Rigid Body Motion. CH#1

→ MECHANICS:- Mechanics is the science of motion. It studies states of rest and motion and the laws governing rest, equilibrium and motion.

* Classical Mechanics:- It deals with macroscopic objects.
* Quantum Mechanics:- It deals with microscopic objects.

→ Divisions of Classical Mechanics:-

1) Mechanics of Particles And Rigid Bodies:- It is based on Newton's laws. Basic concepts & terms are space, time, mass, particle and body, force and energy, velocity, momentum and acceleration.

2) Mechanics of Fluids:- It is also based on Newton's laws and their extension and deals with the behaviour of fluids (Liquids and gases) in motion. Its two well known branches are Hydrodynamics (for liquids) and Aerodynamics (for gases).

3) Mechanics of Elastic Solids:- It deals with the behaviour of solids when they undergo with deformation under force.

4) The Newtonian Mechanics:- The two most fundamental concepts of Newtonian Mechanics are those of particles and rigid body.

* Particle:- A particle is a point mass that is a piece of matter having a definate mass but no size. i.e a geomatrical point. It is clear from the definition that a

particle is an obstraction or idealization. In actual practice any body whose size is very small as compared to the sizes of other bodies being studies along with. It is considered a particle.

★ **Rigid Body:-** A rigid body is defined as a collection of particles s.t the distance b/w every pair of its particle remain unchanged whatever the force acting on it. Clearly like particle a rigid body is also an obstraction or idealization

⇒ **Centre of mass of a System of particles:-**
The centre of mass or centroid of a system of particles is a hypothetical particle s.t if the entire mass of the system we concentrated there, The mechanical property would remain the same. Let there be a system of "n" particles with masses $m_1, m_2, \ldots, m_n$ and at any time "t" theire position vectors are $r_1, r_2, \ldots, r_n$ respectively. Then the position vector of the centre of mass is given by

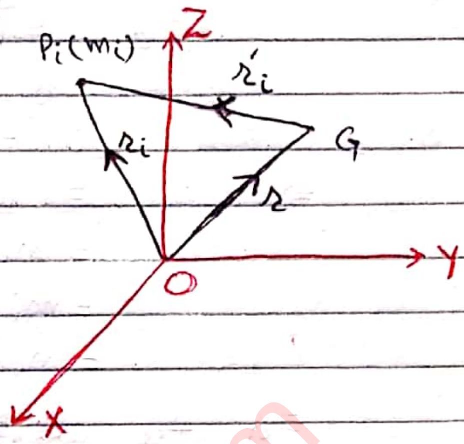
$$r_c = \frac{\sum_i m_i r_i}{\sum_i m_i}$$

For the moving system

$$\dot{r}_c = V_c = \frac{\sum_i m_i \dot{r}_i}{\sum_i m_i}$$

and $$\ddot{r}_c = \dot{V}_c = a_c = \frac{\sum_i m_i \ddot{r}_i}{\sum_i m_i}$$

Now again consider a system of "n" particles with "O" as origin of refrence, G as centroid centre of mass. Let r_i denotes the position vector of ith particle w.r.t O, m_i denotes the mass of ith particle r_i' denotes the position vector of ith particle w.r.t G and r denotes the position vector of G w.r.t O. The by law of vector addition



$$\underline{r} + r_i' = r_i$$

$$\Rightarrow \sum_i m_i \underline{r} + \sum_i m_i r_i' = \sum_i m_i r_i$$

$$\Rightarrow M\underline{r} + \sum_i m_i r_i' = \sum_i m_i r_i \quad \because \sum_i m_i = M \quad \text{---} \circledast$$

where M is the mass of whole system.

$$\text{Now} \quad \underline{r} = \frac{\sum_i m_i r_i}{\sum_i m_i} = \frac{\sum_i m_i r_i}{M}$$

So from $\circledast$ $\quad \cancel{M}\left[\frac{\sum_i m_i r_i}{\cancel{M}}\right] + \sum_i m_i r_i' = \sum_i m_i r_i$

$$\Rightarrow \sum_i \cancel{m_i r_i} + \sum_i m_i r_i' = \sum_i \cancel{m_i r_i}$$

$$\Rightarrow \boxed{\sum_i m_i r_i' = 0}$$

Similarly $\boxed{\sum_i m_i v_i' = 0}$ & $\boxed{\sum_i m_i a_i' = 0}$

Angular Momentum:- Moment of momentum is called Angular momentum it is usually denoted by $\underline{L}$ or λ or $\underline{J}$

If for a system of particles m_i, v_i, r_i denotes the mass, velocity and position vector of the ith particle then its angular momentum $\mathcal{L}_i$ is given by

$$\underline{\mathcal{L}_i} = r_i \times m_i \underline{v_i}$$

Now if angular velocity of the system is $\underline{\omega}$ then $\underline{v_i} = \underline{\omega} \times r_i$

$$\Rightarrow \mathcal{L}_i = r_i \times m_i(\underline{\omega} \times r_i)$$
$$= m_i[r_i \times (\omega \times r_i)]$$
$$= m_i[(r_i \cdot r_i)\underline{\omega} - (r_i \cdot \underline{\omega})\underline{r_i}] \quad \because a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$
$$= m_i[r_i^2 \underline{\omega} - (\bar{r}_i \cdot \underline{\omega}) \underline{r_i}]$$

For the whole system

$$\mathcal{L} = \sum_i m_i[r_i^2 \underline{\omega} - (r_i \cdot \underline{\omega}) r_i]$$

⇒ Parallel axes Theorem for Angular Momentum:-

The total angular momentum of the system w.r.t origin is the sum of angular momentum of the system w.r.t centre of mass and angular momentum of the centre of mass w.r.t origin. i.e with usual meanings

$$\underline{\mathcal{L}} = \underline{\mathcal{L}'} + \underline{\mathcal{L}_c}$$

{ $\underline{\mathcal{L}}$ = Angular momentum of system w.r.t origin

{ $\underline{\mathcal{L}'}$ = A.M of sys w.r.t centre of mass

{ $\underline{\mathcal{L}_c}$ = A.M of centre of mass w.r.t origin.

Proof) Let us consider a system of particles with O as origin of refrence, G as centre of mass of the system and m_i denotes the mass of ith particle. Next let r_i, v_i denotes the position vector & velocity of ith particle w.r.t O, r_i', v_i' denotes the position vector & velocity of ith particle w.r.t G and r_c, v_c denotes the position vector & velocity of G w.r.t O

then $r_i = r_i' + r_c$

$$\Rightarrow \dot{r}_i = \dot{r}_i' + \dot{r}_c \quad \Rightarrow V_i = V_i' + V_c$$

Now $\mathcal{L} = \sum_i m_i [r_i \times v_i]$

$$\mathcal{L} = \sum_i m_i [r_i \times \dot{r}_i] = \sum_i m_i [(r_i' + r_c) \times (\dot{r}_i' + \dot{r}_c)]$$

$$= \sum_i m_i [r_i' \times \dot{r}_i' + r_i' \times \dot{r}_c + r_c \times \dot{r}_i' + r_c \times \dot{r}_c]$$

$$= \sum_i r_i' \times m_i \dot{r}_i' + \sum_i m_i r_i' \times \dot{r}_c + r_c \times \sum_i m_i \dot{r}_i' + (\sum_i m_i) r_c \times \dot{r}_c$$

$$= \sum_i r_i' \times m_i v_i' + 0 \times \dot{r}_c + r_c \times 0 + M r_c \times V_c$$

$$= \sum_i r_i' \times m_i v_i' + M r_c \times V_c$$

$$\mathcal{L} = \mathcal{L}' + \mathcal{L}_c$$

($T = T' + T_c$ H.W) Rigid body motion Translation, Rotation

Question:- Show that $T = T_c + T'$ $[T = K.E]$

Solution

$$T = \frac{1}{2} \sum_i m_i v_i^2$$

$$= \frac{1}{2} \sum_i m_i (v_i' + V_c)^2$$

$$= \frac{1}{2} \sum_i m_i (v_i' + V_c)(v_i' + V_c)$$

$$= \frac{1}{2} \sum_i m_i (v_i'^2 + V_c^2 + 2 V_c v_i')$$

$$= \frac{1}{2} \sum_i m_i v_i'^2 + \frac{1}{2} \sum_i m_i V_c^2 + 2 V_c \frac{1}{2} \sum m_i v_i'$$

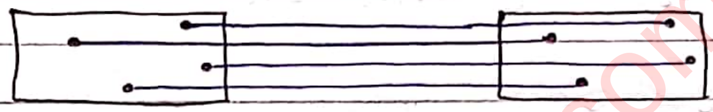
$$= \frac{1}{2} \sum_i m_i v_i'^2 + \frac{1}{2} \sum_i m_i V_c^2 + 0$$

$$= T' + T_c$$

$$\Rightarrow T = T_c + T'$$

⇒ Rigid Body Motion:-

★ Translation:- If during the displacement all particles of the body are displaced by the same amount and the line segments joining the initial & final positions of the particles are represented by || vectors, the displacement is called translation.



★ Rotation:- If during the displacement the points of the rigid body on some line remain fixed, and all others are displaced through the same angle. The displacement is called rotation

★ Linear Velocity:- If δx is the linear displacement of the rigid body in time interval δt then l.v $= \underline{v} = \lim_{\delta t \to 0} \delta x / \delta t$

★ Angular Velocity:- If $\delta\theta$ is the angular displacement of rigid body in time interval δt, then angular speed $= \lim_{\delta t \to 0} \frac{\delta\theta}{\delta t}$. If $\hat{e}$ denotes unit vector in direction of axis rotation the

Angular Velocity $= \underline{\omega} = \omega . \hat{e} = \dot{\theta}\hat{e}$.

Relation b/w $\underline{v}$ & $\underline{\omega}$

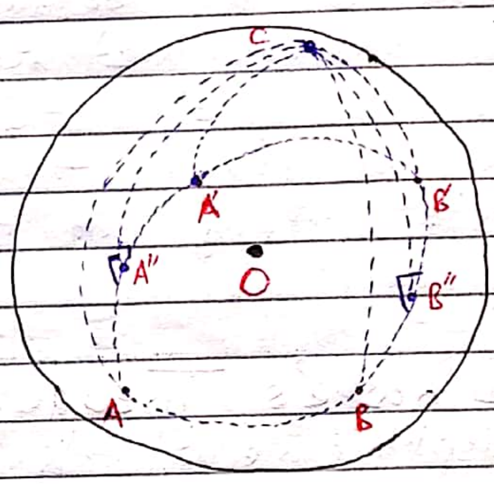
$$\underline{v} = \underline{\omega} \times \underline{r}$$

⇒ **Euler's Theorem:-** It states that the most general displacement of a rigid body fixed at a point is equivallent to a single rotation about some axis through that point.

Proof) Since size and sphere of the body is not specified so we specify it by a sphere with fixed point O. Take O as centre of the sphere.

Let A and B be two fixed points on the sphere. As O is fixed so when the body moves only the points A and B suffer displacement.

Let A' & B' be the new positions of the points A and B after an infintesimal time interval δt.



We join (A,B), (A',B') by great circular arcs. We also join (A,A') & (B,B') by great circular arcs. Let A'' & B'' be the mid points of AA' & BB' respectively. Through A'' & B'' draw axis at right angle which meet at the point C on the sphere. Join C with A, A', B & B' by great circular arcs. Now consider the correspondence of two spherical triangles CAA'' & $\Delta CA'A''$

(i) $m\angle CA''A = m\angle CA''A'$ $\because$ both are right angle

(ii) $m\,AA'' = m\,A''A'$ $\because$ A'' is mid point of AA'

(iii) $m\,CA' = m\,CA''$

$\Rightarrow \Delta CAA'' \cong \Delta CA'A''$

$\Rightarrow m\,CA \cong m\,CA'$

Similarly for triangles $\triangle CAB$ & $\triangle CA'B'$

$m\,CB = m\,CB'$.

$m\,CA = m\,CA'$, $m\,AB = m\,A'B'$ by def of rigid body

$\Rightarrow \triangle CAB \cong \triangle CA'B'$

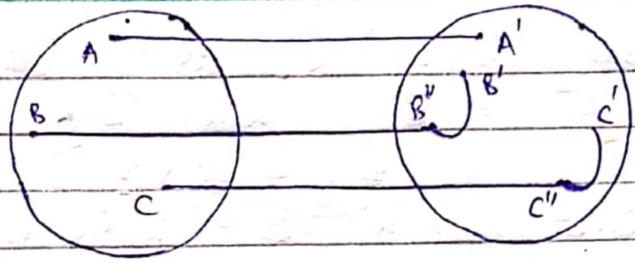
Thus the portion of rigid body lying in $\triangle CAB$ has moved to $\triangle CA'B'$

In this process O and C remained fixed although the later was at rest only instantaneously. Therefore body has undergone a rotation about the axis OC

Remark. The axis OC is not fixed and at different moments there will be different ones of rotation, all passing through the fixed point O. So because of the reason OC is called instantaneous axes of rotation. If the rotation is considered in a plane then the axis of rotation will always be the same & $\perp$ to the plane.

$\Rightarrow$ **Chasle's Theorem:-** The most general displacement of a rigid body free to move without any restriction is equivallent translation & rotation.

Proof (Explanation):- Consider a general displacement in which the body is not constrain to turn about a fix point. Let A, B & C be the initial positions of three non-collinear particles



of the body and A', B', C' be the positions after the displacement, takes place in two steps. The translation which takes $A \rightarrow A'$, $B \rightarrow B''$ & $C \rightarrow C''$. The rotation about A' which carries $B'' \longrightarrow B'$ & $C'' \rightarrow C'$. In this case A is called base point. It is clear that we can choose any other point as base point.

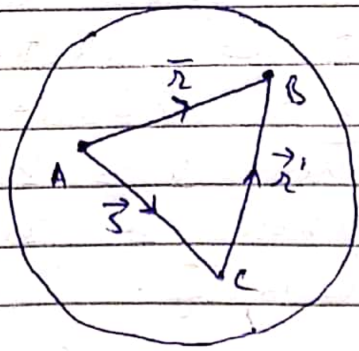
Question:- Show that the translation involved is altered by the change of base point but rotation is independent of base point.

Solution

Let us consider a rigid body moving in a general manner. Let us choose a particle A as base point and let us denote its velocity by V_A. During the time interval dt the displacement of the body is composed of a translation $V_A dt$ & a rotation $\omega\, dt$ about A. where $\underline{\omega}$ is the angular velocity of the body the displacement of any particle B of the body is

$$\underline{dr} = V_A dt + \omega\, dt \times \underline{r}$$

where $\underline{r}$ is the position vector of B w.r.t A. Thus the velocity of B is given by



$$\frac{d\vec{r}}{dt} = V_A + \underline{\omega} \times \underline{r}$$

$$\Rightarrow V_B = V_A + \underline{\omega} \times \underline{r} \qquad \text{---} (1)$$

Now choose C as an other point s.t
$\overline{AC} = \vec{s}$ then by ①

$$V_C = V_A + \underline{\omega} \times \underline{s} \quad \text{——— ②}$$

Subtracting ② from ①

$$V_B - V_C = \underline{\omega} \times \underline{r} - \underline{\omega} \times \underline{s}$$
$$= \underline{\omega} \times (\underline{r} - \underline{s})$$
$$= \underline{\omega} \times \underline{r}'.$$

$$\Rightarrow V_B = V_C + \underline{\omega} \times \underline{r}' \quad \text{——— ③}$$

which is the velocity of B refered to C as base point.

Comparing ① & ③ we have that ω is the independent of the choice of base point.

Question:- Define Screw Motion:- Show that the general motion of a rigid body can be regarded as a screw motion. What is the pitch of the screw motion?

Solution Screw Motion:- In screw motion the direction of axis of rotation is same as the direction of translation. In other words the linear velocity $\underline{V}$ and angular velocity ω has the same direction.

Motion of Rigid Body:- Let O be a fixed point in the rigid body with velocity V w.r.t a co-ordinate system fixed in space & O' be another arbitrary point of the body with velocity V' w.r.t the same system.

Then $$\underline{V}' = \underline{V} + \underline{\omega} \times \underline{r} \quad \text{——— ①}$$

where ω is the instantaneous angular velocity.

In general V' & $\underline{\omega}$ are not parallel. However we can choose O' in such a way that v' is parallel to ω. then by ①

$$\underline{\omega} \times v' = \underline{\omega} \times \underline{v} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

$$0 = \underline{\omega} \times \underline{v} + [(\underline{\omega}\cdot\underline{r})\underline{\omega} - (\underline{\omega}\cdot\underline{\omega})\underline{r}]$$

$$= \underline{\omega} \times \underline{v} + (\underline{\omega}\cdot\underline{r})\underline{\omega} - \omega^2 \underline{r}$$

$$\underline{r} = \frac{\underline{\omega} \times \underline{v}}{\omega^2} + \frac{\underline{\omega}\cdot\underline{r}}{\omega^2}\underline{\omega}$$

which is the equation of straight line through the point $\frac{\underline{\omega} \times \underline{v}}{\omega^2}$ & parallel to ω. this line is called central axis or axis of rotation or axis of screw. This shows that rigid body motion can be regarded as screw motion.

