ADVANCED TOPOLOGY-I

Prepared By:- Muhammad Tahir Wattoo (03448563284)
M.S. MATH From CIIT Islamabad

TOPOLOGY:-

Let 'X' be a non empty set and $'\tau'$ be a collection of subsets of 'X'. Then $'\tau'$ is called topology if

(i) φ and X belongs to τ.
(ii) The intersection of any two sets in $'\tau'$ belongs to τ.
(iii) The union of any number of sets in $'\tau'$ belongs to τ.

The members of τ are then called τ-open sets or simply open sets (and compliment of open sets is called a closed set). X together with τ i.e. (X,τ) is called a topological space.

The set 'X' is called its ground set and the element of 'X' is called its points.

- φ and X are always open as well as closed (clopen).
- Neighborhood of a point $x \in X$ is a set 'N' s.t. $x \in O \subseteq N$ where O is an open set.
- An open set is neighborhood of each of its points.
- Each point of a topological space has at least one neighborhood and that is X.
- A point of a topological space may have more than one neighborhood.

Example:-

Let X= {a, b, c, d}

P(X)= φ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c},

τ_1 = {φ, X, {a}, {c}, {a, c}, {b, c}, {a, b, c}}

τ_2 = {φ, X, {b}, {d}, {b, c}, {b, d} {b, c, d}}

τ_1 *and* τ_2 satisfy all the conditions of a topological space.

Interior of a Set:-

Let (X, τ) be topological space and 'A' is a non-empty subset of 'X'. A point $x\epsilon A$ is an interior point of 'A' if there exits an open neighborhood O $s.t.\ x\epsilon O \subseteq A$.

Example:-

Let X=R

τ is the collection of all possible open intervals of R and φ. Then τ is a topology on R. This topology is called usual topology on R or standard topology on R.

A= [0, 1]

X=0∈A.

Here 0ϵA but not interior point of A. 1ϵA but not interior point of A. All other points of A are interior points of A.

B= (0, 1)

Every point of B is interior point o B.

PREPAIRED BY
MUHAMMAD TAHIR
M.S. MATHEMATICS

Note:-

- Every point of an open set is an interior point of that set.
- Interior of a set is a collection of all interior points of that set and is denoted by $Int(A)$.
- A set 'A' is open if and only if $Int(A) = A$.
- $Int(A) \subseteq A$.

Limit Point of a Set:-

Let (X, τ) be a topological space and 'A' is a subset of 'X'. A point x ϵ X is called a limit point of A if every open neighborhood of 'x' contains a point of A other than x. i.e. ∀ uϵ N(x); A∩u - {x}≠φ.

- Limit point of a set may not be member of that set.
- A set is closed if it contains all of its limit points.
- Collection of all limits points of 'A' is called derived set of A and it is usually denoted by A^d

Closure of a Set:-

Let (X, τ) be a topological space and A⊆X then closure of 'A' is denoted by $Cl(A)$ and is defined by $Cl(A)=$ A∪A^d

- A is closed iff $A = Cl(A)$.
- A ⊆$Cl(A)$.

Exterior Point:-

Let (X, τ) be topological space and A⊆X. Then xϵX is said to be an exterior point of A if x is an interior point of Á. i.e. x is said to be exterior point of A if there exit some open set 'u' such that x ϵ u ⊆ Á.

OR x is exterior point of A if there exit open set u containing x such that u∩A=φ.

Boundary Point:-

Let (X, τ) be a topological space an A subset of X then xϵX is said to be boundary point of A if x is neither the interior point of A nor the interior point of Á. In other words x ϵ X is said to be boundary point of A⊆X if for every open set u containing x, u∩A ≠ φ and u∩Á ≠ φ.

Dense Set:-

Let (X, τ) be topological space and A⊆X, then A is called dense in X if Ā=X.

Example:-

Let X = {1, 2, 3, 4, 5} and τ= {φ, X, {1}, {2}, {1,2}} .

Let A= {1, 2}

Closed sets of X are {X, φ, {2,3,4,5}, {1,3,4,5}, {3,4,5}}.

Closed super set of A is X only. Therefore Ā=X.

⇨ A is dense in X.

PREPAIRED BY:-

MUHAMMAD TAHIR

M.S Mathematics

COMSATS Institute of Information technology Islamabad.

0344-8563284

SEMI OPEN SETS AND SEMI CONTINUITY IN TOPOLOGICAL SPACES

This paper was published by American Mathematical Monthly vol.70 N-I (Jan 1963): pages (36-41).

Semi-Open Sets:-

Let (X, τ) be topological space, a subset U of X is said to be semi-open in X if there exit an open set O in X such that,

$$O \subseteq U \subseteq Cl(O)$$

Example:-

X = {a, b, c, d} and τ= {φ, X, {a}, {b}, {a, b}}.

Let A={a, c}.

Here closed sets are {φ, X, {b, c, d}, {a, c, d}}.

$Cl(\{a\})$= {a, c, d} , $Cl(\{b\})$= {b, c, d} and $Cl(\{a,b\})$= X.

As 'A' is an open set and

{a} ⊆ {a, c} ⊆ {a, c, d} =$Cl(\{a\})$ => {a} ⊆ {a, c} ⊆ $Cl(\{a\})$

=> {a, c} is a semi-open set.

- Every open set is also a semi open set.
- A semi open set may not be an open set.

Equivalently:-

A sub set ‘u’ of X is semi open in X if and only if $u \subseteq Cl[Int(u)]$

Proof:-

Let ‘u’ be a semi open in X.

Put $Int(u) = O$ ------------------------ (1)

And $Int(u) \subseteq u$ (obvious) => $O \subseteq u \subseteq Cl(O)$ ------ by definition

⇨ $U \subseteq Cl[Int(u)]$ By (1)

Conversely,

Let $u \subseteq Cl[Int(u)]$

Since $Int(u) \subseteq u$ => $Int(u) \subseteq u \subseteq Cl[Int(u)]$

i.e. $v \subseteq u \subseteq Cl(v)$, where v is open in X.

⇨ u is semi open in X.

Note:-

- Collection of all semi open sets in X is denoted by SO(X).
- The compliment of a semi-open set is called a semi closed set.
- Collection of all semi closed sets in X is denoted by SC(X).

Example:-

Let X=R with the usual topology on R.

Let E=(0,1), Then Cl(E)=[0,1].

If A=[0,1) , B=(0,1] , C=[0,1], Then each A, B, and C are semi-open in X.

Note:-

C=[0,1] is a closed set which is semi-open as well. This means closed set can be semi open as well [but open sets are always semi-open]

(0,1)⊆C=[0,1]⊆Cl(0,1)=[0,1] That's why C is semi-open.

Example:-

Let X=R with usual topology and let

A=$(\frac{1}{2},1)\cup(\frac{1}{4},\frac{1}{2})\cup(\frac{1}{8},\frac{1}{4})\cup$...........$\cup(1/2^m,1/2^{m+1})\cup$......

And B=$\{0\}\cup(\frac{1}{2},1)\cup(\frac{1}{4},\frac{1}{2})\cup(\frac{1}{8},\frac{1}{4})\cup$...........$\cup(1/2^m,1/2^{m+1})\cup$..........

- ⇨ A is an open set. Since A is union of open intervals and every open interval is a open set and union of any number of open sets is a open set.

Here A=(0,1) and Cl(A)=[0,1] and B=[0,1]

- ⇨ A ⊆ B ⊆ Cl(A).
- ⇨ B is a semi-open set.

A is open so is semi open. In this case B is neither open nor closed (but is semi-open)

Example:-

Let X be the Euclidean Plane R^2 with usual topology.

Let E be the set suh that.

E=$\left\{(x,y): \begin{matrix} 1 < x < 2 \\ 1 < y < 2 \end{matrix}\right.$, Then, Cl(E)=$\left\{(x,y): \begin{matrix} 1 \le x \le 2 \\ 1 \le y \le 2 \end{matrix}\right.$

Then semi open sets are,

A= $\left\{(x,y): \begin{matrix} 1 < x \le 2 \\ 1 \le y \le 2 \end{matrix}\right.$, B= $\left\{(x,y): \begin{matrix} 1 \le x \le 2 \\ 1 < y \le 2 \end{matrix}\right.$

C= $\left\{(x,y): \begin{matrix} 1 < x \le 2 \\ 1 < y < 2 \end{matrix}\right.$, D= $\left\{(x,y): \begin{matrix} 1 \le x < 2 \\ 1 \le y \le 2 \end{matrix}\right.$

F= $\left\{(x,y): \begin{matrix} 1 \le x \le 2 \\ 1 \le y < 2 \end{matrix}\right.$, G= $\left\{(x,y): \begin{matrix} 1 < x \le 2 \\ 1 < y \le 2 \end{matrix}\right.$

And so on (so many semi open sets are available).

Theorem 2:-

Let (X, τ) be a topological space and $\{A_\alpha : \alpha \epsilon \nabla\}$ be any collection of semi-open sets in X. Then $U_{\alpha\epsilon\nabla}\, A_\alpha$ is semi open in X. (i.e. union of any number of semi-open sets is semi-open in X).

Proof:-

Since A_α is semi open in X $\forall\ \alpha\epsilon\nabla$

Therefore there exit an open set O_α in X such that.

$$O_\alpha \subseteq A_\alpha \subseteq Cl(O_\alpha)\ \forall \alpha\epsilon\nabla$$

⇨ $U_{\alpha\epsilon\nabla} O_\alpha \subseteq U_{\alpha\epsilon\nabla} A_\alpha \subseteq U_{\alpha\epsilon\nabla} Cl(O_\alpha) = Cl(U_{\alpha\epsilon\nabla} O_\alpha)$

⇨ $O \subseteq U_{\alpha\epsilon\nabla} A_\alpha \subseteq Cl(A)$ $\quad Since\ O_{\alpha\epsilon\nabla} = O\ and\ O\ is\ open\ set$

⇨ $U_{\alpha\epsilon\nabla} A_\alpha$ Is semi-open set in X.

Theorem 3:-

Let (X, τ) be a topological space and A is a semi-open subset of X. Suppose A ⊆ B ⊆ Cl(A), then prove that B is also a semi-open in X.

Proof:-

Since A is semi open in X,

Therefore there exit an open set O in X s.t.

$$O \subseteq A \subseteq Cl(O).$$

Now, $O \subseteq A \subseteq B$ -------------- (1) by supposition $A \subseteq B \subseteq Cl(A)$

Now, $A \subseteq Cl(O)$ Since $O \subseteq A \subseteq Cl(O)$

⇨ $Cl(A) \subseteq Cl[Cl(O)] = Cl(O)$

⇨ $Cl(A) \subseteq Cl(O)$ --------------------------- (2)

Again, $B \subseteq Cl(A)$ Since $A \subseteq B \subseteq Cl(A) by\ given$

⇨ $Cl(B) \subseteq Cl[Cl(A)]$

⇨ $Cl(B) \subseteq Cl(A)$ ------------------- (3)

By relation 1, 2, 3 we get

$O \subseteq A \subseteq B \subseteq Cl(B) \subseteq Cl(A) \subseteq Cl(O)$ $Since\ B \subseteq Cl(B) always\ true$

⇨ $O \subseteq B \subseteq Cl(O)$

This proves that B is a semi-open set.

Theorem 4:-

Let (X, τ) be a topological space then,

(1) $\tau \subseteq SO(X)$ (just by def.)

(2) For $A \in SO(X)$ and $A \subseteq B \subseteq Cl(A)$, then $B \in SO(X)$ (already proved)

Theorem 5:-

Let $\beta = \{B_\alpha : \alpha \epsilon \nabla\}$ be a collection of sets in X s.t.

(1) $\tau \epsilon \beta$ (2) If B ϵ β and B ⊆ D ⊆ Cl(B) then D ϵ β, Then SO(X) ⊆ β

Proof:-

Let Aϵ SO(X),

Then by definition there exit an open set O ϵ τ such that

Then by condition 1 O ϵ β

So by condition 2 A ϵ β

⇨ SO(X) ⊆ β (proved)

Statement Continued:- Furthermore SO(X) is the smallest class of sets in X

Suppose GO(X) be another class of sets satisfying (1) and (2) such that

GO(X) ⊆ SO(X) ------------------------ (3)

Let A^* ϵ SO(X) Then there exit $O^* \epsilon\ \tau$ such that

$O^* \subseteq A^* \subseteq Cl(O^*)$------------------------ (a)

Then by (2) O^* ϵ GO(X) and $O^* \subseteq A^* \subseteq Cl(O^{*)}$

⇨ A^*ϵ GO(X)

⇨ SO(X) ⊆ GO(X) ------------------------(4)

So GO(X) = SO(X) by equation (3) and (4)

Hence SO(X) is the smallest class of sets satisfying conditions 1 and 2.

Relative Topology (OR) Subspace Topology:-

Let (X, τ) be topological space and Y be a subspace of X. Then the collection $\tau_y = \{U \cap Y : U \in \tau\}$ is a topology on Y. This topology is called relative topology.

Note:-

If τ_y is a relative topology on Y then (Y, τ_y) is subspace of (X, τ_x).

Theorem 6:-

Let (X, τ) be topological space and A ⊆ Y ⊆ X, where Y is a subspace of X. Let A ϵ SO(X) then prove that A ϵ SO(Y).

Proof:-

Since A ϵ SO(X), Therefore there exit an open set O in X s.t.

$$O \subseteq A \subseteq Cl_x(O)$$

⇨ $O \cap Y \subseteq A \cap Y \subseteq Y \cap Cl_x(O)$

⇨ $O \subseteq A \subseteq Cl_y(O)$, where O is open in Y.

⇨ A is semi-open in Y.

i.e. A ϵ SO(Y)

Lemma 1:-

Let (X, τ) be a topological space and O is open in X. prove that Cl(O) – O is nowhere dense in X.

Proof:-

$E \subseteq (X, \tau)$ Is nowhere dense in X
If $Int[Cl(E)] = \varphi$

We have to prove

$$Int[Cl\{Cl(O) - O\}] = \varphi$$

Now, $Int[Cl\{Cl(O) - O\}] = Int[Cl\{Cl(O) \cap (X - O)$

$$\subseteq Int[Cl\{Cl(O) \cap Cl(X - O)$$

⇨ $Int[Cl\{Cl(O) \cap (X - O)\}] \subseteq Int[Cl(O) \cap (X - O)]$ Since X-O is closed.

$= Int[Cl(O)] \cap Int(X - O)$

$= Int[Cl(O)] \cap (X - Cl(O))$

$= \varphi$

⇨ $Int[Cl\{Cl(O) - O\}] = \varphi$

⇨ $Cl(O) - O$ is nowhere dense in X. (proved)

Theorem 7:-

Let (X, τ) be topological space and A ϵ SO(X). Then A = O ∪ B, where.

(1) O ϵ τ
(2) O ∩ B = φ and
(3) B is nowhere dense.

Proof:-

Given A is semi open in X. Then by definition there exit an open set O in the X such that. $O \subseteq A \subseteq Cl(O)$

But $A = O \cup (A - O)$

Let B = A\O, Then clearly A = O∪B, where

(1) O ϵ τ (2) O∩B = φ

The only thing we need to prove is that B is nowhere dense set.

Now, $B = A \setminus O \subseteq Cl(O) \setminus O$, Since $A \subseteq Cl(O)$

⇨ $Int[Cl(B)] \subseteq Int[Cl\{Cl(O) - O\}]$

Since O is open, therefore $Cl(O) - O$ is nowhere dense and hence,

$Int[Cl\{Cl(O) - O\}] = \varphi$

⇨ $Int[Cl(B)] \subseteq \varphi$
⇨ $Int[Cl(B) = \varphi$
⇨ B is nowhere dense in X.

Remark:-

The converse of theorem 7 is not true in general, that is, In a topological space (X, τ) a set 'A' is written as $A = O \cup B$, where O is open, B is nowhere dense and $O \cap B = \varphi$. Then A may not be semi-open.

Example:-

Let X = R with usual topology.

Let $A = \{x \in R: 0 < x < 1\} \cup \{2\}$. Then

(1) $A = O \cup B$, where $O = (0,1)\epsilon\tau$ and
(2) $B = \{2\}$
(3) $O \cap B = \varphi$

Now we show that B is nowhere dense.

Consider, $Int[Cl(B)] = Int[Cl\{2\}] = Int\{2\} = \varphi$

⇨ B is nowhere dense.

Now if we let $O = (0,1)$ $Then\ O \subseteq A\ \ But\ A \nsubseteq Cl(O)$

Hence we cannot find an open set satisfying the relation $O \subseteq A \subseteq Cl(O)$

⇨ $A \notin SO(X)$

Remark:-

The converse of theorem 7 is false

Disconnected Set: - In a Topological space (X, τ) a subset a of X is disconnected if it can be expressed as union of two non-empty disjoint open sets.

even when connectedness is imposed upon 'A'.

Example:-

Let $X = R^2$ with usual topology (open discs or open rectangles whose sides are parallel to coordinate axis form basis for τ.

