin x - 5x + 2 = 0$

& interval b/w 0.6 & 0.4

$$\Rightarrow f(x_1) = \sin x - 5x + 2$$

$x_0 = 0.6$ $\quad$ $x_1 = 0.4$

$f(x_0) = -0.4354$, $f(x_1) = 0.3894$

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Secant formula is

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{---(*)}$$

1st Iteration:- Put $n = 1$ in (*)

$$\Rightarrow x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{0.6(0.3894) - 0.4(-0.4354)}{0.3894 - (-0.4354)}$$

$$= \frac{0.4078}{0.8448} \Rightarrow x_2 = 0.4827$$

$$f(x_2) = \sin(0.4827) - 5(0.4827) + 2$$

$$= 0.0507$$

2nd Iteration:- Put $n = 2$ in (*)

$$\Rightarrow x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{0.4(0.0507) - 0.4827(0.3894)}{0.0507 - 0.3894}$$

$$= \frac{-0.1677}{-0.3387} \Rightarrow 0.4951$$

$$f(x_3) = \sin(0.4951) - 5(0.4951) + 2$$

$$= -0.0004$$

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3rd Iteration:- Put $n=3$ in (*)

$$\Rightarrow x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{0.4827(-0.0004) - 0.4951(0.0507)}{-0.0004 - 0.0507}$$

$$= \frac{-0.025295}{-0.0511} \Rightarrow x_4 = 0.4950$$

$$f(x_4) = \sin(0.4950) - 5(0.4950) + 2$$

$$= 0.0000$$

So Required root is 0.4950

Q4:- Given $x^4 - x - 10 = 0$

$$\Rightarrow f(x) = x^4 - x - 10$$

Secant formula is

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{---(*)}$$

Let $x_0 = 1.8$ & $x_1 = 1.9$

$\Rightarrow f(x_0) = -1.3024$ & $f(x_1) = 1.1321$

1st Iteration:- Put $n=1$ in (*)

$$\Rightarrow x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{1.8(1.1321) - 1.9(-1.3024)}{1.1321 - (-1.3024)}$$

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$$\Rightarrow x_2 = \frac{4.5123}{2.4345} \Rightarrow x_2 = 1.8535$$

$$f(x_2) = (1.8535)^4 - 1.8535 - 10$$

$$= -0.0511$$

2nd Iteration:- Put $n=2$ in (*)

$$\Rightarrow x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{1.9(-0.0511) - 1.8535(1.1321)}{-0.0511 - 1.1321}$$

$$= \frac{-2.19544}{-1.1832} \Rightarrow x_3 = 1.8555$$

$$\Rightarrow f(x_3) = -0.0021$$

3rd Iteration:- Put $n=3$ in (*)

$$\Rightarrow x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{1.8535(-0.0021) - 1.8555(-0.0511)}{-0.0021 - (-0.0511)}$$

$$= \frac{0.090924}{0.0490} \Rightarrow x_4 = 1.8556$$

$$f(x_4) = (1.8556)^4 - 1.8556 - 10$$

$$= 0.0004$$

So Approximate root is 1.8556

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Q 5:- Given $\sin x = \frac{x}{2}$

$\Rightarrow \sin x - \frac{x}{2} = 0$

$\Rightarrow f(x) = \sin x - \frac{x}{2}$

Now Secant formula is

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{---(*)}$$

Let $x_0 = 0.5$ $x_1 = 1$

$\Rightarrow f(x_0) = 0.2294$ & $f(x_1) = 0.3415$

1st Iteration:- Put $n=1$ in (*)

$$\Rightarrow x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{0.5(0.3415) - 1(0.2294)}{0.3415 - 0.2295}$$

$$= \frac{-0.05865}{0.1121} \Rightarrow x_2 = -0.5232$$

& $f(x_2) = \sin(-0.5232) - \left[\frac{-0.5232}{2}\right]$

$= -0.2381$

2nd Iteration:- Put $n=2$ in (*)

$$\Rightarrow x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{1(-0.2181) - (-0.5232)(0.3415)}{-0.2181 - 0.3415}$$

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$$\Rightarrow x_3 = \frac{-0.0594}{-0.5796} \Rightarrow x_3 = 0.1025$$

$$f(x_3) = (0.1025)^4 \sin(0.1025) - \frac{0.1025}{2}$$

$$= 0.0511$$

3rd Iteration:- Put $n = 3$ in (A)

$$\Rightarrow x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{-0.5232(0.0511) - 0.1025(-0.2381)}{0.0511 - (-0.2381)}$$

$$= \frac{-0.0023}{0.2892} = -0.0078$$

$$f(x_4) = \sin(-0.0078) - \left[\frac{-0.0078}{2}\right]$$

$$= -0.0039$$

4th Iteration:- Put $n = 4$ in (A)

$$\Rightarrow x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$$

$$= \frac{0.1025(-0.0039) - (-0.0078)(.0511)}{-0.0039 - .0511}$$

$$= \frac{-0.0000}{-0.055} \Rightarrow x_5 = 0$$

$$f(x_5) = \sin(0) - \frac{0}{2} = 0$$

So Required root is 0

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Q6:- Given $f(x) = \cos x \cosh x - 1$

Secant formula is

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{---(A)}$$

Let $x_0 = -0.7$, $x_1 = 0.4$

$f(x_0) = -0.04$ $\quad f(x_1) = 0.0043$

1st Iteration:- Put $n = 1$ in (A)

$$\Rightarrow x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{-0.7(0.0043) - 0.4(-0.04)}{0.0043 - (-0.04)}$$

$$= \frac{0.013}{0.0443} \Rightarrow x_2 = 0.2934$$

$$f(x_2) = \cos(0.2934)\cosh(0.2934) - 1$$

$$= -0.0012$$

2nd Iteration:- Put $n = 2$ in (A)

$$\Rightarrow x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{0.4(-0.0012) - 0.2934(0.0043)}{-0.0012 - 0.0043}$$

$$= \frac{-0.0017}{-0.0055} \Rightarrow x_3 = 0.3091$$

$$f(x_3) = \cos(0.3091)\cosh(0.3091) - 1$$

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$f(x_3) = -0.0015$

3rd Iteration:- Put $n=3$ in (R)

$$\Rightarrow x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{0.2934(-0.0015) - 0.3091(-0.0012)}{-0.0015 - (-0.0012)}$$

$$= \frac{-0.0000}{-0.0003} \Rightarrow x_4 = 0$$

$f(x_4) = \cos(0)\cosh(0) - 1$

$= 0$

So Required Root is 0

★ ★

Theorem:- Prove that the fixed point iteration method converges

Proof:-

Let $x=a$ be the root of $f(x)=0$, x_n & x_{n+1} be the two approximation to this root such that e_n and e_{n+1} be the corresponding errors then

$x_n - a = e_n$ & $x_{n+1} - a = e_{n+1}$

$\Rightarrow x_n = a + e_n$ & $x_{n+1} = a + e_{n+1}$

Now we write $f(x)=0$ in the form $x = g(x)$ s.t $|g'(x)| < 1$. Then we have the following

$f(a) = 0$, $a = g(a)$ & $x_{n+1} = g(x_n)$

Put $x=a$ ∴

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Now $a+e_{n+1} = g(a+e_n)$

$\Rightarrow a+e_{n+1} = g(a) + e_n g'(a) + \frac{e_n^2}{2!} g''(a) + \cdots$

$\Rightarrow a+e_{n+1} = g(a) + e_n g'(a)$ Neglecting square & High powers of e_n

$\Rightarrow e_{n+1} = e_n g'(a)$ [$\because g(a) = a$]

$\Rightarrow |e_{n+1}| = |e_n||g'(a)| < |e_n|(1)$

$\Rightarrow |e_{n+1}| < |e_n|$

$\Rightarrow$ Absolute error in every next approximation is less than the absolute error in previous approximation.

$\Rightarrow$ Method converges.

* Order of Convergence:-

If e_n denotes the absolute error in n^{th} iteration and e_{n+1} denotes the absolute error in $(n+1)^{th}$ iteration, then p is said to be order of convergence if

$$e_{n+1} = \alpha e_n^p$$

i.e $e_{n+1} = k e_n^p$ for some k.

Theorem:- Prove that order of convergence of fixed point iteration method is linearly convergent i.e fixed point iteration method is linearly constant.

Proof:- As given equation

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$$f(x) = 0 \quad \text{——— ①}$$

By Iteration method we write it as

$$x = g(x) \quad \text{——— ②}$$

Setting iteration as

$$x_{n+1} = g(x_n) \quad \text{——— ③}$$

Let $x = a$ be the root of equation ①

$$\therefore f(a) = 0$$

Also from ② $\quad a = g(a) \quad \text{——— ④}$

Let e_n be error in n^{th} term and e_{n+1} be error in $(n+1)^{th}$ term, then

$$x_n - a = e_n \quad \& \quad x_{n+1} - a = e_{n+1}$$

$$\Rightarrow x_n = e_n + a \quad \& \quad x_{n+1} = e_{n+1} + a$$

Put in eqn ③

$$e_{n+1} + a = g(a + e_n)$$

By Taylor series

$$a + e_{n+1} = g(a) + e_n g'(a) + \frac{e_n^2}{2!} g''(a) + \cdots$$

$$= a + e_n g'(a) \qquad \because g(a) = a$$

$\because$ Neglecting H.P of e_n

$$a + e_{n+1} = a + e_n g'(a)$$

$$\Rightarrow e_{n+1} = e_n g'(a)$$

$$\Rightarrow e_{n+1} = g'(a) e_n^{(1)}$$

$$\Rightarrow e_{n+1} = K e_n \quad \text{where } g'(a) = K$$

Which shows that order of convergence of fixed point iteration

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method is one (OR)
Iteration method is Linearly convergent (Proved)

★ ★ ★

Theorem- Prove that order of convergence of Newton Raphson method is two. i.e Newton Raphson method is quadratically convergent.

Proof) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$\Rightarrow \alpha + e_{n+1} = \alpha + e_n - \frac{f(a+e_n)}{f'(a+e_n)}$

$\Rightarrow e_{n+1} = e_n - \frac{f(a) + e_n f'(a) + \frac{e_n^2}{2!} f''(a) + \cdots}{f'(a) + e_n f''(a) + \frac{e_n^2}{2!} f'''(a) + \cdots}$

$\Rightarrow e_{n+1} = e_n - \frac{e_n f'(a)}{f'(a) + e_n f''(a)}$ $\because f(a) = 0$, $\because$ Neglect H.P of e_n.

$= e_n - \frac{e_n f'(a)}{f'(a)\left[1 + e_n \frac{f''(a)}{f'(a)}\right]}$

$= e_n - e_n \left[1 + e_n \frac{f''(a)}{f'(a)}\right]^{-1}$

$= e_n - e_n \left[1 - e_n \frac{f''(a)}{f'(a)}\right]$ $\because$ Neglecting H.P of e_n

$= e_n - e_n + e_n^2 \left[\frac{f''(a)}{f'(a)}\right] = e_n^2 \left[\frac{f''(a)}{f'(a)}\right]$

$\Rightarrow e_n = K \cdot e_n^2$, where $K = \left[\frac{f''(a)}{f'(a)}\right]$

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⇒ Newton Raphson method has order of convergence 2 i.e Newton Raphson method is quadratically convergent.

* * *

Theorem: Prove that Newton Raphson method is linearly convergent if ∃ multiple roots.

Proof:-

By Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{---------- (1)}$$

Let $x=a$ be root of $f(x)=0$, Then $f(a)=0$. Now with usual meanings

$e_n = x_n - a$, $e_{n+1} = x_{n+1} - a$

$\Rightarrow x_n = e_n + a$, $x_{n+1} = e_{n+1} + a$

Then from (1)

$$e_{n+1} + a = e_n + a - \frac{f(e_n+a)}{f'(e_n+a)}$$

$$e_{n+1} = e_n - \frac{f(a+e_n)}{f'(a+e_n)}$$

$$= e_n - \frac{f(a) + e_n f'(a) + e_n^2/2! f''(a) + \cdots}{f'(a) + e_n f''(a) + e_n^2/2! f'''(a) + \cdots}$$

Now if $x=a$ be a double root then $f(a)=0$ also $f'(a)=0$ then

$$e_{n+1} = e_n - \frac{e_n^2/2! f''(a) + \cdots}{e_n f''(a) + e_n^2/2! f'''(a) + \cdots}$$

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$$\Rightarrow e_{n+1} = e_n - \frac{e_n}{e_n}\left[\frac{e_n/2!\, f''(a) + \cdots}{f''(a) + \frac{e_n}{2!} f'''(a) + \cdots}\right]$$

$$= e_n - \frac{e_n}{2}\left[\frac{f''(a) + \cdots}{f''(a) + \frac{e_n}{2} f'''(a) + \cdots}\right]$$

$$= e_n - \frac{e_n}{2}\left[\frac{f''(a)}{f''(a)}\right] \quad \text{Neglecting Remaining Terms}$$

$$= e_n - \frac{e_n}{2} = \frac{e_n}{2}$$

$$\Rightarrow e_{n+1} = \frac{1}{2} e_n^{(1)}$$

Continuing in this way if $x = \alpha$ is a tripple root then

$$e_{n+1} = \frac{1}{3} e_n^1$$

Also for multipe root n is given $e_{n+1} = \frac{1}{n} e_n$

which shows that order of convergence of Newton Raphson for multiple roots is linear.

Question:- Show that Newton Raphson method converges if

$$|f(x)| < \frac{|f'(x)|^2}{|f''(x)|}$$

Solution:-

By Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{---------- (1)}$$

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By fix point iteration method

$$x_{n+1} = g(x_n) \quad \text{——— ②}$$

From equ ① and ②

$$g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow g(x) = x - \frac{f(x)}{f'(x)}$$

$$\Rightarrow g'(x) = 1 - \frac{[f'(x)f'(x) - f(x)f''(x)]}{(f'(x))^2}$$

$$= \frac{(f'(x))^2 - (f'(x))^2 + f(x)f''(x)}{(f'(x))^2}$$

$$|g'(x)| = \left|\frac{f(x)f''(x)}{(f'(x))^2}\right|$$

for convergence $|g'(x)| < 1$

$$\Rightarrow \left|\frac{f(x)f''(x)}{(f'(x))^2}\right| < 1$$

$$\Rightarrow |f(x)| < \frac{|f'(x)|^2}{|f''(x)|}$$

Proved.

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Question: Discuss the error analysis of secant method.

Solution:

By secant method

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})} \quad \text{---------- (1)}$$

Let 'a' be the exact root of equation $f(x) = 0$, then $f(a) = 0$

Also let $x_{n-1} - a = e_{n-1}$

$x_n - a = e_n$

$\Rightarrow x_{n-1} = a + e_{n-1}$ & $x_n = a + e_n$,

and $x_{n+1} = a + e_{n+1}$

Using all these values in (1) we get

$$a + e_{n+1} = a + e_n - \frac{[a + e_n - a - e_{n-1}] f(a + e_n)}{f(a + e_n) - f(a + e_{n-1})}$$

$$e_{n+1} = e_n - \frac{[e_n - e_{n-1}]\left[f(a) + e_n f'(a) + e_n^2/2! f''(a) + \cdots\right]}{\left[f(a) + e_n f'(a) + e_n^2/2! f''(a) + \cdots\right] - \left[f(a) + e_{n-1} f'(a) + \frac{e_{n-1}^2}{2!} f''(a) + \cdots\right]}$$

As $f(a) = 0$ we get

$$e_{n+1} = e_n - \frac{[e_n - e_{n-1}] e_n f'(a)\left[1 + \frac{e_n}{2} \frac{f''(a)}{f'(a)} + \cdots\right]}{[e_n - e_{n-1}]\left[f'(a) + \frac{e_n^2 - e_{n-1}^2}{2!} f''(a) + \cdots\right]}$$

$$= e_n - \frac{[e_n - e_{n-1}]\left[e_n f'(a) + \frac{e_n^2}{2!} f''(a) + \cdots\right]}{[e_n - e_{n-1}]\left[f'(a) + \frac{e_n + e_{n-1}}{2!} f''(a) + \cdots\right]}$$

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$$e_{n+1} = e_n - \frac{e_n f'(a)\left[1 + \frac{e_n}{2}\frac{f''(a)}{f'(a)} + \cdots\right]}{f'(a)\left[1 + \frac{e_n + e_{n-1}}{2!}\cdot\frac{f''(a)}{f'(a)} + \cdots\right]}$$

$$= e_n - e_n\left[1 + \frac{e_n}{2}\frac{f''(a)}{f'(a)} + \cdots\right]\left[1 + \frac{e_n + e_{n-1}}{2}\cdot\frac{f''(a)}{f'(a)} + \cdots\right]^{-1}$$

$$= e_n - e_n\left[1 + \frac{e_n}{2}\frac{f''(a)}{f'(a)}\right]\left[1 - \frac{e_n + e_{n-1}}{2}\frac{f''(a)}{f'(a)}\right]$$ Using Binom

$$= e_n - e_n + \frac{e_n^2 + e_n e_{n-1}}{2}\frac{f''(a)}{f'(a)} - \frac{e_n^2}{2}\frac{f''(a)}{f'(a)}$$

$$= \frac{e_n^2}{2}\frac{f''(a)}{f'(a)} + \frac{e_n e_{n-1}}{2}\frac{f''(a)}{f'(a)} - \frac{e_n^2}{2}\frac{f''(a)}{f'(a)}$$

Neglecting Remaining

$$\Rightarrow e_{n+1} = \frac{1}{2}\frac{f''(a)}{f'(a)} e_n e_{n-1}$$

$$\Rightarrow e_{n+1} = K \cdot e_n e_{n-1}$$

where $K = \frac{1}{2}\frac{f''(a)}{f'(a)}$

$$\Rightarrow e_{n+1} \propto e_n e_{n-1}$$

This shows that in every iteration error in proportional to the product of errors in two previous iterations. Hence Secant method convergence.

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Theorem:- Prove that secant method is fastly convergent than fix point iteration method and slowly convergent than Newton Raphson method.

Proof:-

We know that order of convergena of fix point iteration method $= 1 = p_1$ and order of convergence of Newton Raphson method $= 2 = p_2$

Let p be the order of convergence of Secant method. Let e_{n-1}, e_n, e_{n+1} be the error of $(n-1)$st, nth and $(n+1)$st approximations to the root then

$$e_{n+1} = k e_n^p \quad \text{---} (1), \qquad e_n = k e_{n-1}^p \quad \text{---} (2)$$

from (2) $$e_{n-1} = \left[\frac{e_n}{k}\right]^{1/p} \quad \text{---} (3)$$

Now as $e_{n+1} \propto e_n e_{n-1} \Rightarrow e_{n+1} = k_1 e_n e_{n-1}$

using (1) and (3) it becomes

$$k e_n^p = k_1 e_n \cdot \frac{e_n^{1/p}}{k^{1/p}}$$

$$\Rightarrow k e_n^p = \frac{k_1}{k^{1/p}} e_n^{1/p + 1}$$

Equating Powers of e_n

$$p = \frac{1}{p} + 1 \Rightarrow p = 1.618$$

$$\Rightarrow p_1 < p < p_2$$

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Hence the proof.

Remarks:-

In Secant formula

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})}$$

If $f(x_n) = f(x_{n-1})$, secant method fails. This shows that it does not converges always. This is a draw back of secant method over the method of false position which always converges, But if secant method converges its rate of convergence is faster that the method of false position.

Question:- Discuss the advantages & Disadvantages of N-R method.

Solution:-

Advantages of Newton Raphson Method:-

(i) It is fast convergent iterative method its convergence is quadratic.

(ii) Newton Raphson method can be used to calculate the complex root of non linear equation.

(iii) Newton's Raphson method is useful in case of large value of $f(x)$. i.e when graph of function is $f(x)$

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while crossing x-axis nearly vertical.

(iv) Newton's method is generally used to improve the result obtain by othed methods.

(v) It is applicable to solution of both algebraic and transcendental function.

Disadvantanges of Newton-Raphson Method:-

(i) If two are more roots are equal or nearly equal, the method is not fastly convergent. In this case the convergence is linear.

(ii) If root is very near to maximum or minimum value of function $f'(x) = 0$ at such point Newton's method fails.

(iii) The evaluation of derivation $f'(x)$ at each step makes the method unpopula computationally and thus is not well adopted to computer usage.

Remark:-

The most important charactristic of order of convergence is that, That if an iteration method has large order of convergence then it will fastly converges to the root. From this we conclude that Newton-Raphson method most rapidly converges to the root.

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→ Matrices & their Applications.

* LU-Decomposition Method (OR) Factorization Method (OR) Triangular Decomposition Method.

To solve a system of linear equations.

* Working Rule:-

We are given to solve a system $A\underline{x} = \underline{b}$, where A is a $n \times n$ matrix

Step I:- Decompose (or Factorize) $A = LU$, where 'L' is a lower triangular and 'U' is an upper triangular matrix. Then given system will become $(LU)\underline{x} = \underline{b}$

$\Rightarrow L(U\underline{x}) = \underline{b}$

Step II:- Put $U\underline{x} = \underline{y}$ then we have $L\underline{y} = \underline{b}$

Step III:- From $L\underline{y} = \underline{b}$ find $\underline{y}$

Step IV:- Use $\underline{y}$ in $U\underline{x} = \underline{y}$ & find $\underline{x}$

In the decomposition $A = LU$, 'L' and 'U' are not unique. To find a unique decomposition, the diagonal entries of 'L' or 'U' are specified by

(i) Dolittle's Method.

(ii) Crout's Method.

(iii) Cholesky's Method.

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1:- DOLITTELE'S METHOD:-

In Dollitle's method, are diagonal entries of "L" are restricted to be one

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

For Inverse of matrix by Dollitle's method

(i) $A = LU$

$\Rightarrow A^{-1} = (LU)^{-1}$

$\Rightarrow A^{-1} = U^{-1} L^{-1}$

(ii) $UA^{-1} = (UU^{-1}) L^{-1}$

$\Rightarrow UA^{-1} = L^{-1}$

(iii) $L L^{-1} = I \quad \Rightarrow L^{-1} = ?$

* * *

Examples:- Solve the system of equation

$$2x_1 + x_2 + 3x_3 = 11$$
$$4x_1 + 3x_2 + 10x_3 = 28$$
$$2x_1 + 4x_2 + 17x_3 = 31$$

By Dollitle's method. Also find A^{-1}

Solution:-

Here $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

Let $A = LU$

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where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$, $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

Comparing corresponding elements of matrix we get

$\boxed{u_{11} = 2}$, $\boxed{u_{12} = 1}$, $\boxed{u_{13} = 3}$,

$l_{21}u_{11} = 4 \Rightarrow l_{21}(2) = 4 \Rightarrow \boxed{l_{21} = 2}$,

$l_{21}u_{12} + u_{22} = 3 \Rightarrow 2(1) + u_{22} = 3 \Rightarrow \boxed{u_{22} = 1}$,

$l_{21}u_{13} + u_{23} = 10 \Rightarrow 2(3) + u_{23} = 10 \Rightarrow \boxed{u_{23} = 4}$,

$l_{31}u_{11} = 2 \Rightarrow l_{31}(2) = 2 \Rightarrow \boxed{l_{31} = 1}$,

$l_{31}u_{12} + l_{32}u_{22} = 4 \Rightarrow (1)(1) + l_{32}(1) = 4 \Rightarrow \boxed{l_{32} = 3}$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 17 \Rightarrow (1)(3) + (3)(4) + u_{33} = 17$

$\Rightarrow 15 + u_{33} = 17 \Rightarrow \boxed{u_{33} = 2}$,

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we have $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

Now Put

$$Ux = y \quad \text{---} ①$$

Then $Ax = b \Rightarrow (LU)x = b \Rightarrow L(Ux) = b$

$$\Rightarrow Ly = b$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ 2y_1 + y_2 \\ y_1 + 3y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$\Rightarrow y_1 = 11$, $2y_1 + y_2 = 28 \Rightarrow y_2 = 6$,

$y_1 + 3y_2 + y_3 = 31 \Rightarrow 11 + 3(6) + y_3 = 31 \Rightarrow y_3 = 2$

$\therefore y = [11 \quad 6 \quad 2]^t$

Now from ① $Ux = y$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \\ 2 \end{bmatrix}$$

$\Rightarrow 2x_1 + x_2 + 3x_3 = 11$ ——— (i)

$x_2 + 4x_3 = 6$ ——— (ii)

$2x_3 = 2$ ——— (iii)

From above equations

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$x_3 = 1$, $x_2 = 6-4(1) = 2$

$2x_1 = 11-2-3(1) = 6$

$\Rightarrow x_1 = 3$

So solution vector is $(3,2,1)^t$

Also for A^{-1}:- First we find L^{-1}

As $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

$LL^{-1} = L^{-1}L = I$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ l'_{21} & 1 & 0 \\ l'_{31} & l'_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2+l'_{21} & 1 & 0 \\ 1+3l'_{21}+l'_{31} & 3+l'_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing corresponding elements we have

$2 + l'_{21} = 0 \Rightarrow \boxed{l'_{21} = -2}$

$3 + l'_{32} = 0 \Rightarrow \boxed{l'_{32} = -3}$

$1 + 3(-3) + l'_{31} = 0 \Rightarrow \boxed{l'_{31} = 5}$

$$\Rightarrow L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

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Now $UA^{-1} = L^{-1}$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2a_{11}+a_{21}+3a_{31} & 2a_{12}+a_{22}+3a_{32} & 3a_{13}+a_{23}+3a_{33} \\ a_{21}+4a_{31} & a_{22}+4a_{32} & a_{23}+4a_{33} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

$\Rightarrow 2a_{31} = 5$, $2a_{32} = -3$, $2a_{33} = 1$

$\Rightarrow \boxed{a_{31} = 5/2}$ $\Rightarrow \boxed{a_{32} = -3/2}$ $\Rightarrow \boxed{a_{33} = 1/2}$

$a_{21} + 4a_{31} = -2$ | $a_{22} + 4a_{32} = 1$ | $a_{23} + 4a_{33} = 0$

$\Rightarrow a_{21} + 4(\frac{5}{2}) = -2$ | $\Rightarrow a_{22} + 4(\frac{-3}{2}) = 1$ | $\Rightarrow a_{23} + 4(\frac{1}{2}) = 0$

$\Rightarrow a_{21} + 10 = -2$ | $\Rightarrow a_{22} - 6 = 1$ | $\Rightarrow a_{23} + 2 = 0$

$\Rightarrow \boxed{a_{21} = -12}$ | $\Rightarrow \boxed{a_{22} = 7}$ | $\Rightarrow \boxed{a_{23} = -2}$

$2a_{11} + a_{21} + 3a_{31} = 1$ | $2a_{12} + a_{22} + 3a_{32} = 0$

$\Rightarrow 2a_{11} + (-12) + 3(\frac{5}{2}) = 1$ | $\Rightarrow 2a_{12} + 7 + 3(\frac{-3}{2}) = 0$

$\Rightarrow 2a_{11} + \frac{15}{2} = 13$ | $\Rightarrow 2a_{12} + 7 - \frac{9}{2} = 0$

$\Rightarrow \boxed{a_{11} = 11/4}$ | $\Rightarrow \boxed{a_{12} = -1/3}$

$3a_{13} + a_{23} + 3a_{33} = 0 \Rightarrow 3a_{13} + (-2) + 3(\frac{1}{2}) = 0$

$\Rightarrow 3a_{13} - 2 + \frac{3}{2} = 0$

$\Rightarrow \boxed{a_{13} = 1/4}$

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$$\Rightarrow A^{-1} = \begin{bmatrix} 11/4 & -5/4 & 1/4 \\ -12 & 7 & -2 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$ Answere

* * *

2) Crout's Method:- In crout's method, the upper triangular matrix U has unit diagonal elements. i.e

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

For Inverse of Matrix by Crout's Method:-

(i) $A = LU$

(ii) $A^{-1} = (LU)^{-1} \Rightarrow A^{-1} = U^{-1}L^{-1}$

$A^{-1}L = U^{-1}(L^{-1}L) \Rightarrow A^{-1}L = U^{-1}$

(iii) $UU^{-1} = U^{-1}U = I \Rightarrow U^{-1} = ?$

* * *

Example:- Solve following system of equations

$$2x_1 + x_2 + 3x_3 = 11$$
$$4x_1 + 3x_2 + 10x_3 = 28$$
$$2x_1 + 4x_2 + 17x_3 = 31$$

By Crout's method. Also find A^{-1} by using Crout's method.

Solution:-

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$$A \quad x = b$$

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Let $LU = A$

$$\Rightarrow \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

Comparing corresponding elements of matrix

$\boxed{l_{11} = 2}$, $\boxed{l_{21} = 4}$, $\boxed{l_{31} = 2}$

$l_{11}u_{12} = 1$
$\Rightarrow 2(u_{12}) = 1$
$\Rightarrow \boxed{u_{12} = 1/2}$

$l_{11}u_{13} = 3$
$\Rightarrow 2(u_{13}) = 3$
$\Rightarrow \boxed{u_{13} = 3/2}$

$l_{21}u_{12} + l_{22} = 3$
$\Rightarrow 4(\frac{1}{2}) + l_{22} = 3$
$\Rightarrow 2 + l_{22} = 3$
$\Rightarrow \boxed{l_{22} = 1}$

$l_{31}u_{12} + l_{32} = 4$
$\Rightarrow 2(\frac{1}{2}) + l_{32} = 4$
$\Rightarrow 1 + l_{32} = 4$
$\Rightarrow \boxed{l_{32} = 3}$

$l_{21}u_{13} + l_{22}u_{23} = 10$
$\Rightarrow 4(\frac{3}{2}) + 1(u_{23}) = 10$
$\Rightarrow 6 + u_{23} = 10$
$\Rightarrow \boxed{u_{23} = 4}$

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 17 \Rightarrow 2(\frac{3}{2}) + 3(4) + l_{33} = 17$
$\Rightarrow 3 + 12 + l_{33} = 17 \Rightarrow \boxed{l_{33} = 2}$

So

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1/2 & 3/2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

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Let $L\underline{y} = \underline{b}$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2y_1 \\ 4y_1 + y_2 \\ 2y_1 + 3y_2 + 2y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$\Rightarrow \boxed{y_1 = 11/2}$, $4y_1 + y_2 = 28 \Rightarrow \boxed{y_2 = 6}$,

$2y_1 + 3y_2 + 2y_3 = 31 \Rightarrow 2(\frac{11}{2}) + 3(6) + 2y_3 = 31$

$\Rightarrow 11 + 18 + 2y_3 = 31 \Rightarrow \boxed{y_3 = 1}$

So $\underline{y} = \left[\frac{11}{2}, 6, 1\right]^t$

Now $U\underline{x} = \underline{y}$

$$\Rightarrow \begin{bmatrix} 1 & 3/2 & 3/2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11/2 \\ 6 \\ 1 \end{bmatrix}$$

$\Rightarrow x_1 + \frac{1}{2}x_2 + \frac{3}{2}x_3 = 11/2$ ——— (i)

$x_2 + x_3 = 6$ ——— (ii)

$x_3 = 1$ ——— (iii)

$\Rightarrow x_2 + 4(1) = 6 \Rightarrow x_2 = 2$,

$x_1 + \frac{1}{2}(2) + \frac{3}{2}(1) = \frac{11}{2} \Rightarrow x_1 = \frac{11}{2} - \frac{5}{2}$

$x_1 = 3$

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So solution vector $x = [3, 2, 1]^t$

Now for A^{-1}:- First we calculate U^{-1}

As $UU^{-1} = U^{-1}U = I$

$$\Rightarrow \begin{bmatrix} 1 & u_{12}' & u_{13}' \\ 0 & 1 & u_{23}' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 3/4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1/2 + u_{12}' & 3/4 + 4u_{12}' + u_{13}' \\ 0 & 1 & 4 + u_{23}' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow u_{12}' + \frac{1}{2} = 0 \Rightarrow \boxed{u_{12}' = -\frac{1}{2}}$,

$\frac{3}{4} + 4u_{12}' + u_{13}' = 0 \Rightarrow \frac{3}{4} + 4(-\frac{1}{2}) + u_{13}' = 0 \Rightarrow \frac{3}{4} - 2 + u_{13}' = 0$

$\Rightarrow \boxed{u_{13}' = 5/4}$,

$4 + u_{23}' = 0 \Rightarrow \boxed{u_{23}' = -4}$

So $$U^{-1} = \begin{bmatrix} 1 & -1/2 & 5/4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

Now $$A^{-1}L = U^{-1}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & 5/4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2a_{11} + 4a_{12} + 2a_{13} & a_{12} + 3a_{13} & 2a_{13} \\ 2a_{21} + 4a_{22} + 2a_{23} & a_{22} + 3a_{23} & 2a_{23} \\ 2a_{31} + 4a_{32} + 2a_{33} & a_{32} + 3a_{33} & 2a_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & 5/4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

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$\Rightarrow \boxed{a_{13} = \frac{1}{4}}$, $a_{12} + 3a_{13} = -\frac{1}{2} \Rightarrow \boxed{a_{12} = -5/4}$

$\boxed{a_{23} = -2}$, $a_{22} + 3a_{23} = 1 \Rightarrow \boxed{a_{22} = 7}$

$\boxed{a_{33} = 1/2}$, $a_{32} + 3a_{33} = 0 \Rightarrow \boxed{a_{32} = -3/2}$

$2a_{11} + 4a_{12} + 2a_{13} = 1 \Rightarrow 2a_{11} + 4(\frac{-5}{4}) + 2(\frac{1}{4}) = 1 \Rightarrow \boxed{a_{11} = \frac{11}{4}}$

$2a_{21} + 4a_{22} + 2a_{23} = 0 \Rightarrow 2a_{21} + 4(7) + 2(-2) = 0 \Rightarrow \boxed{a_{21} = -12}$

$2a_{31} + 4a_{32} + 2a_{33} = 0 \Rightarrow 2a_{31} + 4(\frac{-3}{2}) + 2(\frac{1}{2}) = 0 \Rightarrow \boxed{a_{31} = 5/2}$

So $A^{-1} = \begin{bmatrix} 11/4 & -5/4 & 1/4 \\ -12 & 7 & -2 \\ 15/2 & -3/2 & 1/2 \end{bmatrix}$

★★★

3) CHOLESKY'S METHOD:-

In cholesky's method, the diagonal entries of L and U are same, then for this method

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, \quad U = \begin{bmatrix} l_{11} & u_{12} & u_{13} \\ u_{21} & l_{22} & 0 \\ u_{31} & u_{32} & l_{33} \end{bmatrix}$$

$$U = \begin{bmatrix} l_{11} & u_{12} & u_{13} \\ 0 & l_{22} & u_{23} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Note:- If A is symmetric then

$A = LL^T \because U = L^T$, $A^{-1} = (L^{-1})^T L^{-1}$

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Example:- Solve by Cholesky's Method

$$2x_1 + x_2 + 3x_3 = 11$$
$$4x_1 + 3x_2 + 10x_3 = 28$$
$$2x_1 + 4x_2 + 17x_3 = 31$$

Solution:-

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$$A \quad \underline{x} = \underline{b}$$

Let $A = LU$

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, $U = \begin{bmatrix} l_{11} & u_{12} & u_{13} \\ 0 & l_{22} & u_{23} \\ 0 & 0 & l_{33} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & u_{12} & u_{13} \\ 0 & l_{22} & u_{23} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} l_{11}^2 & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}l_{11} & l_{21}u_{12} + l_{22}^2 & l_{21}u_{13} + l_{22}u_{23} \\ l_{31}l_{11} & l_{31}u_{12} + l_{32}l_{22} & l_{31}u_{13} + l_{32}u_{23} + l_{33}^2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

Comparing corresponding elements we get

$l_{11}^2 = 2$
$\Rightarrow \boxed{l_{11} = \sqrt{2}}$

$l_{11}u_{12} = 1$
$\Rightarrow \sqrt{2}\,u_{12} = 1$
$\Rightarrow \boxed{u_{12} = 1/\sqrt{2}}$

$l_{11}u_{13} = 3$
$\Rightarrow \sqrt{2}\,u_{13} = 3$
$\Rightarrow \boxed{u_{13} = 3/\sqrt{2}}$

$l_{21}l_{11} = 4$
$\Rightarrow l_{21}\sqrt{2} = 4$
$\Rightarrow \boxed{l_{21} = 4/\sqrt{2}}$

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$l_{31} l_{11} = 2$

$\Rightarrow l_{31}(\sqrt{2}) = 2$

$\Rightarrow \boxed{l_{31} = \sqrt{2}}$

$l_{21} u_{12} + l_{22}^2 = 3$

$\Rightarrow \frac{4}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) + l_{22}^2 = 3$

$\Rightarrow 2 + l_{22}^2 = 3$

$\Rightarrow \boxed{l_{22} = 1}$

$l_{21} u_{13} + l_{22} u_{23} = 10$

$\Rightarrow \left(\frac{4}{\sqrt{2}}\right)\left(\frac{3}{\sqrt{2}}\right) + 1(u_{23}) = 10$

$\Rightarrow 6 + u_{23} = 10$

$\Rightarrow \boxed{u_{23} = 4}$

$l_{31} u_{12} + l_{32} l_{22} = 4$

$\Rightarrow \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) + l_{32}(1) = 4$

$\Rightarrow 1 + l_{32} = 4$

$\Rightarrow \boxed{l_{32} = 3}$

$l_{31} u_{13} + l_{32} u_{23} + l_{33}^2 = 17$

$\Rightarrow \sqrt{2}\left(\frac{3}{\sqrt{2}}\right) + 3(4) + l_{33}^2 = 17$

$\Rightarrow 3 + 12 + l_{33}^2 = 17$

$\Rightarrow l_{33}^2 = 2$

$\Rightarrow \boxed{l_{32} = \sqrt{2}}$

$$\Rightarrow L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 4/\sqrt{2} & 1 & 0 \\ \sqrt{2} & 03 & \sqrt{2} \end{bmatrix} \text{ \& } U = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 3/\sqrt{2} \\ 0 & 1 & 4 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

Now $Ly = b$

$$\Rightarrow \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 4/\sqrt{2} & 1 & 0 \\ \sqrt{2} & 03 & \sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$\Rightarrow \sqrt{2}\, y_1 = 11 \Rightarrow y_1 = 11/\sqrt{2}$

$4/\sqrt{2}\, y_1 + y_2 = 28 \Rightarrow \frac{4}{\sqrt{2}}\left(\frac{11}{\sqrt{2}}\right) + y_2 = 28 \Rightarrow y_2 = 6$

$\sqrt{2}\, y_1 + 03\, y_2 + \sqrt{2}\, y_3 = 31 \Rightarrow \sqrt{2}\left(\frac{11}{\sqrt{2}}\right) + 03(6) + \sqrt{2}\, y_3 = 31$

$\Rightarrow y_3 = 2/\sqrt{2}$

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So $y = [11/\sqrt{2}, 6, 2/\sqrt{2}]^t$

Now $Ux = y$

$$\Rightarrow \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 3/\sqrt{2} \\ 0 & 1 & 4 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11/\sqrt{2} \\ 6 \\ \sqrt{2} \end{bmatrix}$$

$$\sqrt{2}x_1 + \frac{1}{\sqrt{2}}x_2 + \frac{3}{\sqrt{2}}x_3 = \frac{11}{\sqrt{2}} \quad \text{——(i)}$$

$$x_2 + 4x_3 = 6 \quad \text{——(ii)}$$

$\sqrt{2}x_3 = \sqrt{2} \Rightarrow x_3 = 1$ Put in (ii)

$\Rightarrow x_2 + 4(1) = 6 \Rightarrow x_2 = 2$

From (i) $\sqrt{2}x_1 + \frac{1}{\sqrt{2}}(2) + \frac{3}{\sqrt{2}}(1) = \frac{11}{\sqrt{2}}$

$\Rightarrow \sqrt{2}x_1 + \sqrt{2} + \frac{3}{\sqrt{2}} = \frac{11}{\sqrt{2}} \Rightarrow \sqrt{2}x_1 = \frac{11}{\sqrt{2}} - \sqrt{2} - \frac{3}{\sqrt{2}}$

$\Rightarrow \sqrt{2}x_1 = \frac{6}{\sqrt{2}} \Rightarrow x_1 = 3$

So solution vector is $x = [3, 2, 1]^T$

* * *

Question:- Let $A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$ (i) Find A^{-1} by Cholesky's Method

(ii) Hence solve system of equations

$x_1 + 2x_2 + 6x_3 = 13$

$2x_1 + 5x_2 + 15x_3 = 15$

$6x_1 + 15x_2 + 46x_3 = 19$

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Solution:- As

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$$

$A = A^T$

$\therefore$ A is symmetric. Then

$A = LL^T$, where $L^T = U$

$$\Rightarrow \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 l_{22}^2 & l_{21}l_{31}+l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21}+l_{32}l_{22} & l_{31}^2+l_{32}^2+l_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$$

Comparing corresponding elements

$l_{11}^2 = 1$

$\Rightarrow \boxed{l_{11} = 1}$

$l_{11}l_{21} = 2$

$\Rightarrow (1)\, l_{21} = 2$

$\Rightarrow \boxed{l_{21} = 2}$

$l_{11}l_{31} = 6$

$\Rightarrow (1)\, l_{31} = 6$

$\Rightarrow \boxed{l_{31} = 6}$

$l_{21}^2 + l_{22}^2 = 5$

$\Rightarrow (2)^2 + l_{22}^2 = 5$

$\Rightarrow 4 + l_{22}^2 = 5$

$\Rightarrow \boxed{l_{22} = 1}$

$l_{21}l_{31} + l_{22}l_{32} = 15$

$\Rightarrow 2(6) + 1(l_{32}) = 15$

$12 + l_{32} = 15$

$\Rightarrow \boxed{l_{32} = 3}$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 46$

$\Rightarrow (6)^2 + (3)^2 + l_{33}^2 = 46$

$\Rightarrow 36 + 9 + l_{33}^2 = 46$

$\Rightarrow l_{33}^2 = 1$

$\Rightarrow \boxed{l_{33} = 1}$

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So $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix}$, $L^T = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

As we know that $LL^{-1} = L^{-1}L = I$

Next we calculate L^{-1}

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ l_{21}' & l_{22}' & l_{23}' \\ l_{31}' & l_{32}' & l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ 2l_{11}'+l_{21}' & 2l_{12}'+l_{22}' & 2l_{13}'+l_{23}' \\ 6l_{11}'+3l_{21}'+l_{31}' & 6l_{12}'+3l_{22}'+l_{32}' & 6l_{13}'+3l_{23}'+l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing each element corresponding to matrix we get

$\boxed{l_{11}' = 1}$, $\boxed{l_{12}' = 0}$, $\boxed{l_{13}' = 0}$

$2l_{11}'+l_{21}' = 0 \Rightarrow 2(1)+l_{21}' = 0 \Rightarrow \boxed{l_{21}' = -2}$

$2l_{12}'+l_{22}' = 1 \Rightarrow 2(0)+l_{22}' = 1 \Rightarrow \boxed{l_{22}' = 1}$

$2l_{13}'+l_{23}' = 0 \Rightarrow 2(0)+l_{23}' = 0 \Rightarrow \boxed{l_{23}' = 0}$

$6l_{11}'+3l_{21}'+l_{31}' = 0 \Rightarrow 6(1)+3(-2)+l_{31}' = 0 \Rightarrow \boxed{l_{31}' = 0}$

$6l_{12}'+3l_{22}'+l_{32}' = 0 \Rightarrow 6(0)+3(1)+l_{32}' = 0 \Rightarrow \boxed{l_{32}' = -3}$

$6l_{13}'+3l_{23}'+l_{33}' = 1 \Rightarrow 6(0)+3(0)+l_{33}' = 1 \Rightarrow \boxed{l_{33}' = 1}$

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$$\therefore L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$A = LL^T \Rightarrow A^{-1} = (LL^T)^{-1}$$

$$\Rightarrow A^{-1} = (L^T)^{-1} L^{-1} \Rightarrow A^{-1} = (L^{-1})^T L^{-1}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4 & -2 & 0 \\ -2 & 1+9 & -3 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

(ii) $x_1 + 2x_2 + 6x_3 = 13$

$2x_1 + 5x_2 + 15x_3 = 15$

$6x_1 + 15x_2 + 46x_3 = 19$

$$A\underline{x} = \underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 13 \\ 15 \\ 19 \end{bmatrix}$$

$$= \begin{bmatrix} 65 - 30 + 0 \\ -26 + 150 - 57 \\ 0 - 45 + 17 \end{bmatrix} = \begin{bmatrix} 35 \\ 67 \\ -26 \end{bmatrix}$$

$$\Rightarrow \underline{x} = [35, 67, -26]^T$$

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* LDU Factorization:-

LDU Factorization is an other method to factorize any (square) matrix. To express $A = LDU$, where L and U are lower and upper Triangular matrices with unit entries in diagonal and D is a diagonal matrix

To find inverse by expressing $A = LDU$ is

$$A = LDU \Rightarrow A^{-1} = (LDU)^{-1} = U^{-1}D^{-1}L^{-1}$$

$$UA^{-1} = D^{-1}L^{-1}$$

where L^{-1} is given by $LL^{-1} = I$.

