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MATHEMATICAL PHYSICS

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Book:- Mathematical Physics
By Dr. Khalid Latif Mir

⇒ Sturm Liouville System OR SL System

* Jaques Charles Francois sturm (1803 – 1855) was born in switzerland and then moved to Paris
* Joseph Liouville (1809 – 1882) was a professor in paris and worked in complex analysis, differential geometry & no. theory.

⇒ SL Equation:- A 2nd order linear homogenous differential equation of the form

$$\frac{d}{dx}\left\{P(x)\frac{dy}{dx}\right\} + q(x)\,y(x) + \lambda\, r(x)\, y(x) = 0$$

is called Sturm Liouville's equation if $P(x), P'(x), q(x), r(x)$ are all real and continuous over an interval $[a, b]$. If we define an operator

$$\mathcal{L} = \frac{d}{dx}\left[P(x)\frac{d}{dx}\right] + q(x)$$

Then above equation takes the form

$$\mathcal{L}\, y(x) + \lambda\, r(x)\, y(x) = 0$$

Examples:- ① The Legender differential Equation

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$$

is a SL equation with $p(x) = 1 - x^2$, $q(x) = 0$, $r(x) = 1$ & $\lambda = n(n+1)$

② The Airy's differential equation

$$\frac{d^2y}{dx^2} + xy = 0$$

is also a SL equation. It can be written as

$$\frac{d}{dx}\left\{1 \cdot \frac{dy}{dx}\right\} + 0 \cdot y + 1 \cdot x \cdot y$$

Here $p(x) = 1$, $q(x) = 0$, $r(x) = x$ & $\lambda = 1$

⇒ SL System:- An SL equation with some suitable initial and boundry value conditions is called SL system.

⇒ **Regular SL Equation:-** An SL equation is said to be regular over interval $[a,b]$ if the co-efficients $p(x)$ and $r(x)$ do not vanish over $[a,b]$

Examples:- ① The Legender differential equation
$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n(n+1)y=0$$
is not regular on $[a,b]$ iff $\pm 1 \in [a,b]$. Because here $p(x)=1-x^2$ which vanishes in any interval containing ± 1.

② The equation $u''+\lambda u=0$ is a regular SL equation over any interval $[a,b]$ because $u''+\lambda u=0$ can be written as
$$\frac{d}{dx}\left\{1.\frac{du}{dx}\right\}+0.u+\lambda.1.u=0$$
$$\Rightarrow p(x)=1 \ , \ q(x)=0 \ , \ r(x)=1$$
Here we note that $p(x)$ and $r(x)$ can never vanish on any interval $[a,b]$

⇒ **Regular SL System:-** A regular SL equation along with some suitable end point conditions is called regular SL system. In general if $\mathcal{L}(u)+\lambda r(x)u=0$ is a regular SL equation on $[a,b]$ the end point conditions are of the form
$$\alpha u(a)+\alpha' u'(a)=0$$
$$\beta u(b)+\beta' u'(b)=0$$

⇒ **Singular SL Equation:-** An SL equation which is not regular is called singular SL equation.

Examples:- ① The differential equation

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$$ is a singular SL equation on $[0,1]$

② The differential equation

$$\frac{d^2y}{dx^2} + xy = 0$$ is singular SL equation on $[0,2]$

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⇒ **Eigen values & Eigen functions of an SL System:-** If in an SL system

$$\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} + q(x)y + \lambda r(x)y = 0$$

corresponding to a value of "λ" there exist some non-zero solution $y(x)$ of the system then that perticular value of λ is called an eigen value of the system and the corresponding solution $y(x)$ is called eigen function of the system.

e.g.- The system $u'' + \lambda u = 0$ with $u(0) = 1$ and $u'(0) = -2$

Then for $\lambda = 4$

$u'' + 4u = 0 \Rightarrow u = c_1 \sin 2x + c_2 \cos 2x$

$u(0) = 1 \Rightarrow c_2 = 1$

& $u'(0) = -2 \Rightarrow 2c_1 = -2 \Rightarrow c_1 = -1$

$\Rightarrow u = -1 \cdot \sin 2x + 1 \cdot \cos 2x$

$\Rightarrow u = \cos 2x - \sin 2x$

Hence here $\lambda = 4$ is the eigen value and $u = \cos 2x - \sin 2x$ is the corresponding eigen solution.