<u>Pitch of the Screw Motion</u>:- Since

$$v' = \underline{v} + \underline{\omega} \times \underline{r}$$

$$\Rightarrow \underline{\omega}\cdot\underline{v}' = \underline{\omega}\cdot\underline{v} + \underline{\omega}\cdot(\underline{\omega} \times \underline{r})$$

$$\omega\cdot\underline{v}' = \underline{\omega}\cdot\underline{v} \Rightarrow \underline{\omega}\cdot\underline{v}' \text{ is caler}$$

$$\Rightarrow \frac{\underline{\omega}\cdot\underline{v}'}{\omega^2} \text{ is scaler } \& \ \frac{\underline{\omega}\cdot\underline{v}'}{\omega^2} = \frac{\omega v' \cos(0)}{\omega^2} = \frac{\omega v'}{\omega^2}$$

$$= v'/\omega$$

$$= \frac{\text{Displacement Per second}}{\text{Rotation Per second.}}$$

$$= \frac{\text{Distance}}{\text{Angle.}}$$

This ratio is called pitch of the screw. i.e distance through which a particle moves per unit angle is called pitch of the screw.

* Displacement of a rigid body in Terms of Angular displacement:-

We know that $\underline{v} = \omega \times r$

$$\Rightarrow \lim_{\delta t \to 0} \frac{\delta \underline{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\delta \theta}{\delta t} \hat{a} \times \underline{r}$$

$$\Rightarrow \frac{\delta \underline{r}}{\delta t} = \frac{\delta \theta}{\delta t} \hat{a} \times \underline{r}$$

$$\Rightarrow \delta \underline{r} = \delta \theta \hat{a} \times \underline{r}$$

which gives linear displacement due to angular displacement $\delta \theta$.

Question:- Show that the finit rotation of a rigid body do not commute but infinitesimal rotation commute. Also show that angular velocities satisfy vector law of addition.

Solution We consider displacement of a rigid body through angles $\delta\theta_1$ & $\delta\theta_2$ about axes specified by $\hat{a}_1$ & $\hat{a}_2$ respectively.

Let $\underline{r}$ be position vector of a particle P of the body. When the body is rotated through $\delta\theta_1$ then new position vector of P becomes

$r_1 = \underline{r} +$ displacement due to $\delta\theta_1$

$$\Rightarrow r_1 = \underline{r} + \delta\theta_1 \hat{a}_1 \times \underline{r} \quad \text{--- (1)}$$

Next body is given angular displacement $\delta\theta_2$ about axes specified by $\hat{a}_2$ then

$$r_{12} = r_1 + \delta\theta_2 \hat{a}_2 \times r_1$$

$$\Rightarrow \underline{r}_{12} = \underline{r} + \delta\theta_1 \hat{a}_1 \times \underline{r} + \delta\theta_2 \hat{a}_2 \times (\underline{r} + \delta\theta_1 \hat{a}_1 \times \underline{r})$$

$$\underline{r}_{12} = \underline{r} + \delta\theta_1 \hat{a}_1 \times \underline{r} + \delta\theta_2 \hat{a}_2 \times \underline{r} + \delta\theta_2 \delta\theta_1 \hat{a}_2 \times (\hat{a}_1 \times \underline{r})$$

$$= [\underline{r} + (\delta\theta_1 \hat{a}_1 + \delta\theta_2 \hat{a}_2) \times \underline{r}] + (\delta\theta_1 \delta\theta_2 \hat{a}_2 \times (\hat{a}_1 \times \underline{r}))$$

Since rotations are infinitesimal so 2nd term on R.H.S can be neglected

$$\Rightarrow \underline{r}_{12} = \underline{r} + (\delta\theta_1 \hat{a}_1 + \delta\theta_2 \hat{a}_2) \times \underline{r}$$

Now performing the same rotations in the inverse order we get

$$\underline{r}_{21} = \underline{r} + (\delta\theta_2 \hat{a}_2 + \delta\theta_1 \hat{a}_1) \times \underline{r}$$

For infinitesimal rotations

$\Rightarrow \underline{r}_{12} = \underline{r}_{21}$ i.e infinitesimal rotation of a rigid body commute.

Now let δt be the time interval during which rotations have been taken place

then
$$\frac{\underline{r}_{12} - \underline{r}}{\delta t} = \frac{\delta\theta_1}{\delta t} \hat{a}_1 \times \underline{r} + \frac{\delta\theta_2}{\delta t} \hat{a}_2 \times \underline{r}$$

when $\delta t \to 0$
$$\underline{v} = \underline{\omega}_1 \times \underline{r} + \underline{\omega}_2 \times \underline{r}$$
$$= (\underline{\omega}_1 + \underline{\omega}_2) \times \underline{r}$$

Which shows angular velocities satisfy vector law of addition.

Question:- A body is pivoted at a point O and is rotating at the rate of 90 radian per second about a fixed line in the direction of the vector $(-1, 2, 2)$ to an observer looking in the direction of this vector. The sence of rotation of body is clockwise. Find velocity of P with position vector $(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3})$

Solution Here $\hat{e} = \dfrac{-\hat{i} + 2\hat{j} + 2\hat{k}}{3}$ $\qquad \because \underline{\omega} = \omega \hat{e}$

So direction of angular velocity is along $\hat{e} = \frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$

$$\Rightarrow \underline{\omega} = \omega \hat{e}$$
$$= 90 \cdot \frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$$
$$\underline{\omega} = 30(-\hat{i} + 2\hat{j} + 2\hat{k})$$

Here $\underline{r} = \overline{OP} = \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$

So
$$\underline{V}_P = \underline{\omega} \times \underline{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -30 & 60 & 60 \\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} \end{vmatrix}$$

$$\underline{V}_P = \underline{\omega} \times \underline{r} = \hat{i}(-20-40) - \hat{j}(10-20) + \hat{k}(-20-20)$$
$$= \hat{i}(-60) - \hat{j}(-10) + \hat{k}(-40)$$
$$\Rightarrow \underline{V}_P = 10(-6\hat{i} + \hat{j} - 4\hat{k})$$

Question:- A rigid body is rotating about a fixed origin O. The points A(0, −1, 2) and B(2, 0, 0) are moving with velocities (7, −2, −1) and (0, 6, −4) respectively (The units are in meters and seconds) Find the angular velocity of the body.

Solution We denote position vector of A with r_A and of B with r_B. And also we denote velocity of A by $\underline{V}_A$ and velocity of B by $\underline{V}_B$. Also let
$$\underline{\omega} = (\omega_1, \omega_2, \omega_3)$$

Then
$$r_A = (0, -1, 2)$$
$$V_A = (7, -2, -1)$$

So $V_A = \underline{\omega} \times r_A$

$$\Rightarrow V_A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ 0 & -1 & 2 \end{vmatrix}$$

$$\Rightarrow (7,-2,-1) = (2\omega_2+\omega_3\,,\,-2\omega_1\,,\,-\omega_1)$$

$$\Rightarrow 2\omega_2+\omega_3 = 7\,,\quad -2\omega_1 = -2\,,\quad -\omega_1 = -1$$

$$\Rightarrow \boxed{\omega_1 = 1}\,,\quad 2\omega_2+\omega_3 = 7$$

Also $V_B = (0,6,-4)$

$$V_B = \underline{\omega}\times r_B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ 2 & 0 & 0 \end{vmatrix}$$

$$\Rightarrow V_B = \hat{i}(0-0) - \hat{j}(-2\omega_3) + \hat{k}(-2\omega_2)$$

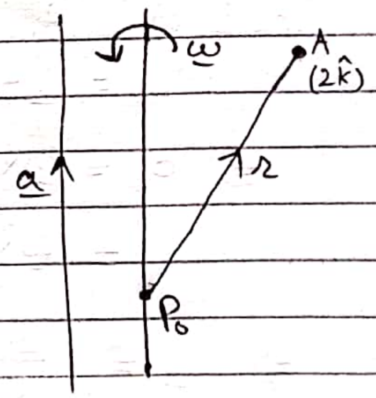
$$\Rightarrow (0,6,-4) = 2\omega_3\hat{j} - 2\omega_2\hat{k}$$

$$\Rightarrow 2\omega_3 = 6 \quad \& \quad -2\omega_2 = -4$$

$$\Rightarrow \omega_3 = 3 \quad \& \quad \omega_2 = 2$$

Hence $\underline{\omega} = (1,2,3)$

Question:- A particle describes a circle about the line whose equation in vector form is given by $\underline{r} = (3+\lambda)\hat{i} + 2\hat{j} + (1-\lambda)\hat{k}$ with angular speed ω radians per second. If at time $t=0$ the particle is at the point $2\hat{k}$. Find its velocity.



Solution: The equation of the line about which the particle revolves is

$$\underline{r} = (3+\lambda)\hat{i} + 2\hat{j} + (1-\lambda)\hat{k}$$

$$= (3\hat{i}+2\hat{j}+\hat{k}) + \lambda(\hat{i}-\hat{k})$$

$$= r_0 + \lambda\,\underline{a}$$

This shows that the axis of rotation passes through the fixed point P_0 with position vector $r_0 = 3\hat{i}+2\hat{j}+\hat{k}$ and parallel

to the vector $\underline{a} = \hat{i} - \hat{k}$
So unit vector in direction of ω is

$$\hat{e} = \frac{1}{\sqrt{2}}(\hat{i} - \hat{k}) \Rightarrow \underline{\omega} = \frac{\omega}{\sqrt{2}}(\hat{i} - \hat{k})$$

Since the position vector of the particle at point A at $t = 0$ is $2\hat{k}$

Now $\overline{P_0A} = \overline{OA} - \overline{OP_0}$

$$= 2\hat{k} - (3\hat{i} + 2\hat{j} + \hat{k})$$

$$= -3\hat{i} - 2\hat{j} + k$$

So $\underline{V} = \underline{\omega} \times \overline{P_0A}$ .... ?

$$\Rightarrow \underline{V} = -\sqrt{2}\,\omega(\hat{i} - \hat{j} + \hat{k})$$

Question:- A rigid body S has spin ω and a particle Q of S has velocity $\underline{V}$. Show that every particle P of S with velocity vector parallel to $\underline{\omega}$ lies on the line

$$\overline{QP} = (\underline{\omega} \times \underline{V})/\omega^2 + \mu \underline{\omega}$$

Solution Given that velocity of Q is $\underline{V}$.
Let $\underline{V}'$ be the velocity of P s.t $\underline{V}'$ is parallel to $\underline{\omega}$ then

$$\underline{V}' = \underline{V} + \underline{\omega} \times \underline{r} \quad \& \quad \underline{\omega} \times \underline{V}' = 0, \quad \underline{r} = \overline{QP}$$

$$\Rightarrow 0 = \underline{\omega} \times \underline{V} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

$$= \underline{\omega} \times \underline{V} + (\underline{\omega} \cdot \underline{r})\underline{\omega} - (\underline{\omega} \cdot \underline{\omega})\underline{r}$$

$$\Rightarrow \underline{r} = \frac{\underline{\omega} \times \underline{V}}{\omega^2} + \frac{\underline{\omega} \cdot \underline{r}}{\omega^2}\,\underline{\omega}$$

$$\Rightarrow \overline{QP} = \frac{\underline{\omega} \times \underline{V}}{\omega^2} + \mu \underline{\omega} \quad \text{--- (1)}$$

Here $\mu = \dfrac{\underline{\omega} \cdot \underline{r}}{\omega^2}$

This show that every particle P of S with velocity vector (V') parallel to ω lies on the line ① with $u = \frac{\omega \cdot r}{\omega^2}$ as an orbitrary parameter.

Question:- The instantaneous velocities of particles at point $(a,0,0)$, $(0, a/\sqrt{3}, 0)$, $(0,0,2a)$ of a rigid body are $(u,0,0)$, $(u,0,v)$ and $(u+v, -\sqrt{3}\,v, v/2)$ respectively w.r.t a rectangular co-ordinate sys find the magnitude and direction of spin of the body and the point at which the central axis cuts the xz-plane.

Solution Let $A=(a,0,0)$, $B=(0,\frac{a}{\sqrt{3}},0)$, $C=(0,0,2a)$

We denote corresponding co-ordinate position vectors by r_A, r_B and r_C and corresponding velocities by V_A, V_B and V_C

Take A as refrence point Then

r_1 = Position vector of B w.r.t $A = r_B - r_A = [-a, \frac{a}{\sqrt{3}}, 0]$

r_2 = " " " C " " $" = r_C - r_A = [-a, 0, 2a]$

Then
$$V_1 = V_B - V_A = (0,0,v)$$
$$V_2 = V_C - V_A = (v, -\sqrt{3}\,v, v/2)$$

Now $V_1 = \omega \times r_1$ & $V_2 = \omega \times r_2$

So
$$V_1 = \omega_1 \times r_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ -a & a/\sqrt{3} & 0 \end{vmatrix}$$

$$\Rightarrow (0,0,v) = \left(\frac{-a}{\sqrt{3}}\omega_3, -a\omega_3, \frac{a\omega_1}{\sqrt{3}} + a\omega_2\right)$$

$$\Rightarrow \boxed{\omega_3 = 0},$$

$$\text{\&}\quad a\omega_1 + a\omega_2\sqrt{3} = \sqrt{3}\,v \quad \text{———} ①$$

and $v_2 = \underline{\omega} \times r_2$

$$\Rightarrow \left(v, -\sqrt{3}v, \frac{v}{2}\right) = (2a\omega_2, -a\omega_3 + 2a\omega_1, a\omega_2)$$

$$\Rightarrow \boxed{\omega_2 = \frac{v}{2a}}$$

So by ① $\quad a\omega_1 + a(v/2a)\sqrt{3} = \sqrt{3}v$

$$\Rightarrow a\omega_1 = \sqrt{3}v - \frac{\sqrt{3}v}{2} = \frac{\sqrt{3}v}{2}$$

$$\Rightarrow \boxed{\omega_1 = \frac{\sqrt{3}v}{2a}}$$

So $\quad \underline{\omega} = \left(\frac{\sqrt{3}v}{2a}, \frac{v}{2a}, 0\right) \Rightarrow |\underline{\omega}| = \frac{v}{a}$

6 Let Q be the refrence point and P be any point on the central axis then

$$\overline{QP} = \frac{\underline{\omega} \times V_Q}{\omega^2} + \lambda \underline{\omega} \quad \text{——— ②}$$

Now

$$\underline{\omega} \times V_Q = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\sqrt{3}v}{2a} & v/2a & 0 \\ u & 0 & 0 \end{vmatrix}$$

$$\left(\because V_Q = V_A = (u, 0, 0)\right)$$

$$\Rightarrow \underline{\omega} \times V_Q = \frac{-uv}{2a}\hat{k}$$

$$\Rightarrow \frac{\underline{\omega} \times V_Q}{\omega^2} = -\frac{ua}{2v}\hat{k}$$

$\left[\omega^2 = v^2/a^2 \quad \Rightarrow \frac{-uv}{2a} \Big/ v^2/a^2 = \frac{-uv}{2a} \times \frac{a^2}{v^2}\right.$

So $\quad \overline{QP} = -\frac{ua}{2v}\hat{k} + \lambda\left[\frac{\sqrt{3}v}{2a}, \frac{v}{2a}, 0\right]$

$$= \left(\frac{\sqrt{3}v}{2a}\lambda, \frac{v}{2a}\lambda, -\frac{ua}{2v}\right)$$

The central axis meet the XZ-plane where $y=0$

$$\Rightarrow \frac{\lambda v}{2a} = 0 \Rightarrow \lambda = 0$$

Question:- The points $(a, 2a, -a)$, $(-a, -a, a)$ and (a, a, a) of a rigid body have instantaneous velocities $\left(\frac{\sqrt{3}v}{2}, 0, \frac{\sqrt{3}v}{2}\right)$, $\left(\frac{-v}{\sqrt{3}}, 0, \frac{-v}{\sqrt{3}}\right)$ and $\left(0, \frac{-v}{\sqrt{3}}, \frac{v}{\sqrt{3}}\right)$ respectively w.r.t a rectangular coordinate system. Show that the body has line through the origin having direction cosines $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

Solution:- We denote the points by A, B, C their position vectors by r_A, r_B & r_C and their velocities by V_A, V_B & V_C. We choose A as refrence point then

r_1 = P.V of B w.r.t A $= r_B - r_A = (-2a, -3a, 2a)$

& r_2 = " " C " " $= r_C - r_A = (0, -a, 2a)$

$$V_1 = V_B - V_A = \left(\frac{-5v}{2\sqrt{3}}, 0, \frac{-5v}{2\sqrt{3}}\right)$$

$$V_2 = V_C - V_A = \left(\frac{-\sqrt{3}}{2}v, \frac{-v}{\sqrt{3}}, \frac{-v}{2\sqrt{3}}\right)$$