Then find L^{-1} we have A^{-1}

***

Example:- Obtain LDU factorization of matrix

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Solution:- $LDU = A$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

& $D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$

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$$\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} d_{11} & d_{11}u_{12} & d_{11}u_{13} \\ d_{11}\ell_{21} & d_{11}\ell_{21}u_{12}+d_{22} & d_{11}\ell_{21}u_{13}+d_{22}u_{23} \\ d_{11}\ell_{31} & d_{11}\ell_{31}u_{12}+d_{22}\ell_{32} & d_{11}\ell_{31}u_{13}+d_{22}\ell_{32}u_{23}+d_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

By Comparing we have

$\boxed{d_{11}=1}$, $d_{11}u_{12}=-1 \Rightarrow (1)u_{12}=-1 \Rightarrow \boxed{u_{12}=-1}$

$d_{11}u_{13}=-2 \Rightarrow (1)u_{13}=-2 \Rightarrow \boxed{u_{13}=-2}$

$d_{11}\ell_{21}=1 \Rightarrow (1)\ell_{21}=1 \Rightarrow \boxed{\ell_{21}=1}$

$d_{11}\ell_{31}=1 \Rightarrow (1)\ell_{31}=1 \Rightarrow \boxed{\ell_{31}=1}$

$d_{11}\ell_{21}u_{12}+d_{22}=2 \Rightarrow (1)(1)(-1)+d_{22}=2 \Rightarrow \boxed{d_{22}=3}$

$d_{11}\ell_{21}u_{13}+d_{22}u_{23}=1 \Rightarrow (1)(1)(-2)+3u_{23}=1 \Rightarrow \boxed{u_{23}=1}$

$d_{11}\ell_{31}u_{12}+d_{22}\ell_{32}=3 \Rightarrow (1)(1)(-1)+3\ell_{32}=3 \Rightarrow \boxed{\ell_{32}=4/3}$

$d_{11}\ell_{31}u_{13}+d_{22}\ell_{32}u_{23}+d_{33}=-1 \Rightarrow (1)(1)(-2)+(3)(\frac{4}{3})(1)+d_{33}=-1$

$\Rightarrow \boxed{d_{33}=-3}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$L \cdot D \cdot U = A$

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Example:- Find inverse of matrix A (define in previous example) by expressing in form of LDU.

Solution:-

(*) Working Rule:

(i) Express $A = LDU$ (ii) From $LL^{-1} = I$ find L^{-1}

(iii) Find D^{-1} (iv) From $UA^{-1} = D^{-1}L^{-1}$ find A^{-1}

In previous example we express $A = LDU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4/3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Next find L^{-1}, where $LL^{-1} = L^{-1}L = I$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4/3 & 1 \end{bmatrix} \begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ l_{21}' & l_{22}' & l_{23}' \\ l_{31}' & l_{32}' & l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ l_{11}' + l_{21}' & l_{12}' + l_{22}' & l_{13}' + l_{23}' \\ l_{11}' + \frac{4}{3}l_{21}' + l_{31}' & l_{12}' + \frac{4}{3}l_{22}' + l_{32}' & l_{13}' + \frac{4}{3}l_{23}' + l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow l_{11}' = 1$, $l_{12}' = 0$, $l_{13}' = 0$

$l_{11}' + l_{21}' = 0 \Rightarrow 1 + l_{21}' = 0 \Rightarrow l_{21}' = -1$

$l_{21}' + l_{22}' = 1 \Rightarrow 0 + l_{22}' = 1 \Rightarrow l_{22}' = 1$

$l_{13}' + l_{23}' = 0 \Rightarrow 0 + l_{23}' = 0 \Rightarrow l_{23}' = 0$

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$l_{11}' + \frac{4}{3} l_{21}' + l_{31}' = 0 \Rightarrow 1 + \frac{4}{3}(-1) + l_{31}' = 0$

$\Rightarrow \underline{l_{31}' = 1/3}$

$l_{12}' + \frac{4}{3} l_{22}' + l_{32}' = 0 \Rightarrow 0 + \frac{4}{3}(1) + l_{32}' = 0$

$\Rightarrow \underline{l_{32}' = -4/3}$

$l_{13}' + \frac{4}{3} l_{23}' + l_{33}' = 1 \Rightarrow 0 + \frac{4}{3}(0) + l_{33}' = 1 \Rightarrow \underline{l_{33}' = 1}$

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1/3 & -4/3 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}$$

$$UA^{-1} = D^{-1}L^{-1}$$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1/3 & -4/3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}' - a_{21}' - 2a_{31}' & a_{12}' - a_{22}' - 2a_{32}' & a_{13}' - a_{23}' - 2a_{33}' \\ a_{21}' + a_{31}' & a_{22}' + a_{32}' & a_{23}' + a_{33}' \\ a_{31}' & a_{32}' & a_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1/3 & 0 \\ -1/9 & 4/9 & -1/3 \end{bmatrix}$$

$\Rightarrow \underline{a_{31}' = -1/9}$, $\underline{a_{32}' = 4/9}$, $\underline{a_{33}' = -1/3}$

$a_{21}' + a_{31}' = -\frac{1}{3} \Rightarrow a_{21}' = \frac{-1}{3} + \frac{1}{9} \Rightarrow \underline{a_{21}' = -2/9}$

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$a_{22}' + a_{32}' = 1/3 \Rightarrow a_{22}' = \frac{1}{3} - \frac{4}{9} \Rightarrow \underline{a_{22}' = -1/9}$

$a_{23}' + a_{33}' = 0 \Rightarrow a_{23}' = 0 + 1/3 \Rightarrow \underline{a_{23}' = 1/3}$

$a_{11}' - a_{21}' - 2a_{31}' = 1 \Rightarrow a_{11}' - (\frac{-2}{9}) - 2(\frac{-1}{9}) = 1 \Rightarrow \underline{a_{11}' = +5/9}$

$a_{12}' - a_{22}' - 2a_{32}' = 0 \Rightarrow a_{12}' - (\frac{-1}{9}) - 2(\frac{4}{9}) = 0 \Rightarrow \underline{a_{12}' = 7/9}$

$a_{13}' - a_{23}' - 2a_{33}' = 0 \Rightarrow a_{13}' - \frac{1}{3} - 2(\frac{-1}{3}) = 0 \Rightarrow \underline{a_{13}' = -1/3}$

$$\Rightarrow A^{-1} = \begin{bmatrix} 5/9 & 7/9 & -1/3 \\ -2/9 & -1/9 & 1/3 \\ -1/9 & 4/9 & -1/3 \end{bmatrix}$$

* * *

Assignment

Q1): Factorization method apply on matrix in a form $A = LU$, where L is lower triangular and U is upper triangular

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

Q2): Solve the following system of equation by LU-decomposition by crout's method

$$3x_1 + 2x_2 + x_3 = 10$$
$$x_1 + 4x_2 - x_3 = 6$$
$$x_1 + 2x_2 + 5x_3 = 20$$

Q3): Find inverse of the following matrix

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$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 9 \\ -3 & 4 & -1 \end{bmatrix}$$ by expressing A in a form of LU-decomposition by Dolittle's Method.

Q4:- Solve the system of equations by factorizing the coefficient matrix approximately

$$x_1 + 2x_2 - x_3 = 5$$
$$2x_1 + 3x_2 - 5x_3 = 13$$
$$3x_1 + 8x_2 + 6x_3 = 13$$

Q5):- Write the matrix

$$A = \begin{bmatrix} 1.25 & 0.8 & 1.1 \\ 4.5 & 1.1 & 0.7 \\ 0.4 & 1.4 & 0.6 \end{bmatrix}$$ as a product of an upper and a lower triangular matrices & hence invert it

Q6): Solve by Cholesky's method of system of equations

$$3x_1 + 2x_2 + x_3 = 10$$
$$x_1 + 4x_2 + x_3 = 6$$
$$2x_1 + x_2 + 4x_3 = 12$$

Q7): Solve the following system of equations

$$2x_1 + 0x_2 + x_3 = 4$$
$$-3x_1 + 4x_2 - 2x_3 = -3$$
$$x_1 + 7x_2 - 5x_3 = 6$$

by (i) Dolittle's Method
(ii) Crout's Method
(iii) Cholesky's Method.

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Solutions

Q 1:-

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

Let $A = LU$, where L & U are lower & upper Triangular matrix respectively

So

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

Compairing corresponding elements of matrix

$\boxed{u_{11} = 1}$, $\boxed{u_{12} = 1}$, $\boxed{u_{13} = 1}$

$l_{21}u_1 = 4 \Rightarrow l_{21}(1) = 4 \Rightarrow \boxed{l_{21} = 4}$

$l_{21}u_{12} + u_{22} = 3 \Rightarrow 4(1) + u_{22} = 3 \Rightarrow \boxed{u_{22} = -1}$

$l_{21}u_{13} + u_{23} = -1 \Rightarrow 4(1) + u_{23} = -1 \Rightarrow \boxed{u_{23} = -5}$

$l_{31}u_{11} = 3 \Rightarrow l_{31}(1) = 3 \Rightarrow \boxed{l_{31} = 3}$

$l_{31}u_{12} + l_{32}u_{22} = 5 \Rightarrow 3(1) + l_{32}(-1) = 5 \Rightarrow \boxed{l_{32} = -2}$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 3 \Rightarrow 3(1) + (-2)(-5) + u_{33} = 3 \Rightarrow \boxed{u_{33} = -10}$

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$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \quad \& \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

Therefore A is factorized in upper & Lower Triangular matrices As

$$A = L \quad U$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

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Q2:- System of equations is

$$3x_1 + 2x_2 + x_3 = 10$$
$$x_1 + 4x_2 - x_3 = 6$$
$$x_1 + 2x_2 + 5x_3 = 20$$

$$\Rightarrow \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & -1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 20 \end{bmatrix}$$

$$A \quad \underline{x} = \underline{b}$$

Let $A = LU$

Then By Crout's method

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \quad \& \quad U = \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

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$$\Rightarrow \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & -1 \\ 1 & 2 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & -1 \\ 1 & 2 & 5 \end{bmatrix}$$

Comparing corresponding elements

$\boxed{l_{11}=3}$, $l_{11}u_{12}=2 \Rightarrow 3(u_{12})=2 \Rightarrow \boxed{u_{12}=2/3}$

$\boxed{l_{21}=1}$, $l_{21}u_{12}+l_{22}=4 \Rightarrow 1(\frac{2}{3})+l_{22}=4 \Rightarrow \boxed{l_{22}=\frac{10}{3}}$

$\boxed{l_{31}=1}$, $l_{31}u_{12}+l_{32}=2 \Rightarrow 1(\frac{2}{3})+l_{32}=2 \Rightarrow \boxed{l_{3}=4/3}$

$l_{11}u_{13}=1 \Rightarrow 3u_{13}=1 \Rightarrow \boxed{u_{13}=1/3}$

$l_{21}u_{13}+l_{22}u_{23}=-1 \Rightarrow 1(\frac{1}{3})+\frac{10}{3}u_{23}=-1$

$\Rightarrow \boxed{u_{23}=-2/5}$

$l_{31}u_{13}+l_{32}u_{23}+l_{33}=5 \Rightarrow 1(\frac{1}{3})+\frac{4}{3}(\frac{-2}{5})+l_{33}=5$

$\Rightarrow \boxed{l_{33}=\frac{26}{5}}$

$$\Rightarrow L = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 10/3 & 0 \\ 1 & 4/3 & 26/5 \end{bmatrix} \quad \& \quad U = \begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $L\underline{y} = \underline{b}$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 1 & 10/3 & 0 \\ 1 & 4/3 & 26/5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 20 \end{bmatrix}$$

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$\Rightarrow 3y_1 = 10 \Rightarrow \boxed{y_1 = 10/3}$

$y_1 + \frac{10}{3}y_2 = 6 \Rightarrow \frac{10}{3} + \frac{10}{3}y_2 = 6$

$\Rightarrow \boxed{y_2 = 4/5}$

$y_1 + \frac{4}{3}y_2 + \frac{26}{5}y_3 = 20 \Rightarrow \frac{10}{3} + \frac{4}{5}\left(\frac{4}{5}\right) + \frac{26}{5}y_3 = 20$

$\frac{26}{5}y_3 = 20 - \frac{10}{3} - \frac{16}{15} \Rightarrow \frac{26}{5}y_3 = 78/5$

$\Rightarrow \boxed{y_3 = 3}$

So $y = \left[\frac{10}{3}, \frac{4}{5}, 3\right]^t$

Now $Ux = y$

$$\Rightarrow \begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 4/5 \\ 3 \end{bmatrix}$$

$\Rightarrow x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 = \frac{10}{3}$ ——— ①

$x_2 - \frac{2}{5}x_3 = \frac{4}{5}$ ——— ②

$\boxed{x_3 = 3}$

from ② $x_2 - \frac{2}{5}(3) = 4/5 \Rightarrow x_2 - \frac{6}{5} = \frac{4}{5}$

$\Rightarrow \boxed{x_2 = 2}$

from ① $x_1 + \frac{2}{3}(2) + \frac{1}{3}(3) = \frac{10}{3} \Rightarrow x_1 + \frac{4}{3} + \frac{1}{3} = \frac{10}{3}$

$\Rightarrow \boxed{x_1 = 1}$

So $x = [1, 2, 3]^t$

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Q 3:-

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 9 \\ -3 & 4 & -1 \end{bmatrix}$$

Let $A = LU$

Then By Doolittle's method

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \& \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 9 \\ -3 & 4 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ l_{21}U_{11} & l_{21}U_{12}+U_{22} & l_{21}U_{13}+U_{23} \\ l_{31}U_{11} & l_{31}U_{12}+l_{32}U_{22} & l_{31}U_{13}+l_{32}U_{23}+U_{33} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 9 \\ -3 & 4 & -1 \end{bmatrix}$$

Comparing corresponding elements

$\boxed{U_{11} = 1}$, $\boxed{U_{12} = -2}$, $\boxed{U_{13} = 3}$

$l_{21}U_{11} = 2 \Rightarrow l_{21}(1) = 2 \Rightarrow \boxed{l_{21} = 2}$

$l_{21}U_{12} + U_{22} = -1 \Rightarrow 2(-2) + U_{22} = -1$

$\Rightarrow \boxed{U_{22} = 3}$

$l_{21}U_{13} + U_{23} = 9 \Rightarrow 2(3) + U_{23} = 9 \Rightarrow \boxed{U_{23} = 3}$

$l_{31}U_{11} = -3 \Rightarrow l_{31}(1) = -3 \Rightarrow \boxed{l_{31} = -3}$

$l_{31}U_{12} + l_{32}U_{22} = 4 \Rightarrow (-3)(-2) + 3l_{32} = 4$

$\Rightarrow \boxed{l_{32} = \frac{-2}{3}}$

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$l_{31}u_{13} + l_{32}u_{23} + u_{33} = -1 \Rightarrow (-3)(3) + \left(\frac{-2}{3}\right)(3) + u_{33} = -1$

$\Rightarrow \boxed{u_{33} = 10}$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -2/3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 10 \end{bmatrix}$$

Now for Inverse:-

First we calculate L^{-1}

As $LL^{-1} = L^{-1}L = I$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ l_{21}' & 1 & 0 \\ l_{31}' & l_{32}' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2+l_{21}' & 1 & 0 \\ -3-\frac{2}{3}l_{21}'+l_{31}' & -\frac{2}{3}+l_{32}' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing elements

$2 + l_{21}' = 0 \Rightarrow \boxed{l_{21}' = -2}$

$-3 - \frac{2}{3}l_{21}' + l_{31}' = 0 \Rightarrow -3 - \frac{2}{3}(-2) + l_{31}' = 0$

$\Rightarrow l_{31}' = 3 - 4/3 \Rightarrow \boxed{l_{31}' = 5/3}$

$-2/3 + l_{32}' = 0 \Rightarrow \boxed{l_{32}' = 2/3}$

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$$\Rightarrow L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5/3 & 2/3 & 1 \end{bmatrix}$$

No $UA^{-1} = L^{-1}$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5/3 & 2/3 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_{11}-2a_{21}+3a_{31} & a_{12}-2a_{22}+3a_{32} & a_{13}-2a_{23}+3a_{33} \\ 3a_{21}+3a_{31} & 3a_{22}+3a_{32} & 3a_{23}+3a_{33} \\ 10a_{31} & 10a_{32} & 10a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5/3 & 2/3 & 1 \end{bmatrix}$$

comparing corresponding elements

$10a_{31} = 5/3 \Rightarrow \boxed{a_{31} = 1/6}$

$10a_{32} = 2/3 \Rightarrow \boxed{a_{32} = 1/15}$

$10a_{33} = 1 \Rightarrow \boxed{a_{33} = 1/10}$

$3a_{21} + 3a_{31} = -2 \Rightarrow 3a_{21} + 3(\frac{1}{6}) = -2 \Rightarrow 3a_{21} + \frac{1}{2} = -2$

$\Rightarrow \boxed{a_{21} = -5/6}$

$3a_{22} + 3a_{32} = 1 \Rightarrow 3a_{22} + 3(\frac{1}{15}) = 1 \Rightarrow \boxed{a_{22} = 4/15}$

$3a_{23} + 3a_{33} = 0 \Rightarrow 3a_{23} + 3(\frac{1}{10}) = 0 \Rightarrow \boxed{a_{23} = -1/10}$

$a_{11} - 2a_{21} + 3a_{31} = 1 \Rightarrow a_{11} - 2(-5/6) + 3(1/6) = 1$

$\Rightarrow a_{11} + \frac{5}{3} + \frac{1}{2} = 1 \Rightarrow \boxed{a_{11} = -7/6}$

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$a_{12} - 2a_{22} + 3a_{32} = 0 \Rightarrow a_{12} - 2(\frac{4}{15}) + 3(\frac{1}{15}) = 0$

$\Rightarrow a_{12} - \frac{8}{15} + \frac{3}{15} = 0 \Rightarrow \boxed{a_{12} = \frac{1}{3}}$

$a_{13} - 2a_{23} + 3a_{33} = 0 \Rightarrow a_{13} - 2(-\frac{1}{10}) + (\frac{3}{10})3 = 0$

$\Rightarrow a_{13} + \frac{2}{10} + \frac{3}{10} = 0 \Rightarrow \boxed{a_{13} = -\frac{1}{2}}$

$\therefore$ $A^{-1} = \begin{bmatrix} -7/6 & 1/3 & -1/2 \\ -5/6 & 4/15 & -1/10 \\ 1/6 & 1/15 & 1/10 \end{bmatrix}$.

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Q4:- System of equations is

$$x_1 + 2x_2 - x_3 = 5$$
$$2x_1 + 3x_2 - 5x_3 = 13$$
$$3x_1 + 8x_2 + 6x_3 = 13$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -5 \\ 3 & 8 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 13 \end{bmatrix}$$

$A \quad \underline{x} = \underline{b}$

Let $A = LU$

where L & U are lower & upper triangular matrices.

Let

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

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$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -5 \\ 3 & 8 & 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -5 \\ 3 & 8 & 6 \end{bmatrix}$$

Comparing corresponding elements

$\boxed{u_{11}=1}$, $\boxed{u_{12}=2}$, $\boxed{u_{13}=-1}$

$l_{21}u_{11}=2 \Rightarrow l_{21}(1)=2 \Rightarrow \boxed{l_{21}=2}$

$l_{21}u_{12}+u_{22}=3 \Rightarrow 2(2)+u_{22}=3 \Rightarrow \boxed{u_{22}=-1}$

$l_{21}u_{13}+u_{23}=-5 \Rightarrow 2(-1)+u_{23}=-5 \Rightarrow \boxed{u_{23}=-3}$

$l_{31}u_{11}=3 \Rightarrow l_{31}(1)=3 \Rightarrow \boxed{l_{31}=3}$

$l_{31}u_{12}+l_{32}u_{22}=8 \Rightarrow 3(2)+l_{32}(-1)=8$

$\Rightarrow \boxed{l_{32}=-2}$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=6 \Rightarrow 3(-1)+(-2)(-3)+u_{33}=6$

$\Rightarrow \boxed{u_{33}=3}$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{bmatrix}$$

Let $Ly = b$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 13 \end{bmatrix}$$

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$\Rightarrow \boxed{y_1 = 5}$

$2y_1 + y_2 = 13 \Rightarrow 2(5) + y_2 = 13$

$\Rightarrow \boxed{y_2 = 3}$

$3y_1 - 2y_2 + y_3 = 13 \Rightarrow 3(5) - 2(3) + y_3 = 13$

$\Rightarrow \boxed{y_3 = 4}$

$\Rightarrow \quad y = [5, 3, 4]^t$

Now

$$Ux = y$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 - x_3 = 5$ ——— ①

$-x_2 - 3x_3 = 3$ ——— ②

$3x_3 = 4$

$\Rightarrow \boxed{x_3 = 4/3}$

from ②

$-x_2 - 3(\frac{4}{3}) = 3 \Rightarrow \boxed{x_2 = -7}$

from ①

$x_1 + 2(-7) - \frac{4}{3} = 5 \Rightarrow \boxed{x_1 = 61/3}$

Now for Inverse:-

First we calculate L^{-1}

As $LL^{-1} = L^{-1}L = I$

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Let $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21}' & 1 & 0 \\ l_{31}' & l_{32}' & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ l_{21}' & 1 & 0 \\ l_{31}' & l_{32}' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2+l_{21}' & 1 & 0 \\ 3-2l_{21}'+l_{31}' & -2+l_{32}' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By comparing corresponding elements

$2 + l_{21}' = 0 \Rightarrow \boxed{l_{21}' = -2}$

$-2 + l_{32}' = 0 \Rightarrow \boxed{l_{32}' = 2}$

$3 - 2l_{21}' + l_{31}' = 0 \Rightarrow 3 - 2(-2) + l_{31}' = 0$

$\Rightarrow \boxed{l_{31}' = -7}$

$$\Rightarrow L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

Now $UA^{-1} = L^{-1}$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

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$$\Rightarrow \begin{bmatrix} a_{11}+2a_{21}-a_{31} & a_{12}+2a_{22}-a_{32} & a_{13}+2a_{23}-a_{33} \\ -a_{21}-3a_{31} & -a_{22}-3a_{32} & -a_{23}-3a_{33} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

Compairing corresponding elements

$3a_{31} = -7 \Rightarrow \boxed{a_{31} = -7/3}$

$3a_{32} = 2 \Rightarrow \boxed{a_{32} = \frac{2}{3}}$

$3a_{33} = 1 \Rightarrow \boxed{a_{33} = 1/3}$

$-a_{21} - 3a_{31} = -2$

$-a_{21} - 3(-7/3) = -2$

$-a_{21} + 7 = -2$

$\boxed{a_{21} = 9}$

$-a_{22} - 3a_{32} = 1 \Rightarrow -a_{22} - 3(\frac{2}{3}) = 1$

$\Rightarrow -a_{22} - 2 = 1$

$\boxed{a_{22} = -3}$

$-a_{23} - 3a_{33} = 0 \Rightarrow -a_{23} - 3(\frac{1}{3}) = 0 \Rightarrow -a_{23} - 1 = 0$

$\boxed{a_{23} = -1}$

$a_{11} + 2a_{21} - a_{31} = 1 \Rightarrow a_{11} + 2(9) + 7/3 = 1$

$\Rightarrow a_{11} = 1 - 7/3 - 18 \Rightarrow \boxed{a_{11} = -58/3}$

$a_{12} + 2a_{22} - a_{32} = 0 \Rightarrow a_{12} + 2(-3) - 2/3 = 0$

$\boxed{a_{12} = 20/3}$

$a_{13} + 2a_{23} - a_{33} = 0 \Rightarrow a_{13} + 2(-1) - 1/3 = 0$

$\Rightarrow \boxed{a_{13} = 7/3}$

$\therefore$

$$A^{-1} = \begin{bmatrix} -58/3 & 20/3 & 7/3 \\ 9 & -3 & -1 \\ -7/3 & 2/3 & 1/3 \end{bmatrix}$$

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Q5:-

$$A = \begin{bmatrix} 1.25 & 0.8 & 1.1 \\ 4.5 & 1.1 & 0.7 \\ 0.4 & 1.4 & 0.6 \end{bmatrix}$$

Let $A = LU$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1.25 & 0.8 & 1.1 \\ 4.5 & 1.1 & 0.7 \\ 0.4 & 1.4 & 0.6 \end{bmatrix}}_{A}$$

$$\Rightarrow \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ l_{21}U_{11} & l_{21}U_{12}+U_{22} & l_{21}U_{13}+U_{23} \\ l_{31}U_{11} & l_{31}U_{12}+l_{32}U_{22} & l_{31}U_{13}+l_{32}U_{23}+U_{33} \end{bmatrix} = \begin{bmatrix} 1.25 & 0.8 & 1.1 \\ 4.5 & 1.1 & 0.7 \\ 0.4 & 1.4 & 0.6 \end{bmatrix}$$

Now compairing the corresponding elements

$\Rightarrow$ $\boxed{U_{11} = 1.25}$, $\boxed{U_{12} = 0.8}$, $\boxed{U_{13} = 1.1}$

$l_{21}U_{11} = 4.5 \Rightarrow l_{21}(1.25) = 4.5 \Rightarrow \boxed{l_{21} = 3.6}$

$l_{21}U_{12} + U_{22} = 1.1 \Rightarrow (3.6)(0.8) + U_{22} = 1.1 \Rightarrow \boxed{U_{22} = -1.78}$

$l_{21}U_{13} + U_{23} = 0.7 \Rightarrow (3.6)(1.1) + U_{23} = 0.7 \Rightarrow \boxed{U_{23} = -3.26}$

$l_{31}U_{11} = 0.4 \Rightarrow l_{31}(1.25) = 0.4 \Rightarrow \boxed{l_{31} = 0.32}$

$l_{31}U_{12} + l_{32}U_{22} = 1.4 \Rightarrow (0.32)(0.8) + l_{32}(-1.78) = 1.4$

$\Rightarrow \boxed{l_{32} = -0.643}$

$l_{31}U_{13} + l_{32}U_{23} + U_{33} = 0.6 \Rightarrow (0.32)(1.1) + (-0.643)(-3.26) + U_{33} = 0.6$

$\Rightarrow 0.352 + 2.096 + U_{33} = 0.6 \Rightarrow \boxed{U_{33} = -1.85}$

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So A can be written as the product of a lower and an upper triangular matrices As

$$\begin{bmatrix} 1 & 0 & 0 \\ 3.6 & 1 & 0 \\ 0.32 & -0.643 & 1 \end{bmatrix} \begin{bmatrix} 1.25 & 0.8 & 1.1 \\ 0 & -1.78 & -3.26 \\ 0 & 0 & -1.85 \end{bmatrix} = \begin{bmatrix} 1.25 & 0.8 & 1.1 \\ 4.5 & 1.1 & 0.7 \\ 0.4 & 1.4 & 0.6 \end{bmatrix}$$

$$L \qquad U \qquad = A$$

Now for Inverse:-

First we calculate L^{-1}

As $LL^{-1} = L^{-1}L = I$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 3.6 & 1 & 0 \\ 0.32 & -0.643 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ l_{21}' & 1 & 0 \\ l_{31}' & l_{32}' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L \qquad L^{-1} \qquad = I$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 3.6+l_{21}' & 1 & 0 \\ 0.32-0.643l_{21}'+l_{31}' & -0.643+l_{32}' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow 3.6+l_{21}' = 0 \Rightarrow \boxed{l_{21}' = -3.6}$

$-0.643+l_{32}' = 0 \Rightarrow \boxed{l_{32}' = 0.643}$

$0.32-0.643l_{21}'+l_{31}' = 0 \Rightarrow 0.32-0.643(-3.6)+l_{31}' = 0$

$\Rightarrow \boxed{l_{31}' = -2.64}$

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So

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3.6 & 1 & 0 \\ -2.64 & 0.643 & 1 \end{bmatrix}$$

Now

$$UA^{-1} = L^{-1}$$

$$\Rightarrow \begin{bmatrix} 1.25 & 0.8 & 1.1 \\ 0 & -1.78 & -3.26 \\ 0 & 0 & -1.85 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3.6 & 1 & 0 \\ -2.64 & 0.643 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1.25a_{11}+0.8a_{21}+1.1a_{31} & 1.25a_{12}+0.8a_{22}+1.1a_{32} & 1.25a_{13}+0.8a_{23}+1.1a_{33} \\ -1.78a_{21}-3.26a_{31} & -1.78a_{22}-3.26a_{32} & -1.78a_{23}-3.26a_{33} \\ -1.85a_{31} & -1.85a_{32} & -1.85a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -3.6 & 1 & 0 \\ -2.64 & 0.643 & 1 \end{bmatrix}$$

$\Rightarrow -1.85\,a_{31} = -2.64 \Rightarrow$ $\boxed{a_{31} = 1.43}$

$-1.85\,a_{32} = 0.643 \Rightarrow$ $\boxed{a_{32} = -0.348}$

$-1.85\,a_{33} = 1 \Rightarrow$ $\boxed{a_{33} = -0.54}$

$-1.78\,a_{21} - 3.26\,a_{31} = -3.6 \Rightarrow -1.78\,a_{21} - 3.26(1.43) = -3.6$

$\Rightarrow -1.78\,a_{21} - 4.662 = -3.6 \Rightarrow$ $\boxed{a_{21} = -0.6}$

$-1.78\,a_{22} - 3.26\,a_{32} = 1 \Rightarrow -1.78\,a_{22} - 3.26(-0.348) = 1$

$-1.78\,a_{22} + 1.134 = 1 \Rightarrow$ $\boxed{a_{22} = 0.075}$

$-1.78\,a_{23} - 3.26\,a_{33} = 0 \Rightarrow -1.78\,a_{23} - 3.26(-0.54) = 0$

$\Rightarrow -1.78\,a_{23} + 1.764 = 0 \Rightarrow$ $\boxed{a_{23} = 0.99}$

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$1.25a_{11} + 0.8a_{21} + 1.1a_{31} = 1 \Rightarrow 1.25a_{11} + 0.8(-0.6) + 1.1(1.43) = 1$

$\Rightarrow 1.25a_{11} - 0.48 + 1.573 = 1 \Rightarrow \boxed{a_{11} = -0.075}$

$1.25a_{12} + 0.8a_{22} + 1.1a_{32} = 0 \Rightarrow 1.25a_{12} + 0.8(0.075) + 1.1(-0.348) = 0$

$\Rightarrow 1.25a_{12} + 0.06 - 0.383 = 0 \Rightarrow \boxed{a_{12} = 0.26}$

$1.25a_{13} + 0.8a_{23} + 1.1a_{33} = 0 \Rightarrow 1.25a_{13} + 0.8(0.99) + 1.1(-0.541) = 0$

$\Rightarrow 1.25a_{13} + 0.792 - 0.595 = 0 \Rightarrow \boxed{a_{13} = -0.158}$

$\Rightarrow$

$$A^{-1} = \begin{bmatrix} -0.075 & 0.26 & -0.158 \\ -0.6 & 0.075 & 0.99 \\ 1.43 & -0.348 & -0.541 \end{bmatrix}$$

It is Approximate Inverse.

★ ★ ★

Q6:- System of equations is

$$3x_1 + 2x_2 + x_3 = 10$$
$$x_1 + 4x_2 + x_3 = 6$$
$$2x_1 + x_2 + 4x_3 = 12$$

$\Rightarrow$

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 1 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 12 \end{bmatrix}$$

$A \quad \underline{x} = \underline{b}$

Let $A = LU$

Then by Cholesky's method

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$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ & } U = \begin{bmatrix} l_{11} & u_{12} & u_{13} \\ 0 & l_{22} & u_{23} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & u_{12} & u_{13} \\ 0 & l_{22} & u_{23} \\ 0 & 0 & l_{33} \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 1 \\ 2 & 1 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} l_{11}^2 & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}l_{11} & l_{21}u_{12}+l_{22}^2 & l_{21}u_{13}+l_{22}u_{23} \\ l_{31}l_{11} & l_{31}u_{12}+l_{32}l_{22} & l_{31}u_{13}+l_{32}u_{23}+l_{33}^2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 1 \\ 2 & 1 & 4 \end{bmatrix}$$

Comparing corresponding elements

$l_{11}^2 = 3 \Rightarrow \boxed{l_{11} = \sqrt{3}}$

$l_{11}u_{12} = 2 \Rightarrow \sqrt{3}\,u_{12} = 2 \Rightarrow \boxed{u_{12} = \frac{2}{\sqrt{3}}}$

$l_{11}u_{13} = 1 \Rightarrow \sqrt{3}\,u_{13} = 1 \Rightarrow \boxed{u_{13} = 1/\sqrt{3}}$

$l_{21}l_{11} = 1 \Rightarrow l_{21}(\sqrt{3}) = 1 \Rightarrow \boxed{l_{21} = 1/\sqrt{3}}$

$l_{21}u_{12} + l_{22}^2 = 4 \Rightarrow \left(\frac{1}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right) + l_{22}^2 = 4$

$\Rightarrow \frac{2}{3} + l_{22}^2 = 4 \Rightarrow \boxed{l_{22} = \sqrt{10/3}}$

$l_{21}u_{13} + l_{22}u_{23} = 1 \Rightarrow \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + \sqrt{\frac{10}{3}}\,u_{23} = 1$

$\Rightarrow \sqrt{\frac{10}{3}}\,u_{23} = \frac{2}{3} \Rightarrow \boxed{u_{23} = \frac{\sqrt{30}}{15}}$

$l_{31}l_{11} = 2 \Rightarrow l_{31}\sqrt{3} = 2 \Rightarrow \boxed{l_{31} = 2/\sqrt{3}}$

$l_{31}u_{12} + l_{32}l_{22} = 1 \Rightarrow \frac{2}{\sqrt{3}}\left(\frac{2}{\sqrt{3}}\right) + \sqrt{\frac{10}{3}}\,l_{22} = 1$

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$$\Rightarrow \sqrt{\frac{10}{3}}\, l_{32} = \frac{-1}{3} \quad \Rightarrow \boxed{l_{32} = -1/\sqrt{30}}$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33}^2 = 4 \Rightarrow \frac{2}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) + \frac{-1}{\sqrt{30}}\left(\frac{\sqrt{30}}{15}\right) + l_{33}^2 = 4$$

$$\Rightarrow \frac{2}{3} - \frac{1}{15} + l_{33}^2 = 4 \quad \Rightarrow \quad l_{33} = \sqrt{17/5}$$

$$\Rightarrow L = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ \frac{1}{\sqrt{3}} & \sqrt{\frac{10}{3}} & 0 \\ \frac{2}{\sqrt{3}} & \frac{-1}{\sqrt{30}} & \sqrt{\frac{17}{5}} \end{bmatrix}, \quad U = \begin{bmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{10}{3}} & \frac{\sqrt{30}}{15} \\ 0 & 0 & \sqrt{\frac{17}{5}} \end{bmatrix}$$

Let $L\underline{y} = \underline{b}$

$$\Rightarrow \begin{bmatrix} \sqrt{3} & 0 & 0 \\ \frac{1}{\sqrt{3}} & \sqrt{\frac{10}{3}} & 0 \\ \frac{2}{\sqrt{3}} & \frac{-1}{\sqrt{30}} & \sqrt{\frac{17}{5}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 12 \end{bmatrix}$$

$$\Rightarrow \sqrt{3}\, y_1 = 10 \quad \Rightarrow \quad \boxed{y_1 = 10/\sqrt{3}}$$

$$\frac{1}{\sqrt{3}} y_1 + \sqrt{\frac{10}{3}}\, y_2 = 6 \Rightarrow \frac{1}{\sqrt{3}}\left(\frac{10}{\sqrt{3}}\right) + \sqrt{\frac{10}{3}}\, y_2 = 6$$

$$\Rightarrow \frac{10}{3} + \sqrt{\frac{10}{3}}\, y_2 = 6 \Rightarrow \sqrt{\frac{10}{3}}\, y_2 = 8/3$$

$$\Rightarrow \boxed{y_2 = 8/\sqrt{30}}$$

$$\frac{2}{\sqrt{3}} y_1 - \frac{1}{\sqrt{30}} y_2 + \sqrt{\frac{17}{5}}\, y_3 = 12$$

$$\Rightarrow \frac{2}{\sqrt{3}}\left(\frac{10}{\sqrt{3}}\right) - \frac{1}{\sqrt{30}}\left(\frac{8}{\sqrt{30}}\right) + \sqrt{\frac{17}{5}}\, y_3 = 12$$

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$$\Rightarrow \frac{20}{3} - \frac{8}{30} + \sqrt{\frac{17}{5}}\, y_3 = 12 \quad \Rightarrow \sqrt{\frac{17}{5}}\, y_3 = 28/5$$

$$\Rightarrow \boxed{y_3 = \frac{28}{\sqrt{85}}}$$

$$\Rightarrow y = \left[\frac{10}{\sqrt{3}}, \frac{8}{\sqrt{30}}, \frac{28}{\sqrt{85}}\right]^t$$

Now $Ux = y$

$$\Rightarrow \begin{bmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{\sqrt{10}}{\sqrt{3}} & \frac{\sqrt{30}}{15} \\ 0 & 0 & \sqrt{\frac{17}{5}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{10}{\sqrt{3}} \\ 8/\sqrt{30} \\ 28/\sqrt{85} \end{bmatrix}$$

$$\Rightarrow \sqrt{3}x_1 + \frac{2}{\sqrt{3}}x_2 + \frac{1}{\sqrt{3}}x_3 = \frac{10}{\sqrt{3}} \quad \text{——— ①}$$

$$\sqrt{\frac{10}{3}}x_2 + \frac{\sqrt{30}}{15}x_3 = \frac{8}{\sqrt{30}} \quad \text{——— ②}$$

$$\sqrt{\frac{17}{5}}x_3 = \frac{28}{\sqrt{85}} \Rightarrow x_3 = \frac{28}{\sqrt{85}} \times \frac{5}{\sqrt{17}}$$

$$\Rightarrow \boxed{x_3 = \frac{28}{17}}$$

From ② $\sqrt{\frac{10}{3}}x_2 + \frac{\sqrt{30}}{15}x_3 = \frac{8}{\sqrt{30}}$

$$\Rightarrow \sqrt{\frac{10}{3}}x_2 + \frac{\sqrt{30}}{15}\left(\frac{28}{17}\right) = \frac{8}{\sqrt{30}}$$

$$\Rightarrow \sqrt{\frac{10}{3}}x_2 + \frac{28\sqrt{30}}{255} = \frac{8}{\sqrt{30}}$$

$$\Rightarrow \boxed{x_2 = \frac{8}{17}}$$

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from ① $\sqrt{3}x_1 + \frac{2}{\sqrt{3}}x_2 + \frac{1}{\sqrt{3}}x_3 = \frac{10}{\sqrt{3}}$

$\Rightarrow \sqrt{3}x_1 + \frac{2}{\sqrt{3}}\left(\frac{8}{17}\right) + \frac{1}{\sqrt{3}}\left(\frac{28}{17}\right) = \frac{10}{\sqrt{3}}$

$\Rightarrow \boxed{x_1 = 42/17}$

So $x = \left[\frac{42}{17}, \frac{8}{17}, \frac{28}{17}\right]^T$

* * *

Q 7:- System of equations is

$2x_1 + 0x_2 + x_3 = 4$

$-3x_1 + 4x_2 - 2x_3 = -3$

$x_1 + 7x_2 - 5x_3 = 6$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ -3 & 4 & -2 \\ 1 & 7 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 6 \end{bmatrix}$$

$A \quad x = b$

Let $A = LU$

i) Dolittle's method:-

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -3 & 4 & -2 \\ 1 & 7 & -5 \end{bmatrix}$$

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$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -3 & 4 & -2 \\ 1 & 7 & -5 \end{bmatrix}$$

Now by compairing corresponding entries

$\Rightarrow \boxed{u_{11}=2}$, $\boxed{u_{12}=0}$, $\boxed{u_{13}=1}$

$l_{21}u_{11}=-3 \Rightarrow l_{21}(2)=-3 \Rightarrow \boxed{l_{21}=-3/2}$

$l_{21}u_{12}+u_{22}=4 \Rightarrow l_{21}(0)+u_{22}=4 \Rightarrow \boxed{u_{22}=4}$

$l_{21}u_{13}+u_{23}=-2 \Rightarrow -3/2(1)+u_{23}=-2 \Rightarrow \boxed{u_{23}=-1/2}$

$l_{31}u_{11}=1 \Rightarrow l_{31}(2)=1 \Rightarrow \boxed{l_{31}=1/2}$

$l_{31}u_{12}+l_{33}u_{22}=7 \Rightarrow 0+l_{32}(4)=7 \Rightarrow \boxed{l_{32}=7/4}$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=-5 \Rightarrow \frac{1}{2}(1)+\frac{7}{4}(\frac{-1}{2})+u_{33}=-5$

$$\boxed{u_{33}=-37/8}$$

$$\Rightarrow L=\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 1/2 & 7/4 & 1 \end{bmatrix}, \quad U=\begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & -1/2 \\ 0 & 0 & -3/8 \end{bmatrix}$$

Let $Ly = b$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ 1/2 & 7/4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 6 \end{bmatrix}$$

$\Rightarrow \boxed{y_1=4}$, $-\frac{3}{2}y_1+y_2=-3 \Rightarrow -\frac{3}{2}(4)+y_2=-3$

$\Rightarrow \boxed{y_2=3}$

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$\frac{1}{2}y_1 + \frac{7}{4}y_2 + y_3 = 6 \Rightarrow \frac{1}{2}(4) + \frac{7}{2}(3) + y_3 = 6$

$2 + \frac{21}{4} + y_3 = 6 \Rightarrow \boxed{y_3 = -5/4}$

So $y = \begin{bmatrix} 4 , 3 , -5/4 \end{bmatrix}^T$

Now

$Ux = y$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & -1/2 \\ 0 & 0 & -37/8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -5/4 \end{bmatrix}$$

$\Rightarrow 2x_1 + x_3 = 4$ ——— ①

$4x_2 - \frac{1}{2}x_3 = 3$ ——— ②

$-\frac{37}{8}x_3 = -5/4 \Rightarrow x_3 = \frac{-5}{4} \times \frac{8}{-37}$

$\Rightarrow \boxed{x_3 = 10/37}$

from ②

$4x_2 - \frac{1}{2}x_3 = 3 \Rightarrow 4x_2 - \frac{1}{2}\left(\frac{10}{37}\right) = 3$

$\Rightarrow 4x_2 = \frac{116}{37} \Rightarrow \boxed{x_2 = 29/37}$

from ① $2x_1 + x_3 = 4 \Rightarrow 2x_1 + \frac{10}{37} = 4$

$\Rightarrow 2x_1 = \frac{138}{37} \Rightarrow \boxed{x_1 = \frac{69}{37}}$

$\Rightarrow x = \begin{bmatrix} \frac{69}{37} , \frac{29}{37} , \frac{10}{37} \end{bmatrix}^T$

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ii) Crout's method:-

$$L=\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, \quad U=\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 2 & 0 & 1 \\ -3 & 4 & -2 \\ 1 & 7 & -5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}=\begin{bmatrix} 2 & 0 & 1 \\ -3 & 4 & -2 \\ 1 & 7 & -5 \end{bmatrix}$$

$\Rightarrow \boxed{l_{11}=2}$ $\quad \boxed{l_{21}=-3}$ $\quad \boxed{l_{31}=1}$