Example:- Find eigen values and eigen functions of the regular SL system $u'' + \lambda u = 0$ with $u(0) = 0$ & $u(\pi) = 0$

Solution:-

Given system is $u'' + \lambda u = 0$,
$u(0) = 0$, $u(\pi) = 0$

Now corresponding characteristic equation is
$D^2 + \lambda = 0 \Rightarrow D^2 = -\lambda \Rightarrow D = \pm i\sqrt{\lambda}$

$\Rightarrow u(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

Now $u(0) = 0 \Rightarrow A\cos(0) + B\sin(0) = 0$
$\Rightarrow A(1) + B(0) = 0 \Rightarrow A = 0$
$\Rightarrow u(x) = B\sin\sqrt{\lambda}\,x$

Next $u(\pi) = 0 \Rightarrow B\sin\sqrt{\lambda}\,\pi = 0$
$\Rightarrow B = 0$ or $\sin\sqrt{\lambda}\,\pi = 0$

But $B \neq 0$ $\because$ If $B = 0$ then $u(x) = 0$ which is not true. So $B \neq 0$ then $\sin\sqrt{\lambda}\pi = 0$
$\Rightarrow \sqrt{\lambda}\pi = n\pi$, where $n = 0, 1, 2, 3, \ldots$

but note that if $n = 0$, then $\lambda = 0$
then $u'' + 0 \cdot u = 0 \Rightarrow u'' = 0$
$\Rightarrow u' = c_1 \Rightarrow u = c_1 x + c_2$

and then $u(0) = 0 \Rightarrow c_1(0) + c_2 = 0$
$\Rightarrow c_2 = 0$
$\Rightarrow u(x) = c_1 x$

Now $u(\pi) = 0 \Rightarrow c_1(\pi) = 0 \Rightarrow c_1 = 0$
$\Rightarrow u = 0$

which is not possible. So $n \neq 0$
$\Rightarrow \sqrt{\lambda} = n$, $n = 1, 2, 3, \ldots$
$\Rightarrow \lambda = n^2$, $n = 1, 2, 3, \ldots$
$\Rightarrow \lambda = \lambda_n = n^2$, $n = 1, 2, 3$ are eigenvalues

of the system $U_n(x) = B_n \sin nx$, $n = 1, 2, 3, \ldots$ are corresponding eighen functions or eigen solutions of the system.

——————★★——————★★

Example:- Determine Eigen values of the system $u'' + \lambda u = 0$ with $u(0) = u(\pi)$ and $u'(0) = 2u'(\pi)$

Solution:-

Given system is $u'' + \lambda u = 0$

Auxiliary equation is

$D^2 + \lambda = 0 \Rightarrow D^2 = -\lambda \Rightarrow D = \pm i\sqrt{\lambda}$

$\Rightarrow u(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

Now $u(0) = u(\pi)$

$\Rightarrow A\cos(0) + B\sin(0) = A\cos\sqrt{\lambda}\pi + B\sin\sqrt{\lambda}\pi$

$\Rightarrow A = A\cos\sqrt{\lambda}\pi + B\sin\sqrt{\lambda}\pi$

$\Rightarrow (\cos\sqrt{\lambda}\pi - 1)A + (\sin\sqrt{\lambda}\pi)B = 0$ —————— ①

Now $u'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$

& $u'(0) = 2u'(\pi)$

$\Rightarrow -A\sqrt{\lambda}\sin\sqrt{\lambda}(0) + B\sqrt{\lambda}\cos\sqrt{\lambda}(0) = -2A\sqrt{\lambda}\sin\sqrt{\lambda}\pi + 2B\sqrt{\lambda}\cos\sqrt{\lambda}\pi$

