TOPOLOGY

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TOPICS:-

- Introduction to Topological Structures.
- Topological Groups.
- Connected Spaces/ Path Connected Spaces/ Locally Connected Spaces.
- Compact Spaces/ Locally Compact Spaces/ Lindelof Spaces.
- Homeomorphism.
- Separation Axioms.
- N-Spheres.
- Normal Spaces.
- Urysohn Lemma.
- Manifolds.
- Introduction to Dimension Theory.

BOOKS:-

- General Topology by John L. Kelley.
- Introduction to Topology and Modern Analyses by George F. Simmon.
- Foundations of Topology by C. Wayne Patty.
- Topology by James R. Munkres.
- Topology by James Dugundji.

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Topology:-

Let X be non empty set and 'τ' be the collection of subsets of X. Then τ is called topology if,

(i) Φ and X belongs to τ.
(ii) The intersection of any two sets in τ belongs to τ.
(iii) The union of any number of sets in τ belongs to τ.

The members of τ are then called τ-open sets OR simply open sets. (And compliment of open set is called a closed set). X together with τ $i.e. (X, \tau)$ is called a topological space.

The set X is called its ground set and the elements of X are called its points.

Neighborhood of a Point:-

If (X, τ) is a topological space and A ⊆ X, Then x ϵ A, There exit $u \epsilon \tau\ s.t\ x \epsilon u \subseteq A$. Then A is neighborhood of x.

- Neighborhood of a point is a set.
- Every open set is neighborhood of each of its points.
- A point of X can have more than one neighborhood.
- Every point of X has at least one neighborhood and that is X.
- Collection of all neighborhoods of a point is called neighborhood system of that point.

Base for a Topology:-

Let X be any set, then a subset β of subsets of X is a base for a topology if ∀ $x \epsilon X$, there exit B ϵ β such that $x \epsilon B$.

For B_1 *and* $B_2 \epsilon \beta$ *and* $x \epsilon B_1 \cap B_2$, Then there exit $B_3 \epsilon \beta$ such that $x \epsilon B_3 \subseteq B_1 \cap B_2$

$\cup B_i = X$

Concepts:-

(i) We are given with the topology τ and need to construct the base for τ.

(ii) We are given with the ground set and need to construct a base and generate corresponding topology.

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Base For a Topology:-

A collection β of a topology τ is called base fot topology if every member of τ is union of members of β.

Example:-

$$\text{Let } X = \{a, b, c, d, e\} \quad \& \quad \tau = \{\varphi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$$

$$\beta_1 = \{\{a, b\}, \{c, d\}, X\} \quad \& \quad \beta_2\{\{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$$

Both β_1 & β_2 are the basis for τ.

Now consider,

$\tau_2 = \{\varphi, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d, e\}, \{d, e\}, \{a, d, e\}, X\}$

$$\beta_3 = \{\{a\}, \{b, c\}, \{d, e\}, \{b, c, d, e\}\} \quad \& \quad \beta_4 = \{\{a\}, \{b, c\}, \{d, e\}\}$$

Then each β_3 *and* β_4 are basis for τ_2

Note:-

Members of the base for a topology are called basic open sets and their compliments are called basic closed sets.

Example:-

If X s a non empty set then the collection of all singletons of X is a base for a discrete topology.

$e.g.\ \beta = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}$ for same above X.

Theorem:-

A collection β of a topology τ is a base for τ if and only if for any points $x \in u \in \tau$, there exit $B\epsilon\beta$ such that $x\epsilon B \subseteq \beta$.

Note:-

- Consider the usual topology on the real line $\mathbb{R}$. Then a base β for this topology is the set of all open intervals, that is,

$$\beta = \{(a, b): a, b \in \mathbb{R} \; and \; a < b\}$$

- Consider the usual topology on the plane $\mathbb{R}^2$ then each of the following collections is a base for the usual topology on $\mathbb{R}^2$.

1. The collection β_1 of all open discs in $\mathbb{R}^2$.
2. The collection β_2 of all open rectangles bounded by the sides, which are parallel to the co-ordinate axis.

Definition:-

A collection β of sub sets of a non empty set X is a base for some topology τ on X if β meet the following requirements,

(1) If A, B $\epsilon\,\beta$ and $x \in A \cap B$, then there exit $C \in \beta$ such that, $x \in C \subseteq A \cap B$.

(2) Union of B=X $\forall B\epsilon\beta$

Base at a Point:-

Let x be any point of a topological space (X, τ). A collection β_x of open sets containing x is called a base at the point x. This is also called local base.

- A collection β_x is base at the point x, if for every open set u containing x, there exit $B \in \beta_x$ such that $x \in B \subseteq u$.

Theorem:-

A collection β of open sets in a topological space (X,τ) is a base for τ if and only if β contains base at each point of X.

Example:-

Let $X = \{a,b,c,d,e\}$ &

$$\tau = \left\{ \begin{matrix} \varphi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{c,d,e\}, \{a,c,d,e,\}, \\ \{b,c,d,e\}, X \end{matrix} \right\}$$

$$\beta_a = \{\{a\}\} \quad , \quad \beta_b = \{\{b\}\} \quad , \quad \beta_c = \{\{c\}\}$$

$$\beta_e = \{\{c,d,e\}\} \quad \& \quad \beta_d = \{\{c,d,e\}\}$$

And $\beta = \{\{a\},\{b\},\{c\},\{c,d,e\}\}$ ∵containing base at each point of X.

Example:-

The set of all open discs with centre x is a base at x with respect to the plane.

Example:-

In discrete topological space X, the base at each point is {{x}}.

Sub Base:-

Let τ be a topology on X, then the collection $\boldsymbol{S}$ of members τ is called sub base for τ if and only if finite intersection of members of $\boldsymbol{S}$ form base for τ.

Example:-

Let $X = \{a, b, c, d, e\}$

$\tau = \{\varphi, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d, e\}, \{d, e\}, \{a, d, e\}, X\}$

Let $\mathcal{S} = \{\{a\}, \{a, b, c\}, \{b, c, d, e\}, \{a, d, e\}\}$

$\beta = \{\{a\}, \{a, d, e\}, \{b, c, d, e\}, \{a, d, e\}, \{b, c\}, \{d, e\}\}$ then clearly β is the base for τ.

$\Longrightarrow \mathcal{S}$ is a sub base.

Question 1:-

Let $X = \{a, b, c, d\}$; sow that $\beta = \{\{b, c\}, \{c, d\}\}$ cannot be base for any topology on X.

Solution:-

(1) Let $\beta_1 = \{b, c\}$ & $\beta_2 = \{c, d\} \epsilon \beta$

$$\beta_1 \cap \beta_2 = \{c\}$$

Let $c \epsilon \beta_1 \cap \beta_2$; then there does not exit any set $\beta_3 \epsilon \beta$ such that,

$$c \epsilon \beta_3 \subseteq \beta_1 \cap \beta_2$$

⇨ β is not base for any topology.

(2) Also union of $B_i \epsilon \beta = \{b, c, d\} \neq X$

⇨ β is not base for any topology.

Question 2:-

Find a sub base (with a few members as possible) for each of the following topology on $X = \{a, b, c\}$

(1) $\tau_1 = \{\varphi, \{a\}, X\}$

(2) $\tau_2 = \{\varphi, \{a\}, \{b, c\}, X\}$

(3) $\tau_3 = \{\varphi, \{a\}, \{a, b\}, \{b, c\}, X\}$

Solution:-

(1) Sub base for $\tau_1 = \mathcal{S}_1 = \{\varphi, \{a\}, X\}$

(2) Sub base for $\tau_2 = \mathcal{S}_2 = \{\{a\}, \{b, c\}\}$

(3) Sub base for $\tau_3 = \mathcal{S}_3 = \{\{a\}, \{a, b\}, \{a, c\}\}$

Quetion3:-

Let $X = \{a, b, c, d, e\}$ generate topologies for,

$\mathcal{S}_1 = \{\{a, b\}, \{c, d\}\ X\}$ & $\mathcal{S}_2 = \{\{a, b\}, \{b, c, d\}, \{d, e\}, X\}$

Solution:-

(1) $\mathcal{S}_1 = \{\{a, b\}, \{c, d\}\ , X\}$

Base for $\mathcal{S}_1 = \beta_1 = \{\varphi, X, \{a, b\}, \{c, d\}\}$

And topology for $\mathcal{S}_1 = \tau_1 = \{\varphi, X, \{c, d\}, \{a, b\}, \{a, b, c, d\}\}$

(2) $\mathcal{S}_2 = \{\{a, b\}, \{b, c, d\}, \{d, e\}, X\}$

Base for $\mathcal{S}_2 = \beta_2 = \{\varphi, X, \{a, b\}, \{b, c, d\}, \{d, e\}, \{b\}, \{d\}\}$

And topology $= \tau_2 = \{\varphi, X, \{a, b\}, \{b, c, d\}, \{d, e\}, \{b\}, \{d\}, \{a, b, c, d\},$

$\{a, b, d, e\}, \{a, b, d\}, \{b, c, d, e\}, \{b, e, d\}, \{b, d\}\}$

Continuous Function:-

Let (X, τ_x) and (Y, τ_y) be topological spaces and $f: X \longrightarrow Y$ is a function. Let $x_\circ \in X$ then f is said to be continuous at $x_\circ$ if for each open set V containing $f(x_o)$ there exit an open set U in X such that $x_o \epsilon U$ & $f(U) \subseteq V$.

Then f is said to be continuous on X if f is continuous at each point of X.

Example:-

Let $X = \{a, b, c\}$; $\tau_x = \{X, \varphi, \{a\}, \{a, b\}\}$ and $Y = \{1,2\}$; $\tau_y = \{\varphi, Y, \{1\}, \{2\}\}$. Let $f: X \longrightarrow Y$ be defined by $f(a) = 1$, $f(b) = 1$ & $f(c) = 2$. Then f is continuous at $x = a, b$ but not continuous at c.

Verification:-

Let $x = b$ ⇨ $f(x) = f(b) = 1$

$$Let\ V = \{1\},\ Then\ \ U = \{a, b\}$$

Now, $b \epsilon U$ & $f(U) = f(\{a, b\}) = 1 \subseteq V$

⇨ f is continuous at $x = b$

Similarly for $x = a$.

Now let $x = c$ ⇨ $f(x) = f(c) = 2$

And the open set $V = \{2\}$. Now $x = c \epsilon U = X$

⇨ $f(U) = f(X) = Y = \{1,2\} \nsubseteq V$
⇨ f is not continuous at c.

Example:-

Let X be an indiscrete topological space and let Y be discrete topological space, then every function $f: X \longrightarrow Y$ which is not constant is discontinuous at all points of X.

Example:-

Let X be an arbitrary topological space and let Y be an indiscrete topological space. Then every function $f: X \longrightarrow Y$ is continuous at all points of X.

Continuous Function (Alternate Definition):-

A function $f: X \longrightarrow Y$ is continuous on X if $f^{-1}(V)$ is open in X for every open set V of Y.

Product Topology/Box Topology:-

Let (X, τ_x) and (Y, τ_y) be topological spaces, The product topology (Box topology) on $X \times Y$ is the topology having as basis the collection β of all sub sets of the for $U \times V$, where U is an open subset of X and V is an open subset of V.

Result:-

If β is a basis for topology of X and γ is a basis for topology of Y, then the collection $\mathfrak{D} = \{B \times C : B \epsilon \beta, and\ C \epsilon \gamma\}$ is a basis for the product topology of $X \times Y$.

Example:-

Let $X = \{a, b, c\}$; $\tau_x = \{\varphi, \{a\}, \{b\}, \{a, b\}, X\}$

Let $Y = \{1,2,3,4\}$; $\tau_y = \{\varphi, \{1,2\}, \{3,4\}, Y\}$

Then we know,

$$X \times Y = \{(a, 1), (a, 2), (a, 3), (a, 4), (b, 1), (b, 2), (b, 3), (b, 4), (c, 1),$$

$(c,2)\,,(c,3)\,,(c,4)\}$

$$\beta_{\tau_x} = \{\{a\},\{b\},X\}\,; \quad \gamma_{\tau_y} = \{\{1,2\},\{3,4\}\}$$

Let $U_1 = \{a\}\,, U_2 = \{b\} \,\epsilon\, \beta_{\tau_x}$ & $V_1 = \{1,2\}\,,\ V_2 = \{3,4\} \,\epsilon\, \gamma_{\tau_y}$

➪ $\mathfrak{D} = \{U_1 \times V_1,\ U_2 \times V_1,\ U_1 \times V_2, U_2 \times V_2,\ X \times V_1,\ X \times V_2\}$
➪ $\mathfrak{D} = \{\{a\} \times \{1,2\}, \{b\} \times \{1,2\}, \{a\} \times \{3,4\}, \{b\} \times \{3,4\}, \{a,b,c\} \times \{1,2\},$

$\{a,b,c\} \times \{3,4\}\}$

➪ $\mathfrak{D} = \{\{(a,1),(a,2)\},\{(b,1),(b,2)\},\{(a,3),(a,4)\},\{(b,3),(b,4)\},$

$\{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\},\{(a,3),(a,4),(b,3),$

$(b,4)\,,(c,3),(c,4)\}\}$

Is the base for $X \times Y$.

Note:-

Let $X_1, X_2, X_3, \dots, X_n$ be a finite family of sets. Then the Cartesian product,

$$\prod_{\alpha=1}^{n} X_\alpha = X_1 \times X_2 \times X_3 \times \cdots \times X_n$$
$$= \{x = (x_1, x_2, \dots, x_n) : x_\alpha \epsilon X_\alpha \forall\ \alpha = 1,2,\dots,n\}$$

For arbitrary product, Let $\{X_\alpha : \alpha \,\epsilon\, \nabla\}$ be an arbitrary family of sets, then the Cartesian product is given by, $\prod_{\alpha\epsilon\nabla} X_\alpha$.

Projection Mapping:-

The function $p_i : \prod_{\alpha\epsilon\nabla} X_\alpha \longrightarrow X_i\ ;\ \ i \,\epsilon\, \nabla$ defined by,

$p_i(x) = x_i$ is called a projection mapping of $\prod_{\alpha\epsilon\nabla} X_\alpha$ onto X_i.

❖ Every projection mapping is surjection (onto).

Definition:-

Let $\{X_\alpha : \alpha \in \nabla\}$ be an arbitrary family of topological spaces, and let $\beta \in \nabla$, then $p_\beta : \prod_{\alpha\in\nabla} X_\alpha \longrightarrow X_\beta$ defined by,

$p_\beta^{-1}(U_\beta) = \prod_{\alpha\in\nabla} B_\alpha$; where $B_\alpha = U_\beta$ $for\ \alpha = \beta$ & $B_\alpha = X_\alpha$ $for\ \alpha \neq \beta$

Then, $\mathcal{S}_\alpha = \{p_\alpha^{-1}(U_\beta) : U_\beta \in \tau_\alpha\}$ and let $\boldsymbol{\mathcal{S}} = \bigcup_{\alpha\in\nabla} \mathcal{S}_\alpha$ be a sub base for topology τ on $\prod_{\alpha\in\nabla} X_\alpha$.