$l_{11}u_{12}=0 \Rightarrow 2(u_{12})=0 \Rightarrow \boxed{u_{12}=0}$

$l_{11}u_{13}=1 \Rightarrow 2(u_{13})=1 \Rightarrow \boxed{u_{13}=\frac{1}{2}}$

$l_{21}u_{12}+l_{22}=4 \Rightarrow -3(0)+l_{22}=4 \Rightarrow \boxed{l_{22}=4}$

$l_{21}u_{13}+l_{22}u_{23}=-2 \Rightarrow -3(\frac{1}{2})+4u_{23}=-2$

$\Rightarrow 4u_{23}=-\frac{1}{2} \Rightarrow \boxed{u_{23}=-\frac{1}{8}}$

$l_{31}u_{12}+l_{32}=7 \Rightarrow 1(0)+l_{32}=7 \Rightarrow \boxed{l_{32}=7}$

$l_{31}u_{13}+l_{32}u_{23}+l_{33}=-5 \Rightarrow 1(\frac{1}{2})+7(-\frac{1}{8})+l_{33}=-5$

$\Rightarrow l_{33}=-5+\frac{13}{2}-\frac{1}{2} \Rightarrow \boxed{l_{33}=\frac{-37}{8}}$

$$\Rightarrow L=\begin{bmatrix} 2 & 0 & 0 \\ -3 & 4 & 0 \\ 1 & 7 & \frac{-37}{8} \end{bmatrix}, \quad U=\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{8} \\ 0 & 0 & 1 \end{bmatrix}$$

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Let $Ly = b$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ -3 & 4 & 0 \\ 1 & 7 & -37/8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 6 \end{bmatrix}$$

$\Rightarrow 2y_1 = 4 \Rightarrow \boxed{y_1 = 2}$

$-3y_1 + 4y_2 = -3 \Rightarrow -3(2) + 4y_2 = -3$

$\Rightarrow \boxed{y_2 = 3/4}$

$y_1 + 7y_2 - \frac{37}{8}y_3 = 6 \Rightarrow 2 + 7(\frac{3}{4}) - \frac{37}{8}y_3 = 6$

$\Rightarrow \frac{-37}{8}y_3 = -5/4 \Rightarrow y_3 = -5/4 \times \frac{-8}{37}$

$\Rightarrow \boxed{y_3 = \frac{10}{37}}$

So $y = \left[2, 3/4, \frac{10}{37}\right]^T$

Now $Ux = y$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3/4 \\ 10/37 \end{bmatrix}$$

$\Rightarrow \boxed{x_3 = 10/37}$

$x_2 - \frac{1}{8}x_3 = \frac{3}{4} \Rightarrow x_2 - \frac{1}{8}(\frac{10}{37}) = 3/4$

$\Rightarrow \boxed{x_2 = \frac{29}{37}}$

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$x_2 - \frac{1}{8}x_3 = \frac{3}{4} \Rightarrow x_2 - \frac{1}{8}\left(\frac{10}{37}\right) = \frac{3}{4}$

$\Rightarrow \boxed{x_2 = 29/37}$

$x_1 + \frac{1}{2}x_3 = 2 \Rightarrow x_1 + \frac{1}{2}\left(\frac{10}{37}\right) = 2$

$\boxed{x_1 = 69/37}$

$\Rightarrow x = \left[\frac{69}{37}, \frac{29}{37}, \frac{10}{37}\right]$

iii) Cholesky's method:-

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ 0 & 0 & l_{33} \end{bmatrix}, \quad U = \begin{bmatrix} l_{11} & u_{12} & u_{13} \\ 0 & l_{22} & u_{23} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ 0 & 0 & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & u_{12} & u_{13} \\ 0 & l_{22} & u_{23} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -3 & 4 & -2 \\ 1 & 7 & -5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} l_{11}^2 & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}l_{11} & l_{21}u_{12} + l_{22}^2 & l_{21}u_{13} + l_{22}u_{23} \\ l_{31}l_{11} & l_{31}u_{12} + l_{32}l_{22} & l_{31}u_{13} + l_{32}u_{23} + l_{33}^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -3 & 4 & -2 \\ 1 & 7 & -5 \end{bmatrix}$$

$\Rightarrow l_{11}^2 = 2 \Rightarrow \boxed{l_{11} = \sqrt{2}}$

$l_{11}u_{12} = 0 \Rightarrow \sqrt{2}\,u_{12} = 0 \Rightarrow \boxed{u_{12} = 0}$

$l_{11}u_{13} = 1 \Rightarrow \sqrt{2}\,u_{13} = 1 \Rightarrow \boxed{u_{13} = 1/\sqrt{2}}$

$l_{21}u_{12} + l_{22}^2 = 4 \Rightarrow \frac{-3}{\sqrt{2}}(0) + l_{22}^2 = 4 \Rightarrow \boxed{l_{22} = 2}$

$l_{31}u_{12} + l_{32}l_{22} = 7 \Rightarrow l_{31}(0) + l_{32}(2) = 7 \Rightarrow \boxed{l_{32} = 7/2}$

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$l_{21}u_{11} = -3 \Rightarrow l_{21}(\sqrt{2}) = -3 \Rightarrow \boxed{l_{21} = \frac{-3}{\sqrt{2}}}$

$l_{31}u_{12} + l_{32}u_{22} = -2 \Rightarrow \left(\frac{-3}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 2\cdot u_{23} = -2$

$\Rightarrow -\frac{3}{2} + 2u_{23} = -2 \Rightarrow \boxed{u_{23} = -1/4}$

$l_{31}u_{11} = 1 \Rightarrow l_{31}\sqrt{2} = 1 \Rightarrow \boxed{l_{31} = 1/\sqrt{2}}$

$l_{31}u_{13} + l_{32}u_{23} + l_{33}^2 = -5 \Rightarrow \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \frac{7}{2}\left(\frac{-1}{4}\right) + l_{33}^2 = -5$

$\Rightarrow l_{33}^2 = \frac{-37}{8} \Rightarrow \boxed{l_{33} = \sqrt{\frac{-37}{8}}}$

$$\Rightarrow L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{-3}{\sqrt{2}} & 2 & 0 \\ \frac{1}{\sqrt{2}} & 7/2 & \sqrt{\frac{-37}{8}} \end{bmatrix}, \quad U = \begin{bmatrix} \sqrt{2} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 2 & -1/4 \\ 0 & 0 & \sqrt{\frac{-37}{8}} \end{bmatrix}$$

Let $Ly = b$

$$\Rightarrow \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{-3}{\sqrt{2}} & 2 & 0 \\ \frac{1}{\sqrt{2}} & 7/2 & \sqrt{\frac{-37}{8}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 6 \end{bmatrix}$$

$\sqrt{2}\,y_1 = 4 \Rightarrow y_1 = \frac{4}{\sqrt{2}}$

$\frac{-3}{\sqrt{2}}y_1 + 2y_2 = -3 \Rightarrow \frac{-3}{\sqrt{2}}\left(\frac{4}{\sqrt{2}}\right) + 2y_2 = -3$

$\Rightarrow -6 + 2y_2 = -3 \Rightarrow y_2 = 3/2$

$\frac{1}{\sqrt{2}}y_1 + \frac{7}{2}y_2 + \sqrt{\frac{-37}{8}}\,y_3 = 6$

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$$\Rightarrow \frac{1}{\sqrt{2}}\left(\frac{4}{\sqrt{2}}\right)+\frac{7}{2}\left(\frac{3}{2}\right)+\sqrt{\frac{-37}{8}}\,y_3 = 6$$

$$\Rightarrow 2+\frac{21}{4}+\sqrt{\frac{-37}{8}}\,y_3 = 6$$

$$\Rightarrow \sqrt{\frac{-37}{8}}\,y_3 = -5/4$$

$$\Rightarrow y_3 = \sqrt{\frac{-200}{592}}$$

Now $Ux = y$

$$\Rightarrow \begin{bmatrix} \sqrt{2} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 2 & -1/4 \\ 0 & 0 & \sqrt{-37/8} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{2} \\ 3/2 \\ \sqrt{-200/592} \end{bmatrix}$$

$$\Rightarrow \sqrt{\frac{-37}{8}}\,x_3 = \sqrt{\frac{-200}{592}} \Rightarrow \boxed{x_3 = \frac{10}{37}}$$

$$2x_2 - \frac{1}{4}x_3 = \frac{3}{2} \Rightarrow 2x_2 - \frac{1}{4}\left(\frac{10}{37}\right) = 3/2$$

$$\Rightarrow 2x_2 = 58/37 \qquad \boxed{x_2 = 29/37}$$

$$\sqrt{2}\,x_1 + \frac{1}{\sqrt{2}}x_3 = \frac{4}{\sqrt{2}} \Rightarrow \sqrt{2}\,x_1 + \frac{1}{\sqrt{2}}\left(\frac{10}{37}\right) = \frac{4}{\sqrt{2}}$$

$$\Rightarrow x_1 + \frac{1}{2}\left(\frac{10}{37}\right) = 4/2$$

$$\Rightarrow \boxed{x_1 = \frac{69}{37}}$$

So $x = \left[\frac{69}{37}, \frac{29}{37}, \frac{10}{37}\right]^T$

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★ INDIRECT METHOD:- OR -: Iterative Method.

There are two methods to solve system of linear equations indirectly.

(i) Jacobi Iterative Method.

(ii) Gauss Seidal Method.

i) ~~Jacobi~~ Jacobi Iterative Method:-

Consider the linear system

$$A\underline{x} = \underline{b}$$

Rearrange the equations so that diagonal elements and equations are diagonally dominent.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \longrightarrow (i)$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \longrightarrow (ii)$$

$$a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = b_3 \longrightarrow (iii)$$

$$\cdots\cdots\cdots\cdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \longrightarrow (n)$$

Find x_1 from equ ①, x_2 from equ ②, x_3 from equ ③ & continuing from equ (n) find value of x_n

$$\Rightarrow x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n\right]$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n\right]$$

& so on $\quad x_n = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{nn-1}x_{n-1}\right]$

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In general we write it as

$$x_i = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j=1 \\ i\neq i}}^{n} a_{ij}\, x_j\right], \quad i = 1, 2, 3, \cdots n$$

Setting Iteration.

$$x_1^{(K+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(K)} - a_{13}x_3^{(K)} - \cdots - a_{1n}x_n^{(K)}\right]$$

$$x_2^{(K+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(K)} - a_{23}x_3^{(K)} - \cdots - a_{2n}x_n^{(K)}\right]$$

$$x_3^{(K+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(K)} - a_{32}x_2^{(K)} - \cdots - a_{3n}x_n^{(K)}\right]$$

$$x_n^{(K+1)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(K)} - a_{n2}x_2^{(K)} - \cdots - a_{nn-1}x_{n-1}^{(K)}\right]$$

In general we write it as setting iteration

$$x_i^{(K+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}\, x_j^{(K)}\right], \quad i = 1, 2, \cdots, n$$

is the required formula for Jacobi's method. Starting from an initial guess, successive solution vector are computed until convergence has been obtained.

★ Condition of Convergence for a Linear System of equation [Convergence Criteria For Jacobi's Method] The condition of convergence for system of linear

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equations is that linear system is diagonally dominent. The linear system of equations is said to be diagonally dominant if in each equation the absolute value of diagonal co-efficient is greater than or equal to the sum of absolute values of off-digonal co-efficients and is atleast strictly greater than one off diagonal coefficient. i.e

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \text{for all } i$$

$$\text{or} \quad |a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \text{for atleast on } i$$

The condition is sufficient but not necessary condition for convergence. that is, if the co-efficients satisfy this condition then method must converge. If not, the convergence is not generated. It may not and probably will not converge.

* *

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Example:- Solve System of equations by Jacobi's Method.

$$3x_1 + 10x_2 + 2x_3 = 29$$
$$x_1 + 2x_2 + 10x_3 = 35$$
$$10x_1 + 3x_2 + x_3 = 19$$

Solution:-

First we write the given system so as it becomes diagonally dominate.

$$\Rightarrow \quad 10x_1 + 3x_2 + x_3 = 19$$
$$3x_1 + 10x_2 + 2x_3 = 29$$
$$x_1 + 2x_2 + 10x_3 = 35$$

Now it is diagonally dominent.

Now the system can be written as.

$$x_1 = \frac{1}{10}[19 - 3x_2 - x_3]$$
$$x_2 = \frac{1}{10}[29 - 3x_1 - 2x_3]$$
$$x_3 = \frac{1}{10}[35 - x_1 - 2x_2]$$

Now Setting Iterations.

$$\left.\begin{aligned} x_1^{(K+1)} &= \frac{1}{10}[19 - 3x_2^{(K)} - x_3^{(K)}] \\ x_2^{(K+1)} &= \frac{1}{10}[29 - 3x_1^{(K)} - 2x_3^{(K)}] \\ x_3^{(K+1)} &= \frac{1}{10}[35 - x_1^{(K)} - 2x_2^{(K)}] \end{aligned}\right\} \longrightarrow (*)$$

Let $x_1^{(0)} = 1$ & $x_2^{(0)} = 1$ & $x_3^{(0)} = 1$ be initial guess

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1st Iteration:-

Put $k=0$ in (*)

$$\Rightarrow x_1^{(1)} = \frac{1}{10}\left[19 - 3x_2^{(0)} - x_3^{(0)}\right] = \frac{1}{10}\left[19 - 3(1) - 1\right] = 1.5$$

$$x_2^{(1)} = \frac{1}{10}\left[29 - 3x_1^{(0)} - 2x_3^{(0)}\right] = \frac{1}{10}\left[29 - 3(1) - 2(1)\right] = 2.4$$

$$x_3^{(1)} = \frac{1}{10}\left[35 - x_1^{(0)} - 2(x_2^{(0)})\right] = \frac{1}{10}\left[35 - 1 - 2(1)\right] = 3.2$$

2nd Iteration:- Put $k=1$ in (*)

$$\Rightarrow x_1^{(2)} = \frac{1}{10}\left[19 - 3x_2^{(1)} - x_3^{(1)}\right] = \frac{1}{10}\left[19 - 3(2.4) - 3.2\right] = 0.86$$

$$x_2^{(2)} = \frac{1}{10}\left[29 - 3x_1^{(1)} - 2x_3^{(1)}\right] = \frac{1}{10}\left[29 - 3(1.5) - 2(3.2)\right] = 1.81$$

$$x_3^{(2)} = \frac{1}{10}\left[35 - x_1^{(1)} - 2(x_2^{(1)})\right] = \frac{1}{10}\left[35 - 1.5 - 2(2.4)\right] = 2.87$$

3rd Iteration:- Put $k=2$ in (*)

$$\Rightarrow x_1^{(3)} = \frac{1}{10}\left[19 - 3x_2^{(2)} - x_3^{(2)}\right] = \frac{1}{10}\left[19 - 3(1.81) - 2.87\right] = 1.07$$

$$x_2^{(3)} = \frac{1}{10}\left[29 - 3x_1^{(2)} - 2x_3^{(2)}\right] = \frac{1}{10}\left[29 - 3(0.86) - 2(2.87)\right] = 2.068$$

$$x_3^{(3)} = \frac{1}{10}\left[35 - x_1^{(2)} - 2x_2^{(2)}\right] = \frac{1}{10}\left[35 - 0.86 - 2(1.81)\right] = 3.052$$

4th Iteration:- Put $k=3$ in (*)

$$\Rightarrow x_1^{(4)} = \frac{1}{10}\left[19 - 3x_2^{(3)} - x_3^{(3)}\right] = \frac{1}{10}\left[19 - 3(2.068) - 3.052\right] = 0.9744$$

$$x_2^{(4)} = \frac{1}{10}\left[29 - 3x_1^{(3)} - 2x_3^{(3)}\right] = \frac{1}{10}\left[29 - 3(1.07) - 2(3.052)\right] = 1.9686$$

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$$x_3^{(4)} = \frac{1}{10}\left[35 - x_1^{(3)} - 2x_2^{(3)}\right] = \frac{1}{10}\left[35 - 1.07 - 2(2.068)\right] = 2.9794$$

Now Consider the Table

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	1	-1	1
1	1.5	2.4	3.2
2	0.86	1.81	2.87
3	1.07	2.068	3.052
4	0.9744	1.9686	2.9794
5	1.0115	2.0118	3.0088
6	0.9956	1.9948	2.9965
7	1.0019	2.0020	3.0015
8	0.9993	1.9991	2.9994
9	1.0003	2.0003	3.0002
10	0.9999	1.9999	2.9999
11	1	2	3
12	1	2	3

Hence Solution vector is $[1, 2, 3]^T$

***

Example:- Consider

$$4x - y + z = 7$$
$$4x - 8y + z = -21$$
$$-2x + y + 5z = 15$$

Solve by Jacobi's method.

Solution:-

Rewrite these equations we have

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$$x = \frac{1}{4}[7 + y - z]$$

$$y = \frac{1}{8}[21 + 4x + z]$$

$$z = \frac{1}{5}[15 + 2x - y]$$

Setting iteration is

$$\left.\begin{aligned} x^{(K+1)} &= \frac{1}{4}[7 + y^{(K)} - z^{(K)}] \\ y^{(K+1)} &= \frac{1}{8}[21 + 4x^{(K)} + z^{(K)}] \\ z^{(K+1)} &= \frac{1}{5}[15 + 2x^{(K)} - y^{(K)}] \end{aligned}\right\} \quad \text{---} \circledR$$

Let Guess vector is $[1, 2, 2]^T$

1st Iteration:- Put $K=0$ in ®

$$\Rightarrow x^{(1)} = \frac{1}{4}[7 + y^{(0)} - z^{(0)}] = \frac{1}{4}[7 + 2 - 2] = 1.75$$

$$y^{(1)} = \frac{1}{8}[21 + 4x^{(0)} + z^{(0)}] = \frac{1}{8}[21 + 4(1) + 2] = 3.375$$

$$z^{(1)} = \frac{1}{5}[15 + 2x^{(0)} - y^{(0)}] = \frac{1}{5}[15 + 2(1) - 2] = 3$$

2nd Iteration:- Put $K=1$ in ®

$$\Rightarrow x^{(2)} = \frac{1}{4}[7 + y^{(1)} - z^{(1)}] = \frac{1}{4}[7 + 3.375 - 3] = 1.8438$$

$$y^{(2)} = \frac{1}{8}[21 + 4x^{(1)} + z^{(1)}] = \frac{1}{8}[21 + 4(1.75) + 3] = 3.875$$

$$z^{(2)} = \frac{1}{5}[15 + 2x^{(1)} - y^{(1)}] = \frac{1}{5}[15 + 2(1.75) - 3.375] = 3.025$$

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3rd Iteration:- Put $k=2$ in (*)

$$x^{(3)} = \frac{1}{4}\left[7+y^{(2)}-z^{(2)}\right] = \frac{1}{4}\left[7+3.875-3.025\right] = 1.9625$$

$$y^{(3)} = \frac{1}{8}\left[21+4x^{(2)}+z^{(2)}\right] = \frac{1}{8}\left[21+4(1.8438)+3.025\right] = 3.9250$$

$$z^{(3)} = \frac{1}{5}\left[15+2x^{(2)}-y^{(2)}\right] = \frac{1}{5}\left[15+2(1.8438)-3.875\right] = 2.9625$$

Now consider Table

K	$x^{(k)}$	$y^{(k)}$	$z^{(k)}$
0	1	2	2
1	1.75	3.375	3
2	1.8438	3.875	3.025
3	1.9625	3.9250	2.9625
4	1.9906	3.9766	3
5	1.9941	3.9953	3.0009
6	1.9939	3.9972	2.9986
7	1.9997	3.9968	3.1978
8	1.9498	4.0246	3.0005
9	2.0060	3.9750	3.1798
10	1.9488	4.0255	3.0074
11	2.0045	3.9753	2.9744
12	2.0002	3.9991	3.0067
13	1.998	4.0009	3.1987
14	1.9506	4.0239	2.9991
15	2.0062	3.9752	2.9759
16	2	4	3
17	2	4	3

So solution vector $= [2, 4, 3]^T$

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ii) Gauss Seidal Method:-

Consider the linear system $Ax = b$. Re arrange equations so that diagonal elements are not zero and equation are diagonally dominent. Consider system of equations.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3 \\ \vdots \quad\quad \vdots \quad\quad \cdots \quad\quad \vdots \quad\quad &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \quad \text{---} (1)$$

Find x_1 from 1st equation, x_2 from 2nd equation & so on find x_n. The system (1) can be written as

$$x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n\right]$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n\right]$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$x_n = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{nn-1}x_{n-1}\right]$$

Setting iteration we get

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \cdots - a_{1n}x_n^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - \cdots - a_{2n}x_n^{(k)}\right]$$

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$$x_3^{(k+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)} - a_{34}x_4^{(k)} - \cdots - a_{3n}x_n^{(k)}\right]$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$x_n^{(k+1)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - \cdots - a_{nn-1}x_{n-1}^{(k+1)}\right]$$

In general it also can be written as

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right]$$

$$i = 1, 2, 3, \cdots, n$$

Starting from an initial guess, successive solution vectors are computed until convergence have been obtained.

Remark:- Guass Seidal method converges twice as fast as Jacobi's method.

* * *

Example:- Solve by Guass Seidal method

$$10x_1 + 3x_2 + x_3 = 19$$
$$3x_1 + 10x_2 + 2x_3 = 29$$
$$x_1 + 2x_2 + 10x_3 = 35$$

Solution:-

Equations are diagonally dominent

$$\Rightarrow \quad x_1 = \frac{1}{10}\left[19 - 3x_2 - x_3\right]$$

$$x_2 = \frac{1}{10}\left[29 - 3x_1 - 2x_3\right]$$

$$x_3 = \frac{1}{10}\left[35 - x_1 - 2x_2\right]$$

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Setting Iterations are

$$x_1^{(k+1)} = \frac{1}{10}\left[19 - 3x_2^{(k)} - x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{10}\left[29 - 3x_1^{(k+1)} - 2x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{10}\left[35 - x_1^{(k+1)} - 2x_2^{(k+1)}\right] \quad \text{---------- (*)}$$

Let $x^0 = [0, 0, 0]^T$ be initial guess.

1st Iteration:- Put $k=0$ in (*)

$$\Rightarrow x_1^{(1)} = \frac{1}{10}\left[19 - 3x_2^{(0)} - x_3^{(0)}\right] = \frac{1}{10}[19 - 3(0) - 0] = 1.9$$

$$x_2^{(1)} = \frac{1}{10}\left[29 - 3x_1^{(1)} - 2x_3^{(0)}\right] = \frac{1}{10}[29 - 3(1.9) - 2(0)] = 2.33$$

$$x_3^{(1)} = \frac{1}{10}[35 - 1.9 - 2(2.33)] = 2.844$$

2nd Iteration:- Put $k=1$ in (*)

$$\Rightarrow x_1^{(2)} = \frac{1}{10}\left[19 - 3x_2^{(1)} - x_3^{(1)}\right] = \frac{1}{10}[19 - 3(2.33) - 2.844] = 0.9166$$

$$x_2^{(2)} = \frac{1}{10}\left[29 - 3x_1^{(2)} - 2x_3^{(1)}\right] = \frac{1}{10}[29 - 3(0.9166) - 2(2.844)] = 2.0562$$

$$x_3^{(2)} = \frac{1}{10}\left[35 - x_1^{(2)} - 2x_2^{(2)}\right] = \frac{1}{10}[35 - 0.9166 - 2(2.0562)] = 2.9971$$

3rd Iteration:- Put $k=2$ in (*)

$$\Rightarrow x_1^{(3)} = \frac{1}{10}\left[19 - 3x_2^{(2)} - x_3^{(2)}\right] = \frac{1}{10}[19 - 3(2.0562) - 2.9971] = 0.9834$$

$$x_2^{(3)} = \frac{1}{10}\left[29 - 3x_1^{(3)} - 2x_3^{(2)}\right] = \frac{1}{10}[29 - 3(0.9834) - 2(2.9971)] = 2.0056$$

$$x_3^{(3)} = \frac{1}{10}\left[35 - x_1^{(3)} - 2x_2^{(3)}\right] = \frac{1}{10}[35 - 0.9834 - 2(2.0056)] = 3.0005$$

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4th Iteration:- Put $k=3$ in (*)

$$\Rightarrow x_1^{(4)} = \frac{1}{10}\left[19 - 3x_2^{(3)} - x_3^{(3)}\right] = \frac{1}{10}\left[19 - 3(2.0056) - 3.0005\right]$$
$$= 0.9983$$

$$x_2^{(4)} = \frac{1}{10}\left[29 - 3x_1^{(4)} - 2x_3^{(3)}\right] = \frac{1}{10}\left[29 - 3(0.9983) - 2(3.0005)\right]$$
$$= 2.0004$$

$$x_3^{(4)} = \frac{1}{10}\left[35 - x_1^{(4)} - 2x_2^{(4)}\right] = \frac{1}{10}\left[35 - 0.9983 - 2(2.0004)\right]$$
$$= 3$$

5th Iteration:- Put $k=4$ in (*)

$$\Rightarrow x_1^{(5)} = \frac{1}{10}\left[19 - 3x_2^{(4)} - x_3^{(4)}\right] = \frac{1}{10}\left[19 - 3(2.0004) - 3\right]$$
$$= 0.9999$$

$$x_2^{(5)} = \frac{1}{10}\left[29 - 3x_1^{(5)} - 2x_3^{(4)}\right] = \frac{1}{10}\left[29 - 3(0.9999) - 2(3)\right] = 2$$

$$x_3^{(5)} = \frac{1}{10}\left[35 - x_1^{(5)} - 2x_2^{(5)}\right] = \frac{1}{10}\left[35 - 0.9999 - 2(2)\right] = 3$$

6th Iteration:- Put $k=5$ in (*)

$$\Rightarrow x_1^{(6)} = \frac{1}{10}\left[19 - 3x_2^{(5)} - x_3^{(5)}\right] = \frac{1}{10}\left[19 - 3(2) - 3\right] = 1$$

$$x_2^{(6)} = \frac{1}{10}\left[29 - 3x_1^{(6)} - 2x_3^{(4)}\right] = \frac{1}{10}\left[29 - 3(1) - 2(3)\right] = 2$$

$$x_3^{(6)} = \frac{1}{10}\left[35 - x_1^{(6)} - 2x_2^{(6)}\right] = \frac{1}{10}\left[35 - 1 - 2(2)\right] = 3$$

Hence solution vector is

$$\underline{x} = [1, 2, 3]^T$$

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Example:- Solve by Guass Seidal method.

$$10x_1 + x_2 + 2x_3 = 44$$
$$x_1 + 2x_2 + 10x_3 = 61$$
$$2x_1 + 10x_2 + x_3 = 51$$

Solution:- As equations are not diagonally dominent, Re-arranging we have

$$10x_1 + x_2 + 2x_3 = 44$$
$$2x_1 + 10x_2 + x_3 = 51$$
$$x_1 + 2x_3 + 10x_3 = 61$$

$$\Rightarrow x_1 = \tfrac{1}{10}[44 - x_2 - 2x_3]$$

$$x_2 = \tfrac{1}{10}[51 - 2x_1 - x_3]$$

$$x_3 = \tfrac{1}{10}[61 - x_1 - 2x_2]$$

Iterative equations are

$$\left.\begin{aligned} x_1^{(K+1)} &= \frac{1}{10}\left[44 - x_2^{(K)} - 2x_3^{(K)}\right] \\ x_2^{(K+1)} &= \frac{1}{10}\left[51 - 2x_1^{(K+1)} - x_3^{(K)}\right] \\ x_3^{(K+1)} &= \frac{1}{10}\left[61 - x_1^{(K+1)} - 2x_2^{(K+1)}\right] \end{aligned}\right\} \quad \text{---- (R)}$$

Let initial vector is $x^{(0)} = [1, 1, 1]^T$

1st Iteration:- Put $K = 0$ in (R)

$$\Rightarrow x_1^{(1)} = \frac{1}{10}\left[44 - x_2^{(0)} - 2x_3^{(0)}\right] = \frac{1}{10}[44 - 1 - 2(1)]$$

$$= 4.1$$

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$$x_2^{(1)} = \frac{1}{10}[51 - 2x_1^{(1)} - x_3^{(0)}] = \frac{1}{10}[51 - 2(4.1) - 1] = 4.18$$

$$x_3^{(1)} = \frac{1}{10}[61 - x_1^{(1)} - 2x_2^{(1)}] = \frac{1}{10}[61 - 4.1 - 2(4.18)] = 4.854$$

2nd Iteration:- Put $k=1$ in ⊛

$$\Rightarrow x_1^{(2)} = \frac{1}{10}[44 - x_2^{(1)} - 2x_3^{(1)}] = \frac{1}{10}[44 - 4.18 - 2(4.854)] = 3.0112$$

$$x_2^{(2)} = \frac{1}{10}[51 - 2x_1^{(2)} - x_3^{(1)}] = \frac{1}{10}[51 - 2(3.0112) - 4.854] = 4.0124$$

$$x_3^{(2)} = \frac{1}{10}[61 - x_1^{(2)} - 2x_2^{(2)}] = \frac{1}{10}[61 - 3.0112 - 2(4.0124)] = 4.9964$$

3rd Iteration:- Put $k=2$ in ⊛

$$\Rightarrow x_1^{(3)} = \frac{1}{10}[44 - x_2^{(2)} - 2x_3^{(2)}] = \frac{1}{10}[44 - 4.0124 - 2(4.9964)] = 2.9995$$

$$x_2^{(3)} = \frac{1}{10}[51 - 2x_1^{(3)} - x_3^{(2)}] = \frac{1}{10}[51 - 2(2.9995) - 4.9964] = 4.0005$$

$$x_3^{(3)} = \frac{1}{10}[61 - x_1^{(3)} - 2x_2^{(3)}] = \frac{1}{10}[61 - \underset{2.9995}{4.0005} - 2(4.0005)] = 5$$

4th Iteration:- Put $k=3$ in ⊛

$$\Rightarrow x_1^{(4)} = \frac{1}{10}[44 - x_2^{(3)} - 2x_3^{(3)}] = \frac{1}{10}[44 - 4.0005 - 2(5)] = 3$$

$$x_2^{(4)} = \frac{1}{10}[51 - 2x_1^{(4)} - x_3^{(3)}] = \frac{1}{10}[51 - 2(3) - 5] = 4$$

$$x_3^{(4)} = \frac{1}{10}[61 - x_1^{(4)} - 2(x_2)^{(4)}] = \frac{1}{10}[61 - 3 - 2(4)] = 5$$

5th Iteration:- Put $k=4$ in ⊛

$$x_1^{(5)} = \frac{1}{10}[44 - x_2^{(4)} - 2x_3^{(4)}] = \frac{1}{10}[44 - 4 - 2(5)] = 3$$

$$x_2^{(5)} = \frac{1}{10}[51 - 2x_1^{(5)} - x_3^{(4)}] = \frac{1}{10}[51 - 2(3) - 5] = 4$$

$$x_3^{(5)} = \frac{1}{10}[61 - x_1^{(5)} - 2x_2^{(5)}] = \frac{1}{10}[61 - 3 - 2(4)] = 5$$

So required vector is $\underline{x} = [3, 4, 5]^T$

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ASSIGNMENT

Q1:- Explain Jacobi's method to solve iterative method.

Q2:- Explain Guass Seidal method and write advantages of Gauss Seidal method.

Q3:- Write general iterative formula for both Jacobi's and Gauss Seidal methods for solving a system of linear equation.

Q4:- Describe the convergence criteria for a system of linear equation.

Q5:- Solve by Jacobi's method. Carry out three iterations. Re arrange equation for convergence.

$$5x_1 + 2x_2 + x_3 = 14$$
$$x_1 + x_2 + 8x_3 = 20$$
$$x_1 + 5x_2 - x_3 = 10$$

Q6:- Apply 3 iteration of Gauss Seidal method

$$8x_1 + x_2 - x_3 = 8$$
$$3x_1 - x_2 + 9x_3 = 12$$
$$x_1 - 7x_2 + 2x_3 = -4$$

Q7:- Apply 4 iteration of Gauss Seidal method after arranging equations

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for convergence.

$$x_1 + 2x_2 + 5x_3 = 2$$
$$6x_1 - x_2 - x_3 = 3$$
$$2x_1 + 7x_3 - x_3 = -3$$

Q0:-

Apply Jacobi's method

$$4x_1 + x_2 - x_3 = 0$$
$$x_1 - 4x_2 + 2x_3 = 13$$
$$2x_1 + x_2 + 5x_3 = 10$$

★ ——— ★ ——— ★

SOLUTION

Q no 1 to 4:- Already Solved.

★ ——— ★ ——— ★

Q5:- As equations are not diagonally dominent so we re-arrange equations and make them diagonally dominent.

By Jacobi Method

$$\Rightarrow \quad 5x_1 + 2x_2 + x_3 = 14$$
$$x_1 + 5x_2 - x_3 = 10$$
$$x_1 + x_2 + 8x_3 = 20$$

$$\Rightarrow \quad x_1 = \frac{1}{5}[14 - 2x_2 - x_3]$$
$$x_2 = \frac{1}{5}[10 - x_1 + x_3]$$
$$x_3 = \frac{1}{8}[20 - x_1 - x_2]$$

Iterative equations are

$$x_1^{(K+1)} = \frac{1}{5}[14 - 2x_2^{(K)} - x_3^{(K)}]$$
$$x_2^{(K+1)} = \frac{1}{5}[10 - x_1^{(K)} + x_3^{(K)}] \quad \text{(★)}$$

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$$x_3^{(k+1)} = \frac{1}{8}\left[20 - x_1^{(k)} - x_2^{(k)}\right] \longrightarrow (*)$$

Let $x^{(0)} = [0,0,0]$ be initial guess

1st Iteration:- Put $k=0$ in (*)

$$\Rightarrow x_1^{(1)} = \frac{1}{5}\left[14 - 2x_2^{(0)} - x_3^{(0)}\right] = \frac{1}{5}[14 - 2(0) - 0] = 2.8$$

$$x_2^{(1)} = \frac{1}{5}\left[10 - x_1^{(0)} + x_3^{(0)}\right] = \frac{1}{5}[10 - 0 + 0] = 2$$

$$x_3^{(1)} = \frac{1}{8}\left[20 - x_1^{(0)} - x_2^{(0)}\right] = \frac{1}{8}[20 - 0 - 0] = 2.5$$

2nd Iteration:- Put $k=1$ in (*)

$$x_1^{(2)} = \frac{1}{5}\left[14 - 2x_2^{(1)} - x_3^{(1)}\right] = \frac{1}{5}[14 - 2(2) - 2.5] = 1.5$$

$$x_2^{(2)} = \frac{1}{5}\left[10 - x_1^{(1)} + x_3^{(1)}\right] = \frac{1}{5}[10 - 2.8 + 2.5] = 1.94$$

$$x_3^{(2)} = \frac{1}{8}\left[20 - x_1^{(1)} - x_2^{(1)}\right] = \frac{1}{8}[20 - 2.8 - 2] = 1.9$$

3rd Iteration:- Put $k=2$ in (*)

$$x_1^{(3)} = \frac{1}{5}\left[14 - 2x_2^{(2)} - x_3^{(2)}\right] = \frac{1}{5}[14 - 2(1.94) - 1.9] = 1.644$$

$$x_2^{(3)} = \frac{1}{5}\left[10 - x_1^{(2)} + x_3^{(2)}\right] = \frac{1}{5}[10 - 1.5 + 1.9] = 2.08$$

$$x_3^{(3)} = \frac{1}{8}\left[20 - x_1^{(2)} - x_2^{(2)}\right] = \frac{1}{8}[20 - 1.5 - 1.94] = 2.07$$

Now consider the table

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	2.8	2	2.5
2	1.5	1.94	1.9
3	1.644	2.08	2.07
4	1.544	2.0852	2.0345

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K	$x_1^{(K)}$	$x_2^{(K)}$	$x_3^{(K)}$
5	1.5590	2.0981	2.0464
6	1.5515	2.0975	2.0429
7	1.5524	2.0983	2.0439
8	1.5519	2.0983	2.0437
9	1.5519	2.0984	2.0437
10	1.5519	2.0984	2.0437

So Required Solution vector is

$$\underline{x} = [1.5519, 2.0984, 2.0437]^T$$

* ——— * ——— *

Q6:- First we rearrange the equations so that they become diagonally dominent

By Guass Seidal Method

$\Rightarrow$

$$8x_1 + x_2 - x_3 = 8$$
$$x_1 - 7x_2 + 2x_3 = -4$$
$$3x_1 - x_2 + 9x_3 = 12$$

$\Rightarrow$ $x_1 = \frac{1}{8}[8 - x_2 + x_3]$

$x_2 = -\frac{1}{7}[-4 - x_1 - 2x_3] \Rightarrow x_2 = \frac{1}{7}[4 + x_1 + 2x_3]$

$x_3 = \frac{1}{9}[12 - 3x_1 + x_2]$

Iterative equations are

$$x_1^{(K+1)} = \frac{1}{8}[8 + x_2^{(K)} + x_3^{(K)}]$$
$$x_2^{(K+1)} = \frac{1}{7}[4 - x_1^{(K+1)} + 2x_3^{(K)}]$$
$$x_3^{(K+1)} = \frac{1}{9}[12 - 3x_1^{(K+1)} + x_2^{(K+1)}]$$

$\longrightarrow$ (*)

Let $x^{(0)} = [1, 1, 1]$ be initial guess

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1st Iteration:- Put $K=0$ in (*)

$$\Rightarrow x_1^{(1)} = \frac{1}{8}\left[8 - x_2^{(0)} - x_3^{(0)}\right] = \frac{1}{8}[8-1+1] = 1$$

$$x_2^{(1)} = \frac{1}{7}\left[4 - x_1^{(1)} + 2x_3^{(0)}\right] = \frac{1}{7}[4-1+2(1)] = 0.7143$$

$$x_3^{(1)} = \frac{1}{9}\left[12 - 3x_1^{(1)} + x_2^{(1)}\right] = \frac{1}{9}[12-3(1)+0.7143] = 1.0794$$

2nd Iteration:- Put $K=1$ in (*)

$$\Rightarrow x_1^{(2)} = \frac{1}{8}\left[8 - x_2^{(1)} + x_3^{(1)}\right] = \frac{1}{8}[8-0.7143+1.0794] = 1.0456$$

$$x_2^{(2)} = \frac{1}{7}\left[4 - x_1^{(2)} + 2x_3^{(1)}\right] = \frac{1}{7}[4-1.0456+2(1.0794)] = 0.7305$$

$$x_3^{(2)} = \frac{1}{9}\left[12 - 3x_1^{(2)} + x_2^{(2)}\right] = \frac{1}{9}[12-3(1.0456)+0.7305] = 1.066$$

3rd Iteration:- Put $K=2$ in (*)

$$x_1^{(3)} = \frac{1}{8}\left[8 - x_2^{(2)} + x_3^{(2)}\right] = \frac{1}{8}[8-0.7305+1.066] = 1.042$$

$$x_2^{(3)} = \frac{1}{7}\left[4 - x_1^{(3)} + 2x_3^{(2)}\right] = \frac{1}{7}[4-1.042+2(1.066)] = 0.7271$$

$$x_3^{(3)} = \frac{1}{9}\left[12 - 3x_1^{(3)} + x_2^{(3)}\right] = \frac{1}{9}[12-3(1.042)+0.7271] = 1.0668$$

Now Consider the Table

K	$x_1^{(K)}$	$x_2^{(K)}$	$x_3^{(K)}$
0	1	1	1
1	1	0.7143	1.0794
2	1.0456	0.7305	1.066
3	1.042	0.7271	1.0668
4	1.0425	0.7273	1.0666
5	1.0424	0.7273	1.0666
6	1.0424	0.7273	1.0666

So Solution is $X = [1.0424, 0.7273, 1.0666]^T$

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Q 7:- By Guass Seidal Method

As equations are not diagonally dominent. So we Re-arrange the equations

$$\Rightarrow \quad 6x_1 - x_2 - x_3 = 3$$
$$2x_1 + 7x_2 - x_3 = -3$$
$$x_1 + 2x_2 + 5x_3 = 2$$

$$\Rightarrow \quad x_1 = \frac{1}{6}[3 + x_2 + x_3]$$
$$x_2 = \frac{1}{7}[-3 - 2x_1 + x_3]$$
$$x_3 = \frac{1}{5}[2 - x_1 - 2x_2]$$

Iterative equations are

$$\left.\begin{aligned} x_1^{(k+1)} &= \frac{1}{6}\left[3 + x_2^{(k)} + x_3^{(k)}\right] \\ x_2^{(k+1)} &= \frac{1}{7}\left[-3 - 2x_1^{(k+1)} + x_3^{(k)}\right] \\ x_3^{(k+1)} &= \frac{1}{5}\left[2 - x_1^{(k+1)} - 2x_2^{(k+1)}\right] \end{aligned}\right\} \longrightarrow (*)$$

Let $x^{(0)} = [0, 0, 0]^T$ be initial guess

1st Iteration:- Put $k = 0$ in (*)

$$x_1^{(1)} = \frac{1}{6}\left[3 + x_2^{(0)} + x_3^{(0)}\right] = \frac{1}{6}[3 + 0 + 0] = 0.5$$

$$x_2^{(1)} = \frac{1}{7}\left[-3 - 2x_1^{(1)} + x_3\right] = \frac{1}{7}[-3 - 2(0.5) + 0] = -0.5714$$

$$x_3^{(1)} = \frac{1}{5}\left[2 - x_1^{(1)} - 2(x_2^{(1)}\right] = \frac{1}{5}[2 - 0.5 - 2(-0.5714)] = 0.5286$$

2nd Iteration:- Put $k = 1$ in (*)

$$x_1^{(2)} = \frac{1}{6}\left[3 + x_2^{(1)} + x_3^{(1)}\right] = \frac{1}{6}[3 - 0.5714 + 0.5286] = 0.4929$$

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$$x_2^{(2)} = \frac{1}{7}\left[-3-2x_1^{(2)}+x_3^{(1)}\right] = \frac{1}{7}\left[-3-2(0.4929)+0.5286\right] = -0.4939$$

$$x_3^{(2)} = \frac{1}{5}\left[2-x_1^{(2)}-2x_2^{(2)}\right] = \frac{1}{5}\left[2-0.4929-2(-0.4939)\right] = 0.499$$

3rd Iteration:- Put $K=2$ in (*)

$$x_1^{(3)} = \frac{1}{6}\left[3+x_2^{(2)}+x_3^{(2)}\right] = \frac{1}{6}\left[3+(-0.4939)+0.499\right] = 0.5009$$

$$x_2^{(3)} = \frac{1}{7}\left[-3-2x_1^{(3)}+x_3^{(2)}\right] = \frac{1}{7}\left[-3-2(0.5009)+0.499\right] = -0.5004$$

$$x_3^{(3)} = \frac{1}{5}\left[2-x_1^{(3)}-2x_2^{(3)}\right] = \frac{1}{5}\left[2-0.5009-2(-0.5004)\right] = 0.5$$

4th Iteration:- Put $K=3$ in (*)

$$x_1^{(4)} = \frac{1}{6}\left[3+x_2^{(3)}+x_3^{(3)}\right] = \frac{1}{6}\left[3-0.5004+0.5\right] = 0.4999$$

$$x_2^{(4)} = \frac{1}{7}\left[-3-2x_1^{(4)}+x_3^{(3)}\right] = \frac{1}{7}\left[-3-2(0.4999)+0.5\right] = -0.5$$

$$x_3^{(4)} = \frac{1}{5}\left[2-x_1^{(4)}-2x_2^{(4)}\right] = \frac{1}{5}\left[2-0.4999-2(-0.5)\right] = 0.5$$

5th Iteration:- Put $K=4$ in (*)

$$\Rightarrow x_1^{(5)} = \frac{1}{6}\left[3+x_2^{(4)}+x_3^{(4)}\right] = \frac{1}{6}\left[3+(-0.5)+0.5\right] = 0.5$$

$$x_2^{(5)} = \frac{1}{7}\left[-3-2x_1^{(5)}+x_3^{(4)}\right] = \frac{1}{7}\left[-3-2(0.5)+.5\right] = -0.5$$

$$x_3^{(5)} = \frac{1}{5}\left[2-x_1^{(5)}-2(x_2)^{(5)}\right] = \frac{1}{5}\left[2-0.5-2(-0.5)\right] = 0.5$$

So Solution vector is

$$\underline{x} = \left[0.5, -0.5, 0.5\right]^T$$

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Q8:- Jacobi's Method

$$4x_1 + x_2 - x_3 = 0$$
$$x_1 - 4x_2 + 2x_3 = 13$$
$$2x_1 + x_2 + 5x_3 = 10$$

$$\Rightarrow \quad x_1 = \frac{1}{4}[-x_2 + x_3]$$
$$x_2 = \frac{-1}{4}[13 - x_1 - 2x_3]$$
$$x_3 = \frac{1}{5}[10 - 2x_1 - x_2]$$

Iterative equations are.

$$x_1^{(k+1)} = \frac{1}{4}[-x_2^{(k)} + x_3^{(k)}]$$
$$x_2^{(k+1)} = \frac{-1}{4}[13 - x_1^{(k)} - 2x_3^{(k)}]$$
$$x_3^{(k+1)} = \frac{1}{5}[10 - 2x_1^{(k)} - x_2^{(k)}]$$

Let $x^{(0)} = [0, 0, 0]$ be initial guess

1st Iteration:- Put $k = 0$ in (*)

$$x_1^{(1)} = \frac{1}{4}[-x_2^{(0)} + x_3^{(0)}] = \frac{1}{4}(0 + 0) = 0$$
$$x_2^{(1)} = \frac{-1}{4}[13 - x_1^{(0)} - 2x_3^{(0)}] = \frac{-1}{4}[13 - 0 - 2(0)] = -3.25$$
$$x_3^{(1)} = \frac{1}{5}[10 - 2x_1^{(0)} - x_2^{(0)}] = \frac{1}{5}[10 - 2(0) - 0] = 2$$

2nd Iteration:- Put $k = 1$ in (*)

$$x_1^{(2)} = \frac{1}{4}[-x_2^{(1)} + x_3^{(1)}] = \frac{1}{4}[-(-3.25) + 2] = 1.3125$$
$$x_2^{(2)} = \frac{-1}{4}[13 - x_1^{(1)} - 2x_3^{(1)}] = \frac{-1}{4}[13 - 0 - 2(2)] = -2.25$$

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$$x_3^{(2)} = \frac{1}{5}\left[10 - 2x_1^{(1)} - x_2^{(1)}\right] = \frac{1}{5}\left[10 - 2(0) - (-3.25)\right] = 2.65$$

3rd Iteration:- Put $k=2$ in (*)

$$x_1^{(3)} = \frac{1}{4}\left[-x_2^{(2)} + x_3^{(2)}\right] = \frac{1}{4}\left[-(-2.25) + 2.65\right] = 1.225$$

$$x_2^{(3)} = -\frac{1}{4}\left[13 - x_1^{(2)} - 2x_3^{(2)}\right] = -\frac{1}{4}\left[13 - 1.3125 - 2(2.65)\right] = -1.5969$$

$$x_3^{(3)} = \frac{1}{5}\left[10 - 2x_1^{(2)} - x_2^{(2)}\right] = \frac{1}{5}\left[10 - 2(1.3125) - (-2.225)\right] = 1.925$$

Now Consider the Table.