Let $A = \{(x, y): 0 < x < 1, and\ 0 < y < 1\} \cup \{(x, 0): 1 \le x \le 2\}$

We note that $A = O \cup B, where\ O = \{(x, y): 0 < x < 1, and\ 0 < y < 1\} \epsilon\ \tau$

And $B = \{(x, 0): 1 \le x \le 2\}\ and\ O \cap B = \varphi$

And B is nowhere dense because $Int\{Cl[1,2]\} = \varphi$

And A is connected because it is not disconnected.

Moreover $A \notin SO(X)$ $Since\ O \subseteq A \nsubseteq Cl(O)$

Theorem 8:-

Let (X, τ) be a topological space and A = O∪B, where

① O ≠ φ is open ② A is connected and ③ $B^d = \varphi$, where B^d is derived set of B. Then prove that A ϵ SO(X)

Proof:-

$$A = O \cup B \implies O \subseteq A$$

The only thing we need to prove is that $A \subseteq Cl(O)$

OR $O \cup B \subseteq Cl(O)$ <u>OR</u> We need to show $B \subseteq Cl(O)$, Since O ⊆ Cl(O) obvious

Assume contrary, $B \nsubseteq Cl(O)$

Let $B = B_1 \cup B_2, where$

$B_1 \subseteq Cl(O) and\ B_2 \subseteq X - Cl(O) \because B \neq \varphi$

Now, $A = O \cup B = O \cup (B_1 \cup B_2)$

- ⇨ $A = (O \cup B_1) \cup B_2$

And $O \cup B_1 \neq \varphi \because O \neq \varphi$ *and* $B_2 \neq \varphi \because B_2 \nsubseteq Cl(O)$

And $O \cup B_1 \subseteq Cl(O)$ *and* $B_2 \subseteq B_2$, *a closed set*

$$B_2 \cap Cl(O) = \varphi$$

- ⇨ $O \cup B_1$ *and* B_2 *constitute a partition for* A.
- ⇨ A is disconnected.

Which is not true, so our supposition is wrong and hence

$$B \subseteq Cl(O) \Longrightarrow O \cup B \subseteq Cl(O) \Longrightarrow A \subseteq Cl(O)$$

- ⇨ $O \subseteq A \subseteq Cl(O)$
- ⇨ A ϵ SO(X) $\because$ O is open. (proved)

Remark 4:-

It is not true that the components of a semi-open set are semi open.

Example 4:-

Let X = R and $A = \{0\} \cup \left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{4}, \frac{1}{2}\right) \cup \left(\frac{1}{8}, \frac{1}{4}\right) \cup \ldots \ldots \cup \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) \cup \ldots \ldots$

Then A is semi-open and {0} is a component of A, But {0} is not semi-open in X.

$$A - \{0\} \subseteq A \subseteq Cl[A - \{0\}]$$

$A - \{0\}$ *is open set* $\because A - \{0\}$ *is union of open sets.*

- ⇨ A is semi-open $\because$ open set ⊆ A ⊆Cl(open set)

{0} is a component of A but {0} is neither open nor semi-open.

Remark 5:-

(1) In general the compliment of a semi-open set may not be semi- open.
(2) Intersection of two semi-open sets may not be semi-open.

Example:-

- Let X = R with usual topology.

We consider, A = [0, 1] ϵ SO(X) and B = [1, 2] ϵ SO(X)

- A∩B = {1} ∉SO(X)
- Let X = [0, 1]

$$A = \left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{4}, \frac{1}{2}\right) \cup \left(\frac{1}{8}, \frac{1}{4}\right) \cup \dots \dots \cup \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) \cup \dots \dots$$

⇨ A ϵ SO(X) and $\acute{A} = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots \dots\}$ ∉ SO(X)

Theorem 9:-

Let (X,τ_x) and (Y,τ_y) be topological spaces. Let f: X⟶Y be continuous and open mapping. Let A ϵ SO(X), prove that f(A) ϵ SO(Y).

Proof:-

Since A ϵ SO(X), Therefore there exit an open set O and nowhere dense set B such that $A = O \cup B : O \cap B = \varphi \; and \; B \subseteq Cl(O) - O$

$Cl(O) - O \subseteq Cl(O)$

Now, $O \subseteq A = O \cup B$

⇨ $f(O) \subseteq f(A) = f(O \cup B)$
⇨ $= f(O) \cup f(B)$
⇨ $\subseteq f(O) \cup fCl(O) \qquad \because B \subseteq Cl(O)$

⇨ $= Cl[f(O)]$

⇨ $f(O) \subseteq f(A) \subseteq Cl[f(O)]$

$\because f[Cl(O)] = Cl[f(O)]$
$\& f(O) \subseteq Cl[f(O)]$
⇨ $f(O) \cup f[Cl(O)] = Cl[f(O)]$

Since f is open, therefore f(O) is open in Y and hence $f(A) \epsilon SO(Y)$

Remark 6:-

f Must be open in theorem 9, otherwise for $A \in SO(X)$; $f(A)$ may not be semi-open in Y.

Example 5:-

Let X = Y = R with usual topology. Let $f: X \longrightarrow Y$ be defined by f(x) = 1∀ x ϵ X. Then X is semi-open in X but f(X) is not semi open in Y.

Solution:-

1:- Since f(x) = 1 ∀ x ϵ X.

Therefore f is a constant function and every constant function is continuous. Therefore f is a continuous function.

2:-Let 'u' be any open set in X, Then f (u) = {1} $\notin \tau_y$.

This gives that f is not an open function.

Now X is open and hence semi-open. But f(X) = {1}.

Since {1} contains no open set therefore {1} cannot be semi-open in Y.

Lemma 2:-

Let τ be the collection of open sets in the topological space X. Then prove that $\tau = IntSO(X)$.

Proof:-

Let O ϵ τ.

Therefore O is an open set.

⇨ $O\epsilon SO(X)$ $\because O\ is\ open$

And since $O = Int(O)$ $\because O\ is\ open$

⇨ $O\epsilon Int\ SO(X)$

⇨ $\tau \subseteq Int\ SO(X)$ -------------------------- ❶

Conversely,

Let $O\epsilon Int\ SO(X)$

Then $O = Int(A) for\ some\ A\epsilon SO(X)$

And thus, O ϵ τ $\because$ Int of any set is open.

⇨ $Int\ SO(X) \subseteq \tau$ ------------------------- ❷

From ❶ and ❷

$\tau = Int\ SO(X)$.

Theorem 10:-

Let $\tau\ and\ \tau^*$ be two topologies for X. Suppose $SO(X,\tau) \subseteq SO(X,\tau^{*)}$. $Then\ \tau \subseteq \tau^*$.

Proof:-

$SO(X,\tau) \subseteq SO(X,\tau^*)$

⇨ $Int[SO(X,\tau)] \subseteq Int[SO(X,\tau^*)]$

⇨ $\tau \subseteq \tau^*$

$\because Int[SO(X,\tau)] are\ open\ sets\ in\ \tau$

$Int[SO(X,\tau^*)] are\ open\ sets\ in\ \tau^*$

Corollary 1:-

Let τ and τ^* be two topologies for X. Suppose $SO(X,\tau) = SO(X,\tau^*)$ *then* $\tau = \tau^*$

Remark 7:-

It is interesting to note that converse of theorem 10 is false in general.

Example 6:-

$(x,y), [x,y), (x,y], [x,y] \epsilon \tau$

& $[x,y), [x,y] \epsilon \tau^*$

Let X = R

$\tau = \{(x,y): x < y\}$

& $\tau^* = \{[x,y): x < y\}$

Then $\tau \subseteq \tau^*$ But $SO(X,\tau) \nsubseteq SO(X,\tau^*)$

$\because (x,y] \epsilon SO(X,\tau)$ *but* $(x,y] \notin SO(X,\tau^*)$

Basis:-

$\forall\, x \epsilon X\ \exists\, B \in \beta$

Such that $x \in B$

B_1 *and* $B_2 \epsilon \beta, x \epsilon B_1 \cap B_2$

Then there exit B_3 *such that*

$x \epsilon B_3 \subseteq B_1 \cap B_2$

$\cup B_i = X$

- Let β and γ are two basis such that β is basis for (X,τ_x) and γ is a basis for (Y,τ_y) then $\beta \times \gamma = \{B \times C: B \epsilon \beta, C \epsilon \gamma\}$

❖ There can be construct more than one basis corresponds to each topology but there is only one topology corresponds to each basis.

Theorem 11:-

Let $(X, \tau_1) and\ (X, \tau_2)$ be topological spaces and $X = X_1 \times X_2$ be topological product. Let $A_1 \epsilon SO(X_1) and\ A_2 \epsilon SO(X_2)$. Then prove that $A_1 \times A_2 \epsilon SO(X_1 \times X_2)$

Proof:-

We have $A_i = O_i \times B_i$; $i = 1,2$

Where O_i is open in X_i ; $i = 1,2$

And B_i is nowhere dense in X_i ; $i = 1,2$

And $O_i \cap B_i = \varphi\ \forall i = 1,2$

Further,

$$B_i \subseteq Cl(O_i) - O_i \ ; \quad i = 1,2$$

Now, $A_1 \times A_2 = (O_1 \cup B_1) \times (O_2 \cup B_2)$

⇨ $= (O_1 \times O_2) \cup (O_1 \times B_2) \cup (B_1 \times O_2) \cup (B_1 \times B_2)$ --- *

⇨ $\subseteq (O_1 \times O_2) \cup [Cl(O_1) \times Cl(O_2)] \cup [Cl(O_1) \times O_2] \cup [Cl(O_1) \times Cl(O_2)]$ $\because B_1 \subseteq Cl(O_1), B_2 \subseteq Cl(O_2)$

⇨ $= Cl(O_1) \times Cl(O_2)$

⇨ $O_1 \times O_2 \subseteq A_1 \times A_2 \subseteq Cl(O_1) \times Cl(O_2) = Cl(O_1 \times O_2)$ $from\ *$

Since $O_1 \times O_2$ is open in the product space,

Therefore $A_1 \times A_2 \epsilon SO(X_1 \times X_2)$

Remark 8:-

If $A \epsilon SO(X_1 \times X_2)$ then in general we cannot write $A = A_1 \times A_2$, where $A_1 \epsilon SO(X_1) and\ A_2 \epsilon SO(X_2)$.

Example 7:-

Let $X = R^2$ with usual topology.

Let $A = \{(x,y): 0 < x < 1, 0 < y < 1\} \cup (1,1)$

Then A is semi-open in R×R. But we cannot find two sets $A_1 and\ A_2\ s.t.$
$A = A_1 \times A_2$ and $A_1 \epsilon SO(R) and\ A_2 \epsilon SO(R)$

Semi-Continuous Function:-

Let $(X, \tau_x) and\ (Y, \tau_y)$ be topological spaces and $f: X \longrightarrow Y$ be a single valued function then $'f'$ is said to be semi-continuous if and only if, for each open set V in Y, $f^{-1}(V)$ is semi-open in X.

Remark 9:-

Every continuous function is semi-continuous as well but a semi-continuous function may not be continuous.

Example 8:-

Let $X = Y = [0,1]$ with usual topology and $f: X \longrightarrow Y$ defined by,

$$f(x) = \begin{cases} 1 & if\ 0 \leq x \leq \frac{1}{2} \\ 0 & if\ \frac{1}{2} \leq x \leq 1 \end{cases}$$

This is a semi-continuous function but not a continuous function.

Let V be an open set in Y,

$$V = \begin{cases} 1\epsilon V, 0 \notin V & \Longrightarrow & f^{-1}(V) = \left[0, \frac{1}{2}\right] \epsilon SO(X) \\ 0\epsilon V, 1 \notin V & \Longrightarrow & f^{-1}(V) = \left(\frac{1}{2}, 1\right] \epsilon SO(X) \\ 0 \notin V, 1 \notin V & \Longrightarrow & f^{-1}(V) = \varphi \epsilon \tau_x \\ 0\epsilon V, 1\epsilon V & \Longrightarrow & f^{-1}(V) = [0,1] \epsilon \tau_x \end{cases}$$

Theorem 12:-

$Let\ (X, \tau_x) and\ (Y, \tau_y)$ Be topological spaces and $f: X \longrightarrow Y$ be a single valued function, then $'f'$ is semi-continuous if and only if for $f(p)\epsilon V$, there exit an $A\epsilon SO(X) s.t. p\epsilon A\ and\ f(A) \subseteq V$.

Proof:-

Let $f(p)\epsilon V \epsilon \tau_y$

⇨ There exit an $A_p \epsilon SO(X) s.t.\ p\epsilon A_p and\ f(A_p) \subseteq V$

We have to prove that f is semi-continuous.

For this we show that $f^{-1}(V)\epsilon SO(X)$

Now, $f(p)\epsilon V \quad \Longrightarrow \quad p\epsilon f^{-1}(V)$

By hypothesis there exit an $A_p \epsilon SO(X) s.t.\ p\epsilon A_p\ and\ f(A_p) \subseteq V$

⇨ $p\epsilon A_p \subseteq f^{-1}f(A_p) \subseteq f^{-1}(V) \qquad \because A \subseteq f^{-1}f(A) \& ff^{-1}(A) \subseteq A$

⇨ $p\epsilon A_p \subseteq f^{-1}(V)$

Thus $f^{-1}(V) = \cup_{p\epsilon f^{-1}(V)} A_p$

Since arbitrary union of semi-open sets is semi-open, therefore

$$f^{-1}(V) = \cup_{p\epsilon f^{-1}(V)} A_p\ is\ semi-open$$

⇨ f is semi-continuous

Conversely,

Let $f: X \longrightarrow Y$ be semi-continuous

Let $f(p) \epsilon V \epsilon \tau_y$

⇨ $p \epsilon f^{-1}(V) \epsilon SO(X)$ $\quad \because f$ *is semi continuous* & $V \epsilon \tau_y; f^{-1} \epsilon SO(X)$

Let $f^{-1}(V) = A$

$i.e.\ \ p \epsilon A \ \ and \ \ f(A) = ff^{-1}(V) \subseteq V$

⇨ $p \epsilon A \ \ and \ \ f(A) \subseteq V \ \ $ (*This completes the proof*)

Theorem 13:-

Let $(X, \tau_x) and\ (Y, \tau_y)$ be topological spaces. Let $f: X \longrightarrow Y$ be a semi-continuous function and Y be 2nd axioms space. Let P be the set of discontinuities of $'f'$ then prove that P is op 1st category.

Proof:-

> **2nd Axioms Space: -** A topological space (X, τ_x) is said to be 2nd axioms space if it has countable basis.

> **First Category: -** A set is of 1st category if it is countable union of nowhere dense sets.

Given, ❶ ***f*** is semi-continuous.

❷ (Y, τ_y) is 2nd axioms space.

❸ $P \subseteq X: P$ is set of discontinuities of f.

We have to prove that P is of 1st category.

⇨ $P = \cup_{countable}\ G_\alpha \ \ ant\ Int[Cl(G_\alpha) = \varphi$

Let p ϵ P, Let $f(p) \epsilon O_{ip} \subseteq (Y, \tau_y)$, where O_{ip} the countable union of basic open sets because (Y, τ_y) is a 2nd axioms space.

Now if O is open in X such that $p \epsilon O$,

Then $f(O) \nsubseteq O_{ip}$ because $'f'$ is discontinuous at $p \epsilon P$.

Now, since f is semi-continuous, therefore there exit

$A_{ip} \epsilon SO(X, p)\ s.t.\ p \epsilon A_{ip} \ \ and \ \ f(A_{ip}) \subseteq O_{ip}$

As A_{ip} is semi-open in X, therefore, there exit $U_{ip} \ \ and \ \ B_{ip} \ \ s.t.$

$A_{ip} = B_{ip} \cup U_{ip},\quad where\ U_{ip}$ is open in X and B_{ip} is nowhere dense in X.

Moreover, $B_{ip} \subseteq Cl(U_{ip}) - U_{ip}$

Thus, $p \epsilon B_{ip}$ a nowhere dense set. $\because p \notin open\ set\ i.e. U_{ip}$

- ⇨ $P \subseteq \cup_{p \epsilon P} B_{ip}$
- ⇨ P is of 1st category.

Remark 10:-

The converse of theorem 13 is false in general.

Example 9:-

Let $X = (0,1]\ and\ X^* = [0,1]$

Let $f: X \longrightarrow X^* = \begin{cases} 0, & x\ is\ irrationan \\ \frac{1}{q}, & x\ is\ rational = Q \end{cases}$

Where $Q = \left\{\frac{p}{q}; p, q \epsilon Z, q \neq 0, (p,q) = 1\right\}$

Then, f is continuous at irrationals and discontinuous at rational.

Hence the set of discontinuities is of 1st category [∵ the set of rational is countable set.]

Consider $u = \left(\frac{1}{2}, 1\right] \epsilon X^*$ is open as $0 \notin u$

⇨ $f^{-1}(u) = f^{-1}\left(\frac{1}{2}, 1\right]$ = sub set of rational b/w (0, 1]

And we cannot find an open set O in X such that

O ⊆ sub set of rational between (0, 1] ⊆ Cl(O)

⇨ f is not semi-continuous.