Then τ is called product topology and the topological space $(\prod_{\alpha\in\nabla} X_\alpha, \tau)$ is called the product space.

Previous Knowledge

Semi Group:-

A non-empty set S is said to be semi group under the binary operation $*$ if,

(i) (S, $*$) is closed.
(ii) $*$ is associated in S.

Then we write $(S,*)$ is a semi group.

$e.g. (N, +), (E,\bullet)$ are semi groups.

Group:-

A non empty set G is said to be group under the binary operation $*$ if,

(i) $(G,*)$ is closed $i.e. \forall\ a, b \in G \implies (a * b) \in G$

(ii) $(G,*)$ is associated $i.e. \forall\, a,b,c \in G \implies (a*b)*c = a*(b*c)$
(iii) Identity element under the binary operation $*$ exit in G. That is there exit an element $e \in G\ s.t.\ \forall\, a \in G,\ e*a = a*e = a$.
(iv) Every element of G has its inverse under the binary operation $*$ in G. That is $\forall\, a \in G$ there exit some $b \in G\ s.t.\ a*b = b*a = e$. Where e is identity element in G.

Then we write $(G,*)$ is a group.

Abelian Group:-

A group $(G,*)$ is said to be commutative group or abelian group if "$*$" is commutative in G.

Sub Group:-

Let G be a group and H is a non empty sub set of G, then H is said to be sub group of G if H itself is group under the induced (same) binary operation of G. then we write it as $H \leq G$.

Remark:-

Every group G has at least two sub groups namely the identity element $\{e\}$ and group G itself. These two groups of G are called trivial or improper subgroups of G. Any other sub group of G is called proper or non-trivial sub group of G.

Cyclic Group:-

A group G generates by a single element say a is called cyclic group. In this case "a" is called generator of G. and we write it as $G = \langle a \rangle$.

Left Coset:-

Let G be a group and $x \epsilon G$, then a set defined by

$xH = \{xh \epsilon G: h \epsilon H\}$ is called left coset of H in G determined by x, where H is sub set of G.

Similarly $Hx = \{hx: h \epsilon H\}$ is said to be right coset of H in G.

Normal Sub Group:-

A sub group H of a group G is said to be normal sub group if and only if for all $g \epsilon G \implies gH = Hg$. Then we write $H \underline{\Delta} G$.

Factor Group OR Quotent Group:-

Let G be group and $H \underline{\Delta} G$. Then the collection of all left (or right) cosets of H in G is called factor group or quotient group. It is denoted by G\H $i.e.$ $G\backslash H = \{gh: g \epsilon G\}$.

Homomorphism:-

Let $(G,*)$ & $(G^{'},\bullet)$ be two groups then a function,

$\varphi: (G,*) \longrightarrow (G^{'},\bullet)$ is said to be group homomorphism or simply homomorphism if φ preserves the binary operations of both G & $G^{'}$, that is for all $a, b \epsilon G$; $\varphi(a * b) = \varphi(a) \bullet \varphi(b)$.

Kernal of φ:-

Let $\varphi: G \longrightarrow G^{'}$ be a group homomorphism. Then kernel of φ is denoted and defined by $ker\varphi = \{g \epsilon G: \varphi(g) = e^{'};\ e^{'}$ is identity of $G^{'}\}$

Permutation Group:-

Let $X \neq \varphi$, then any bijection mapping from X⟶X is called permutation on X.

Then the set of all permutations on X is denoted by S_x forms a non abelian group under the binary operation of compositions of functions. If $O(X) = n$, then $O(S_x) = n!$ and in this case S_x is denoted by S_n.

Note:-

$\forall\ a \in G$, where $(G,*)$ is a group,

- $H = aH$; where H is a sub group of G.
- $O(H) = O(aH)$
- In general we know that $aH \neq Ha$.
- Either cosets of a sub group H of a group G are similar or disjoint.
- $\cup_{\forall \alpha \in G}\ aH = G$.
- $Let\ \ \varphi: G \longrightarrow G'$ be a homomorphism,
  (a) $\varphi(e) = e'$. (b) $\forall\ a \in G;\ \varphi(a^{-1}) = \{\varphi(a)\}^{-1}$

Left Action:-

Let $(S,*)$ be a semi group, for a fixed element $a \in S$, the mapping $l_a: S \longrightarrow S$ defined by, $l_a(x) = a * x$ is called left action.

Right Action:-

Let $(S,*)$ be a semi group, for a fixed element $a \in S$, the mapping $R_a: S \longrightarrow S$ defined by, $R_a(x) = x * a$ is called right action.

Inversion Mapping:-

If $(G,*)$ is a group then a mapping $I: G \longrightarrow G$ defined by, $I(x) = x^{-1}$ is called an inversion mapping.

Note:-

If $(G,*)$ is a group, then left action is called **"Left Translation"** and right action is called **"Right Translation".**

Right Topological Semi-Group:-

A right topological semi group consists of a,

(i) Semi group $(S,*)$.
(ii) Topology τ on S.
(iii) For all $a \in S$, the right action $R_a: S \longrightarrow S$ is a continuous mapping.

Left Topological Semi-Group:-

A left topological semi group consists of a,

(i) Semi group $(S,*)$.
(ii) Topology τ on S.
(iii) For all $a \in S$, the left action $l_a: S \longrightarrow S$ is a continuous mapping.

Right Topological Group:-

A right topological group consists of a,

(i) A group $(G,*)$.
(ii) Topology τ on G.
(iii) For each $a \in G$, the right translation $R_a: G \longrightarrow G$ is continuous.

Left Topological Group:-

A left topological group consists of a,

(i) A group $(G,*)$.
(ii) Topology τ on G.
(iii) For each $a \epsilon G$, the left translation $l_a: G \longrightarrow G$ is continuous.

Para-topological Group:-

A para-topological group consists of a,

(i) A group $(G,*)$.
(ii) Topology τ on G.
(iii) Multiplication mapping $m: G \times G \longrightarrow G$ defined by, $m\big((x,y)\big) = x * y$ is continuous with respect to the product topology on $G \times G$.

Semi-topological Group:-

A semi-topological group consists of a,

(i) A group $(G,*)$.
(ii) Topology τ on G.
(iii) Left translation $l_a: G \longrightarrow G$ and right translation $R_a: G \longrightarrow G$ are continuous for each fixed $a \epsilon G$.

- Continuity defined in para-topological group is called "jointly continuity" (∵ x and y vary independently). And the continuity defined in semi topological group is called "separately continuity" (∵ a is fixed).
- Jointly continuous is always separately continuous (that is if $m: G \times G \longrightarrow G$ is continuous then $l_a: G \longrightarrow G$ & $R_a: G \longrightarrow G$ is

continuous for each fixed $a \epsilon G$). But a separately continuous may not be jointly continuous.

Topological Group:-

A topological group $(G,*,\tau)$ $[((G,*),\tau)]$ is a group $(G,*)$ together with topology τ defined on G that satisfies the following two properties,

(i) The multiplication mapping $m: G \times G \longrightarrow G$ defined by $m((x,y)) = x * y$ is continuous, where $G \times G$ is a product topology.

(ii) The inversion mapping $I: G \longrightarrow G$ defined by $I(x) = x^{-1} \; \forall \, x \, \epsilon \, X$, is continuous.

Remark:-

(i) Is equivalently to: Whenever $W \subseteq G$ is an open sub set and $(x * y) \epsilon W$, there exit open set U containing 'x' and V containing 'y' in G such that $U * V \subseteq W$.

(ii) Is equivalently to: Whenever W is an open set in G containing x^{-1}, there exit an open set U in G containing 'x' such that, $I(U) \subseteq W \quad \Longrightarrow \quad U^{-1} \subseteq W$.

Note:-

Let $(G,*)$ be a group then for $A \subseteq G$; $A^{-1} = \{x \,\epsilon\, G: x^{-1} \,\epsilon\, A\}$. And for any sub set $A \subseteq G$, $B \subseteq G$; $A * B = \{a * b: a \,\epsilon\, A \text{ and } b \,\epsilon\, B\}$.

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Topological Group (alternate):-

Let $(G,\circledast)$ be a group endowed with the topology τ on G, then $(G,\circledast,\tau)$ is called a topological group if for each $x,y\,\epsilon\, G$, the multiplication mapping $m: G\times G \longrightarrow G$ defined by $m\big((x,y)\big) = x\times y^{-1}$ is continuous.

Topological Group (alternate):-

Let $(G,\circledast)$ be a group endowed with the topology τ on G, then $(G,\circledast,\tau)$ is called a topological group if for each $x,y\,\epsilon\, G$, and for each open set W in G containing $x\circledast y^{-1}$, there exit open sets U and V in G containing x *and* y^{-1} respectively such that, $U\circledast V^{-1}\subseteq G$.

Example:-

Let $G=\{\pm i,\pm 1\}$, Then (G, ⊛) is a group,

$\tau=\big\{\varphi, G,\{1\},\{-1\}.\{1,-1\}\big\}$

Verification:-

Let $x=i,\quad y=-i \quad \Longrightarrow \quad y^{-1}=i$

➩ $x\bullet y^{-1}=(i)(i)=-1$

Let $x\circledast y\,\epsilon\, W=\{-1\}$, Let $U=G,\ V=G$ (∵ only chance of G).

➩ $V^{-1}=G^{-1}=G$ & $U\bullet V^{-1}=G\bullet G=G\nsubseteq W=\{-1\}$.

➩ $(G,\circledast,\tau)$ is not topological group.

Example:-

Let $G=\{\pm 1,\pm i\}$, then (G, •) is a group and let,

$\tau=\big\{\varphi, G,\{1,-1\},\{i,-i\}\big\}$

Verification:-

Let $x = i,\quad y = -i \quad \Longrightarrow \quad y^{-1} = i$

⇨ $x \bullet y^{-1} = (i)(i) = -1$

Let $x \circledast y = x \bullet y^{-1} = -1 \in W = \{1, -1\}$

Now $x = i \in U = \{i, -i\}$ & $y = -i \in V = \{i, -i\}$ & $V^{-1} = \{-i, i\}$

⇨ $U \bullet V^{-1} = \{i, -i\} \bullet \{-i, i\} = \{1, -1\} \subseteq W$.

Similarly we can verify every pair of point and will notice $U \bullet V^{-1} \subseteq W$

⇨ $(G, \circledast, \tau)$ is a topological group.

EXERCISE

Question:-

Every open set W in the product space can be expressed in product of open sets U and V in the component space.

Example:-

Let $G = \{1, 3, 5, 7\}$ & $\tau = \{\varphi, G, \{1\}, \{1,3,5\}\}$ then:

1) $(G, \odot)_8$ is a group.
2) (G, τ) is a topological group.
3) Also show that $(G, \odot, \tau)$ is not a topological group.

Example:-

$G = \{1, \omega, \omega^2\}$, $\tau = \{\varphi, 1, G\}$ verify for topological group.

Question:-

Show that the following are topological groups;

1) $(Z, +, \tau_z)$; τ_z is relative topology with usual topology on $\mathbb{R}$.
2) $(\mathbb{R}, +)$ with usual topology on $\mathbb{R}$.
3) $(\mathbb{R}_+, \bullet, \tau_{\mathbb{R}_+})$ is a topological group. (1^{st} show $(\mathbb{R}, +)$ is a group.

Note:-

(1) Let $(G, *)$ is a group and $A, B \subseteq G$ then,
$A * B = \{a * b : a \in A \;\&\; b \in B\}$

(2) $A^{-1} = \{a^{-1} \in G : a \in A\}$

(3) The group with discrete topology is always a topological group and this topological group is simply called discrete group.

(4) Let $(G, *)$ be a group endowed with indiscrete topology then it is always a topological group.

Lemma:-

Let $(G, \bullet, \tau)$ be a topological group then:

1) The map $g \mapsto g^{-1}$ is a homeomorphism of G onto itself.
2) Fix $g_\circ \in G$ then the map $g \mapsto g_\circ g,\ g \mapsto g g_\circ$ are homeomorphism of G onto itself.

Proof:-

(1) Let us define $i: G \longrightarrow G$ by $i(g) = g^{-1}\ \forall\ g \in G$

i is one to one:-

Let $g_1, g_2 \in G$ & $i(g) = g^{-1}$

⇨ $g_1^{-1} = g_2^{-1}$ ⇨ $g_1^{-1} g_1 = g_2^{-1} g_1$

⇨ $e = g_2^{-1} g_1$ ⇨ $g_2 e = g_2 g_2^{-1} g_1$

➯ $g_2 = eg_1$ ➯ $g_2 = g_1$
➯ i is one to one.

i is onto:-

Let $g \in G$ $(codomain)$

Then, by definition there exit $g^{-1} \in G$ $s.t$ $i(g^{-1}) = (g^{-1})^{-1} = g$

➯ i is onto.

Now by definition, $i: G \longrightarrow G$ is continuous

Now, $ii^{-1} = id$ (identity mapping)

Since i is continuous, therefore, $i^{-1}: G \longrightarrow G$ is continuous.

Hence, $i: G \longrightarrow G$ is homeomorphism.

(2) We have to prove that in a topological group every left translation (right translation) is homeomorphism.

Let $(G, \bullet, \tau)$ be topological group.

Let $g_\circ \in G$ be a fixed point then, $l_{g_\circ}: G \longrightarrow G$ is defined by;

$$l_{g_\circ}(g) = g_\circ g \quad \forall\, g \in G$$

$l_{g_\circ}$ is one to one:-

Let $g_1, g_2 \in G$ $s.t.$ $l_{g_\circ}(g_1) = l_{g_\circ}(g_2)$

➯ $g_1 = g_2$ $\because G$ is a group and left (right) cancellation law holds
➯ $l_{g_\circ}$ is one to one.

$l_{g_\circ}$ is onto:-

Let $y \in G$ $(codomain)$ $s.t.$ $l_{g_\circ}(x) = y$ for some $x \in G$.

➯ $g_\circ x = y$ $\Longrightarrow$ $g_\circ^{-1} g_\circ x = g_\circ^{-1} y$

➩ $ex = g_\circ^{-1}y$ ➩ $x = g_\circ^{-1}y$

Since G is a group and $g_\circ, y \in G$

➩ $g_\circ^{-1} \in G$ & $g_\circ^{-1}y \in G$ ➩ $x \in G$
➩ $l_{g_\circ}: G \longrightarrow G$ is on to

Now, 1st we show that $l_{g_\circ}: G \longrightarrow G$ is continuous.

Let $x \in G$ be an arbitrary point, and let W be an open set containing g.

Since $(G,\bullet,\tau)$ is a topological group, therefore there exit open sets U & V containing g & x respectively in G $s.t.$ $UV \subseteq W$

➩ $gV \subseteq UV \subseteq W$ $\because g \in U$
➩ $l_g(V) = gV \subseteq W$ $\because gV = l_g(V)$
➩ l_g is continuous.

Now we show that $l_g^{-1}: G \longrightarrow G$ is continuous.