K	$x_1^{(K)}$	$x_2^{(K)}$	$x_3^{(K)}$
0	0	0	0
1	0	-3.25	2
2	1.3125	-2.25	2.65
3	1.225	-1.5969	1.925
4	0.8805	-1.9813	1.8294
5	0.9527	-2.1152	2.0441
6	1.0398	-1.9898	2.042
7	1.008	-1.9691	1.9820
8	0.9878	-2.007	1.9906
9	0.9994	-2.0078	2.0063
10	1.0035	-1.9969	2.0017
11	0.9997	-1.9983	1.998
12	0.9991	-2.0011	1.9999
13	1.0002	-2.0003	2.0006
14	1.0002	-1.9997	2
15	1	-2	2
16	1	-2	2

So the Solution

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★ NORM OF A MATRIX :-

For a matrix A, norm of A is denoted by $\|A\|$ and it is a real number such that the following conditions are satisfied

(i) $\|A\| \geq 0$ (ii) $\|A\| = 0 \Leftrightarrow A = 0$

(iii) $\|K \cdot A\| = |K| \, \|A\|$

(iv) $\|A+B\| \leq \|A\| + \|B\|$

(v) $\|AB\| \leq \|A\| \, \|B\|$

Some Useful Norms :-

1) Column Norm (OR) 1 - Norm :-

$$\|A\|_c = \|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$

Example

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 8 & -6 \\ 1 & -3 & 9 \end{bmatrix}$$

$\|A\|_1 = \max(|1|+|3|+|1|, |2|+|8|+|-3|, |5|+|-6|+|9|)$

$= \max(1+3+1, 2+8+3, 5+6+9)$

$= \max(5, 13, 20)$

$= 20$

$\Rightarrow \|A\|_1 = 20$

2) Row Norm (OR) ∞ - Norm :-

$$\|A\|_r = \|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

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e.g $\|A\|_2 = \|A\|_\infty \overset{Max}{=} (1+2+5, 3+8+6, 1+3+9)$

$= Max(8, 17, 13)$

$= 17$

3) Eucledian Norm OR 2-Norm:-

$$\|A\|_2 = \sqrt{\sum_{i,j} |a_{ij}|^2}$$

e.g $$\|A\|_2 = \sqrt{|a_{11}|^2 + |a_{12}|^2 + |a_{13}|^3 + |a_{21}|^2 + |a_{22}|^2 + |a_{23}|^3 + |a_{31}|^2 + |a_{32}|^2 + |a_{33}|^2}$$

$$= \sqrt{1+4+25+9+64+36+1+9+81}$$

$$= \sqrt{230} = 15.1658$$

★ ——— ★ ——— ★

★ Condition Number:- If 'A' is a $n \times n$ non-sigular matrix then condition number of 'A' is denoted & defined by

$$Cond(A) = K(A) = \|A\| . \|A^{-1}\|$$

Example:-

If $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 5 \\ 1 & -1 & 0 \end{bmatrix}$

then find $K_1(A)$, $K_2(A)$, $K_\infty(A)$

Solution

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As $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 5 \\ 1 & -1 & 0 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1 & -1/5 & -2/5 \\ 1 & -1/5 & -7/5 \\ -1 & 2/5 & 4/5 \end{bmatrix}$

(i) $\|A\|_1 = \text{Max}(2+3+1,\ 0+2+1,\ 1+5+0)$

$= \text{Max}(6, 3, 6)$

$= 6$

$\|A^{-1}\|_1 = \text{Max}(1+1+1,\ \frac{1}{5}+\frac{1}{5}+\frac{2}{5},\ \frac{2}{5}+\frac{7}{5}+\frac{4}{5})$

$= \text{Max}(3,\ 4/5,\ 13/5)$

$= 3$

$\Rightarrow K_1(A) = \|A\|_1 \cdot \|A^{-1}\|_1$

$= 6 \times 3 = 18$

(ii) $\|A\|_\infty = \text{Max}(2+0+1,\ 3+2+5,\ 1+1+0)$

$= \text{Max}(3, 10, 2)$

$= 10$

$\|A^{-1}\|_\infty = \text{Max}(1+\frac{1}{5}+\frac{2}{5},\ 1+\frac{1}{5}+\frac{1}{5},\ 1+\frac{2}{5}+\frac{4}{5})$

$= \text{Max}(1.6,\ 1.4,\ 2.2)$

$= 2.6$

$\Rightarrow K_\infty(A) = 10 \times 2.6 = 26$

(iii) $\|A\|_2 = \sqrt{\sum_{i,j} |a_{ij}|^2}$

$= \sqrt{4+0+1+9+4+25+1+1+0} = \sqrt{45}$

$\|A^{-1}\|_2 = \sqrt{1+\frac{1}{25}+\frac{4}{25}+1+\frac{1}{25}+\frac{49}{25}+1+\frac{4}{25}+\frac{16}{25}} = \sqrt{6}$

$K_2(A) = \sqrt{45} \cdot \sqrt{6} = 3\sqrt{30} = 16.4317$

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Theorem:- Condition number of a matrix is always greater than or equal to one.

Proof:- Let 'A' be an nxn non-singular matrix then

$$A \cdot A^{-1} = I$$

Now $\|I\| = \|A A^{-1}\|$

Next $1 \leq \|I\| = \|A A^{-1}\|$

$$\leq \|A\| \|A^{-1}\| = K(A)$$

$$\Rightarrow K(A) \geq 1$$

★ ——— ★ ★ ——— ★

★ **Importance of Condition Number:-**
Condition number of a matrix is a measure of condition of the matrix. Large condition number of matrix indicates ill condition of the system of linear equation $Ax = b$.
Small condition number of matrix indicates the system is well condition or stable.

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Example:- Consider the system

$$1.01x_1 + 0.99x_2 = 2$$
$$0.99x_1 + 1.01x_2 = 2$$

Then $x = [1, 1]^T$ is the solution. But if it is written as

$$1.01x_1 + 0.99x_2 = 2$$
$$0.99x_1 + 1.01x_2 = -2$$

then $x = [100, -100]^T$ is its solution. Determine it is stable or ill condition.

Solution:-

Here $A = \begin{bmatrix} 1.01 & 0.99 \\ 0.99 & 1.01 \end{bmatrix}$

$|A| = 1.0201 - 0.9801 \Rightarrow |A| = 0.0400$

$$A^{-1} = \begin{bmatrix} 25.250 & -24.75 \\ -24.75 & 25.25 \end{bmatrix}$$

$\|A\|_1 = Max(1.01+0.99, 0.99+1.01)$

$= Max(2, 2)$

$= 2$

$\|A^{-1}\|_1 = Max(25.25 + 24.75, 24.75+25.25)$

$= Max(50, 50)$

$= 50$

$K_1(A) = \|A\| \|A^{-1}\|$

$= 2 \times 50 = 100$

And

$\|A\|_\infty = Max(1.01+0.99, 0.99+1.01)$

$= Max(2, 2) = 2$

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$\|A^{-1}\|_\infty = \text{Max}(25-25+24.75, 24-75+25-25)$

$= 50$

$K_\infty(A) = 2 \times 50 = 100$

Condition number is large so system is ill condition.

Remark:- A system is supposed to be ill condition when one or more of following indicators are presents.

1) Small change in coefficients of matrix A or in elements of b results a large change in solution.
2) Matrix A posses a very small determinent as compared to the average of its elements.
3) If $\text{Cond}(A) = \|A\|\|A^{-1}\|$ is large.

★ ——— ★ ——— ★

★ Error & Residual of An Approximation.

Consider the linear system $A\underline{x} = \underline{b}$. Let $\underline{x}$ be the exact solution and $\underline{x}^{(a)}$ be the approximate solution of system then

$$e = x - x^{(a)}$$

is called error and $r = A\underline{x} - A\underline{x}^{(a)}$ is called residual of an approximation

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The quantities $\frac{\|r\|}{\|b\|}$ and $\frac{\|e\|}{\|x\|}$ are called relative residual and relative error of the approximation solution $\underline{x}^{(a)}$

★ ——— ★ ★ ★ ——— ★

Example:- Let $1.01x_1 + 0.99x_2 = 2$

$0.99x_1 + 1.01x_2 = 2$

Then $\underline{x} = [1, 1]^T$ is exact solution and $\underline{x} = [1.01, 1.01]^T$ is approximate solution. Then find error & residual.

Solution:-

$$\underline{e} = \underline{x} - \underline{x}^{(a)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1.01 \\ 1.01 \end{bmatrix}$$

$$e = \begin{bmatrix} -0.01 \\ -0.01 \end{bmatrix}$$

And $\quad \underline{r} = A\underline{x} - Ax^{(a)}$

$$\Rightarrow \underline{r} = b - Ax^{(a)} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2.02 \\ 2.02 \end{bmatrix}$$

$$\Rightarrow r = \begin{bmatrix} -0.02 \\ -0.02 \end{bmatrix}$$

Here $\underline{e}$ and $\underline{r}$ both are small now if we take $\underline{x}^{(a)} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Then $e = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $r = \begin{bmatrix} -0.02 \\ -0.02 \end{bmatrix}$

Here error is relatively large to the residual but, as the system is ill condition, so all this indicates that error and residual does not indicate whatever the system is ill condition or well condition (stable)

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★ An Important Result:-

Now consider $Ax = b$, then

$e = x - x^{(a)}$ ($\because x^{(a)}$ is approximate solution)

and

$$r = Ax - Ax^{(a)}$$

$$= A(x - x^{(a)})$$

$$\Rightarrow r = Ae \quad \text{——— (1)}$$

$$\Rightarrow e = A^{-1}r \quad \text{——— (2)}$$

From (1) $\|r\| = \|Ae\|$

$$\leq \|A\|\|e\| \quad \text{——— (3)}$$

From (2) $\|e\| = \|A^{-1}r\|$

$$\leq \|A^{-1}\|\|r\| \quad \text{——— (4)}$$

From (3) $\dfrac{\|r\|}{\|A\|} \leq \|e\| \quad \text{——— (5)}$

From (4) & (5)

$$\frac{\|r\|}{\|A\|} \leq \|e\| \leq \|A^{-1}\|\|r\| \quad \text{——— (6)}$$

Now if we choose $x^{(a)} = 0$ then

$$r = Ax - Ax^{(a)}$$

$$= Ax - 0 = Ax$$

$$\Rightarrow r = b$$

and

$$e = x - x^{(a)}$$

$$e = x$$

Then (6) becomes $\dfrac{\|b\|}{\|A\|} \leq \|x\| \leq \|A^{-1}\|\|b\|$

$$\Rightarrow \frac{\|A\|}{\|b\|} \geq \frac{1}{\|x\|} \geq \frac{1}{\|A^{-1}\|\|b\|}$$

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$$\Rightarrow \frac{1}{\|A^{-1}\|\|b\|} \leq \frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|} \quad \text{——— (7)}$$

Multiply (6) by (7)

$$\frac{1}{\|A\|\|A^{-1}\|} \cdot \frac{\|z\|}{\|b\|} \leq \frac{\|e\|}{\|x\|} \leq \|A\|\|A^{-1}\| \frac{\|z\|}{\|b\|}$$

$$\Rightarrow \frac{1}{K(A)} \frac{\|z\|}{\|b\|} \leq \frac{\|e\|}{\|x\|} \leq K(A) \frac{\|z\|}{\|b\|} \quad \text{——— (8)}$$

Now if $K(A) \approx 1$ i.e system is stable. then from (8)

$$\frac{\|z\|}{\|b\|} \leq \frac{\|e\|}{\|x\|} \leq \frac{\|z\|}{\|b\|}$$

$$\Rightarrow \frac{\|z\|}{\|b\|} \approx \frac{\|e\|}{\|x\|}$$

Hence if the system is stable then relative residual and relative error are approximately equal.

——★★——★——★★——

★ Eigen Values & Eigen Vectors:-

Def

'A' is a square matrix over R, then $\lambda \in R$ is said to be eigen value of A if for some non-zero column vector V $AV = \lambda V$

Example:-

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow 5 = \lambda$ is eigen value of that matrix.

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Remark:- If $AV = \lambda V$, $V \neq 0$, then λ is called eigen value of A and V is called eigen vector of A corresponding to λ. An eigen value is also called characteristic root or latent root and eigen vector is also called characteristic vector or latent vector.

** ——— * ——— **

Theorem:- If A is invertible & $AB = 0$ then $B = 0$

Proof:- As A is invertible so A^{-1} exist.

Thus $AB = 0$

$\Rightarrow A^{-1}(AB) = A^{-1}.0 \Rightarrow (A^{-1}A)B = 0$

$\Rightarrow IB = 0 \Rightarrow B = 0$

Remark:-

(i) From above theorem if $B \neq 0$, then A is not invertible

(ii) Now $AV = \lambda V$

$\Rightarrow AV - \lambda V = 0$

$\Rightarrow (A - \lambda I)V = 0$

$\Rightarrow A - \lambda I$ is singular $\because V \neq 0$

$\Rightarrow |A - \lambda I| = 0$

** ——— * ——— **

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★ POWER METHOD:-

It is an iterative method for finding dominant eigen value and eigen vector of a matrix.

i) **Dominant Eigen Value:-** An eigen value λ of square matrix A is said to be dominant eigen value if $|\lambda|$ has largest value as compared to absolute value of the other eigen values of A. Then corresponding eigen vector is called dominant eigen vector.

ii) **Normalized Vector:-** An eigen vector $\underline{x}$ is said to be normalized if the coordinate of largest magnitude is 1. for example $[1,0,0]^T$, $[1,1,1]^T$, $[0,1,1]^T$ are all normalized vectors.

Working Rule for Power Method:- To find dominant eigen value and corresponding eigen vector of matrix take $\underline{x}^0$ is initial vector which is normalized vector like $\underline{x}_0 = [1,1,1]^T$

$A\underline{x}_0 = c_1\underline{x}_1$ where x_1 is normalized

$A\underline{x}_1 = c_2\underline{x}_2$ where x_2 is normalized

$A\underline{x}_2 = c_3\underline{x}_3$ where x_3 " "

⋮ ⋮

$A\underline{x}_n = c_{n+1}x_{n+1}$ where x_{n+1} is normalized

Then $\lambda = c_{n+1}$ is dominant eigen value

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and x_{n+1} is the corresponding eigen vector.
The iterative process will be stopped when $x_{n+1} = x_n$

Working Rule for Finding Absolute Least Eigen Value:

Step I:- If λ_1 is the dominant eigen value and x_1 be corresponding dominant eigen vector then find $B = A - \lambda_1 I$

Step II:- Find dominant eigen value of B say it is λ_2 (By taking -ve common)

Step III:- Absolute least eigen value of $A = \lambda_1 + \lambda_2$

★ ——— ★ ——— ★

Examples- Find dominant and absolute least eigen pair of $A = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}$

Solution:-

Taking initial guess vector $x_0 = [1, 1]^T$

Absolute سب سے بڑی ہے جس کو value

$$Ax_0 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix} = 10\begin{bmatrix} 0.7 \\ 1 \end{bmatrix} = \lambda_1 x_1$$

$$Ax_1 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0.7 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.5 \\ 9.4 \end{bmatrix} = 9.4\begin{bmatrix} 0.585 \\ 1 \end{bmatrix} = \lambda_2 x_2$$

$$Ax_2 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0.585 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.925 \\ 9.17 \end{bmatrix} = 9.17\begin{bmatrix} 0.537 \\ 1 \end{bmatrix} = \lambda_3 x_3$$

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$$A\underline{x}_3 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0.537 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.605 \\ 9.074 \end{bmatrix} = 9.074\begin{bmatrix} 0.516 \\ 1 \end{bmatrix} = \lambda_4 \underline{x}_4$$

$$A\underline{x}_4 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0.516 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.581 \\ 9.032 \end{bmatrix} = 9.032\begin{bmatrix} 0.507 \\ 1 \end{bmatrix} = \lambda_5 \underline{x}_5$$

$$A\underline{x}_5 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0.507 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.535 \\ 9.014 \end{bmatrix} = 9.014\begin{bmatrix} 0.503 \\ 1 \end{bmatrix} = \lambda_6 \underline{x}_6$$

$$A\underline{x}_6 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0.503 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.515 \\ 9.006 \end{bmatrix} = 9.006\begin{bmatrix} 0.5013 \\ 1 \end{bmatrix} = \lambda_7 \underline{x}_7$$

$$A\underline{x}_7 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0.5013 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.506 \\ 9.0026 \end{bmatrix} = 9.0026\begin{bmatrix} 0.500 \\ 1 \end{bmatrix} = \lambda_8 \underline{x}_8$$

$$A\underline{x}_8 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.5 \\ 9 \end{bmatrix} = 9\begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \lambda_9 \underline{x}_9$$

we see that $\underline{x}_8 = \underline{x}_9$, so stop iteration & $\lambda = 9$ is dominant eigen value and $[0.5, 1]^T$ is dominant eigen vector.

For absolute least eigen value

calculate $B = A - \lambda I$ $\quad \because \lambda$ is dominant eigen value

$B = A - 9I$

$$= \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

Let Guess vector $\underline{x}_0 = [1 \quad 1]^T$

$$B\underline{x}_0 = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \lambda_1 \underline{x}_1$$

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$$B\underline{x}_1 = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -5 \\ 2.5 \end{bmatrix} = -5\begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \lambda_2 \underline{x}_2$$

$$B\underline{x}_2 = \begin{bmatrix} 4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -5 \\ 2.5 \end{bmatrix} = -5\begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \lambda_3 \underline{x}_3$$

So $\lambda = -5$ is dominant eigen value of B.

Absolute least eigen value of $A = \lambda_1 + \lambda_2$

$\Rightarrow A = 9 + (-5)$

$= 4$ Ans.

★ ——— ★ ——— ★

Example:- using power method to find the dominant & smallest eigen value & corresponding eigen vector of matrix

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution:-

Let $y_0 = [1, 0, 1]^T$ be initial guess

$$A\underline{y}_0 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0.25 \\ -0.25 \end{bmatrix} = \lambda_1 \underline{y}_1$$

$$A\underline{y}_1 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0.25 \\ -0.25 \end{bmatrix} = \begin{bmatrix} 4.25 \\ 0.25 \\ 0.25 \end{bmatrix} = 4.25\begin{bmatrix} 1 \\ 0.059 \\ 0.059 \end{bmatrix} = \lambda_2 \underline{y}_2$$

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$$A\underline{y}_2 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.059 \\ 0.059 \end{bmatrix} = \begin{bmatrix} 4.059 \\ 0.177 \\ -0.059 \end{bmatrix} = 4.059 \begin{bmatrix} 1 \\ 0.044 \\ -0.015 \end{bmatrix} = \lambda_3 \underline{y}_3$$

$$A\underline{y}_3 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.044 \\ -0.015 \end{bmatrix} = \begin{bmatrix} 4.044 \\ 0.073 \\ 0.015 \end{bmatrix} = 4.044 \begin{bmatrix} 1 \\ 0.018 \\ .004 \end{bmatrix} = \lambda_4 \underline{y}_4$$

$$A\underline{y}_4 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.018 \\ .004 \end{bmatrix} = \begin{bmatrix} 4.018 \\ 0.032 \\ -0.004 \end{bmatrix} = 4.018 \begin{bmatrix} 1 \\ .008 \\ 0 \end{bmatrix} = \lambda_5 \underline{y}_5$$

$$A\underline{y}_5 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ .008 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.008 \\ 0.016 \\ 0 \end{bmatrix} = 4.008 \begin{bmatrix} 1 \\ .004 \\ 0 \end{bmatrix} = \lambda_6 \underline{y}_6$$

$$A\underline{y}_6 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ .004 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.004 \\ .008 \\ 0 \end{bmatrix} = 4.004 \begin{bmatrix} 1 \\ .002 \\ 0 \end{bmatrix} = \lambda_7 \underline{y}_7$$

$$A\underline{y}_7 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ .002 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.002 \\ .004 \\ 0 \end{bmatrix} = 4.002 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_8 \underline{y}_8$$

$$A\underline{y}_8 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_9 \underline{y}_9$$

$$A\underline{y}_9 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_{10} \underline{y}_{10}$$

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$\Rightarrow \lambda_1 = 4$ is dominant eigen value and corresponding eigen vector is $[1, 0, 0]^T$

For smallest eigen value calculate

$B = A - \lambda I$ $\quad \because \lambda$ is dominant eigen value

$\Rightarrow B = A - 4I$

$$= \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -5 \end{bmatrix}$$

Let $x_0 = (1, 1, 1)^T$

$$\Rightarrow Bx_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix} = -5 \begin{bmatrix} -0.2 \\ 0.2 \\ 1 \end{bmatrix} = c_1 x_1$$

$$Bx_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} -0.2 \\ 0.2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.6 \\ -5 \end{bmatrix} = -5 \begin{bmatrix} -0.04 \\ -0.12 \\ 1 \end{bmatrix} = c_2 x_2$$

$$Bx_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} -0.04 \\ -0.12 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.12 \\ 1.24 \\ -5 \end{bmatrix} = -5 \begin{bmatrix} 0.02 \\ -0.25 \\ 1 \end{bmatrix} = c_3 x_3$$

$$Bx_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} 0.02 \\ -0.25 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.25 \\ 1.5 \\ -5 \end{bmatrix} = -5 \begin{bmatrix} 0.05 \\ -0.3 \\ 1 \end{bmatrix} = c_4 x_4$$

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$$B\underline{x}_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} 0.05 \\ -0.3 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.3 \\ 1.06 \\ -5 \end{bmatrix} = -5 \begin{bmatrix} 0.06 \\ -0.32 \\ 1 \end{bmatrix} = \underline{c}_5 x_5$$

$\Rightarrow \lambda_2 = -5$ is dominant eigen value of B.

Smallest eigen value of $A = \lambda_1 + \lambda_2$

$\Rightarrow A = 4 + (-5) = -1$

Corresponding eigen vector is $[0.06, -0.32, 1]^T$

★★★

Example:- Use power method to compute the largest eigen value & corresponding eigen vector of matrix $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$

Solution:-

Let $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$

Take $\underline{y}_0 = [1, 0, 0]^T$ be initial guess.

Then by power method

$$A\underline{y}_0 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix} = \lambda_1 \underline{y}_1$$

$$A\underline{y}_1 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ -4 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \lambda_2 \underline{y}_2$$

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$$A\underline{y}_2 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 5.2 \\ -5.2 \end{bmatrix} = 5.6\begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \lambda_3 \underline{y}_3$$

$$A\underline{y}_3 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \begin{bmatrix} 5.86 \\ 5.72 \\ -5.72 \end{bmatrix} = 5.86\begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \lambda_4 \underline{y}_4$$

$$A\underline{y}_4 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \begin{bmatrix} 5.96 \\ 5.92 \\ -5.92 \end{bmatrix} = 5.96\begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \lambda_5 \underline{y}_5$$

$$A\underline{y}_5 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \begin{bmatrix} 5.98 \\ 5.96 \\ -5.96 \end{bmatrix} = 5.98\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \lambda_6 \underline{y}_6$$

$$A\underline{y}_6 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ -6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \lambda_7 \underline{y}_7$$

$$A\underline{y}_7 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ -6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

So dominant or largest eigen value is $\lambda = 6$ & corresponding eigen vector is $[1, 1, -1]^T$

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SEC II

⇒ Linear Operators**

An operator P is said to be linear if for all f_1, f_2 and α_1, α_2 .

$$P(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 P(f_1) + \alpha_2 P(f_2)$$

Where P is defined as space 'f' of functions and $f_1, f_2 \in f$ and $\alpha_1, \alpha_2 \in R$

In Numerical Analysis the following linear operators are defined.

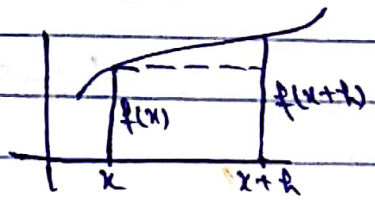
i) Shift operator.
ii) Forward Difference operator.
iii) Backward Difference operator.
iv) Centeral Difference operator.
v) Average or Mean Value operator.

1) Shift Operator:-

$$E f(x) = f(x+h)$$

$$E^2 f(x) = E[E f(x)] = E f(x+h)$$
$$= f(x+h+h) = f(x+2h)$$

Similarly $E^n f(x) = f(x+nh)$



2) Forward Difference Operator:-

Let the values $x_0, x_0+h, x_0+2h, \ldots, x_0+nh$ be corresponding indipendents variables and $f(x_0), f(x_0+h), f(x_0+2h), \ldots, f(x_0+nh)$

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be corresponding dependent variables of function $f(x)$.
The values of x are known as arguments and $f(x)$ are called entries, where h be a common difference b/w intervals.

Now forward difference operator is defined as

$$\Delta f(x) = f(x+h) - f(x)$$

which is of order 1.

Similarly $\Delta f(x+h) = f(x+2h) - f(x+h)$

$$\Delta f(x+nh) = f(x+(n+1)h) - f(x+nh)$$

In 2nd order forward difference operator is defined as

$$\Delta^2 f(x) = \Delta[f(x+h) - f(x)]$$

$$= \Delta f(x+h) - \Delta f(x)$$

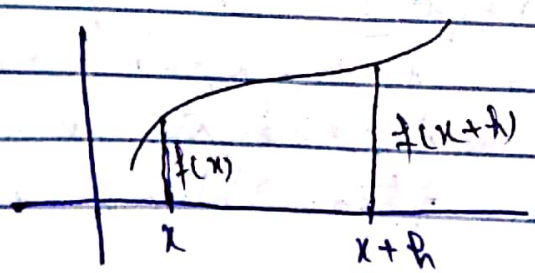
$$= f(x+2h) - f(x+h) - [f(x+h) - f(x)]$$

$$= f(x+2h) - f(x+h) - f(x+h) + f(x)$$

$$= f(x+2h) - 2f(x+h) + f(x)$$

Similarly

$$\Delta^3 f(x) = f(x+3h) - 3f(x+2h) + 3f(x+h) + f(x)$$



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3) Backward Difference Operator:-

It is denoted & defined as

$\nabla f(x) = f(x) - f(x-h)$ $\quad \nabla$ = nebla

&

$\nabla f(x+h) = f(x+h) - f(x)$

$\nabla f(x+2h) = f(x+2h) - f(x+h)$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$\nabla f(x+nh) = f(x+nh) - f(x+(n-1)h)$

4) Central Difference Operator:-

The 1st order central difference operator is defined as

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

5) Average Operator:-

Average operator denoted by μ & defined as

$$\mu f(x) = \frac{f(x+h/2) + f(x-h/2)}{2}$$

★★ ——— ★ ——— ★★

★ Relations with Shift Operator:-

1) Relation b/w forward difference operator & shift operator:-

$\Delta f(x) = f(x+h) - f(x) = Ef(x) - f(x)$

$= (E-1) f(x)$

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$\Rightarrow \quad \Delta = E - 1 \quad$ or $\quad E = \Delta + 1$

ii) Relation b/w Backward Difference Operator & Shift Operator:-

$$\nabla f(x) = f(x) - f(x-h)$$

$$\Rightarrow \nabla f(x) = f(x) - E^{-1} f(x)$$

$$= (1 - E^{-1}) f(x)$$

$$\Rightarrow \nabla = 1 - E^{-1} \quad \Rightarrow E^{-1} = 1 - \nabla$$

$$\Rightarrow E = (1 - \nabla)^{-1}$$

(iii) Relation b/w Central Difference Operator & Shift operator:-

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

$$\Rightarrow \delta f(x) = E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x)$$

$$= (E^{\frac{1}{2}} - E^{-\frac{1}{2}}) f(x)$$

$$\Rightarrow \delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} \quad \& \quad \delta^2 = (E^{\frac{1}{2}} - E^{-\frac{1}{2}})^2$$

$$\Rightarrow \delta^2 = E + E^{-1} - 2$$

(iv) Relation b/w Average Operator & Shift Operator:

$$\mu f(x) = \frac{f(x + h/2) + f(x - \frac{h}{2})}{2}$$

$$= \frac{E^{\frac{1}{2}} f(x) + E^{-\frac{1}{2}} f(x)}{2}$$

$$= \frac{(E^{\frac{1}{2}} + E^{-\frac{1}{2}}) f(x)}{2}$$

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$$\Rightarrow \mu = \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}$$

Formulaes:- $\Delta = E - 1$, $E = 1 + \Delta$

$\nabla = 1 - E^{-1}$, $E = (1 - \nabla)^{-1}$

$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

$\mu = \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}$

* * *

Question:- Prove the following identities.

(i) $E = e^{hD}$; $D = d/dx$

$$Ef(x) = f(x+h)$$
$$= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$
$$= f(x) + hDf(x) + \frac{h^2}{2!}D^2 f(x) + \frac{h^3}{3!}D^3 f(x) + \cdots$$
$$Ef(x) = \left[1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right] f(x)$$
$$\Rightarrow E = 1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots$$
$$\Rightarrow E = e^{hD}, \text{ where } D = \frac{d}{dx} \quad \text{(Proved)}$$

(ii) $\Delta \sin k = 2\sin(h/2)\cos(k + h/2)$

L.H.S $\Delta \sin k = \sin(k+h) - \sin k$

$$= 2\cos\left(\frac{k+h+k}{2}\right)\sin\left(\frac{k+h-k}{2}\right)$$
$$= 2\cos\left(\frac{2k+h}{2}\right)\sin\left(\frac{h}{2}\right)$$
$$= 2\sin\left(\frac{h}{2}\right)\cos(k + h/2) \quad \text{Proved.}$$

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(iii) $\Delta - \nabla = \Delta\nabla = \nabla\Delta = \delta^2$

$\Delta\nabla = (E-1)(1-E^{-1})$

$= E - 1 - 1 + E^{-1} = E - 2 + E^{-1}$ ——— ①

$\Delta - \nabla = (E-1) - (1-E^{-1})$

$= E - 1 - 1 + E^{-1} = E - 2 + E^{-1}$ ——— ②

$\nabla\Delta = (1-E^{-1})(E-1)$

$= E - 1 - 1 + E^{-1} = E - 2 + E^{-1}$ ——— ③

$\delta^2 = (E^{\frac{1}{2}} - E^{-\frac{1}{2}})^2$

$= E + E^{-1} - 2$ ——— ④

from ① to ④

$\Delta - \nabla = \Delta\nabla = \nabla\Delta = \delta^2$

iv) $\sum_{K=0}^{n-1} \Delta^2 f_K = \Delta f_n - \Delta f_0$

$\Delta^2 f_K = \Delta(\Delta f_K)$

$= \Delta(f_{K+1} - f_K)$

$= \Delta f_{K+1} - \Delta f_K$ ——— ①

Put $K = 0, 1, 2, \cdots n-1$ in ①

$\Rightarrow \sum_{K=0}^{n-1} \Delta^2 f_K = (\Delta f_1 - \Delta f_0) + (\Delta f_2 - \Delta f_1) + (\Delta f_3 - \Delta f_2)$

$+ \cdots + (\Delta f_n - \Delta f_{n-1})$

$= \Delta f_n - \Delta f_0$

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v) $\Delta^n e^{ax+b} = e^{x+b}(e^{ah}-1)^n$

$\Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b}$

$= e^{ax+ah+b} - e^{ax+b}$

$= e^{ax+b}(e^{ah}-1)$ ——(i)

$\Delta^2 e^{ax+b} = \Delta(\Delta e^{ax+b})$

$= \Delta\left[e^{ax+b}(e^{ah}-1)\right]$ from (i)

$= (e^{ah}-1)\left[\Delta e^{ax+b}\right]$

$= (e^{ah}-1)\left[e^{ax+b}(e^{ah}-1)\right]$

$= e^{ax+b}(e^{ah}-1)^2$

Continuing in this way we have

$\Delta^n e^{ax+b} = e^{ax+b}(a^{ah}-1)^n$ Proved.

(vi) $\frac{\Delta^2}{E}(\sin x) = 2\sin x(\cos h - 1)$

$\frac{\Delta^2}{E}(\sin x) = \Delta^2 E^{-1}(\sin x)$

$= \Delta^2 \sin(x-h)$

$= \Delta\left[\Delta \sin(x-h)\right]$

$= \Delta\{\sin x - \sin(x-h)\}$

$= \Delta \sin x - \Delta \sin(x-h)$

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$$\Rightarrow \frac{\Delta^2}{E}(\sin x) = \sin(x+h) - \sin x - \sin x + \sin(x-h)$$

$$= \sin(x+h) - 2\sin x + \sin(x-h)$$

$$= 2\sin\left(\frac{x+h+x-h}{2}\right)\cos\left(\frac{x+h-x+h}{2}\right) - 2\sin x$$

$$= 2\sin x \cos h - 2\sin x$$

$$= 2\sin x(\cos h - 1) \quad \text{proved.}$$

(vii) $\Delta(f_k g_k) = f_{k+1}\Delta g_k + g_k \Delta f_k$

L.H.S $\Delta(f_k g_k) = f_{k+1} g_{k+1} - f_k g_k$

$$= f_{k+1} g_{k+1} - f_{k+1} g_k + f_{k+1} g_k - f_k g_k$$

$$= f_{k+1}(g_{k+1} - g_k) + g_k(f_{k+1} - f_k)$$

$$= f_{k+1}\Delta g_k + g_k \Delta f_k \quad \text{Proved}$$

(viii) $\Delta\left(\frac{f_k}{g_k}\right) = \frac{g_k \Delta f_k - f_k \Delta g_k}{g_k g_{k+1}}$

L.H.S $\Delta\left(\frac{f_k}{g_k}\right) = \frac{f_{k+1}}{g_{k+1}} - \frac{f_k}{g_k}$

$$= \frac{f_{k+1} \cdot g_k - f_k \cdot g_{k+1}}{g_{k+1} g_k}$$

$$= \frac{f_{k+1} g_k - f_k g_k + f_k g_k - f_k g_{k+1}}{g_{k+1} g_k}$$

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$$\Rightarrow \Delta\left(\frac{f_k}{g_k}\right) = \frac{g_k(f_{k+1}-f_k) - f_k(g_{k+1}-g_k)}{g_{k+1}\,g_k}$$

$$= \frac{g_k\,\Delta f_k - f_k\,\Delta g_k}{g_{k+1}\,g_k} = \text{R.H.S}$$

(ix) $hD = \log(1+\Delta) = -\log(1-\nabla)$

As we know that $e^{hD} = E$

$\Rightarrow hD = \log(E)$

$\Rightarrow hD = \log(1+\Delta)$ ——— ①

Again $e^{hD} = E$

$\Rightarrow hD = \log E = \log(1-\nabla)^{-1}$

$= -\log(1-\nabla)$ ——— ②

from ① & ②

$$hD = \log(1+\Delta) = -\log(1-\nabla)$$

(x) $\Delta^k f_i = \sum_{r=0}^{k} (-1)^{k+r} \binom{k}{r} f_{i+r}$

L.H.S $\Delta^k f_i = (E-1)^k f_i$

$= (-1)^k (1-E)^k f_i$

$= (-1)^k \left\{ \sum_{r=0}^{k} \binom{k}{r} (1)^{k-r} (-E)^r \right\} f_i$

$\because$ by binomial theorem

$$(a+b)^n = \sum_{r=0}^{k} \binom{n}{r} a^{n-r} b^r$$

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$$\Rightarrow \Delta^K f_i = \left\{ \sum_{r=0}^{K} (-1)^{K-r} \binom{K}{r} (-1)^r E^r \right\} f_i$$

$$= \sum_{r=0}^{K} (-1)^{K+r} \binom{K}{r} f_{i+r}$$

$$= R.H.S$$

$\because E^n f(x) = f(x+nh)$

$\Rightarrow E^r f_i = f_{i+r}$

(xi) $\Delta^2 y_K = \nabla^2 y_{K+2}$

L.H.S $\Delta^2 y_K = (E-1)^2 y_K$

$$= \left[\sum_{i=0}^{2} \binom{2}{i} E^{2-i} (-1)^i \right] y_K$$

$$= \left[\sum_{i=0}^{2} (-1)^i \binom{2}{i} E^{2-i} \right] y_K$$

$$= \sum_{i=0}^{2} (-1)^i \binom{2}{i} y_{K+2-i}$$

R.H.S $= \nabla^2 y_{K+2}$

$$= (1-E^{-1})^2 y_{K+2}$$

$$= \left[\sum_{i=0}^{2} \binom{2}{i} (1)^{2-i} (-E)^{-i} \right] y_{K+2}$$

$$= \left[\sum_{i=0}^{2} (-1)^i \binom{2}{i} E^{-i} \right] y_{K+2}$$

$$= \sum_{i=0}^{2} (-1)^i \binom{2}{i} y_{K+2-i}$$

$\Rightarrow$ L.H.S $=$ R.H.S

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(xii) $e^x = \frac{\Delta^2}{E} e^x \cdot \frac{E e^x}{\Delta^2 e^x}$

R.H.S $\frac{\Delta^2 e^x}{E} \cdot \frac{E e^x}{\Delta^2 e^x}$

$= \Delta^2 E^{-1} e^x \cdot E e^x / \Delta^2 e^x$

$= \Delta^2 e^{x-h} \cdot E e^x / \Delta^2 e^x$

$= \Delta^2 e^x e^{-h} \cdot \frac{E e^x}{\Delta^2 e^x} = e^{-h} E e^x$

$= e^{-h} e^{x+h}$

$= e^x = $ L.H.S proved.

(xiii) $\mu\delta = \frac{1}{2} \Delta E^{-1} + \frac{1}{2} \Delta = \frac{1}{2}(\Delta + \nabla)$

$\mu\delta = \left[\frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}\right]\left[E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right]$

$= \frac{E - E^{-1}}{2}$ ——— ① $\because (x-a)(x+a) = x^2 - a^2$

$\frac{1}{2} \Delta E^{-1} + \frac{1}{2} \Delta = \frac{\Delta E^{-1} + \Delta}{2}$

$= \frac{-(1-E)E^{-1} + (E-1)}{2}$

$= \frac{1 - E^{-1} + E - 1}{2} = \frac{E - E^{-1}}{2}$ ——— ②

$\frac{1}{2}(\Delta + \nabla) = \frac{1}{2}(E - 1 + 1 - E^{-1}) = \frac{E - E^{-1}}{2}$ ——— ③

from ①, ② & ③ proved.

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(xiv) $\Delta - \frac{1}{2}\delta^2 = \delta\sqrt{1+\delta^2/4}$

L.H.S $\Delta - \frac{1}{2}\delta^2$

$= E - 1 - \frac{1}{2}\left[E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right]^2$

$= E - 1 - \frac{1}{2}\left[E + E^{-1} - 2\right]$

$= E - 1 - \frac{E}{2} - \frac{E^{-1}}{2} + 1 = \frac{E - E^{-1}}{2}$

R.H.S $\delta\sqrt{1+\delta^2/4}$

$= \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\sqrt{\frac{1 + \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2}{4}}$

$= \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\sqrt{\frac{4 + E + E^{-1} - 2}{4}}$

$= \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\sqrt{\frac{E + E^{-1} + 2}{4}}$

$= \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\sqrt{\frac{\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)^2}{(2)^2}}$

$= \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\left(\frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}\right) = \frac{E - E^{-1}}{2}$

$\Rightarrow$ L.H.S = R.H.S

(xv) $hD = \sinh^{-1}(\mu\delta)$

Rewrite problem we have

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$\sinh(hD) = RS$

$= \left(\frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}\right)\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)$

$= \frac{E - E^{-1}}{2} \qquad \because E = e^{hD}$

$= \frac{e^{hD} - e^{-hD}}{2}$

$= \sinh(hD) = R.H.S$

(xvi) $\mu = \left(1 - \frac{\nabla}{2}\right)(1 - \nabla)^{-\frac{1}{2}}$

R.H.S $\left(1 - \frac{\nabla}{2}\right)(1 - \nabla)^{-\frac{1}{2}}$

$= \frac{2 - (1 - E^{-1})}{2} \cdot \{1 - E^{-1}\}^{-\frac{1}{2}}$

$= \frac{2 - 1 + E^{-1}}{2}\left(1 - 1 + E^{-1}\right)^{-\frac{1}{2}}$

$= \frac{1 + E^{-1}}{2}\left[E^{\frac{1}{2}}\right] = \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2} = \mu$

(xvii) $\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)(1 + \Delta)^{\frac{1}{2}} = 2 + \Delta$

L.H.S $= \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)(1 + \Delta)^{\frac{1}{2}}$

$= \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)(1 + E - 1)^{\frac{1}{2}} = \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)(E)^{\frac{1}{2}}$

$= E + E^{-\frac{1}{2} + \frac{1}{2}} = E + E^{0} = 1 + \Delta + E^{0}$

$= 1 + \Delta + 1$

$= 2 + \Delta = R.H.S$

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(xiii) $\mu(f_k g_k) = \mu f_k \mu g_k + \frac{1}{4}\delta f_k \delta g_k$

L.H.S $= \mu(f_k g_k)$

$= \left(\frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}\right)(f_k g_k)$

$= \frac{1}{2}\left[E^{\frac{1}{2}}(f_k g_k)\right] + \frac{1}{2}\left[E^{-\frac{1}{2}}(f_k g_k)\right]$

$= \frac{1}{2} f_{k+\frac{1}{2}} g_{k+\frac{1}{2}} + \frac{1}{2} f_{k-\frac{1}{2}} g_{k-\frac{1}{2}}$

R.H.S $= \mu f_k \mu g_k + \frac{1}{4}\delta f_k \delta g_k$

$= \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}(f_k) \cdot \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2} \cdot g_k + \frac{1}{4}(E^{\frac{1}{2}} - E^{-\frac{1}{2}}) f_k \delta g_k$

$= \left[\frac{1}{2}E^{\frac{1}{2}} f_k + \frac{1}{2}E^{-\frac{1}{2}} f_k\right]\left[\frac{1}{2}E^{\frac{1}{2}} g_k + \frac{1}{2}E^{-\frac{1}{2}} g_k\right] + \left[\frac{1}{2}E^{\frac{1}{2}} f_k - \frac{1}{2}E^{-\frac{1}{2}} f_k\right]\left[\frac{1}{2}E^{\frac{1}{2}} g_k - \frac{1}{2}E^{-\frac{1}{2}} g_k\right]$

$= \left(\frac{1}{2} f_{k+\frac{1}{2}} + \frac{1}{2} f_{k-\frac{1}{2}}\right)\left(\frac{1}{2} g_{k+\frac{1}{2}} + \frac{1}{2} g_{k-\frac{1}{2}}\right) + \left(\frac{1}{2} f_{k+\frac{1}{2}} - \frac{1}{2} f_{k-\frac{1}{2}}\right)\left(\frac{1}{2} g_{k+\frac{1}{2}} - \frac{1}{2} g_{k-\frac{1}{2}}\right)$

$= \frac{1}{4}(f_{k+\frac{1}{2}} g_{k+\frac{1}{2}}) + \frac{1}{4}(f_{k+\frac{1}{2}} g_{k-\frac{1}{2}}) + \frac{1}{4}(f_{k-\frac{1}{2}} g_{k+\frac{1}{2}}) + \frac{1}{4}(f_{k-\frac{1}{2}} g_{k-\frac{1}{2}}) + \frac{1}{4}(f_{k+\frac{1}{2}} g_{k+\frac{1}{2}}) - \frac{1}{4}(f_{k+\frac{1}{2}} g_{k-\frac{1}{2}}) - \frac{1}{4}(f_{k-\frac{1}{2}} g_{k+\frac{1}{2}}) + \frac{1}{4}(f_{k-\frac{1}{2}} g_{k-\frac{1}{2}})$

$= \frac{1}{2} f_{k+\frac{1}{2}} g_{k+\frac{1}{2}} + \frac{1}{2} f_{k-\frac{1}{2}} g_{k-\frac{1}{2}}$ proved.