Theorem 14:-

Let $f_i : X_i \longrightarrow X_i^*$ be semi-continuous. Let $f : X_1 \times X_2 \longrightarrow X_1^* \times X_2^*$ be defined as $f : \big((x_1, x_2)\big) = \big(f_1(x_1), f_2(x_2)\big)$. Then prove that

$f : X_1 \times X_2 \longrightarrow X_1^* \times X_2^*$ is semi-continuous.

Proof:-

Given $f_1 : X_1 \longrightarrow X_1^*$ *and* $f_2 : X_2 \longrightarrow X_2^*$ are semi-continuous functions.

Let u *and* v are open sets such that $u \subseteq X_1^*$ *and* $v \subseteq X_2^*$

As f_1 *and* f_2 are semi-continuous,

Therefore, $f_1^{-1}(u) \epsilon SO(X_1)$ *and* $f_2^{-1}(v) \epsilon SO(X_2)$

$i.e.$ Inverse images of open sets are semi-open.

Now let, $u \times v \subseteq X_1^* \times X_2^*$

We have to prove $f^{-1}(u \times v) \epsilon SO(X_1 \times X_2)$

Now, $f^{-1}(u \times v) = f_1^{-1}(u) \times f_2^{-1}(v)$

$\epsilon\ SO(X_1) \times SO(X_2)$

$\epsilon\ SO(X_1 \times X_2)$

⇨ $f^{-1}(u \times v)\ \epsilon\ SO(X_1 \times X_2)$

⇨ $f : X_1 \times X_2 \longrightarrow X_1^* \times X_2^*$ is semi continuous.

Theorem 15:-

Let $h : X \longrightarrow X_1 \times X_2$ be semi continuous, where X, X_1 *and* X_2 are topological spaces. Let $f_i : X \longrightarrow X_1$ be defined as follows. For $x \epsilon X$; $h(x) = (x_1, x_2)$. Let $f_i(x) = x_i$ then $f_i : X \longrightarrow X_i$ is semi-continuous for $i = 1,2$.

Proof:-

$h : X \longrightarrow X_i$ is semi-continuous.

Let O_1 *be open in* X_1. Then $O_1 \times X_2$ *is open in* $X_1 \times X_2$

And hence, $h^{-1}(O_1 \times X_2)$ is semi-open in X.

But $f_1^{-1}(O_1) = h^{-1}(O_1 \times X_2) \in SO(X)$

⇨ f_1 is semi-continuous.

Similarly for f_2.

Remark 11:-

The converse of theorem 15 is generally false.

Example 10:-

Let $X = X_1 = X_2 = [0,1]$

$$f_1 : X \longrightarrow X_1 = \begin{cases} 1 & if\ 0 \le x \le \frac{1}{2} \\ 0 & if\ \frac{1}{2} < x \le 1 \end{cases}$$

$$f_2 : X \longrightarrow X_2 = \begin{cases} 1 & if\ 0 \le x < \frac{1}{2} \\ 0 & if\ \frac{1}{2} \le x < 1 \end{cases}$$

PREPAIRED BY
MUHAMMAD TAHIR
M.S. MATHEMATICS

Then,

$f_i : X \longrightarrow X_i$ is semi-continuous but $h(x) = [f_1(x), f_2(x)] : X \longrightarrow X_1 \times X_2$ is not semi-continuous.

Remark 12:-

Composition of two semi-continuous functions is not a semi-continuous function.

f is said to be continuous at $x = x_\circ$ if $\forall\ \varepsilon > 0\ \exists$ a $\delta > 0\ s.t.$ $| f(x) - f(x_\circ) | < \varepsilon$ whenever $| x - x_\circ) < \delta$

Example 11:-

$$Let\ X = X_1 = X_2 = [0,1]$$

$$f_1 : X \longrightarrow X_1 = \begin{cases} x & if\ \ 0 \le x \le \frac{1}{2} \\ 0 & if\ \frac{1}{2} < x \le 1 \end{cases}$$

$$f_2 : X_1 \longrightarrow X_2 = \begin{cases} 0 & if\ \ 0 \le x < \frac{1}{2} \\ 1 & if\ \frac{1}{2} \le x \le 1 \end{cases}$$

$$Now, \qquad (f_2 \circ f_1)^{-1}(x) = (f_1^{-1} \circ f_2^{-1})(x)$$

Let $u \epsilon X_2$; $0 \epsilon u\ and\ 1 \notin u \implies f_2^{-1}(u) = \left[0, \frac{1}{2}\right]$

⇨ $(f_1^{-1} \circ f_2^{-1})(u) = f_1^{-1}\{f_2^{-1}(u)] = f_1^{-1}\left(\left[0, \frac{1}{2}\right]\right) = X \quad open$

Now $0 \notin u\ and\ 1 \in u \implies f_2^{-1}(u) = \left[\frac{1}{2}, 1\right]$

⇨ $(f_1^{-1} \circ f_2^{-1})(u) = f_1^{-1}\{f_2^{-1}(u)\} = f_1^{-1}\left[\frac{1}{2}, 1\right] = \left\{\frac{1}{2}, 0\right\} \notin SO(X)$

⇨ Composition of two semi-continuous functions is not a semi-continuous.

Remark 13:-

The algebraic sum and product of semi-continuous functions are not in general semi-continuous.

Theorem 16:-

Let $f_n: M \longrightarrow M^*, where\ M\ and\ M^*$ are metric spaces with metrics $d\ and\ d^*$, be semi-continuous for $n = 1,2,3,4, \dots \dots$, and let $f_\circ: M \longrightarrow M^*$ be the uniform limit of $\{f_n\}\ then\ f_\circ: M \longrightarrow M^*$ is semi-continuous.

Proof:-

Let $O^*\ be\ open\ in\ M^*\ and\ f_\circ(x)\epsilon O^*$.

As (M^*, d^*) be metric spaces then there exit $\eta > 0\ s.t.$

$$f_\circ(x)\ \epsilon\ S^*_\eta(f_\circ(x) \subseteq\ O^*$$

As $f_\circ: M \longrightarrow M^*$is uniform limit of $\{f_n\}$, then for $\varepsilon = \eta/2$ there exit $n^*\ s.t.$

$$d^*(f^*_n(x), f_\circ(x) < \frac{\eta}{2} \quad \forall\ x \in M$$

⇨ $f^*_n(x) \in S^*_{\frac{\eta}{2}}(f_\circ(x)) \subseteq O^*$

As f^*_n is semi-continuous, then by a well known theorem there exit $A\ \epsilon\ SO(X)$ such that $x\ \epsilon\ A\ and\ f^*_n(A) \subseteq S^*_{\frac{\eta}{2}}\{f_\circ(x)\}$

Theorem will be prove if we show $f_\circ(A) \subseteq O^*$

Let $y\ \epsilon\ A$, then

$$d^*[f_\circ(y), f_\circ(x)] \leq d^*[f_\circ(y), f^*_n(y)] = d[f^*_n(y), f_\circ(x) < \frac{\eta}{2} + \frac{\eta}{2} = \eta$$

⇨ $f_\circ(A) \subseteq S^*_\eta\{f_\circ(x)\} \subseteq O^*$

⇨ $f_\circ$ is semi-continuous (proved)

PREPAIRED BY:-
MUHAMMAD TAHIR
M.S Mathematics
0344-8563284

SEMI-CONTINUOUS MAPPINGS

This course was established in 1973 by "Nota di Takashi Noire" and published by "Academia Nazionale Dei Lincei"

Introduction:-

In 1963 N-Levine defined a subset A of a topological space 'X' to be semi-open if there exit an open set u in X such that $u \subseteq A \subseteq Cl(O)$, where $Cl(u)$ denotes the closure of u. He also defined a mapping f of a topological space X into a topological space Y to be semi-continuous if for every open set V in Y, $f^{-1}(V)$ is a semi-open set in X. The purpose of present note is to give a generalization of the following two theorems and to investigate some properties of semi-open sets and semi-continuous mappings.

❖ Theorem A:-

Let X_1 *and* X_2 be topological spaces. If A_i is a semi-open set in X_i for $i = 1,2$; then $A_1 \times A_2$ is a semi-open set in the product space $X_1 \times X_2$.

❖ Theorem B:-

Let X_i *and* Y_i be topological spaces and $f_i: X_i \longrightarrow Y_i$ be semi-continuous mapping for $i = 1,2$. Then a mapping $f: X_1 \times X_2 \longrightarrow Y_1 \times Y_2$ defined by putting $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$ is semi-continuous.

Semi-Open Sets

Lemma 1:-

If U is open and A is a semi-open set, then U∩A is semi-open.

Proof:-

As A ϵ SO(X), Then there exit an open set O in X such that,

$$O \subseteq A \subseteq Cl(O)$$

⇨ $U \cap O \subseteq U \cap A \subseteq U \cap Cl(O) \subseteq Cl(U \cap O)$

Since U∩A is open in X and $U \cap O \subseteq U \cap A \subseteq \overline{(U \cap O)}$

⇨ $U \cap A \,\epsilon\, SO(X)$ $(proved)$

Theorem 1:-

Let A and $X_\circ$ be subsets of X such that $A \subseteq X_\circ$ and $X_\circ \,\epsilon\, SO(X)$, then A ϵ SO(X) if and only if $A \,\epsilon\, SO(X_\circ)$

Proof:-

As $A \subseteq X_\circ$ and $X_\circ \epsilon\, SO(X)$.

So $X_\circ$ is a subspace of X by a well known theorem.

Hence, $A \,\epsilon\, SO(X_\circ)$

So we need only to prove that $A \,\epsilon\, SO(X)$

Let $A \,\epsilon\, SO(X_\circ)$,

Then by definition there exit an open set $U_\circ$ in $X_\circ$ $s.t.$

$$U_\circ \subseteq A \subseteq Cl(U_\circ)$$

Since $U_\circ$ *in* $X_\circ$, then there exit an open set U in X such that $U_\circ = U \cap X_\circ$

⇨ $U \cap X_\circ \subseteq A \subseteq Cl(U \cap X_\circ)$

Since U is open and $X_\circ$is semi-open so $U \cap X_\circ$is semi-open in X

⇨ $A \epsilon SO(X)$ $(proved)$

Lemma 2:-

A is semi-open if and only if $Cl(A) = Cl\{Int(A)\}$

Proof:-

Suppose A is semi-open then by a well known theorem

$$A \subseteq Cl\{Int(A)\}$$

⇨ $Cl(A) \subseteq Cl\{Cl(IntA)\} = Cl\{Int(A)\}$

⇨ $Cl(A) \subseteq Cl\{Int(A)\}$ ---------------------- ①

$$As \quad Int(A) \subseteq A$$

⇨ $Cl\{Int(A)\} \subseteq Cl(A)$ ----------------------- ②

By relation ① and ② we get

$$Cl(A) = Cl\{Int(A)\}$$

Conversely,

Let, $Cl(A) = Cl\{Int(A)\}$

To prove A is semi-open.

As $Int(A) \subseteq A \subseteq Cl(A)$

⇨ $Int(A) \subseteq A \subseteq Cl\{Int(A)\}$ $\because Cl(A) = Cl\{Int(A)\}$

As $Int(A)$ is open set and $Int(A) \subseteq A \subseteq Cl\{Int(A)\}$

⇨ A is semi-open $(proved)$

Lemma 3:-

Let $\{X_\alpha : \alpha \epsilon \beta\}$ be any family of topological spaces and $\prod X_\alpha$ denotes the product space, then

① $Int \prod A_\alpha = \prod Int A_\alpha$ $if\ A_\alpha = X_\alpha$ Except for finite $\alpha \epsilon \beta$ $and \prod Int A_\alpha \neq \varphi$.

② $Cl \prod A_\alpha = \prod Cl A_\alpha$

Proof:-

❶ $As\ A_\alpha = X_\alpha$ Except for a finite α ϵ β.

So the result is obvious for all $A_\alpha = X_\alpha$

So we prove this lemma just for finite case,

As $Int(A_\alpha)$ is open in X_α $\forall\ \alpha = 1,2,3 \dots, n$

$$So \quad \prod_{\alpha=1}^{n} Int(A_\alpha) \ \ is\ open\ in \prod_{\alpha=1}^{n} X_\alpha$$

$$Also \quad \prod_{\alpha=1}^{n} Int(A_\alpha) \subseteq \prod_{\alpha=1}^{n} A_\alpha$$

⇨ $\prod_{\alpha=1}^{n} Int(A_\alpha) \subseteq Int \prod_{\alpha=1}^{n} A_\alpha$ ------------------------ ①

Now, Let $(x_1, x_2, x_3, \dots, x_n)\ \epsilon\ Int \prod_{\alpha=1}^{n} A_\alpha$

As $Int \prod_{\alpha=1}^{n} A_\alpha\ is\ open\ in\ \prod_{\alpha=1}^{n} X_\alpha$

⇨ There exit open set U_α in X_α $\forall\ \alpha = 1,2,3,\dots,n\ \ s.t.$

$$(x_1, x_2, \dots, x_n)\epsilon \prod_{\alpha=1}^{n} U_\alpha \subseteq Int \prod_{\alpha=1}^{n} A_\alpha \subseteq \prod_{\alpha=1}^{n} X_\alpha$$

Since $U_\alpha\ \epsilon\ A_\alpha\ \forall\ \alpha = 1,2,3,\dots,n$; It follows that $x_\alpha\ \epsilon\ Int(A_\alpha)\ \forall\ \alpha = 1,2,3,\dots,n$

⇨ $(x_1, x_2, x_3, \dots, x_n)\ \epsilon \prod_{\alpha=1}^{n} IntA_\alpha$

⇨ $Int \prod_{\alpha=1}^{n} A_\alpha \subseteq \prod_{\alpha=1}^{n} Int(A_\alpha)$ ----------------------- ②

From equation ① and ② $Int \prod_{\alpha=1}^{n} A_\alpha = \prod_{\alpha=1}^{n} Int(A_\alpha)$

❷As $A_\alpha = X_\alpha$ except for finite α ϵ β and X_α are topological spaces,

So result obviously for all $A_\alpha = X_\alpha$.

So we prove this lemma just for finite case,

As $A_\alpha \subseteq Cl(A_\alpha)\ \forall\ \alpha = 1,2,3,\dots,n$

⇨ $\prod_{\alpha=1}^{n} A_\alpha \subseteq \prod_{\alpha=1}^{n} Cl(A_\alpha)$

Also, $(\prod_{\alpha=1}^{n} X_\alpha) \setminus \prod_{\alpha=1}^{n} Cl(A_\alpha) = \ \cup_{\alpha=1}^{n} (X_\alpha \times (X_\omega \setminus \overline{A_\omega})\ )$

$$\alpha \neq \omega,\ \ 1 \leq \alpha, \omega \leq n$$

Which is open in $\prod_{\alpha=1}^{n} X_\alpha$

⇨ $\prod_{\alpha=1}^{n} ClA_\alpha$ is closed and so $Cl \prod_{\alpha=1}^{n} A_\alpha \subseteq \prod_{\alpha=1}^{n} Cl(A_\alpha)$ -------- ③

Now let, $(x_1, x_2, x_3, \dots, x_n)\epsilon\ \prod_{\alpha=1}^{n} Cl(A_\alpha)$

Let, $\omega\ be\ a\ neighborhood\ of\ (x_1, x_2, \dots, x_n) in\ \prod_{\alpha=1}^{n} X_\alpha$

Then there exit open set U_α in $X_\alpha\ \forall\ \alpha = 1,2,3,\dots,n\ \ s.t.$

$$(x_1, x_2, \dots x_n) \epsilon \prod_{\alpha=1}^{n} U_\alpha \subseteq \omega$$

Then, $x_\alpha \epsilon U_\alpha \; \forall \alpha = 1,2,3, \dots, n$

But $X_\alpha \epsilon \overline{A_\alpha} \quad \forall \alpha = 1,2,3, \dots, n$

And $U_\alpha \cap A_\alpha \neq \varphi \quad \forall \alpha = 1,2,3, \dots, n$

Since $\prod_{\alpha=1}^{n} U_\alpha \subseteq \omega$ and we know that $\omega \cap \prod_{\alpha=1}^{n} A_\alpha \neq \varphi$

⇨ $(x_1, x_2, \dots, x_n) \epsilon \, Cl \prod_{\alpha=1}^{n} A_\alpha$ ---------------- ④

From equation ③ and ④ we have

$$\prod_{\alpha=1}^{n} Cl(A_\alpha) = Cl \prod_{\alpha=1}^{n} A_\alpha$$

Lemma 4:-

If A is a non-empty semi-open set, then $Int(A) \neq \varphi$

Proof:-

Since A is semi-open,

Then, $Cl(A) = Cl\{Int(A)\}$

Suppose, $Int(A) = \varphi$

Then, $Cl(A) = \varphi$

⇨ $A = \varphi$

$Cl(A) = Cl(\varphi) \quad \because IntA = \varphi$ ⇨ $Cl(A) = \varphi \quad \because \overline{\varphi} = \varphi$ ⇨ $A = \varphi \quad \because Cl(\varphi) = \varphi$