Since, $l_g^{-1} = l_{g^{-1}}$ & $l_{g^{-1}}: G \longrightarrow G$ is continuous for fixed $g^{-1} \in G$

Hence, $l_g^{-1}: G \longrightarrow G$ is continuous.

This proves that $l_g: G \longrightarrow G$ is homeomorphism.

Now, Let $(G,\bullet,\tau)$ be topological group.

Let $g_\circ \in G$ be a fixed point then, $R_{g_\circ}: G \longrightarrow G$ is defined by;

$$R_{g_\circ}(g) = gg_\circ \quad \forall\, g \in G$$

$R_{g_\circ}$ is one to one:-

Let $g_1, g_2 \in G$ $s.t.$ $R_{g_\circ}(g_1) = R_{g_\circ}(g_2)$

➩ $g_1 = g_2$ $\because G$ is a group and left (right) cancellation law holds
➩ $R_{g_\circ}$ is one to one.

$R_{g_\circ}$ is onto:-

Let $y \in G$ (codomain) $s.t.$ $R_{g_\circ}(x) = y$ for some $x \in G$.

⇨ $xg_\circ = y$ ⇨ $xg_\circ g_\circ^{-1} = yg_\circ^{-1}$
⇨ $xe = yg_\circ^{-1}$ ⇨ $x = yg_\circ^{-1}$

Since G is a group and $g_\circ, y \in G$

⇨ $g_\circ^{-1} \in G$ & $g_\circ^{-1}y \in G$ ⇨ $x \in G$

That is for $y \in G$ (codomain) we have find $x \in G$ (domain) $s.t.$ $R_{g_\circ}(x) = y$

⇨ $R_{g_\circ}: G \longrightarrow G$ is on to

Now, 1st we show that $R_{g_\circ}: G \longrightarrow G$ is continuous.

Let $x \in G$ be an arbitrary point, and let W be an open set containing g.

Since $(G,\bullet,\tau)$ is a topological group, therefore there exit open sets U & V containing g & x respectively in G $s.t.$ $VU \subseteq W$

⇨ $Vg \subseteq VU \subseteq W$ $\because g \in U$
⇨ $R_g(V) = Vg \subseteq W$ $\because Vg = R_g(V)$
⇨ R_g is continuous.

Now we show that $R_g^{-1}: G \longrightarrow G$ is continuous.

Since, $R_g^{-1} = R_{g^{-1}}$ & $R_{g^{-1}}: G \longrightarrow G$ is continuous for fixed $g^{-1} \in G$

Hence, $R_g^{-1}: G \longrightarrow G$ is continuous.

This proves that $R_g: G \longrightarrow G$ is homeomorphism.

Theorem:-

Let $(G,\ast\ \tau)$ be a topological group and A,B are subsets of G. If A is open and B is arbitrary then $A \ast B$ & $B \ast A$ are open.

Proof:-

Since A is open subset of G and $R_b: G \longrightarrow G$ is homeomorphism,

Therefore, R_b is an open mapping.

⇨ $R_b(A) = A * b$ is an open set.
⇨ $A * B = \bigcup_{b \epsilon B} A * B$ is open.

Now, Since A is open subset of G and $l_b: G \longrightarrow G$ is homeomorphism,

Therefore, l_b is an open mapping.

⇨ $l_b(A) = b * A$ is an open set.
⇨ $B * A = \bigcup_{b \epsilon B} b * A$ is open.

Theorem:-

Let $(G, \bullet, \tau)$ be a topological group and β_e is open base at the identity e of G then:

1. For every $U \epsilon\, \beta_e$, $\exists\ V \epsilon\, \beta_e\ s.t.\ V^2 \subseteq U.$
2. For every $U \epsilon\, \beta_e$, $\exists\ V \epsilon\, \beta_e\ s.t.\ V^{-1} \subseteq U.$
3. For every $U \epsilon\, \beta_e$, and every $x \,\epsilon\, U$, $\exists\ V \epsilon\, \beta_e\ s.t.\ Vx \subseteq U.$
4. For every $U \epsilon\, \beta_e$, and $x \,\epsilon\, G$, $\exists\ V \epsilon\, \beta_e\ s.t.\ xVx^{-1} \subseteq U.$
5. For every $U, V \epsilon\, \beta_e$, $\exists\ \omega \,\epsilon\, \beta_e\ s.t.\ \omega \subseteq U \cap V.$

Proof:-

❶ Let $U \,\epsilon\, \beta_e$ $i.e.$ $e \,\epsilon\, U$

Now $m: G \times G \longrightarrow G$ is continuous,

Thus for every open set U in G containing $'e'$ there exit an open set W in $G \times G$ containing $(e, e)\ s.t.\ m(W) \subseteq U.$

Now every open set $'W'$ in $G \times G$ can be expressed in the form $V_1 \times V_2$ and in this case $V \times V$, $V \in \beta_e$ such that,

$m(W) = m(V \times V) = V^2 \subseteq U$.

∵ Both sets G, G therefore, V×V.

∵ W containing (e, e)

⇨ V, V containing (e, e)

⇨ $V \in \beta_e$

❷ Let $U \in \beta_e$ i.e. $e \in U$.

Since inverse mapping $i: G \longrightarrow G$ is continuous, therefore for each U containing $i(e) = e^{-1} = e$, *there exit* $V \in G$ containing e such that,

$$i(V) = V^{-1} \subseteq U.$$

❸ Since $(G, \bullet, \tau)$ is a topological group,

Therefore $r_x: G \longrightarrow G$ is homeomorphism.

Thus for each $U \in \beta_e$ and for each $x \in U$, there exit $V \in \beta_e$ such that,

$$r_x(V) \subseteq U \quad or \quad Vx \subseteq U.$$

❹ Since $(G, \bullet, \tau)$ is a topological group,

Therefore $l_x: G \longrightarrow G$ & $r_x: G \longrightarrow G$ are homeomorphism.

⇨ l_x , r_x^{-1} are continuous.

⇨ $r_x^{-1} \circ l_x : G \longrightarrow G$ is continuous.

Thus for each $U \in \beta_e$ and for each $x \in G$, there exit $V \in \beta_e$ s.t.

$$(r_x^{-1} \circ l_x)(V) \subseteq U \quad ⇨ \quad r_x^{-1}\big(l_x(V)\big) \subseteq U.$$

⇨ $r_x^{-1}(xV) \subseteq U$ ⇨ $xVx^{-1} \subseteq U$.

❺ Let $U, V \in \beta_e$,

Since β_e is base at e, therefore, by definition of open base at e, there exit $\omega \in \beta_e$ s.t. $\omega \subseteq U \cap V$.

Theorem:-

Prove that every topological group $(G,\bullet,\tau)$ is a regular space.

Proof:-

> Theorem:-A topological space (X,τ) is regular iff for each open set U in X and $x\epsilon U$ there exit an open set V in X $s.t.$
>
> $x \epsilon V \subseteq Cl(V) \subseteq U.$

Let U be an open neighborhood of the identity $e \epsilon G$.

Then there exit $V \epsilon \beta_e$ $s.t.$ $V^2 \subseteq U$.

Let $x \epsilon Cl(V)$ ⇨ $Vx \cap V \neq \varphi$

⇨ $a_1 x = a_2$, $where,\ a_1, a_2 \epsilon V$

⇨ $x = a_1^{-1} a_2 \epsilon VV^{-1} = VV = V^2$ $\because a_1, a_2 \epsilon V$

⇨ $x \epsilon V^2 \subseteq U$ $\because V^2 \subseteq U.$

⇨ $x \epsilon U$ ⇨ $Cl(V) \subseteq U.$

⇨ $x \epsilon V \subseteq Cl(V) \subseteq U.$

Hence for each $x \epsilon U$, there exit open set V in X $s.t.$

$$x \epsilon V \subseteq Cl(V) \subseteq U.$$

⇨ G is a regular space.

Theorem:-

Let $(G,\bullet,\tau)$ be a right topological space, prove that for any base $\beta_g = \{ug: u \epsilon \beta_e\}$ is base of τ at g.

Proof:-

Let β_e be the base of the space (G,τ) at e.

We prove that $\beta_g = \{ug: u \epsilon \beta_e\}$ is a base of the space (G,τ) $at\ 'g'$

Let W be any open set in (G,τ) containing g.

Since right translation $r_g : G \longrightarrow G$ is continuous, therefore there exit an open set $u \in \beta_e \ s.t. \ r_g(u) \subseteq W$

⇨ $ug \subseteq W : \ u \in \beta_e$ ⇨ $\bigcup_{g \in W} ug = W ; \ u \in \beta_e$

⇨ Every open set W containing 'g' can be expressed as union of $ug : u \in \beta_e$.

This proves that $\beta_g = \{ug : u \in \beta_e\}$ is base at 'g'.

Theorem:-

Suppose that a subgroup 'H' of a right topological space $(G, \bullet, \tau)$ contains a non-empty open subset of G, then prove that H is open in G.

Proof:-

Let u be a non-empty open subset of $G \ s.t. \ u \subseteq H$.

We prove that H is open.

Let $a \in H$ be a fixed element.

Since right translation $r_a : G \longrightarrow G$ is homeomorphism,

⇨ $r_a : G \longrightarrow G$ is an open mapping.

Hence $r_a(u) = ua$ is an open subset of H.

⇨ $ua \subseteq H$ ⇨ $\bigcup_{a \in H} ua = H$

Since each ua is open, therefore H is open.

Theorem:-

Let $f : (G, \bullet, \tau_G) \longrightarrow (H, \bullet, \tau_H)$ be a homeomorphism of left topological groups. If f is continuous at the neutral element e_G of G, then f is continuous on G.

Proof:-

Let $x \epsilon G$ be arbitrary point.

Let O be an open set containing $y = f(x)$ in H.

Since left translation $l_y : H \longrightarrow H$ is homeomorphism of H.

This implies there exit an open set V containing e_H $s.t.$ $l_y(V) \subseteq O$

$$Or \quad yV \subseteq O$$

Since f is continuous at e_G, therefore there exit some open set u in G containing e_G $s.t.$ $f(u) \subseteq V$.

Again, since $l_x : G \longrightarrow G$ is homeomorphism, therefore the set $l_x(u) = xu$ is open in G containing x.

Consider $f(xu) = f(x)f(u) = yV \subseteq O$ $\quad \because f(x) = y$ & $f(u) \subseteq V$

⇨ f is continuous at x.

⇨ f is continuous at G.

Assignment No. 1:-

Let $(G,\bullet)$ be a group and β_e be a family of subsets of G satisfying conditions:

1. For every $U \epsilon \beta_e$, $\exists$ $V \epsilon \beta_e$ $s.t.$ $V^2 \subseteq U$.
2. For every $U \epsilon \beta_e$, $\exists$ $V \epsilon \beta_e$ $s.t.$ $V^{-1} \subseteq U$.
3. For every $U \epsilon \beta_e$, and every $x \epsilon U$, $\exists$ $V \epsilon \beta_e$ $s.t.$ $Vx \subseteq U$.
4. For every $U \epsilon \beta_e$, and $x \epsilon G$, $\exists$ $V \epsilon \beta_e$ $s.t.$ $xVx^{-1} \subseteq U$.
5. For every $U, V \epsilon \beta_e$, $\exists$ $\omega \epsilon \beta_e$ $s.t.$ $\omega \subseteq U \cap V$.
6. $\{e\} = \bigcap_{B \epsilon \beta_e} B$

Prove that the family $\beta_u = \{Bu : u \epsilon \beta_e\}$ is a base for a T_1-topology τ on G.

Further prove that with this topology τ, $(G,\bullet,\tau)$ is a topological group.

Proof:-

Let β_e be a family of subsets of G such that conditions (i) ⟶ (iv) holds.

Let τ be the family of all subsets w of G. satisfying the condition.

For each $x \epsilon W$, there is $U \epsilon \beta_e$ $s.t.$ $Ux \subseteq W$

(i) $\varphi, G \epsilon \boldsymbol{\tau}$

(ii) Let γ be the collection of open sets $i.e.$ $\gamma = \{\omega_i : \omega_i \epsilon \tau\}$. Then

$$x \epsilon \bigcup_{i \epsilon \nabla} \omega_i \quad \Rightarrow \quad x \epsilon \omega_i \quad \text{for some } i$$

So, there is $U \epsilon \beta_e$ such that $Ux \subseteq \omega_i$ $\because$ $\omega_i \epsilon \tau$

⇨ $Ux \subseteq \bigcup_{i \epsilon \nabla} \omega_i$

⇨ Arbitrary union of open sets is open.

(iii) Let $\omega_1, \omega_2 \epsilon \tau$ and put $\omega = \omega_1 \cap \omega_2$

We have to prove $\omega \epsilon \tau$.

Take any $x \epsilon \omega$. There exit $U_1 \epsilon \beta_e$ & $U_2 \epsilon \beta_e$ such that $U_1 x \subseteq \omega_1$ and $U_2 x \subseteq \omega_2$.

From (v) it follows that, there is $U \epsilon \beta_e$ such that $U \subseteq U_1 \cap U_2$

Then $Ux \subseteq U_1 x \cap U_2 x \subseteq \omega_1 \cap \omega_2 = \omega$

⇨ $Ux \subseteq \omega_1 \cap \omega_2 = \omega$

Hence $\omega \epsilon \tau$ and τ is a topology on G.

Now, Let $x \epsilon G$ and $U \epsilon \beta_e$

Take any $y \epsilon Ux$, $then$ $yx^{-1} \epsilon U$

By (iii) there is an element $V \epsilon \beta_e$ such that $Vyx^{-1} \subseteq U$ ⇨ $Vy \subseteq Ux$

Hence $Ux \epsilon \tau$

So for each $x \epsilon G$ and $U \epsilon \beta_e$, $Ux \epsilon \tau$.

Property (iii) and above line implies the family $\beta_u = \{Ua : a \epsilon G, U \epsilon \beta_e\}$ is a base for the topology τ.

Hence $\tau \epsilon \tau_{\beta_e}$

Now, we are to show that the multiplication in G is jointly continuous with respect to the topology τ.

Let a & b be arbitrary elements of G, and O be any element of τ such that $ab \epsilon O$, then there exit $\omega \epsilon \beta_e$ such that $\omega ab \subseteq O$

It suffices to find $U \epsilon \beta_e$ and $V \epsilon \beta_e$ such that $UaVb \subseteq \omega ab$

Or equivalently $UaV \subseteq \omega a$ or $U(aVa^{-1}) \subseteq \omega$

Now we are going to choose U and V

By (i) choose $U \epsilon \beta_e$ $s.t.$ $U^2 \subseteq \omega$

By (iv) choose $V \epsilon \beta_e$ $s.t.$ $aVa^{-1} \subseteq U$

Then we have

$$U(aVa^{-1}) \subseteq UU \subseteq U^2 \subseteq \omega$$

⇨ $U(aVa^{-1}) \subseteq U^2 \subseteq \omega$

⇨ $U(aVa^{-1}) \subseteq \omega$ ⇨ $UaVb \subseteq \omega ab$

Thus the multiplication in G is continuous with respect to the topology τ.