Let $\underline{\omega}$ be the angular velocity of the rigid body then

$$V_1 = \underline{\omega} \times r_1 \quad \& \quad V_2 = \underline{\omega} \times r_2$$

Now

$$V_1 = \underline{\omega} \times r_1$$

$$\Rightarrow \left(\frac{-5v}{2\sqrt{3}}, 0, \frac{-5v}{2\sqrt{3}}\right) = (2a\omega_2 + 3a\omega_3, -2a\omega_3 - 2a\omega_1, -3a\omega_1 + 2a\omega_2)$$

$$\Rightarrow 2a\omega_2 + 3a\omega_3 = \frac{-5v}{2\sqrt{3}} \quad \text{——— (1)}$$

$$-2a\omega_3 - 2a\omega_1 = 0 \quad \text{——— (2)}$$

$$-3a\omega_1 + 2a\omega_2 = \frac{-5v}{2\sqrt{3}} \quad \text{——— (3)}$$

Now $V_2 = \omega \times r_2$

$$\Rightarrow \left(\frac{-\sqrt{3}}{2}v, \frac{-v}{\sqrt{3}}, \frac{-v}{2\sqrt{3}}\right) = (2a\omega_2 + a\omega_3, 2a\omega_1, -a\omega_1)$$

$$\Rightarrow 2a\omega_2 + a\omega_3 = \frac{-\sqrt{3}}{2}V \quad \text{--- (4)}$$

$$-a\omega_1 = \frac{-V}{2\sqrt{3}} \quad \Rightarrow \boxed{\omega_1 = \frac{V}{2a\sqrt{3}}}$$

By (2) $-2a\omega_3 - 2a\left(\frac{V}{2a\sqrt{3}}\right) = 0$

$$\Rightarrow \boxed{\omega_3 = \frac{-V}{2a\sqrt{3}}}$$

By (4) $2a\omega_2 + a\left(\frac{-V}{2a\sqrt{3}}\right) = \frac{-\sqrt{3}}{2}V$

$$\Rightarrow 2a\omega_2 = \frac{-\sqrt{3}}{2}V + \frac{V}{2\sqrt{3}}$$

$$= \frac{-3V + V}{2\sqrt{3}} = \frac{-V}{\sqrt{3}}$$

$$\Rightarrow \boxed{\omega_2 = \frac{-V}{2a\sqrt{3}}}$$

So $\underline{\omega} = \left(\frac{V}{2a\sqrt{3}}, \frac{-V}{2a\sqrt{3}}, \frac{-V}{2a\sqrt{3}}\right)$

$$\Rightarrow |\omega| = V/2a$$

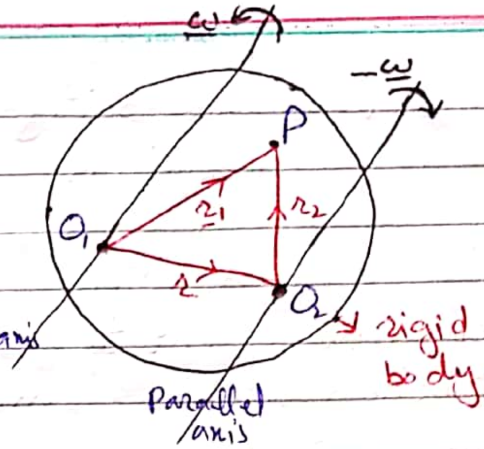
Now $\hat{\omega} = \frac{\omega}{|\omega|} = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

This shows that direction cosine of the line are

$$\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

Question- Show that equal & opposite rotations of a rigid body about distinct parallel axes are equivalent to a translation of the body.

Solution:- Let O_1 & O_2 be the two fixed points of the rigid body. Let two parallel axis passes through O_1 and O_2. Let the body rotates about the axis through O_1 & O_2 with a $\underline{\omega}$ and $-\underline{\omega}$ respectively. Let P be any arbitrary point of the body with $\overline{O_1P} = \underline{r}_1$, $\overline{O_2P} = r_2$ and $O_1O_2 = r$



Let V_p denotes the velocity of P then

$$\underline{V}_p = (\underline{\omega} \times \underline{r}_1) + (-\underline{\omega} \times r_2)$$

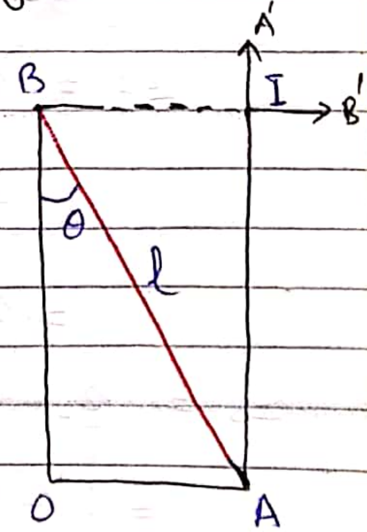
$$= \underline{\omega} \times (\underline{r}_1 - \underline{r}_2) = \underline{\omega} \times \underline{r}$$

$$\Rightarrow \underline{V}_p = \underline{\omega} \times \underline{r}$$

which is independent of P i.e this result is true for all particles of the rigid body Hence the rigid body is translated.

Question:- A rod AB moves in a vertical plane s.t its ends A, B are always in contact with the ground and a vertical wall. Find angular velocity of rod.

Solution Let OA be along the ground and OB be along the vertical wall as shown in the figure. Let l be the length of the rod and it makes an angle θ with the wall



clearly velocity of the end point A is along OA & B is along OB.

Question:- A rigid body receives three successive rotations about three mutually perpendicular intersecting lines fixed in space. Each rotation being through a right angle and the sences being cyclic. Find the axis and magnitude of the single equivallent rotation.

Solution Let the three mutually perpendicular axis be x-axis, y-axis and z-axis

Let $\underline{r}$ be the position vector of the body initially. and after 1st, 2nd, 3rd rotations the new position vectors be $\underline{r}_1$, $\underline{r}_2$ and $\underline{r}_3$ respectively

Then $\underline{r}_1 = \underline{r} + \delta\theta\,\hat{a} \times \underline{r}$

$= \underline{r} + \frac{\pi}{2}\hat{i} \times \underline{r}$

$\underline{r}_2 = \underline{r}_1 + \frac{\pi}{2}\hat{j} \times \underline{r}_1$

$\underline{r}_3 = \underline{r}_2 + \frac{\pi}{2}\hat{k} \times \underline{r}_2$

Now $\underline{r}_2 = \underline{r}_1 + \frac{\pi}{2}\hat{j} \times \underline{r}_1$

$= \left(\underline{r} + \frac{\pi}{2}\hat{i} \times \underline{r}\right) + \frac{\pi}{2}\hat{j} \times \left(\underline{r} + \frac{\pi}{2}\hat{i} \times \underline{r}\right)$

$= \underline{r} + \frac{\pi}{2}\hat{i} \times \underline{r} + \frac{\pi}{2}\hat{j} \times \underline{r} - \left(\frac{\pi}{2}\right)^2 \hat{k} \times \hat{r}$

$\underline{r}_3 = \underline{r}_2 + \frac{\pi}{2}\hat{k} \times \underline{r}_2$

$= \left(\underline{r} + \frac{\pi}{2}\hat{i} \times \underline{r} + \frac{\pi}{2}\hat{j} \times \underline{r} - \left(\frac{\pi}{2}\right)^2 \hat{k} \times \underline{r}\right)$

$$+\frac{\pi}{2}\hat{k}\times\left(\underline{r}+\frac{\pi}{2}\hat{i}\times\underline{r}+\frac{\pi}{2}\hat{j}\times\underline{r}-\left(\frac{\pi}{2}\right)^2\hat{k}\times\underline{r}\right)$$

$$\Rightarrow \underline{r}_3=\underline{r}+\frac{\pi}{2}\hat{i}\times\underline{r}+\frac{\pi}{2}\hat{j}\times\underline{r}-\left(\frac{\pi}{2}\right)^2\hat{k}\times\underline{r}+\frac{\pi}{2}\hat{k}\times\underline{r}+\left(\frac{\pi}{2}\right)^2\hat{k}\times(\hat{i}\times\underline{r})+\left(\frac{\pi}{2}\right)^2\hat{k}\times(\hat{j}\times\underline{r})-\left(\frac{\pi}{2}\right)^3\hat{k}\times(\hat{k}\times\underline{r})$$

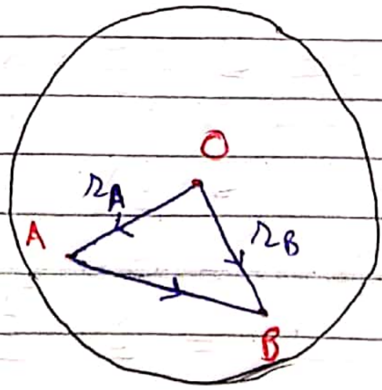
$$\Rightarrow \underline{r}_3=x\hat{i}+y\hat{j}+z\hat{k}+\frac{\pi}{2}(0+y\hat{k}-z\hat{j})+\frac{\pi}{2}(-x\hat{k}+0+z\hat{i})-\left(\frac{\pi}{2}\right)^2(x\hat{j}-\hat{i}y+0)+\frac{\pi}{2}(\hat{j}x-y\hat{i}+0)+\left(\frac{\pi}{2}\right)^2\hat{k}\times(0+\hat{k}y-\hat{j}z)+\left(\frac{\pi}{2}\right)^2\hat{k}\times(-x\hat{k}+0+z\hat{i})-\left(\frac{\pi}{2}\right)^3\cdot(0)$$

$$\Rightarrow \underline{r}_3=x\hat{i}+y\hat{j}+z\hat{k}+\frac{\pi}{2}(y\hat{k}-z\hat{j})+\frac{\pi}{2}(-x\hat{k}+z\hat{i})-\left(\frac{\pi}{2}\right)^2(x\hat{j}-\hat{i}y)+\frac{\pi}{2}(\hat{j}x-y\hat{i})+\left(\frac{\pi}{2}\right)^2(0+\hat{i}z)+\left(\frac{\pi}{2}\right)^2(-0+z\hat{j})-0$$

$$\Rightarrow \underline{r}_3=\left[x+\frac{\pi}{2}z+\left(\frac{\pi}{2}\right)^2y-\left(\frac{\pi}{2}\right)y+\left(\frac{\pi}{2}\right)^2z\right]\hat{i}+\left[y-\frac{\pi}{2}z-\left(\frac{\pi}{2}\right)^2x+\frac{\pi}{2}x+\left(\frac{\pi}{2}\right)^2z\right]\hat{j}+\left[z+\frac{\pi}{2}y-\frac{\pi}{2}x\right]\hat{k}$$

Question:- A wheel is rolling without slipping along a plane with angular velocity $\underline{\omega}$. A and B are taken at different distances from the centre on two different spots. Show that at any time the velocity of A relative to B is $\underline{\omega}\times\overline{AB}$ in the direction $\perp$ to $\overline{AB}$.

Solution: Let $\underline{V}$ be the velocity of the origin O and r_A, r_B denotes the position vectors of A & B w.r.t O. Also let



velocities of A and B be V_A and V_B w.r.t O

Then $\underline{V}_A = \underline{V} + \underline{\omega} \times \underline{r}_A$

$\underline{V}_B = \underline{V} + \underline{\omega} \times \underline{r}_B$

The velocity of B relative to A given by

$$V_{AB} = V_B - V_A$$

$$= \underline{V} + \underline{\omega} \times \underline{r}_B - \underline{V} - \underline{\omega} \times \underline{r}_A$$

$$= \underline{\omega} \times (\underline{r}_B - \underline{r}_A)$$

$$\Rightarrow V_{AB} = \underline{\omega} \times \overline{AB}$$

This shows that at any time the velocity of A relative to B is $\underline{\omega} \times \overline{AB}$ and is in direction $\perp$ to $\overline{AB}$

Question:- State and prove principle of angular momentum.

Solution

Statement:- The rate of change of angular momentum about a point 'O' is equal to the torque of force acting on the particle.

Proof:- Let "m" be mass and at any time t, $\underline{r}$ be position vector. $\underline{V}$ be velocity and $\underline{\mathcal{L}}$ be the angular momentum of the particle then

$$\underline{\mathcal{L}} = \underline{r} \times m\underline{V}$$

Diff both sides w.r.t t

$$\frac{d\underline{\mathcal{L}}}{dt} = \frac{d\underline{r}}{dt} \times m\underline{V} + \underline{r} \times \frac{m\,d\underline{V}}{dt}$$

$$= \underline{V} \times m\underline{V} + \underline{r} \times m\underline{a}$$

$$= 0 + \underline{r} \times m\underline{a}$$

$$= \underline{r} \times \underline{F}$$

$$\Rightarrow \frac{d\underline{\mathcal{L}}}{dt} = \tau$$ which is required proof.

Question:- Find the relation between kinetic energy and angular momentum for a system of particles.

Solution

Consider a system of 'n' particles with angular velocity $\underline{\omega}$ of the system. Then for the ith particle

$$\underline{V}_i = \underline{\omega} \times \underline{r}_i$$

Now

$$K.E = \frac{1}{2} m_i V_i^2$$

For the whole system $K.E = \frac{1}{2} \sum m_i V_i^2$

$$\Rightarrow K.E = \frac{1}{2} \sum m_i (\underline{\omega} \times \underline{r}_i)^2$$

$$= \frac{1}{2} \sum m_i (\underline{\omega} \times \underline{r}_i) \cdot (\underline{\omega} \times \underline{r}_i)$$

$$= \frac{1}{2} \sum m_i (\underline{\omega} \cdot \underline{r}_i \times (\underline{\omega} \times \underline{r}_i)) \qquad \because a.b \times c = a \times b.c$$

$$= \frac{1}{2} \sum m_i (\underline{\omega} \cdot \underline{r}_i \times \underline{V}_i)$$

$$= \frac{1}{2} \underline{\omega} \cdot \sum m_i \underline{r}_i \times \underline{V}_i$$

$$= \frac{1}{2} \underline{\omega} \cdot \sum r_i \times m_i V_i$$

$$= \frac{1}{2} \underline{\omega} \cdot \sum \mathcal{L}_i$$

$$\Rightarrow K.E = \frac{1}{2} \underline{\omega} \cdot \underline{\mathcal{L}}$$

which is required relation.

This is called rotational kinetic energy of the system.

Question- If a rigid body is turning about a fixed point O and OXYZ are rectangular axis and if the components of velocity of particles with coordinates (1,0,0) are (0,2,5) then find the components in the direction of the x-axis of the velocity of the particle with coordinate (0,0,1)

Solution Let $\underline{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

Now $\underline{V} = \underline{\omega} \times \underline{r}$

$$\Rightarrow (0,2,5) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ 1 & 0 & 0 \end{vmatrix}$$

$$(0,2,5) = (0, \omega_z, -\omega_y)$$

$$\Rightarrow \omega_z = 2 \quad , \quad \omega_y = -5$$

Now again

$$V_x\hat{i} + V_y\hat{j} + V_z\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & -5 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow V_x\hat{i} + V_y\hat{j} + V_z\hat{k} = -5\hat{i} - \omega_x\hat{j} + 0.\hat{k}$$

$$\Rightarrow V_x = -5$$

This shows that the component of velocity of the particles with coordinates (0,0,1) in the direction of x-axis is -5

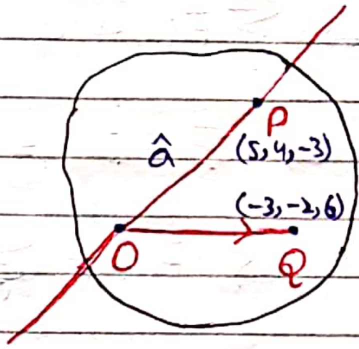
Question:- A rigid body with a fixed point O is rotating at a point of 50 rpm about an axis with direction ~~of~~ OP, where P has coordinates (5, 4, −3) ~~in~~ inches measured from O. Find the velocity of Q with coordinates (-3,-2,6) inched at this constant

Solution

Given $\omega = 50$ rpm

$\Rightarrow \underline{\omega} = \frac{50}{60}$ rps

$= \frac{5}{6}$ r/sec



Given $\overline{OP} = 5\hat{i} + 4\hat{j} - 3\hat{k}$

$$\hat{a} = \frac{\overline{OP}}{|OP|} = \frac{1}{5\sqrt{2}}(5\hat{i}+4\hat{j}-3\hat{k})$$

So $$\underline{\omega} = \omega\hat{a} = \frac{5}{6} \times \frac{1}{5\sqrt{2}}(5\hat{i}+4\hat{j}-3\hat{k})$$

$$= \frac{1}{6\sqrt{2}}(5\hat{i}+4\hat{j}-3\hat{k})$$

Now $r = \overline{OQ} = -3\hat{i} - 2\hat{j} + 6\hat{k}$

Hence $V_Q = \omega \times \overline{OQ}$

$$= \frac{3}{\sqrt{5}}\hat{i} - \frac{7}{2\sqrt{2}}\hat{j} + \frac{1}{3\sqrt{2}}\hat{k}$$

Question: Derive first equation of motion for a system of particles.