Which is a contradiction, and hence $Int(A) \neq \varphi$

Theorem 2:-

Let $\{X_\alpha : \alpha \epsilon \beta\}$ be any family of topological space, $X = \prod X_\alpha$ the product space and $A = \prod_{j=1}^{n} A_{\alpha_j} \times \prod_{\alpha \neq \alpha_j} X_\alpha$, a non-empty subset of X, where n is a positive integer. Then $A_{\alpha_j} \epsilon SO\left(X_{\alpha_j}\right)$ for each $j(1 \leq j \leq n)$ if and only if $A \epsilon SO(X)$

Proof:-

Suppose $A_{\alpha_j} \epsilon SO\left(X_{\alpha_j}\right) \quad \forall j(1 \leq j \leq n)$

Since $A \neq \varphi$ *this implies* $A_{\alpha_j} \neq \varphi \quad \forall j(1 \leq j \leq n)$

As $A_{\alpha_j} \epsilon SO\left(X_{\alpha_j}\right)$ *So* $Int\left(A_{\alpha_j}\right) \neq \varphi \quad (\because A_{\alpha_j} \neq \varphi)$

Thus $\prod_{j=1}^{n} Int(A_{\alpha_j}) \times \prod_{\alpha \neq \alpha_j} X_\alpha \neq \varphi$

Now, $Cl\{Int(A)\} = \prod_{j=1}^{n} Cl\left\{Int\left(A_{\alpha_j}\right)\right\} \times \prod_{\alpha \neq \alpha_j} X_\alpha$

$= \prod_{j=1}^{n} Cl(A_{\alpha_j}) \times \prod_{\alpha \neq \alpha_j} X_\alpha \qquad \because A_{\alpha_j} \epsilon SO(X_{\alpha_j})$

⇨ $Cl\{Int(A)\} = Cl(A)$

⇨ $A \epsilon SO(X)$

Conversely,

Let $A \epsilon SO(X)$

Then, $Int(A) \neq \varphi \qquad \because A \neq \varphi$

As $Int(A) \subseteq \prod_{j=1}^{n} Int(A_{\alpha_j}) \times \prod_{\alpha \neq \alpha_j} X_\alpha$

So $\prod_{j=1}^{n} Int(A_{\alpha_j}) \times \prod_{\alpha \neq \alpha_j} X_\alpha \neq \varphi$

Since, $A \epsilon SO(X)$ so by a well known theorem,

$\prod_{j=1}^{n} Cl\left\{Int\left(A_{\alpha_j}\right)\right\} \times \prod_{\alpha \neq \alpha_j} X_\alpha = Cl\{Int(A)\} = Cl(A)$

$$= \prod_{j=1}^{n} Cl(A_{\alpha_j}) \times \prod_{\alpha \neq \alpha_j} X_\alpha$$

⇨ $Cl\left(Int\left(A_{\alpha_j}\right)\right) = Cl\left(A_{\alpha_j}\right) \quad \forall\, j(1 \leq j \leq n)$

⇨ $A_{\alpha_j} \in SO(X_\alpha) \quad \forall\, j(1 \leq j \leq n)$

Semi-Continuous Mapping

Theorem 3:-

If $f: X \longrightarrow Y$ is a semi-continuous mapping and $X_\circ$ is an open set in X, then restriction $f \big| X_\circ : X_\circ \longrightarrow Y$ is semi-continuous.

Proof:-

Since f is a semi-continuous mapping,

⇨ For any open set V in Y, $f^{-1}(V)$ is semi-open in X.

Since $X_\circ$is open. So $f^{-1}(V) \cap X_\circ$ is semi open in X.

Therefore, $(f \big| X_\circ)^{-1}(V) = f^{-1}(V) \cap X_\circ$ is semi-open in $X_\circ$.

⇨ $f \big| X_\circ$ Is semi-continuous.

Remark:-

In above theorem if $X_\circ \in SO(X)$ then $f \big| X_\circ$ is not always semi-continuous.

Example:-

Let $X = Y = [0,1]$ with usual topology and $X_\circ = [\frac{1}{2}, 1]$

Let, $f: X \longrightarrow Y$ be mapping as follows,

$$f(x) = \begin{cases} 1 & if\ 0 \leq x \leq \frac{1}{2} \\ 0 & if\ \frac{1}{2} < x \leq 1 \end{cases}$$

Then f is semi-continuous.

However, $\left(\frac{1}{2}, 1\right]$ is open in Y and $f^{-1}\left(\left(\frac{1}{2}, 1\right]\right) \cap X_\circ = \left\{\frac{1}{2}\right\} \notin SO(X_\circ)$

Therefore, $f \big| X_\circ$ is not semi-continuous.

Theorem 4:-

Let $f: X \longrightarrow Y$ be a mapping and $\{A_\alpha : \alpha \epsilon \beta\}$ semi-open cover for X $i.e. A_\alpha \ \epsilon \ SO(X)$ for each $\alpha \ \epsilon \ \beta$ and $\bigcup_{\alpha \epsilon \beta} A_\alpha = X.$ if the restriction $f \big| A_\alpha : A_\alpha \longrightarrow Y$ is semi-continuous for each $\alpha \epsilon \beta,$ then f is semi-continuous.

Proof:-

Suppose V is an arbitrary open set in Y, then for each α ϵ β we have

$$(f \big| A_\alpha)^{-1}(V) = f^{-1}(V) \cap A_\alpha \ \epsilon \ SO(A_\alpha)$$

Because $f \big| A_\alpha$ is semi-continuous. Hence by a well known theorem,

$$f^{-1}(V) \cap A_\alpha \ \epsilon \ SO(X) \ \ for\ each\ \alpha \epsilon \beta$$

As union of any number of semi-open sets is semi-open so,

$$\bigcup_{\alpha\epsilon\beta}[f^{-1}(V)\cap A_\alpha] = f^{-1}(V)\,\epsilon\, SO(X)$$

⇨ f is semi-continuous.

Theorem 5:-

Let $\{X_\alpha : \alpha\epsilon\beta\}$ & $\{Y_\alpha : \alpha\epsilon\beta\}$ be any two families of topological spaces with the same index set β. For each $\alpha\epsilon\beta$, *Let* $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ be a mapping. Then a mapping $f : \prod X_\alpha \longrightarrow \prod Y_\alpha$ defined by,

$f(x_\alpha) = (f_\alpha(x_\alpha)$ Is semi-continuous if and only if f_α is semi-continuous for each $\alpha\epsilon\beta$.

Proof:-

Let f_α is semi-continuous for each $\alpha\epsilon\beta$

Suppose V is the basic open set of the topology of $\prod Y_\alpha$.

Then there are $\alpha_j\epsilon\beta\ (1 \le j \le n)$ and open sets V_{α_j} in Y_{α_j} $s.t.$

$$V = \prod_{j=1}^{n} V_{\alpha_j} \times \prod_{\alpha\neq\alpha_j} Y_\alpha$$

Since f_{α_j} is semi-continuous. So $f_{\alpha_j}^{-1}\left(V_{\alpha_j}\right)$ is semi open X_{α_j} for each $j\ (1 \le j \le n)$

If there exit α_j $s.t.$ $f_{\alpha_j}^{-1}\left(V_{\alpha_j}\right) = \varphi$

Then, $f^{-1}(V) = \prod_{j=1}^{n} f_{\alpha_j}^{-1}\left(V_{\alpha_j}\right) \times \prod_{\alpha\neq\alpha_j} X_\alpha = \varphi$

Hence $f^{-1}(V)$ is semi-open in $\prod X_\alpha$.

If $f_{\alpha_j}^{-1}\left(V_{\alpha_j}\right) \neq \varphi$ for each $j(1 \le j \le n)$

Then, $f^{-1}(V) = \prod_{j=1}^{n} f_{\alpha_j}^{-1}\left(V_{\alpha_j}\right) \times \prod_{\alpha \neq \alpha_j} X_\alpha \neq \varphi$

Hence by a well known theorem, $f^{-1}(V)$ is semi open in $\prod X_\alpha$.

Now, for any open set ω in Y there exit a family $\{Y_\lambda : \lambda \epsilon \Delta\}$ of basic open sets such that $\omega = \bigcup_{\lambda \epsilon \Delta} V_\lambda$

Hence by a well known theorem,

$f^{-1}(\omega) = \bigcup_{\lambda \epsilon \Delta} f^{-1}(V_\lambda)$ is semi-open in $\prod X_\alpha$.

⇨ F is semi-continuous.

Conversely,

Let f is semi-continuous.

Let for each fixed $\alpha \epsilon \beta$,

Let $p_\alpha : \prod Y_r \longrightarrow Y_\alpha$ be the projection.

Suppose V_α is the arbitrary open set in Y_α,

Then, $p_\alpha^{-1}(V_\alpha) = V_\alpha \times \prod_{r \neq \alpha} Y_r$ is open in $\prod Y_r$.

Since f is semi-continuous then,

$$f^{-1}[p_\alpha^{-1}(V_\alpha)] = f_\alpha^{-1}(V_\alpha) \times \prod_{r \neq \alpha} X_r$$

Is semi-continuous in $\prod X_r$

If $f_\alpha^{-1}(V_\alpha) = \varphi$ then it is obvious that f_α is semi-continuous.

If $f_\alpha^{-1}(V_\alpha) \neq \varphi$

Then, $f_\alpha^{-1}(V_\alpha) \times \prod_{r \neq \alpha} X_r \neq \varphi$

Hence by a well known theorem,

$f_\alpha^{-1}(V_\alpha)$ is semi open in X_α

⇨ f_α is semi-continuous $\forall\ \alpha\epsilon\beta$

Theorem 6:-

Let $\{X_\alpha : \alpha\epsilon\beta\}$ be any family of topological spaces. If $f: X \longrightarrow \prod X_\alpha$ is semi-continuous mapping, then $p_\alpha{}_\circ f: X \longrightarrow X_\alpha$ is semi-continuous, where p_α is projection of $\prod X_r$ onto X_α.

Proof:-

Let for a fixed $\alpha\epsilon\beta$,

Suppose U_α is an arbitrary open set in X_α then,

$p_\alpha^{-1}(U_\alpha)$ is open in $\prod X_\alpha$.

Since f is semi-continuous, we have

$$f^{-1}[p_\alpha^{-1}(U_\alpha)] = (p_\alpha{}_\circ f)^{-1}(U_\alpha)\ \epsilon\ SO(X)$$

⇨ $p_\alpha{}_\circ f$ is semi-continuous.

Theorem 7:-

If $f: X \longrightarrow Y$ is an open and semi-continuous mapping, then $f^{-1}(B) \in SO(X)$ for every $B\epsilon SO(Y)$.

Proof:-

For an arbitrary $B\epsilon SO(Y)$,

There exit an open set V in Y such that,

$$V \subseteq B \subseteq Cl(V)$$

Since f is open and continuous,

⇨ $f^{-1}(V) \subseteq f^{-1}(B) \subseteq f^{-1}Cl(V) \subseteq Cl\{f^{-1}(V)\}$

Since f is semi-continuous and V is an open set in Y,

⇨ $f^{-1}(V) \in SO(X)$

Hence $f^{-1}(B)$ is semi-open in X.

- The composition mapping of two semi-continuous mappings is not always semi-continuous.

Corollary:-

Let X, Y and Z are three topological spaces. If $f: X \longrightarrow Y$ is an open and semi-continuous mapping and $g: Y \longrightarrow Z$ is semi-continuous mapping, then $g_o f: X \longrightarrow Z$ is semi-continuous.

Proof:-

Since $g: Y \longrightarrow Z$ is semi-continuous.

Then for any open set V in Z $g^{-1}(V) \in SO(Y)$

And since f is open and semi-continuous, then by theorem 7.

$$f^{-1}\{g^{-1}(V)\} \in SO(X)$$

⇨ $(f^{-1}{}_o g^{-1})(V) \in SO(X)$
⇨ $(g_o f)^{-1}(V) \in SO(X)$
⇨ $g_o f$ Is semi-continuous.

Semi-Topological Properties

This course was established in 1973, by "S.Gene Crosseley and S.K Hildebrand" and published by "Texas Journal Math (1973).

Introduction:-

In [1] Norman Levine defined a semi-open set in a topological space as a set A such that there exit an open set O so that $O \subseteq A \subseteq Cl(O)$. He also defined a function to be semi-continuous if and only if the inverse of open sets is semi-open. Also in [1], among others, the following two results were obtained.

Theorem 0.1:-

Let (X, τ) be topological space then,

1. $\tau \subseteq SO(X)$, where SO(X) denotes the class of semi-open sets in (X, τ)
2. For A ϵ SO(X, τ) and $A \subseteq B \subseteq \overline{A}$, Then B ϵ SO(X, τ).

Theorem 0.2:-

Let $f: X \longrightarrow Y$ be a continuous and open mapping, where X and Y are topological spaces. Let A ϵ SO(X), Then $f(A) \in SO(Y)$

In [2] the author defined a set to be semi-closed if and only if its compliment is semi-open. Semi-closure and semi-interior were defined in a manner analogous to closure and interior. Also in [2], among others, the following four results were established.

Theorem 0.3:-

In a topological space all non-void semi-open sets must contain semi-open set.

Proof:-

Let (X, τ) be a topological space and A ϵ SO(X) be a semi-open set such that A ≠ φ.

Then there exit an open set O in X such that,

$$O \subseteq A \subseteq Cl(O)$$

Then O must non empty $i.e. O \neq \varphi$

Because if $O = \varphi$

⇨ $Cl(O) = \varphi$ $\quad\because \overline{\varphi} = \varphi$

And in this case $A \nsubseteq \overline{O}$ $\quad\because A \neq \varphi$

⇨ $O \neq \varphi$

Hence $A \neq \varphi$ is semi-open set must contain a non-empty open set.

Semi-Interior of a Set:-

Let (X, τ) be a topological space and $A \neq \varphi$ is a subset of X. Then semi-interior of A is denoted by $sInt(A)$ or $A_{\circ}$ and is the union of all semi-open sets contained in A.

Note:-

(1) $sInt(A) is\ a\ semi-open\ set.$

(2) $sInt(A)\ is\ the\ largest\ semi-$open set contained in A.

Semi-Interior Point:-

Let (X, τ) be topological space and $A \subseteq X$. A point $x \epsilon A$ is called semi-interior point of A if there exit a semi-open set u in X $s.t.$ $x \epsilon u \subseteq A$.

Note:-

(1) Collection of all semi-interior points of A is called $sInt(A)$

(2) If $A \epsilon SO(X)$, then every point of A is semi-interior point of A. Because $\forall x \epsilon A, \quad x \epsilon A \subseteq A$.

Semi-Closure of a Set:-

Let (X, τ) be a topological space and A is a non-void subset of X. Then semi-closure of A is denoted by $sCl(A)$ OR $\underline{A}$ and is the intersection of all semi-closed sets containing A.

Note:-

(1) $sCl(A)$ Is a semi-closed set.

(2) $sCl(A)$ Is the smallest semi-closed set containing A.

(3) $Int(A) \subseteq sInt(A) \subseteq A \subseteq sCl(A) \subseteq Cl(A)$

Semi-Limit Point:-

Let (X, τ) be a topological space and A is a subset of X, a point $x \epsilon X$ is called semi-limit point of A if for each semi-open set u containing x, we have $u \cap A \neq \varphi, \quad u \cap (A - \{x\}) \neq \varphi$.

Note:-

A is semi-closed if A contains all semi-limit points.

Theorems 0.4:-

1. A is semi-open if and only if $A_{\circ} = A$
2. A is semi-closed if and only if $\underline{A} = A$

Proof:-

❶Let A be a semi-open set in X,

Then $A \subseteq A_{o} \quad But \quad A_{o} \subseteq A \quad (always)$

⇨ $A = A_{o}$

Conversely,

Let $A = A_{o} \quad (semi - open)$

Since A_{o}s semi-open, therefore A is semi-open.

❷Let A be a semi-closed set in X,

Then $\underline{A} \subseteq A \qquad But\ \underline{A} \subseteq A \quad (always)$

⇨ $A = \underline{A}$

Conversely,

PREPARED BY:-
MUHAMMAD TAHIR
M.S Mathematics
COMSATS Institute of Information technology Islamabad.

Let $A = \underline{A}$

Since $\underline{A}$ is semi-closed, therefore A is semi-closed.

Theorem 0.5:-

If A is open and S is semi-open, then $A \cap S$ is semi-open.

Proof:-

Let S be semi-open in X, Then there exit an open set $O \epsilon X$ such that,

$$O \subseteq S \subseteq Cl(O)$$

⇨ $O \cap A \subseteq S \cap A \subseteq Cl(O) \cap A \subseteq Cl(O \cap A)$

Since $O \cap A$ is open in X and $O \cap A \subseteq S \cap A \subseteq Cl(O \cap A)$

⇨ S∩A is semi-open in X.

Theorem 0.6:-

Let (X, τ) be a topological space and $A \subseteq X$, then prove that

$$\underline{[X - (\overline{A} - A)]} = X$$

Proof:-

L.H.S

$\overline{A} - A$ Contains no semi-interior points.