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Assignment

Show that:-

i) $\Delta \log x_k = \log(1 + h/x_k)$

ii) $\nabla \log x_k = -\log(1 - h/x_k)$

iii) $\Delta\sqrt{f_k} = \dfrac{\Delta f_k}{\sqrt{f_k} + \sqrt{f_{k+1}}}$

iv) $(1+\Delta)(1-\nabla) = 1$

v) $\Delta + \nabla = \dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta}$

vi) $\Delta\left[\dfrac{1}{f(x)}\right] = \dfrac{-\Delta f(x)}{f(x) f(x+1)}$

vii) $\dfrac{\Delta}{E} \sin x = 2\cos(x - h/2)\sin h/2$

viii) $\Delta = \delta E^{1/2} = E^{1/2}\delta$

ix) $\mu^2 = 1 + \delta^2/4$

x) $\mu\left(\dfrac{f_k}{g_k}\right) = \dfrac{\mu(f_k)\mu(g_k) - \frac{1}{4}(\delta(f_k)\,\delta(g_k))}{g_{k-\frac{1}{2}} \cdot g_{k+\frac{1}{2}}}$

xi) $\Delta(1+\Delta)^{-1/2} = \nabla(1-\nabla)^{-1/2} = \delta$

★ ——— ★ ★ ——— ★

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Solutions

(i) $\Delta \log x_k = \log\left(1 + h/x_k\right)$

L.H.S $= \Delta \log x_k$

$= \log(x_k + h) - \log x_k$

$= \log\left(\frac{x_k + h}{x_k}\right) = \log\left[\frac{x_k}{x_k} + \frac{h}{x_k}\right]$

$= \log\left(1 + h/x_k\right) = $ R.H.S

(ii) $\nabla \log x_k = -\log\left(1 - h/x_k\right)$

L.H.S $= \nabla \log x_k$

$= \log(x_k) - \log(x_k - h)$

$= -\left[\log(x_k - h) - \log(x_k)\right]$

$= -\left[\log\left(\frac{x_k - h}{x_k}\right)\right] = -\log\left[\frac{x_k}{x_k} - \frac{h}{x_k}\right]$

$= -\log\left[1 - h/x_k\right]$

(iii) $\Delta\sqrt{f_k} = \frac{\Delta f_k}{\sqrt{f_k} + \sqrt{f_{k+1}}}$

L.H.S $= \Delta\sqrt{f_k} = \Delta f_k^{1/2}$

$= (f_{k+1})^{1/2} - (f_k)^{1/2}$

$= (f_{k+1})^{1/2} - (f_k)^{1/2} \times \frac{(f_{k+1})^{1/2} + (f_k)^{1/2}}{(f_{k+1})^{1/2} + (f_k)^{1/2}}$

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$$= \frac{\left[(f_{k+1})^{1/2}\right]^2 - \left[(f_k)^{1/2}\right]^2}{(f_{k+1})^{1/2} + (f_k)^{1/2}}$$

$$= \frac{f_{k+1} - f_k}{(f_{k+1})^{1/2} + (f_k)^{1/2}}$$

$$= \frac{\Delta f_k}{\sqrt{f_k} + \sqrt{f_{k+1}}} = R.H.S$$ proved

(iv) $(1+\Delta)(1-\nabla) = 1$

$L.H.S = (1+E-1)(1-1-E^{-1})$

$= (E)(E^{-1}) = E^{1-1} = E^0$

$= 1 = R.H.S$ proved.

(v) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

$\Delta + \nabla = (E-1) + (1-E^{-1})$

$= E - 1 + 1 - E^{-1}$

$= E - E^{-1}$ ——— (1)

$\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \frac{E-1}{1-E^{-1}} - \frac{1-E^{-1}}{E-1}$

$= \frac{(E-1)^2 - (1-E^{-1})^2}{(1-E^{-1})(E-1)}$

$= \frac{(E^2+1-2E) - (1+E^{-2}-2E^{-1})}{E-1-1+E^{-1}}$

$= \frac{E^2+1-2E-1-E^{-2}+2E^{-1}}{-2+E+E^{-1}}$

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$$\Rightarrow \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \frac{-2(E-E^{-1}) + E^2 - E^{-2}}{-2+E+E^{-1}}$$

$$= \frac{-2(E-E^{-1}) + (E+E^{-1})(E-E^{-1})}{-2+E+E^{-1}}$$

$$= \frac{E-E^{-1}[-2+E+E^{-1}]}{[-2+E+E^{-1}]}$$

$$= E - E^{-1} \quad \text{---------- (2)}$$

from (1) & (2) L.H.S = R.H.S

(vi) $\Delta\left(\frac{1}{f(x)}\right) = \frac{-\Delta f(x)}{f(x) f(x+1)}$

$$L.H.S = \Delta\left(\frac{1}{f(x)}\right) = \Delta f(x)^{-1}$$

$$= f(x+1)^{-1} - f(x)^{-1}$$

$$= \frac{1}{f(x+1)} - \frac{1}{f(x)} = \frac{f(x) - f(x+1)}{f(x) f(x+1)}$$

$$= \frac{-[f(x+1) - f(x)]}{f(x) f(x+1)]}$$

$$= \frac{-\Delta f(x)}{f(x) f(x+1)} = R.H.S$$

vii) $\frac{\Delta}{E} \sin x = 2\cos\left(x - \frac{h}{2}\right) \sin\left(\frac{h}{2}\right)$

$$L.H.S = \frac{\Delta}{E} \sin x = \Delta E^{-1} \sin x$$

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$$= \Delta \sin(x-h)$$

$$= \sin(x-h+h) - \sin(x-h)$$

$$= \sin(x) - \sin(x-h)$$

$$= 2\cos\left(\frac{x+x-h}{2}\right)\sin\left(\frac{x-x+h}{2}\right)$$

$$= 2\cos\left(\frac{2x-h}{2}\right)\sin\left(\frac{h}{2}\right)$$

$$= 2\cos\left(x-\frac{h}{2}\right)\sin\left(\frac{h}{2}\right) = \text{L.H.S}$$

(viii) $\Delta = \delta E^{1/2} = E^{1/2}\delta$

$$\Delta = E - 1 \quad \text{——— ①}$$

$$\delta E^{1/2} = \left[E^{1/2} - E^{-1/2}\right]E^{1/2} = E - E^{0}$$

$$= E - 1 \quad \text{——— ②}$$

$$E^{1/2}\delta = E^{1/2}\left[E^{1/2} - E^{-1/2}\right] = E - E^{0}$$

$$= E - 1 \quad \text{——— ③}$$

From ① to ③ proved.

(ix) $\mu^2 = 1 + \delta^2/4$

$$\mu^2 = \left[\frac{E^{1/2} + E^{-1/2}}{2}\right]^2 = \frac{E + 1 + 1 + E^{-1}}{4}$$

$$= \frac{2 + E + E^{-1}}{4} \quad \text{——— ①}$$

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Now $1+\frac{\delta^2}{4} = 1+\frac{\left(E^{1/2}-E^{-1/2}\right)^2}{4}$

$= 1+\frac{E+E^{-1}-2E^0}{4} = 1+\frac{E+E^{-1}-2}{4}$

$= \frac{4+E+E^{-1}-2}{4}$

$= \frac{2+E+E^{-1}}{4}$ ——— ②

from ① & ② L.H.S = R.H.S

(xi) $\Delta(1+\Delta)^{-1/2} = \nabla(1-\nabla)^{-1/2} = \delta$

$\Delta(1+\Delta)^{-1/2} = (E-1)(1+E-1)^{-1/2}$

$= (E-1)(E^{-1/2})$

$= E^{1/2}-E^{-1/2} = \delta$ ——— ①

& $\nabla(1-\nabla)^{-1/2} = (1-E^{-1})(1-1+E^{-1})^{-1/2}$

$= (1-E^{-1})(E^{-1})^{-1/2}$

$= E^{1/2}-E^{-1/2} = \delta$ ——— ②

from ① & ② proved

(x) $\mu\left(\frac{f_k}{g_k}\right) = \frac{\mu(f_k)\mu(g_k)-\frac{1}{4}\delta(f_k)\delta(g_k)}{g_{k-\frac{1}{2}}\cdot g_{k+\frac{1}{2}}}$

$R.H.S = \frac{\mu(f_k)\mu(g_k)-\frac{1}{4}\left[\delta(f_k)\delta(g_k)\right]}{g_{k-\frac{1}{2}}\cdot g_{k+\frac{1}{2}}}$

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$$= \left(\frac{f_{k+1/2} + f_{k-1/2}}{2}\right)\left(\frac{g_{k+1/2} + g_{k-1/2}}{2}\right) - \frac{1}{4}\left[\left(f_{k+1/2} - f_{k-1/2}\right)\left(g_{k+1/2} - g_{k-1/2}\right)\right]$$

$$g_{k-1/2}\, g_{k+1/2}$$

$$= \frac{\frac{1}{4} f_{k+1/2} g_{k+1/2} + \frac{1}{4} f_{k+1/2} g_{k-1/2} + \frac{1}{4} f_{k-1/2} g_{k+1/2} + \frac{1}{4} f_{k-1/2} g_{k-1/2} - \frac{1}{4} f_{k+1/2} g_{k+1/2} + \frac{1}{4} f_{k+1/2} g_{k-1/2} + \frac{1}{4} f_{k-1/2} g_{k+1/2} - \frac{1}{4} f_{k-1/2} g_{k-1/2}}{g_{k-1/2}\, g_{k+1/2}}$$

$$= \frac{\frac{1}{2} f_{k+1/2} g_{k-1/2} + \frac{1}{2} f_{k-1/2} g_{k+1/2}}{g_{k-1/2}\, g_{k+1/2}}$$

$$= \frac{1}{2}\frac{f_{k+\frac{1}{2}}}{g_{k+1/2}} + \frac{1}{2}\frac{f_{k-1/2}}{g_{k-1/2}}$$

$$= \frac{1}{2}\left[\frac{f_{k+1/2}}{g_{k+1/2}} + \frac{f_{k-1/2}}{g_{k-1/2}}\right]$$

$$= u\left[\frac{f_k}{g_k}\right] = \text{L.H.S}$$

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⇒ Interpolation:-

It is the technique of estimating the value of a function $y=f(x)$ for any intermediate value of the indipendent variable 'x' when the value of function corresponding to a number of values of x are given.

Example:-

If given for function $y=f(x)$

x :	0	1	2	3	4
f(x):	3	7	5.6	11	15

Then to find $f(x)$; $0<x<4$ is called interpolation.

⇒ Extrapolation:-

The process of estimating the value of function $y=f(x)$ outside the range of given values is called extrapolation.

Example:- From above example to calculate $f(-0.05)$ or $f(4.5)$ etc is called extra

⇒ Arguments & Entries:-

If for a function $y=f(x)$ for value of indipendent variable x say $x_0, x_1, x_2, \ldots, x_n$. It is given $f(x_0)$, $f(x_1)$, $f(x_2)$, $\ldots$, $f(x_n)$ Then $x_0, x_1, x_2, \ldots, x_n$ are called arguments & $f(x_0)$, $f(x_1)$,

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$\ldots, f(x_n)$ are called entries.

⇒ Equally & Unequally Spaced Arguments:-

If $x_0, x_1, x_2, \ldots, x_n$ are argument for the variable x s.t $x_k - x_{k-1} = h \ \forall \ k = 1, 2, 3, \ldots, n$ then these arguments are said to be equally space arguments, where h is called step size or Interval difference. If the arguments are not equally spaced then they called unequally spaced.

⇒ Interpolating Polynomial:-

Suppose we are given the value of function $f(x)$ at certain points $x_0, x_1, x_2, \ldots, x_n$ and we want to estimate $f(\alpha)$ for some $\alpha \in [x_0, x_n]$. Then we assume that function can be approximated by a polynomial. This polynomial is called interpolating polynomial.

⇒ Forward Difference Formula:-

we know that

$$\Delta f(x) = f(x+h) - f(x)$$

Then we note that

$$\Delta^2 f(x) = \Delta(\Delta f(x))$$

$$= \Delta[f(x+h) - f(x)]$$

$$= f(x+2h) - f(x+h) - f(x+h) + f(x)$$

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$$\Rightarrow \Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

If we denote $f(x_0)$ by y_0, $f(x_1)$ by y_1, ..., $f(x_n)$ by y_n then

$$\Delta y_0 = y_1 - y_0$$

$$\Delta^2 y_0 = \Delta(\Delta y_0)$$

$$= \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$$

$$\Delta^3 y_0 = \Delta(\Delta^2 y_0) = \Delta[\Delta y_1 - \Delta y_0]$$

$$= \Delta^2 y_1 - \Delta^2 y_0$$

In general

$$\Delta y_x = y_{x+1} - y_x$$

$$\Delta^2 y_x = \Delta y_{x+1} - \Delta y_x$$

$$\Delta^3 y_x = \Delta^2 y_{x+1} - \Delta^2 y_x$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\Delta^n y_x = \Delta^{n-1} y_{x+1} - \Delta^{n-1} y_x$$

⇒ Forward Difference Table :-

x	y	1st Difference Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		$y_1 - y_0 = \Delta y_0$			
x_1	y_1		$\Delta y_1 - \Delta y_0 = \Delta(y_1 - y_0) = \Delta\Delta y_0 = \Delta^2 y_0$		
		$y_2 - y_1 = \Delta y_1$		$\Delta^3 y_0$	
x_2	y_2		$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$		$\Delta^4 y_0$
		$y_3 - y_2 = \Delta y_2$		$\Delta^3 y_1$	
x_3	y_3		$\Delta y_3 - \Delta y_2 = \Delta^2 y_2$		
		$y_4 - y_3 = \Delta y_3$			
x_4	y_4				

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Interpolating Polynominal

Types

Equally spaced Data

(Methods)

(i) Newton's Formula (Forward & Backward)
(ii) Gauss Formula (Forward & Backward)
(iii) Stirling's Formula
(iv) Bessel's Formula
(v) Everett's Formula

Unequally Spaced Date

(Methods)

(i) Aitken's Formula
(ii) Lagranges's Formula
(iii) Newton's Divided difference Formula.

Remark:-

(i) We use Formulae of unequally spaced data to calculate equally spaced data interpolating polynomial but equally spaced formulae only applicable on equally spaced data.

(ii) Newton's difference formula is also called Gregory newton's difference formula.

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⇒ FOR Equally Spaced Data:-

1) (ii) Newton's Forward Difference Formula:-

If $x_0, x_1, x_2, \ldots, x_n$ are equally spaced, then for any x

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \ldots\ldots$$

where x_0 = 1st argument, $u = \frac{x-x_0}{h}$

$\therefore h$ is step size.

Proof:- Given $u = \frac{x-x_0}{h}$

$$\Rightarrow uh = x - x_0 \quad \Rightarrow x = x_0 + uh$$

$$\Rightarrow f(x) = f(x_0 + uh)$$

$$= E^u f(x_0) \qquad \because Ef(x) = f(x+h)$$

$$= (1+\Delta)^u f(x_0) \qquad \because E = 1 + \Delta$$

$$\Rightarrow f(x) = \left[1 + \Delta u + \frac{u(u-1)}{2!}\Delta^2 + \frac{u(u-1)(u-2)}{3!}\Delta^3 + \ldots\right] f(x_0)$$

$$= f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \ldots\ldots$$

(Proved)

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Note:- This formula is used when estimated values lies in beginning of given data.

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Question:- $\sin 45^\circ = 0.707$, $\sin 50^\circ = 0.7760$, $\sin 55^\circ = 0.8192$, $\sin 60^\circ = 0.8660$ find $\sin 52^\circ$ by Newton's forward differ formula.

Solution:-

x:	45	50	55	60
$f(x)$:	0.707	0.7760	0.8192	0.8660

find $\sin 52^\circ = ?$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	0.707			
		0.0589		
50°	0.7760		−0.0057	
		0.0532		−0.0007
55°	0.8192		−0.0064	
		0.0468		
60	0.8660			

$x_0 = 45$, $x = 52$, $h = 5$

$$u = \frac{x - x_0}{h} = \frac{52 - 45}{5} = \frac{7}{5} = 1.4$$

Now by Newton's forward difference formula

$$\Rightarrow f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0)$$

$$f(52) = 0.707 + (1.4)(0.0589) + \frac{1.4(1.4-1)}{2!}(-0.0057) + \frac{(1.4)(0.4)(-0.6)}{3!}(-0.0007)$$

$$\Rightarrow f(52) = 0.7880$$

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Question:- $\sqrt[3]{29} = 3.0727$, $\sqrt[3]{30} = 3.1074$, $\sqrt[3]{27} = 3.000$, $\sqrt[3]{28} = 3.0367$. find $\sqrt[3]{26}$ using N.F.D formula.

Solution

As $f(x) = \sqrt[3]{x}$

x :	27	28	29	30
$f(x)$:	3.000	3.0367	3.0727	3.1074

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
27	3.00			
		0.0367		
28	3.0367		−0.0011	
		0.0358		0
29	3.0727		−0.0011	
		0.0347		
30	3.1074			

As $x = 26$, $x_0 = 27$, $h = 1$

$$u = \frac{x - x_0}{h} = \frac{26 - 27}{1} = -1$$

By Newton's forward difference formula.

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0)$$

$$f(26) = 3.00 + (-1)(0.0367) + \frac{(-1)(-2)}{2!}(-0.0011) + \frac{(-1)(-2)(-3)}{3!}(0)$$

$$= 3.00 - 0.0367 - 0.0011$$

$$= 2.9620$$

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Question:- Find polynomial which satisfying following data.

x	1	2	3	4
$f(x)$	1	-1	1	-1

Solution:-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1			
		-2		
2	-1		4	
		2		8
3	1		-4	
		-2		
4	-1			

$x_0 = 1$, x is valueable

$h = 1$

$$u = \frac{x - x_0}{1} = \frac{x-1}{1} = x-1$$

By Newton's forward difference formula

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \cdots$$

$$= 1 + (x-1)(-2) + \frac{(x-1)(x-2)}{2}(4) + \frac{(x-1)(x-2)(x-3)}{3!}(-8) + \cdots$$

$$= 1 - 2x + 2 + 2(x^2 - 2x - x + 2) - \frac{4}{3}(x^3 - 6x^2 + 11x - 6)$$

$$= 1 - 2x + 2 + 2x^2 - 4x - 2x + 4 - \frac{4}{3}x^3 + 8x^2 - \frac{44}{3}x - 8$$

$$= -\frac{4}{3}x^3 + 10x^2 - \frac{68}{3}x + 11$$

$$= \frac{1}{3}(-4x^3 + 30x^2 - 68x + 33)$$

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Question: Given

x:	2	4	6	8	10
$f(x)$:	15	23	7	29	38

find $f(2.3)$ & $f(x)$

Solution:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
2	15				
		8			
4	23		-24		
		-16		62	
6	7		38		-113
		22		-51	
8	29		-13		
		9			
10	38				

$x = 2.3, \quad x_0 = 2, \quad h = 2$

(i) $u = \frac{x - x_0}{2} = \frac{2.3 - 2}{2} = \frac{0.3}{2} = 0.15$

Now $f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(x_0) + \cdots$

$f(2.3) = 15 + (0.15)(8) + \frac{(0.15)(0.15-1)}{2}(-24) + \frac{(0.15)(0.15-1)(0.15-2)}{3!}(62) + \frac{(0.15)(0.15-1)(0.15-2)(0.15-3)}{4!}(-113)$

$= 15 + 1.2 + 1.53 + 2.44 + 3.17$

$= 23.34$

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(ii) Here $u = \frac{x-x_0}{h} = \frac{x-2}{2}$

Hence $f(x) = 15 + \left(\frac{x-2}{2}\right)(8) + \left(\frac{x-2}{2}\right)\left(\frac{x-2}{2}-1\right)(-24)$

$$+ \frac{\left(\frac{x-2}{2}\right)\left(\frac{x-2}{2}-1\right)\left(\frac{x-2}{2}-2\right)}{6}(6) + \frac{\left(\frac{x-2}{2}\right)\left(\frac{x-2}{2}-1\right)\left(\frac{x-2}{2}-2\right)\left(\frac{x-2}{2}-3\right)}{24}(-113)$$

* * *

(ii) **1) Newton's Backward Difference Formula:-**

If for a function f, its value at $x_0, x_1, x_2, \ldots, x_n$ are $f(x_0), f(x_1), f(x_2), \ldots f(x_n)$ then for any value of x

$$f(x) = f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n) + \ldots$$

Proof:- Now $u = \frac{x-x_n}{h}$, where x_n is last argument and h is interval differe.

$$\Rightarrow x = x_n + uh$$

$$f(x) = f(x_n + uh)$$

$$= E^u f(x_n)$$

$$= [(1-\nabla)^{-1}]^u f(x_n)$$

$$= (1-\nabla)^{-u} f(x_n) \quad \because E = (1-\nabla)^{-1}$$

$$= \left[1 + u\nabla + \frac{u(u+1)}{2!}\nabla^2 + \frac{u(u+1)(u+2)}{3!}\nabla^3 + \ldots\right] f(x_n)$$

$$\Rightarrow f(x) = f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n) + \ldots$$

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Note:- This formula is used when estimating value lies at the end of the given data

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Question:- The population of town is given below. Estimate population for year 1965

Year	1931	1941	1951	1961	1971
Population	46	66	81	93	101

Solution

x	y	1st diff.	2nd diff.	3rd diff.	4th diff.
1931	46				
		20			
1941	66		−5		
		15		2	
1951	81		−3		−3
		12		−1	
1961	93		−4		
		8			
1971	101				

$x_n = 1971$, $x = 1965$

$h = 10$

$$u = \frac{x - x_n}{h} = \frac{1965 - 1971}{10} = \frac{-6}{10}$$

$$\Rightarrow u = -0.6$$

Now By Newton's Backward diff. formula

$$f(x) = f(x_n) + u \nabla f(x_n) + \frac{u(u+1)}{2!} \nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}$$

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$$\nabla^3 f(x_n) + \frac{u(u+1)(u+2)(u+3)}{4!} \nabla^4 f(x_n)$$

$+ \ldots\ldots$

$$\therefore \quad f(1965) = 101 + (-0.6)(8) + \frac{(-0.6)(0.4)}{2!}(-4)$$
$$+ \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!}(-1) +$$
$$\frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{4!}(-3)$$
$$= 101 + (-4.8) + 0.48 + 0.0560 + 0.1008$$
$$= 96.837 \qquad \text{Ans.}$$

Question:- By constructing difference table, find 7th term as well as general term of sequence 0, 0, 2, 6, 12, 20

Solution:-

It is easy to calculate general term by using Newton's forward or Newton's back ward difference formula.

The given difference Table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	[0]					
		[0]				
2	0		[2]			
		2		[0]		
3	2		2		[0]	
		4		0		[0]
4	6		2		[0]	
		6		[0]		
5	12		[2]			
		[8]				
6	[20]					

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By Newton's forward difference table

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \cdots$$

$$f(x) = 0 + (x-1)(0) + \frac{(x-2)(x-1)}{2}(2) + 0 \qquad \because u = \frac{x-x_0}{h}, \; u = \frac{x-1}{1}$$

$$= 0 + 0 + (x-2)(x-1)$$

$$= x^2 - 3x + 2$$

which is general term of sequence

To find 7^{th} term put $x = 7$

$$f(x) = x^2 - 3x + 2$$

$$= 49 - 21 + 2 = 30$$

By Newton's Backward difference formula

$x_n = 6 \qquad h = 1$

$$u = \frac{x - x_n}{1} = x - 6$$

$$f(x) = f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n) + \cdots$$

$$= 20 + (x-6)(8) + \frac{(x-6)(x-5)}{2!}(2) + \frac{(x-6)(x-5)(x-4)}{3!}(0)$$

$$= 20 + 8x - 48 + x^2 - 11x + 30$$

$$= x^2 - 3x + 2$$

which is general term of sequence

Put $x = 7$

$$f(7) = 49 - 21 + 2$$

$$= 30$$

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Question:- Find a polynomial which satisfies $y = |x|$ at $x = -2, -1, 0, 1, 2$ by Newton's Backward difference formula.

Solution:-

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
-2	2				
		-1			
-1	1		0		
		-1		2	
0	0		2		4
		1		-2	
1	1		0		
		1			
2	2				

By Newton's Backward difference formula

$$f(x) = f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n)$$

$$= 2 + (x-2)(1) + \frac{(x-2)(x-1)}{2!}(0) + \frac{(x-2)(x-1)(x)}{3!}(-2) + \frac{(x-2)(x-1)(x)(x+1)}{4!}(4)$$

$$= 2 + x - 2 - \frac{1}{3}x^3 + x^2 - \frac{2}{3}x - \frac{1}{6}x^4 + \frac{1}{3}x^3 + \frac{1}{6}x^2 - \frac{1}{3}x$$

$$\Rightarrow f(x) = -\frac{1}{6}x^4 + \frac{7}{6}x^2$$

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Theorem:- The nth forward (backward) difference of a polynomial of degree 'n' is constant and all higher orders differences are zero.

Proof:- Let $f(x)$ be a polynomial of degree 'n'

i.e $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$

then

1st order forward difference is

$$\Delta f(x) = f(x+h) - f(x) \quad \text{---------(i)}$$

As $f(x+h) = a_0 + a_1(x+h) + a_2(x+h)^2 + a_3(x+h)^3 + \cdots + a_n(x+h)^n$

$= a_0 + a_1(x+h) + a^2(x^2+2xh+h^2) + a_3(x^3+3x^2h+3xh^2+h^3) + \cdots + a_n(x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + h^n)$

Put $f(x+h)$ and $f(x)$ in (i), we have

$\Delta f(x) = a_0 + a_1(x+h) + a_2(x+h)^2 + a_3(x+h)^3 + \cdots + a_n(x+h)^n - a_0 - a_1x - a_2x^2 - a_3x^3 - \cdots - a_nx^n$

$\Delta f(x) = a_1(x+h-x) + a_2((x+h)^2 - x^2) + a_3[(x+h)^3 - x^3] + \cdots + a_n[(x+h)^n - x^n]$

$\Delta f(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots + b_{n-1}x^{n-1}$

Now

$\Delta^2 f(x) = \Delta(\Delta f(x))$

$= \Delta(b_0 + b_1x + b_2x^2 + \cdots + b_{n-1}x^{n-1})$

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$$= C_0 + C_1 x + C_2 x^2 + \cdots + C_{n-2} x^{n-2}$$

Hence

1st Forward difference of $f(x)$ = Polynomial of degree $n-1$

2nd " " " " = " " " $n-2$

3rd " " " " = " " " $n-3$

⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮

n^{th} " " " " = " " " $n-n = 0$

$\Rightarrow n^{th}$ forward difference of $f(x)$ is a Polynomial of degree zero which is constant.

$$\Delta^n f(x) = C$$

$$\Delta^{n+1} f(x) = \Delta(\Delta^n f(x))$$

$$= \Delta(C) = C - C = 0$$

$$\Delta^{n+2} f(x) = \Delta(\Delta^{n+1} f(x))$$

$$= \Delta(0) = 0 - 0 = 0$$

Hence all forward differences of order greater than n are zero

Similarly: It is easy to prove for backward difference.

$\Rightarrow$

Let $f(x)$ be a polynomial of degree 'n' i.e $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Then 1st order backward difference is

$$\nabla f(x) = f(x) - f(x-h) \quad \text{---------- (i)}$$

Now $f(x-h) = a_0 + a_1(x-h) + a_2(x-h)^2 + a_3(x-h)^3 + \cdots + a_n(x-h)^n$

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$\Rightarrow f(x-h) = a_0 + a_1(x-h) + a_2(x^2 - 2xh + h^2)$
$+ a_3(x^3 + h^3 - 3x^2h + 3xh^2) + \cdots$
$+ a_n\left(x^n + n(-x)^{n-1}h + \frac{n(n-1)}{2!}(-x)^{n-2}h^2 + \cdots h^n\right)$

Put $f(x+h)$ & $f(x)$ in eqn (i) we have

$\nabla f(x) = [a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n] - [a_0$
$+ a_1(x-h) + a_2(x^2 - 2hx + h^2) + a_3(x^3 + h^3 -$
$3x^2h + 3xh^2) + \cdots + a_n[x^n + n(-x)^{n-1}h$
$+ \frac{n(n-1)}{2!}(-x)^{n-2}h^2 + \cdots + h^n)$

$\nabla f(x) = a_1(x - x + h) + a_2(x^2 - x^2 + 2hx - h^2)$
$+ a_3(x^3 - x^3 + h^3 + 3x^2h - 3xh^2) + \cdots$
$+ a_n\left[x^n - x^n - n(-x)^{n-1}h - \frac{n(n-1)}{2!}(-x)^{n-2}h^2\right.$
$\left. + \cdots + h^n\right]$

$= a_1h + a_2\left(2h - \frac{h^2}{x}\right)x + a_3\left(\frac{h^3}{x^2} + 3h - \frac{3h^2}{x}\right)x^2 + \cdots + a_n\left(nh - \frac{n(n-1)}{2!}\right.$
$\left.\frac{(-x)^{n-2}h^2}{x^{n-1}} + \cdots + \frac{h^n}{x^{n-1}}\right)x^{n-1}$

$\nabla f(x) = b_0 + b_1x + b_2x^2 + \cdots + b_{n-1}x^{n-1}$

Similarly 2nd order difference is given by

$\nabla^2 f(x) = \nabla(\nabla f(x))$
$= \nabla(b_0 + b_1x + b_2x^2 + \cdots + b_{n-1}x^{n-1})$
$= c_0 + c_1x + c_2x^2 + \cdots + c_{n-2}x^{n-2}$

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Hence 1st order backward diff of $f(x)$ = Polynomial of degree $n-1$
2nd " " " " = " " " $n-2$
3rd " " " " = " " " $n-3$
⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮
n^{th} " " " " = " " " $n-n=0$

n^{th} backward difference of $f(x)$ is polynomi of degree zero which is constant.

$$\nabla^n f(x) = C$$

$$\nabla^{n+1} f(x) = \nabla[\nabla^n f(x)] = \nabla(C)$$
$$= C - C = 0$$

$$\nabla^{n+2} f(x) = \nabla[\nabla^{n+1} f(x)] = \nabla(0)$$
$$= 0 - 0 = 0$$

Hence all the backward difference of order greater than n are zero

***

⇒ Centeral Difference Table:-

Central difference is defined as

$$\delta[f(x)] = f(x + h/2) - f(x - h/2)$$

or

$$\delta f(x) = f(x+h) - f(x-h)$$

In the case when step size is $2h$ instead of h

Central difference Table is

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x	$f(x)$	Ist diff	2^{nd} diff.	3^{rd} diff	4^{th} diff
x_{-2}	y_{-2}				
		Δy_{-2}			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		
		Δy_{-1}		$\Delta^3 y_{-2}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$	
x_1	y_1		$\Delta^2 y_0$		
		Δy_1			
x_2	y_2				

(ii) Guass Central Difference Formula:-

(I) Guass forward formula:-

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots$$

where

$$u = \frac{x - x_0}{h}$$

Proof:- By Newton's forward difference formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots \quad \text{——— (i)}$$

Now as $\Delta^{n+1} y_{i-1} = \Delta(\Delta^n y_{i-1})$

$$= \Delta^n (y_{i-1+1} - y_{i-1})$$

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$$= \Delta^n (y_i - y_{i-1})$$

$$\Rightarrow \Delta^{n+1} y_{i-1} = \Delta^n y_i - \Delta^n y_{i-1}$$

$$\Delta^n y_i = \Delta^n y_{i-1} + \Delta^{n+1} y_{i-1}$$

From above

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

&

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$$= \Delta^3 y_{-1} + \Delta^4 y_{-2} + \Delta^5 y_{-2}$$

using all these values in (i) we have

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}[\Delta^2 y_{-1} + \Delta^3 y_{-1}] + \frac{u(u-1)(u-2)}{2!}[\Delta^3 y_{-1} + \Delta^4 y_{-2} + \Delta^5 y_{-2}] + \cdots$$

$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \left[\frac{u(u-1)}{2!} + \frac{u(u-1)(u-2)}{3!}\right]\Delta^3 y_{-1} + \cdots$$

$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \cdots$$

which is required Gauss forward formula for equal interval.

★★★

Note:- Gauss forward formula is used when no. of entries below the line (central line) is greater then above central line.

★ ★★★ ★

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Question:- Find value of y corresponding to $x=5$ from table.

x	2	4	6	8	10
y	0	0	1	0	0

By Gauss forward formula.

Solution:-

The difference table is given by

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_2	2	0				
			0			
x_1	4	0		1		
	→5		1		−3	
x_0	6	1		−2		6
			−1		3	
x_1	8	0		1		
			0			
x_2	10	0				

central line

Put $u = \frac{x-x_0}{h} = \frac{x-6}{2}$

at $x=5$, $u = \frac{5-6}{2} = -0.5$

By Gauss forward formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2}$$

$$f(5) = 1 + (-0.5)(-1) + \frac{(-0.5)(-1.5)}{2}(-2) + \frac{(-0.5)(-1.5)(0.5)}{3!}(3) + \frac{(-0.5)(-1.5)(0.5)(-2.5)}{4!}(6)$$

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$\Rightarrow f(5) = 1 + 0.5 - 0.75 + 0.19 - 0.23$

$= 0.71$

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Question:- Find values of y corresponding to $x = 4.7$ from table by Gauss forward formula

x :	3	4	5	6
y :	6	24	60	120

Solution:-

The difference table is given by

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	
x_{-1}	3	6				
			18			
x_0	4 (4.7)	24		18		central line
			36		6	
x_1	5	60		24		
			60			
x_2	6	120				

$u = \frac{x - x_0}{h} = \frac{x-4}{1}$

At $x = 4.7$, $u = 4.7 - 4 = 0.7$

By Gauss forward formula

$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \cdots$

$f(4.7) = 24 + (0.7)(36) + \frac{(0.7)(-0.3)}{2!}(18) + \frac{(0.7)(-0.3)(1.7)}{3!}(6)$

$= 46.95$ Ans

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(II) Gauss Backward Formula:-

$$f(x) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2} + \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots$$

where $u = \frac{x - x_0}{h}$

Proof :- By Newton's forward difference formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots \quad \text{——— (1)}$$

As $\Delta^{n+1} y_{i-1} = \Delta(\Delta^n y_{i-1})$

$= \Delta^n(\Delta y_{i-1})$

$= \Delta^n(y_{i-1+1} - y_{i-1})$

$= \Delta^n y_i - \Delta^n y_{i-1}$

$\Delta^n y_i = \Delta^n y_{i-1} + \Delta^{n+1} y_{i-1}$

$\Rightarrow \Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$

& $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$

$= \Delta^3 y_{-1} + \Delta^4 y_{-2} + \Delta^5 y_{-2}$

Also $\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$

Using all these values in eqn (1) we have

$$f(x) = y_0 + u(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{u(u-1)}{2!}\left[\Delta^2 y_{-1} + \Delta^3 y_{-2} + \Delta^4 y_{-2}\right]$$

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$$+\frac{u(u-1)(u-2)}{3!}(\Delta^3 y_{-2}+2\Delta^4 y_{-2}+\Delta^5 y_{-3}+\Delta^6 y_{-3})+\cdots$$

$$=y_0+u\Delta y_{-1}+\left[u+\frac{u(u-1)}{2!}\right]\Delta^2 y_{-1}+\left[\frac{u(u-1)}{2!}+\frac{u(u-1)(u-2)}{3!}\right]\Delta^3 y_{-2}+\cdots$$

$$=y_0+u\Delta y_{-1}+u\left[1+\frac{u-1}{2!}\right]\Delta^2 y_{-1}+\frac{u(u-1)}{2!}\left[1+\frac{u-2}{3}\right]\Delta^3 y_{-2}+\cdots$$

$$=y_0+u\Delta y_{-1}+\frac{u(u+1)}{2!}\Delta^2 y_{-1}+\frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2}+\cdots$$ is required formula

★★★

Question:- Find the population of town for 1974 of data by Gauss Backward formula.

Year :	1939	1949	1959	1969	1979	1989
Population:	12	15	20	27	39	52

Solution:-

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	
1939	12						
		3					
1949	15		2				
		5		0			
1959	20		2		3		
		7		3		−10	central line.
x_0 1969	27		5		−7		
→1974 ?		12		−4			
1979	39		1				
		13					
1989	52						

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Taking $x_0 = 1969$, $x = 1974$, $h = 10$

$$u = \frac{x - x_0}{h} = \frac{1974 - 1969}{10} = 0.5$$

By Gauss Backward formula

$$f(x) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2} + \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots$$

$$f(1974) = 27 + 0.5(7) + \frac{(0.5)(1.5)(5)}{2!} + \frac{(0.5)(1.5)(-0.5)(3)}{3!} + \frac{(0.5)(-0.5)(1.5)(2.5)}{4!}(-1) + \frac{0.5(1.5)(-0.5)(2.5)(-1.5)}{5!}(-10)$$

$$= 27 + 3.5 + 1.875 - 0.1875 + 0.2743 - 0.1172$$

$$= 32.345 \quad \text{Ans.}$$

★

Question :- Find Polynomial Satisfying value

x :	2	4	6	8	10
y :	-2	1	3	8	20

By using Gauss formula.

Solution :-

x	y	Δ	Δ^2	Δ^3	Δ^4
2	-2				
		3			
4	1		-1		
		2		4	
x_0 6	3		3		0
		5		4	
8	8		7		
		12			
10	20				

central line.

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$$u = \frac{x - x_0}{h} = \frac{x-6}{2} \quad \text{---(i)}$$

(i) Gauss Backward formula:-

$$f(x) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2} + \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots$$

Put $u = \frac{x-6}{2}$

$$f(x) = 3 + u(2) + \frac{1}{2}(u^2+u)(3) + \frac{1}{6}(u)(u-1)(4) + 0$$

$$= 3 + 2u + \frac{3}{2}u^2 + \frac{3}{2}u + \frac{2}{3}u^3 - \frac{2}{3}u + 0$$

$$= \frac{2}{3}u^3 + \frac{3}{2}u^2 + \frac{17}{6}u + 3 + 0$$

$$= \frac{2}{3}\frac{(x-6)^3}{8} + \frac{3}{2}\frac{(x-6)^2}{4} + \frac{17}{6}\frac{(x-6)}{2} + 3$$

$$f(x) = \frac{1}{12}x^3 - \frac{9}{8}x^2 + \frac{71}{12}x - 10$$

which is required polynomial.

(ii) Gauss forward formula:-

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{1}{3!}\frac{(u)(u-1)(u+1)}{3}\Delta y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots$$

$$= 3 + u(5) + \frac{u(u-1)}{2}(3) + \frac{u(u^2-1)}{6}(4) + 0$$

$$= \frac{2}{3}u^3 + \frac{3}{2}u^2 + \frac{17}{6}u + 3$$

$$f(x) = \frac{2}{3}\frac{(x-6)^3}{8} + \frac{3}{2}\frac{(x-6)^2}{4} + \frac{17}{6}\frac{(x-6)}{2} + 3$$

using (i)

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$$\Rightarrow f(x) = \frac{1}{12}(x^3 - 18x^2 + 108x - 216) + \frac{3}{8}(x^2 + 36 - 12x)$$
$$+ \frac{17}{12}(x - 6) + 3$$
$$= \frac{1}{12}x^3 - \frac{9}{8}x^2 + \frac{71}{12}x - 10$$

EXERCISE

(i) :- Given Data, Find interpolation polynomial using
(i) Newton's Forward diff formula.
(ii) Newton's Backward diff formula.
Also find $f(0.25)$

x:	0.1	0.2	0.3	0.4	0.5
y:	1.4	1.56	1.76	2.0	2.28

(ii) :- Estimate number of students for 1953 from data given below

Year :	1948	1950	1952	1954
No. of Stud:	50	79	102	113

(iii) :- Find Polynomial fitting following data.

x:	-3	-2	-1	0	1
y:	16	7	4	1	-8

Hence calculate y corresponding $x = -1.5$

(iv) Write $y = \sin\frac{\pi x}{2}$ as polynomial when value consider with at $x = 0, 1, 2, 3$

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(v) :- Use Newton's forward difference formula to produce a third degree polynomial satisfying following data.

x	0	2	4	6
f(x)	40	48	88	−224

(vi) Interpolate by means of Gauss's forward formula for f(32) given

$f(25) = 0.2707$, $f(30) = 0.3027$

$f(35) = 0.3386$, $f(40) = 0.3794$

(vii) :- The pressure of wind 'P' corresponding to velocity 'V' is given by following data. Estimate when V=25 find P=?