Solution

Consider a system of "n" particles. Let F_i denotes the total external force on the ith particle and let R_{ij} denotes the total internal force on the ith particle due to jth particle. $(i \neq j)$

Now for the ith particle equation of motion can be written as

$$F_i + \sum_{j=1}^{n} R_{ij} = m_i \ddot{r}_i \quad \text{——— (1)}$$

For the whole system

$$\sum_{i=1}^{n} F_i + \sum_{i=1}^{n}\sum_{j=1}^{n} R_{ij} = \sum_{i=1}^{n} m_i \ddot{r}_i$$

$$\Rightarrow \sum_{i=1}^{n} F_i + 0 = \sum_{i=1}^{n} m_i \ddot{r}_i$$

$$\Rightarrow \sum_i F_i = \sum_i m_i (\ddot{r}_i' + \ddot{r}_c)$$

$$\Rightarrow F = \sum_i m_i \ddot{r}_i' + \sum_i m_i \ddot{r}_c$$

$$= 0 + \sum_i m_i \ddot{r}_c$$

$$\Rightarrow F = m\ddot{r}_c \quad \text{——} \quad (2)$$

$\Rightarrow$ Total external force = Mass x Acceleration of centroid

Also $F = m\ddot{r}_c = \frac{d}{dt}(m\dot{r}_c) = \frac{d}{dt}(P)$

$$\Rightarrow F = \frac{d}{dt}(P)$$

$\Rightarrow$ F = Rate of change of momentum of centroid

(2) is required equ of motion.

Question- Derive 2nd equation of motion for angular momentum.

Solution For system of n particles

$$F_i + \sum_j R_{ij} = m_i \ddot{r}_i$$

$$\Rightarrow r_i \times F_i + \sum_j r_i \times R_{ij} = r_i \times m_i \ddot{r}_i$$

$$\Rightarrow \sum_i r_i \times F_i + \sum_i \sum_j r_i \times R_{ij} = \sum_i r_i \times m_i \ddot{r}_i$$

$$\Rightarrow \sum_i r_i \times F_i + 0 = \sum_i r_i \times m_i \ddot{r}_i$$

$$\Rightarrow \sum_i r_i \times F_i = \frac{d}{dt}\left(\sum_i r_i \times m_i v_i\right)$$

$$\Rightarrow \sum_i r_i \times F_i = \frac{d}{dt}(\underline{\mathcal{L}}) = \underline{\mathcal{L}}^{\circ}$$

$\Rightarrow \underline{\mathcal{L}}^{\circ}$ = Total momentum of the external force

$\Rightarrow \underline{\mathcal{L}}^{\circ}$ = Total external force

$= \underline{N}$

$\Rightarrow \quad \underline{\mathcal{L}}^{\circ} = \underline{N}$

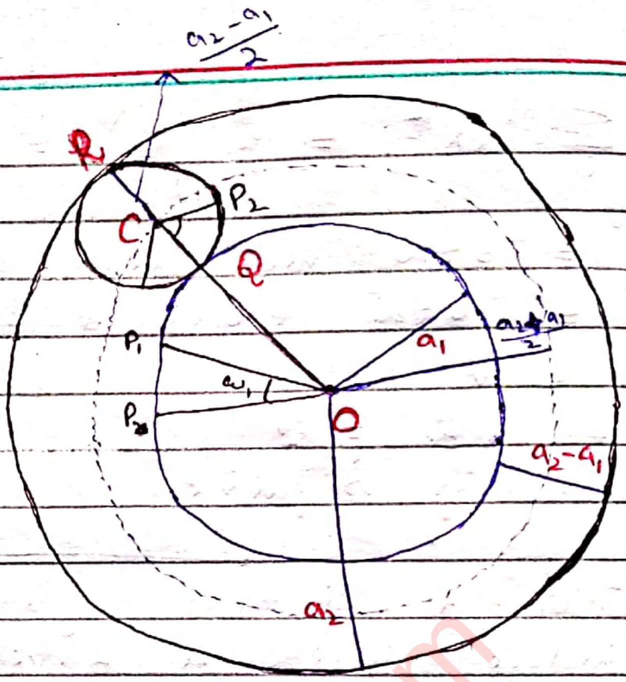
which is required equation.

Question:- A circular cylinder with radius a_1 is inclosed inside a coaxial hollow circular ~~cy~~ cylinder of radius a_2 $(a_2 > a_1)$. The space b/w them contains ball bearing of radius $\frac{1}{2}(a_2 - a_1)$. The inner and outer cylinders are made to turn with constant angular speeds ω_1 and ω_2 respectively. Show that if there is no slipping. The angular speed of a ball is $(\omega_2 a_2 - \omega_1 a_1)/(a_2 - a_1)$ and that the centre of a ball moves in a circle with angular speed $(\omega_1 a_1 + \omega_2 a_2)/a_1 + a_2$ what is the length of the circular arc on either cylinder with which the ball ~~the~~ is in contact in unit time.

Solution:- The problem is explained in the figure. The rotations are supposed to be clockwise. Let ω_b be the angular speed of the ball bearing, ω_c be the angular speed of the radius OC, V_Q be the linear speed of the point Q, at which a ball bearing is in contact with the inner cylinder.

Also let V_R be the linear

speed of the point R, at which a ball bearing is in contact with the outer cylinder.



Now using the formula

$$\underline{v} = \underline{\omega} \times \underline{r}$$

$$V_q = \omega_1 a_1$$

$$V_2 = \omega_2 a_2 = V_q + \omega_b QR$$

$$\Rightarrow V_2 = \omega_1 a_1 + \omega_b QR$$

$$\Rightarrow \omega_2 a_2 = \omega_1 a_1 + \omega_b (a_2 - a_1)$$

$$\Rightarrow \omega_b = \frac{\omega_2 a_2 - \omega_1 a_1}{a_2 - a_1}$$

The period of rotation of a ball is

$$r = \frac{2\pi}{|\omega_b|} = \frac{2\pi(a_2 - a_1)}{|\omega_2 a_2 - \omega_1 a_1|}$$

Now V_c = Velocity of $C = V_q + \omega_b QC$

$$= \omega_1 a_1 + \frac{\omega_2 a_2 - \omega_1 a_1}{a_2 - a_1} \cdot \frac{1}{2}(a_2 - a_1)$$

$$= \frac{\omega_1 a_1 + \omega_2 a_2}{2}$$

So $\omega_c = \frac{V_c}{OC} = \frac{\omega_1 a_1 + \omega_2 a_2}{a_1 + a_2}$

where $OC = OR - CR$

$$= a_2 - \frac{1}{2}(a_2 - a_1)$$

$$= \frac{1}{2}(a_1 + a_2)$$

Hence C moves in a circle with periodic time

$$T' = \frac{2\pi}{|\omega_c|} = \frac{2\pi(a_1+a_2)}{|\omega_1 a_1 + \omega_2 a_2|}$$

Length of Circular Arc:- We suppose that the radii of inner and outer cylinders are initially in vertical positions, i.e OP_1 and CO_2 are vertical at $t=0$
Consequently P_1, P_2 coinside with P at $t=0$. In unit time the radius OP_1 will move through an angular distance ω_1. Now OQC is in the line joining centers of the coaxial cylinders and the ball bearing and the centre C has angular velocity ω_c, The radius OQ will move through an angular distance ω_c. The arc on the inner cylinder with which the ball bearing is in contact unit time is

$$P_1Q = a_1|\omega_c - \omega_1| = \frac{a_1 a_2|\omega_2 - \omega_1|}{a_2 + a_1}$$

Moment And Product Of Inertia Of A Rigid Body***

⇒ **Moment of Inertia:-** If we want to stop a rotating wheel, we have to apply a force, Similarly a stationary wheel resists being put into rotation. This shows that the wheel is inert to change its states of rotation or of rest. This type of inertness or inertia is possessed by almost all the rotating bodies and is called the rotation inertia or moment of Inertia.

The moment of inertia of a rigid body depends upon the particular axis, about which it is rotating, and on the mass of the body and also upon the distribution of its mass w.r.t axis of rotation.

Question. Define
(a)(i) Moment of Inertia (ii) Radius of gyration
(b) State & Prove Parallel axes theorem.

Solution Moment of Inertia:- If r is the perpendicular distance of any element of a body of mass m from any given line, then the quantity $\sum_i m_i r_i^2$ is called the moment of inertia of the body about the given line. In other words moment of inertia can thus be obtained

"Take each element of the body multiply it by the square of its $\perp$ distance from the given line and add together all the quantities, thus obtained".

Radius of Gyration:- If moment of inertia of a body is "mk^2" where k is not in the polar form, because it is dimensionless, then k is called the radius of gyration or swing radius about the given line.

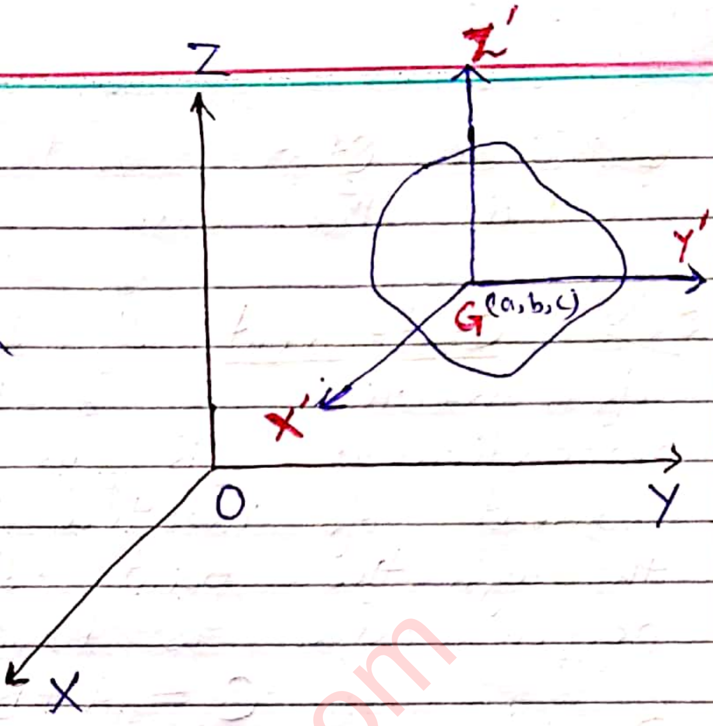
e.g Moment of Inertia of a rod is $\frac{1}{3} m a^2$ then radius of gyration is $\sqrt{\frac{a^2}{3}} = \frac{a}{\sqrt{3}}$

Parallel Axes Theorem:-

Statement:- The moment of inertia of any body about any axis is equal to the moment of inertia about a parallel axes through the centre of mass together with the product of the whole mass and square of the distance between the axes i.e

$$I = I' + Md^2$$

Proof:- Let a body be situated at a point $P(x,y,z)$ and suppose we want to find moment of inertia of body about z-axis. Let $G(a,b,c)$ be the centre of mass of the body. Throug $G(a,b,c)$ draw a rectangular coordinate system $GX'Y'Z'$ parallel to $OXYZ$.



Then $x = x' + a$

$y = y' + b$

$z = z' + c$

Then moment of inertia about z-axis =

$$I = \sum_i m_i (x_i^2 + y_i^2)$$

$$= \sum_i m_i \left[(x'+a)^2 + (y'+b)^2 \right]$$

$$= \sum_i m_i \{ x'^2 + a^2 + 2ax' + y'^2 + b^2 + 2by' \}$$

$$= \sum_i m_i (x'^2 + y'^2) + \sum_i m_i (a^2 + b^2) + 2a \sum_i m_i x' + 2b \sum_i m_i y'$$

$$= \sum_i m_i (x'^2 + y'^2) + \left(\sum_i m_i\right)\left(\sqrt{a^2 + b^2}\right)^2 + 2a(0) + 2b(0)$$

$$= I' + Md^2$$

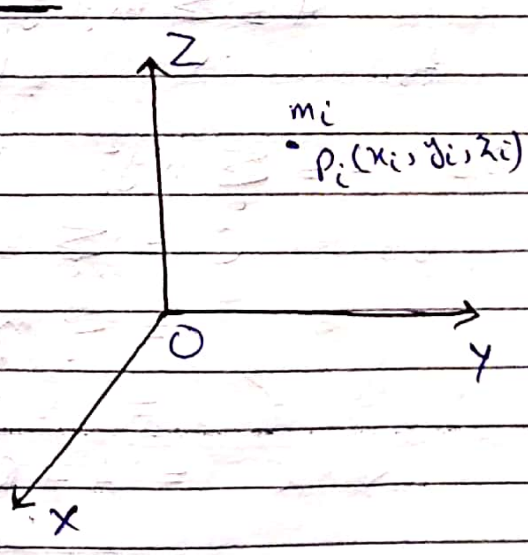
$$\Rightarrow I = I' + Md^2$$

$\Rightarrow$

M.O.I about x-axis

$$= I_{xx} = m_i \left(\sqrt{y_i^2 + z_i^2}\right)^2$$

$$= m_i (y_i^2 + z_i^2)$$



For whole system

$$I_{xx} = \sum_i m_i (y_i^2 + z_i^2)$$

$$I_{yy} = \sum_i m_i (x_i^2 + z_i^2)$$

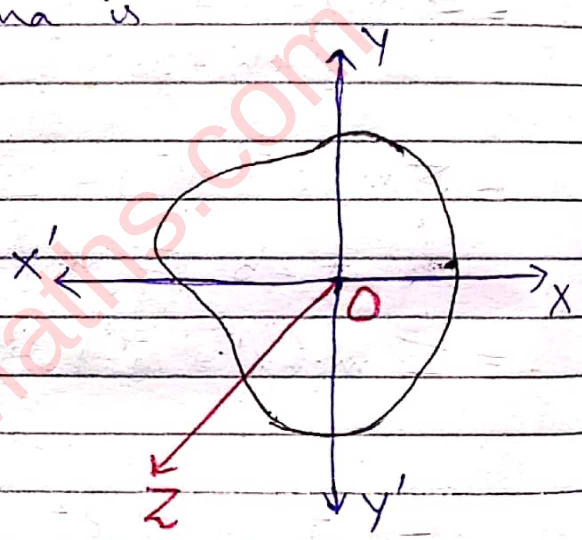
$$I_{zz} = \sum_i m_i (x_i^2 + y_i^2)$$

Question:- State and Prove "Perpendicular Axis Theorem:-

Solution

Statement:- If A and B are moment of inertia of a lamina about two perpendicular axis in its plane. Then its moment of inertia about the line through the point of intersection and perpendicular to lamina is

$$C = A + B$$



Proof:-

Taking the two ⊥ lines as X-axis and Y-axis and the line through point of intersection and ⊥ to lamina as Z-axis. We have

$$A = \sum_i m_i(y_i^2 + z_i^2) \quad \& \quad B = \sum_i m_i(z_i^2 + x_i^2)$$

As in XY plane $z = 0$

So $A = \sum_i m_i y_i^2$ and $B = \sum_i m_i x_i^2$

Now $C = \sum_i m_i(x_i^2 + y_i^2)$

$$= \sum_i m_i x_i^2 + \sum_i m_i y_i^2$$

$$= B + A$$

$$\Rightarrow C = A + B$$

Question:- Calculate Parallel axes theorem in terms of radius of gyration in the form $K^2 = K_c^2 + d^2$

Solution

Let I be the moment of inertia of the body about any axis l, I_c be the moment of inertia of the body about line l_c through centre of mass of the body and parallel to l and d be the perpendicular distance between both the axes, then if M represents the mass of the body then by parallel axes theorem

$$I = I_c + Md^2 \quad \text{———} \quad (1)$$

Now let "k" and "k_c" be the swing radii about "l" and "l_c" respectively, then

$$I = Mk^2 \quad \& \quad I_c = Mk_c^2$$

$\therefore$ (1) gives $Mk^2 = Mk_c^2 + Md^2$

$$\Rightarrow k^2 = k_c^2 + d^2$$

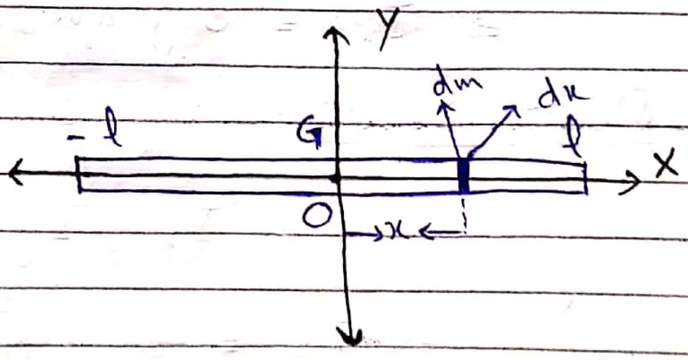
As required.