⇨ $sInt(\overline{A} - A) = \varphi$

⇨ $X - sInt(\overline{A} - A) = X$

⇨ $sCl[X - (\overline{A} - A) = X$

PREPAIRED BY
MUHAMMAD TAHIR WATTOO
M.S. MATHEMATICS
CIIT ISLAMABAD
03448563284

⇨ $[X-(\overline{A}-A]=X$

Irresolute Function:-

Let (X,τ_x) and (Y,τ_y) be topological spaces. A function $f: X \longrightarrow Y$ is called irresolute if $f^{-1}(B)$ is semi-open in X for every semi-open set B in Y.

Theorem 1.1:-

Let $f:(X,\tau_x) \longrightarrow (Y,\tau_y)$ be continuous and open, then

$$f^{-1}(\overline{A}) = \overline{f^{-1}(A)}$$

Proof:-

$f: X \longrightarrow Y$ is continuous and open.

Let A be any subset of Y.

⇨ $\overline{A}$ Is a closed set of Y.
⇨ $f^{-1}(\overline{A})$ Is a closed subset of X.

As $A \subseteq \overline{A}$

⇨ $f^{-1}(A) \subseteq f^{-1}(\overline{A})$ $\quad \because f$ *is continuous*
⇨ $\overline{[f^{-1}(A)]} \subseteq \overline{[f^{-1}(\overline{A})]} = f^{-1}(\overline{A})$ $\quad \because f^{-1}(\overline{A})$ *is closed*
⇨ $\overline{[f^{-1}(A)} \subseteq f^{-1}(\overline{A})$ -------------------------- ❶

As f is open,

⇨ Image of every open set is open under f.

Let (X,τ_x) *and* (Y,τ_y) be two topological spaces, A function $f: X \longrightarrow Y$ is continuous iff for every A⊆X $f(\overline{A}) \subseteq \overline{f(A)}$

⇨ f^{-1} is a continuous function.

Then by a well known theorem for every A⊆Y,

$f^{-1}(\overline{A}) \subseteq \overline{f^{-1}(A)}$ ------------------------ ❷

By relation ❶ and ❷

$$f^{-1}(\overline{A}) = \overline{f^{-1}(A)}$$

Theorem 1.2:-

Let $f:(X,\tau_x) \longrightarrow (Y,\tau_y)$ be continuous and open then f is irresolute.

Proof:-

Let $A \epsilon SO(Y)$

Then by definition there exit $O \epsilon \tau_y$ such that,

$$O \subseteq A \subseteq Cl(O)$$

⇨ $f^{-1}(O) \subseteq f^{-1}(A) \subseteq f^{-1}(\overline{A}) = \overline{[f^{-1}(O)]}$ $\quad \because f \text{ is continuous } \& \text{ open}$

As O is open,

- ⇨ $f^{-1}(O)$ is open because f is continuous.
- ⇨ $f^{-1}(O) \subseteq f^{-1}(A) \subseteq \overline{(f^{-1}(O)}$
- ⇨ $f^{-1}(A) \epsilon SO(X)$
- ⇨ f is irresolute function.

Example 1.1:-

A continuous irresolute function need not be open.

Proof:-

Let $X = \{a, b, c\}$,

$\tau = \{\varphi, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $\tau^* = \{\varphi, \{a\}, \{a, b\}, X\}$

Let $f: (X, \tau_x) \longrightarrow (Y, \tau_y)$ be defined by $f(x) = x \quad \forall\, x \epsilon X$

Then this function is continuous and irresolute but not an open function.

See,

$f^{-1}(\varphi) = \varphi \in \tau \quad \Longrightarrow f^{-1}(\varphi) \text{ is open}$

$f^{-1}(\{a\}) = \{a\} \quad \text{open in } (X, \tau)$

$f^{-1}(\{a, b\}) = \{a, b\} \quad \text{open in } (X, \tau)$

$f^{-1}(X) = X \quad \text{open}$

As inverse image of every open set is open,

⇨ f is continuous.

Now, $P(X) = \{\varphi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{c, b\}, \{a, b, c\}\}$

Closed sets of (X, τ) are $\{X, \{b, c\}, \{c\}, \{b\}, \varphi\}$

Now, $Cl(\varphi) = \varphi, \quad Cl(X) = X, \quad Cl\{a\} = X$

$Cl\{a, b\} = X, \quad and \quad Cl\{a, c\} = X$

⇨ SO(X, τ)= $\{\varphi, \{a\}, \{a, b\}, \{a, c\}, X\}$

Now, closed sets of (X, τ^*) are $\{\varphi, X, \{b, c\}, \{c\}]$

⇨ $SO(X, \tau_x) = \{\varphi, \{a\}, \{a, b\}, \{a, c\}, X\}$

And, $f^{-1}(\varphi) = \varphi \in SO(X, \tau)$

$f^{-1}(\{a\}) = \{a\} \in SO(X, \tau)$

$f^{-1}(\{a,b\}) = \{a,b\} \in SO(X,\tau)$

$f^{-1}(\{a,c\}) = \{a,c\} \in SO(X,\tau)$

$f^{-1}(X) = X \in SO(X)$

As inverse image of every semi-open set is semi open,

⇨ f is irresolute.

Now as $\{a,c\}$ is open in (X, τ)

⇨ $f(\{a,c\}) = \{a,c\} \notin SO(X,\tau^*)$
⇨ Image of every open set is not open.
⇨ f is not open.

Theorem 1.3:-

Let C(X, Y), SC(X, Y) and I(X, Y) denote respectively, the classes of continuous, semi continuous and irresolute functions from X to Y, where X and Y are topological spaces. Then,

$$C(X,Y) \subseteq SC(X,Y) \quad and \quad I(X,Y) \subseteq SC(X,Y)$$

Proof:-

❶ Let $f \in C(X,Y)$

⇨ f is irresolute function.
⇨ Inverse image of every open set (say A) of Y is open in X.
⇨ $f^{-1}(A)$ is open in X.

As every open set is also semi-open,

⇨ $f^{-1}(A)$ is semi-open in X.
⇨ Inverse image of every open set of Y is semi-open in X.
⇨ $f \in SC(X,Y)$

⇨ $C(X,Y) \subseteq SC(X,Y)$

❷ Let $g \in I(X,Y)$

⇨ g is irresolute function.
⇨ Inverse image of every semi-open set (say B) of Y is semi-open in X.
⇨ $f^{-1}(B)$ is semi-open in X.

As all open sets of Y ⊆ semi-open sets of Y

⇨ Inverse image of every open set (say B) is semi-open in X.
⇨ g is semi-continuous.
⇨ $g \in SC(X,Y)$
⇨ $I(X,Y) \subseteq SC(X,Y)$ (proved)

Theorem 1.4:-

A function $f:(X,\tau_x) \longrightarrow (Y,\tau_y)$ is irresolute if and only if, for every semi-closed subset H of Y, $f^{-1}(H)$ is semi-closed in X.

Proof:-

Let $f: X \longrightarrow Y$ be irresolute.

Let H ϵ SC(Y), then $Y - H$ is semi0open in Y.

Or, $f^{-1}(Y-H) = f^{-1}(Y) - f^{-1}(H) = X - f^{-1}(H)$ $\because f^{-1}(Y) = X$

⇨ $X - f^{-1}(H)$ is semi open in X. $\because f$ is irresolute.
⇨ $f^{-1}(H)$ is semi closed in X.

Conversely,

Let $f^{-1}(H)$ is semi-closed in X, for every semi-closed set H in Y.

We have to prove that f is irresolute.

As $B \in SO(Y)$

⇨ $(Y - B) \epsilon\, SC(Y)$

⇨ $f^{-1}(Y - B) \epsilon\, S(X)$ $\quad \because f^{-1}(H) \epsilon SC(X)\ \forall\ H \epsilon SC(Y)$

⇨ $f^{-1}(Y) - f^{-1}(B) \epsilon\, SC(X)$

⇨ $X - f^{-1}(B) \epsilon\, SC(X)$

⇨ $f^{-1}(B) \epsilon\, SC(X)$

⇨ f is irresolute. (proved).

Theorem 1.5:-

A function $f: S \longrightarrow T$, where S and T are topological spaces is irresolute if and only if for every subset A of S, $f(\underline{A}) \subseteq \underline{f(A)}$

Proof:-

Let $f: S \longrightarrow T$ be irresolute function.

Let $A \epsilon S$, Then $\underline{f(A)} \epsilon\, SC(T)$

⇨ $f^{-1}\left[\underline{f(A)}\right]$ is semi-close in S. $\quad \because f$ is irresolute.

Now, $A \subseteq f^{-1}f(A) \subseteq f^{-1}\left[\underline{f(A)}\right]$ $\quad \because f(A) \subseteq \underline{f(A)}$

⇨ $\underline{A} \subseteq sClf^{-1}[\underline{f(A)}] = f^{-1}\underline{f(A)}$ $\quad \because f^{-1}f(A)$ is semi closed.

⇨ $f(\underline{A}) \subseteq f\left[f^{-1}\underline{f(A)}\right] \subseteq \underline{f(A)}$

⇨ $f(\underline{A}) \subseteq \underline{f(A)}$

Conversely,

Assume that $f(\underline{A}) \subseteq \underline{f(A)}$

We have to prove that f is irresolute.

Let $H \epsilon\, SC(T)$

Then $f\left[\underline{f^{-1}(H)}\right] \subseteq \underline{ff^{-1}(H)} \subseteq \underline{H} \subseteq H$ $\quad \because H$ *is semi closed*

Now, $\underline{f^{-1}(H)} \subseteq f^{-1}f[\underline{f^{-1}(H)}] \subseteq f^{-1}(\underline{H}) = f^{-1}(H)$

⇨ $\underline{f^{-1}(H)} \subseteq f^{-1}(H)$ $\quad$ *But* $f^{-1}(H) \subseteq \underline{f^{-1}(H)}$ *always.*
⇨ $f^{-1}(H) = \underline{f^{-1}(H)}$
⇨ $f^{-1}(H) \in SC(S)$
⇨ f is irresolute.

Theorem 1.6:-

Let (X, τ_x) *and* (Y, τ_y) be topological spaces. A function $f: X \longrightarrow Y$ is irresolute if and only if for all B⊆Y, $\underline{f^{-1}(B)} \subseteq f^{-1}(\underline{B})$.

Proof:-

Assume that f is irresolute.

Let B be any subset of Y. Then $\underline{B} \in SC(Y)$,

Hence, $f^{-1}(\underline{B}) \in SC(X)$

But we know $B \subseteq \underline{B}$

⇨ $f^{-1}(B) \subseteq f^{-1}(\underline{B})$
⇨ $SC(f^{-1}(B)) \subseteq SC\left(f^{-1}(\underline{B})\right) = f^{-1}(\underline{B})$
⇨ $\underline{f^{-1}(B)} \subseteq f^{-1}(\underline{B})$

Conversely,

Let, $\underline{f^{-1}(B)} \subseteq f^{-1}(\underline{B})$ for every subset B of Y.

We will prove that f is irresolute.

For this we will show that the inverse image of semi-closed set is semi-closed.

Let $B \in SC(Y)$ $then,$ $\underline{B} = B$

By hypothesis, $\underline{\left(f^{-1}(B)\right)} \subseteq f^{-1}\left(\underline{B}\right) = f^{-1}(B)$

And $f^{-1}(B) \subseteq \underline{\left(f^{-1}(B)\right)} \subseteq f^{-1}\left(\underline{B}\right) \subseteq f^{-1}(B)$

⇨ $f^{-1}(B) \subseteq \underline{\left(f^{-1}(B)\right)} \subseteq f^{-1}(B)$
⇨ $f^{-1}(B) = \underline{f^{-1}(B)}$
⇨ $f^{-1}(B) \in SC(X)$
⇨ f is irresolute.

Theorem 1.7:-

Let (X, τ_x) and (Y, τ_y) and (Z, τ_z) be topological spaces. If $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are both irresolute then $g_\circ f: X \longrightarrow Z$ is irresolute.

Proof:-

Let $B \in SO(Z)$

⇨ $g^{-1}(B)$ is semi open in Y $\because$ g is irresolute.

Now as $g^{-1}(B) \in SO(Y)$ and f is irresolute from X⟶Y

⇨ $f^{-1}\left(g^{-1}(B)\right) \in SO(X)$
⇨ $f^{-1}\left(g^{-1}(B)\right) = (g_\circ f)^{-1}(B) \in SO(X)$

Now as $B \in SO(Z)$ and $(g_\circ f)^{-1}(B) \in SO(X)$

⇨ $g_\circ f$ is irresolute from X⟶Z.

Pre-Semi-Open Function:-

Let X and Y be topological spaces, a function $f: X \longrightarrow Y$ is said to be pre-semi-open if and only if, for all $A \epsilon SO(X)$, $f(A) \epsilon SO(Y)$.

Theorem 1.8:-

Let (X, τ_x) *and* (Y, τ_y) be topological spaces. If $f: X \longrightarrow Y$ is continuous and open, then f is irresolute and pre semi open.

Proof:-

Let $f: X \longrightarrow Y$ be continuous and open mapping.

To prove that f is irresolute.

Consider a semi open set B in Y. Then there exit an open se u in Y such that,

$$u \subseteq B \subseteq Cl(u)$$

⇨ $f^{-1}(u) \subseteq f^{-1}(B) \subseteq f^{-1}Cl(u) = Cl\big(f^{-1}(u)\big) \qquad \because f \text{ is cont \& open}$

Since f is continuous, therefore $f^{-1}(u)$ is open in X and

$$f^{-1}(u) \subseteq f^{-1}(B) \subseteq Cl\big(f^{-1}(u)\big)$$

⇨ $f^{-1}(B) \epsilon SO(X)$

⇨ f is irresolute.

Now we prove that f is pre semi open.

Let $A \epsilon SO(X)$

⇨ There exit an open set O in X such that,

$$O \subseteq A \subseteq Cl(O)$$

⇨ $f(O) \subseteq f(A) \subseteq f[Cl(O)] \subseteq Cl[f(O)]$ $\quad \because f$ *is continuous*.

⇨ $f(O) \subseteq f(A) \subseteq Cl[f(O)]$

Since f is open mapping, therefore $f(O)$ is open in Y.

Hence, $f(A) \in SO(Y)$ this implies f is pre semi open.

Semi-Homeomorphism:-

Let (X, τ_x) *and* (Y, τ_y) be topological spaces. X and Y are said to be semi-homeomorphism if and only if there exit a function $f: X \longrightarrow Y$ such that,

(1) f is bijective (2) f is irresolute

(3) f is pre semi open.

Theorem 1.9:-

Let (X, τ_x) *and* (Y, τ_y) be topological spaces. If $f: X \longrightarrow Y$ is homeomorphism then f is semi homeomorphism.

Proof:-

Let $f: X \longrightarrow Y$ be homeomorphism, then

1. f is bijective 2. f is continuous 3. f is open.

Since f is continuous and open bijection,

Therefore it is irresolute and pre semi open bijection.

Hence f is semi-homeomorphism.

Example 1.2:-

A semi-homeomorphism need not be homeomorphism.

Solution:-

(See example 1.1)

Remark 1.1:-

Image of T_o space under semi homeomorphism may not be a T_o space.

Remark 1.2:-

The image of a T_1 space under a semi homeomorphism is not necessarily a T_1-space.

Example 1.4:-

Let X=(R×R), where R denote the set of real numbers and let,

$\tau_1 = \{\varphi$, Together with all subsets of X whose compliments are subsets of a finite number of lines parallel to the x-axis}

Note that, $SO(X, \tau_1) = \tau_1$

And let. $\tau_2 = \{\varphi$, Together with all subsets of X whose compliments are a finite number of lines parallel to x-axis}

Note that, $SO(X, \tau_2) = SO(X, \tau_1)$

Furthermore, defining $f: (X, \tau_1) \longrightarrow (X, \tau_2)\ by\ f(p) = p\ for\ p \epsilon X,$

We see that f is a semi-homeomorphism.

Observe that (X, τ_1) is a T_1 space where (X, τ_2) is not.

Theorem 1.10:-

If $f: X \longrightarrow Y$ is a semi homeomorphism, then $\underline{f^{-1}(B)} = f^{-1}(\underline{B})$ for all B subset of Y.

Proof:-

$f: X \longrightarrow Y$ is semi homeomorphism.

⇨ f *is* 1) bidective. 2) irresolute. 3) pre semi open.

Let B be any subset of Y.

Then $\underline{B} \in SC(Y)$,

Hence, $f^{-1}(\underline{B}) \in SC(X)$

As we know that $B \subseteq \underline{B}$

⇨ $f^{-1}(B) \subseteq f^{-1}(\underline{B})$

⇨ $sCl(f^{-1}(B)) \subseteq sCl\left(f^{-1}(\underline{B})\right) = f^{-1}(\underline{B})$ $\quad \because f^{-1}(\underline{B}) \in SC(X)$

⇨ $\underline{f^{-1}(B)} \subseteq f^{-1}(\underline{B})$ ------------------------ ❶

As f is semi homeomorphism,

⇨ f is pre semi open and bijective.