$\Rightarrow B\sqrt{\lambda} = -2A\sqrt{\lambda}\sin\sqrt{\lambda}\pi + 2B\sqrt{\lambda}\cos\sqrt{\lambda}\pi$

$\Rightarrow B = -2A\sin\sqrt{\lambda}\pi + 2B\cos\sqrt{\lambda}\pi$

$\Rightarrow (2\sin\sqrt{\lambda}\pi)A + (1 - 2\cos\sqrt{\lambda}\pi)B = 0$ —————— ②

For non trivial values of A & B

$$\begin{vmatrix} \cos\sqrt{\lambda}\pi - 1 & \sin\sqrt{\lambda}\pi \\ 2\sin\sqrt{\lambda}\pi & 1 - 2\cos\sqrt{\lambda}\pi \end{vmatrix} = 0$$

$$\Rightarrow (\cos\sqrt{\lambda}\pi - 1)(1 - 2\cos\sqrt{\lambda}\pi) - 2\sin^2\sqrt{\lambda}\pi = 0$$

$$\Rightarrow \cos\sqrt{\lambda}\pi - 2\cos^2\sqrt{\lambda}\pi - 1 + 2\cos\sqrt{\lambda}\pi - 2\sin^2\sqrt{\lambda}\pi = 0$$

$$\Rightarrow 3\cos\sqrt{\lambda}\pi - 1 - 2(\cos^2\sqrt{\lambda}\pi + \sin^2\sqrt{\lambda}\pi) = 0$$

$$\Rightarrow 3\cos\sqrt{\lambda}\pi - 1 - 2(1) = 0$$

$$\Rightarrow 3\cos\sqrt{\lambda}\pi = 3 \quad \Rightarrow \cos\sqrt{\lambda}\pi = 1$$

$$\Rightarrow \sqrt{\lambda} = 2n, \quad n = 0, 1, 2, 3, \dots$$

$\Rightarrow \lambda = \lambda_n = 4n^2, \ n = 0, 1, 2, 3, \dots$ are eigen values and $u_n(x) = A_n\cos(2nx) + B_n\sin(2nx)$ are corresponding eigen function.

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⇒ Orthogonality of Eigenfunction of an SL System:-

Two eigen functions $u(x)$ and $v(x)$ of an SL system $\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} + q(x)y + \lambda r(x)y = 0$ are orthogonal on $[a, b]$ w.r.t a weight function $w(x)$ if

$$\int_a^b u(x)\, v(x)\, w(x)\, dx = 0$$

Theorem:- Prove that any two eigen functions of a Regular SL system corresponding to two distinct eigen values are orthogonal w.r.t $r(x)$.

Proof:-

Let us consider a Regular SL system.

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\, r(x)u = 0 \quad \text{over } [a,b]$$

with

$$\alpha u(a) + \alpha' u'(a) = 0 \;,\; \beta u(b) + \beta' u'(b) = 0$$

Let λ_m and λ_n be two distinct eigen values of the system and $u_m(x)$ and $u_n(x)$ be the corresponding eigen functions of the system. Then we have

$$\frac{d}{dx}\left\{p(x)\frac{du_m}{dx}\right\} + q(x)u_m(x) + \lambda_m r(x)u_m(x) = 0 \longrightarrow (1)$$

$$\left.\begin{array}{l}\alpha u_m(a) + \alpha' u_m'(a) = 0 \;,\; \beta u_m(b) + \beta' u_m'(b) = 0\end{array}\right\} \longrightarrow (2)$$

and

$$\frac{d}{dx}\left\{p(x)\frac{du_n}{dx}\right\} + q(x)u_n(x) + \lambda_n r(x)u_n(x) = 0 \longrightarrow (3)$$

$$\left.\begin{array}{l}\alpha u_n(a) + \alpha' u_n'(a) = 0 ,\; \beta u_n(b) + \beta' u_n'(b) = 0\end{array}\right\} \longrightarrow (4)$$