Question:- Calculate Moment of Inertia of a thin uniform rod of length $2l$ for different cases.

Solution

About Y-axis:-

Let us consider a rod of length $2l$, mass m, lying on x-axis s.t its centroid centre of mass G is at origin.



In order to find out its m.o.i about Y-axis we cut a small element of mass dm & of length dx and at a distance x from Y-axis. Then by definition of moment of inertia

$$dI_y = dm \times x^2 = dm\, x^2$$

where $dm = \frac{m}{2l} \times dx = \frac{m\,dx}{2l}$

So $dI_y = \frac{m}{2l} x^2\, dx$

$$\Rightarrow I_y = \int_{-l}^{l} \frac{m}{2l} x^2\, dx$$

$$= \frac{m}{2l} \cdot 2 \int_0^l x^2\, dx = \frac{1}{3} m l^2$$

$$\Rightarrow I_y = \frac{1}{3} m l^2$$

About x-axis:- Let r be the distance of small element from X-axis, then

$$dI_x = r^2\, dm$$

$$= (0)^2\, dm = 0 \qquad \because r = 0$$

$$\Rightarrow dI_x = 0$$

$$\Rightarrow I_x = 0 \int_{-l}^{l} dm \Rightarrow I_x = 0$$

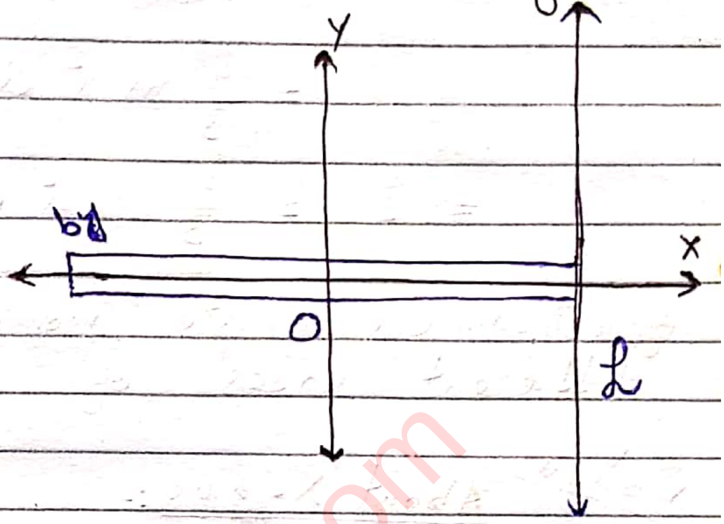
About z-axis:- By ⊥ axis theorem

$$I_z = I_x + I_y = 0 + \frac{1}{3} m l^2$$

$$\Rightarrow I_z = \frac{1}{3} m l^2$$

About a line ∥ to Y-axis and toching one end of rod:-

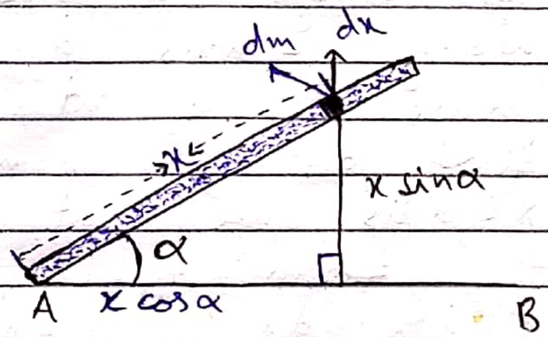
Let $\mathcal{L}$ be the line as shown in the figure, then by ∥ axis theorem



$$I_L = I_Y + Md^2$$
$$= \frac{1}{3} ml^2 + ml^2$$
$$\Rightarrow I_L = \frac{4}{3} ml^2$$

Question- Show that moment of inertia of a uniform rigid rod of mass m and length 2a about an axis through an end making angle α with rod is $\frac{4}{3} m a^2 \sin^2 \alpha$.

Solution Let AB be the line about which moment of inertia is to calculate.



In order to find out its moment of inertia we cut a small element of mass dm, length dx and at a distance x from end A. then

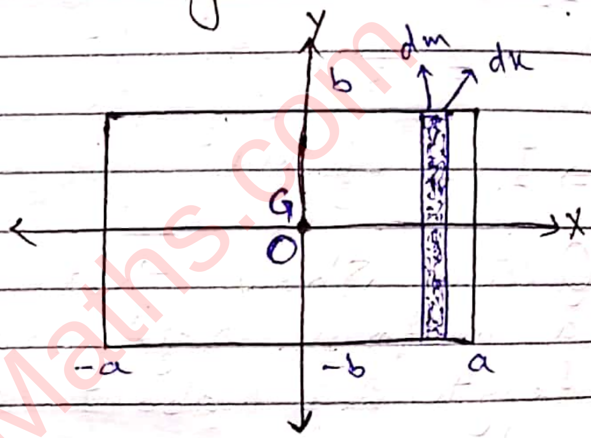
$$dI_{AB} = dm \cdot (x \sin\alpha)^2$$
$$= \frac{m}{2a} dx \cdot (x \sin\alpha)^2$$
$$I = \int_0^{2a} \frac{m}{2a} x^2 \sin^2\alpha \, dx$$
$$= \frac{m}{2a} \sin^2\alpha \cdot \left.\frac{x^3}{3}\right|_0^{2a}$$

$$I = \frac{m}{2a}\sin^2\alpha \cdot \frac{1}{3}[8a^3 - 0]$$

$$= \frac{4}{3} m a^2 \sin^2\alpha$$

$$\Rightarrow I = \frac{4}{3} m a^2 \sin^2\alpha$$

Question:- Calculate moment of inertia for different cases of rectangular lamina?



Solution: About X-axis:-

Let us consider a rectangular lamina of mass m, of length 2a, of width 2b and is placed in such a way that its length is along x-axis, width is along y-axis and its centroid centre of mass G is at origin. In order to find out its moment of inertia about X-axis we cut a small element in the form of a rod, of length 2b, width dx, mass dm and parallel to y-axis.

Then moment of inertia of small element about X-axis

$$= dI_x = \frac{1}{3} dm\, b^2$$

where $dm = \frac{m}{4ab} \times 2b\, dx$

$$= \frac{m}{2a} dx$$

$$\Rightarrow dI_x = \frac{mb^2}{6a} dx \Rightarrow I_x = \frac{mb^2}{6a}\int_{-a}^{a} dx$$

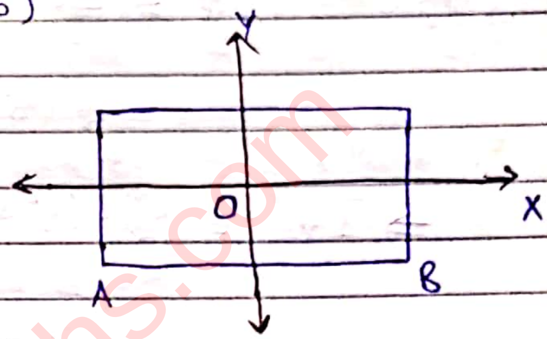
$$\Rightarrow I_x = \frac{1}{3} m b^2$$

Similarly $I_y = \frac{1}{3} m a^2$

& by perpendicular axis theorem

$$I_z = \frac{1}{3} m b^2 + \frac{1}{3} m a^2$$

$$\Rightarrow I_z = \frac{1}{3} m (a^2 + b^2)$$



About line AB:-

By parallel axis theorem

$$I_{AB} = I_x + m b^2$$

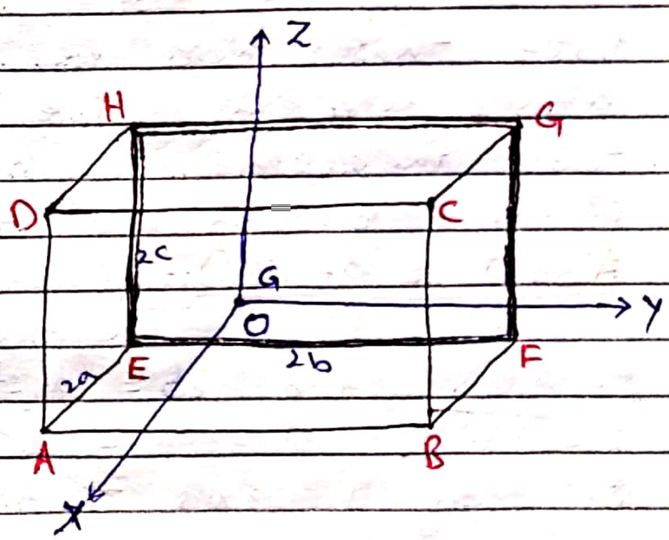
$$= \frac{1}{3} m b^2 + m b^2$$

$$\Rightarrow I_{AB} = \frac{4}{3} m b^2$$

Question. Calculate Moment of Inertia of $\boxed{\text{Parallelopiped}}$ for different cases.
cuboid

Solution

About X-axes:-



Let us consider a parallelopiped of mass m, of edges 2a, 2b and 2c, with centroide centre of mass at 'O' i.e origin and is placed in such a way the the edges 2a, 2b and 2c are along X-axis, Y-axis and Z-axis respectively. In order to find out the m.o.i of this parallelopiped about x-axis we cut a small element in the form of a rectangle with edges 2b and 2c and width of the rectangle be dx. and mass of dm then

$$dm = \frac{m}{8abc} \times 4bc\,dx = \frac{m}{2a}dx$$

Now $dI_x = \frac{1}{3}dm(b^2+c^2)$

$$= \frac{1}{3}(b^2+c^2)\cdot\frac{m}{2a}dx$$

$$\Rightarrow I_x = \frac{m}{6a}(b^2+c^2)\int_{-a}^{a}dx$$

$$= \frac{m}{6a}(b^2+c^2)(2a)$$

$$\Rightarrow I_x = \frac{1}{3}m(b^2+c^2)$$

Similarly $I_y = \frac{1}{3}m(a^2+c^2)$

$$I_z = \frac{1}{3}m(a^2+b^2)$$

Note that here we can not use ⊥ axis theorem i.e $I_z = I_x + I_y$. Because this theorem is valid only for plane bodies i.e lamina.

<u>About Edge AD:-</u>

The edge AD is ∥ to z-axis and d = distance b/w the z-axis and AD $= \sqrt{a^2+b^2}$

So by ∥ axis theorem

$$I_{AD} = I_z + md^2$$

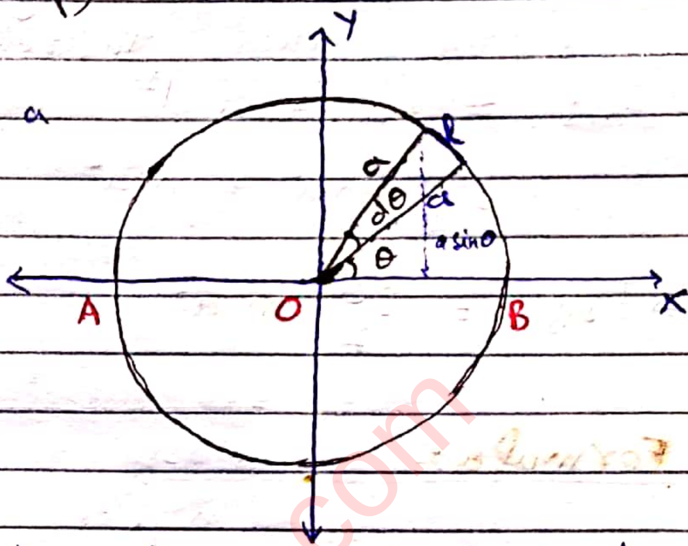
$$= \frac{1}{3}m(a^2+b^2) + m(a^2+b^2)$$

$$\Rightarrow I_{AD} = \frac{4}{3}m(a^2+b^2)$$

Question:- Calculate moment of inertia of a circular ring (Hoop)

Solution:

Let us consider a ring of mass m, radius "a" and with centre at origin. Let its diameter AB be taken along X-axis.



i.e $I_{AB} = I_{xx}$

In order to find out moment of inertia of this ring we cut a small of mass dm and of length l, then

$$dm = \frac{m}{2\pi a} \times l$$

Now $l = r\theta \Rightarrow l = a(d\theta) = a d\theta$

So

$$dm = \frac{m}{2\pi a} \cdot a d\theta = \frac{m}{2\pi} d\theta$$

Now $dI_{xx} = dm(a\sin\theta)^2$

$$= a^2 \sin^2\theta \cdot \frac{m}{2\pi} d\theta$$

$$\Rightarrow I_{xx} = \frac{ma^2}{2\pi} \int_0^{2\pi} \sin^2\theta \, d\theta$$

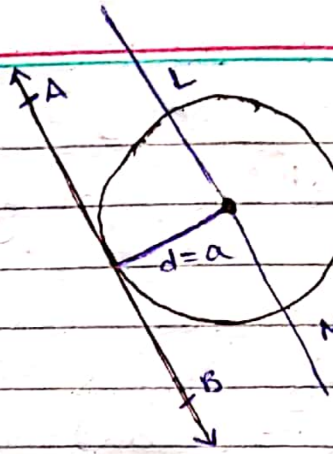
$$I_{xx} = \frac{1}{2} m a^2$$

Similarly $I_{yy} = \frac{1}{2} ma^2$ and by $\perp$ axes theorem

$$I_{zz} = I_{xx} + I_{yy} = \frac{1}{2} ma^2 + \frac{1}{2} ma^2$$

$$\Rightarrow I_{zz} = ma^2$$

About a line tangent to ring:- Let AB be the tangent line and taken LM be those diameter of ring which is ∥ to AB. Then by parallel axes theorem



$$I_{AB} = I_{LM} + md^2$$

$$= \frac{1}{2}ma^2 + ma^2$$

$$\Rightarrow I_{AB} = \frac{3}{2}ma^2$$

Formulas:-

	Mass	Area
	m	$4ab$
	dm	$2b\,dx$

This is for rectangular Lamina

$$dm : m = 2b\,dx : 4ab$$

$$dm \times 4ab = m \times 2b\,dx$$

$$\Rightarrow dm = \frac{m}{4ab} \times 2b\,dx$$

$$= \frac{m}{2a}\,dx$$

Mass of body

Length of body

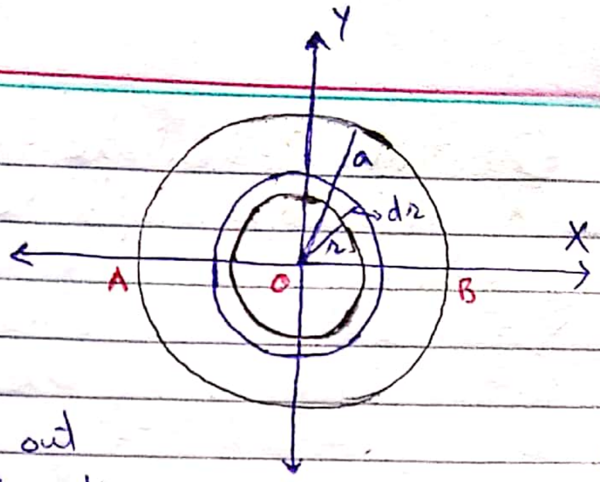
Length of small element

In general.

$$dm = \frac{\text{Mass of whole body}}{\text{Volume / Area / Length of whole body}} \times \text{V/A/L of small element}$$

Question:- Calculate Moment of inertia of a circular disc.