⇨ Image of every semi open set is semi open under f

⇨ f^{-1} is irresolute.

Then by theorem 1.5 for every $B \in Y$

$f^{-1}(\underline{B}) \subseteq \underline{f^{-1}(B)}$ ------------------------------ ❷

From equation ❶ and ❷

$$f^{-1}(\underline{B}) = \underline{f^{-1}(B)}$$

Corollary 1.1:-

If $f: X \longrightarrow Y$ is semi homeomorphism, then $\underline{f(B)} = f(\underline{B})$ for all $B \subseteq X$.

Proof:-

$f: X \longrightarrow Y$ is semi homeomorphism.

⇨ f is 1) bijective. 2) irresolute. 3)pre semi open

Let $B \subseteq X$ then $\underline{f(B)} \in SC(Y)$

⇨ $f^{-1}[\underline{f(B)}]$ is semi closed in X. ∵ f is irresolute.

Now, $B \subseteq f^{-1}(f(B)) \subseteq f^{-1}(\underline{f(B)}$ ∵ $f(B) \subseteq \underline{f(B)}$

⇨ $\underline{B} \subseteq sCl\left(f^{-1}(f(B))\right) = f^{-1}\underline{f(B)}$ ∵ $f^{-1}\underline{f(B)}$ *is semi closed*.

⇨ $f(\underline{B}) \subseteq f\left[f^{-1}(\underline{f(B)})\right] \subseteq \underline{f(B)}$

⇨ $f(\underline{B}) \subseteq \underline{f(B)}$ ---------------------------------- ❶

Since f is bijective and irresolute,

⇨ f^{-1} is exit and also irresolute.

Then by theorem 1.6, for $B \epsilon X$

$\underline{f(B)} \subseteq f(\underline{B})$ -- -- ❷

From relation ❶ and ❷

$$\underline{f(B)} = f(\underline{B})$$

Corollary 1.2:-

If $f: X \longrightarrow Y$ is semi homeomorphism, then $f(B_\circ) = \big(f(B)\big)_\circ$ for all $B \subseteq X$.

Proof:-

$$B_\circ = \big(X - \underline{(X - B)}\big)$$

Thus, $f(B_\circ) = f\left[X - \underline{(X - B)}\right]$

$$= \left[Y - f\underline{(X - B)}\right]$$

$$= \left[Y - \underline{f(X - B)}\right] \qquad \because f \text{ is irresolute}$$

$$= \left[Y - \underline{[Y - f(B)}\right]$$

⇨ $f(B_\circ) = [f(B)]_\circ$

Corollary 1.3:-

If $f: X \longrightarrow Y$ is semi homeomorphism, then $f^{-1}(B_\circ) = \big(f^{-1}(B)\big)_\circ$ for all B ⊆ Y.

Proof:-

As $f: X \longrightarrow Y$ is semi homeomorphism,

⇨ $f^{-1}: Y \longrightarrow X$ is irresolute (bijective and pre semi open)

Let B ⊆ Y, $B_\circ = [Y - (\underline{Y - B})]$

Thus, $f^{-1}(B_\circ) = f^{-1}[Y - (\underline{Y - B})]$

$= [X - f^{-1}(\underline{Y - B})]$

$= \left[X - \left[\underline{f^{-1}(Y - B)}\right]\right]$ $\because f^{-1}$ *is irresolute*

$= \left[X - \left[\underline{X - f^{-1}(B)}\right]\right]$

⇨ $f^{-1}(B_\circ) = [f^{-1}(B)]_\circ$ (proved)

Theorem 1.11:-

$(\underline{A})_\circ = \varphi$ if and only if A is nowhere dense set.

Proof:-

Let A is nowhere dense set.

As we know that, $A^\circ \subseteq A_\circ \subseteq A \subseteq \underline{A} \subseteq \overline{A}$,

As A is nowhere dense set,

⇨ $(\overline{A})^\circ = \varphi$. This implies, $\overline{A}$ contains no open set.

⇨ $\underline{A}$ Contains no open set. $\because \underline{A} \subseteq \overline{A}$

⇨ $\underline{A}$ Contains no semi open set.

⇨ $(\underline{A})_\circ = \varphi$.

Conversely,

Let, $(\underline{A})_\circ = \varphi$

We know by a well known theorem, (theorem 0.7)

$$(\overline{A})^{\circ} \subseteq (\underline{A})_{\circ}$$

Since $(\underline{A})_{\circ} = \varphi$ *this implies* $(\overline{A})^{\circ} \subseteq \varphi$

⇨ $(\overline{A})^{\circ} = \varphi$
⇨ A is nowhere dense set.

Theorem 1.12:-

If $f: X \longrightarrow Y$ is a semi homeomorphism and A ⊆ X is nowhere dense in X. Then $f(A)$ is nowhere dense in Y.

Proof:-

As A is nowhere dense in X. Then by theorem 1.11

$$(\underline{A})_{\circ} = \varphi$$

We have to show $\left(\underline{f(A)}\right)_{\circ} = \varphi$

As $f: X \longrightarrow Y$ is semi homeomorphism,

⇨ $\underline{f(A)} = f(\underline{A})$

⇨ $\left[\underline{f(A)}\right]_{\circ} = [f(\underline{A})]_{\circ} = f(\underline{A})_{\circ}$ $\quad\because$ *corollary* 1.2

$= f(\varphi)$

⇨ $\left[\underline{f(A)}\right]_{\circ} = \varphi$
⇨ $f(A)$ is nowhere dense set.

Semi-Topological Properties:-

A property which is preserved under semi homeomorphism is said to be a semi-topological property.

PREPAIRED BY:-

MUHAMMAD TAHIR

M.S Mathematics

COMSATS Institute of Information technology Islamabad.

0344-8563284

SEMI WEAKLY CONTINUOUS MAPPINGS

This course was established in 1985 by “T. Noiri and B. Ahmad” and was published by “Kyungpook Math Journal vol.25, No.2 page 123-126.

Weakly Continuous Function:-

Let (X, τ_x) *and* (Y, τ_y) be topological spaces. A function $f: X \longrightarrow Y$ is said to be weakly continuous at X if for each $x \in X$ and for each open set V containing $f(x)$, there exit $u \in SO(X, \tau)$ such that $f(u) \subseteq Cl(V)$.

Almost Continuous Function:-

Let (X, τ_x) *and* (Y, τ_y) be topological spaces. A function $f: X \longrightarrow Y$ is said to be almost continuous if for each $x \in X$ and for each open set V containing $f(x)$, there exit a semi-open set u in X containing x such that $f(u) \subseteq Int[Cl(V)]$

Note:-

Almost continuous function is also weakly continuous,

$$\because Int[Cl(V)] \subseteq Cl(V)$$

But converse is not true in general.

Semi-Weakly continuous Function:-

Let $(X, \tau_x) and (Y, \tau_y)$ are topological spaces , a function $f: X \longrightarrow Y$ is said to be semi-weakly continuous function (s.w.c) at X if for each $x \epsilon X$ and for each open set V containing $f(x)$ there exit $u \epsilon SO(X)$ such that, $f(u) \subseteq sCl(V)$

Note:-

- Semi-continuous $\longrightarrow$ Semi-weakly continuous $\longrightarrow$ Weakly-continuous.
- Almost continuous $\longrightarrow$ Weakly-continuous.

Example:-

Let X = Y = R,

Let τ be the usual topology on X and σ be the countable topology on Y. Then the identity mapping $f: X \longrightarrow Y$ is semi-weakly continuous but not semi-continuous.

Theorem 1:-

Let (X, τ_x) *and* (Y, τ_y) be topological spaces. A mapping $f: X \longrightarrow Y$ is semi weakly continuous if and only if for every open set V in Y,

$$f^{-1}(V) \subseteq sInt\left[f^{-1}\left(sCl(V)\right)\right]$$

Proof:-

Let $x \epsilon X$ and V be an open set containing $f(x)$, satisfying the relation,

$$f^{-1}(V) \subseteq sInt[f^{-1}sCl(V)]$$

We will prove that f is semi weakly continuous.

Put $u = sInt[f^{-1}(sCl(v))]$,

Then, $x \in u \in SO(X, x)$

⇨ $u = sInt[f^{-1}(sCl(V))] \subseteq f^{-1}(sCl(V))$
⇨ $f(u) \subseteq ff^{-1}[sCl(V)] \subseteq sCl(V)$
⇨ $f(u) \subseteq sCl(V)$
⇨ f is semi weakly continuous.

Conversely,

Let $f: X \longrightarrow Y$ be semi weakly continuous.

Let $x \epsilon X$ and V be an open set containing $f(x)$.

⇨ $x \in f^{-1}(V)$

By hypothesis (f is semi weakly continuous), there exit a semi open set u in X containing x such that $f(u) \subseteq sCl(V)$

⇨ $x \in u \subseteq f^{-1}[sCl(V)]$
⇨ $u = sInt(u)$ $\quad \because u$ is open

$\subseteq sInt[f^{-1}(sCl(V))]$

⇨ $x \in sInt[f^{-1}(sCl(V))]$
⇨ $f^{-1}(V) \subseteq sInt[f^{-1}(sCl(V))]$

Theorem 2:-

Let (X, τ_x) *and* (Y, τ_y) are topological spaces. A function $f: X \longrightarrow Y$ be a function and $g: X \longrightarrow X \times Y$ be the graph mapping of f given by, $g(x) = (x, f(x))$ for every $x \epsilon X$. If g is semi weakly continuous, then f is semi weakly continuous.

Proof:-

Let $x \epsilon X$ and V be an open set containing $f(x)$.

⇨ X×V containing $(x, f(x)) = g(x)$.

Since g is semi weakly continuous, therefore there exit $u \epsilon SO(X, x)\ s.t.$

$$g(u) \subseteq sCl(X \times V) = sCl(X) \times sCl(V)$$

$$= X \times sCl(V)$$

Or $(u, f(u)) \subseteq X \times sCl(V)$ $\quad \because g(x) = (x, f(x))\ so\ g(u) = (u, f(u))$

⇨ $f(u) \subseteq sCl(V)$ $\quad \because g\ is\ graph\ of\ f.$

⇨ f is semi weakly continuous.

Theorem 3:-

Let $(X, \tau_x)\ and\ (Y, \tau_y)$ be topological spaces and if $f: (X, \tau_x) \longrightarrow (Y, \tau_y)$ is semi weakly continuous mapping and Y is housdorff space. Theen the graph G(f) is a semi closed set of X×Y.

Proof:-

Let $(x, y) \notin G(f)$

We will show that (x, y) is not semi limit point of $G(f)$.

Now, since $(x, y) \notin G(f)$ $\quad so\ \ y \neq f(x)$

Since Y is a T_2-space therefore there exit open sets W and V in Y such that,

$$f(x) \epsilon W; \quad y \epsilon V \quad and \quad W \cap V = \varphi$$

Since f is semi weakly continuous, therefore there exit a $u \epsilon SO(X, x)$

Such that $f(u) \subseteq sCl(W)$

Since $V \cap W = \varphi$

⇨ $V \cap sCl(W) = \varphi$
⇨ $V \cap f(u) = \varphi \quad \because f(u) \subseteq sCl(W)$
⇨ $(U \times V) \cap G(f) = \varphi$

Where $U \times V \epsilon SO\big(X \times Y, (x, y)\big)$

⇨ (x, y) is not semi limit point of $G(f)$.
⇨ $G(f)$ contains all of its semi limit points.
⇨ $G(f)$ is semi closed set of X×Y.

Semi-Connected Space (s-Connected Space):-

A topological space (X, τ_x) is said to be semi connected space if it cannot be expressed as union of two non-empty disjoint semi open sets.

Note:-

- Every semi connected space is connected.
- A connected space may not be semi connected.

Example:-

$$X = \{a, b, c\}$$

$$\tau = \{\varphi, X, \{a\}, \{b\}, \{a, b\}\}$$

It is connected because we cannot write it as union of two non empty disjoint open sets.

Now, $SO(X) = \{\varphi, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\}$

And, $\{a\} \cup \{b,c\} = X$ & $\{a\} \cap \{b,c\} = \varphi$

- ⇨ This is semi disconnected.
- ⇨ This is not semi connected space.

Theorem 4:-

Let (X, τ_x) is an s-connected space and $f: (X, \tau_x) \longrightarrow (Y, \tau_y)$ is a semi weakly continuous surjection, then Y is connected.

Proof:-

Suppose that Y is disconnected.

⇨ There exit open sets U and V in Y such that,

$$U \cup V = Y \quad \& \quad U \cap V = \varphi$$

⇨ $f^{-1}(Y) = f^{-1}(U \cup V)$

⇨ $X = f^{-1}(U) \cup f^{-1}(V)$ -------------------------- ❶

And $U \cap V = \varphi$

⇨ $f^{-1}(U \cap V) = f^{-1}(\varphi)$

⇨ $f^{-1}(U) \cap f^{-1}(V) = \varphi$ -------------------------- ❷

Since f is onto and $U \neq \varphi$ & $V \neq \varphi$

⇨ $f^{-1}(U) \neq \varphi$ & $f^{-1}(V) \neq \varphi$

Now, since f is semi weakly continuous and U, V are open in Y, therefore,

$$f^{-1}(U) \subseteq sInt[f^{-1}sCl(u)]$$

And $$f^{-1}(V) \subseteq sInt[f^{-1}sCl(v)]$$

⇨ $f^{-1}(U) \subseteq sInt\{f^{-1}(U)\}$ and $f^{-1}(V) \subseteq sInt\{f^{-1}(V)\}$
⇨ $f^{-1}(U) = sInt[f^{-1}(U)]$ and $f^{-1}(V) = sInt\{f^{-1}(V)\}$
⇨ $f^{-1}(U)$ and $f^{-1}(V)$ are semi open sets.

So by relation ❶ and ❷ we arte get that X is semi disconnected.

A contradiction.

Hence, the proof.

PREPAIRED BY:-

MUHAMMAD TAHIR

M.S Mathematics

COMSATS Institute of Information technology Islamabad.

0344-8563284

s-Continuous, s-Open, s-Closed Functions

This course was established in 2001 by "M. Khan (Department of Mathematic, Govt. College Multan-Pakistan) and B. Ahmed (B.Z.U. Multan Pakistan)"

s-Continuous Function:-

A function $f: X \longrightarrow Y$ is said to be s-continuous function (also called strongly semi-continuous) if the inverse image of every semi open set is open.

Note:-

It is known that that an s-continuous function is irresolute, semi continuous and continuous.

Regular Space(*):-

A topological space (X, τ) is said to be regular if for every $x \epsilon X$ and for any closed subset A of X such that $x \notin A$

There exit two open sets U and V such that, $x \epsilon u, \quad A \subseteq V \quad and\ U \cap V = \varphi$

p-Regular Space:-

A topological space (X, τ) is said to be p-regular space if for each semi closed set F and $x \epsilon X - F$, there exit disjoint open sets U and V such that $x \epsilon U \quad and \quad F \subseteq V$

Semi-Regular Space:-

A space (X, τ) is said to be semi-regular if for each semi closed set F and $x \epsilon X - F$ there exit disjoint semi open sets U and V such that $x \epsilon U \quad and \quad F \subseteq V$

- Clearly p-regular space is semi-regular as well as regular but the converse is not true in general.

Example:-

Let $X = \{a, b, c\}$ *and* $\tau = \{\varphi, \{a\}, \{b\}, \{a, b\}, X\}$. Then X is semi regular but not p-regular.

Solution:-

$\tau = \{\varphi, \{a\}, \{b\}, \{a, b\}, X\}$

Closed sets of X=$\{X, \{b, c\}, \{a, c\}, \{c\}, \varphi\}$

$$\overline{\varphi} = \varphi, \quad \overline{X} = X, \quad \overline{\{a\}} = \{a, c\}, \quad \overline{\{b\}} = \{b, c\},$$

$$\overline{\{a, b\}} = X$$

⇨ $SO(X) = \{\varphi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$
⇨ $SC(X) = \{X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}, \varphi\}$

Then for each semi closed set (say F) of X and $x \epsilon X - F$, there exit two disjoint semi open sets (say U and V) such that $x \epsilon U \quad and \quad F \subseteq V$

⇨ (X, τ) is semi regular.

Now for $\{b, c\} \epsilon SC(X)$ and $a \epsilon X - \{b, c\}$ we cannot find two open sets U and V in X such that $a \epsilon U \quad and \; \{b, c\} \subseteq V$

⇨ (X, τ) is not semi regular.

Theorem 1:-

The image of a regular space under a clopen and s-continuous surjection is p-regular space.

Proof:-

Let $F \epsilon SC(Y)$ *and* $y \epsilon Y - F$,

Let $x \epsilon f^{-1}(y)$

Since f is s-continuous therefore by a well known theorem, $f^{-1}(F)$ is closed in X and $x \epsilon X - f^{-1}(F)$.