In particular all right translations of G are continuous and the space $G, \tau)$ is homogeneous.

Let $b \epsilon G$ and $V \epsilon \beta_e$

To show $bV \epsilon \tau$

Take any $y \epsilon bV$ ⇨ $b^{-1}y \epsilon V$

By (iii) there is an element $\omega \epsilon \beta_e$ $s.t.$ $wb^{-1}y \subseteq V$

Also by (iv) there is $U \epsilon \beta_e$ $s.t.$ $b^{-1}Ub \subseteq \omega$

Therefore $b^{-1}Ubb^{-1}y \subseteq \omega b^{-1}y \subseteq V$ ⇨ $b^{-1}Ub \subseteq V$

⇨ $Uy \subseteq bV$ ⇨ $bV \epsilon \tau$

Now, we are to show that $i: G \longrightarrow G$ defined by $i(x) = x^{-1}$ $\forall\, x \epsilon G$ is continuous.

To show this we show that $i^{-1}: G \longrightarrow G$ is an open mapping.

Let $Ua \epsilon \tau$ ⇨ $i^{-1}(Ua) = a^{-1}U^{-1}$

Since for all $b \in G$ and $V \in \beta_e$, $bV \in \tau$

So it suffices to verify that $U^{-1} \in \tau$.

Take arbitrary $x \in U^{-1}$ then $x^{-1} \in U$

By (iii) $Vx^{-1} \subseteq U$ for some $V \in \beta_e$

By applying (ii) choose $\omega \in \beta_e$ s.t. $w^{-1} \subseteq V$

Then $w^{-1}x^{-1} \subseteq Ux^{-1} \subseteq U$ ⇨ $x\omega = (\omega^{-1}x^{-1})^{-1} \subseteq U^{-1}$

And $x\omega$ is an open neighborhood of x in (G,τ) so $U^{-1} \in \tau$

⇨ $a^{-1}U^{-1} \in \tau$

So $(G,\bullet,\tau)$ is a topological group.

Also by (iv) and homogeneity of G imply that the topology τ satisfies the T_1-separation axiom.

This completes the proof.

Theorem:-

Prove that every subgroup H of a right (left) topological group is closed as well.

Proof:-

Let H be an open subgroup of right topological group $(G,\bullet,\tau)$.

Since right translation $r_a: G \longrightarrow G$ is homeomorphism, therefore for each $a \in G$; $r_a(H) = Ha$ is open in G. (Alt)

Let $\gamma = \{Ha : a \in G\}$ then γ forms the partition of G.

⇨ Every element of γ is closed set in G and hence $H = Ha$ is closed in G.

(Alternate) $\bigcup_{a\in G-\{e\}} Ha$ is open and hence,

$\{\bigcup_{a\in G-\{e\}} Ha\}' = H$ *is closed.*

Homogeneous Topological Space:-

A topological space (X,τ) is said to be homogeneous if for each $x \epsilon X$ and for each $y \epsilon X$, there exit a homeomorphism $f: X \longrightarrow X$ $s.t.$ $f(x) = y$.

Theorem:-

Prove that every right topological group $(G,\bullet,\tau)$ is a homogeneous space.

Proof:-

Let $x, y \epsilon G$

Put $z = x^{-1}y \epsilon G$. Then right topological group has the property, there exit a homeomorphism $R_Z: G \longrightarrow G$ and $R_Z(x) = xz$

⇨ $R_Z(x) = x(x^{-1}y) = (xx^{-1})y = ey = y$

⇨ This is a homogeneous space.

Theorem:-

Let $(G,\bullet,\tau)$ be a topological group and H be a subgroup of G, Prove that $Cl(H)$ is a subgroup of G.

Proof:-

Let $g, h \epsilon \overline{H}$

We need to show, (i) $gh \epsilon \overline{H}$ (ii) $h^{-1} \epsilon \overline{H}$

(i) Let U be an open neighborhood of gh.

Let $m: G \longrightarrow G$ be the multiplication mapping which is continuous.

So, $m^{-1}(U)$ is open $G \times G$, and $(g, h) \epsilon\, m^{-1}(U)$

This implies, there exit open set V_1 containing g and V_2 containing h such that, $V_1 \times V_2 \subseteq m^{-1}(U)$

Now, since $g \epsilon V_1$ & $g \epsilon \overline{H}$, therefore $V_1 \cap H \neq \varphi$.

Say $x \in V_1 \cap H$ ⇨ $x \in H$

And $h \in V_2$ & $h \in \overline{H}$, therefore $V_2 \cap H \neq \varphi$

Say $y \in V_2 \cap H$ ⇨ $y \in H$

⇨ $x, y \in H$, Since H is a subgroup therefore $xy \in H$

And since $(x, y) \in m^{-1}(U)$ ⇨ $m(x, y) = xy \in U$

⇨ $U \cap H \neq \varphi$

This proves that $gh \in \overline{H}$

(ii) Now for $h \in \overline{H}$, we show that $h^{-1} \in \overline{H}$

Since inverse mapping: $G \longrightarrow G$ is continuous ($\because G$ is topological group)

Let $h^{-1} \in W$; where W an open neighborhood of is h^{-1}.

Then $i(W) = W^{-1}$ is open subset containing h.

Since $h \in \overline{H}$, therefore $W^{-1} \cap H \neq \varphi$.

Let $z \in W^{-1} \cap H$ ⇨ $z^{-1} \in W \cap H \neq \varphi$ ⇨ $h^{-1} \in \overline{H}$

By (i) and (ii) we conclude that $\overline{H}$ is a subgroup of G.

Theorem:-

If $(G, \bullet, \tau)$ be a topological group and K_1 & K_2 are compact subsets of G then prove that K_1K_2 is compact.

Proof:-

Let $(G, \bullet, \tau)$ be a topological group and K_1 & K_2 are compact subsets of G.

Then $K_1 \times K_2$ is compact subset of $G \times G$. Since $m: G \times G \longrightarrow G$ is continuous and we know that continuous image of a compact set is compact.

Therefore $m(K_1 \times K_2) = K_1K_2$ is compact.

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Connected Spaces

Definition:-

Let (X,τ) be a topological space. A separation of X is a pair U,V of disjoint non-empty open subsets of X whose union is X.

Connected Set:-

A set is connected if it is all in one piece. $i.e.$ it does not comprise on two or more than two separated pieces.

Connected Space:-

A topological space (X,τ) is said to be connected if it cannot be expressed as union of two non empty disjoint open sets.

$i.e.$ there does not exit open sets U,V such that $U \neq \varphi$, $V \neq \varphi$ and $U \cup V = X$, $U \cap V = \varphi$.

Note:-

A space which is not connected is called disconnected.

Note:-

(i) On the real line an interval is one piece, and therefore it is connected.

(ii) The set $(0,1) \cup (2,3)$ consists of two separated pieces and is disconnected.

(iii) Each point on the real line is in one piece $\{x\}$, therefore each point set is connected subset of $\mathbb{R}$.

(iv) $(0,1) - \left\{\frac{1}{2}\right\}$ is disconnected.

(v) Every indiscrete space $\tau = \{\varphi, X\}$ is connected.

(vi) Every discrete space $\tau = P(X)$ is disconnected.

Characterization of connected spaces:-

In a topological space (X,τ) the following are equivalent.

(i) X is connected.

(ii) The only open and closed subsets of X are φ & X.

(iii) There does not exit a continuous mapping $f: X \longrightarrow \{0,1\}$, where topology on $\{0,1\}$ is discrete.

Proof:-

(i)⟶(ii)

Let X is connected.

We prove that φ *and* X are the only clopen sets.

Assume contrary, $i.e.$ there exit a non-empty set A which is open as well closed.

⇨ $A \neq \varphi$, A is open.

Again A is closed gives that $(X - A)$ is non-empty open set such that,

$$A \cup (X - A) = X, \quad A \cap (X - A) = \varphi$$

⇨ X is disconnected.

A contradiction. Hence the only open and closed sets are φ *and* X.

(ii)⟶(iii)

Let φ & X are the only open and closed subsets of X.

We prove that there is no continuous function $f: X \longrightarrow \{0,1\}$ with $\tau = \{\varphi, \{0\}, \{1\}, \{0,1\}\}$.

Assume contrary, $i.e.$ there is a continuous function $g: X \longrightarrow \{0,1\}$ with topology $\tau = \{\varphi, \{0\}, \{1\}, \{0,1\}\}$.

Then $g^{-1}(\{0\}) \neq \varphi$ is open in X. And $g^{-1}(\{1\}) \neq \varphi$ is open in X.

$$g^{-1}(\{0\}) \cup g^{-1}(\{1\}) = g^{-1}(\{0\} \cup \{1\}) = g^{-1}(\{0,1\}) = X$$

Now consider, $g^{-1}(\{0\}) \cap g^{-1}(\{1\}) = g^{-1}(\{0\} \cap \{1\}) = g^{-1}(\varphi) = \varphi$

⇨ There exit other than φ & X open sets which are open as well closed.

This contradicts (ii)

⇨ Our assumption is wrong.

Therefore, no such continuous function exit.

(iii)⟶(i)

Let there be no continuous function $f: X \longrightarrow \{0,1\}$ with discrete topology.

We prove that X is connected.

Assume contrary, $i.e.$ X is disconnected. Say A, B form the disconnection of X $i.e.$ $X = A \cup B$, $A \cap B = \varphi$. And $A \neq \varphi$ open, $B \neq \varphi$ open.

Define $g: X \longrightarrow \{0,1\}$ by

$$g(x) = \begin{cases} 0 & if \quad x \notin A \\ 1 & if \quad x \in A \end{cases}$$

Then g is continuous and onto. A contradiction.

Hence X is connected.

Theorem:-

Prove that the continuous surjection image of a connected space is connected.

Proof:-

Given that, (i) (X, τ) is connected space.

⇨ X cannot be expressed as union of two non-empty disjoint open sets.

(ii) $f: X \longrightarrow Y$ is onto + continuous.

We have to prove Y is connected.

Suppose on contrary that Y is disconnected $i.e.$ $Y = A \cup B$;

$$A \neq \varphi \quad , \quad B \neq \varphi \qquad And \quad A \cap B = \varphi$$

A & B are open in Y.

Since f is continuous so $f^{-1}(A)\ is\ open \neq \varphi$ & $f^{-1}(B)\ is\ open \neq \varphi$

$f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X$ and

$f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\varphi) = \varphi$

⇨ $f^{-1}(A)$ & $f^{-1}(B)$ forms separation of X.

⇨ X is disconnected. A contradiction.

So our supposition is wrong.

⇨ Y is connected.

Disconnected Subspace:-

A subspace Y of a space (X, τ) is disconnected if there eit open sets G, H of X such that,

$Y \cap G \neq \varphi$, $Y \cap H \neq \varphi$ And

$$(Y \cap G) \cup (Y \cap H) = Y \quad \& \quad (Y \cap G) \cap (Y \cap H) = \varphi$$

Exercise:-

Prove that the continuous image of a connected subset is connected.

Proof:-

Let X be a connected space and $f: X \longrightarrow Y$ be a continuous function.

To prove $f(X)$ is connected.

Suppose on the contrary that $f(X)$ is disconnected. Then there exit two open sets (non-empty) $A\ \&\ B\ in\ f(X)$ such that $A \cup B = f(X)$ & $A \cap B = \varphi$

Now as A & B are open in $f(X)$ and f is continuous function.

So $f^{-1}(A)$ & $f^{-1}(B)$ are open in X.

Further $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}\big(f(X)\big) = X$.

And $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\varphi) = \varphi$

⇨ X is disconnected.

Which is a contradiction? Because X is connected.

So our supposition is wrong.

Hence $f(X)$ is connected.

Theorem:-

Prove that a topological space (X, τ) is connected if and only if no-proper non-empty subset of X is both open and closed.

Proof:-

Suppose $A \neq \varphi$ is a proper subset of X which is not both open and closed.

We prove that X is connected.

Suppose on the contrary that X is disconnected.

As X is disconnected then there exit two open sets A and B such that

$$A \cup B = X \quad \& \quad A \cap B = \varphi$$

Now as A is open and also as $A \cup B = X$ & $A \cap B = \varphi$ So by law of compliments $B = A'$

Also B is open implies A' is open. ⇨ A is closed.

⇨ A is both open and closed.

A contradiction, So our supposition is wrong.

Hence X is connected.

Conversely,

Let X is connected.

We have to prove that no-proper subset (non-empty) of X is both open and closed.

Suppose on contrary that a non-empty subset A of X is both open and closed.

Let $B = A'$. As A is closed ⇨ A' is open ⇨ B is open.

⇨ A & B are open in X with,

$$A \cup B = A \cup A' = X \quad And \quad A \cap B = A \cap A' = \varphi$$

⇨ X is disconnected.

A contradiction. So our supposition is wrong.

So there does not exit a proper non-empty subset of X which is both open and closed.

Component of a Topological Space:-

Let (X, τ) be topological space. A maximal connected subset of X is called a component of X.

- A component of X is a connected subset of X which is not properly contained in any bigger connected subset of X.

Note:-

If X itself is connected, then the only component of X is X itself.

Totally Disconnected Space:-

A topological space (X, τ) is said to be totally disconnected if and only if all non-empty subsets (which are not one point subsets) are disconnected.

Example:-

Every discrete space is totally disconnected.

Theorem:-

A totally disconnected space is housdorff.

Proof:-

Let (X, τ) be a totally disconnected space.

Let $x, y \in X$ s.t. $x \neq y$, Then by definition there is a disconnection A, B of X s.t. $x \in A$, $y \in B$ and A, B are open in X.

⇨ X is housdorff.

Example:-

$$\text{Let } X = \mathbb{R}^2 \;\; \& \;\; Y = \{(x,y): y = 0\} \cup \left\{(x,y): x > 0 \text{ and } y = \frac{1}{x}\right\}$$

Then Y is disconnected because,

$$A = \{(x,y): y = 0\}, \quad B = \left\{(x,y): x > 0 \text{ and } y = \frac{1}{x}\right\}$$

Then $A \cap B = \varphi$, $A \cup B = Y$.

Theorem:-

Let (X, τ) be topological space. If the sets C & D forms a separation of X and if Y is connected space of X, then Y lies entirely within either C or D.

Proof:-

Since C and D are both open in X, the set $C \cap Y$ and $D \cap Y$ are open in Y.

These two sets are disjoint and there union is Y.

If they both are non-empty, then they will form a separation of Y.

But Y is connected, therefore either $C \cap Y = \varphi$ or $D \cap Y = \varphi$

If $C \cap Y = \varphi$ then $Y \subseteq D$ else $Y \subseteq C$. (Proved)

Theorem:-

Prove that the union of a collection of connected subspaces of a topological space (X, τ) that have a point in common is connected.

Proof:-

Let $\{A_\alpha : \alpha \in \nabla\}$ be a collection of connected subspaces of a topological space (X, τ) .