Solution: About Diameter:- Let us consider a circular disc of mass m, of radius "a" and with



centre at origin.
Let AB be its diameter and is taken along X-axis. So

$I_{AB} = I_{xx}$

In order to find out moment of inertia of disc we cut a small ~~diam~~ element in the form of a ring of radius 'r' and of mass dm and let 'dr' be the width of ring, then

$$dm = \frac{m}{\pi a^2} \times 2\pi r\, dr = \frac{2mr}{a^2} dr$$

$$\text{So } dI_{xx} = \frac{1}{2} dm\, r^2 = \frac{1}{2} r^2 \cdot \frac{2mr}{a^2} dr$$

$$= \frac{1}{a^2} m r^3 dr$$

$$\Rightarrow I_{xx} = \frac{m}{a^2} \int_0^a r^3 dr$$

$$= \frac{m}{a^2}\left(\frac{a^4}{4}\right) = \frac{1}{4} m a^2$$

$$\Rightarrow I_{xx} = \frac{1}{4} m a^2$$

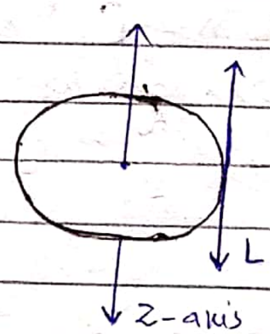
Similarly $I_{yy} = \frac{1}{4} ma^2$ and by $\perp$ axes theorem

$$I_z = \frac{1}{4} ma^2 + \frac{1}{4} m a^2$$

$$\Rightarrow I_{zz} = \frac{1}{2} ma^2$$

About tangent to surface & $\parallel$ to z-axis:-

Let L be the line then

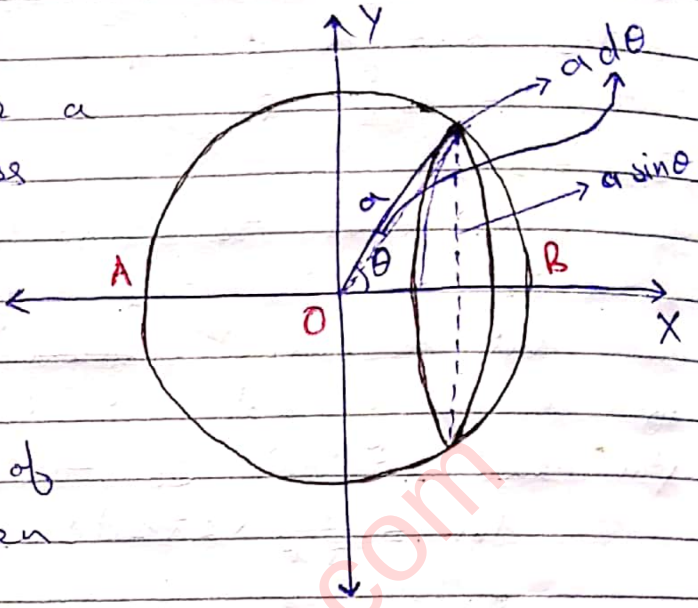


$$I_L = I_z + m a^2$$

$$= \frac{1}{2} ma^2 + ma^2$$

$$I_L = \frac{3}{2} m a^2$$

Question:- Calculate moment of Inertia of a hollow sphere about its diameter.



Solution Let us consider a hollow sphere of mass m, of radius "a".
Let AB be the diameter of the sphere, then along X-axis. Let centre of the sphere be taken along X-axis.

In order to find out moment of inertia we cut a small element in the form of a ring of mass dm then, width $ad\theta$ & radius $a\sin\theta$,

$$dm = \frac{m}{4\pi a^2} 2\pi (a\sin\theta)(ad\theta)$$

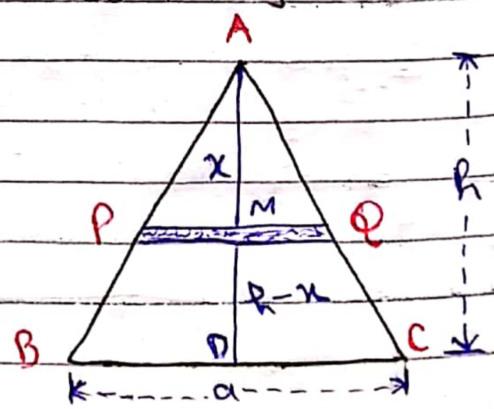
$$= \frac{m}{2} \sin\theta \, d\theta$$

Now $dI_{AB} = dm.(a\sin\theta)^2 = \frac{ma^2}{2} \sin^3\theta \, d\theta$

$$\Rightarrow I_{AB} = \int_0^{\pi} \frac{1}{2} ma^2 \sin^3\theta \, d\theta$$

$$\Rightarrow I_{AB} = \frac{2}{3} ma^2$$

Question:- Calculate Moment of Inertia of a triangular lamina about its base.



Solution Let us consider a triangular lamina of mass m, of height h and of base 'a'. In order to find out moment of inertia of triangular lamina ABC we cut a small element in the form of a rod at a distance $h-x$ from the base and parallel to base. Let mass of small element be dm, thickness of dx and length be PQ. then

$$dI_{BC} = 0 + dm.(h-x)^2 \qquad \boxed{I = I' + Md^2}$$

$$\Rightarrow dI_{BC} = dm(h-x)^2$$

Now $dm = \frac{m}{\frac{1}{2}ah} \times \overline{PQ}\, dx$

$$= \frac{2m}{ah} \times PQ\, dx$$

Now from the similar triangle ABC and $\triangle APQ$

$$\frac{PQ}{BC} = \frac{AM}{AD} \Rightarrow \frac{PQ}{a} = \frac{x}{h}$$

$$\Rightarrow PQ = \frac{ax}{h}$$

So $dm = \frac{2m}{ah} \times \frac{ax}{h} dx$

$$= \frac{2mx}{h^2} dx$$

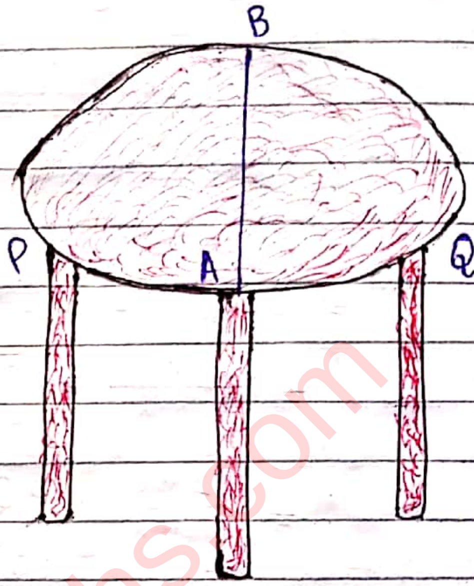
Thus $dI_{BC} = \frac{2mx}{h^2}(h-x)^2 dx$

$$\Rightarrow I_{BC} = \frac{2m}{h^2}\int_0^h x(h-x)^2 dx$$

$$\Rightarrow I_{BC} = \frac{1}{6} m h^2$$

Question:- Calculate moment of Inertia of a stool about any diameter of disc s.t " Plate is a thin circular disc of mass M and of radius "a" with three legs equally space round the rim of the disc and the legs are of length 2a and mass "M/4" and are mounted ∥ to each other.

Solution Let the stool be shown in figure. Let AB be the diameter of the plate about which moment of inertia is to be calculated.



Let I_1 be the moment of inertia of the plate disc about AB. I_2 be the moment of inertia of that leg of the table whose upper end touches the line AB and I_3 be the moment of inertia of the rest leg.

Now we know that moment of inertia of a circular disc of radius "a" and mass M about its diameter is $1/4\ Ma^2$

$$\therefore I_1 = \frac{1}{4} M a^2$$

Now using the concept of moment of inertia of a rod of length 2a about one of its end, we have moment of one leg as

$$I_2 = \frac{4}{3}(\text{mass})(\text{half length})^2$$

$$= \frac{4}{3}\left(\frac{M}{4}\right)(a^2) = \frac{1}{3} M a^2$$

Now each rest leg is at the distance d from the line AB

$$\therefore I_3 = \frac{1}{3} Ma^2 + \frac{M}{4} d^2$$ correction at ? Page 115

$$\Rightarrow I_3 = \frac{1}{3}Ma^2 + \frac{M}{4}(a \sin 60^\circ)^2$$

$$= \frac{25}{48} M a^2$$

Thus moment of inertia of the whole table about AB $= I_{AB} = I_1 + I_2 + 2I_3$

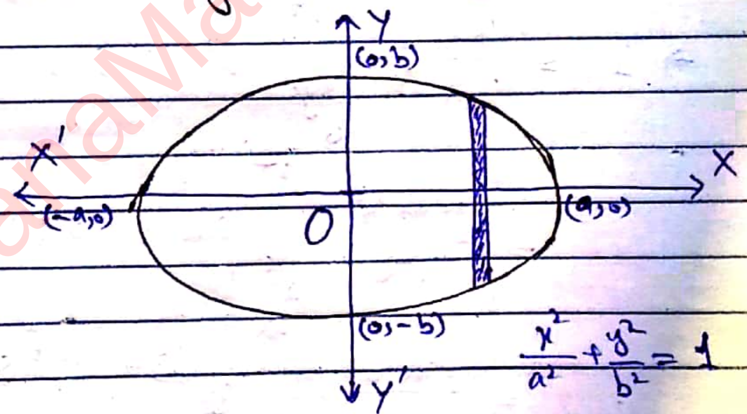
$$\Rightarrow I_{AB} = \frac{1}{4}Ma^2 + \frac{1}{3}Ma^2 + 2\left[\frac{25}{48}Ma^2\right]$$

$$= \frac{39}{24} M a^2$$

$$\Rightarrow I_{AB} = \frac{13}{8} M a^2$$

Question – Calculate moment of inertia of an elliptic disc of axes 2a and 2b about its major axis.

Solution



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let us consider an elliptic disc of mass m, of axes 2a and 2b s.t major axes 2a is taken along X-axes and minor axis 2b is taken along Y-axis.

In order to find out moment of inertia of elliptic disc about X-axis we cut a small element in the form of a rod strip of length 2y of mass dm, of width dx and parallel to y axis.

Now

$$dI_x = \frac{1}{3} dm\, y^2$$

where $dm = \frac{m}{\pi ab} \cdot 2y\, dx$

$\because \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$y^2 = \frac{a^2-x^2}{a^2} \cdot b^2$

$y = \left(\frac{a^2-x^2}{a^2} \cdot b^2\right)^{\frac{1}{2}}$

$\Rightarrow dm = \frac{2m}{\pi ab} \cdot y \, dx$

$\Rightarrow dI_x = \frac{2m}{3\pi ab} y^3 \, dx$

$I_x = \frac{2m}{3\pi ab} \int_{-a}^{a} y^3 \, dx$

$= \frac{4m}{3\pi ab} \int_0^a \left(\frac{a^2-x^2}{a^2} \cdot b^2\right)^{3/2} dx$

Putting $x = a \sin\theta$

$y = b \cos\theta$

then $x = 0 \Rightarrow \theta = 0$

$x = a \Rightarrow \theta = \pi/2$

$\Rightarrow dxdy = a\,dr\,d\theta$

$\Rightarrow I_x = \frac{4mb^2}{3\pi \cancel{a^4}} \int_0^{\pi/2} \cancel{a^3} \cos^3\theta \, \cancel{a} \cos\theta \, d\theta$

$= \frac{4mb^2}{3\pi} \int_0^{2\pi} \cos^4\theta \, d\theta$

$= \frac{4mb^2}{3\pi} \int_0^{2\pi} \frac{1}{8}(3 + \cos 4\theta + 4\cos 2\theta) \, d\theta$

$\Rightarrow I_x = \frac{1}{4} mb^2$

Similarly $I_y = \frac{1}{4} ma^2$

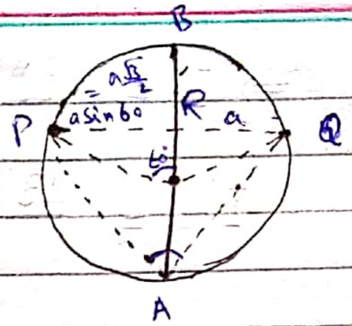
By $\perp$ axes theorem

$I_z = I_x + I_y = \frac{1}{4}mb^2 + \frac{1}{4}ma^2$

$= \frac{1}{4} m(a^2 + b^2)$

Stool

$\Rightarrow$ Now we calculate I_3.
Let P & Q be the that end point of each leg which toches the disc.



Now consider the leg with the end point P. ~~from the diameter |AB|~~
Let |PR| be the distance of P from the diameter AB then $|PR| = a\sin 60 = \frac{\sqrt{3}}{2}a$.
Now let O be the centre of disc & let L be the other end point of the leg whose one end point in P. then
$|PL|$ = Length of the leg $= 2a$
Let S be the mid point of Leg PL then $|PS| = a$
Now ΔPSR is a right angle triangle with $m\angle P = 90^\circ$, $|PS| = a$, $|PR| = \frac{\sqrt{3}}{2}a$

So $$|SR| = \sqrt{|PS|^2 + |PR|^2} = \sqrt{a^2 + \frac{3}{4}a^2}$$

$$= \sqrt{\frac{7}{4}a^2} = \frac{\sqrt{7}}{2}a$$

Now draw a line "ℓ" through S (which is centroid of the leg) and $\parallel$ to AB. Then ℓ is $\perp$ to the leg.
Then I_ℓ = M.O.I of leg a about line ℓ.

$$= \frac{1}{3}\cdot\frac{M}{4}a^2 = \frac{M}{12}a^2$$

$$\Rightarrow I_3 = I_\ell + \frac{M}{4}|SR|^2 = \frac{M}{12}a^2 + \frac{M}{4}\cdot\frac{7}{4}a^2$$

$$= \frac{M}{12}a^2 + \frac{7M}{16}a^2 = \frac{4Ma^2 + 21Ma^2}{48}$$

$$I_3 = \frac{25}{48}Ma^2$$

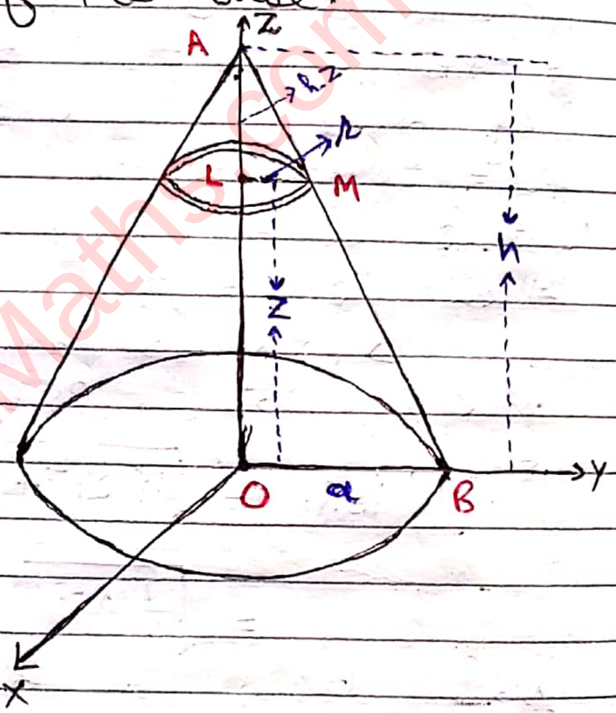
Hence $I = I_1 + I_2 + 2I_3$

$$= \frac{1}{4}Ma^2 + \frac{1}{3}Ma^2 + \frac{50}{48}Ma^2$$

$$\Rightarrow I = \frac{12Ma^2 + 16Ma^2 + 50Ma^2}{48} = \frac{78Ma^2}{48}$$

$$I = \frac{13Ma^2}{8}$$

Question- Calculate Moment of Inertia of a circular cone about.
(i) Its axes of symmetry.
(ii) Any diameter of the base.



Solution Let us consider a right circular cone of mass m, base radius a, of height h. In order to find out m.O.I of the cone we consider a coordinate system OXYZ as shown in the figure.
Here z axes is the axis of symmetry. Now we have to calculate moment of inertia of the cone about z-axis & Y-axis.

(i) In order to calculate moment of inertia we cut a small element in the form of a disc of radius r and of width dz and at a distance 'z' from the base of the cone.

Now $dI_z = \frac{1}{2} dm r^2$

where $dm = \frac{m}{\frac{1}{3}\pi a^2 h} \times \pi r^2 dz$

$$\Rightarrow dm = \frac{3mr^2}{a^2h} dz$$

Now from similar triangles ΔAOB & ΔALM

$$\frac{r}{h-z} = \frac{a}{h} \Rightarrow r = \frac{a}{h}(h-z)$$

$$\text{So} \quad dm = \frac{3m}{a^2h} \cdot \frac{a^2}{h^2}(h-z)^2 dz$$

$$\Rightarrow dI_{zz} = \frac{3m}{2h^3}(h-z)^2 \cdot \frac{a^2}{h^2}(h-z)^2 dz$$

$$= \frac{3ma^2 (h-z)^4 dz}{2h^5}$$

$$\Rightarrow I_{zz} = \frac{3ma^2}{2h^5}\int_0^h (h-z)^4 dz$$

$$= \frac{3ma^2}{2h^5}\left[-\frac{(h-z)^5}{5}\Big|_0^h\right]$$

$$= \frac{3ma^2}{2h^5}\left[-0 + \frac{h^5}{5}\right]$$

$$\Rightarrow I_{zz} = \frac{3}{10} ma^2$$

Moment of Inertia about Y-axis:- For the small element

$$dI_{yy} = \frac{1}{4} dmr^2 + dmz^2$$

$$= \frac{1}{4}\frac{3ma^2}{h^5}(h-z)^4 dz + \frac{3m}{h^3}(h-z)^2 z^2 dz$$

$$I_{yy} = \frac{3ma^2}{4h^5}\int_0^h (h-z)^4 dz + \frac{3m}{h^3}\int_0^h (h-z)^2 z^2 dz$$

$$= \frac{3ma^2}{4h^5}\left(\frac{h^5}{5}\right) + \frac{3m}{h^3}\left(\frac{h^5}{30}\right)$$

$$\Rightarrow I_{yy} = \frac{3}{20} ma^2 + \frac{1}{10} mh^2$$

$$I_{yy} = \frac{1}{20} m(3a^2 + 2h^2)$$

Similarly $I_{xx} = \frac{1}{20} m(3a^2 + 2h^2)$

⇒ **Angular Momentum:-** Moment of momentum is called angular momentum. If m is the mass of the particle and v is the velocity of the particle then its angular momentum

a.m. = Moment of Momentum = $\underline{L} = \underline{r} \times \underline{P}$ ($\underline{P}$ is the momentum of particle which is given by $\underline{P} = m\underline{v}$).