Now $(1)\times u_n(x) - (3)\times u_m(x)$ gives

$$\frac{d}{dx}\{p(x)u_m'(x)\}u_n(x) + \cancel{q(x)u_m(x)u_n(x)} + \lambda_m r(x)u_m(x)u_n(x)$$
$$- \frac{d}{dx}\{p(x)u_n'(x)\}u_m(x) - \cancel{q(x)u_m(x)u_n(x)} - \lambda_n r(x)u_m(x)u_n(x) = 0$$

$$\Rightarrow (\lambda_m - \lambda_n)u_m(x)u_n(x)r(x) = \frac{d}{dx}\{p(x)u_n'(x)\}u_m(x) - \frac{d}{dx}\{p(x)u_m'(x)\}u_n(x)$$

$$= \frac{d}{dx}\{p(x)u_n'(x)u_m(x) - p(x)u_m'(x)u_n(x)\}$$

$$\Rightarrow (\lambda_m - \lambda_n)\int_a^b u_m(x)u_n(x)r(x)\,dx = \int_a^b \frac{d}{dx}\{p(x)u_n'(x)u_m(x) - p(x)u_m'(x)u_n(x)\}dx$$

$$= p(x)\{u_n'(x)u_m(x)\} - p(x)u_m'(x)u_n(x)\Big|_a^b$$

$$= p(x)\{u_n'(x)u_m(x) - u_m'(x)u_n(x)\Big|_a^b$$

$$\Rightarrow (\lambda_m - \lambda_n)\int_a^b u_m(x)u_n(x)r(x)dx = p(b)\{u_n'(b)u_m(b) - u_m'(b)u_n(b)\} - p(a)\{u_n'(a)u_m(a) - u_m'(a)u_n(a)\} \longrightarrow (*)$$

From ② $u_m'(a) = \frac{-\alpha}{\alpha'}u_m(a)$, $u_m'(b) = \frac{-\beta}{\beta'}u_m(b)$

From ④ $u_n'(a) = \frac{-\alpha}{\alpha'}u_n(a)$, $u_n'(b) = \frac{-\beta}{\beta'}u_n(b)$

Then (*) becomes

$$(\lambda_m - \lambda_n)\int_a^b u_m(x)u_n(x)r(x)dx = p(b)\left[\left(\frac{-\beta}{\beta'}\cancel{u_n(b)u_m(b)}\right) - \left(\frac{-\beta}{\beta'}\cancel{u_n(b)u_m(b)}\right)\right] - p(a)\left[\left(\frac{-\alpha}{\alpha'}\cancel{u_n(a)u_m(b)}\right) - \left(\frac{-\alpha}{\alpha'}\cancel{u_n(a)u_m(b)}\right)\right]$$

$$= 0$$

$$\Rightarrow (\lambda_m - \lambda_n)\int_a^b u_m(x)u_n(x)r(x)\,dx = 0$$

As $\lambda_m \neq \lambda_n$ so $\lambda_m - \lambda_n \neq 0$

$$\Rightarrow \int_a^b u_m(x)u_n(x)r(x)\,dx = 0$$

$\Rightarrow u_m(x), u_n(x)$ are orthogonal over $[a, b]$ w.r.t $r(x)$

Theorem:- Prove that eigen values of a regular SL system are real.

Proof:-

Let us consider a regular SL system

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u(x) + \lambda r(x)u(x) = 0 \text{ over } [a,b]$$

with $\alpha u(a) + \alpha' u'(a) = 0$, $\beta u(b) + \beta' u'(b) = 0$

To prove λ is real.