Let $p \in \bigcap_{\alpha \in \nabla} A_\alpha$: We prove that $Y = \bigcup_{\alpha \in \nabla} A_\alpha$ is connected.

Assume contrary, $i.e. Y$ is disconnected.

Let $Y = C \cup D$, where C, D non empty open subsets of are X $s.t.$ $C \cap D = \varphi$

The point $p \in C$ or $p \in D$.

Suppose $p \in C$, Since A_α is connected. It must lie entirely in C $i.e.$

$$A_\alpha \subseteq C \ \forall \alpha \in \nabla \quad \Rightarrow \quad Y = \bigcup_{\alpha \in \nabla} A_\alpha \subseteq C$$

Contradicting the fact that D is non-empty.

Hence Y is connected.

Theorem:-

Let A be a connected sub space of a topological space (X, τ). If $A \subseteq B \subseteq Cl(A)$ then B is also connected sub space of X.

Proof:-

We prove that B is connected.

Assume contrary that is, let B is disconnected.

Hence $B = C \cup D$, where C, D are non empty open sub sets of B and $C \cap D = \varphi$.

Since $A \subseteq B = C \cup D$ and A is connected. Therefore, A must lie entirely in C or D.

Suppose that $A \subseteq C$, then $\bar{A} \subseteq \bar{C}$. Since $\overline{C}$ and D are disjoint.

⇨ B cannot intersect D. This contradicts the fact that $D = \varphi$.

Hence the proof.

Theorem:-

Prove that a finite cartesian product of connected spaces is connected.

Proof:-

We prove that the Cartesian product of two connected spaces is connected.

Let $a \times b \in X \times Y$. Then horizontal slice $X \times b$ is connected being homeomorphic with connected space X.

The vertical slice $x \times Y$ is connected being the homeomorphic with the connected space Y.

This implies, $T_x = (X \times b) \cup (x \times Y)$ is connected.

Because union of connected spaces is connected if they have a point in common. And in this case $x \times b$ is a common point.

Now, $X \times Y = \bigcup_{x \in X} T_x$ is connected.

Theorem:-

Prove that the real line $\mathbb{R}$ with the usual topology is connected.

Proof:-

Assume contrary that is real line $\mathbb{R}$ is disconnected.

Let U, V are non-empty open sets such that $U \cup V = \mathbb{R}$ & $U \cap V = \varphi$.

$i.e.$ (U, V forms a disconnection of $\mathbb{R}$).

Let $x \in U$ & $y \in V$. Assume $x < y$.

Let $G = U \cap [x, y]$ & $H = V \cap [x, y]$

Then $G \cup H = \{U \cap [x, y]\} \cup \{V \cap [x, y]\} = (U \cup V) \cap [x, y]$

$$= \mathbb{R} \cap [x, y] = [x, y]$$

Now G is bounded above by $'y'$ and by the least upper bound property of $\mathbb{R}$, G has least upper bound say $c \in \mathbb{R}$.

Then, $x \leq c \leq y$. We derive a contradiction by showing that $c \notin H$.

Assume that $c \in H$.

Since $x \notin H$ and H is open in $[x, y]$. It implies that there exit $d \in \mathbb{R}$ such that $x < d < c$ and $(d, c] \subseteq H$.

⇨ d is an upper bound of G. And $d < c = l_{ub}(G)$.

A contradiction and thus $c \notin H$.

Similarly we can show that $c \notin G$.

But $c \in [x, y]$ & $G \cup H = [x, y]$

With the final contradiction it follows that $\mathbb{R}$ with the usual topology is connected topological space.

Assignment No. 2:-

Prove that a sub space $X\ of\ \mathbb{R}$ is connected if and only if X is an interval.

Proof:-

Suppose X is connected.

To prove X is an interval.

Assume on contrary that is X is not an interval, then there exit x, y, z such that $x < y < z\ \ and\ \ x, z \in X$ & $y \notin X$.

Now, $]-\infty, y[\ \ and\ \ ]y, \infty[$ are open in $\mathbb{R}$.

⇨ $]-\infty, y[\, \cap X$ & $]y, \infty[\, \cap X$ are open in X with,

$$(\,]-\infty, y[\, \cap X) \cup (\,]y, \infty[\, \cap X) = (\,]-\infty, y[\, \cup\,]y, \infty[\,) \cap X$$
$$= (\mathbb{R} - \{y\}) \cap X = X$$

And $(\,]-\infty, y[\, \cap X) \cap (\,]y, \infty[\, \cap X) = (\,]-\infty, y[\, \cap\,]y, \infty[\,) \cap X$

$$= \varphi \cap X = \varphi$$

⇨ X is disconnected. Which is a contradiction.

So our supposition is wrong. Hence X is an interval.

Conversely,

Suppose X is an interval.

To prove X is an connected.

On the contrary suppose X is disconnected. Then there exit two non-empty open disjoint subsets A and B of X such that $A \cup B = X$ & $A \cap B = \varphi$.

Let $a \in A$ & $b \in B$.

As $A \cap B = \varphi$ ⇨ $a \neq b$.

Let $a < b$. Put $y = Sup([a, b] \cap A)$.

Then by definition of supremum for every $\varepsilon > 0$, there is some point a' in A such that $y - \varepsilon < a'$ ⇨ $y - a' < \varepsilon$ ⇨ $d(y, a') < \varepsilon$

⇨ $a' \in B(y, \varepsilon)$

So every open ball with centre at y contains a point of A different from y.

⇨ y is limit point of A.

As A is also closed so $y \in A$, Similarly $y \in B$.

⇨ $A \cap B \neq \varphi$

Which is a contradiction. so our supposition is wrong.

Hence X is connected.

Path:-

Let (X, τ) be a topological space. Let $x, y \in X$. A path in X from x *to* y is a continuous map $f: [0,1] \longrightarrow X$ such that $f(0) = x$ & $f(1) = y$.

Path Connected Space:-

A topological space (X, τ) is said to be path connected if every pair of points of X can be joined by a path in X.

A sub set 'A' of a topological space X is path connected in X if A is path connected in the subspace topology.

Note:-

Every path connected space is connected. Converse is not true in general, that is a connected space may not be path connected.

Example:-

The unit ball $(B^n \ in \ \mathbb{R}^n)$ is path connected.

Define the unit ball $B^n \ in \ \mathbb{R}^n = X$ by the equation $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}$

Let $x, y \in B^n$. Define a straight line path $f: [0,1] \longrightarrow B^n$ such that

$$f(t) = (1-t)x + ty \ ; \quad 0 \leq t \leq 1$$

⇨ $f(0) = x$, $f(1) = y$ lies in B^n ⇨ $|f(t)| \leq 1$

Note:-

Every open ball $B_d(x, \varepsilon)$ and every closed ball in $\mathbb{R}^n$ are path connected.

Example:-

Punctured Euclidean plane $\mathbb{R}^n - \{0\}$, $for \ n > 1$ is path connected, where '0' is the origin in $\mathbb{R}^n$. If $n > 1$, this space is path connected.

Given x & y different from 0, we can join x & y by a straight line path between them if that path does not pass through the origin. Otherwise we can choose a point z not on the line joining $x \ and \ y$ and take the path from $x \ to \ z$ and then from $z \ to \ y$.

Theorem:-

If (X, τ) is path connected and $f: X \longrightarrow Y$ is a continuous function, then $f(x)$ is path connected.

Proof:-

Let $f(x),\ f(y) \epsilon f(X)$ ⇨ $x, y \epsilon X$ and X is path connected.

This gives that there exit a path $\lambda: [0,1] \longrightarrow X$ such that $\lambda(0) = x,\ \lambda(1) = y$

Since composition of two continuous functions is continuous therefore,

$fo\lambda: [0,1] \longrightarrow f(x)$ is continuous and $fo\lambda\ (0) = f(\lambda(0)) = f(x)$

And $fo\lambda\ (1) = f(\lambda(1)) = f(y)$.

This proves that $f(X)$ is path connected.

⇨ Path connectedness is a topological property.

(∵ Homeomorphic image of a path connected space is path connected).

Theorem:-

If (X, τ) is a path connected space, then prove that it is connected.

Proof:-

Let (X, τ) be a path connected space.

We prove this theorem by contradiction.

Let (X, τ) is disconnected. Then there exit non empty open sets U & V in X such that $U \cap V = \varphi$ & $U \cup V = X$.

Let $a \epsilon U$ & $b \epsilon V$ and let $f: [0,1] \longrightarrow X$ be a path from $a \longrightarrow b$.

Then $f^{-1}(U)$ & $f^{-1}(V)$ are non empty disjoint open sub sets of $[0,1]$ such that $[0,1] = f^{-1}(U) \cup f^{-1}(V)$

This gives that $[0,1]$ is a disconnected set. A contradiction.

Hence (X, τ) is connected.

Theorem:-

Let (X, τ_x) & (Y, τ_y) be path connected spaces. Prove that $(X \times Y, \tau)$ is a path connected space. Where τ is the product topology.

Proof:-

Let (x_1, y_1) & (x_2, y_2) be points of $X \times Y$.

Since (X, τ_x) is path connected, so there must exit some path $\lambda_1 : [0,1] \longrightarrow X$

Such that $\lambda_1(0) = x_1$, $\lambda_1(1) = x_2$

Again (Y, τ_y) is path connected, therefore there must exit some path $\lambda_2 : [0,1] \longrightarrow Y$ such that $\lambda_2(0) = y_1$, $\lambda_2(1) = y_2$.

Now define $(\lambda_1 * \lambda_2) : [0,1] \longrightarrow X \times Y$ by $(\lambda_1 * \lambda_2)(t) = \big(\lambda_1(t), \lambda_2(t)\big)$

Then $\lambda_1 * \lambda_2$ is continuous mapping, also

$$(\lambda_1 * \lambda_2)(0) = \big(\lambda_1(0), \lambda_2(0)\big) = (x_1, y_1) = z_1$$

$$And\ (\lambda_1 * \lambda_2)(1) = \big(\lambda_1(1), \lambda_2(1)\big) = (x_2, y_2) = z_2$$

⇨ $X \times Y$ is path connected with respect to the product topology.

Theorem:-

If $\{A_i : i \epsilon N\}$ is a collection of path connected subsets of a space (X, τ) and $\bigcap_{i\epsilon N} A_i \neq \varphi$. Then $\bigcup_{i\epsilon N} A_i$ is path connected.

Proof:-

Let $x, y \,\epsilon \bigcup_{i\epsilon N} A_i$ ⇨ $x \,\epsilon\, A_{i1}$, $y \,\epsilon\, A_{i2}$

Let $z \,\epsilon \bigcap_{i\epsilon N} A_i \neq \varphi$ ⇨ $z \,\epsilon\, A_{i1}$, $z \,\epsilon\, A_{i2}$

⇨ $x, z \,\epsilon\, A_{i1}$ path connected. Therefore there exit some path

$$\lambda_1 : [0,1] \longrightarrow A_{i1} \quad s.t. \quad \lambda_1(0) = x \ , \ \lambda_1(1) = z$$

Again $z, y \,\epsilon\, A_{i2}$ path connected.

$$⇨ \ \lambda_2 : [0,1] \longrightarrow A_{i2} \quad s.t. \quad \lambda_2(0) = z \ , \ \lambda_2(1) = y$$

Let $\lambda_1 * \lambda_2 : [0,1] \longrightarrow \bigcup_{i\epsilon N} A_i$. Define a function

$$(\lambda_1 * \lambda_2)(t) = \begin{cases} \lambda_1(2t) & ; \quad 0 \leq t \leq \dfrac{1}{2} \\ \lambda_2(2t-1) & ; \quad \dfrac{1}{2} \leq t \leq 1 \end{cases}$$

Then obviously $\lambda_1 * \lambda_2$ is continuous.

Then clearly $(\lambda_1 * \lambda_2)(0) = \lambda_1(0) = x$

& $(\lambda_1 * \lambda_2)(1) = \lambda_2(2-1) = \lambda_2(1) = y$

⇨ $\bigcup_{i\epsilon N} A_i$ is path connected.

Heridity Property:-

Prove that the sub space of a path connected space is path connected.

Proof:-

?

Locally Connected Space:-

A topological space (X,τ) is said to be locally connected at $x \epsilon X$ if for every neighborhood $U\ of\ x$, there is a connected neighborhood $V\ of\ x$ contained in U.

If X is locally connected at each of its points then it is simply called locally connected space.

Locally Path Connected Space:-

A topological space (X,τ) is said to be locally path connected at $x \epsilon X$ if for every neighborhood $U\ of\ x$, there is a path connected neighborhood $V\ of\ x$ contained in U.

If X is locally connected at each of its points then it is simply called locally path connected space.

Example:-

Each interval in the real line is both connected and locally connected.

The subspace $[-1,0) \cup (0,1]$. This subspace of $\mathbb{R}$ is not connected, but it is locally connected.

Theorem:-

Prove that a topological space (X, τ) is locally connected if and only if each component of each open set is open.

Proof:-

Let (X, τ) be a locally connected space.

Let $U \epsilon \tau$. Let C be a component of U.

Let $p \epsilon C$. Since space is locally connected, therefore there is a connected neighborhood $V\ of\ p \ \ s.t. \ \ V \subseteq U$.

If $V \nsubseteq C$, then C is the proper subset of the connected set $V \cup C$, therefore $V \subseteq C$ and hence C is open.

Conversely,

Let each component of each open set is open.

We prove that X is locally connected.

Let $p \epsilon X$ and let U be a neighborhood of p. then the component $V\ of\ U$ that contains p is a connected neighborhood of $p \ \ s.t. \ \ V \subseteq U$.

⇨ (X, τ) is locally connected.

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Compact And Locally Compact Spaces

Covering / Open Cover:-

Let (X,τ) be a topological space. A collection $\{U_\alpha : \alpha \in \nabla\}$ of subsets of X is called cover of X if $X = \bigcup_{\alpha \in \nabla} U_\alpha$.

The cover $\{U_\alpha : \alpha \in \nabla\}$ is called

(i) Open cover if for each $\alpha \in \nabla$, $U_\alpha \in \tau$.
(ii) Countable cover if it is a collection of countable subsets of X.
(iii) Finite cover if ∇ is a finite set.
(iv) If β is a base for τ and $U_\alpha \in \beta$ then the covering $\{U_\alpha : \alpha \in \nabla\}$ is called basic open cover of X.
(v) Similarly if $\mathcal{S}$ is a sub base for τ and $U_\alpha \in \mathcal{S}$, then the covering $\{U_\alpha : \alpha \in \nabla\}$ is called sub basic open cover of X.

Compact Space:-

A topological space (X,τ) is said to be compact if for every open cover $\{U_\alpha : \alpha \in \nabla\}$ of X, there exit a finite subset $\nabla_\circ$ *of* ∇ such that,

$$X = \bigcup_{\alpha \in \nabla_\circ} U_\alpha$$

Heine Boral Property:-

Heine Boral property of the set of real numbers $\mathbb{R}$ is: If X is a closed and bounded subset of $\mathbb{R}$, then any class of open sets of $\mathbb{R}$ whose union contains X has a finite sub class whose union contains X.