Where the moment is taken about O. Now $\underline{L} = \underline{r} \times \underline{P} = \underline{r} \times m\underline{v}$

$= \underline{r} \times m(\underline{\omega} \times \underline{r}) \quad \because \underline{v} = \underline{\omega} \times \underline{r}$

$$\Rightarrow \underline{L} = m[r^2\underline{\omega} - (\underline{r}\cdot\underline{\omega})\underline{r}]$$

⇒ **Product of Inertia:-** If three mutually ⊥ axes OX, OY and OZ are taken and if the coordinates of any element m of the system referred to these axes be x, y & z then the quantities denoted & defined as following are called products of inertia.

$$I_{xy} = -\sum mxy$$
$$I_{yz} = -\sum myz$$
$$I_{zx} = -\sum mzx$$

Note:- Moment of inertia is always +ve but product of inertia may be +ve or −ve.

Question: With usual notations prove that

$$[\underline{L}] = [I][\underline{\omega}]$$

Solution

Consider a system of n particles of masses $m_1, m_2, \dots, m_n$ with position vectors $r_1, r_2, \dots, r_n$ respectively.

Since the angular velocity is constant, so the angular momentum of the ith particle is $\underline{L}_i = m_i[r_i^2 \underline{\omega} - (r_i \cdot \underline{\omega}) r_i]$ &

For the whole system

$$L = \sum_{i=1}^{n} m_i [r_i^2 \underline{\omega} - (r_i \cdot \underline{\omega}) r_i]$$

Now

$$r_i = x_i \hat{i} + y_i \hat{j} + z_i \hat{k}$$
$$\underline{L}_i = L_x \hat{i} + L_y \hat{j} + L_z \hat{k} \quad \&$$
$$\underline{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

So

$$L_x \hat{i} + L_y \hat{j} + L_z \hat{k} = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2 + z_i^2)(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k})$$
$$- (x_i \omega_x + y_i \omega_y + z_i \omega_z)(x_i \hat{i} + y_i \hat{j} + z_i \hat{k})$$
$$= \sum m_i [x_i^2 \omega_x + y_i^2 \omega_x + z_i^2 \omega_x] \hat{i}$$
$$+ [x_i^2 \omega_y + y_i^2 \omega_y + z_i^2 \omega_y] \hat{j} + [x_i^2 \omega_z + y_i^2 \omega_z$$
$$+ z_i^2 \omega_z] \hat{k} - [x_i^2 \omega_x + x_i y_i \omega_y + x_i z_i \omega_z] \hat{i}$$
$$- [x_i y_i \omega_x + y_i^2 \omega_y + z_i y_i \omega_z] \hat{j} - [x_i \omega_x z_i$$
$$+ y_i z_i \omega_y + z_i^2 \omega_z] \hat{k}$$

comparing $\hat{i}$ coordinate on both sides

$$\underline{L}_x = \sum m_i [(\cancel{x_i^2} + y_i^2 + z_i^2] \omega_x - [\cancel{x_i^2 \omega_x} + x_i y_i \omega_y + x_i z_i \omega_z]$$

$$\Rightarrow L_x = \sum m_i [(y_i^2 + z_i^2) \omega_x - x_i y_i \omega_y - x_i z_i \omega_z]$$

$$\Rightarrow L_x = \sum m_i (y_i^2 + z_i^2) \omega_x + (-\sum m_i x_i y_i) \omega_y) +$$
$$(-\sum m_i x_i z_i) \omega_z$$

$$\Rightarrow L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

Similarly

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

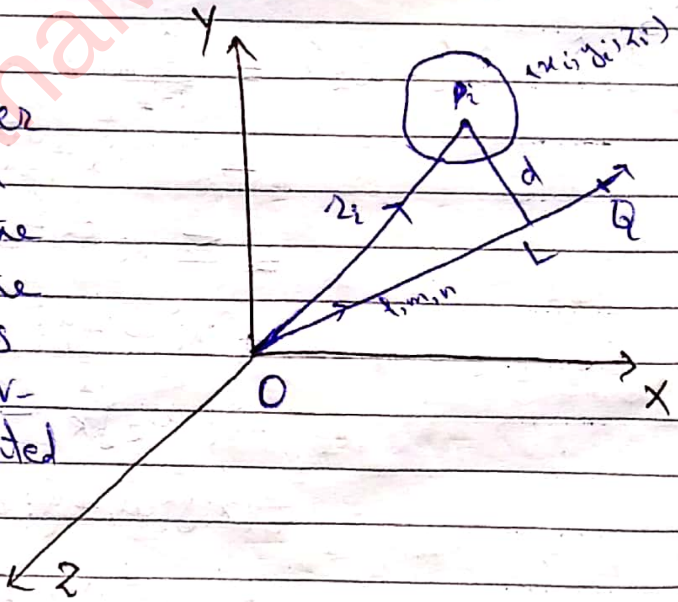
or
$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\Rightarrow [\underline{L}] = [I][\underline{\omega}]$$

⇒ **Dr. Routh's Rule:-** Calculate moment of inertia of a rigid body about a line.

Solution

Let us consider a rigid body such that P_i denotes the ith particle of the body with mass m_i and position vector r_i and situated at a point P (x_i, y_i, z_i).



Let OQ be the line about which moment of inertia is to calculate. Let λ, μ, ν be the direction cosines of the line OQ & $PL = d_i$ be the ⊥ distance of P from line OQ. then

$$d_i^2 = (OP)^2 - (OL)^2$$

$$= (x_i^2 + y_i^2 + z_i^2) - (\lambda x_i + \mu y_i + \nu z_i)^2$$

$$= x_i^2 + y_i^2 + z_i^2 - \lambda^2 x_i^2 - \mu^2 y_i^2 - \nu^2 z_i^2 - 2\lambda\mu x_i y_i - 2\mu\nu y_i z_i - 2\lambda\nu x_i z_i$$

$$= (1-\lambda^2)x_i^2 + (1-\mu^2)y_i^2 + (1-\nu^2)z_i^2 - 2\lambda\mu x_i y_i - 2\mu\nu y_i z_i - 2\nu\lambda x_i z_i$$

As $\lambda^2 + \mu^2 + \nu^2 = 1$ so

$$d_i^2 = (\mu^2+\nu^2)x_i^2 + (\lambda^2+\nu^2)y_i^2 + (\lambda^2+\mu^2)z_i^2 - 2\lambda\mu x_i y_i - 2\mu\nu y_i z_i - 2\lambda\nu x_i z_i$$

$$= (y_i^2+z_i^2)\lambda^2 + (x_i^2+z_i^2)\mu^2 + (x_i^2+y_i^2)\nu^2 - 2\lambda\mu x_i y_i - 2\mu\nu y_i z_i - 2\lambda\nu x_i z_i$$

Now

$$dI_{OQ} = m_i d_i^2$$

For the whole rigid body

$$I_{OQ} = \sum_i m_i d_i^2$$

$$= \sum_i m_i(y_i^2+z_i^2)\lambda^2 + \sum_i m_i(x_i^2+z_i^2)\mu^2 + \sum_i m_i(x_i^2+y_i^2)\nu^2 + 2\lambda\mu\left(-\sum_i m_i x_i y_i\right) + 2\mu\nu\left(-\sum_i m_i y_i z_i\right) + 2\lambda\nu\left(-\sum_i m_i x_i z_i\right)$$

$$\Rightarrow I_{OQ} = \lambda^2 I_{xx} + \mu^2 I_{yy} + \nu^2 I_{zz} + 2\lambda\mu I_{xy} + 2\mu\nu I_{yz} + 2\nu\lambda I_{zx}$$

$$\Rightarrow I_{OQ} = A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu$$

Here A, B, C are moment of inertia about axes and F, G, H are product of inertia about YZ, ZX and XY planes respectively of the rigid body.

⇒ Momental Ellipsoid:-

We know by Dr. Routh's rule the moment of inertia of rigid body about any line l, with direction cosines λ, μ, ν is given by

$$I = I_l = A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\lambda\nu + 2H\lambda\mu \quad \text{①}$$

Now if we choose a point $P(x, y, z)$ on the line l s.t

$$|OP| = \frac{1}{\sqrt{I}}, \text{ then}$$

$$\lambda = \frac{x}{|OP|} = x\sqrt{I}$$

$$\mu = \frac{y}{|OP|} = y\sqrt{I}, \quad \nu = \frac{z}{|OP|} = z\sqrt{I}$$

using these values equ ① becomes

$$I = A(x\sqrt{I})^2 + B(y\sqrt{I})^2 + C(z\sqrt{I})^2 + 2F(y\sqrt{I})(z\sqrt{I}) + 2G(x\sqrt{I})(z\sqrt{I}) + 2H(x\sqrt{I})(y\sqrt{I})$$

$$I = Ax^2I + By^2I + Cz^2I + 2FyzI + 2GxzI + 2HxyI$$

$$\Rightarrow Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy = 1$$

As A, B, C are +ve so above equation represents an ellipsoid. This ellipsoid is called ellipsoid of inertia or momental ellipsoid. The momental ellipsoid gives information about moment of inertia & product of inertia at given point. If P is any point on momental ellipsoid then $|OP| = 1/\sqrt{I}$ or $I = 1/|OP|^2$ i.e moment of inertia of rigid body about any line OP is equal to resiprocal of length of $|OP|$.

⇒ Moment And Products of Inertia In Case of Continuous Distribution of Mass *

In this case

$$I_{xx} = \rho \iiint (y^2+z^2)\, dx\, dy\, dz$$

$$I_{yy} = \rho \iiint (x^2+z^2)\, dx\, dy\, dz$$

$$I_{zz} = \rho \iiint (x^2+y^2)\, dx\, dy\, dz$$

$$I_{xy} = -\rho \iiint xy\, dx\, dy\, dz$$

$$I_{yz} = -\rho \iiint yz\, dx\, dy\, dz$$

$$I_{zx} = -\rho \iiint zx\, dx\, dy\, dz$$

We also denote

$A = I_{xx}$, $B = I_{yy}$, $C = I_{zz}$

$F = I_{yz}$, $G = I_{zx}$, $H = I_{xy}$

MUHAMMAD TAHIR WATTOO

0344-8563284

Question:- State & Prove parallel axes Theorem for Products of inertia.

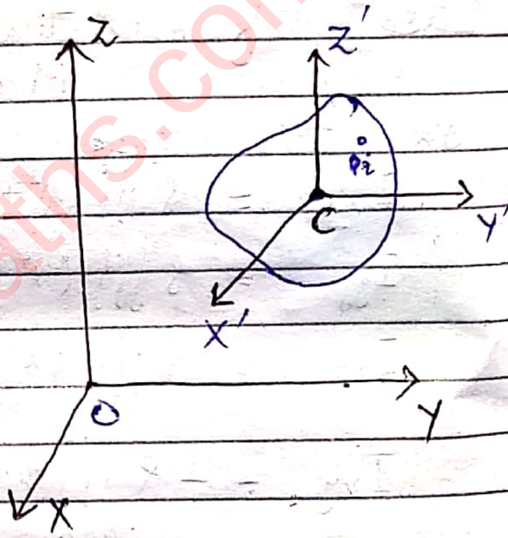
Solution

Statement:- If $CX'Y'Z'$ is the coordinate system through its centroid of the body and parallel to the coordinate system $OXYZ$ then

$$I_{xy} = I_{x'y'} - m\bar{x}\bar{y}$$
$$I_{yz} = I_{y'z'} - m\bar{y}\bar{z}$$
$$I_{zx} = I_{z'x'} - m\bar{z}\bar{x}$$

Proof:-

Let $C(\bar{x}, \bar{y}, \bar{z})$ be the coordinate of the centroid of a rigid body w.r.t the system $OXYZ$. Let us choose another system $CX'Y'Z'$ parallel to $OXYZ$.



Let (x_i, y_i, z_i) and (x_i', y_i', z_i') be the coordinates of the particle P_i of the body w.r.t $OXYZ$ & $CX'Y'Z'$ respectively. If m_i is the mass of particle P_i then

$$I_{xy} = -\sum m_i x_i y_i$$

Now $x_i = x_i' + \bar{x}$

$y_i = y_i' + \bar{y}$ & $z_i = z_i' + \bar{z}$

So

$$I_{xy} = -\sum m_i (x_i' + \bar{x})(y_i' + \bar{y})$$

$$= -\sum m_i x_i' y_i' - x \sum m_i x_i' - \bar{y} \sum m_i y_i'$$

$$- \sum m_i \bar{x}\bar{y}$$

As $\sum m_i x_i' = 0$, $\sum m_i y_i' = 0$

So
$$I_{xy} = -\sum m_i x_i' y_i' - 0 - 0 - \sum m_i \bar{x}\bar{y}$$
$$= -\sum m_i x_i' y_i' - \bar{x}\bar{y}\sum m_i$$
$$\Rightarrow I_{xy} = -\sum m_i x_i' y_i' - \bar{x}\bar{y}m$$
$$\Rightarrow I_{xy} = I_{x'y'} - m\bar{x}\bar{y}$$

Similarly
$$I_{yz} = I_{y'z'} - m\bar{y}\bar{z}$$
$$I_{zx} = I_{z'x'} - m\bar{z}\bar{x}$$

Question:- State & Prove $\parallel$ axes theorem for inertia matric OR state & Prove generalised parallel axes theorem.

Solution

Statement:- With usual notations generalised parallel axes theorem is given by
$$I_{ij} = I'_{ij} + M r_c^2 \delta_{ij} - M X_i Y_j$$

Proof:-

We now prove that the whole inertia matrix at a point can be related to the same at another point of the body. As we know that the Tensor form of inertia matrix is given by
$$I_{ij} = \sum_{\ell} m_\ell(\delta_{ij} r_\ell^2 - x_{\ell i} x_{\ell j}) \quad \text{——— (i)}$$

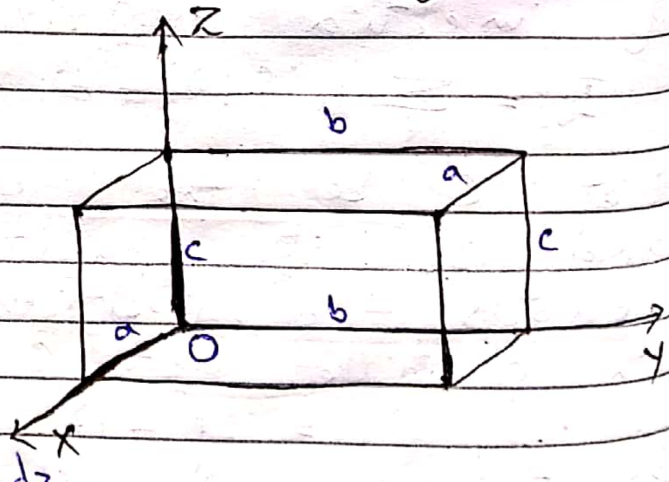
Let C be the centre of mass and r_c its position vector. Let r_ℓ' denotes the position vector of the ℓth particle w.r.t C. Then
$$r_\ell = r_\ell' + r_c$$

In terms of components
$$r_{\ell i} = r'_{\ell i} + r_{ci}$$

Question:- Calculate inertia matrix of a uniform solid rectangular box (Parallelopiped or cuboid) at or on of its corners.

Solution

Let the box be shown in figure with axes along its edges a, b & c respectively.