V:	10	20	30	40
P:	1.1	2	4.4	7.9

Solutions

Q1 :- Difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	1.4	0.16			
0.2	1.56	0.2	0.04	0	
0.3	1.76	0.24	0.04	0.04	0.04
0.4	2.0	0.28	0.08		
0.5	2.28				

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formula

(i) Newton Forward Difference

$x_0 = 0.1$, $h = 0.1$

$$u = \frac{x - x_0}{h} = \frac{x - 0.1}{0.1}$$

$$\Rightarrow u = 10x - 1$$

By formula

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots$$

$$= 1.14 + (10x-1)(0.16) + \frac{(10x-1)(10x-2)}{2}(0.04) + 0 + \frac{(10x-1)(10x-2)(10x-3)(10x-4)}{24}(0.04)$$

$$= 1.14 + (10x-1)(0.16) + \frac{(100x^2 - 20x - 10x + 2)}{2}(0.04) + \frac{(100x^2-20x-10x+2)(100x^2-40x-30x+12)}{24}(0.04)$$

$$= 1.4 + 1.6x - 0.16x + 2x^2 - 0.6x + 0.04 + \frac{(100x^2-30x+2)(100x^2-70x+2)}{24}(0.04)$$

$$= 1.4 + 1.6x - 0.16 + 2x^2 - 0.6x + 0.04 + \left[\frac{10000x^4 - 7000x^3 + 1200x^2 - 3000x^3 + 2100x^2 - 360x + 200x^2}{24} - \frac{140x + 24}{24}\right](0.04)$$

$$= 1.4 + 1.6x - 0.16 + 2x^2 - 0.6x + 0.04 + 16.67x^4 - 11.67x^3 + 2x^2 - 5x^3 + 3.5x^2 - 0.6x + 0.33x^2$$

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$$-0.23x + 0.04$$

$$\Rightarrow y = 16.67x^4 - 16.67x^3 + 7.83x^2 + 0.17x + 1.32$$

$$f(0.25) = 0.0651 - 0.2605 + 0.4894 + 0.0425 + 1.32$$

$$= 1.6565$$

(ii) By Backward Differe formula.

$$u = \frac{x - x_n}{h} = \frac{x - 0.5}{0.1}, \text{ where } x_n = 0.5$$

$$\Rightarrow u = \frac{x - 0.5}{0.1} \times \frac{10}{10} \Rightarrow u = 10x - 5$$

By formula

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots$$

$$= 2.28 + (10x-5)(0.28) + \frac{(10x-5)(10x-4)}{2}(0.08) + \frac{(10x-5)(10x-4)(10x-3)}{6}(0.04) + \frac{(10x-5)(10x-4)(10x-3)(10x-2)}{24}(0.04)$$

$$= 2.28 + 2.8x - 1.4 + \left[\frac{100x^2 - 90x + 20}{2}\right](0.08) + \left(\frac{1000x^3 - 1200x^2 + 470x - 60}{6}\right)(0.04) + \left[\frac{1000x^4 - 2000x^3 - 1200x^3 + 2400x^2 + 4700x^2 - 940x - 600x + 120}{24}\right](0.04)$$

$$= 2.28 + 2.8x - 1.4 + 4x^2 - 3.6x + 0.8 + 6.67x^3 - 8x^2 + 3.13x - 0.4 + 16.67x^4 - 23.33x^3 + 11.83x^2 - 2.57x + 0.2$$

$$\Rightarrow y = 16.67x^4 - 16.67x^3 + 7.83x^2 + 0.17x + 1.32$$

$$\therefore f(0.25) = 1.6565$$

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Q2:- Newton difference Table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1948	50			
		29		
1950	79		-6	
		23		-8
1952	102		-14	
		11		
1954	113			

Here $x = 1953$, $x_n = 1954$, $h = 2$

$$u = \frac{x - x_n}{h} = \frac{1953 - 1954}{2}$$

$$u = -0.5$$

By Newton's Backward differe formula

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots$$

$$f(1953) = 113 + \frac{(-0.5)(-0.5+1)}{2}(-14) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{6}(-8) + (-0.5)(11)$$

$$= 113 - 5.5 + 1.75 + 0.5$$

$$= 109.75$$

Thus The students of year 1953 is 109.75

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Question 3:- Difference table is

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
	-3	16				
			-9			
	-2	7		6		
			-3		-6	
$x_0 \rightarrow$	-1	4		0		0
			-3		-6	
	0	1		-6		
			-9			
	1	-8				

Here $x_0 = -1$, $h = 1$

$$u = \frac{x-(-1)}{1} = x+1$$

By Gauss backward formula

$$f(x) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2} + \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2}$$

$$= 4 + (x+1)(-3) + \frac{(x+1)(x+2)}{2}(0) + \frac{(x+1)(x+2)(x)}{6}(-6) + 0$$

$$= 4 + (-3x-3) + 0 + (x^2+3x+2)(x)(-1)$$

$$f(x) = -x^3 - 3x^2 - 5x + 1$$

$$\therefore f(-1.5) = -(-1.5)^3 - 3(-1.5)^2 - 5(-1.5) + 1$$

$$= 3.375 - 6.75 + 7.5 + 1$$

$$= 5.125$$

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Q4:- $y = \frac{\sin \pi x}{2}$, $x = 0, 1, 2, 3$

Difference Table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	0			
		1		
1	1		−2	
		−1		−2
2	0		0	
		−1		
3	−1			

Here $x_0 = 0$, $u = \frac{x-0}{1} \Rightarrow u = x$; $h = 1$

By Newton's forward difference formula.

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 0 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{1}{6}[x(x-1)(x-2)](-2)$$

$$= x + \frac{1}{2}[-2x^2 + 2x] + \frac{1}{6}[-2x^3 + 4x^2 + 2x^2 - 4x]$$

$$= x - x^2 + x - 0.33x^3 + x^2 - 0.67x$$

$$f(x) = -0.33x^3 + 1.33x$$

Q5:- Difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	40			
		8		
2	48		32	
		40		−384
4	88		−352	
		−312		
6	−224			

$x_0 = 0$, $h = 2$ $u = \frac{x - x_0}{2} = \frac{x}{2}$

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$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 40 + \frac{x}{2}(8) + \frac{\frac{x}{2}\left(\frac{x}{2}-1\right)}{2}(32) + \frac{\frac{x}{2}\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)}{6}(-384)$$

$$= 40 + 4x + \left[\frac{x^2-2x}{2}\right]16 + \left[\frac{x^3-4x^2-2x^2+8x}{8}\right](-64)$$

$$= 40 + 4x + 4x^2 - 8x - 8x^3 + 32x^2 + 16x^2 - 64x$$

$$\Rightarrow f(x) = -8x^3 + 52x^2 - 68x + 40$$

Q6:- Differen table is

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
	25	0.2707			
			0.032		
x_0	30	0.3027		0.0039	
			0.0359		−0.029
	35	0.3386		−0.0251	
			0.0108		
	40	0.3494			

Here $x_0 = 30$, $x = 32$, $h = 5$

$$u = \frac{x - x_0}{h} = \frac{32-30}{5} \Rightarrow u = 0.4$$

By Gauss forward formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \cdots$$

$$f(32) = 0.3027 + (0.4)(0.0359) + \frac{(0.4)(0.4-1)}{2}(0.0039) + \frac{(0.4)(-0.6)(1.4)}{6}(-0.029)$$

$$= 0.3027 + 0.0144 - 0.0047 + 0.0016$$

$$f(32) = 0.314$$

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Q7:- Difference Table is

V	P	Δ	Δ^2	Δ^3
10	1.1			
20	2	0.9	1.5	
30	4.4	2.4	1.1	-0.4
40	7.9	3.5		

Here $x_0 = 30$, $x = 25$, $h = 10$

$$u = \frac{x - x_0}{h} = \frac{25 - 30}{10} \Rightarrow u = -0.5$$

By Gauss backward formula

$$f(v) = P_0 + u\Delta P_{-1} + \frac{u(u+1)}{2!}\Delta^2 P_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 P_{-2} + \cdots$$

$$f(25) = 4.4 + (-0.5)(2.4) + \frac{(-0.5)(0.5)}{2}(1.1) + \frac{(-0.5)(0.5)(-1.5)}{6}(-0.4)$$

$$= 4.4 - 1.2 - 0.1375 - 0.025$$

$$= 3.0375$$

$\Rightarrow$ At the velocity, $v = 25$ the wind pressure $P = 3.0375$

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(iv) Sterling's Formula:-

$$f(x) = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

where $u = \frac{x - x_0}{h}$

Proof:- By Gauss forward formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \cdots \longrightarrow (1)$$

By Gauss Backward formula

$$f(x) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2} + \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots \longrightarrow (2)$$

Taking mean of (1) & (2)

$$f(x) = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

which is required formula.

Question:- If $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$ Find y_{35} by sterlings

Solution:-

$$u = \frac{x - x_0}{h} = \frac{35 - 30}{10} \quad \because x_0 = 30$$

$$= 0.5$$

See the table

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	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_{-1}	20	512			
			−73		
x_0	30	439		−20	
			−93		10
x_1	40	346		−10	
			−103		
x_2	50	243			

By Sterlings formula

$$f(x) = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \cdots$$

$$f(35) = 439 + (0.5)\left[\frac{-73-93}{2}\right] + \frac{(0.5)^2}{2!}(-20)$$

$$= 439 - 41.5 - 2.3$$

$$= 395 \quad \text{Ans.}$$

★ ★ ★

Question:- Use Sterlings formula to find $y(28)$ given that

x:	20	25	30	35	40
y:	49225	48316	47236	45926	44306

Solution:- The difference table is

	x	y	Δ	Δ^2	Δ^3	Δ^4
	20	49225				
			−909			
	25	48316		−171		
x_0	30	47236	−1080	−230	−65	−9
			−1310		−74	
	35	45926		−310		
	40	44306	−1620			

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By sterlings formula $u = \frac{x-30}{5} = \frac{28-30}{5}$

$\Rightarrow u = -0.4$

$$f(x) = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2}$$

$$f(28) = 47236 + (-0.4)\left(\frac{-1080-1310}{2}\right) + \frac{(-0.4)^2}{2!}(-230) + \frac{(-0.4)((0.4)^2-1)}{3!}\cdot\frac{(-65-74)}{2} + \frac{1}{4!}(-0.4)^2(.160-1)(-9)$$

$$= 47236 + 478 - 18.4 - 3892 + 0.0504$$

$\Rightarrow f(28) \approx 47692$ Ans

Assignment

Q1:- Use Sterlings formula estimate $y(2.38)$ from table

x :	2.0	2.2	2.4	2.6	2.8
$f(x)$:	-1.7843	-0.5837	0.0123	1.3976	1.9798

Q2:- Use Sterlings formula find polynomial satisfying following data.

x :	20	30	40	50
$f(x)$:	1313	1727	2392	3493

Q3:- Using sterling formula find $\tan 16^\circ$

θ :	0	5°	10°	15°	20°	25°	30°
$\tan\theta$:	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

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Q1:- Difference Table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	−1.7843				
		1.2006			
2.2	−0.5837		−0.6046		
		0.596		1.3939	
x_0 2.4	0.0123		0.7893		−2.9863
		1.3853		−1.5924	
2.6	1.3976		−0.8031		
		0.5822			
2.8	1.9798				

Here $x_0 = 2.4$, $x = 2.38$, $h = 0.2$

$$u = \frac{x - x_0}{h} = \frac{2.38 - 2.4}{0.2}$$

$$\Rightarrow u = -0.1$$

$$f(x_0) = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2}$$

$$f(2.38) = 0.0123 + (-0.1)\left[\frac{1.3854 + 0.596}{2}\right] + \left(\frac{0.01}{2}\right)[0.7893] + \frac{(-0.1)(0.01-1)}{6}\left[\frac{1.3939 - 1.5924}{2}\right] + \frac{(0.01)(0.01-1)}{24}[-2.9863]$$

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$= 0.0123 - 0.0991 + 0.0039 - 0.0016 + 0.0012$

$\Rightarrow f(2.38) = -0.0833$

★★

Q2:-

x	20	30	40	50
$f(x)$	1313	1727	2392	3493

The difference Table

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
20	1313			
		414		
$x_0 \rightarrow$ 30	1727		251	
		665		185
40	2392		436	
		1101		
50	3493			

Let $x_0 = 30$, $u = \frac{x-30}{10}$

By Sterling's formula

$$f(x) = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u^2(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \cdots$$

$$= 1727 + \left[\frac{x-30}{10}\right]\left[\frac{665+444}{2}\right] + \frac{\left[\frac{x-30}{10}\right]}{2!}(251)$$

$$= 1727 + \left(\frac{x-30}{10}\right)(539.5) + \frac{x^2+900-60x}{200}(251)$$

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$$f(x) = 1727 + \frac{539.5x - 16185}{10} + \frac{251x^2 + 225900 - 1506x}{200}$$

$$= 1727 + 53.95x - 1618.5 + 1.255x^2 + 1129.5 - 75.3x$$

$$f(x) = 1.255x^2 - 21.35x + 1238$$

Q3:- See Page 187

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iv) Bessel's Formula:-

$$f(x) = \frac{1}{2}(y_0 + y_1) + (u - \frac{1}{2})\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right] + \frac{u(u-1)(u-\frac{1}{2})}{3!}\Delta^3 y_{-1} + \frac{u(u+1)(u-1)(u-2)}{4!} \cdot \left[\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right] + \dots\dots$$

Proof:- As we know that Gauss forward formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \dots \quad \text{---} (1)$$

As Gauss Backward formula is

$$f(x) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2} + \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2} + \dots\dots \quad \text{---} (2)$$

Shifting origin from x_0 to x_1 in (2) also u will be changed

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As $u = \frac{x - x_0}{h} \Rightarrow u = \frac{x - x_1}{h}$

$u = \frac{x - (x_0 + h)}{h} = \frac{x - x_0 - h}{h}$

$u = \frac{x - x_0}{h} - 1 \Rightarrow u = u - 1$

value of u will be changed to u-1

Put in ② we get

$$f(x) = y_1 + (u-1)\Delta y_0 + \frac{(u-1)u}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u-2)(u+1)}{4!}\Delta^4 y_{-1} + \cdots \quad \text{③}$$

Taking mean of ① & ③

$$f(x) = \frac{y_0 + y_1}{2} + (u - 1/2)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right] + \frac{u(u-1)(u-1/2)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!}\cdot \left[\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right] + \cdots\cdots$$

which is required Besseles formula.

★★★

Question:- Let $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$. Solve by Bessel's formula taking origin at 24 find y_{25}

Solution:-

The difference table is

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	X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	
x_{-1}	20	2854				
			308			
x_0	24	3162		74		
			382		8	center line
x_1	28	3544		66		
			448			
x_2	32	3992				

$$u = \frac{x - x_0}{h} = \frac{25 - 24}{4}$$

$$u = 1/4 \Rightarrow u = 0.25$$

By Bessel's formula

$$f(x) = \frac{y_0 + y_1}{2} + (u - 1/2)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right] + \frac{u(u-1)(u-1/2)}{3!}\Delta^3 y_{-1} + \cdots$$

$$f(25) = \frac{3162 + 3544}{2} + \left[\frac{1}{4} - \frac{1}{2}\right](382) + \frac{(0.25)(0.25-1)}{2!}\left(\frac{74+66}{2}\right) + \frac{(0.25)(0.25-1)\left(\frac{1}{4} - \frac{1}{2}\right)}{3!}(8)$$

$$= 3353 - 95.5 - 6.5625 - 0.0625$$

$$\Rightarrow y_{25} = 3250.875$$

Question:- Use Bessele's formula to find a polynomial satisfying values

x :	2	4	6	8	10	12
y :	-2	1	3	8	20	32

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Solution The difference table is

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
	2	−2					
			3				
	4	1		−1			
			2		4		
x_0	6	3		3		0	
			5		4		−11
x_1	8	8		7		−11	
			12		−7		
	10	20		0			
			12				
	12	32					

$$u = \frac{x - x_0}{h} = \frac{x-6}{2}$$

By Bessel's formula

$$y = \frac{1}{2}(y_0 + y_1) + (u - \tfrac{1}{2})\Delta y_0 + \frac{1}{2!}u(u-1)\left[\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right] + \frac{1}{3!}u(u-1)(u-\tfrac{1}{2})\Delta^3 y_{-1} + \frac{1}{4!}u(u-1)(u+1)(u-2)\cdot\left[\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right] + \frac{1}{5!}u(u-1)(u+1)(u-2)(u-\tfrac{1}{2})\Delta^5 y_{-2} + \cdots$$

using values

$$y = \frac{1}{2}(3+8) + (u-\tfrac{1}{2})(5) + \frac{1}{2!}u(u-1)\left[\frac{3+7}{2}\right] + \frac{1}{3!}u(u-1)(u-\tfrac{1}{2})(4) + \frac{1}{4!}u(u-1)(u+1)(u-2)\left(\frac{0-11}{2}\right) + \frac{1}{5!}u(u-1)(u+1)(u-2)(u-\tfrac{1}{2})(-11)$$

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$$\Rightarrow y = \frac{11}{2} - 5u - 5/2 + 5/2\,u^2 - 5/2\,u + \frac{2}{3}\left(u^3 - \frac{3}{2}u^2 + \frac{1}{2}u\right)$$
$$- \frac{11}{48}(u^4 - 2u^3 - u^2 + 2u) - \frac{11}{120}(u^5 - 5/2\,u^4 + 5/2\,u^2 - u)$$

$$= \frac{11}{2} + 5u - 5/2 + 5/2\,u^2 - 5/2\,u + 2/3\,u^3 - u^2 + 1/3\,u -$$
$$\frac{11}{48}u^4 + \frac{11}{24}u^3 + \frac{11}{48}u^2 - \frac{11}{24}u - \frac{11}{120}u^5 + \frac{11}{48}u^4$$
$$- \frac{11}{48}u^2 + \frac{11}{120}u$$

$$\Rightarrow y = \frac{-11}{120}u^5 + \frac{9}{8}u^3 + \frac{3}{2}u^2 + \frac{37}{15}u + 3$$

using value of u

$$y = \frac{-11}{120}\left(\frac{x-6}{2}\right)^5 + \frac{9}{8}\left(\frac{x-6}{2}\right)^3 + \frac{3}{2}\left(\frac{x-6}{2}\right)^2 + \frac{37}{15}\left(\frac{x-6}{2}\right) + 3$$

$$= \frac{-11}{3840}\left[x^5 - 30x^4 + 360x^3 - 2160x^2 + 6480x - 7776\right]$$
$$+ 9/64\left[x^3 - 18x^2 + 108x - 216\right] + \frac{3}{8}\left[x^2 - 12x + 36\right]$$
$$+ 37/30\left[x - 6\right] + 3$$

$$= \frac{-11}{3840}x^5 + \frac{11}{128}x^4 - \frac{33}{320}x^3 + \frac{99}{16}x^2 - \frac{297}{16}x + \frac{891}{40}$$
$$+ 9/64\,x^3 - 81/32\,x^2 + \frac{243}{16}x - \frac{243}{8} + \frac{3}{8}x^2$$
$$- 9/2\,x + 27/2 + \frac{37}{30}x - 37/5 + 3$$

$$\Rightarrow y = f(x) = \frac{-11}{3840}x^5 + \frac{11}{128}x^4 - \frac{57}{64}x^3 + \frac{129}{32}x^2$$
$$- \frac{797}{120}x + 1$$

Ans.

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v) Laplace - Everett Formula:-

$$f(x) = \left[\binom{u}{1} y_1 + \binom{u+1}{3}\Delta^2 y_0 + \binom{u+2}{5}\Delta^4 y_{-1} + \cdots\right] - \left[\binom{u-1}{1} y_0 + \binom{u}{3}\Delta^2 y_{-1} + \binom{u+1}{5}\Delta^4 y_{-2} + \cdots\right]$$

where $u = \frac{x - x_0}{h}$ $\left[\binom{u}{1}$ denotes ${}^uC_1\right]$

Proof:- Gauss forward formula is

$$y_x = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots$$

It can be written as

$$y_x = y_0 + {}^uC_1\Delta y_0 + {}^uC_2\Delta^2 y_{-1} + {}^{u+1}C_3\Delta^3 y_{-1} + {}^{u+1}C_4\Delta^4 y_{-2} + {}^{u+2}C_5\Delta^5 y_{-2} + \cdots \quad \text{---(1)}$$

Now $\Delta^{2k+1} y_{-k} = \Delta^{2k}(\Delta y_{-k})$

$= \Delta^{2k}(y_{-k+1} - y_{-k})$

Put $k = 0, 1, 2, \ldots$ in

$$\Delta^{2k+1} y_{-k} = \Delta^{2k} y_{-k+1} - \Delta^{2k} y_{-k}$$

$$\Delta y_0 = y_1 - y_0$$

$$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$$

$$\Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2}$$ & so on

Put values in equ(1)

$$y_x = y_0 + {}^uC_1(y_1 - y_0) + {}^uC_2\Delta^2 y_{-1} + {}^{u+1}C_3(\Delta^2 y_0 - \Delta^2 y_{-1})$$

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$$+ {}^{u+1}C_4 \Delta^4 y_{-2} + {}^{u+2}C_5 (\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \cdots$$

$$\Rightarrow y_x = {}^{u-1}C_0 y_0 + {}^{u}C_1 y_1 - {}^{u}C_1 y_0 - \left[{}^{u+1}C_3 - {}^{u}C_2\right]\Delta^2 y_{-1}$$
$$+ {}^{u+1}C_3 \Delta^2 y_0 - \left[{}^{u+2}C_5 - {}^{u+1}C_4\right]\Delta^4 y_{-2} +$$
$$u+2C_5 \Delta^4 y_{-1} + \cdots\cdots$$
$$= -\left[{}^{u}C_1 - {}^{u-1}C_0\right]y_0 + {}^{u}C_1 y_1 - \left[{}^{u+1}C_3 - {}^{u}C_2\right]\Delta^2 y_{-1}$$
$$+ {}^{u+1}C_3 \Delta^2 y_0 - \left[{}^{u+2}C_5 - {}^{u+1}C_4\right]\Delta^2 y_{-2} +$$
$$u+1C_5 \Delta^4 y_{-1} + \cdots\cdots$$

As we know that

$${}^{n}C_2 + {}^{n}C_{2+1} = {}^{n+1}C_{2+1}$$

$$\Rightarrow {}^{n+1}C_{2+1} - {}^{n}C_2 = {}^{n}C_{2+1}$$

By using this identity

$$y_x = - {}^{u-1}C_1 y_0 + {}^{u}C_1 y_1 - {}^{u}C_3 \Delta^2 y_{-1} + {}^{u+1}C_3 \Delta^2 y_0$$
$$- {}^{u+1}C_5 \Delta^2 y_{-1} + {}^{u+2}C_5 \Delta^4 y_{-1} - \cdots$$

$$\Rightarrow y_x = \left[{}^{u}C_1 y_1 + {}^{u+1}C_3 \Delta^2 y_0 + {}^{u+2}C_5 \Delta^4 y_{-1} + \cdots\right]$$
$$- \left[{}^{u-1}C_1 y_0 + {}^{u}C_3 \Delta^2 y_{-1} + {}^{u+1}C_5 \Delta^4 y_{-2} + \cdots\right]$$

which is required formula.

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Question:- Apply Everett formula to obtain value of $y(25)$ given $y(20)=2854$, $y(24)=3162$, $y(28)=3544$, $y(32)=3992$.

Solution:- The difference table is

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	
	20	2854				
			308			
x_0	24	3162		74		After
			382		−8	
x_1	28	3544		66		Before
			448			
	32	3992				

By Laplace Everett formula

$$y_x = \left[{}^{u}C_1 y_1 + {}^{u+1}C_3 \Delta^2 y_0 + {}^{u+2}C_5 \Delta^4 y_{-1}\right] - \left[{}^{u-1}C_1 y_0 + {}^{u}C_3 \Delta^2 y_{-1} + {}^{u+1}C_5 \Delta^4 y_{-2} + \cdots\right]$$

$$u = \frac{x - x_0}{h} = \frac{25-24}{4} = 0.25$$

$$\Rightarrow y_x = u y_1 + \frac{u(u+1)(u-1)}{3!}\Delta^2 y_0 - \frac{u-1}{1} y_0 - \frac{u(u-1)(u-2)}{3!}\Delta^2 y_{-1}$$

$$\Rightarrow y_{25} = 25(3544) + \frac{(0.25+1)(0.25)(0.25-1)}{3!}(66) - (0.25-1)(3162) - \frac{(0.25)(0.25-1)(0.25-2)(74)}{6}$$

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$\Rightarrow y_{25} = 886 - 2.5781 + 2371.5 - 4.047$

$= 3250.9$ Ans.

Question:- Apply Everett formula to following data to find polynomial which satisfies value of y corresponding to each argument.

x:	0	1	2	3	4	5
y:	0	-1	8	135	704	2375

Solution:- The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	0					
		-1				
1	-1		10			
		9		108		
x_0 2	8		118		216	
		127		324		120
x_1 3	135		442		336	
		569		660		
4	704		1102			
		1671				
5	2375					

take $u = \frac{x - x_0}{h} = \frac{x-2}{1} = x - 2$

$$y = \left[{}^{u}C_1 y_1 + {}^{u+1}C_3 \Delta^2 y_0 + {}^{u+2}C_5 \Delta^4 y_{-1} + \cdots\right]$$

$$- \left[{}^{u+1}C_1 y_0 + {}^{u}C_3 \Delta^2 y_{-1} + {}^{u+1}C_5 \Delta^4 y_{-2} + \cdots\right]$$

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using values

$$y = u(135) + \frac{u(u+1)(u-1)}{3!}(442) + \frac{1}{5!}(u+2)(u+1)(u)(u-1)$$
$$(u-2)(336) - (u+1)8 - \frac{u}{3!}(u-1)(u-2)(118) -$$
$$\frac{1}{5!}(u+1)(u)(u-1)(u-2)(u-3)(216)$$

$$= 135u + \frac{221}{3}(u^3-u) + \frac{14}{5}(u^5-5u^3+4u) - 8u + 8$$
$$- \frac{59}{3}(u^3-3u^2+2u) - \frac{9}{5}(u^5-5u^4+5u^3+5u^2-6u)$$

$$= u^5 + 9u^4 + 31u^3 + 50u^2 + 36u + 8$$

Put $u = x-2$

$$f(x) = (x-2)^5 + 9(x-2)^4 + 31(x-2)^3 + 50(x-2)^2$$
$$+ 36(x-2) + 8$$

$$f(x) = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32 + 9(x^4 - 8x^3 +$$
$$24x^2 - 32x + 16) + 31(x^3 - 6x^2 + 12x - 8) +$$
$$50(x^2 + 4 - 4x) + 36x - 72 + 8$$
$$= x^5 + (-10+9)x^4 + (40 - 72 + 31)x^3 +$$
$$(-80 + 50 + 216 - 186)x^2 + (80 - 288 + 372 + 36$$
$$- 200)x + (-32 + 144 - 248 - 72 + 8 + 200)$$

$$\Rightarrow f(x) = x^5 - x^4 - x^3$$

which is required Polynomial

Bessele's formula:- $\frac{1}{2}[y_0 + y_1] + (u - \frac{1}{2})\Delta y_0 + \frac{1}{2!}u(u-1)\left[\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right] + \frac{1}{3!}u(u-1)(u-\frac{1}{2})\Delta^3 y_{-1} + \frac{1}{4!}u(u-1)(u+1)(u-2)$
$\left[\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right] + \frac{1}{5!}u(u-1)(u+1)(u-2)(u-\frac{1}{2})\Delta^5 y_{-2} + \frac{1}{6!}u(u-1)(u+1)(u-2)$
$(u+2)(u-3)\left[\frac{\Delta^6 y_{-2} + \Delta^6 y_{-3}}{2}\right] + \frac{1}{7!}u(u-1)(u+1)(u-2)(u+2)$
$(u+3)(u-\frac{1}{2})\ldots\ldots$

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⇒ FOR Unequally Spaced Data:-

i) Aitken's Method:- First the table for Aitken's Method is given by

x_k	y_k				$x_i - x$
x_0	y_0				$x_0 - x$
x_1	y_1	$P_{0,1}(x)$			$x_1 - x$
x_2	y_2	$P_{0,2}(x)$	$P_{0,1,2}(x)$		$x_2 - x$
x_3	y_3	$P_{0,3}(x)$	$P_{0,1,3}(x)$	$P_{0,1,2,3}(x)$	$x_3 - x$

where

$$P_{0,k}(x) = \frac{1}{x_k - x_0} \begin{vmatrix} y_0 & x_0 - x \\ y_k & x_k - x \end{vmatrix}$$

$$P_{0,1,k}(x) = \frac{1}{x_k - x_1} \begin{vmatrix} P_{0,1}(x) & x_1 - x \\ P_{0,k}(x) & x_k - x \end{vmatrix}$$

$$P_{0,1,2,k}(x) = \frac{1}{x_k - x_2} \begin{vmatrix} P_{0,1,2}(x) & x_2 - x \\ P_{0,1,k}(x) & x_k - x \end{vmatrix}$$

In above table $P_{0,1,2,3}(x)$ is required answer

Question:- Apply Aitken's Iteration to following data to find $f(2)$

x:	0	1	4	6
$f(x)$:	1	-1	1	-1

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Solution

x	y	$P_{0,k}(x)$	$P_{0,1,k}(x)$	$P_{0,1,2,k}(x)$	$x_k - 2$
0	1				$0-2=-2$
1	-1	$\frac{1}{1-0}\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = -3$			$1-2=-1$
4	1	$\frac{1}{4}\begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} = 1$	$\frac{1}{3}\begin{vmatrix} -3 & -1 \\ 1 & 2 \end{vmatrix} = 5/3$		$4-2=2$
6	-1	$\frac{1}{6}\begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} = \frac{1}{3}$	$\frac{1}{5}\begin{vmatrix} -3 & -1 \\ \frac{1}{3} & 4 \end{vmatrix} = \frac{37}{15}$	$\frac{1}{2}\begin{vmatrix} 5/3 & 2 \\ \frac{37}{15} & 4 \end{vmatrix} =$	$6-2=4$

***

Question:- Apply Aitken's method to find $f(3)$

x:	0	1	2	4
y:	1	1	2	5

Solution:- $x = 3$

x	y	$P_{0,k}(x)$	$P_{0,1,k}(x)$	$P_{0,1,2,k}(x)$	$x_k - 3$
0	1				$0-3=-3$
1	1	$\frac{1}{1-0}\begin{vmatrix} 1 & -3 \\ 1 & -2 \end{vmatrix} = 1$			$1-3=-2$
2	2	$\frac{1}{2-0}\begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} = 5/2$	$\frac{1}{2-1}\begin{vmatrix} 1 & -2 \\ 5/2 & -1 \end{vmatrix} = 4$		$2-3=-1$
4	5	$\frac{1}{4-0}\begin{vmatrix} 1 & -3 \\ 5 & 1 \end{vmatrix} = 4$	$\frac{1}{4-1}\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} = 3$	$\frac{1}{4-2}\begin{vmatrix} 4 & -1 \\ 3 & 1 \end{vmatrix} = \frac{7}{2}$	$4-3=1$

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$\Rightarrow y(3) = 7/2$ is required answer.

* * *

Question:- Use Aitken's iteration to following data to find $f(3)$

x	0	1	2	4	5
y	0	16	48	88	0

Solution:- $x = 3$

x	y	$P_{0,k}(x)$	$P_{0,1,k}(x)$	$P_{0,1,2,k}(x)$	$P_{0,1,2,3,k}(x)$	$x_k - x$
5	0	$\frac{1}{5-0}\begin{vmatrix} 5-3 & 1 \\ 0 & 0 \end{vmatrix} = 0$	$\frac{1}{5-1}\begin{vmatrix} 5-3 & 1 \\ 48 & 24 \end{vmatrix} = 24$	$\frac{1}{5-2}\begin{vmatrix} 5-3 & 1 \\ 72 & 72 \end{vmatrix} = 72$	$\frac{1}{5-4}\begin{vmatrix} 5-3 & 1 \\ 72 & 84 \end{vmatrix} = 84$	2
4	88	$\frac{1}{4-0}\begin{vmatrix} 4-3 & 3 \\ 0 & 88 \end{vmatrix} = 66$	$\frac{1}{4-1}\begin{vmatrix} 4-3 & 1 \\ 48 & 66 \end{vmatrix} = 60$	$\frac{1}{4-2}\begin{vmatrix} 4-3 & 1 \\ 24 & 96 \end{vmatrix} = 78$		1
2	48	$\frac{1}{2-0}\begin{vmatrix} 2-3 & 3 \\ 0 & 48 \end{vmatrix} = 72$	$\frac{1}{2-1}\begin{vmatrix} 2-3 & 1 \\ 72 & 48 \end{vmatrix} = 96$			-1
1	16	$\frac{1}{1-0}\begin{vmatrix} 1-3 & 3 \\ 0 & 16 \end{vmatrix} = 48$				-2
0	0					-3

So required $f(3) = 84$ answer.

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(ii) Lagrange Interpolation Formula:-

Let $f(x_0), f(x_1), \cdots, f(x_n)$ be values of function $y = f(x)$ corresponding to arguments $x_0, x_1, x_2, \cdots, x_n$ not necessarily spaced, then

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} f(x_2) + \cdots\cdots + \frac{(x-x_0)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n)$$

Proof

Suppose $f(x)$ is a polynomial of degree n

Let

$$f(x) = (x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)A_0 + (x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)A_1 + (x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)A_2 + \cdots\cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})A_n \quad \text{---------- (1)}$$

where A's are constant to be determined. Put $x = x_0$

$$\Rightarrow f(x_0) = (x_0-x_1)(x_0-x_2)(x_0-x_3)\cdots(x_0-x_n)A_0$$

$$\Rightarrow A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\cdots(x_0-x_n)}$$

Similarly put $x = x_1$ we get

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$$A_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)}$$

Continuing in this way

$$A_n = \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})}$$

Put all A's values in ① we get

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\cdots(x_0-x_n)} f(x_0)$$
$$+ \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} f(x_1)$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} f(x_2)$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} f(x_n)$$

Which is Lagrang's Interpolation formula.

***

Question:- Find polynomial function $f(x)$ given that $f(0)=2$, $f(1)=3$, $f(2)=12$ & $f(3)=35$ Also find $f(5)$

Solution:-

x	0	1	2	3
$f(x)$	2	3	12	35

By Lagrang's formula

$$f(x) = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)}(2) + \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}(3)$$

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$$+\frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)}(12)+\frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)}(35)$$

$$=\frac{x^3-6x^2+11x-6}{-3}+\frac{3(x^3-5x^2+6x)}{2}$$

$$-6(x^3-4x^2+3x)+\frac{35}{6}(x^3-3x^2+2x)$$

$\Rightarrow f(x) = x^3+x^2-x+2$ Required Polyn

$f(5) = (5)^3+(5)^2-5+2$

$= 125+25-5+2$

$= 147$

★——★★★——★

Question:- Given $\log_{10}654 = 2.8156$, $\log_{10}658 = 2.8182$ $\log_{10}659 = 2.8189$, $\log_{10}661 = 2.8202$. Find $\log_{10}656$

Solution:- Let $f(x) = \log_{10}(x)$

x :	654	658	659	661
$f(x)$:	2.8156	2.8182	2.8189	2.8202

Applying Lagrang's formula

$$\text{Log}_{10}656 = f(656) = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)}(2.8156)$$

$$+\frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)}(2.8182)$$

$$+\frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)}(2.8189)$$

$$+\frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)}(2.8202)$$

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$$\Rightarrow f(656) = \frac{-30}{-140}(2.8156) + \frac{30}{12}(2.8182) + \frac{20}{-10}(2.8189) + \frac{12}{42}(2.8202)$$

$$= 0.6033 + 7.0455 - 5.6378 + 0.8058$$

$$= 2.8168$$

★ ——— ★

Theorem:- Let $L_i(x)$ denotes the Lagrang's polynomial of degree less than or equal to n and

$$\prod_{i=0}^{n}(x) = (x-x_0)(x-x_1)\cdots\cdots(x-x_n)$$

$$\prod_{i=0}^{n}{}'(x) = (x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1}) \cdots\cdots (x_i-x_n)$$

then $L_i(x) = \dfrac{\prod_{i=0}^{n}(x)}{(x-x_i)\prod_{i=0}^{n}{}'(x_i)}$, $i = 0,1,2,\cdots n$

Proof:-

From Lagrang's Interpolation formula

$$L_i(x) = \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})\cdots\cdots(x_i-x_{i+1})\cdots\cdots(x_i-x_n)}$$

Multiply & dividing by $(x-x_i)$ we have

$$L_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_i)(x-x_{i+1})\cdots\cdots(x-x_n)}{(x-x_i)(x_i-x_0)(x_i-x_1)\cdots\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots\cdots(x_i-x_n)}$$

$$= \frac{\prod_{i=0}^{n}(x)}{(x-x_i)\prod_{i=0}^{n}{}'(x)}$$

Proved

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Theorem:- With usual meanings of symbol as in previous theorem, prove that

$$\sum_{i=0}^{n} \frac{\prod_{i=0}^{n}(x)}{(x-x_i)\prod_{i=0}^{n}{}'(x)} = 1 \quad \text{i.e} \quad \sum_{i=0}^{n} L_i(x) = 1$$

Proof:- We know that

$$L_i(x) = \frac{\prod_{i=0}^{n}(x)}{(x-x_i)\prod_{i=0}^{n}{}'(x)} \qquad \text{---------- (1)}$$

Now $\prod_{i=0}^{n}(x) = (x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)$

$$\Rightarrow \frac{1}{\prod_{i=0}^{n}(x)} = \frac{1}{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)}$$

$$= \frac{A_0}{(x-x_0)} + \frac{A_1}{(x-x_1)} + \cdots + \frac{A_i}{x-x_i} + \cdots + \frac{A_n}{x-x_n} \qquad \text{---------- (2)}$$

$$\Rightarrow \frac{1}{\prod_{i=0}^{n}(x)} = \sum_{i=0}^{n} \frac{A_i}{(x-x_i)} \qquad \text{---------- (3)}$$

Multiply (2) by $x-x_i$ on both sides

$$\frac{(x-x_i)}{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_i)\cdots(x-x_n)} = \frac{A_0(x-x_i)}{x-x_0} + \frac{A_1(x-x_i)}{x-x_0} + \cdots + \frac{A_i(x-x_i)}{(x-x_i)} + \cdots + \frac{A_n(x-x_i)}{x-x_n}$$

Putting $x = x_i$

$$\frac{1}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} = A_i$$

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$$\Rightarrow A_i = \frac{1}{\prod_{i=0}^{n}{}'(x_i)}$$ Put in ③

$$\frac{1}{\prod_{i=0}^{n}(x)} = \sum_{i=0}^{n} \frac{1}{(x-x_i)\prod_{i=0}^{n}{}'(x_i)}$$

$$\Rightarrow \sum_{i=0}^{n} \frac{\prod_{i=0}^{n}(x)}{(x-x_i)\prod_{i=0}^{n}{}'(x)} = 1$$

From ① $$\sum_{i=0}^{n} L_i(x) = 1$$

---

* Inverse Lagrange's Interpolation

Formula:- If for a function f, has values $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ at $x_0, x_1, x_2, \ldots, x_n$ then for given $f(x)$

$$x = \frac{[f(x)-f(x_1)][f(x)-f(x_2)][f(x)-f(x_3)]\cdots[f(x)-f(x_n)]x_0}{[f(x_0)-f(x_1)][f(x_0)-f(x_2)][f(x_0)-f(x_3)]\cdots[f(x_0)-f(x_n)]}$$

$$+\frac{[f(x)-f(x_0)][f(x)-f(x_2)][f(x)-f(x_3)]\cdots[f(x)-f(x_n)]x_1}{[f(x_1)-f(x_0)][f(x_1)-f(x_2)][f(x_1)-f(x_3)]\cdots[f(x_1)-f(x_n)]}$$

$$+\frac{[f(x)-f(x_0)][f(x)-f(x_1)][f(x)-f(x_3)]\cdots[f(x)-f(x_n)]x_2}{[f(x_2)-f(x_0)][f(x_2)-f(x_1)][f(x_2)-f(x_3)]\cdots[f(x_2)-f(x_n)]}$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots+$$

$$\frac{[f(x)-f(x_0)][f(x)-f(x_1)][f(x)-f(x_2)]\cdots[f(x)-f(x_{n-1})]x_n}{[f(x_n)-f(x_0)][f(x_n)-f(x_1)][f(x_n)-f(x_2)]\cdots[f(x_n)-f(x_{n-1})]}$$

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Example:- Given

x	1	3	4	5
$f(x)$	6	8	15	20

& $f(x) = 18$ Find x

Solution:- Given that

x:	1	3	4	5
$f(x)$:	6	8	15	20

By using Inverse Lagrang's formula

$$x = \frac{(18-8)(18-15)(18-20)}{(6-8)(6-15)(6-20)}(1) + \frac{(18-6)(18-15)(18-20)}{(8-6)(8-15)(8-20)}(3)$$

$$+\frac{(18-6)(18-8)(18-20)}{(15-6)(15-8)(15-20)}(4) + \frac{(18-6)(18-8)(18-15)}{(20-6)(20-8)(20-15)}(5)$$

$$\Rightarrow x = \frac{(10)(3)(-2)(1)}{(-2)(-9)(-14)} + \frac{(12)(3)(-2)(3)}{(2)(-7)(-12)} + \frac{(12)(10)(-2)(4)}{(9)(7)(-5)}$$

$$+\frac{(12)(10)(3)(5)}{(14)(12)(5)}$$

$$= 0.2381 - 1.2857 + 3.0476 + 2.1429$$

$$\Rightarrow x = 4.1429 \quad \text{Ans}$$

★★★

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(iii) Newton's Divided Difference Formula:-

Divided Difference:-

Let $x_0, x_1, x_2, \ldots, x_n$ be the arguments, which are not necessarily equally spaced and a function f has the values $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ over these arguments respectively. Then the quantities

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \ldots\ldots$$

$$f[x_{n-1}, x_n] = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

are called 1st divided difference &

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \text{ etc,}$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} \text{ etc,}$$

$$\vdots \qquad \vdots$$

$$f[x_0, x_1, x_2, \ldots, x_n] = \frac{f[x_1, x_2, x_3, \ldots, x_n] - f[x_0, x_1, x_2, \ldots, x_{n-1}]}{x_n - x_0}$$

are called 2^{nd}, 3^{rd}, ..., n^{th} divided differences.

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Theorem:- Prove the symmetric property of divided difference (OR) Show that divided difference is independent of order of argument.

Proof:-

For 1st Order:-

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

This can be written as

$$f[x_0, x_1] = \frac{f(x_1)}{x_1 - x_0} - \frac{f(x_0)}{x_1 - x_0}$$

$$= \frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1}$$

$$= \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0}$$

$$\Rightarrow f[x_0, x_1] = \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} \longrightarrow (*)$$

Now

$$f[x_1, x_0] = \frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1} \quad (\because \text{by } * \text{ def})$$

$$= \frac{f(x_0)}{x_0 - x_1} - \frac{f(x_1)}{x_0 - x_1}$$

$$= f[x_0, x_1]$$

$$\Rightarrow f[x_0, x_1] = f[x_1, x_0]$$

For 2nd Order :-

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

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$$\Rightarrow f[x_0,x_1,x_2] = \frac{1}{x_2-x_0}\left[\frac{f(x_1)}{x_1-x_2} + \frac{f(x_2)}{x_2-x_1} - \frac{f(x_0)}{x_0-x_1} - \frac{f(x_1)}{x_1-x_0}\right] \quad (\text{by } *)$$

$$= \frac{1}{x_2-x_0}\left[f(x_1)\left(\frac{1}{x_1-x_2} - \frac{1}{x_1-x_0}\right) + \frac{f(x_2)}{x_2-x_1} - \frac{f(x_0)}{x_0-x_1}\right]$$

$$= \frac{1}{x_2-x_0}\left[f(x_1)\left(\frac{x_1-x_0+x_2-x_1}{(x_1-x_2)(x_1-x_0)}\right) + \frac{f(x_2)}{x_2-x_1} - \frac{f(x_0)}{x_0-x_1}\right]$$

$$= \frac{1}{x_2-x_0}\left[\frac{f(x_1)(x_2-x_0)}{(x_1-x_2)(x_1-x_0)} + \frac{f(x_2)}{x_2-x_1} - \frac{f(x_0)}{x_0-x_1}\right]$$

$$= \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} + \frac{f(x_1)}{(x_0-x_0)(x_1-x_2)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)} \quad \text{---------- (A)}$$

Now by using above concept we write $f[x_1,x_2,x_0]$ as

$$f[x_1,x_2,x_0] = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)} + \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}$$

$$= \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)} \quad \text{---------- B}$$

By A & B

$$f[x_0,x_1,x_2] = f[x_1,x_2,x_0]$$

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$$\text{Q } f[x_0, x_2, x_1] = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)}$$

$$= \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}$$

$$= f[x_0, x_1, x_2] \quad \text{Q So on}$$

And Hence

$$f[x_0, x_1, x_2] = f[x_0, x_2, x_1] = f[x_1, x_0, x_2] = f[x_1, x_2, x_0] = f[x_2, x_0, x_1] = f[x_2, x_1, x_0]$$

Similarly by using definition of $f[x_0, x_1, x_2, \cdots, x_n]$

$$f[x_0, x_1, x_2, \cdots, x_n] = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)\cdots(x_2-x_n)}$$

$$+ \cdots\cdots + \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$

It is easy to show that n^{th} divided difference is also symmetric (Proved)

★★★

Theorem:- Prove that nth divided difference of a polynomial of degree 'n' is constant Q all higher difference are zero.