Since X is regular therefore there exit open sets U and V in X such that.

$$x \epsilon U \quad and \quad f^{-1}(f) \subseteq V \quad and\ U \cap V = \varphi$$

Since f is closed, therefore by a well known theorem there exit an open set W of Y such that $F \subseteq W$ *and* $f^{-1}(W) \subseteq V$

Therefore, $U \cap f^{-1}(W) = \varphi \qquad \because U \cap V = \varphi\ \&\ f^{-1}(W) \subseteq V$

And hence, $f(u) \cap V = \varphi$,

Since f is open, so f(u) is open in Y. And $y \epsilon f(u) \quad \because f(x) = y\ \&\ f(x) \epsilon f(u)$

$i.e.$ there exit two open sets $f(u)$ and W in Y such that,

$$F \subseteq W \quad \& \quad y \epsilon f(u)$$

⇨ Y is p-regular.

Theorem 2:-

Let $f: X \longrightarrow Y$ be s-continuous and semi closed surjection with compact point inverses and X is a regular space, then Y is semi regular.

Proof:-

Let $C \in SC(Y)$ and $y \in Y - C$

Since f is s-continuous therefore by a well known theorem $f^{-1}(C)$ is closed in X.

Moreover, the compact sets $f^{-1}(y)$ and $f^{-1}(C)$ are disjoint in a regular space.

As X is regular space, therefore there exit two disjoint open sets F and G in X such that, $f^{-1}(y) \subseteq F$ and $f^{-1}(C) \subseteq G$

Since, f is semi closed then by a well known theorem there exit two semi open sets V and W containing y and C respectively such that,

$$f^{-1}(V) \subseteq F \quad and \quad f^{-1}(W) \subseteq G$$

Since $F \cap G = \varphi$,

⇨ $f^{-1}(V) \cap f^{-1}(W) = \varphi$
⇨ $V \cap W = \varphi$

$i.e.$ for $C \in SC(Y)$ and $y \in Y - C$, there exit two semi open sets V and W in Y such that $y \in V$ and $C \subseteq W$ $and\ V \cap W = \varphi$

⇨ Y is a semi regular space.

Corollary:-

Let $f: X \longrightarrow Y$ be s-continuous and closed surjection with compact point inverses. Then Y is p-regular if x is regular.

Proof:-

Let $C \in SC(Y)$ *and* $y \in Y - C$

Since f is semi continuous, therefore by a well known theorem $f^{-1}(C)$ is closed in X. Moreover, the compact sets $f^{-1}(y)$ *and* $f^{-1}(C)$ are disjoint in a regular space.

As X is regular space, therefore there exit two disjoint open sets F and G in X such that, $f^{-1}(y) \subseteq F$ & $f^{-1}(C) \subseteq G$

Since f is closed surjection, therefore by a well known theorem, there exit two open sets V and W in Y containing y and C respectively such that,

$$f^{-1}(V) \subseteq F \quad \& \quad f^{-1}(W) \subseteq G$$

Since, $F \cap G = \varphi$ *this implies* $f^{-1}(V) \cap f^{-1}(W) = \varphi$

And hence, $V \cap W = \varphi$ *i.e.* for $C \in SC(Y)$ & $y \in Y - C$, there exit two open sets V and W such that $y \in V$ *and* $C \subseteq W$ & $V \cap W = \varphi$

⇨ Y is a p-regular space.

Open Function (*):-

A function f is said to be open function if image of each open set is open.

Semi-Open Function (*):-

A function $f: X \longrightarrow Y$ is said to be semi-open function if image of every open set of X is semi-open in Y.

Pre Semi-Open Function (*):-

Let X and Y be topological spaces, a function $f: X \longrightarrow Y$ is said to be pre-semi-open if and only if for all $A \epsilon SoX$), $f(A) \epsilon SO(Y)$.

s-Open Function:-

A function $f: X \longrightarrow Y$ is said to be s-open if the image of every semi-open set is open.

- ❖ It is known that every s-open function is open, semi-open and pre-semi-open.

Theorem 3:-

For a function $f: X \longrightarrow Y$, the following are equivalent.

1) f is s-open
2) $f[sInt(A)] \subseteq Intf(A)$ $\quad for\ each\ A \subseteq X$
3) $sInt[f^{-1}(B)] \subseteq f^{-1}Int(B)$ $\quad for\ each\ B \subseteq Y$
4) $f^{-1}[Cl(B)] \subseteq sClf^{-1}(B)$ $\quad for\ each\ B \subseteq Y$
5) $f^{-1}[Bd(B)] \subseteq sBd[f^{-1}(B)]$ $\quad for\ each\ B \subseteq Y$

Proof:-

①⟹② Obviously $f[sInt(A)] \subseteq f(A)$

Now $sInt(A)$ is a semi open set in X.

- ⇨ $f[sInt(A)]$ is open in Y $\quad \because f$ is s-open.
- ⇨ $f[sInt(A)]$ is open subset of $f(A)$ in Y, But $Int(A)$ is the largest open set contained in $f(A)$

⇨ $f[sInt(A)] \subseteq Int\, f(A)$

②⟹③ For any B ⊆ Y, put $f^{-1}(B) = A \subseteq X$

Then by ②, $f[sIntf^{-1}(B)] \subseteq Intff^{-1}(B) \subseteq Int(B)$

⇨ $f[sIntf^{-1}(B)] \subseteq Int(B)$
⇨ $sIntf^{-1}(B) \subseteq f^{-1}[Int(B)]$

③⟹④ By 3 $sIntf^{-1}(B) \subseteq f^{-1}[Int(B)]$

⇨ $X - f^{-1}[Int(B)] \subseteq X - sIntf^{-1}(B) = sCl[X - f^{-1}(B)$
⇨ $f^{-1}(Y) - f^{-1}[Int(B)] \subseteq sCl[f^{-1}(Y) - f^{-1}(B)]$ $\quad \because X = f^{-1}(Y)$
⇨ $f^{-1}[Y - Int(B)] \subseteq sClf^{-1}[Y - B]$
⇨ $f^{-1}Cl[Y - B] \subseteq sClf^{-1}[Y - B]$
⇨ $f^{-1}Cl(C) \subseteq sClf^{-1}(C),$ $\quad where\ Y - B = C \in Y$

④⟹⑤ For B ⊆ Y,

$Bd(B) = Cl(B) \cap Cl(Y - B)$ is closed set in Y.

Now, $f^{-1}Bd(B) = f^{-1}Cl(B) \cap f^{-1}Cl(Y - B)$

$\subseteq sClf^{-1}(B) \cap sClf^{-1}(Y - B)$ $\quad by$ ②

$= sClf^{-1}(B) \cap [sClf^{-1}(Y) - sClf^{-1}(B)]$

$= sClf^{-1}(B) \cap [sCl(X) - sClf^{-1}(B)]$

⇨ $f^{-1}Bd(B) \subseteq sClf^{-1}(B) \cap sCl[X - f^{-1}(B)] = sBd(B)$
⇨ $f^{-1}Bd(B) \subseteq sBdf^{-1}(B)$

⑤⟹① Let U be an arbitrary open set in X,

Put $Y - f(U) = B$

Now we show that B is closed in Y.

By 5, $U \cap f^{-1}Bd(B) \subseteq U \cap sBdf^{-1}(B)$

⇨ $f[U \cap f^{-1}Bd(B)] \subseteq f[U \cap sBdf^{-1}(B)]$

Since $f(U) \cap Bd(B) = f[U \cap f^{-1}Bd(B)]$

Therefore we have,

$f(U) \cap Bd(B) \subseteq f[U \cap sBdf^{-1}(B)]$ ---------------------- ❶

$B = Y - f(U)$ gives,

$f^{-1}(B) = f^{-1}[Y - f(U)] = f^{-1}(Y) - f^{-1}f(U)$

$\subseteq X - U \qquad \because U \subseteq f^{-1}f(U) \Longrightarrow (f^{-1}f(U))' \subseteq U'$

⇨ $f^{-1}(B) \subseteq X - U$
⇨ $sClf^{-1}(B) \subseteq sCl(X - U) = X - sInt(U) = X - U \quad \because U$ *is semi open.*
⇨ $sClf^{-1}(B) \subseteq X - U$
⇨ $sClf^{-1}(B) \cap U = \varphi$ -------------------------------- ❷

Now, $U \cap sBdf^{-1}(B) = U \cap \left[sClf^{-1}(B) \cap sCl\left(X - f^{-1}(B)\right)\right]$

$= U \cap sClf^{-1}(B) \cap sCl[X - f^{-1}(B)]$

$= \varphi \cap sCl[X - f^{-1}(B)] \qquad by$ ❷

$= \varphi$

Using $U \cap sBdf^{-1}(B) = \varphi$, ❶ *becomes*

$f(U) \cap Bd(B) \subseteq \varphi$

⇨ $f(U) \cap Bd(B) = \varphi$
⇨ $Bd(B) \subseteq Y - f(U) = B$
⇨ B contains all of its boundary points.
⇨ B is closed.
⇨ $f(U)$ is open in Y.

This proves that f is s-open function.

PREPAIRED BY
MUHAMMAD TAHIR
M.S. MATHEMATICS

Theorem 4:-

For any function $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$, we have,

(1) $g_{\circ}f$ is s-open if f is s-open and g is open.
(2) $g_{\circ}f$ is s-open if f is pre semi open and g is s-open.
(3) $g_{\circ}f$ is open if f is semi open and g is s-open.
(4) $g_{\circ}f$ is pre semi open if f is s-open and g is semi open.

Proof:-

Proves of these statements are obvious by definition.

---

s-Closed Function:-

A function $f: X \longrightarrow Y$ is said to be s-closed if the image o every semi-closed set is closed.

Theorem 5:-

A function $f: X \longrightarrow Y$ is s-closed if and only if $Clf(A) \subseteq f[sCl(A)]$, for each $A \subseteq X$.

Proof:-

Let f is s-closed.

Obviously, $f(A) \subseteq f[sCl(A)]$

Now, $sCl(A)$ is semi closed in X.

⇨ $f[sCl(A)]$ is closed in Y. $\because f \; is \; s - closed$
⇨ $f[scl(A)]$ is closed superset of A.

But $Clf(A)$ is the smallest closed set containing $f(A)$

⇨ $Clf(A) \subseteq f[sCl(A)]$

Conversely,

Let $A \epsilon SC(X)$

We show that $f(A)$ is closed in Y.

By hypothesis, $Clf(A) \subseteq f[sCl(A)] = f(A)$ $\because A\epsilon SC(X)$

⇨ $Clf(A) \subseteq f(A)$ ------------------------------------ ❶

But, $f(A) \subseteq Clf(A)\ (always)$ ----------------------- ❷

By relation ❶ and ❷ $f(A) = Clf(A)$

⇨ $f(A)$ is closed.
⇨ f is s-closed. (The proof)

Theorem 6:-

A surjection function $f: X \longrightarrow Y$ is s-closed if and only if for each subset B in Y and each semi closed set U in X containing $f^{-1}(B)$, there exit an open set V in Y containing B such that, $f^{-1}(V) \subseteq U$.

Proof:-

Let U be an arbitrary open set in X containing $f^{-1}(B)$,

Where B ⊆ Y.

Clearly, $Y - f(X - U) = V\ (say)$ is open in Y.

Since $f^{-1}(B) \subseteq U\ and\ f$ is onto, then simple calculations gives, B ⊆ V.

Moreover, we have

$$f^{-1}(V) \subseteq X - f^{-1}[f(X - U)] \subseteq U$$

⇨ $f^{-1}(V) \subseteq U$

Conversely,

Let F be an arbitrary semi closed set in X and $y \in Y - f(F)$

Then, $f^{-1}(Y) \subseteq f^{-1}[Y - f(F)]$

⇨ $f^{-1}(y) \subseteq X - f^{-1}f(\mathrm{F}) \subseteq X - F$
⇨ $f^{-1}(y) \subseteq X - F$

Since $X - F$ is semi open, therefore there exit an open set V_y containing y such that, $f^{-1}(V_y) \subseteq X - F$.

⇨ $y \in V_y \subseteq Y - f(F)$
⇨ $Y - f(F) = \cup\{V_y : y \in Y - f(F)\}$ is open in Y.
⇨ $f(F)$ is closed in Y.
⇨ f is s-closed. (This completes the proof).

Remark 1:-

If $f: X \longrightarrow Y$ is s-continuous and closed (or irresolute and s-closed) surjection, then using theorem 2.2(iii) [2], one can easily see that the class SC(X) and C(X) (closed sets of X) coincide.

Remark 2:-

In general, an s-open function need not be s-closed.

Example:-

Let $X = \{a, b, c\}$, $\tau_x = \{\varphi, \{a\}, \{b\}, \{a, b\}, X\}$

And $Y = \{a, b, c, d\}$ $\quad and \quad$ $\tau_y = \{\varphi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, Y\}$

Let $f: X \longrightarrow Y$ be an identity function. Then f is s-open but not s-closed.

Remark 3:-

However, for bijection, it is easily seen that the notations of s-open and s-closed coincides. Moreover, f is s-open if and only if f^{-1} is s-continuous.

Proof:-

Let $f: X \longrightarrow Y$ is s-open.

⇨ Image of every semi open set of X is open in Y.

As image of every semi-open set is open under f.

⇨ By a well known theorem f^{-1} is s-continuous. (Since f is s-continuous if inverse image of every semi-open set is open).

Conversely,

Let f^{-1} is s-continuous.

⇨ Image of every semi-open set of X is open in Y under f.
⇨ f is s-open.

s-Closed Space:-

A space X is said to be s-closed if for every semi-open cover of X, there exit a finite subfamily such that the union of their semi-closures cover X.

Compact Space (*):-

A topological space (X, τ) is said to be compact if every open cover for X has a finite sub cover.

Semi-Compact Space:-

A topological space (X, τ) is said to be semi-compact, if for every semi open cover of X, there exit a finite sub family such that there union cover X.

Almost Compact Space:-

A topological space (X, τ) is said to be almost compact if for every open cover of X, there exit a finite sub family such that union of their closures cover X.

Note:-

Every compact space is almost compact, as well as semi-compact.

❖ Moreover, every semi-compact space is s-closed.

Theorem 7:-

The inverse image of an almost compact space under s-open bijection is s-closed.

Proof:-

Let $\{V_\alpha : \alpha \in I\}$ be semi open cover for X.

⇨ $\bigcup_{\alpha \epsilon I} V_\alpha = X$

As V_α are semi-open in X and $f: X \longrightarrow Y$ is s-open.

⇨ $f(V_\alpha): \alpha \epsilon I$ are open in Y.
⇨ $\bigcup_{\alpha \epsilon I} f(V_\alpha) = f(X)$
⇨ $\bigcup_{\alpha \epsilon I} f(V_\alpha) = Y$
⇨ $f(V_\alpha)$ is an open cover for Y.

As Y is almost compact, therefore there exit finite sub family of $\bigcup_{\alpha \epsilon I} f(V_\alpha)$ such that the union of their closures cover Y.

⇨ $\bigcup_{i=1}^{N} Cl[f(V_\alpha)] = Y$
⇨ $Y = \bigcup_{i=1}^{N} Cl[f(V_\alpha)]$
⇨ $f^{-1}(Y) = f^{-1}[\bigcup_{i=1}^{N} Clf(V_\alpha)]$
⇨ $X = f^{-1}[\bigcup_{i=1}^{N} Clf(V_\alpha)] \subseteq f^{-1}[\bigcup_{i=1}^{N} f\{sCl(V_\alpha)\}]$
⇨ $X \subseteq f^{-1} f \bigcup_{i=1}^{N} sCl(V_\alpha) \subseteq \bigcup_{i=1}^{N} sCl(V_\alpha)$
⇨ $X \subseteq \bigcup_{i=1}^{N} sCl(V_\alpha)$

As $\bigcup_{\alpha \epsilon I} (V_\alpha)$ is semi-open cover for X and we have find a finite sub family such that union of their semi closures cover X.

⇨ X is s-closed.

PREPAIRED BY:-

MUHAMMAD TAHIR

M.S Mathematics

COMSATS Institute of Information technology Islamabad.

0344-8563284

s-Regular Space:-

A topological space (X,τ) is said to be s-regular if for each closed set F and $X \epsilon X - F$, there exit semi-open sets U and V in X such that $x \epsilon U, \quad F \subseteq V \quad and \quad U \cap V = \varphi$.

- Every regular space is s-regular.
- Every semi-regular space is s-regular.

Almost Regular Space:-

A topological space (X,τ) is said to be almost regular space if for each regular closed set F and $x \epsilon X - F$, there exit open sets U and V such that, $x \epsilon U, \quad F \subseteq V \quad and \quad U \cap V = \varphi$.

- F is regular closed in (X,τ) if $F = Cl[Int(F)]$
- F is regular open in (X,τ) if $F = Int[Cl(F)]$
- Every regular closed set is closed and semi-open.
- A set which is semi-closed as well as semi open is called semi-regular set.