Then

$$I_{xx} = \rho \iiint (y^2+z^2)\,dx\,dy\,dz$$

$$\Rightarrow I_{xx} = \frac{m}{abc}\int_0^a\int_0^b\int_0^c (y^2+z^2)\,dx\,dy\,dz$$

$$= \frac{m}{abc}\left[\int_0^a dx\int_0^b y^2 dy\int_0^c dz + \int_0^a dx\int_0^b dy\int_0^c z^2 dz\right]$$

$$= \frac{m}{abc}\left[a\cdot\frac{b^3}{3}\cdot c + ab\cdot\frac{c^3}{3}\right]$$

$$= \frac{m}{abc}\left[\frac{ab^3c}{3} + \frac{abc^3}{3}\right]$$

$$= \frac{m}{a\cancel{b}c}\left[\frac{a\cancel{b}c(b^2+c^2)}{3}\right]$$

$$\Rightarrow I_{xx} = \frac{1}{3}m(b^2+c^2)$$

Similarly $I_{yy} = \frac{1}{3}m(a^2+c^2)$ & $I_{zz} = \frac{1}{3}m(a^2+b^2)$

Now $I_{xy} = -\delta\iiint xy\,dx\,dy\,dz$

$$= \frac{-m}{abc}\int_0^a\int_0^b\int_0^c xy\,dx\,dy\,dz$$

$$= \frac{-m}{abc}\int_0^a x\,dx\int_0^b y\,dy\int_0^c dz$$

$$= \frac{-m}{\cancel{abc}}\left(\frac{a^{\cancel{2}}}{2}\right)\left(\frac{b^{\cancel{2}}}{2}\right)(\cancel{c}) = \frac{-1}{4}mab$$

Similarly $I_{yz} = \frac{-1}{4}mbc$, $I_{zx} = \frac{-1}{4}mca$

Hence Inertia matrix is given by

$$I_{ij} = \begin{bmatrix} \frac{1}{3}m(b^2+c^2) & -\frac{1}{4}mab & -\frac{1}{4}mac \\ -\frac{1}{4}mab & \frac{1}{3}m(a^2+c^2) & -\frac{1}{4}mbc \\ -\frac{1}{4}mac & -\frac{1}{4}mbc & \frac{1}{3}m(a^2+b^2) \end{bmatrix}$$

$$\Rightarrow I_{ij} = \frac{1}{12} m \begin{bmatrix} 4(b^2+c^2) & -3ab & -3ac \\ -3ab & 4(a^2+c^2) & -3bc \\ -3ac & -3bc & 4(a^2+b^2) \end{bmatrix}$$

For cube $a = b = c$ then

$$I_{ij} = \frac{1}{12} m \begin{bmatrix} 4(a^2+a^2) & -3a^2 & -3a^2 \\ -3a^2 & 4(a^2+a^2) & -3a^2 \\ -3a^2 & -3a^2 & 4(a^2+a^2) \end{bmatrix}$$

$$\Rightarrow I_{ij} = \frac{1}{12} m a^2 \begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}$$

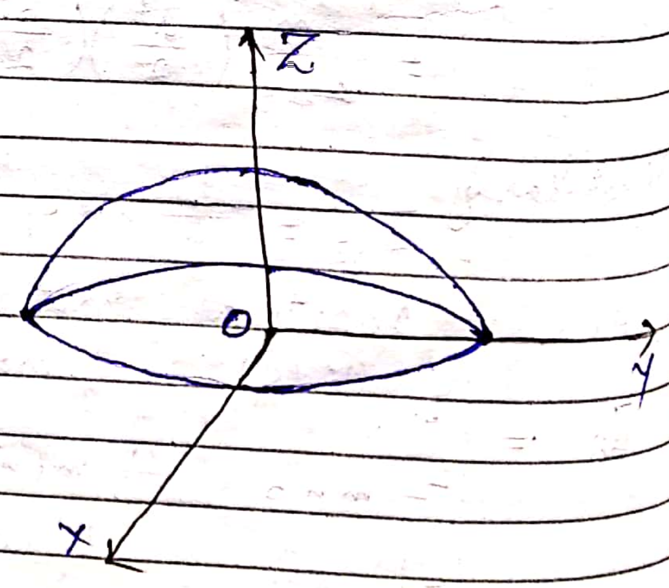
Question. Find the moment of inertia of a uniform hemisphere of mass m and radius a about

(i) Its axes of symmetry

(ii) An axes $\perp$ to the axes of symmetry & passing through the centre of the base.

Also calculate the inertia matrix for solid sphere at its centre.

Solution Let O be the centre of the base and we choose a coordinate system OXYZ. s.t z-axes is the axes of symmetry



and X-axis & Y-axis are the axes $\perp$ of symmetry axes and through O

(i) Now $I_{zz} = \rho \int_V (y^2+x^2)\, dV$

Put $x = r\sin\theta \cos\phi$

$y = r\sin\theta \sin\phi$

$z = r\cos\theta$

$0 \leq r \leq a$, $0 \leq \theta \leq \pi/2$, $0 \leq \phi \leq 2\pi$

& $dx\,dy\,dz = J\, dr\, d\theta\, d\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi$

So $I_{zz} = \dfrac{m}{\frac{2}{3}\pi a^3} \int_0^a \int_0^{\pi/2} \int_0^{2\pi} (r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi)(r^2\sin\theta\, dr\, d\theta\, d\phi)$

$$= \frac{3m}{2\pi a^3} \int_0^a \int_0^{\pi/2} \int_0^{2\pi} r^4 \sin^3\theta\, dr\, d\theta\, d\phi$$

$$= \frac{3m}{2\pi a^3} \int_0^a r^4\, dr \int_0^{\pi/2} \sin^3\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{3m}{2\pi a^3} \left(\frac{a^5}{5}\right)\left(\frac{2}{3}\right)(2\pi) = \frac{2}{5} m a^2$$

(ii) $I_{xx} = \rho \iiint (y^2+z^2)\, dx\, dy\, dz$

$$= \frac{3m}{2\pi a^3} \int_0^a \int_0^{\pi/2} \int_0^{2\pi} (r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta)(r^2\sin\theta\, dr\, d\theta\, d\phi)$$

$$= \frac{3m}{2\pi a^3} \left[\int_0^a r^4\, dr \int_0^{\pi/2} \sin^3\theta\, d\theta \int_0^{2\pi} \sin^2\phi\, d\phi + \int_0^a r^4\, dr \int_0^{\pi/2} \cos^2\theta \sin\theta\, d\theta \int_0^{2\pi} d\phi \right.$$

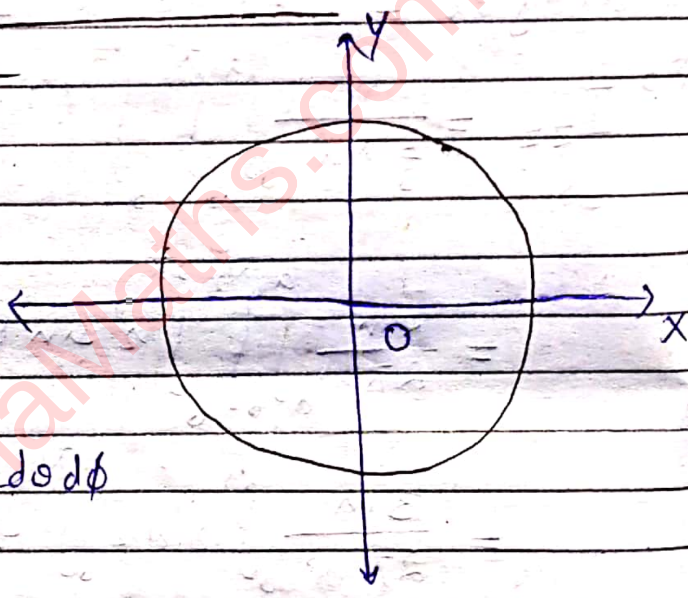
$$I_{xx} = \frac{3m}{2\pi a^3}\left[\frac{a^5}{5}\cdot\frac{2}{3}\cdot\pi + \frac{a^5}{5}\left(\frac{1}{3}\right)(2\pi)\right]$$

$$= \frac{3m}{2\pi a^3}\left[\frac{2\pi a^5}{3\times 5} + \frac{2\pi a^5}{3\times 5}\right]$$

$$\Rightarrow I_{xx} = \frac{2}{5} m a^2$$

Similarly $I_{yy} = \frac{2}{5} m a^2$

Now for Sphere:-



$$I_{xx} = \rho \iiint (y^2+z^2)\, dv$$

$$= \frac{m}{\frac{4}{3}\pi a^3}\int_0^a\int_0^\pi\int_0^{2\pi} (r^2\sin^2\theta \sin^2\phi + r^2\cos^2\theta)(r^2\sin\theta\, dr\, d\theta\, d\phi)$$

$$= \frac{2}{5} m a^2$$

Similarly $I_{yy} = \frac{2}{5} m a^2$ & $I_{zz} = \frac{2}{5} m a^2$

Now $I_{xy} = -\rho \iiint xy\, dx\, dy\, dz$

$$= \frac{-m}{\frac{4}{3}\pi a^3}\int_0^a\int_0^\pi\int_0^{2\pi} (r\sin\theta\cos\phi)(r\sin\theta\sin\phi)(r^2\sin\theta\, dr\, d\theta\, d\phi)$$

$$= \frac{-3m}{4\pi a^3}\int_0^a r^4\, dr \int_0^\pi \sin^3\theta \int_0^{2\pi} \sin\phi\cos\phi\, d\phi$$

$$\Rightarrow I_{xy} = \frac{-3m}{4\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{4}{3}\right)(0) = 0$$

Similarly $I_{yz} = 0$ & $I_{zx} = 0$

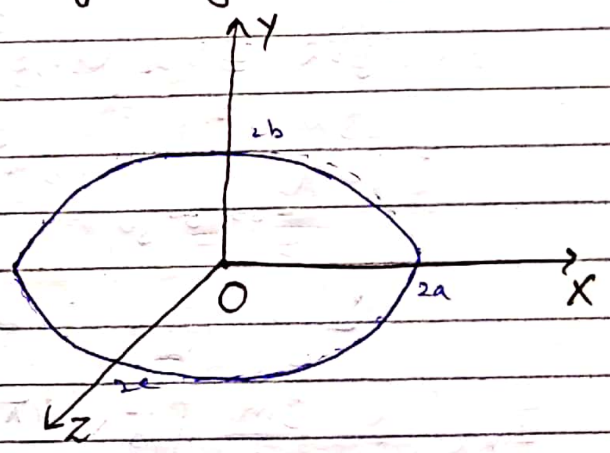
So

$$I_{ij} = \begin{bmatrix} \frac{2}{5}ma^2 & 0 & 0 \\ 0 & \frac{2}{5}ma^2 & 0 \\ 0 & 0 & \frac{2}{5}ma^2 \end{bmatrix}$$

$$\Rightarrow I_{ij} = \frac{2}{5}ma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question:- Find moment of inertia & Product of inertia for the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ w.r.t its axes of symmetry.

Solution consider an ellipsoid of mass m, of axes $2a, 2b$ and $2c$. Consider a co-ordinate system OXYZ through the centre of mass O of ellipsoid s.t $2a$ is along X-axes, $2b$ along Y-axes and $2c$ along z-axes. Then we have the equation



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Now as X-axes is the axis of symmetry so

$$I_{xx} = \rho \iiint (y^2+z^2)\,dx\,dy\,dz$$

Put $x = ax'$, $y = by'$, $z = cz'$

$$\Rightarrow dx\,dy\,dz = abc\,dx'\,dy'\,dz'$$

& $$\frac{(ax')^2}{a^2} + \frac{(by')^2}{b^2} + \frac{(cz')^2}{c^2} = 1$$

$$\Rightarrow x'^2 + y'^2 + z'^2 = 1$$

which is a unit sphere.

And
$$I_{xx} = \frac{m}{\frac{4}{3}\pi abc} \iiint (b^2y'^2 + c^2z'^2)\,abc\;dx\,dy\,dz$$

$$\Rightarrow I_{xx} = \frac{3m}{4\pi} \iiint (b^2y'^2 + c^2z'^2)\,dx\,dy\,dz$$

Put $x' = r\sin\theta\cos\phi$ $\quad 0 \le r \le 1$

$y' = r\sin\theta\sin\phi$ $\quad 0 \le \theta \le \pi$

$z' = r\cos\theta$ $\quad 0 \le \phi \le 2\pi$

$$dx'\,dy'\,dz' = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$I_{xx} = \frac{3m}{4\pi} \int_0^1 \int_0^{\pi} \int_0^{2\pi} (b^2r^2\sin^2\theta\sin^2\phi + c^2r^2\cos^2\theta) \times r^2\sin\theta\,dr\,d\theta\,d\phi$$

By previous question

$$I_{xx} = \frac{3m}{4\pi}\left[b^2 \times \frac{4\pi}{15} + c^2 \times \frac{4\pi}{15}\right]$$

$$= \frac{3m}{4\pi} \times \frac{4\pi}{15}\left[b^2 + c^2\right]$$

$$I_{xx} = \frac{1}{5} m(b^2 + c^2)$$

Similarly $I_{yy} = \frac{1}{5} m(a^2 + c^2)$

$$I_{zz} = \frac{1}{5} m(a^2 + b^2)$$

Now $I_{xy} = -\rho \iiint xy \, dx\, dy\, dz$

$$= -\rho \iiint ax' by' abc \, dx' dy' dz'$$

$$= \frac{-3m}{4\pi} ab \iiint x' y' \, dx' dy' dz'$$

$$= \frac{-3m}{4\pi} ab \int_0^1 \int_0^\pi \int_0^{2\pi} r^4 \sin^3\theta \sin\phi \cos\phi \, dr\, d\theta\, d\phi$$

$$= 0 \quad \because \int_0^{2\pi} \sin\phi \cos\phi \, d\phi = 0$$

Similarly $I_{yz} = 0$ & $I_{zx} = 0$

Question- Show that in matrix notation

$$[\dot{L}] = [\underline{\omega} \times L] + [I][\dot{\underline{\omega}}]$$

Solution As we know that

$$\underline{L} = \sum_i r_i \times (m_i v_i)$$

$$\Rightarrow \frac{dL}{dt} = \frac{d}{dt} \sum_i r_i \times (m_i v_i)$$

$$= \sum_i \dot{r}_i \times (m_i v_i) + \sum_i r_i \times (m_i \dot{v}_i)$$

$$= \sum_i v_i \times (m_i v_i) + \sum_i r_i \times (m_i \dot{v}_i)$$

$$= 0 + \sum_i r_i \times \left(m_i \frac{d}{dt}(\underline{\omega} \times r_i)\right)$$

$$= \sum_i r_i \times [m_i (\underline{\omega} \times \dot{r}_i + \dot{\underline{\omega}} \times r_i)]$$

$$\Rightarrow \dot{L} = \sum_i r_i \times m_i (\omega \times \dot{r}_i) + \sum_i r_i \times m_i (\dot{\omega} \times r_i)$$

$$\Rightarrow \dot{\underline{L}} = \sum_i r_i \times m_i(\underline{\omega} \times v_i) + [I][\dot{\underline{\omega}}]$$

$$= \sum_i r_i \times m_i(\underline{\omega} \times (\underline{\omega} \times r_i)) + [I][\dot{\underline{\omega}}]$$

$$= \sum_i r_i \times m_i[(\underline{\omega} \cdot r_i)\underline{\omega} - (\underline{\omega} \cdot \underline{\omega}) r_i] + [I][\dot{\underline{\omega}}]$$

$$= \sum_i r_i \times m_i(\underline{\omega} \cdot r_i)\underline{\omega} - 0 + [I][\dot{\underline{\omega}}]$$

$$= -\sum_i \underline{\omega} \times m_i(\underline{\omega} \cdot r_i) r_i + [I][\dot{\underline{\omega}}]$$

$$= \sum_i \underline{\omega} \times m_i [r_i \times (\underline{\omega} \times r_i)] + [I][\dot{\underline{\omega}}]$$

$$= \sum_i \omega \times m_i(r_i \times v_i) + [I][\dot{\underline{\omega}}]$$

$$= \underline{\omega} \times \sum_i r_i \times m_i v_i + [I][\dot{\underline{\omega}}]$$

$$= [\underline{\omega} \times \underline{L}] + [I][\dot{\underline{\omega}}]$$

$$\Rightarrow \dot{\underline{L}} = [\underline{\omega} \times \underline{L}] + [\underline{I}][\dot{\underline{\omega}}]$$

Question:- A body of mass $M = 30$ Kg is running with a velocity of 3.0 ~~second~~ meter per second on ground just tangently to a merry go round which is at rest. The boy suddenly jumps on the merry go round. Calculate the angular velocity acquired by the system. The merry go round has a radius of $r = 2.0$ meter and a mass $m = 120$ Kg and its moment of inertia is $120\ Kg/m^2$.

Solution:- The merry-go-round rotates about an axis which we regard as passing through its centre of mass. The moment of inertia I_1 of the boy about the axis of rotation is given by

$$I_1 = Md^2 = 30 \times 2 \times 2$$
$$= 120 \, Kg/m^2$$

The moment of inertia of the merry go round is given, say

$$I_2 = 120 \, Kg/m^2$$

Since the velocity of the boy about the merry go round is 3.0 m/s So this angular velocity about the axis of rotation is therefore.

$$\omega_1 = \frac{v}{d} = \frac{3}{2} = 1.5 \text{ rad/sec}$$

Initially the merry go round is at rest and therefore for its angular velocity $\omega_2 = 0$ we ignore friction & therefore there is no external torque on the system. Hence by the law of conservation of angular momentum

$$I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\omega$$

where ω is the angular velocity of the system when the boy jumps on the merry go round.

On substitution we obtain

$$\omega = \frac{120 \times 1.5 + 0}{120 + 240} = 0.75 \text{ rad/sec}$$