Proof:- Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n$

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be a polynomial of degree n. Now $x_0, x_1, x_2, \ldots, x_n$ be arguments which are equally step size h i.e

$$x_1 - x_0 = h = x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1}$$

Then

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta f(x_0)}{h} \quad -(*)$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{f[x_1, x_2] - f[x_0, x_1]}{2h}$$

$$= \frac{\Delta f(x_1)/h - \Delta f(x_0)/h}{2h} \quad \text{using } (*)$$

$$= \frac{\Delta^2 f(x_0)}{2h^2}$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{3h}$$

$$= \frac{\frac{\Delta^2 f(x_1)}{2h^2} - \frac{\Delta^2 f(x_0)}{2h^2}}{3h}$$

$$= \frac{\Delta^3 f(x_0)}{6h^3} = \frac{\Delta^3 f(x_0)}{3!\, h^3}$$

Hence in general.

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1st divided difference $= \frac{\Delta}{h} = \frac{\Delta}{1!h}$

2nd " " $= \frac{\Delta^2}{2h^2} = \frac{\Delta^2}{2!h^2}$

3rd " " $= \frac{\Delta^3}{6h^3} = \frac{\Delta^3}{3!h^3}$

$\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$

n^{th} divided difference $= \frac{\Delta^n}{n!h^n}$

Hence nth divided difference of $f(x)$ is

$$f(x) = \frac{1}{n!h^n} [n^{th} \text{ forward difference of } f(x)]$$

$$= \frac{1}{n!h^n} [\text{constant}]$$

$$= \text{constant}$$

& $(n+1)^{th}$ divided difference of $f(x)$ is

$$f(x) = \frac{1}{(n+1)!h^{n+1}} [(n+1)^{th} \text{ forward diff of } f(x)]$$

$$= \frac{1}{(n+1)!h^{n+1}} (0)$$

$$= 0$$

Hence it is proved that n^{th} divided difference of polynomial of degree n is constant & all higher differences are zero.

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⇒ Newton's Divided Difference Formula

Let a function f has values $f(x_0)$, $f(x_1)$, $f(x_2)$, ..., $f(x_n)$ at arguments $x_0, x_1, x_2, \ldots, x_n$ which are not necessarily equally spaced then

$$f(x) = f(x_0) + (x-x_0)f[x_0, x_1] + (x-x_0)(x-x_1)\cdot f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)f[x_0, x_1, x_2, x_3] + \ldots + (x-x_0)(x-x_1)(x-x_2)\ldots(x-x_n)f[x_0, x_1, x_2, \ldots, x_n]$$

Proof:- Consider $f[x, x_0] = \dfrac{f(x_0) - f(x)}{x_0 - x}$

$$\Rightarrow (x_0 - x)f[x, x_0] = f(x_0) - f(x)$$

$$\Rightarrow f(x) = f(x_0) + (x - x_0)f[x, x_0] \longrightarrow (i)$$

Next

$$f[x, x_0, x_1] = \frac{f[x_0, x_1] - f[x, x_0]}{x_1 - x}$$

$$\Rightarrow (x_1 - x)f[x, x_0, x_1] = f[x_0, x_1] - f[x, x_0]$$

$$\Rightarrow f[x, x_0] = f[x_0, x_1] + (x - x_1)f[x, x_0, x_1]$$

by $f[x, x_0]$ value in (i)

$$f(x) = f(x_0) + (x - x_0)\left[f[x_0, x_1] + (x - x_1)f[x, x_0, x_1]\right]$$

$$= f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x, x_0, x_1] \longrightarrow (ii)$$

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Also $f[x, x_0, x_1, x_2] = \dfrac{f[x_0, x_1, x_2] - f[x, x_0, x_1]}{x_2 - x}$

$\Rightarrow f[x, x_0, x_1, x_2](x_2 - x) = f[x_0, x_1, x_2] - f[x, x_0, x_1]$

$\Rightarrow f[x, x_0, x_1] = f[x_0, x_1, x_2] + (x - x_2) f[x, x_0, x_1, x_2]$

Putting in (ii)

$f(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1)$
$[f(x_0, x_1, x_2) + (x - x_2) f[x, x_0, x_1, x_2]]$

$= f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$
$+ (x - x_0)(x - x_1)(x - x_2) f[x, x_0, x_1, x_2]$

Continuining in this way we have

$f(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$
$+ (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3] + \cdots$
$+ (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{n-1}) f[x_0, x_1, x_2, \cdots, x_n]$

Now as n^{th} divided difference of a polynomial of degree n is constant and all higher differences are zero, so being $(n+1)$ st divided difference $f[x, x_0, x_1, \cdots, x_n] = 0$

Thus finally

$f(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$
$+ (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3] + (x - x_0)$
$(x - x_1)(x - x_2)(x - x_3) f[x_0, x_1, x_2, x_3, x_4] + \cdots$
$+ (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{n-1}) f[x_0, x_1, x_2, \cdots$
$x_n]$ proved

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Question:- The observed values of function are respectively 168, 120, 72 & 63 at four positions 3, 7, 9 & 10 of indipendent variable. what is best estimate you can given for value of function at position 6 of indipendent variable.

Solution:- Divided difference table is

x	f(x)	1st D.D	2nd D.D	3rd D.D
3	168			
		$\frac{120-168}{7-3}=-12$		
7	120		$\frac{-24+12}{9-3}=-2$	
		$\frac{72-120}{9-7}=-24$		$\frac{5+2}{10-3}=1$
9	72		$\frac{-9+24}{10-7}=5$	
		$\frac{63-72}{10-9}=-9$		
10	63			

Newton's divided difference formula

$$f(x)=f(x_0)+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]$$
$$+(x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3]$$

As $x=6$, $f(x_0)=168$

$$\Rightarrow f(6)=168+(6-3)[-12]+(6-3)(6-7)(-2)+(6-3)(6-7)(6-9)(1)$$
$$=168-36+6+9$$
$$=147 \quad \text{Ans.}$$

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Question:- Find Polynomial satisfying data by Newton's divided difference formula

x :	0	1	2	3
$f(x)$:	1	2	11	34

Solution- Divided difference table is

x	$f(x)$	1st diff.	2nd diff.	3rd diff.
0	1			
		$\frac{2-1}{1-0} = 1$		
1	2		$\frac{9-1}{2-0} = 4$	
		$\frac{11-2}{2-1} = 9$		$\frac{7-4}{3-0} = 1$
2	11		$\frac{23-9}{3-1} = 7$	
		$\frac{34-11}{3-2} = 23$		
3	34			

Newton's D.D formula is

$$f(x) = f(x_0) + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2) f[x_0, x_1, x_2, x_3]$$

$$\Rightarrow f(x) = 1 + (x-0)(1) + (x-0)(x-1)(4) + (x-0)(x-1)(x-2)(1)$$

$$= 1 + x + 4x^2 - 4x + x(x^2 - 3x + 2)$$

$$= x^3 + x^2 - x + 1 \quad \text{Ans.}$$

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Note:- In this question, arguments are equally spaced. Hence we can find $f(x)$ using Newton's forward or backward formula.

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Question:- Apply Newton's divided difference formula to find no. of person getting 6 rupees from the following data

Income Per Day	3	5	7	8	10
No. of Person	180	154	120	110	90

Solution:- Divided difference table is

x	y	1st D.D	2nd D.D	3rd D.D	4th D.D
3	180				
		$\frac{154-180}{5-3}=-13$			
5	154		$\frac{-17+13}{7-3}=-1$		
		$\frac{120-154}{7-5}=-17$		$\frac{2.33+1}{8-3}=0.67$	
7	120		$\frac{-10+17}{8-5}=2.33$		$\frac{-0.47-0.67}{10-3}=-0.16$
		$\frac{110-120}{8-7}=-10$		$\frac{0-2.33}{10-5}=-0.47$	
8	110		$\frac{-10+10}{10-7}=0$		
		$\frac{90-110}{10-8}=-10$			
10	90				

using Newton's divided difference formula given that $x=6$

$$f(x)=f(x_0)+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]+(x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3]+\dots$$

$$f(6)=180+(6-3)(-13)+(6-3)(6-5)(-1)+(6-3)(6-5)(6-7)(0.67)+(6-3)(6-5)(6-7)(6-8)(-0.16)$$

$$=135$$

No. of persons getting Rs. 6 per day is 135

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Assignment

Q1:- Find polynomial satisfying order pair $(-1,33), (0,5), (2,9)$ & $(5,1335)$ by Newton's D.D formula.

Q2:- Find $\ln(9.2)$ from $\ln(8.80) = 2.0794$ $\ln(9.0) = 2.1972$, $\ln(9.5) = 2.2513$ by Newton's divided difference formula.

Q3:- The following tables are given. What is form of function.

(i)

x	0	1	2	5
y	2	3	12	147

(ii)

x	0	1	4	5
$f(x)$	8	11	78	123

Q4:- Find polynomial of lowest possible degree which assumes values 3, 12, 15, −2 when $x = 3, 2, 1, -1$ respectively by using Newton's divided difference formula.

* **

Solutions*

Q1:-

x :	−1	0	2	5
y :	33	5	9	1335

By Newton D.D formula....

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Divided Difference Table.

x	y	1st D.D	2nd D.D	3rd D.D
-1	33			
		$\frac{5-33}{0+1} = -28$		
0	5		$\frac{2+28}{2+1} = 10$	
		$\frac{9-5}{2-0} = 2$		$\frac{88-10}{5+1} = 13$
2	9		$\frac{442-2}{5-0} = 88$	
		$\frac{1335-9}{5-2} = 442$		
5	1335			

Using Newton's divided difference formula

$$f(x) = f(x_0) + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3]$$

$$= 33 + (x+1)[-28] + (x+1)(x)[10] + (x+1)(x)(x-2)[13]$$

$$= 33 - 28x - 28 + (x^2+x)10 + (x^2+x)(x-2)(13)$$

$$= 33 - 28x - 28 + 10x^2 + 10x + (x^3 - 2x^2 + x^2 - 2x)(13)$$

$$= 33 - 28x - 28 + 10x^2 + 10x + 13x^3 - 26x^2 + 13x^2 - 26x$$

$$f(x) = 13x^3 - 3x^2 - 44x + 5$$

Q 2:- x :- 8.80 9.0 9.5

f(x) 2.0794 2.1972 2.2513

D.D table is

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x	y	1st D.D	2nd D.D
8.80	2.0794		
		$\frac{2.1972-2.0794}{9.0-8.80}=0.589$	
9.0	2.1972		$\frac{0.1080-0.589}{9.5-8.8}=-0.6869$
		$\frac{2.2513-2.1972}{9.5-9.0}=0.1080$	
9.5	2.2513		

By Newton's Divided Difference formula

$f(x)=f(x_0)+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]$

$=2.0794+(9.2-8.80)[0.589]+[9.2-8.80][9.2-9][-0.6869]$

$=2.0794+0.2356-0.0550$

$=2.2601$

Q 3 (i)

x	y	1st D.D	2nd D.D	3rd D.D
0	2			
		$\frac{3-2}{1-0}=1$		
1	3		$\frac{9-1}{2-0}=4$	
		$\frac{12-3}{2-1}=9$		$\frac{9-4}{5-0}=1$
2	12		$\frac{45-9}{5-1}=9$	
		$\frac{147-12}{5-2}=45$		
5	147			

By Newton's D.D formula

$f(x)=f(x_0)+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]$

$+(x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3]+\cdots$

$=2+(x-0)(1)+(x-0)(x-1)(4)+(x-0)(x-1)$

$(x-2)(1)$

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$f(x) = 2 + x + 4x^2 - 4x + x^3 - 2x^2 - x^2 + 2x$

$\Rightarrow f(x) = x^3 + x^2 - x + 2$

(ii)

x	y	1st D.D	2nd D.D	3rd D.D
0	8			
		$\frac{11-8}{1-0} = 3$		
1	11		$\frac{16.75-3}{4} = 3.4375$	
		$\frac{78-11}{4-1} = 16.75$		$\frac{7.0625-3.4375}{5} = 0.725$
4	78		$\frac{45-16.75}{5-1} = 7.0625$	
		$\frac{123-78}{5-4} = 45$		
5	123			

By Newton's D.D formula

$f(x) = f(x_0) + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, x_1, x_2]$
$+ (x-x_0)(x-x_1)(x-x_2) f[x_0, x_1, x_2, x_3] + \cdots$

$= 8 + x(3) + x(x-1)(3.4375) + x(x-1)(x-4)(0.725)$

$= 8 + 3x + 3.4375x^2 - 3.4375x + x(x^2 - 5x + 4)(0.725)$

$= 8 + 3x + 3.4375x^2 - 3.4375x + 0.725x^3 - 3.625x^2 + 2.9x$

$\Rightarrow f(x) = 0.725x^3 - 0.1875x^2 + 2.4625x + 8$

Q4:- See on Page 187

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⇒ Numerical Differentiation:-

It is technique of evaluating $f'(x)$ or $f''(x)$ or $f'''(x)$ etc. when for a function $f(x)$, it is given that at some arguments $x_0, x_1, x_2, \ldots, x_n$ has values $f(x_0), f(x_1), \ldots, f(x_n)$.

* **Technique I:-** For any arguments $x_0, x_1, x_2, \ldots, x_n$ and corresponding values $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ to find $f'(a), f''(a), f'''(a), \ldots$ first find $f(x)$ by appropriate method (Like Newton's Formula, Gauss Formula, Lagrange's Formula etc) and then by $f(x)$ find $f'(x)$ & $f'(a), f''(a), \ldots$

No matter that arguments are equally spaced or unequally spaced or $x \in [x_0, x_n]$ or $x \notin [x_0, x_n]$

Example:- Find $f'(x)$ & $f''(x)$ at $x = 1.5$ from table

x :	1	2	3	4
$f(x)$:	4	11	30	67

Solution:- As given

x	1	2	3	4
$f(x)$	4	11	30	67

By Lagrange Interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

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$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$\Rightarrow f(x)=\frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)}(4)+\frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)}(11)$$

$$+\frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)}(30)+\frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}(67)$$

After simplyfing we have

$$f(x) = x^3+3$$

$$\Rightarrow f'(x) = 3x^2 \quad \& \quad f''(x) = 6x$$

$$f'(1.5) = 3(1.5)^2 = 6.75 \quad \& \quad f''(1.5) = 6(1.5) = 9.0$$

2nd Method:- Using Newton's forward formula

x	$f(x)$	Δ	Δ^2	Δ^3
1	4	7		
2	11		12	
		19		6
3	30		18	
		37		
4	67			

$x_0 = 1$, $h = 1$

$$u = \frac{x-x_0}{h} \Rightarrow u = x-1$$

By Newton's forward formula.

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$$f(x) = f(x_0) + u\,\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3}\Delta^3 f(x_0)$$

$$= 4 + (x-1)(7) + \frac{(x-1)(x-2)}{2!}(12) + \frac{(x-1)(x-2)(x-3)}{3!}(6)$$

$f(x) = x^3 + 3$

$f'(x) = 3x^2$ $\qquad$ $f''(x) = 6x$

$f'(1.5) = 6.75$ $\qquad$ $f''(1.5) = 9$ Ans

Note :- If arguments are not equally spaced then construct difference table and use D.D formula.

---

* **Technique II :-** In This technique Numerical Differentiation at a point included in the table of value of function and given arguments should be equally spaced we can find only $f'(x_k)$, $f''(x_k)$, $f'''(x_k)$, ... with $k = 0, 1, 2, \ldots, n$. In This technique two two formulae used which are

(i) Numerical Differentiation by forward Difference operator

(ii) Numerical Differentiation by backward Difference operator.

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(i) Numerical Differentiation by Forward Difference Operator

As by shift operator

$$Ef(a) = f(a+h)$$
$$= f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots$$

∵ By Taylor expension

$$\Rightarrow Ef(a) = f(a) + hDf(a) + \frac{h^2}{2!}D^2f(a) + \frac{h^3}{3!}D^3f(a) + \cdots$$
$$= \left[1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right]f(a)$$

$$Ef(a) = e^{hD}f(a)$$
$$\Rightarrow E = e^{hD} \quad \text{——— (i)}$$

But if we consider forward difference operator

$$E = 1 + \Delta \quad \text{——— (ii)}$$

By eqn (i) & (ii)

$$e^{hD} = 1 + \Delta$$
$$hD = \ln(1+\Delta) \quad \text{∵ taking log on b.s}$$
$$hD = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots$$
$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right]$$
$$D^n = \frac{1}{h^n}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right]^n \longrightarrow (*)$$

When put $n=1$ in (*) we get

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$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right]$$

$$f'(a) = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right] f(a) \quad \text{——①}$$

when $n = 2$

$$f''(a) = D^2 f(a)$$

$$= \frac{1}{h^2}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right]^2 f(a)$$

$$= \frac{1}{h^2}\left[\Delta^2 - \frac{1}{2}\Delta^3 - \frac{1}{2}\Delta^3 + \frac{1}{4}\Delta^4 + \frac{1}{3}\Delta^4 + \frac{1}{3}\Delta^4 - \frac{1}{4}\Delta^5 - \frac{1}{6}\Delta^5 - \frac{1}{6}\Delta^5 - \frac{1}{4}\Delta^5 + \cdots\right] f(a)$$

$$= \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \cdots\right] f(a) \quad \text{——②}$$

when $n = 3$

$$f'''(a) = \frac{1}{h^3}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right]^3 f(a)$$

$$= \frac{1}{h^3}\left[\Delta^3 - \frac{3}{2}\Delta^4 + \frac{7}{4}\Delta^5 - \frac{15}{8}\Delta^6 + \cdots\right] f(a) \quad \text{——③}$$

①, ② & ③ are 1st, 2nd & 3rd derivatives of $f(x)$

* ——— *

Example:- Given data

X :	0	1	2	3	4	5
Y :	3	4	19	84	259	628

Find $f'(1)$ & $f''(1)$

Solution:- The difference table is

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	X	Y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
	0	3					
			1				
→	1	4		14			
			15		36		
	2	19		50		24	
			65		60		0
	3	84		110		24	
			175		84		
	4	259		194			
			369				
	5	628					

Here $a = 1$

The 1st derivative is given by

$$Df(a) = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} - \cdots\right] f(a)$$

$$f'(1) = \frac{1}{h}\left[\Delta f(1) - \frac{\Delta^2 f(a)}{2} + \frac{\Delta^3 f(a)}{3} - \frac{\Delta^4 f(a)}{4} + \cdots\right.$$

$$= \frac{1}{1}\left[15 - \frac{1}{2}(50) + \frac{1}{3}(60) - \frac{1}{4}(24)\right]$$

$$= 15 - 25 + 20 - 6$$

$$f'(1) = 4$$

The 2nd derivative is given by

$$D^2 f(a) = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \cdots\right] f(a)$$

$$f''(1) = \frac{1}{1}\left[50 - 60 + \frac{11}{12}(24)\right]$$

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$\Rightarrow f''(1) = 50 - 60 + 22 = 12$

**

Example:- Calculate 1st three derivative from data

x	0	1	2	3	4	5
y	2	2	8	26	62	128

at $x = 2$

Solution:- The difference table is

	x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
	0	2					
			0				
	1	2		6			
			6		6		
→	2	8		12		0	
			18		6		0
	3	26		18		0	
			36		6		
	4	62		24			
			60				
	5	128					

At $x = 2$ 1st derivative is

$$\therefore f(2) = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots\right] f(2)$$

$$= \frac{1}{1}\left[18 - \frac{1}{2}(18) + \frac{1}{3}(6)\right] = 18 - 9 + 2$$

$$f'(2) = 11$$

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The 2nd derivative at $x=2$

$$f''(2) = \frac{1}{h^2}[\Delta^2 - \Delta^3]f(2)$$

$$= \frac{1}{1}[18-6] \Rightarrow f''(2) = 12$$

3rd derivative at $x=2$

$$f'''(2) = \frac{1}{h^3}[\Delta^3]f(2) = \frac{1}{1}(6)$$

$$= 6$$

** ——— * ——— **

(ii) Numerical Differentiation By Backward Difference Operator

Consider $Ef(a) = f(a+h)$

$$\Rightarrow Ef(a) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots$$

$$= f(a) + hDf(a) + \frac{h^2D^2}{2!}f(a) + \frac{h^3D^3}{3!}f(a) + \cdots$$

$$= \left[1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right]f(a)$$

$$Ef(a) = e^{hD} f(a)$$

$$\Rightarrow E = e^{hD}$$

But $E = (1-\nabla)^{-1}$ ∵ by backward O.O

$$\Rightarrow (1-\nabla)^{-1} = e^{hD}$$

$$\Rightarrow hD = (-1)\ln(1-\nabla)$$

$$= \nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \cdots$$

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$$\Rightarrow D = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \cdots\right]$$

So in general

$$D^n = \frac{1}{h^n}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \cdots\right]^4$$

For $n = 1$

$$D f(a) = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \cdots\right] f(a)$$

For $n = 2$

$$D^2 f(a) = \frac{1}{h^2}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \cdots\right]^2 f(a)$$

$$= \frac{1}{h^2}\left[\nabla^2 + \frac{\nabla^4}{4} + \frac{\nabla^3}{2} + \frac{\nabla^4}{3} + \frac{\nabla^4}{3} + \frac{1}{4}\nabla^5 + \frac{1}{6}\nabla^5 + \frac{1}{6}\nabla^5 + \frac{1}{4}\nabla^5 + \cdots\right] f(a)$$

$$= \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \cdots\right] f(a)$$

For $n = 3$

$$D^3 f(a) = \frac{1}{h^3}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \cdots\right] f(a)$$

$$= \frac{1}{h^3}\left[\nabla^3 + \left(\frac{1}{2}+1\right)\nabla^4 + \left(\frac{1}{3}+\frac{1}{2}+\frac{11}{12}\right)\nabla^5 + \left(\frac{1}{4}+\frac{1}{3}+\frac{11}{24}+\frac{5}{6}\right)\nabla^6 + \cdots\right] f(a)$$

$$= \frac{1}{h^3}\left[\nabla^3 + \frac{3}{2}\nabla^4 + \frac{7}{4}\nabla^5 + \frac{15}{8}\nabla^6 + \cdots\right] f(a)$$

& so on

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Remark:- If we want to calculate the derivative of a function at a point which is given in the beginning of table, we use forward difference formula, and if we wish to calculate derivative near end points of the data, then we use some backward difference formula for differentiation.

** ——— * ——— **

Example:- Find $f'(4)$ & $f''(4)$ from table

x	0	1	2	3	4	5
f(x)	3	4	19	84	259	628

Solution:- The difference table is

x	f(x)	∇	∇^2	∇^3	∇^4	∇^5
0	3					
		1				
1	4		14			
		15		36		
2	19		50		24	
		65		60		0
3	84		110		24	
		175		84		
→ 4	259		194			
		369				
5	628					

Now here $a = 4$

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The first derivative of the function is given by

$$Df(4) = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \cdots\right] f(a)$$

$$= \frac{1}{1}\left[\nabla f(a) + \frac{\nabla^2}{2} f(a) + \frac{\nabla^3}{3} f(a) + \cdots\right]$$

$$= 1\left[175 + \frac{1}{2}(110) + \frac{1}{3}(60) + \frac{1}{4}(24)\right]$$

$$= 175 + 55 + 20 + 6$$

$$f'(4) = 256$$

2nd derivative of function at $x = 4$

$$f''(4) = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \cdots\right] f(a)$$

$$= 1\left[110 + 60 + \frac{11}{12}(24)\right] = 110 + 60 + 22$$

$$f''(4) = 192$$

★★★

Question- Compute first three derivative of $f(x)$ at $x = 2$ from data

x	0	1	2	3	4	5
y	-6	-10	-8	6	38	94

Solution

	x	y	∇	∇^2	∇^3	∇^4	∇^5
	0	-6					
			-4				
	1	-10		6			
			2		6		
→	2	[-8]		12		0	
			[14]		6		0
	3	6		[8]		0	
			32		[6]		
	4	38		24			
			56				
	5	94					

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Here $a = 2$

First derivative is given by

$$Df(a) = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right] f(a)$$

$$Df(2) = \frac{1}{1}\left[\Delta f(2) - \frac{\Delta^2}{2} f(2) + \frac{\Delta^3}{3} f(2) - \cdots\right]$$

$$= 14 - \frac{1}{2}(18) + \frac{1}{3}(6)$$

$$= 14 - 9 + 2 \Rightarrow f'(2) = 7$$

2nd derivative is

$$D^2 f(a) = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \cdots\right] f(a)$$

$$D^2 f(2) = \frac{1}{1}\left[\Delta^2 f(2) - \Delta^3 f(2) + \cdots\right]$$

$$= 14 - 18 \Rightarrow f''(2) = -4$$

3rd derivative is

$$D^3 f(a) = \frac{1}{h^3}\left[\Delta^3 - \frac{3}{2}\Delta^4 + \cdots\right] f(a)$$

$$f''(2) = \frac{1}{1}\left[\Delta^3 f(2)\right]$$

$$f''(2) = 6$$

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Assignment

Q1:- Find 1st three derivative of $f(x)$ at $x = 2.2$

x :	0	2	3	4
$f(x)$:	-2	16	40	82

Q2:- Using backward difference operator formula find three 1st derivative

x :	0	1	2	3	4	5
$f(x)$:	2	2	8	26	62	122

Q3:- Find minimum value of $f(x)$ from table

x :	0.60	0.65	0.70	0.75
$f(x)$:	0.6221	0.6055	0.6138	0.6170

Q4:- Calculate $f'(x)$ and $f''(x)$ at $x = 1.2$ the value of $f(x) = \text{Tan}^{-1}(e^x)$ tabulated below

x :	1	1.1	1.2	1.3	1.4
$\text{Tan}^{-1}(e^x)$:	1.21828	1.24946	1.27824	1.30473	1.32902

Q5:- Find 1st two derivative at $x = 4$

x :	1	2	3	4	5	6	7
$f(x)$:	4	11	20	67	128	219	346

Q6:- Find radius of curvature at 0.6 on curve defined by following

x :	0.0	0.2	0.4	0.6	0.8	1.0	1.2
y :	0.710	1.175	1.811	2.666	3.801	5.292	7.232

Hint:- Find $y'(0.6)$ & $y''(0.6)$ and put in

$$\rho = \frac{\{1 + (y')^2\}^{3/2}}{y''}$$

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Solutions.

Q1:- x : 0 2 3 4

$f(x)$: −2 16 40 82

D.D Table is

x	y	1st D.D	2nd D.D	3rd D.D
0	−2			
		$\frac{16+2}{2-0}=9$		
2	16		$\frac{24-9}{3-0}=5$	
		$\frac{40-16}{3-2}=24$		$\frac{9-5}{4-0}=1$
3	40		$\frac{42-24}{4-2}=9$	
		$\frac{82-40}{4-3}=42$		
4	82			

By Newton's D.D formula

$f(x) = f(x_0)+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]+\cdots$

$= -2 + x(9) + x(x-2)(5) + x(x-2)(x-3)(1)$

$= -2 + 9x + 5x^2 - 10x + x(x^2-5x+6)$

$= -2 + 9x + 5x^2 - 10x + x^3 - 5x^2 + 6x$

$= x^3 + 5x - 2$

$\Rightarrow f(x) = x^3 + 5x - 2$

$f'(x) = 3x^2 + 5 \Rightarrow f'(2\cdot2) = 19.52$

$f''(x) = 6x \Rightarrow f''(2\cdot2) = 13.2$

$f'''(x) = 6 \Rightarrow f'''(2\cdot2) = 6$

* * *

Argument میں space کے بارے derivatives پہلے سے دی گئی ہو تو ہم 2 Technique استعمال کرتے ہوئے Tech 1 سے کسی ایک پر سوال کر سکتا ہے

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Q2:- Difference table :-

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	2					
		0				
1	2		6			
		6		6		
2 →	8		12		0	
		18		6		0
3 →	26		18		0	
		36		6		
4 →	62		24			
		60				
5	122					

By Backward difference operator

$$f'(2) = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \cdots\right] f(2)$$

$$= \frac{1}{h}\left[\nabla f(2) + \frac{1}{2}\nabla^2 f(2)\right]$$

$$= \frac{1}{1}\left[6 + \frac{1}{2}(6)\right] = 6 + 3$$

$$f'(2) = 9$$

$$f''(2) = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 \cdots\right] f''(2)$$

$$= \frac{1}{1}[6] = 6$$

$$f'''(2) = \frac{1}{h^3}\left[\nabla^3 + \cdots\right]$$

$$= 0$$

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Q3:- Difference Table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.60	0.6221			
		−0.0066		
0.65	0.6155		0.0049	
		−0.0017		0
0.70	0.6138		0.0049	
		0.0032		
0.75	0.6170			

By Newton's forward difference operator

$$f(x) = f(x_0) + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

Here $u = \frac{x - 0.60}{0.5} \Rightarrow u = 2x - 1.2$

$$\Rightarrow f(x) = 0.6221 + (2x-1.2)(-0.0066) + \frac{(2x-1.2)(2x-2.2)}{2}(0.0049)$$

$$= 0.6221 + (2x-1.2)(-0.0066) + \frac{4x^2 - 4.4x - 2.4x + 2.64}{2}(0.0049)$$

$$= 0.6221 - 0.0132x + 0.00792 + 0.0098x^2 - 0.01078x - 0.0059x + 0.00647$$

$$\Rightarrow f(x) = 0.0098x^2 - 0.0299x + 0.6365$$

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Q4:- Difference Table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1.21828				
		0.03118			
1.1	1.24946		−0.0024		
		0.02878		0.00011	
→ 1.2	1.27824		−0.00229		0
		0.02649		0.00011	
1.3	1.30473		−0.00218		
		0.02431			
1.4	1.32904				

By Newton's forward difference operator

$$f'(1.2) = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right] f(1.2)$$

$$= \frac{1}{0.1}\left[0.02649 - \frac{(-0.00218)}{2}\right]$$

$$= \frac{1}{0.1}[0.02649 - 0] = 0.2649$$

$$f''(1.2) = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \cdots\right] f(1.2)$$

$$= \frac{1}{(0.1)^2}[-0.00218] = \frac{1}{0.01}[-0.00218]$$

$$= -0.218$$

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Q5:- Difference Table is

x	f(x)	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1	4						
		7					
2	11		2	?			
		9		36			
3	20		38		-60		
		47		-24		100	
→ 4	67		14		40		-150
		→ 61		16		-50	
5	128		→ 30		-10		
		91		→ 06			
6	219		36				
		127					
7	346						

By Backward difference operator

$$f'(4) = \frac{1}{h}\left[\nabla - \frac{\nabla^2}{2} + \frac{\nabla^3}{3} - \frac{\nabla^4}{4} + \cdots\right] f(4)$$

$$= \frac{1}{1}\left[\nabla f(4) - \frac{1}{2}\nabla^2 f(4) + \frac{1}{3}\nabla^3 f(4)\right]$$

$$= \frac{1}{1}\left[61 - \frac{30}{2} + \frac{6}{3}\right] = 61 - 15 + 2$$

$$= 48$$

$$f''(4) = \frac{1}{h^2}\left[\nabla^2 - \nabla^3 + \frac{11}{12}\nabla^4 - \cdots\right] f(4)$$

$$= \frac{1}{1}\left[30 - 6\right]$$

$$= 24$$

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Q6:- Differen table is

	x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
	0	0.710						
			0.465					
	0.2	1.175		0.171				
			0.636		0.048			
	0.4	1.811		0.219		0.013		
			0.855		0.061		0.002	
→	0.6	2.666		0.28		0.015		0
			1.135		0.076		0.002	
	0.8	3.801		0.356		0.017		
			1.491		0.093			
	1	5.292		0.449				
			1.94					
	1.2	7.232						

$$y'(0.6) = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3}\right] f(0.6)$$

$$= \frac{1}{0.2}\left[1.135 - \frac{0.356}{2} + \frac{0.093}{3}\right]$$

$$= \frac{1}{0.2}[0.988] \Rightarrow y'(0.6) = 4.94$$

$$y''(0.6) = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^3 + \cdots\right] f(0.6)$$

$$= \frac{1}{(0.2)^2}[0.356 - 0.093] \Rightarrow y''(0.6) = 6.575$$

$$\rho = \frac{[1+(y')^2]^{3/2}}{y''} \Rightarrow \rho = \frac{[1+(4.94)^2]^{3/2}}{6.575}$$

$$\rho = \frac{[1+24.4036]^{3/2}}{6.575} \Rightarrow \rho = 19.4736$$

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⇒ Numerical Integration:-

Sometimes it is either difficult or impossible to evaluate the definate integral by using well known analytic method e.g.

$$\int_0^{\pi} \frac{x + \tan^{-1}(x)}{\ln(1+\sin x) + e^{-x^2}}\, dx$$

In such cases we use numerical integration which gives us an approximation to the value of defined integration. There are several method available for numerical integration, but the most commonly used methods be clasified in two groups

(i) Newton - Cote's Rule
(ii) Gauss - Quadrature Rule

Numerical Integration

Newton Cotes Rule	Gauss Quadrature Rule
When $n=1$	
Trapezoidal Rule	(i) Two - point formula
When $n=2$	
Simpson's $\frac{1}{3}$ Rule	
When $n=3$	(ii) Three - point formula
Simpson's $\frac{3}{8}$ Rule	
When $n=4$	
Boole's Rule	
When $n=6$	
Weddle's Rule	

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(i) Newton-Cotes Rule:-

Let $y=f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ or $[f(x_0), f(x_1), f(x_2), \ldots, f(x_n)]$ equally spaced arguments at $x_0, x_1, x_2, \ldots, x_n$ respectively

Let $x_1 - x_0 = x_2 - x_1 = h$

Then $x_1 = x_0 + h$

$x_2 = x_1 + h = x_0 + h + h = x_0 + 2h$

$x_3 = x_2 + h = x_0 + 2h + h = x_0 + 3h$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$x_n = x_0 + nh$

where $u = \dfrac{x - x_0}{h}$

$\Rightarrow x = x_0 + uh \Rightarrow dx = h\,du$

When $x = x_0$, $u = 0$; $x = x_n$, $u = \dfrac{x_n - x_0}{h} = \dfrac{nh}{h} = n$

Now $\displaystyle\int_{x_0}^{x_n} f(x)\,dx = \int_0^n f(x_0 + uh)\,h\,du$

$\displaystyle = h\int_0^n f(x_0 + uh)\,du$

$\displaystyle = h\int_0^n (1+\Delta)^u f(x_0)\,du$ $\quad \because E^n f(x_0) = (1+\Delta)^n f(x_0)$

$\Rightarrow \displaystyle\int_{x_0}^{x_n} f(x)\,dx = h\int_0^n \left[f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \cdots\right] du$

$\because$ by Newton's forward difference.

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$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{(u^2-u)}{2}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{6}\Delta^3 y_0 + \frac{(u^4-6u^3+11u^2-6u)}{24}\Delta^4 y_0 + \cdots\cdots\right] du$$

$$= h\left[y_0 u\Big|_0^n + \Delta y_0 \frac{u^2}{2}\Big|_0^n + \frac{\Delta^2 y_0}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Big|_0^n + \frac{\Delta^3 y_0}{6}\left(\frac{u^4}{4} - u^3 + u\right)\Big|_0^n + \frac{\Delta^4 y_0}{24}\left(\frac{u^5}{5} - \frac{6}{4}u^4 + \frac{11u^3}{3} - \frac{6u^2}{2}\right)\Big|_0^n + \cdots\cdots\right]$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left[\frac{n^3}{3} - \frac{n^2}{2}\right]\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 - n^2\right)\Delta^3 y_0 + \frac{1}{24}\left(\frac{n^5}{5} - \frac{3}{2}n^4 + \frac{11}{3}n^3 - 3n^2\right)\Delta^4 y_0 + \cdots\right]$$

This is called Newton's cote's formula.

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1) Trapezoidal Rule:-

In order to evaluate $\int_a^b f(x)\,dx$. Let us divide $[a,b]$ into 'n' sub intervals each of length 'h' where $h = \frac{b-a}{n}$, then

$P = \{x_0, x_1, x_2, \cdots\cdots, x_n\}$ is partition of $[a,b]$ with $x_k - x_{k-1} = h \;\forall\, k$

Now for all

$$\int_{x_0}^{x_n} f(x) = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots\cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx \quad \text{——} (\star)$$

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1st we consider $\int_{x_0}^{x_1} f(x)\,dx$

In this case $f(x)$ is polynomial of degree 1. So This has 1st forward difference is constant and all higher order differences are zero

Now put $n=1$ in Newton Cote's formula

$$\int_{x_0}^{x_1} f(x) = h\left[y_0 + \frac{1}{2}\Delta y_0 + 0 + 0 + \cdots + 0\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] \quad \because \Delta y_0 = y_1 - y_0$$

$$= \frac{h}{2}\left[2y_0 + y_1 - y_0\right]$$

$$= \frac{h}{2}(y_0 + y_1)$$

Similarly; $\int_{x_1}^{x_2} f(x)\,dx = \frac{h}{2}(y_1 + y_2)$

$$\int_{x_2}^{x_3} f(x)\,dx = \frac{h}{2}(y_2 + y_3)$$

.....

$$\int_{x_{n-1}}^{x_n} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Put in ⊛ we get

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1}) + y_n\right]$$

which is Trapezoidal Rule

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2) Simpson's (1/3) Rule :-

In order to evaluate $\int_a^b f(x)\,dx$. Let divide $[a,b]$ into 'n' subintervals where **n should be even.**

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx \quad \text{——— (A)}$$

$\therefore n=2$

1st we find $\int_{x_0}^{x_2} f(x)\,dx$

In This case $f(x)$ is polynomial of degree 2. So that's 1st & 2nd forward differences are constant & higher order differences are zero.

Put $n=2$ in Newton's Cote's formula

$$\int_{x_0}^{x_2} f(x)\,dx = h\left[2y_0 + \frac{4}{2}\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - \frac{4}{2}\right)\Delta^2 y_0 + 0 + 0 + \cdots\right]$$

$$= h\left[2y_0 + 2\Delta y_0 + \frac{1}{3}\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= h\left[2y_0 + 2y_1 - 2y_0 + \frac{1}{3}y_2 - \frac{2}{3}y_1 + \frac{1}{3}y_0\right]$$

$$= \frac{h}{3}(6y_1 + y_2 - 2y_1 + y_0)$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly

$$\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

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$$\int_{x_4}^{x_6} f(x)\,dx = \frac{h}{3}(y_4 + 4y_5 + y_6)$$

.....

$$\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

So by ⊛

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\Big[y_0 + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + \underbrace{2(y_2 + y_4 + y_6 + \cdots + y_{n-2})}_{**} + y_n\Big]$$

3) Simpson's (3/8) Rule :-

In order to evaluate $\int_a^b f(x)\,dx$. Let us denotes (n is multiple of 3) and divide $[a, b]$ into n intervals

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + \cdots + \int_{x_{n-3}}^{x_n} f(x)\,dx \quad \text{⊛}$$

Consider $\int_{x_0}^{x_3} f(x)\,dx$ to find that integral

Put $n = 3$ in Newton Cote's formula.

we have

$$\int_{x_0}^{x_3} f(x)\,dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= \frac{3h}{8}\left[8y_0 + 12\Delta y_0 + 6\Delta^2 y_0 + \Delta^3 y_0\right]$$

$$= \frac{3h}{8}\left[8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}\left[8y_0 + 12y_1 - 12y_0 + 6y_2 - 12y_1 + 6y_0 + 3y_1 + y_3 - 3y_2 - y_0\right]$$

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$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly $\int_{x_3}^{x_6} f(x)\,dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$

$$\int_{x_6}^{x_9} f(x)\,dx = \frac{3h}{8}[y_6 + 3y_7 + 3y_8 + y_9]$$

.....

$$\int_{x_{n-3}}^{x_n} f(x)\,dx = \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Put in (*) we have

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}[y_0 + 3(y_1 + y_2 + y_4 + y_5 + y_7 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3}) + y_n]$$

which is Simpson's $(\frac{3}{8})$ Rule.

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4) Boole's Rule:-

To evaluate $\int_a^b f(x)\,dx$ we partitioned interval $[a,b]$, where $n=4$ (or 5 points)

$f(x)$ is a polynomial of degree 4 and all differences of order higher then 4 are zero.

Put $n=4$ in Newton Cote's formula

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_4} f(x)\,dx + \int_{x_4}^{x_8} f(x)\,dx + \cdots + \int_{x_{n-4}}^{x_n} f(x)\,dx \quad \text{(*)}$$

First find $\int_{x_0}^{x_4} f(x)\,dx$ (Then put $n=4$ in N.C.F)

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$$\int_{x_0}^{x_4} f(x)\,dx = 4h\left[y_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 + \frac{2}{3}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0\right]$$

$$= 4h\left[y_0 + 2(y_1 - y_0) + \frac{5}{3}(y_2 - 2y_1 + y_0) + \frac{2}{3}(y_3 - 3y_2 + 3y_1 - y_0) + \frac{7}{90}(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)\right]$$

$$= \frac{2}{45}h\left[90y_0 + 18(y_1 - y_0) + 150(y_2 - 2y_1 + y_0) + 60(y_3 - 3y_2 + 3y_1 - y_0) + 7(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)\right]$$

$$= \frac{2}{45}h\left[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4\right]$$

Similarly $\int_{x_4}^{x_8} f(x)\,dx = \frac{2h}{45}\left[7(y_4 + \cdots) + 32y_5 + 12y_6 + 32y_7 + 7y_8\right]$

.....

$$\int_{x_{n-4}}^{x_n} f(x)\,dx = \frac{2h}{45}\left[7y_{n-4} + 32y_{n-3} + 12y_{n-2} + 32y_{n-1} + 7y_n\right]$$

Put all in ⊛ we get

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{2}{45}h\left[7(y_0 + y_n) + 32(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 12(y_2 + y_6 + y_{10} + \cdots + y_{n-2}) + 14(y_4 + y_8 + y_{12} + \cdots + y_{n-4})\right]$$

Which is required Boole's Formula.

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5) Weddle's Rule:-

In order to evaluate $\int_a^b f(x)\,dx$. Let divide $[a,b]$ into sub-intervals

$$P = \{x_0, x_6, x_{12}, \dots, x_n\}$$

For all

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_6} f(x)\,dx + \int_{x_6}^{x_{12}} f(x)\,dx + \dots + \int_{x_{n-6}}^{x_n} f(x)\,dx \quad \text{——— } \circledast$$

First find $\int_{x_0}^{x_6} f(x)\,dx$

In this case $f(x)$ is polynomial of degree 6 and all forward difference of greater then 6 are zero

Put $n=6$ in Newton's Cote's formula

$$\int_{x_0}^{x_6} f(x)\,dx = h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{3}{10}\Delta^6 y_0\right]$$

$$= \frac{3h}{10}\left[20y_0 + 60\Delta y_0 + 90\Delta^2 y_0 + 80\Delta^3 y_0 + 41\Delta^4 y_0 + 11\Delta^5 y_0 + \Delta^6 y_0\right]$$

$$= \frac{3h}{10}\left[20y_0 + 6(y_1 - y_0) + 90(y_2 - 2y_1 + y_0) + 8(y_3 - 3y_2 + 3y_1 - y_0) + 41(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0) + 11(y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0) + (y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0)\right]$$

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$$= \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

Similarly $\int_{x_6}^{x_{12}} f(x)dx = \frac{3h}{10}[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$

.....

$$\int_{x_{n-6}}^{x_n} f(x)dx = \frac{3h}{10}[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n]$$

Put in (*) we have

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{10}[(y_0 + y_2 + y_4 + \cdots + y_n) + 2(y_6 + y_{12} + y_{18} + \cdots + y_{n-6}) + 5(y_1 + y_5 + y_7 + y_{11} + \cdots + y_{n-1}) + 6(y_3 + y_9 + y_{15} + \cdots + y_{n-3})]$$

Which is required Weddle's Rule.

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Question :- Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using

(i) Trapezoidal Rule with 5 points.
(ii) Simpson's Rule with 5 points.
(iii) Boole's Rule.
Compare result with exact value.