Finite Intersection Property:-

A collection $\mathcal{C} = \{C_\alpha : \alpha \epsilon \nabla\}$ of sub sets of (X, τ) is said to have the finite intersection property if every finite sub collection $\{C_1, C_2, \dots, C_n\}$ of $\mathcal{C}$ the intersection $C_1 \cap C_2 \cap \dots \cap C_n \neq \varphi$.

Theorem:-

Let (X, τ) be a topological space, then X is compact if and only if for every collection $\mathcal{C}$ of closed sets of X having finite intersection property, has non empty intersection.

Proof:-

Let X be a compact space and let $\mathcal{C} = \{C_\alpha : \alpha \epsilon \nabla\}$ be a collection of closed sets of X having finite intersection property.

To prove $\bigcap_{\alpha \epsilon \nabla} C_\alpha \neq \varphi$

Assume contrary, that is $\bigcap_{\alpha \epsilon \nabla} C_\alpha = \varphi$

$$\Rightarrow \left(\bigcap_{\alpha \epsilon \nabla} C_\alpha\right)' = \varphi' \quad \Rightarrow \quad \bigcup_{\alpha \epsilon \nabla} C_\alpha' = X$$

Now, $\{C_\alpha : \alpha \epsilon \nabla\}$ is the collection of closed sets, so $\{C_\alpha' : \alpha \epsilon \nabla\}$ is a collection of open sets with $\bigcup_{\alpha \epsilon \nabla} C_\alpha' = X$ $\Rightarrow$ $\{C_\alpha' : \alpha \epsilon \nabla\}$ is an open cover of X.

As X is compact, So this open cover has a finite sub cover say

$\{C_{\alpha_1}', C_{\alpha_2}', \dots, C_{\alpha_n}'\}$ that is $X = \bigcup_{i=1}^{n} C_{\alpha_i}'$

$$\Rightarrow \quad X' = \left(\bigcup_{i=1}^{n} C_{\alpha_i}'\right)' \quad \Rightarrow \quad \varphi = \bigcap_{i=1}^{n} (C_\alpha')'$$

$$\Rightarrow \quad \varphi = \bigcap_{i=1}^{n} C_{\alpha_i}$$

$\Rightarrow$ $\{C_{\alpha_1}, C_{\alpha_2}, \dots, C_{\alpha_n}\}$ is a finite sub collection of $\{C_\alpha : \alpha \epsilon \nabla\}$ with empty intersection.

$\Rightarrow$ $\{C_\alpha : \alpha \epsilon \nabla\}$ does not satisfies finite intersection property, which is a contradiction, So our supposition is wrong and hence

$$\bigcap_{\alpha \epsilon \nabla} C_\alpha \neq \varphi$$

Conversely,

Let every collection of closed sets with finite intersection property has non empty intersection.

To prove X is compact.

Let $\{U_\alpha : \alpha \in I\}$ be an open cover for X. $i.e.$ $\bigcup_{\alpha\in I} U_\alpha = X$.

$$\Rightarrow \left(\bigcup_{\alpha\in I} U_\alpha\right)' = X' \quad \Rightarrow \quad \bigcap_{\alpha\in I} U_\alpha' = \varphi$$

$\Rightarrow$ $\{U_\alpha' : \alpha \in I\}$ is a collection of closed sets with empty intersection then by given hypothesis $\{U_\alpha' : \alpha \in I\}$ does not satisfies finite intersection property. Then there exit a finite sub collection $\{U_{\alpha_1}', U_{\alpha_2}', \ldots, U_{\alpha_n}'\}$ with empty intersection.

$$\Rightarrow \bigcap_{i=1}^{n} U_{\alpha_i}' = \varphi \quad \Rightarrow \quad \left(\bigcap_{i=1}^{n} U_{\alpha_i}'\right)' = \varphi'$$

$$\Rightarrow \bigcup_{i=1}^{n} U_{\alpha_i} = X$$

$\Rightarrow$ $\{U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, \ldots, U_{\alpha_n}\}$ is a finite open sub cover for X.

$\Rightarrow$ X is compact.

Alternate,

Assume that for any finite sub set ∇_o of ∇, we have

$$X \neq \bigcup_{\alpha\in\nabla_o} U_\alpha$$

$\Rightarrow$ $\bigcap_{\alpha\in\nabla_o} U_\alpha' \neq \varphi$ $i.e.$ $\{U_\alpha' : \alpha \in \nabla_o\}$ where ∇_o is a finite subset of ∇ and U_α' is closed subset of X $\forall\, \alpha \in \nabla_o$.

By given condition $\bigcap_{\alpha\in\nabla} U_\alpha' \neq \varphi$ $\Rightarrow$ $\bigcup_{\alpha\in\nabla} U_\alpha \neq X$, A contradiction.

Hence X is compact.

Theorem:-

The continuous image of a compact space is compact/ Compactness is a topological property/ Hmeomorphic image of a compact space is compact.

Proof:-

Let X be a compact space and $f: X \longrightarrow Y$ be a continuous function.

To prove $f(X)$ is compact.

Let $\{U_\alpha : \alpha \epsilon I\}$ be an open cover for $f(X)$ $i.e.$ $f(X) = \bigcup_{\alpha \epsilon I} U_\alpha$

As U_α is open in $f(X)$ and f is continuous, so $f^{-1}(U_\alpha)$ is an open set in X.

Also U_α is open in $f(X)$ and $f(X)$ is a subspace of Y so there exit an open set V_α $in\, Y$ such that, $U_\alpha = V_\alpha \cap f(X)$

$$\Rightarrow U_\alpha \subseteq V_\alpha \quad \Rightarrow \bigcup_{\alpha \epsilon I} U_\alpha \subseteq \bigcup_{\alpha \epsilon I} V_\alpha \quad \Rightarrow f(X) \subseteq \bigcup_{\alpha \epsilon I} V_\alpha$$

$$\Rightarrow X \subseteq f^{-1}\left(\bigcup_{\alpha \epsilon I} V_\alpha\right) \quad \Rightarrow X \subseteq \bigcup_{\alpha \epsilon I} f^{-1}(V_\alpha) \subseteq X$$

$$\Rightarrow X = \bigcup_{\alpha \epsilon I} f^{-1}(V_\alpha)$$

As V_α is open in Y and f is continuous so $f^{-1}(V_\alpha)$ is open in X.

$\Rightarrow \{f^{-1}(V_\alpha : \alpha \epsilon I\}$ is an open cover for X.

As X is compact so there exit a finite sub cover say $\{f^{-1}(V_1), f^{-1}(V_2), \ldots, f^{-1}(V_n)\}$ of $\{f^{-1}(V_\alpha : \alpha \epsilon I\}$ that is

$$X = \bigcup_{i=1}^{n} f^{-1}(V_i) \quad \Rightarrow X = f^{-1}\left(\bigcup_{i=1}^{n} V_i\right) \quad \Rightarrow f(X) \subseteq \bigcup_{i=1}^{n} V_i$$

$$\Rightarrow f(X) \subseteq \left(\bigcup_{i=1}^{n} V_i\right) \cap f(X) \quad \Rightarrow f(X) \subseteq \bigcup_{i=1}^{n} (V_i \cap f(X))$$

$$\Rightarrow f(X) \subseteq \bigcup_{i=1}^{n} U_i \subseteq f(X) \quad \Rightarrow f(X) = \bigcup_{i=1}^{n} U_i$$

⇨ $f(X)$ is compact.

Theorem:-

Every closed sub space of a compact space is compact/ Closed Heridity property.

Proof:-

Let X be a compact space and Y be a closed subspace of X.

To prove Y is compact.

Let $\{U_\alpha : \alpha \in I\}$ be an open cover for Y. As U_α is an open set in Y and Y is a sub space of X so $U_\alpha = V_\alpha \cap Y$ where V_α is an open set in X.

Now,

$$U_\alpha = V_\alpha \cap Y \quad \Rightarrow \quad U_\alpha \subseteq V_\alpha \bigcup_{\alpha \in I} U_\alpha \subseteq \bigcup_{\alpha \in I} V_\alpha$$

$$\Rightarrow \quad Y \subseteq \bigcup_{\alpha \in I} V_\alpha \quad \Rightarrow \quad X = Y \cup Y' \subseteq \bigcup_{\alpha \in I} V_\alpha \cup Y' \subseteq X$$

$$\Rightarrow \quad X = \bigcup_{\alpha \in I} V_\alpha \cup Y'$$

As Y is closed in X. ⇨ Y' is an open set in X.

⇨ $\{Y', V_\alpha : \alpha \in I\}$ is an open cover for X.

As X is compact so this open cover has a finite sub cover say $\{Y', V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n}\}$. therefore $X = \bigcup_{i=1}^{n} V_{\alpha_i} \cup Y'$

Now,

$$Y \subseteq X = \bigcup_{i=1}^{n} V_{\alpha_i} \cup Y' \quad \Rightarrow \quad Y \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$$

$$\Rightarrow \quad Y \cap Y \subseteq \left(\bigcup_{i=1}^{n} V_{\alpha_i} \right) \cap Y \quad \Rightarrow \quad Y \subseteq \bigcup_{i=1}^{n} (V_{\alpha_i} \cap Y)$$

$$\Rightarrow \quad Y \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \subseteq Y \quad \subseteq \quad Y = \bigcup_{i=1}^{n} U_{\alpha_i}$$

$\Rightarrow$ $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ is a finite sub cover for Y.

$\Rightarrow$ Y is compact.

Theorem:-

Prove that every compact sub set of a Housdorff space is closed.

Proof:-

Let X be a T_2-Space and C be a compact sub set (sub-space) of X.

To prove C is closed. For this we have to prove C' is an open set.

If $C' = \varphi$ $\Rightarrow$ C' is an open set.

If $C' \neq \varphi$, Let $x \in C'$ $\Rightarrow$ $x \notin C$

As we know that in a T_2-Space any point and a disjoint compact sub space of X can be separated by open sets, then there exit two open sets say U_x & V_x such that $x \in U_x$ & $C \subseteq V_x$ *and* $U_x \cap V_x = \varphi$

$$\Rightarrow \{x\} \subseteq U_x \quad \Rightarrow \quad \bigcup_{x\in C'} \{x\} \subseteq \bigcup_{x \in C'} U_x$$

$$\Rightarrow C' \subseteq \bigcup_{x\in C'} U_x \subseteq C' \quad \Rightarrow \quad C' = \bigcup_{x\in C'} U_x$$

$\Rightarrow$ C' is an open set. $\Rightarrow$ C is closed.

Theorem:-

Let (X, τ) be a compact space and (Y, U) be a Housdorff space and $f: X \longrightarrow Y$ is a continuous function then prove that f is closed mapping.

Proof:-

To prove f is closed mapping.

As we know that the continuous image of a compact space is compact.

As f is continuous and C be a compact then by above theorem $f(C)$ is compact.

Also we know that the compact subspace of a Housdorff space is closed

⇨ $f(C)$ is closed in Y.

Also closed subset of a compact set is compact.

As C is closed sub set of X that is compact space.

Hence C is closed in X.

⇨ $f(C)$ is closed in Y.

⇨ f is closed mapping.

Hence the proof.

Theorem:-

Let (X,τ) be a compact space. Let (Y,U) be a Housdorff space and let f is continuous surjection then prove that f is a quotient mapping.

Proof:-

?

Continuous mapping may not be a quotient mapping, where as every quotient mapping is continuous.

Theorem:-

Let (X,τ) and (Y,U) be compact spaces, then prove that $X \times Y$ is compact.

Proof:-

Let $G = \{U : U \epsilon \tau_{X\times Y}\}$ be an open cover for X×Y.

For each $x \epsilon X$, G is a cover of $\{x\} \times \{Y\}$ by sets that are open in $X \times Y$.

There is a neighborhood $V_x\ of\ x$ and a finite sub collection $F_x\ of\ G$ such that

$$V_x \times Y \subseteq \bigcup_{x\epsilon F_x} U$$

Since $\{V_x : x \epsilon X\}$ cover X. Since X is compact therefore there exit a finite sub set $F\ of\ X$ such that $\{V_x : x \epsilon F\}$ cover X.

Let $\wp = \bigcup_{x\epsilon F} U_x$.

Since $\wp$ is finite union of finite collection of G. This proves that $X \times Y$ is compact.

Note:-

(i) All finite spaces are compact.
(ii) Every co-finite space is compact. (Co-finite Space:- if $u \epsilon \tau$ then u' is finite)
(iii) $\mathbb{R}$ with the usual topology is not compact.
(iv) No infinite discrete space is compact.
(v) A subset of a compact space need not to be compact. (If closed then compact).

Theorem:-

Any finite union of compact sub sets of a space X is compact.

Proof:-

We prove the theorem for only two sub sets of a space X.

Let A & B be compact sub sets of X. We prove that $A \cup B$ is compact sub set of X.

Let $\{U_\alpha : \alpha \epsilon \nabla\}$ be a cover of $A \cup B$ by sets $U_{\alpha's}$ open in X. $i.e.$

$$A \cup B = \bigcup_{\alpha \epsilon \nabla} U_\alpha$$

$$\Rightarrow \quad A \subseteq \bigcup_{\alpha \epsilon \nabla} U_\alpha \quad \& \quad B \subseteq \bigcup_{\alpha \epsilon \nabla} U_\alpha$$

Since A is a compact sub set of X, therefore there exit a finite sub set ∇_o of ∇ such that,

$$A \subseteq \bigcup_{\alpha \epsilon \nabla_o} U_\alpha - - - - - - - - - - - \quad (1)$$

Again by compactness of B, there exit a finite sub set ∇_1 of ∇ such that,

$$B \subseteq \bigcup_{\alpha \epsilon \nabla_1} U_\alpha - - - - - - - - - - - \quad (2)$$

Let $\nabla_2 = \nabla_0 \cup \nabla_1$ then ∇_2 is a finite sub set of ∇ and

$$A \cup B \subseteq \bigcup_{\alpha \epsilon \nabla_2} U_\alpha$$

$\Rightarrow$ $A \cup B$ is compact.

Example:-

Let $X = \mathbb{R}$ and $F = \{\dots, (-\infty, -2), (-\infty, -1), (-\infty, 0), (-\infty, 1), (-\infty, 2), \dots\}$. Then F has finite intersection property.

Theorem:-

The following are equivalent in a topological space (X, τ)

(i) X is compact.

(ii) Every class of closed sets with empty intersection has a finite sub class with empty intersection.

Proof:-

Let X is compact and let $\{C_\alpha : \alpha \epsilon I\}$ be a class of closed sets in X with $\bigcap_{\alpha\epsilon I} C_\alpha = \varphi$.

$$\Rightarrow \left(\bigcap_{\alpha\epsilon I} C_\alpha = \varphi\right)' = \varphi' \quad \Rightarrow \quad \bigcup_{\alpha\epsilon I} C_\alpha{}' = X$$

As C_α is closed. This implies that C_α' is open in X.