Semi Compact/s-Compact Space:-

A topological space (X,τ) is called s-compact if for every cover $\{U_\alpha : \alpha \epsilon \nabla\}$ of X by sets $U_\alpha \epsilon SO(X)$, there exit a finite subset $\nabla_\circ$ of ∇ such that $X = \bigcup_{\alpha \epsilon \nabla_\circ} U_\alpha$

Theorem:-

Let (X,τ) be a topological space, prove that an s-compact set A and disjoint regular closed set B in an s-regular space can be separated by semi-open sets.

Proof:-

Let $a \epsilon A$

Since X is s-regular and B is a regular closed set such that $a \epsilon X - B$,

Therefore there exit semi-open sets $G_\alpha \text{ and } H_\alpha \text{ s.t.}$

$$a \epsilon G_\alpha; \quad B \subseteq H_\alpha \quad \text{and} \quad G_\alpha \cap H_\alpha = \varphi$$

Clearly, $\{G_\alpha : \alpha \epsilon A\}$ is a cover of A by semi-open sets of X.

Since A is s-compact, therefore there exit a finite sub collection (say)

$$G_{\alpha_1}, G_{\alpha_2}, G_{\alpha_3}, \dots \dots, G_{\alpha_n} \quad s.t.$$

$$A \subseteq \bigcup_{i=1}^{n} G_{\alpha_i} = G \epsilon SO(X)$$

Now corresponding to these $\alpha_i; i = 1,2,3, \dots, n$ we have H_{α_i} $s.t.$ $B \subseteq H_{\alpha_i}$ for each $i = 1,2,3, \dots, n$

⇨ $B \subseteq H_{\alpha_1} \cap H_{\alpha_2} \cap \dots \cap H_{\alpha_n}$
⇨ $B = sInt(B) \subseteq sInt\left[H_{\alpha_1} \cap H_{\alpha_2} \cap \dots \cap H_{\alpha_n}\right] \quad \because B \text{ is semi} - \text{open}$

$= H$

⇨ $B \subseteq H \epsilon SO(X);$ H is semi open.

Consequently, G and H are required disjoint semi-open sets.

Completely Continuous Function:-

A function $f:(X,\tau_x)\longrightarrow(Y,\tau_y)$ is said to be completely continuous if $f^{-1}(V)\,\epsilon\,RO(X)$ for each open set V in Y.

Note:-

$f:(X,\tau_x)\longrightarrow(Y,\tau_y)$ is completely continuous if and only if $f^{-1}(V)\,\epsilon\,RC(X)$ for each closed set B in Y.

Theorem:-

Let $f:X\longrightarrow Y$ be a completely continuous and semi closed surjection with s-compact point inverses, if X is s-regular then Y is s-regular.

Proof:-

Let X be s-regular.

Let F be a closed set and $y\,\epsilon\,Y-F, then\ f^{-1}(V)\,\epsilon\,RC(X)\ \&\ f^{-1}(V)$ is s-compact.

Clearly, $f^{-1}(y)\notin f^{-1}(F)$

Since X is s-regular, therefore there exit semi-open sets $U_y\ and\ U_F$ in X such that $f^{-1}(y)\,\epsilon\,U_y\ \&\ f^{-1}(F)\subseteq U_F\quad and\quad U_y\cap U_F=\varphi$

Since f is semi closed preserving, therefore there exit semi open sets $V_y\ and\ V_F\ \ s.t.\quad y\,\epsilon\,V_y\ \ and\ \ F\subseteq V_F$

And $f^{-1}(V_y)\subseteq U_y\quad and\quad f^{-1}(V_F)\subseteq U_F\quad and\quad U_y\cap U_F=\varphi$

Gives $V_y\cap V_F=\varphi$

This proves that Y is s-regular.

Theorem:-

Let (X, τ_x) be a topological space then X is s-regular if and only if for each open set V containing $x \in X$, there exit a semi open set U containing x such that $x \in U \subseteq sCl(U) \subseteq V$.

Proof:-

Let (X, τ_x) be s-regular space and V is an open set containing x $i.e. x \in V$.

⇨ $x \notin X - V \quad (closed\ set)$

Since space is s-regular, therefore there exit $U, L \in SO(X)$ $s.t.$

$$x \in U \ , \quad X - V \subseteq L$$

⇨ $X - L \subseteq V \quad and \quad U \cap L = \varphi$
⇨ $U \subseteq X - L \quad (semi - closed)$
⇨ $sCl(U) \subseteq X - L \quad \because X - L$ is semi closed.

Thus, $\quad x \in U \subseteq sCl(U) \subseteq X - L \subseteq V$

⇨ $x \in U \subseteq sCl(U) \subseteq V \quad (proved)$

Conversely,

We prove that X is s-regular.

Let F be a closed subset of X and $x \notin F \quad \Longrightarrow \quad x \in X - F$,

Where $X - F$ is open in X.

By hypothesis, there exit a semi-open set U in X containing x such that,

$$x \in U \subseteq sCl(U) \subseteq X - F$$

⇨ $x \in U \quad and \quad F \subseteq X - sCl(U) \quad (semi - open\ set)$

Let $V = X - sCl(U)$,

Then, $x \epsilon U$, $F \subseteq V$ *and* $U \cap V = \varphi$

⇨ X is s-regular.

Theorem:-

Let $f: (X, \tau_x) \longrightarrow (Y, \tau_y)$ be a continuous and semi closed preserving surjection. If f is s-regular then Y is s-regular.

Proof:-

Let X be s-regular.

Let U be an open set in Y such that $y \epsilon U$

Let $x \epsilon f^{-1}(y)$. Now $f^{-1}(U)$ is open in X and $x \epsilon f^{-1}(U)$

Since X is s-regular, therefore there exit $V \epsilon SO(X, x)\ s.t.$

$$x \epsilon V \subseteq sCl(V) \subseteq f^{-1}(U)$$

⇨ $f(x) \epsilon f(V) \subseteq fsCl(V) \subseteq ff^{-1}(U) \subseteq U$

Where $f(V)$ is semi-open and, $sCl[f(U)] \subseteq f[sCl(V)]$

Thus, $y \epsilon f(V) \subseteq sCl[f(V)] \subseteq f[sCl(V)] \subseteq U$

⇨ $y \epsilon f(V) \subseteq sCl[f(V)] \subseteq U$

His proves that Y is s-regular.

Prove That:-

$$sBd[sBd\{sBd(A)\}] = sBd[sBd(A)]$$

Proof:-

$$sBd[sBd\{sBd(A)\}] = sCl[sBd\{sBd(A)\}] \cap sCl[X - sBd\{sBd(A)\}]$$

$$= sBd[sBd(A)] \cap sCl[X - sBd\{sBd(A)\}] \text{ ------ } ❶$$

Consider, $X - sBd[sBd(A)] = X - [sCl\{sBd(A)\} \cap sCl\{X - sBd(A)\}]$

$$= X - [sBd(A) \cap sCl\{X - sBd(A)\}]$$

$$\because sBd(A) \text{ is semi} - \text{closed}$$

$$= [X - sBd(A)] \cup [X - sCl\{X - sBd(A)\}]$$

Now, $sCl[X - sBd\{sBd(A)\}] = sCl[\{X - sBd(A)\} \cup \{X - sCl(X - sBd(A))\}]$

$$= sCl[X - sBd(A)] \cup sCl[X - sCl\{X - sBd(A)\}]$$

$$= D \cup sCl(X - D) = X$$

Where, $D = sCl[X - sBd(A)]$

⇨ $sCl[X - sBd\{sBd(A)\}] = X$ ----------------------- ❷

By equation ❶ and ❷

$$sBd[sBd\{sBd(A)\}] = sBd[sBd(A)] \cap X$$

⇨ $sBd[sBd\{sBd(A)\}] = sBd[sBd(A)]$ *(proved)*

---

s-Closed Space:-

A topological space (X, τ) is said to be s-closed if for every cover $\{V_\alpha : \alpha \in \nabla\}$ of X by sets V_α semi open in X for each $\alpha \in \nabla$, there exit a finite subset $\nabla_\circ$ *of* ∇ *s.t.* $X = \cup_{\alpha \in \nabla_\circ} sCl(V_\alpha)$

S-Closed Space:-

A topological space (X, τ) is said to be S-closed if for each covering $\{V_\alpha : \alpha \in \nabla\}$ of X by semi-open sets of X, there exit a finite subset $\nabla_\circ$ of ∇ *s.t.* $X = \cup_{\alpha \in \nabla_\circ} Cl(V_\alpha)$

Note:-

Every S-closed space is s-closed and every s-closed space is s-compact and every s-compact space is compact.

s-Regular Space:-

(Already defined)

Theorem:-

A topological space (X, τ) is s-closed if and only if every proper semi-regular subset of X is s-closed relative to X.

Proof:-

Let (X, τ) be s-closed space. And $G \subseteq F$ be a proper semi-regular subset of X.

We prove that G is s-closed relative to X.

Let $\{V_\alpha : \alpha \in \nabla\}$ be a cover for G, where $V_\alpha \in SO(X)\ \forall\ \alpha \in \nabla$

⇨ $G \subseteq \bigcup_{\alpha \in \nabla} V_\alpha$

⇨ $X = \bigcup_{\alpha \in \nabla} V_\alpha \cup (X - G),\ where\ \ X - G \in SO(X)$

Since X is s-closed, therefore there exit a finite sub set $\nabla_\circ\ of\ \nabla\ \ s.t.$

$$X = \bigcup_{\alpha \in \nabla_\circ} sCl(V_\alpha) \cup sCl(X - G)$$

⇨ $G \subseteq \bigcup_{\alpha \in \nabla_\circ} sCl(V_\alpha)$

⇨ G is s-closed relative to X.

Conversely,

Let every proper semi-regular subset of X be s-closed relative to X.

We prove that X is s-closed.

Let $\{V_\alpha : \alpha \epsilon \nabla\}$ be a cover for X by sets semi-open in X.

For some $\beta \epsilon \nabla$, $sCl(V_\beta) \epsilon SR(X)$

Let $G = sCl(V_\beta) \epsilon SR(X)$

⇨ $X - G \epsilon sR(X)$

By hypothesis, $X - G$ is s-closed relative to X.

Since, $X - G \subseteq \cup \{V_\alpha : \alpha \epsilon \nabla\}$

Hypothesis ⟹ $X - G = \bigcup_{\alpha \epsilon \nabla_\circ} sCl(V_\alpha)$ $\quad$ *for some finite set* $\nabla_\circ$ *of* ∇

⇨ $X = \bigcup_{\alpha \epsilon \nabla_\circ} sCl(V_\alpha) \cup sCl(V_\beta)$

$= \bigcup_{\alpha \epsilon \nabla_\circ \cup \{\beta\}} sCl(V_\alpha)$

This proves that X is s-closed space.

Exercise:-

Let A and B be subsets of a topological space (X, τ) such that $A \subseteq B \subseteq X$ and $B \epsilon SO(X)$. If A is s-closed relative to X then prove that A is s-closed relative to B.

Proof:-

Let $\{V_\alpha : \alpha \epsilon \nabla\}$ be a cover for A, where $V_\alpha \epsilon SO(B)$ $\forall\ \alpha \epsilon \nabla$

⇨ $A \subseteq \bigcup_{\alpha \epsilon \nabla} V_\alpha$

As $B \epsilon SO(X) \implies A \subseteq \bigcup_{\alpha \epsilon \nabla} V_\alpha$ $\quad s.t.\ V_\alpha \epsilon SO(X)\ \forall\ \alpha \epsilon \nabla$

As A is s-closed relative to X, therefore there exit a finite subset $\nabla_{\circ}$ $of\ \nabla$ such that, $A = \cup_{\alpha \epsilon \nabla_{\circ}} sCl(V_{\alpha})$

⇨ $A \cap B = \cup_{\alpha \epsilon \nabla_{\circ}} sCl(V_{\alpha}) \cap B$

⇨ $A = \cup_{\alpha \epsilon \nabla_{\circ}} sCl_B(V_{\alpha})$, $\quad where \ \ V_{\alpha} \epsilon SO(B)$

⇨ A is s-closed relative to B.

PREPAIRED BY:-

MUHAMMAD TAHIR

M.S Mathematics

COMSATS Institute of Information technology Islamabad.

0344-8563284

Almost Open Mapping:-

A mapping $f:(X,\tau_x) \longrightarrow (Y,\tau_y)$ is said to be almost open if for every open set U of Y,

$$f^{-1}[Cl(U)] \subseteq Cl[f^{-1}(U)]$$

Note:-

- Every open mapping is almost open mapping. The converse is not true in general.
- Composition of two almost open mappings is not almost open mapping in general.

Example:-

Let $X = Y = Z = \{a,b,c\}$

$$\tau_x = \{\varphi,\{a\},\{a,b\},\{a,c\},X\}, \quad \tau_y = \{\varphi,\{a\},\{a,b\},Y\}$$

$$\tau_z = \{\varphi,\{c\},Z\}$$

$f: X \longrightarrow Y$ be identity mapping.

$g: Y \longrightarrow Z$ be defined by $g(a) = b, g(b) = c, g(c) = c$

Then f & g are almost open mappings but $g_{\circ}f$ is not almost open.

Almost Closed Mapping:-

A mapping $f:(X,\tau_x) \longrightarrow (Y,\tau_y)$ is said to be almost closed if for every closed set V of Y,

$$Int[f^{-1}(V)] \subseteq f^{-1}[Int(V)]$$

Theorem:-

Let $f:(X,\tau_x) \longrightarrow (Y,\tau_y)$ be almost open mappings, prove that $g_\circ f$ is almost open if g is continuous.

Proof:-

Let U be an open set of Z.

As $g:Y \longrightarrow Z$ is continuous so $g^{-1}(U)$ is open in Y.

Now as $f:X \longrightarrow Y$ is almost open mapping and $g^{-1}(U)$ is open in Y.

⇨ $f^{-1}[Cl\{g^{-1}(U)\}] \subseteq Cl[f^{-1}\{g^{-1}(U)\}]$ ------------------ ⊛

Since $g:Y \longrightarrow Z$ is almost open mapping and U is open in Z.

⇨ $g^{-1}[Cl(U)] \subseteq Cl[g^{-1}(U)]$
⇨ $f^{-1}\{g^{-1}Cl(U)\} \subseteq f^{-1}[Cl\{g^{-1}(U)\}]$

Put in equation ⊛ implies.

$$f^{-1}\{g^{-1}Cl(U)\} \subseteq f^{-1}[Clg^{-1}(U)] \subseteq Cl[f^{-1}g^{-1}(U)]$$

⇨ $f^{-1}[g^{-1}Cl(U)] \subseteq Cl\left[f^{-1}\left(g^{-1}(U)\right)\right]$
⇨ $(f^{-1}{}_\circ g^{-1})Cl(U) \subseteq Cl(f^{-1}{}_\circ g^{-1})(U)$
⇨ $(g_\circ f)^{-1}Cl(U) \subseteq Cl(g_\circ f)^{-1}(U)$

Now as U is open set in Z and,

$$(g_\circ f)^{-1}Cl(U) \subseteq Cl[(g_\circ f)^{-1}(U)]$$

⇨ $g_\circ f$ is an almost open mapping.

s-Normal Space:-

A topological space (X, τ) is said to be s-normal if for every pair of disjoint closed sets A and B of X, there exit disjoint semi-open sets U and V such that, $A \subseteq U$, $B \subseteq V$

Note:-

A ⊆ X is semi closed in X $iff \quad Int[Cl(A)] = Int(A)$

Theorem:-

Let $f: X \longrightarrow Y$ be a continuous semi-closed function. If X is normal then Y is s-normal.

Proof:-

Let F_1 and F_2 be disjoint closed sets of Y.

Since f is continuous therefore $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are disjoint closed sets of X.

As X is normal, therefore there exit disjoint open sets U_1 and U_2 in X such that, $f^{-1}(F_1) \subseteq U_1 \quad \& \quad f^{-1}(F_2) \subseteq U_2 \quad and \quad U_1 \cap U_2 = \varphi$

Since f is semi-closed, therefore there exit two semi open sets V_1 and V_2 in Y containing F_1 and F_2 respectively such that,

$$f^{-1}(V_1) \subseteq U_1 \quad and \quad f^{-1}(V_2) \subseteq U_2$$

Since, $U_1 \cap U_2 = \varphi$

⇨ $f^{-1}(V_1) \cap f^{-1}(V_2) = \varphi$
⇨ $V_1 \cap V_2 = \varphi$

That is for two disjoint closed sets F_1 and F_2 of Y there exit two semi-open sets V_1 and V_2 in Y such that $F_1 \subseteq V_1$ *and* $F_2 \subseteq V_2$ *and* $V_1 \cap V_2 = \varphi$

⇨ Y is s-normal-

Semi T_2-Space:-

A topological space (X, τ_x) is said to be semi T_2-space if for $x_1, x_2 \epsilon X\ s.t. x_1 \neq x_2$, there exit semi open sets U and V of X such that,

$$x_1 \epsilon U \quad \& \quad x_2 \epsilon V \qquad and \qquad U \cap V = \varphi$$

PREPAIRED BY:-

MUHAMMAD TAHIR

M.S Mathematics

COMSATS Institute of Information technology Islamabad.

0344-8563284