Solution:- $a = 0$, $b = 1$, $n = 4$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

$$f(x) = \frac{1}{1+x^2}$$

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x	0	0.25	0.5	0.75	1
$f(x)$	1	0.941	0.8	0.64	0.5

1) Trapezoidal Rule:-

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{0.25}{2}\left[1 + 2(0.941 + 0.8 + 0.64) + 0.5\right]$$

$$= \frac{0.25}{2}(6.282)$$

$$= 0.78275$$

2) Simpson's (1/3) Rule:-

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3) + 2y_2 + y_4\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{0.25}{3}\left[1 + 4(0.941 + 0.64) + 2(0.8) + 0.5\right]$$

$$= 0.78533$$

3) Boole's Rule:-

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}\left[7(y_0 + y_n) + 32(y_1 + y_3) + 12y_2\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{2(0.25)}{45}\left[7(1 + 0.5) + 32(0.941 + 0.64) + 12(0.8)\right]$$

$$= 0.785466$$

Exact Value:- $\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_0^1$

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$= \text{Tan}^{-1}(1) - \text{Tan}^{-1}(0)$

$= \pi/4 = 0.78539$

Error in Simpson's Rule is smallest. So Simpson's Rule is better than other rules

* ——— **

Example:- Apply Simpson's $\frac{1}{3}$ rule to find approximate value of π from $\int_0^1 \frac{1}{1+x^2}\,dx$ taking 4 intervals.

Solution:-

$$I = \int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \text{Tan}^{-1}(x)\Big|_0^1$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\Rightarrow I = \frac{\pi}{4} \Rightarrow \pi = 4I$$

Now we find approximate value of I using Simpson's (1/3) Rule

Take $a = 0$, $b = 1$, $n = 4$

$$h = \frac{b-a}{4} = \frac{1-0}{4} = 0.25$$

So table is

x :	0	0.25	0.5	0.75	1.0
f(x) :	1	0.9412	0.80	0.640	0.50

Now By Simpson's 1/3 Rule

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$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[f_0 + f_4 + 4(f_1+f_3) + 2f_2\right]$$

$$= \frac{0.25}{3}\left[1 + .5 + 4(0.9412 + 0.64) + 2(0.80)\right]$$

$$\Rightarrow \quad I = 0.7854$$

So approximate value of π is

$$\pi = 4I \quad \Rightarrow \quad \pi = 4(0.7854)$$

$$\Rightarrow \pi = 3.1416$$

Example:- Evaluate $\int_0^{\pi/2} \sin x\,dx$ using Boole's Rule.

Solution:-

Here $a = 0$, $b = \pi/2$

$f(x) = \sin x$

Take $n = 4$

$$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{4} = \pi/8$$

So table is

x	0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$
$f(x)$	0	0.3827	0.7071	0.9239	1

By Boole's Rule :-

$$\int_0^{\pi/2} \sin x\,dx = \frac{2}{45}h\left[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4\right]$$

$$= \frac{2}{45}\left(\frac{\pi}{8}\right)\left[7(0) + 32(0.3827) + 12(0.7071) + 32(0.9239) + 7(1)\right.$$

$$= \frac{\pi}{180}(57.2964) = 1.000 \quad \text{Ans.}$$

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Example:- Use Boole's formula and 3/8 Simpson Rule in combination to evaluate integral $\int_0^{0.7} f(x)\,dx$ when $f(x)$ is given by

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
y	0	0.001	0.399	0.893	0.1559	0.2449	0.3497	0.4705

Solution:- $a = 0$, $b = 0.7$, $h = 0.1$ (as from table)

The given function is

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x:	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
y:	0	0.001	0.399	0.893	0.1559	0.2449	0.3997	0.4705

In Simpson's $\frac{3}{8}$ Rule $n=3$ and in boole's Rule $n=4$.

So there are two possibilities

(i) From x_0 to x_3 Apply ~~Boo~~ Simpson's 3/8 and x_3 to x_7 apply Boole's Rule and then add

(ii) From x_0 to x_4 apply Boole's Rule and x_4 to x_7 apply Simpson's 3/8 rule and then add it

$$\int_0^{0.7} f(x)\,dx = \frac{2h}{45}\left[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4\right] + \frac{3h}{8}\left[y_4 + 3(y_5 + y_6) + y_7\right]$$

$$= \frac{2(0.1)}{45}\left[7(0) + 32(0.001) + 12(0.399) + 32(0.893) + 7(0.1579)\right] + \frac{3(0.1)}{8}\left[0.1579 + 3(0.2449 + 0.3497) + 0.4705\right]$$

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$$\int_0^{0.7} f(x)\,dx = \frac{0.2}{45}(4.4737) + \frac{0.3}{8}(2.4122)$$

$$= 0.0199 + 0.0905$$

$$= 0.1104$$

*****

ASSIGNMENT.

Q1:- Evaluate $\int_0^1 (1+x+e^x)\,dx$ using trapezoidal rule for 8 trapeziums upto 5 decimal places.

Q2:- Evaluate $\int_0^{\pi} \frac{\sin x}{x}\,dx$ for $n=6$ by Simpson's Rule.

Q3:- Integrate function $f(x)$ given by following table (i) using 5 points simpson's Rule (ii) Boole's Rule

x :	1	1.5	2.0	2.5	3.0
$f(x)$:	1	0.44	0.25	0.16	0.11

Q4:- Evaluat $\int_0^1 \frac{1}{1+x^4}\,dx$ using
(i) Trapezoidal rule
(ii) Simpson's 3/8 rule
compare result with exact value & which of two method is superior and why.

Q5:- Use 7 points weddel's rul to

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evaluate $\int_0^{\pi/2} \sin x \, dx$ using 6 digits accuracy.

SOLUTIONS

Q1:-

Here $a=0$, $b=1$, $n=8$

$$\Rightarrow h = \frac{b-a}{n} = \frac{1-0}{8} = 0.125$$

So $x_0 = 0$, $x_1 = 0.125$, $x_2 = 0.25$

$x_3 = 0.375$, $x_4 = 0.5$, $x_5 = 0.625$

$x_6 = 0.75$, $x_7 = 0.875$, $x_8 = 1$

$$f(x) = 1 + x + e^x$$

So table is

x	0	0.125	0.25	0.375	0.5	0.625
$f(x)$	2	2.25815	2.53403	2.83	3.14872	3.49246

x	0.75	0.875	1
$f(x)$	3.867	4.27386	4.71828

Now by Trapezoidal rule

$$\int_0^1 f(x)\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + \cdots + y_7) + y_8\right]$$

$$\Rightarrow \int_0^1 (1 + x + e^x)\,dx = \frac{0.125}{2}\left[2 + 2(2.25815 + 2.53403 + 2.83 + 3.14872 + 3.49246 + 3.867 + 4.27386) + 4.71828\right]$$

$$= 3.22052$$

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Q2:- $\int_0^{\pi} \frac{\sin x}{x}$

$a = 0$, $b = \pi$, $n = 6$

$\Rightarrow h = \pi/6$ So

$a_0 = 0$, $a_1 = \pi/6$, $a_2 = \pi/3$, $a_3 = \pi/2$

$a_4 = \frac{2\pi}{3}$, $a_5 = 5/6\,\pi$, $a_6 = \pi$

$f(x) = \frac{\sin x}{x}$

x :	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$f(x)$:	1	0.95493	0.82699	0.63662	0.4135	0.19099	0

Now by Simpson's $\frac{1}{3}$ Rule

$$\int_0^{\pi} \frac{\sin x}{x} = \frac{h}{3}\left[y_0 + y_6 + 4y_1 + 4y_3 + 4y_5 + 2y_2 + 2y_4\right]$$

$$= \frac{\pi/6}{3}\left[1 + 0 + 4(0.95493 + 0.63662 + 0.19099) + 2(0.82699 + 0.4135)\right]$$

$$= 1.851993304$$

Eg Now by Simpson's $\frac{3}{8}$ Rule

$$\int_0^{\pi} \frac{\sin x}{x} = \frac{3\pi/6}{8}\left[y_0 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3 + y_6\right]$$

$$= \frac{3\pi}{6} \times \frac{1}{8}\left[1 + 3(0.95493 + 0.82699 + 0.4135 + 0.19099) + 2(0.63662) + 0\right]$$

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$$\Rightarrow \int_0^{\pi} \frac{\sin x}{x} = 1.852061154$$

Q3:-

x :	1	1.5	2	2.5	3
$f(x)$:	1	0.44	025	0.16	0.11

Here $a = 1$, $b = 3$ & $h = 0.5$

1) By Simpson's $\frac{1}{3}$ Rule

$$\int_a^b f(x)\,dx = \frac{h}{3}[y_0 + 4(y_1 + y_3) + 2y_2 + y_4]$$

$$= \frac{0.5}{3}[1 + 4(0.44 + 0.16) + 2(0.25) + 0.11]$$

$$= 0.668333$$

2) Now by Boole's Rule

$$\int_a^b f(x)\,dx = \frac{2h}{45}[7(y_0 + y_4) + 32(y_1 + y_3) + 12y_2]$$

$$= \frac{2(0.5)}{45}[7(1 + 0.11) + 32(0.44 + 0.66) + 12(0.25)]$$

$$= 0.666$$

Q4:- $\int_0^1 \frac{1}{1+x^4}\,dx$

Here $a = 0$, $b = 1$. Let $n = 6$

$$\Rightarrow h = \frac{1}{6} = 0.166667$$

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$a_0 = 0$, $a_1 = 0.166667$, $a_2 = 0.333334$, $a_3 = 0.5$

$a_4 = 0.666668$, $a_5 = 0.833335$, $a_6 = 1$

x :	0	0.166667	0.333334	0.5	0.666668
$f(x)$:	1	0.999229	0.987805	0.9411765	0.835050
x :	0.833335	1			
$f(x)$:	0.664647	0.5			

(i) By Trapezoidal Rule

$$\int_0^1 \frac{1}{1+x^4}\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \cdots + y_5) + y_6\right]$$

$$= 0.86298631$$

(ii) By Simpson's $\frac{3}{8}$ Rule

$$\int_0^1 \frac{1}{1+x^4} = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3\right]$$

$$= \frac{3(\frac{1}{6})}{8}\left[(1+0.5) + 3(0.999229 + 0.987805 + 0.835050 + 0.664647) + 2(0.9411765)\right]$$

$$= 0.865160853$$

Exact Value :-

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Q5:- $\int_0^{\pi/2} \sin x \, dx$

Here $a = 0$, $b = \pi/2$, $n = 6$

$\Rightarrow h = \pi/12$

$\therefore x_0 = 0$, $x_1 = \pi/12$, $x_2 = \pi/6$, $x_3 = \pi/4$

$x_4 = \pi/3$, $x_5 = \frac{5\pi}{12}$, $x_6 = \pi/2$

x :	0	$\pi/2$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/2$
$f(x)$:	0	0.258819	0.5	0.70710678	0.8660254	0.9659258

$\pi/2$

1

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Now by Weddle's Rule

$$\int_0^{\pi/2} \sin x\, dx = \frac{3h}{10}\left[(y_0+y_2+y_4+y_6)+5(y_1+y_5)+6y_3\right]$$

$$= \frac{3\pi/12}{10}\left[(0+0.5+0.8660254+1)+5(0.258819+0.9659258)+6(0.7071067)\right]$$

$$= 0.9995463055$$

——————***

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ii) Gauss Quadrature Formula:-

Gauss investigated that in the quadrature formulae, the equally spaced arguments make the accuracy of quadrature formula limited. He derived a quarature formula in which the limit of integration are from -1 to $+1$, if not then these can be made by putting

$$x = \frac{(b-a)t + (a+b)}{2}$$

$$\Rightarrow t = \frac{2x-a-b}{b-a}$$

at $x = a$, $t = \frac{2a-a-b}{b-a} = \frac{a-b}{b-a} = -1$

at $x = b$, $t = \frac{2b-a-b}{b-a} = \frac{b-a}{b-a} = +1$

* Gauss Two point Quadrature Formula:-

Let $I = \int_a^b f(x)\,dx$

Put $x = \frac{(b-a)t + (b+a)}{2}$

$$\Rightarrow dx = \frac{b-a}{2}\,dt$$

when $x = a$, $t = -1$

$x = b$, $t = 1$

In This case

$$I = \int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{(b-a)t+(b+a)}{2}\right)\left(\frac{b-a}{2}\right)dt$$

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} F(t)\,dt \qquad \text{---------- (1)}$$

where

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$$F(t) = f\left[\frac{(b-a)t+(b+a)}{2}\right]$$

Let $\int_{-1}^{1} F(t)\,dt = C_1 F(t_1) + C_2 F(t_2)$ ——— (2)

Two points are t_1 and t_2.

Where C_1, C_2, t_1 and t_2 are unknown and $C_1 \neq 0$, $C_2 \neq 0$ and $t_1 \neq t_2$

Now as, cubic polynomial can be constructed at the most by using (4) unknown quantities so, to determine 4 unknown quantities.

We take $F(t) = 1, t, t^2, t^3$

When $F(t) = 1$:- Then

$$\int_{-1}^{1} F(t)\,dt = C_1 F(t_1) + C_2 F(t_2)$$

$$\Rightarrow \int_{-1}^{1} 1\,dt = C_1(1) + C_2(1)$$

$$\Rightarrow \boxed{2 = C_1 + C_2} \quad \text{——— (3)}$$

When $F(t) = t$:- Then (2) gives

$$\int_{-1}^{1} t\,dt = C_1 t_1 + C_2 t_2$$

$$\Rightarrow \boxed{0 = C_1 t_1 + C_2 t_2} \quad \text{——— (4)}$$

When $F(t) = t^2$:- Then (2) gives

$$\int_{-1}^{1} t^2\,dt = C_1 t_1^2 + C_2 t_2^2$$

$$t^3/3\Big|_{-1}^{1} = C_1 t_1^2 + C_2 t_2^2$$

$$\Rightarrow \boxed{2/3 = C_1 t_1^2 + C_2 t_2^2} \quad \text{——— (5)}$$

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When $F(t) = t^3$:- Then ② gives

$$\int_{-1}^{1} t^3\,dt = c_1 t_1^3 + c_2 t_2^3$$

$$t^4/4 \Big|_{-1}^{1} = c_1 t_1^3 + c_2 t_2^3$$

$$\Rightarrow \boxed{0 = c_1 t_1^3 + c_2 t_2^3} \longrightarrow ⑥$$

Now multiplying ④ by t_1^2 and and then subtract from equ ⑥

$$c_1 t_1^3 + c_2 t_2^3 = 0$$

$$c_1 t_1^3 + c_2 t_2 t_1^2 = 0$$

$$c_2 t_2 [t_2^2 - t_1^2] = 0$$

$$\Rightarrow c_2 t_2 (t_2 - t_1)(t_2 + t_1) = 0$$

$\Rightarrow c_2 = 0$ or $t_2 = 0$ or $t_2 = t_1$ or $t_2 = -t_1$

but $c_2 \neq 0$ & $t_2 \neq t_1$

So $t_2 = 0$ or $t_2 = -t_1$

When $t_2 = 0$ from ④ $c_1 t_1 = 0$

$\Rightarrow c_1 = 0$ or $t_1 = 0$

but $c_1 \neq 0$ So $t_1 = 0$

But $t_1 = t_2$ which is not possible

Hence $t_2 \neq 0$

Hence finally $t_2 = -t_1$

From ⑤

$$c_1 t_1^2 + c_2 (-t_1)^2 = 2/3$$

$\Rightarrow (c_1 + c_2) t_1^2 = 2/3 \Rightarrow 2t_1^2 = 2/3$ by ②

$\Rightarrow t_1 = \pm 1/\sqrt{3}$

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When $t_1 = -1/\sqrt{3}$ then $t_2 = 1/\sqrt{3}$

When $t_2 = -1/\sqrt{3}$ then $t_1 = 1/\sqrt{3}$

Next put $t_2 = -t_1$ in eqn (4)

$$c_1 t_1 - c_2 t_1 = 0$$

$$(c_1 - c_2) t_1 = 0 \Rightarrow c_1 - c_2 = 0 \quad \because t_1 \neq 0$$

$$\Rightarrow c_1 = c_2$$

From (3) $c_1 + c_2 = 2 \Rightarrow 2c_1 = 2 \quad \because c_1 = c_2$

$$\Rightarrow c_1 = 1 \Rightarrow c_2 = 1$$

So from (2) $$\int_{-1}^{1} F(t)\,dt = (1) F\left(\frac{1}{\sqrt{3}}\right) + F\left(\frac{-1}{\sqrt{3}}\right)$$

Hence $$I = \int_a^b f(x)\,dx = \frac{b-a}{2}\left[F\left(\frac{1}{\sqrt{3}}\right) + F\left(\frac{-1}{\sqrt{3}}\right)\right]$$

Which is required Gauss two point Quadrature Formula.

★★★

Question:- Evaluate $\int_0^{\pi/2} \sin x\,dx$ by two-point Gauss formula.

Solution:- $$I = \int_0^{\pi/2} \sin x\,dx$$

$a = 0$, $b = \pi/2$

Put $$x = \frac{(b-a)t + (b+a)}{2}$$

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$$\Rightarrow x = \frac{(\pi/2 - 0)t + (\pi/2 + 0)}{2} = \frac{\pi/2\, t + \pi/2}{2}$$

$$dx = \frac{\pi}{4} dt$$

When $x = 0$; $t = -1$

When $x = \pi/4$; $t = 1$

$$I = \int_{-1}^{1} \sin\left(\frac{\pi/2\, t + \pi/2}{2}\right) \frac{\pi}{4} dt$$

$$= \frac{\pi}{4} \int_{-1}^{1} F(t)\, dt \quad \text{where } F(t) = \sin\left(\frac{\pi}{4}t + \frac{\pi/2}{2}\right)$$

$$I = \frac{\pi}{4}\left[F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{\pi}{4}\left[\sin\left[\frac{\pi/2\left(\frac{-1}{\sqrt{3}}\right) + \pi/2}{2}\right] + \sin\left[\frac{\pi/2\left(\frac{1}{\sqrt{3}}\right) + \pi/2}{2}\right]\right]$$

$$= \frac{\pi}{4}\left[\sin(0.3319) + \sin(1.2388)\right]$$

$$= \frac{\pi}{4}\left[0.3258 + 0.94541\right]$$

$$= 0.99848$$

★★★

Question:- Evaluate $\int_0^1 e^{-x} dx$ using two point Gauss Quadrature formula.

Solution:-

$$I = \int_0^1 e^{-x} dx$$

where $a = 0$, $b = 1$

Put $x = \frac{(b-a)t + (b+a)}{2}$

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$\Rightarrow x = \frac{1t+1}{2}$ $\Rightarrow x = \frac{t+1}{2}$

$dx = \frac{1}{2}dt$

When $x = 0$; $t = -1$

" $x = 1$; $t = 1$

$$I = \int_{-1}^{1} e^{-\left(\frac{t+1}{2}\right)} \cdot \frac{1}{2}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-\left[\frac{t+1}{2}\right]}\, dt$$

By Gauss Quadrature formula

$$I = \frac{1}{2}\int_{-1}^{1} F(t)\, dt, \text{ where } F(t) = e^{-\left[\frac{t+1}{2}\right]}.$$

$$= \frac{1}{2}\left[F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{2}\left[e^{-\left[\frac{\frac{1}{\sqrt{3}}+1}{2}\right]} + e^{-\left[\frac{\frac{1}{\sqrt{3}}+1}{2}\right]}\right]$$

$$= \frac{1}{2}\left[e^{-0.2113} + e^{-0.7887}\right]$$

$$= \frac{1}{2}[0.8095 + 0.4544]$$

$I = 0.6320$ Ans.

★★ ——— ★ ——— ★★★

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★ Gauss Three-point Quadrature Formula:-

$$\int_{-1}^{1} F(x)\,dx = \frac{1}{9}\left[5F\left(\frac{-\sqrt{3}}{\sqrt{5}}\right) + 8F(0) + 5F\left(\frac{\sqrt{3}}{\sqrt{5}}\right)\right]$$

(Not include its proof)

Question:- Apply Gauss Qudrature for 3-pts evaluate $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$

Solution:-

$$I = \int_{-1}^{1} \frac{1}{1+x^2}\,dx$$

$$= \int_{-1}^{1} F(x)\,dx, \text{ where } F(x) = \frac{1}{1+x^2}$$

$$I = \frac{1}{9}\left[5F\left(\frac{-\sqrt{3}}{\sqrt{5}}\right) + 8F(0) + 5F\left(\frac{\sqrt{3}}{\sqrt{5}}\right)\right]$$

$$= \frac{1}{9}\left[5\cdot\frac{1}{1+3/5} + 8(1) + 5\cdot\frac{1}{1+3/5}\right]$$

$$= \frac{1}{9}\left[5(0.625) + 8(1) + 5(0.625)\right]$$

$$= \frac{1}{9}(14.25)$$

$$\Rightarrow I = 1.5833 \quad \text{Ans.}$$

★ ——— ★

Question:- Apply 3-pt Gaussian Formula for $I = \int_{2}^{3} \frac{1}{2-\sin x^2}\,dx$

Solution:- $I = \int_{2}^{3} \frac{1}{2-\sin x^2}\,dx$

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where $a=2$, $b=3$

Put $x = \frac{(b-a)t + (b+a)}{2}$

$\Rightarrow x = \frac{(3-2)t+5}{2}$ $\Rightarrow x = \frac{t+5}{2}$

$dx = \frac{1}{2}dt$

When $x=2$; $t=-1$

When $x=3$; $t=1$

$$I = \int_{-1}^{1} \frac{1}{2-\sin\left(\frac{t+5}{2}\right)^2}\frac{1}{2}dt$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{1}{2-\sin\left(\frac{t+5}{2}\right)^2}dt \longrightarrow (*)$$

$$\int_{-1}^{1} \frac{1}{2-\sin\left(\frac{t+5}{2}\right)^2}dt = \frac{1}{9}\left[F\left(-\sqrt{\tfrac{3}{5}}\right)\cdot 5 + 8F(0) + 5F\left(\sqrt{\tfrac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[\frac{5}{2-\sin\left[\frac{-\sqrt{3/5}+5}{2}\right]^2} + 8\left(\frac{1}{2}\right) + \frac{5}{2-\sin\left[\frac{\sqrt{3/5}+5}{2}\right]^2}\right]$$

$$= \frac{1}{9}\left[\frac{5}{2-0.7340} + \frac{8}{2} + \frac{5}{2-0.0633}\right]$$

$$= \frac{1}{9}\left[3.9494 + 4 + 2.5817\right] = \frac{1}{9}\left[10.5311\right]$$

$= 1.1701$ put in $(*)$

$I = \frac{1}{2}(1.1701) = 0.5851$ Ans.

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Assignment.

Apply 2 & 3 pts Gaussian formula.

(i) $I = \int_0^1 \frac{dx}{1+x^2}$

(ii) $I = \int_0^1 \frac{\sin x}{x}\, dx$

(iii) $I = \int_0^1 x\, dx$

(iv) $I = \int_{-1}^{1} (1-x^2)^{2/3} \cos x\, dx.$

Solutions

Q1:- $I = \int_0^1 \frac{dx}{1+x^2}$

Put $x = \frac{(b-a)t + (b+a)}{2}$ where $a=0, b=1$

$\Rightarrow x = \frac{t+1}{2} \Rightarrow dx = \frac{1}{2} dt$

when $x = 0$; $t = -1$

when $x = 1$; $t = 1$

$$\Rightarrow I = \int_{-1}^{1} \frac{1}{1+\left(\frac{t+1}{2}\right)^2} \cdot \frac{1}{2}\, dt$$

$$= \int_{-1}^{1} \frac{1}{1+\left(\frac{t+1}{2}\right)^2} \cdot \frac{1}{2}\, dt = \frac{1}{2} \int_{-1}^{1} \frac{1}{1+\left(\frac{t+1}{2}\right)^2} dt$$

$$F(t) = \frac{1}{1+\left(\frac{t+1}{2}\right)^2}$$

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(i) By Gauss Two Point method

$$I = \frac{1}{2}\left[F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$\Rightarrow I = \frac{1}{2}\left[\frac{1}{1+0.044658} + \frac{1}{1+0.622008}\right]$$

$$= 0.786885$$

(ii) By Gauss three Point method

$$I = \frac{1}{2}\cdot\frac{1}{9}\left[5F\left(-\sqrt{\frac{3}{5}}\right) + 8F(0) + 5F\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{2}\cdot\frac{1}{9}\left[5\left(\frac{1}{1+0.0127067}\right) + 8\left(\frac{1}{1+0.25}\right) + 5\left(\frac{1}{1+0.787298}\right)\right]$$

$$= 0.785267063$$

★★

Q2:- $$I = \int_0^1 \frac{\sin x}{x}$$

Put $x = \frac{(1-0)t + 1}{2} \Rightarrow x = \frac{t+1}{2}$

$\Rightarrow dx = \frac{1}{2}dt$

when $x = 0$, $t = -1$

$x = 1$, $t = 1$

$$\Rightarrow I = \int_{-1}^{1} \frac{\sin\left(\frac{t+1}{2}\right)}{\frac{t+1}{2}} \cdot \frac{1}{2}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{\sin\left(\frac{t+1}{2}\right)}{\frac{t+1}{2}}\, dt$$

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$$\Rightarrow I = \frac{1}{2}\int_{-1}^{1} \frac{2\sin\left(\frac{t+1}{2}\right)}{t+1}\,dt$$

$$\Rightarrow I = \int_{-1}^{1} \frac{\sin\left(\frac{t+1}{2}\right)}{t+1}\,dt$$

(i) By Gauss Two Point formula

$$I = F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{0.2097555}{0.42265} + \frac{0.709420}{1.577550}$$

$$= 0.946041$$

(ii) By Gauss Three Point formula

$$I = \frac{1}{9}\left[5F\left(-\sqrt{\frac{3}{5}}\right) + 8F(0) + 5F\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[5\left(\frac{0.112463}{0.2254}\right) + 8\left(\frac{0.47942553}{1}\right) + 5\left(\frac{0.775368}{1.7746}\right)\right]$$

$$= 0.946086$$

Q4:- $$\int_{-1}^{1} (1-x^2)^{2/3} \cos x\,dx$$

$$F(x) = (1-x^2)^{2/3}\cos x$$

(i) By Gauss two point formula

$$I = F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)$$

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$\Rightarrow I = 0.639446 + 0.639446$

$= 1.2788928$

(ii) By Gauss three point formula

$$I = \frac{1}{9}\left[5F\left(-\sqrt{\frac{3}{5}}\right) + 8F(0) + 5F\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[5(0.388) + 8(1) + 5(0.388)\right]$$

$$= 1.32$$

Q3:-

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⇒ Error Analysis:-

In Numerical Integration; we study many methods to evaluate problem and we demonstrated through examples that some methods are superior to others but we did not give any theoratical justification.

Now we shall calculate the amount of error in each method, for this we shall use Taylor's Series.

Note:-

Error = X − A , where

X = Exact value & A = Approximate Value.

* —— * —— * —— * —— *

1) Error Analysis of Trapezoidal Rule:-

; n = 1

$E = X - A$

Let us consider $I = \int_{x_0}^{x_1} f(x)\,dx$ then

X = Exact value $= F(x_1) - F(x_0)$, where

$F'(x) = f(x)$

A = Value by Trapezoidal Rule

$= \frac{h}{2}[y_0 + y_1] = \frac{h}{2}[f(x_0) + f(x_1)]$

$\text{Error} = X - A = [F(x_1) - F(x_0)] - \frac{h}{2}[f(x_0) + f(x_1)]$

$= [F(x_0 + h) - F(x_0)] - \frac{h}{2}[f(x_0) + f(x_0 + h)]$

$= [F(x_0) + hF'(x_0) + \frac{h^2}{2!}F''(x_0) + \frac{h^3}{3!}F'''(x_0) + \cdots - F(x_0)]$

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$$-\frac{h}{2}\left[f(x_0)+f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\cdots\right]$$

$$=hf(x_0)+\frac{h^2}{2}f'(x_0)+\frac{h^3}{6}f''(x_0)+\cdots-hf(x_0)$$

$$-\frac{h^2}{2}f'(x_0)-\frac{h^3}{4}f''(x_0)-\cdots$$

$$=-\frac{h^3}{12}f''(x_0)$$ Neglecting remaining

$$\Rightarrow E=\frac{-1}{12}h^3f''(x_0)$$

★ ——— ★ ——— ★★ ——— ★ ——— ★

2) Error Analysis of Simpson's $\frac{1}{3}$ Rule:- ; n=2

$$E=X-A$$

$$=\int_{x_0}^{x_2}f(x)dx-\frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]$$

$$E=I_1-I_2 \longrightarrow (1)$$

$$I_1=\int_{x_0}^{x_2}f(x)dx \qquad \text{Let } f(x)=F'(x)$$

$$=\int_{x_0}^{x_2}F'(x)dx = F(x)\Big|_{x_0}^{x_2}$$

$$=F(x_2)-F(x_0)$$

$$=F(x_0+2h)-F(x_0)$$

$$=F(x_0)+2hF'(x_0)+\frac{4h^2}{2!}F''(x_0)+\frac{8h^3}{3!}F'''(x_0)+\cdots\cdots-F(x_0)$$ ∵ By Taylor series.

$$I_1=2hf(x_0)+2h^2f'(x_0)+\frac{4}{3}h^3f''(x_0)+\frac{2}{3}h^4f'''(x_0)+\frac{4}{15}h^5f^{iv}(x_0)+\cdots$$

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$$I_2 = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_0+h) + f(x_0+2h)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\left\{f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \cdots\right\} + f(x_0) + 2hf'(x_0) + \frac{4h^2}{2!}f''(x_0) + \frac{8h^3}{3!}f'''(x_0) + \frac{16h^4}{4!}f^{iv}(x_0) + \cdots\right]$$

$$I_2 = 2hf(x_0) + 2h^2f'(x_0) + \frac{4}{3}h^3f''(x_0) + \frac{2}{3}h^4f'''(x_0) + \frac{5}{18}h^5f^{iv}(x_0) + \cdots$$

Put I_1 & I_2 in ①

$$\Rightarrow E = 2hf(x_0) + 2h^2f'(x_0) + \frac{4}{3}h^3f''(x_0) + \frac{2}{3}h^4f'''(x_0) + \frac{4}{15}h^5f^{iv}(x_0) + \cdots - 2hf(x_0) - 2h^2f'(x_0) - \frac{4}{3}h^3f''(x_0) - \frac{2}{3}h^4f'''(x_0) - \frac{5}{18}h^5f^{iv}(x_0) - \cdots$$

$$E = \frac{-1}{90}h^5f^{iv}(x_0) \quad \because \text{by neglecting higher terms.}$$

★ ★ ★ ★

3) Error Analysis of Simpson's $\frac{3}{8}$ Rule:-

; $n = 3$

$$E = X - A$$

$$= \int_{x_0}^{x_3} f(x)\,dx - \frac{3h}{8}\left\{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right\}$$

$$E = I_1 - I_2 \quad \text{——— ①}$$

$$I_1 = \int_{x_0}^{x_3} f(x)\,dx \qquad \text{Let } f(x) = F'(x)$$

$$\Rightarrow I_1 = \int_{x_0}^{x_3} F'(x)\,dx$$

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$$\Rightarrow I_1 = F(x)\Big|_{x_0}^{x_3} = F(x_3) - F(x_0)$$

$$= F(x_0+3h) - F(x_0)$$

$$= F(x_0) + 3hF'(x_0) + \frac{(3h)^2}{2!}F''(x_0) + \frac{(3h)^3}{3!}F'''(x_0) + \frac{(3h)^4}{4!}F^4(x_0) + \frac{(3h)^5}{5!}F^5(x_0) + \cdots - F(x_0)$$

$$I_1 = 3hf(x_0) + \frac{9h^2}{2}f'(x_0) + \frac{9h^3}{2}f''(x_0) + \frac{27h^4}{8}f'''(x_0) + \frac{81}{40}h^5f^4(x_0) + \frac{81}{80}h^6f^5(x_0) + \cdots$$

$$I_2 = \frac{3h}{8}\{f(x_0) + 3f(x_0+h) + 3f(x_0+2h) + f(x_0+3h)\}$$

$$= \frac{3h}{8}\{f(x_0) + 3\{f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \frac{h^4}{4!}f^4(x_0) + \frac{h^5}{5!}f^5(x_0) + \cdots\} + 3\{f(x_0) + 2hf'(x_0) + \frac{(2h)^2}{2!}f''(x_0) + \frac{(2h)^3}{3!}f'''(x_0) + \frac{(2h)^4}{4!}f^4(x_0) + \frac{(2h)^5}{5!}f^5(x_0) + \cdots\} + f(x_0) + 3hf'(x_0) + \frac{(3h)^2}{2!}f''(x_0) + \frac{(3h)^3}{3!}f'''(x_0) + \frac{(3h)^4}{4!}f^4(x_0) + \frac{(3h)^5}{5!}f^5(x_0) + \cdots\}$$

$$= 3hf(x_0) + \frac{9}{2}h^2f'(x_0) + \frac{9}{2}h^3f''(x_0) + \frac{27}{8}h^4f'''(x_0) + \frac{33}{16}h^5f^4(x_0) + \cdots$$

Put the value of I_1 & I_2 in eq ①

$$\Rightarrow E = \frac{81}{40}h^5f^4(x_0) - \frac{33}{16}h^5f^4(x_0) + \cdots$$

$$= \frac{-3}{80}h^5f^4(x_0)$$

Required.

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4) Error Analysis of Boole's Rule:- ; n=4

$$E = X - A$$

$$= \int_{x_0}^{x_4} f(x)\,dx - \frac{2h}{45}\left\{7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right\}$$

$$E = I_1 - I_2 \longrightarrow (1)$$

Now

$$I_1 = \int_{x_0}^{x_4} f(x)\,dx$$

$$= F(x_4) - F(x_0) \qquad \because f(x) = F'(x)$$

$$= F(x_0 + 4h) - F(x_0)$$

$$= F(x_0) + 4hF'(x_0) + \frac{(4h)^2}{2!}F''(x_0) + \frac{(4h)^3}{3!}F'''(x_0) + \frac{(4h)^4}{4!}F^{iv}(x_0) + \frac{(4h)^5}{5!}F^5(x_0) + \frac{(4h)^6}{6!}F^6(x_0) + \frac{(4h)^7}{7!}F^7(x_0) + \frac{(4h)^8}{8!}F^8(x_0) + \frac{(4h)^9}{9!}F^9(x_0) + \cdots - F(x_0)$$

∵ By Taylor series.

$$I_1 = 4f(x_0) + 8h^2f'(x_0) + \frac{32}{3}h^3f''(x_0) + \frac{32}{3}h^4f'''(x_0) + \frac{128}{15}h^5f^4(x_0) + \frac{256}{45}h^6f^5(x_0) + \frac{1024}{315}h^7f^6(x_0) + \frac{512}{315}h^8f^7(x_0) + \cdots$$

Now

$$I_2 = \frac{2h}{45}\left[7f(x_0) + 32f(x_0+h) + 12f(x_0+2h) + 32f(x_0+3h) + 7f(x_0+4h)\right]$$

$$= \frac{2h}{45}\left[7f(x_0) + 32\left\{f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \right.\right.$$

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$$\frac{h^3}{3!}f'''(x_0)+\frac{h^4}{4!}f^4(x_0)+\frac{h^5}{5!}f^5(x_0)+\frac{h^6}{6!}f^6(x_0)+\frac{h^7}{7!}f^7(x_0)+\cdots\cdots\Big\}+2\Big\{f(x_0)+2hf'(x_0)+\frac{(2h)^2}{2!}f''(x_0)+\frac{(2h)^3}{3!}f'''(x_0)+\frac{(2h)^4}{4!}f^4(x_0)+\frac{(2h)^5}{5!}f^5(x_0)+\frac{(2h)^6}{6!}f^6(x_0)+\cdots\cdots\Big\}+32\Big\{f(x_0)+3hf'(x_0)+\frac{(3h)^2}{2!}f''(x_0)+\frac{(3h)^3}{3!}f'''(x_0)+\frac{(3h)^4}{4!}f^4(x_0)+\frac{(3h)^5}{5!}f^5(x_0)+\frac{(3h)^6}{6!}f^6(x_0)+\cdots\Big\}+7\Big\{f(x_0)+4hf'(x_0)+\frac{(4h)^2}{2!}f''(x_0)+\frac{(4h)^3}{3!}f'''(x_0)+\frac{(4h)^4}{4!}f^4(x_0)+\frac{(4h)^5}{5!}f^5(x_0)+\frac{(4h)^6}{6!}f^6(x_0)+\cdots\Big\}$$

$$I_2=4f(x_0)+8h^2f'(x_0)+\frac{32}{3}h^3f''(x_0)+\frac{32}{3}h^4f'''(x_0)+\frac{128}{15}h^5f^4(x_0)+\frac{256}{45}h^6f^5(x_0)+\frac{88}{27}h^7f^6(x_0)+\cdots$$

Put the value of I_1 & I_2 in eqn (1)

$$\Rightarrow E=\frac{1024}{315}h^7f^6(x_0)-\frac{88}{27}h^7f^6(x_0)+\cdots\cdots$$

$$=-\frac{8}{945}h^7f^6(x_0)$$ neglecting higher term

$$\Rightarrow \text{Error}=\frac{-8}{945}h^7f^6(x_0)$$

★★★ ★★★

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(5) Error Analysis of Weddle's Rule:- n=6

$$E = X - A$$

$$= \int_{x_0}^{x_6} f(x)\,dx - \frac{3h}{10}\left\{ f(x_0) + 5f(x_1) + f(x_2) + 6f(x_3) + f(x_4) + 5f(x_5) + f(x_6) \right\}$$

$$E = I_1 - I_2 \qquad \text{---------- (1)}$$

$$I_1 = \int_{x_0}^{x_6} f(x)\,dx$$

$$= F(x_6) - F(x_0) \qquad \because f(x) = F'(x)$$

$$= F(x_0+6h) - F(x_0)$$

$$= F(x_0) + 6hF'(x_0) + \frac{(6h)^2}{2!}F''(x_0) + \frac{(6h)^3}{3!}F'''(x_0) + \frac{(6h)^4}{4!}F^4(x_0) + \frac{(6h)^5}{5!}F^5(x_0) + \frac{(6h)^6}{6!}F^6(x_0) + \frac{(6h)^7}{7!}F^7(x_0) + \cdots\cdots - F(x_0) \qquad \because \text{by Taylor series.}$$

$$I_1 = 6hf(x_0) + 18h^2 f'(x_0) + 36h^3 f''(x_0) + 54h^4 f'''(x_0) + \frac{324}{5}h^5 f^4(x_0) + \frac{324}{5}h^6 f^5(x_0) + \frac{1944}{35}h^7 f^6(x_0) + \cdots\cdots$$

Now

$$I_2 = \frac{3h}{10}\left[f(x_0) + 5f(x_0+h) + f(x_0+2h) + 6f(x_0+3h) + f(x_0+4h) + 5f(x_0+5h) + f(x_0+6h) \right]$$

$$= \frac{3h}{10}\left[f(x_0) + 5\left\{ f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \right.\right.$$

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$$\frac{h^3}{3!}f'''(x_0)+\frac{h^4}{4!}f^4(x_0)+\frac{h^5}{5!}f^5(x_0)+\frac{h^6}{6!}f^6(x_0)+\cdots\Big\}$$

$$+\Big\{f(x_0)+2hf'(x_0)+\frac{(2h)^2}{2!}f''(x_0)+\frac{(2h)^3}{3!}f'''(x_0)+$$

$$\frac{(2h)^4}{4!}f^4(x_0)+\frac{(2h)^5}{5!}f^5(x_0)+\frac{(2h)^6}{6!}f^6(x_0)+\cdots\Big\}$$

$$+6\Big\{f(x_0)+3hf'(x_0)+\frac{(3h)^2}{2!}f''(x_0)+\frac{(3h)^3}{3!}f'''(x_0)+$$

$$\frac{(3h)^4}{4!}f^4(x_0)+\frac{(3h)^5}{5!}f^5(x_0)+\frac{(3h)^6}{6!}f^6(x_0)+\cdots\Big\}$$

$$+\Big\{f(x_0)+4hf'(x_0)+\frac{(4h)^2}{2!}f''(x_0)+\frac{(4h)^3}{3!}f'''(x_0)+$$

$$\frac{(4h)^4}{4!}f^4(x_0)+\frac{(4h)^5}{5!}f^5(x_0)+\frac{(4h)^6}{6!}f^6(x_0)+\cdots\Big\}$$

$$+5\Big\{f(x_0)+5hf'(x_0)+\frac{(5h)^2}{2!}f''(x_0)+\frac{(5h)^3}{3!}f'''(x_0)+$$

$$\frac{(5h)^4}{4!}f^4(x_0)+\frac{(5h)^5}{5!}f^5(x_0)+\frac{(5h)^6}{6!}f^6(x_0)+\cdots\Big\}$$

$$+\Big\{f(x_0)+6hf'(x_0)+\frac{(6h)^2}{2!}f''(x_0)+\frac{(6h)^3}{3!}f'''(x_0)+$$

$$\frac{(6h)^4}{4!}f^4(x_0)+\frac{(6h)^5}{5!}f^5(x_0)+\frac{(6h)^6}{6!}f^6(x_0)+\cdots\Big\}\Big]$$

$$I_2 = 6hf(x_0)+18h^2f'(x_0)+36h^3f''(x_0)+54h^4f'''(x_0)$$
$$+\frac{324}{5}h^5f^4(x_0)+\frac{324}{5}h^6f^5(x_0)+\frac{1167}{20}h^7f^6(x_0)+\cdots$$

Put value of I_1 & I_2 in eqn (1)

$$\Rightarrow h=\frac{1944}{35}h^7f^6(x_0)-\frac{1111}{20}h^7f^6(x_0)+\cdots$$

$$h=\frac{-1}{140}h^7f^6(x_0)$$ is required result

∵ neglecting higher terms

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$x_0 \rightarrow$

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	0						
		0.0875					
5	0.0875		0.0013				
		0.0888		0.0002			
10	0.1763		0.0015		0.0009		
		0.0903		0.0011		0.0009	
15	0.2679		0.0026		0.0020		0.0011
		0.0929		0.0031			
20	0.3640		0.0057				
		0.0986					
25	0.4663		0.0088				
		0.1074					
30	0.5774						

$$f(x) = y_0 + u\left[\Delta y_0 + \frac{\Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1)(u^2-2)}{5!}\left[\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right] + \frac{u^2(u^2-1)(u^2-2)}{6!}\Delta^6 y_{-3}$$

$$\Rightarrow f(16) = 0.2679 + (0.2)\left[\frac{0.0916 + 0.0961}{2}\right] + \frac{(0.2)^2}{2}(0.0045) + \frac{(0.2)[(0.2)^2-1]}{6}\left[\frac{0.0017 + 0.0017}{2}\right] + 0 + \frac{(0.2)[(0.2)^2-1][(0.2)^2-4]}{120}\left[\frac{-0.002 + 0.009}{2}\right] + \frac{(0.2)^2[(0.2)^2-1][(0.2)^2-2]}{120}(0.0011)$$

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$= 0.2679 + 0.0019 + 0.0009 - 0.0005 + .0001$

$f(16) = 0.2707$

⇒ ★★——★★★——★★

Q4 Divided difference table is

x	y	1st D.D	2nd D.D	3rd D.D
3	3			
		$\frac{12-3}{2-3} = -9$		
2	12		$\frac{-3+9}{1-3} = -3$	
		$\frac{15-12}{1-2} = -3$		$\frac{-3.83+3}{-1-3} = 0.2075$
1	15		$\frac{8.5+3}{-1-2} = -3.83$	
		$\frac{-2-15}{-1-1} = 8.5$		
-1	-2			

By Newton's DD formula

$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + \cdots\cdots$

$= 3 + (x-3)(-9) + (x-3)(x-2)(-3) + (x-3)(x-2)(x-1)(0.2075)$

$= 3 - 9x + 27 + (-3)(x^2 - 5x + 6) + (0.2075)(x^2 - 5x + 6)(x-1$

$= 3 - 9x + 27 - 3x^2 + 15x - 18 + 0.2075(x^3 - 5x^2 + 6x - x^2 + 5x - 6)$

$= 3 - 9x + 27 - 3x^2 + 15x - 18 + 0.2075x^3 - 1.0375x^2 + 1.245x - 0.2075x^2 + 1.0375x - 1.245$

$\Rightarrow f(x) = 0.2075x^3 - 4.245x^2 + 8.2825x + 10.755$

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