$\Rightarrow \{C_\alpha' : \alpha \epsilon I\}$ is an open cover for X. As X is compact so there exit a finite sub cover say $\{C_{\alpha_1}', C_{\alpha_2}', \dots, C_{\alpha_n}'\}$ of $\{C_\alpha' : \alpha \epsilon I\}$ that is,

$$\bigcup_{i=1}^{n} C_{\alpha_i}{}' = X \quad \Rightarrow \quad \left(\bigcup_{i=1}^{n} C_{\alpha_i}{}'\right)' = X'$$

$$\Rightarrow \bigcap_{i=1}^{n} C_{\alpha_i} = \varphi$$

Conversely,

Suppose in a topological space X each class $\{C_\alpha : \alpha \in I\}$ of closed sets with empty intersection has a finite sub class with empty intersection.

To prove X is compact.

Let $\{U_\alpha : \alpha \in I\}$ be an open cover for X $i.e.$

$$X = \bigcup_{\alpha \in I} U_\alpha \quad \Rightarrow \quad X' = \left(\bigcup_{\alpha \in I} U_\alpha\right)'$$

$$\Rightarrow \quad \varphi = \bigcap_{\alpha \in I} U_\alpha{}'$$

As U_α is open then $U_\alpha{}'$ is closed set so $\{U'_\alpha : \alpha \in I\}$ is class of closed sets with empty intersection. So by given condition there exit a finite sub class say $\{U'_{\alpha_1}, U'_{\alpha_2}, \dots, U'_{\alpha_n}\}$ with empty intersection.

$$\Rightarrow \quad \bigcap_{i=1}^{n} U_{\alpha_i}{}' = \varphi \quad \Rightarrow \quad \left(\bigcap_{i=1}^{n} U_{\alpha_i}{}'\right)' = \varphi'$$

$$\Rightarrow \quad \bigcup_{i=1}^{n} U_{\alpha_i} = X$$

⇨ $\{U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, \dots, U_{\alpha_n}\}$ is a finite open sub cover for X.

⇨ X is compact.

Lindelof Space:-

A topological space (X, τ) is called a Lindelof space if every open cover of X contains a countable sub covering of X.

Note:-

Every compact space is Lindelof space, where as a lindelof space may not be a compact space.

Locally Compact Space:-

A topological space (X,τ) is said to be locally compact space at $p \epsilon X$ if there exit an open set U and a compact set K of X such that $p \epsilon U \subseteq K$.

The space (X,τ) is said to be locally compact if it is locally compact at each point of X.

Note:-

Ever compact space is locally compact, where as a locally compact space may not be compact.

Example:-

$\mathbb{R}$ with the usual topology is locally compact but not compact.

Theorem:-

Let (X,τ) be a housdorff space and $p \epsilon X$. Then X is locally compact at p if and only if there is a neighborhood $U \, of \, p$ such that $\overline{U}$ is compact.

Proof:-

Let X be locally compact at $p \epsilon X$.

Then there exit a compact set K and an open set U containing U such that $p \epsilon U \subseteq K$.

Since K is compact sub set of the housdorff space X, therefore K is closed $i.e.\ K = \overline{K}$.

Now, $U \subseteq K \quad \Rightarrow \quad \overline{U} \subseteq \overline{K} = K$

Since $\overline{U}$ is a closed sub set of a compact set K, therefore $\overline{U}$ itself is compact.

Conversely,

Since $p \epsilon U \subseteq \overline{U}$, where $\overline{U}$ is compact.

Therefore X is locally compact at p.

Theorem:-

Every closed sub space of a locally compact housdorff space is locally compact.

Proof:-

?

Theorem:-

Let (X, τ) be a topological space then X is locally compact Housdorff if and only if there exit a space Y satisfying the following conditions,

1) X is sub space of Y.
2) The set $(Y - X)$ consists of a single point.
3) Y is compact housdorff space.

If Y & $Y^{'}$ are two spaces satisfying these conditions then there is a homeomorphism of Y & $Y^{'}$ that equals the identity map on X.

Proof:-

Let X be a locally compact housdorff space.

Step I:- First we prove the uniqueness of Y.

Let Y & $Y^{'}$ be two spaces satisfying conditions 1 to 3. Define $h: Y \longrightarrow Y^{'}$ by $h(p) = q, \ where \ \ p \, \epsilon \, Y - X \ \ \& \ \ q \, \epsilon \, Y^{'} - X$

And $h(x) = x \quad \forall \ x \, \epsilon \, X$.

To prove h is homeomorphism it is sufficient to prove that h is an open mapping.

Case 1:- Let U be an open sub set of Y $s.t.$ $p \notin U$, Then by definition of h, $h(U) = U$.

Now U is open in Y and $U \subseteq X$.

⇨ $U = U \cap X$ is open in X.

Now Y' is housdorff space. This implies Y' is a T_1-space so singleton sets are closed in Y. Hence $\{q\}$ is closed in Y'. Thus $Y' - \{q\} = X$ is open in Y'.

This proves that U is open in Y'.

Case 2:- Let U be an open sub set of Y $s.t.$ $p \in U$.

Since $Y - U = C$ is closed in Y.

Since Y is compact, therefore C is compact sub space of Y and $C \subseteq X$.

Therefore C is compact sub space of X.

Now X is sub space of Y' so C is compact sub space of Y'. Because Y' is housdorff, therefore C is closed sub space of Y'. And hence $Y' - C$ is open in Y'.

Now, $h(U) = h(Y - C) = h(Y) - h(C) = Y' - C$ is open in Y'.

In both cases h is homeomorphism.

Step II:- Now we construct Y.

Case 1:- Let $p \notin X$ and $Y = X \cup \{p\}$

Topologizing Y: Let open sets of Y consists of

1) All sets U open in X and
2) All sets of the form $Y - C$, where C is compact sub set of X.

Now we verify that this collection satisfy all conditions of topology on Y.

(i) φ is open in X (type 1), therefore φ is open in Y.

$Y = Y - \varphi$, therefore Y is open in Y.

(ii) Intersection:- Let U_1, U_2 (type 1) open sets in Y, then $U_1 \cap U_2$ is open in X and hence open in Y.

Let $(Y - C_1) \cap (Y - C_2) = Y - (C_1 \cap C_2)$, which is open in Y.

Let $U_1 \cap (Y - C) = U_1 \cap (X - C)$, where C is closed in X and hence $X - C$ is open in X, therefore $U_1 \cap (Y - C)$ is open.

(iii) Arbitrary Union:- $\cup U_\alpha = U$ is open in Y (for U_α open of type 1)

$\cup \left(Y - C_\beta\right) = Y - \cap C_\beta = Y - C$ is open of type 2.

$\cup U_\alpha \cup \left[\cup \left(Y - C_\beta\right)\right] = U \cup (Y - C) = Y - (C - U)$ open of type 2

This shows that the collection of sub sets of type1 and type2 form topology on Y.

Step III:- Now we show that X is a sub space of Y.

First we show that $X \cap U$ is open in X for every open set U in Y.

(i) If U is open of type1 then $U = U \cap X$ is open in X.
(ii) If $(Y - C)$ is open of type2, then $Y - C \cap X = X - C$ is open in X.

Conversely, only sets open in X a set of type1. This gives that X is a sub space of Y.

Step IV:- To show Y is compact.

Let $\mathbb{A}$ be an open cover of Y, then $\mathbb{A}$ contains an open set $Y - C$ (type2) (to contain $\{p\}$)

Now, $X \cap V_\alpha$; (V_α are all open sets of $\mathbb{A}$ different from $Y - C$) is covering of C by sets open in X.

Since C is compact, so there exit a finite sub set ∇_0 such that

$$C \subseteq \bigcup_{\alpha \epsilon \nabla_0} (X \cap V_\alpha) \subseteq \bigcup_{\alpha \epsilon \nabla_0} V_\alpha$$

$$\Rightarrow \quad C \cup (Y - C) = Y = \left(\bigcup_{\alpha \epsilon \nabla_0} V_\alpha\right) \cup (Y - C)$$

$\Rightarrow$ Y is compact.

Step V:- To show Y is housdorff.

$Let\ x, y \epsilon Y$ such that $x \neq y$.

If $x, y \in X$. Since X is housdorff space, therefore there exit open sets $U, V\ in\ X$ such that $x \in U,\ \ y \in V\ \ \&\ \ U \cap V = \varphi$.

Since X is open in Y, $So\ U, V$ are open in Y.

This proves that Y is housdorff.

Now. If $x \in X\ \ \&\ \ y = p \in Y - X$

We can choose a compact set $C\ in\ X$ containing a neighborhood U of x, then $U\ \&\ Y - C$ are open sub sets of Y containing $x\ \&\ y$ respectively and

$$U \cap (Y - C) = \varphi$$

⇨ Y is housdorff.

Conversely,

Let Y be satisfying conditions 1 to 3.

We prove that X is locally compact housdorff space.

Obviously X is housdorff space being a sub space of a housdorff space.

Let $x \in X$, we show that X is locally compact at x.

Choose disjoint open sets $U\ \&\ V$ containing of Y such that

$$x \in U\ ,\quad p = y \in V$$

Then $C = Y - C$ is closed in Y, so it is compact sub space of Y.

Since $C \subseteq X$, so C is compact sub space of X.

⇨ X is locally compact at x.

Hence X is locally compact.

One Point Compactification:-

If Y is a housdorff space and X is a proper sub space of Y such that $\overline{X} = Y$, then Y is said to be compactification of X.

If $(Y - X)$ is a single point, then Y is called one point compactification of X.

Normal Space:-

A topological space (X, τ) is said to be normal if for every pair of disjoint closed sets A, B of X there ext disjoint open sets U, V such that

$$A \subseteq U \quad , \quad B \subseteq V$$

T_4-Space:-

A normal, T_1-Space is called T_4-Space.

- A regular T_1-space is called T_3-space.
- Regular space may not be T_2-space, but T_3-space is T_2-space.
- Normal space may not be a regular space, but T_4-space ⇨ T_3-space ⇨ regular space, T_2-space ⇨ T_1-space ⇨ T_0-space.

Theorem:-

A T_4-Space is a Regular space.

Proof:-

Let (X, τ) be a T_4-space, then X is a normal space as well as T_1-space.

Let F be a closed sub set of X and $x \in X - F$.

Since X is T_1-space, therefore $\{x\}$ and F are disjoint closed sets.

Sine X is normal, therefore there exit disjoint open sets U & V such that

$$\{x\} \subseteq U \quad , \quad F \subseteq V \quad ; \quad (U \cap V = \varphi)$$

$$Or \quad x \in U \quad , \quad F \subseteq V$$

⇨ (X, τ) is a regular space.

Theorem:-

A topological space (X, τ) is normal if and only if for each closed set F and open set H containing F, there exit an open set G such that

$$F \subseteq G \subseteq Cl(G) \subseteq H$$

Proof:-

Let (X, τ) is a normal space.

Let $F \subseteq H$, where F is closed and H is open. Then $X - H$ is closed and $F \cap (X - H) = \varphi$.

Since X is normal, therefore there exit disjoint open sets G & U such that

$$F \subseteq G \quad \& \quad X - H \subseteq U$$

$$G \cap H = \varphi \quad \Rightarrow \quad G \subseteq X - U$$

$$And \quad X - H \subseteq U \quad \Rightarrow \quad X - U \subseteq H$$

Thus, $F \subseteq G \subseteq X - U \subseteq H$

$$Or \quad F \subseteq G \subseteq Cl(G) \subseteq Cl(X - U) \subseteq X - U \subseteq H$$

$$\Rightarrow \quad F \subseteq G \subseteq Cl(G) \subseteq H$$

Conversely,

Let F_1 & F_2 be two disjoint closed sets, then $F_1 \subseteq X - F_2$, where $X - F_2$ is open.

By hypothesis, there exit an open set G such that $F_1 \subseteq G \subseteq Cl(G) \subseteq X - F_2$

$$\Rightarrow \quad Cl(G) \subseteq X - F_2 \subseteq F_2 \subseteq X - Cl(G) = open$$

Now, $G \cap \left(X - Cl(G)\right) = \varphi$

$\Rightarrow$ (X, τ) is normal.

Theorem:-

Prove that continuous closed surjection image of a normal space is normal (Homework).

Proof:-

Let X be a normal space and $Y = f(X)$ is its closed continuous image.

To prove Y is normal.

Let A_1 & B_1 be two disjoint closed sets of Y.

As f is continuous so inverse image of each closed set is closed.

So, $f^{-1}(A_1)$ and $f^{-1}(B_1)$ are closed sets in X.

Let $A = f^{-1}(A_1)$ and $B = f^{-1}(B_1)$

Now, $A \cap B = f^{-1}(A_1) \cap f^{-1}(B_1) = f^{-1}(A_1 \cap B_1) = f^{-1}(\varphi) = \varphi$

So A & B are disjoint closed sets in X. As X is normal so there exit two open sets U & V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \varphi$

As U and V are open sets in X so U' and V' are closed sets in X.

As f is closed function,

⇨ $f(U')$ and $f(V')$ are closed sets in Y.

Put $U_1 = [f(U')]' = Y - f(U')$ & $V_1 = [f(V')]'$

Then U_1 and V_1 are open sets in Y. Next we show that $A_1 \subseteq U_1$

Let $x \in A_1 = f(A)$ ⇨ $x \in f(A)$ ⇨ $f^{-1}(x) \in A \subseteq U$

⇨ $f^{-1}(x) \in U$ ⇨ $f^{-1}(x) \notin U'$

⇨ $x \notin f(U')$ ⇨ $x \in [f(U')]' = U_1$

⇨ $A_1 \subseteq U_1$

Similarly we can show that $B_1 \subseteq V_1$

Now, $U_1 \cap V_1 = [f(U')]' \cap [f(V')]' = [f(U') \cup f(V')]'$

$= [f(U' \cup V')]' = [f(X)]' = Y'$

⇨ $U_1 \cap V_1 = \varphi$

⇨ $Y = f(X)$ is normal.

Theorem:-

Prove that the closed sub space of a normal space is normal (Homework).

Proof:-

Let X be a normal space and Y be a closed sub space of X.

To prove Y is normal.

Let A & B be two closed sets in Y such that $A \cap B = \varphi$

As A & B are closed in Y and Y is a sub space of X. So there exit two closed sub sets A_1 and B_1 in X such that $A = A_1 \cap Y$, $B = B_1 \cap Y$

As $A \cap B = \varphi \quad \Rightarrow \quad A_1 \cap B_1 = \varphi$

As A_1 & B_1 are closed sets in X, $A_1 \cap B_1 = \varphi$ and X is normal.

So there exit two open sets V_1 & U_1 *in* X such that

$$A_1 \subseteq U_1 \quad , \quad B_1 \subseteq V_1 \quad and \quad U_1 \cap V_1 = \varphi$$

$$put \quad U = U_1 \cap Y \quad , \quad V = V_1 \cap Y$$

Then U *and* V are open sets in Y.

As $A_1 \subseteq U_1 \quad \Rightarrow \quad A_1 \cap Y \subseteq U_1 \cap Y \quad \Rightarrow \quad A \subseteq U$

Also $B_1 \subseteq V_1 \quad \Rightarrow \quad B_1 \cap Y \subseteq V_1 \cap Y \quad \Rightarrow \quad B \subseteq V$

And $U \cap V = (U_1 \cap Y) \cap (V_1 \cap Y) = (U_1 \cap V_1) \cap Y = \varphi \cap Y = \varphi$

$\Rightarrow \quad U \cap V = \varphi$

$\Rightarrow \quad Y$ is normal.

Theorem:-

Prove that every discrete space with at least two points is normal (Homework).

Proof:-

Let X be a discrete space with at least two points.

To prove X is normal.

As X is discrete, so each sub set of X is closed as well as open.

Let A & B are two closed sub sets of X. such that $A \cap B = \varphi$

$$Put \quad U = A \quad and \quad V = B$$

Then U & V are open in X because X is discrete.

Also $A \subseteq U$, $B \subseteq V$ *and* $U \cap V = \varphi$

⇨ X is normal.

Urysohn Lemma:-

Let (X, τ) be a normal space. If F_1 , F_2 are any disjoint closed sets in X, then there exit a continuous mapping $f: X \longrightarrow [0,1]$ with $f(F_1) = 0$ & $f(F_2) = 1$.

Proof:-

Step 1:- $F_1 \cap F_2 = \varphi$, where F_1 & F_2 are closed sub sets of X.

⇨ $F_1 \subseteq X - F_2$; and $X - F_2$ is an open set.

Since X is normal space, therefore there exit an open set $U_{1/2}$ such that

$$F_1 \subseteq U_{1/2} \subseteq Cl\left(U_{1/2}\right) \subseteq X - F_2$$

Step 2:- Now $U_{1/2}$ and $X - F_2$ are open sub sets containing closed sets F_1 and $Cl\left(U_{1/2}\right)$ respectively.

Therefore there exit open sets $U_{1/4}$ and $U_{3/4}$ such that

$$F_1 \subseteq U_{1/4} \subseteq Cl\left(U_{1/4}\right) \subseteq U_{1/2} \subseteq Cl\left(U_{1/2}\right) \subseteq U_{3/4} \subseteq Cl\left(U_{3/4}\right) \subseteq X - F_2$$

Step 3:- Let $d = \frac{m}{2^n}$; $n = 1,2,3, \dots$ *and* $m = 1,2, \dots, 2^n - 1$

(Numbers of the form of d are called "dyadic rational numbers")

If we continuous this process, we obtain an open set U_d such that $d_1 < d_2$

$$⇨ \ F_1 \subseteq U_{d_1} \subseteq Cl(U_{d_1}) \subseteq U_{d_2} \subseteq Cl(U_{d_2}) \subseteq X - F_2$$

Step 4:- We define $f: X \longrightarrow [0,1]$ as $f(x) = 0$ *if* $x \in U_d$

And $f(x) = Sup\{d : x \notin U_d\}$

It is clear that $f(F_1) = 0$ *and* $f(F_2) = 1$

Step 5:- We shoe that f is continuous.

Consider the family $\{[0,a),(a,1]\ ; 0<a<1\}$ which is a sub base for $[0,1]$

We simply show that $f^{-1}([0,a))$ *and* $f^{-1}((a,1])$ are open.

We note that $f(x)<a \iff x \in U_d$ *for some* $d<a$

This gives that $f^{-1}([0,a)) = \{x : f(x)<a\} = \cup\, U_d$

Again $f^{-1}((a,1]) = \{x : f(x)>a\} = \cup_{d>a}\big(X - Cl(U_d)\big)$;

Which are open sets.

This proves that f is continuous.

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Assignment Number 3

Paracompactness:-

A T_2-Space (X, τ) is pare compact if every open covering $\mathcal{A}$ of X has a locally finite open refinement β that covers X.

Refinement:-

Let X be a set and let $\mathcal{A}$ and β be covers of X, we say $\mathcal{A}$ is refinement of β or $\mathcal{A}$ refines β if for each $U \epsilon \mathcal{A}$ there exit $V \epsilon \beta$ such that $U \subseteq V$.

Locally Finite:-

Let (X, τ) be a topological space. A collection $\mathcal{A}$ of sub sets of X is said to be locally finite in X if every point of X has a neighborhood that intersects only finitely many elements of $\mathcal{A}$.

$e.g.$ $\mathcal{A} = \{(n, n+2): n \epsilon Z\}$ is locally finite in the topological space $\mathbb{R}$.

σ-Locally Finite:-

A collection β of sub sets of topological space (X, τ) is σ-locally finite if,

$$\beta = \bigcup_{n \epsilon N} B_n ,$$

Where B_n is a locally finite collection of sub sets of X.

Theorem:-

Every regular Lindelof space is Paracompact.

Proof:-

Let (X,τ) be a regular lindelof space.

To prove X is paracompact.

Let $\mathfrak{U}$ be an open cover of X. Since X is lindelof, then there exit a countable sub collection $V \, of \, \mathfrak{U}$ that covers X.

➯ V is an open σ-locally finite refinement of $\mathfrak{U}$.

Since (X,τ) is regular, by theorem it is paracompact.

Theorem:-

Let (X,τ) be a regular topological space, then following are equivalent.

(a) (X,τ) is paracompact.
(b) Every open cover of X has a open σ-locally finite refinement.
(c) Every open cover of X has locally finite refinement.
(d) Every open cover of X has a closed locally finitr refinement.

Proof:-

(a)⟶(b)

Let (X,τ) is paracompact.

To prove every cover of X has an open σ-locally finite refinement.

Let $\mathcal{A}$ be an open cover of X then there exit an open locally finite refinement β of $\mathcal{A}$.

➯ β is an open σ-locally finite refinement.

(b)⟶(c)

Let every open cover of X has an open σ-locally finite refinement.

To prove every open cover of X has a locally finite refinement .

Let $\mathfrak{U}$ be an open cover of X, then there exit an open σ-locally finite refinement $q\ of\ \mathfrak{U}$ so that $q = \bigcup_{n \epsilon N} q_n$, where q_n is locally finite.

For each $n \epsilon N$, let $\omega_n = \cup \{V: V \epsilon q_n\}$

Then $\{\omega_n : n \epsilon N\}$ is an open cover of X.

For each $n \epsilon N$ let $A_n = \omega_n - \bigcup_{i=1}^{n-1} \omega_i$. Then $\{A_n : n \epsilon N\}$ is refinement of $\{\omega_n : n \epsilon N\}$. Let $x \epsilon X$ and let n_x be the smallest number of $\{n \epsilon N, x \epsilon \omega_n\}$ then $x \epsilon A_{n_x}$ and hence $\{A_n : n \epsilon N\}$ covers X.

Also ω_{n_x} is neighborhood of x that does not intersect A_n for any $n > n_x$ and so $\{A_n : n \epsilon N\}$ is locally finite.

Let $\mathcal{A} = \{A_n \cap V : n \epsilon N\ and\ V \epsilon q_n\}$

Since q is refinement of $\mathfrak{U}$ this imply $\mathcal{A}\ refins\ \mathfrak{U}$.

Let $x \epsilon X$, Since $\{A_n : n \epsilon N\}$ is locally finite so there exit a neighborhood $M\ of\ x$ that intersects only a finite number of members of $A_{n_1}, A_{n_2}, \dots, A_{n_k}$ of $\{A_n : n \epsilon N\}$ for each $i = 1,2, \dots, k$ there exit a neighborhood n_x of x that intersects only a finite number of members of q_n. Then $M \cap \bigcap_{i=1}^{k} N x_i$ is a neighborhood of x that intersects only a finite number of members of $\mathcal{A}$, therefore $\mathcal{A}$ is locally finite.

⇨ $\mathcal{A}$ is locally finite refinement of $\mathfrak{U}$.

(c)⟶(d)

Let every open cover of X has a locally finite refinement.

To prove, every open cover of X has a closed locally finite refinement.

Let $\mathfrak{U}$ be an open cover of X.

For each $x \epsilon X$, let $U_x \epsilon \mathfrak{U}\quad s.t.\quad x \epsilon U_x$

Since (X, τ) is regular therefore there exit a neighborhood V_x of X such that $\overline{V_x} \subseteq U_x$ then $\{V_x : x \epsilon X\}$ is an open cover of X, so by assumption this open cover has a locale finite refinement $\{A_\alpha : \alpha \epsilon \nabla\}$ then $\{\overline{A_\alpha} : \alpha \epsilon \nabla\}$ is also locally finite.

For $\alpha \epsilon \nabla \;\exists\; x \epsilon X \quad s.t. \quad A_\alpha \subseteq V_x$, therefore since $\overline{V_x} \subseteq U_x$ for each $x \epsilon X$, $\overline{A_\alpha} \subseteq U_x$ thus $\{\overline{A_\alpha}: \alpha \epsilon \nabla\}$ is closed locally finite refinement of $\mathfrak{U}$.

(d)⟶(a)

Let every open cover of X has a closed locally finite refinement.

To prove (X, τ) is paracompact.

Let $\mathfrak{U}$ be an open cover of X, then there exit a closed locally finite refinement $\boldsymbol{\mathcal{A}}$ of $\mathfrak{U}$.

For each $x \epsilon X$, let V_x be a neighborhood of x that intersects only a finite number of members of $\boldsymbol{\mathcal{A}}$. Then $\{V_x: x \epsilon X\}$ is an open cover of X.

So there exit a closed locally finite refinement $\mathcal{C}$ of $\{V_x: x \epsilon X\}$.

For each $A \epsilon \boldsymbol{\mathcal{A}}$ let $A^* = X - \cup\{C \epsilon \mathcal{C}: A \cap C = \varphi\}$

Since $\mathcal{C}$ is locally finite

$$\Rightarrow \quad \overline{\cup\{C \epsilon \mathcal{C} : A \cap C = \varphi\}} = \cup\{\overline{C} \epsilon \mathcal{C}: A \cap C = \varphi\}$$

But each member of $\mathcal{C}$ is closed, so

$$\cup\{\overline{C} \epsilon \mathcal{C}: A \cap C = \varphi\} = \cup\{C \epsilon \mathcal{C}: A \cap C = \varphi\}$$

Since $\cup\{C \epsilon \mathcal{C}: A \cap C = \varphi\}$ is closed so A^* is open.

For each $A \epsilon \boldsymbol{\mathcal{A}} : A \subseteq A^* \quad \Rightarrow \quad \{A^* : A \epsilon \boldsymbol{\mathcal{A}}\}$ is cover of X.

We claim that $\{A^* : A \epsilon \boldsymbol{\mathcal{A}}\}$ is locally finite.

Let $x \epsilon X$, there exit a neighborhood $W \; of \; x$ that intersects only a finite number of members of $C_1, C_2, \dots, C_n \; of \; \mathcal{C}$.

Since $\mathcal{C}$ covers X,

$$W \subseteq \bigcup_{i=1}^{n} C_i$$

Therefore $W \cap A^* \neq \varphi$

Then there exit $k(1 \leq k \leq n) \;\; s.t. \;\; C_k \cap A^* \neq \varphi$

But $C_k \cap A^* \neq \varphi \quad \Rightarrow \quad C_k \cap A = \varphi$

Since each C_i intersects only a finite number of members of $\boldsymbol{\mathcal{A}}$, $W \cap A^* = \varphi$ for all, but a finite number of members of $\{A^* : A \in \boldsymbol{\mathcal{A}}\}$.

Therefore $\{A^* : A \in \boldsymbol{\mathcal{A}}\}$ is locally finite.

Now for each $A \in \boldsymbol{\mathcal{A}}$, choose $U_A \in \mathfrak{U}$ $s.t.$ $A \subseteq U_A$ then $\{A^* \cap U_A : A \in \boldsymbol{\mathcal{A}}\}$ is an open locally finite refinement of $\mathfrak{U}$.

This implies X is paracompact.

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Topological Manifolds

Second Countable Space:-

A topological space (X, τ) is said to be 2nd countable provided there are countable basis for τ.

Note:-

Every 2nd countable space is 1st countable. A space which is 1st countable may not be 2nd countable.

- Every countable infinite space is 1st countable. $\beta_x = \{\{x\}\}$

First Countable Space:-

A topological space (X, τ) is said to be 1st countable if it has a countable base at each of its point.

Manifolds:-

A topological n-dimensional manifolds or manifolds is a 2nd countable T_2 -space in which each point has a neighborhood $i.e.$ homeomorphic to the open disc $U^n = \{x \epsilon R^n : |x| < 1\}$

- A 1-manifold is called a curve.
- A 2-manifold is called a surface.

Note:-

U^n can not be homeomorphic to U^m unless $n = m$.

For each $n \epsilon N$, R^n is n-manifold.

Example:-

The square $\{(x, y) \in R^2 : -1 \le x \le 1 \,\&\, -1 \le y \le 1\}$ is 2-manifolds.

Introduction To Dimension Theory

Dimension of a topological space is defined in three different ways.

1) Small inductive dimension : ind(X)
2) Large inductive dimension :Ind(X)
3) Covering dimension : dim(X)

Note:-

Three dimension function coincides in the class of separable spaces $i.e.\ ind(X) = Ind(X) = \dim(X)$.

Small Inductive Dimension Of Topological Space:-

Let (X, τ) be a regular topological space, ind(X) is called small inductive dimension of X which is an integer larger or equal to -1 or the "infinity number (∞)" ; where

$$ind(X): X \longrightarrow \{-1,0,1,2, \dots\} \cup \{\infty\}$$

Is defined as

$(mu1)$ $ind(X) = -1$ if and only if $X = \varphi$.

$(mu2)$ $ind(X) =\leq n,$ where n can be 0,1,2,… if for every point $x \epsilon X$ and Each neighborhood $V \subseteq X$ of the point $x,$ there exit an open set $U \subseteq X \quad s.t. \quad x \epsilon U \subseteq V \quad \& \quad ind\big(F_r(U)\big) \leq n-1$

$(mu3)$ $ind(X) = n$ if $ind(X) \leq n$ & $ind(X) > n-1$

$(mu4)$ $ind(X) = \infty$ if $ind(X) > n$ for $n = -1,0,1,2, \dots$

Note:-

Small inductive dimension is also called Menger Urysohn dimension.

Note:-

If the regular space (X, τ_x) and (Y, τ_y) are homomorphic, then $ind(X) = ind(Y)$. **Example:-**

$$X = \{a, b, c, d\}\ ,\ \ \tau = \{\varphi, X, \{a, b\}, \{c, d\}\} --- regular$$

Closed sets $\varphi, X, \{a, b\}, \{c, d\}$

Since (X, τ) is regular and $X \neq \varphi$ ⇨ $ind(X) \neq -1$

Let $x = a$ ⇨ $V = \{a, b\} = U$

$$F_r(U) = Cl(U) \cap Cl(X - U) = \{a. b\} \cap \{c, d\} = \varphi$$

$$\Rightarrow\ ind(F_r(U)) = -1 = 0 - 1 \quad \Rightarrow\ n = 0$$

$$\Rightarrow\ ind(X) = 0$$

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