Assume $\lambda = \lambda_m + i\lambda_n$ and let

$u_x = u_m(x) + i u_n(x)$ be the corresponding eigen function. Then we have

$$\frac{d}{dx}\left\{p(x)\frac{d}{dx}(u_m+iu_n)\right\}+q(x)(u_m+iu_n)+(\lambda_m+i\lambda_n)(u_m+iu_n)\,r(x)=0$$

$$\frac{d}{dx}\left\{p(x)u_m'(x)\right\}+q(x)u_m(x)+(\lambda_m u_m-\lambda_n u_n)r(x)=0 \longrightarrow (1)$$

& $$\frac{d}{dx}\left\{p(x)u_n'(x)\right\}+q(x)u_n(x)+(\lambda_m u_n+\lambda_n u_m)r(x)=0 \longrightarrow (2)$$

Real
Img

Also

$$\left.\begin{array}{l}\alpha u_m(a)+\alpha' u_m'(a)=0,\ \alpha u_n(a)+\alpha' u_n'(a)=0\\ \beta u_m(b)+\beta' u_m'(b)=0,\ \beta u_n(b)+\beta' u_n'(b)=0\end{array}\right\}\rightarrow (3)$$

(1) $\times u_n(x)$ − (2) $\times u_m(x)$ gives

$$\frac{d}{dx}\left\{p(x)\left[u_m'(x)u_n(x)-u_n'(x)u_m(x)\right]\right\}+\left[\lambda_m \cancel{u_m(x)}\,u_n(x)-\lambda_n u_n^2(x)\,r(x)\right]-\left[\cancel{\lambda_m u_n(x)u_m(x)}+\lambda_n u_m^2(x)\,r(x)\right]=0$$

$$\lambda_n\left[u_m^2(x)+u_n^2(x)\right]r(x)=\frac{d}{dx}\left\{p(x)\left(u_m'(x)u_n(x)-u_n'(x)u_m(x)\right)\right\}$$

Next Integrate

$$\Rightarrow \lambda_n\int_a^b\left[u_m^2(x)+u_n^2(x)\right]r(x)\,dx=0$$

$$\Rightarrow \lambda_n=0 \quad \because r(x)\neq 0 \ \&\ u(x)\neq 0$$

$$\Rightarrow \lambda=\lambda_m+i(0) \quad\Rightarrow\quad \lambda=\lambda_m$$

$\Rightarrow \lambda$ is real.

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⇒ Periodic Function:- A real valued function f is said to be periodic function of "λ" iff λ is the least +ve real number s.t $f(x) = f(x+\lambda)$, for all x.

e.g:- (i) $\sin x$, $\cos x$ are periodic function of period 2π

(ii) $\tan x$ is a period function of π.

⇒ Periodic SL System:- An SL equation

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda r(x)u = 0$$

defined over $[a,b]$ is said to form periodic SL system provides $p(a) = p(b)$ and end point conditions are of the form $u(a) = u(b)$ and $u'(a) = u'(b)$

Theorem:- Prove that eigen functions corresponding to distinct eigen values of a periodic SL system are orthogonal w.r.t weight function $r(x)$

Proof:-

Consider a periodic SL system

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda r(x)u = 0$$

defined over $[a, b]$ with $u(a) = u(b)$, $u'(a) = u'(b)$

Let $u_1(x)$, $u_2(x)$ be eigen functions of the system corresponding to distinct eigen values λ_1 and λ_2 respectively then

$$\frac{d}{dx}\left\{p(x)\frac{du_1}{dx}\right\} + q(x)u_1(x) + \lambda_1 r(x)u_1(x) = 0 \longrightarrow (1)$$

$$u_1(a) = u_1(b), \quad u_1'(a) = u_1'(b) \longrightarrow (2)$$

& $\frac{d}{dx}\left\{p(x)\frac{du_2}{dx}\right\} + q(x)u_2(x) + \lambda_2 r(x) u_2(x) \longrightarrow ③$

$u_2(a) = u_2(b) \quad , \quad u_2'(a) = u_2'(b)] \longrightarrow ④$

Now ① $\times u_2(x)$ — ③ $\times u_1(x)$ gives

$$\frac{d}{dx}\left\{p(x)u_1'(x)\right\}u_2(x) + q(x)u_1(x)u_2(x) + \lambda_1 r(x)u_1(x)u_2(x) = 0$$

$$\frac{d}{dx}\left\{p(x)u_2'(x)\right\}u_1(x) + q(x)u_1(x)u_2(x) + \lambda_2 r(x)u_1(x)u_2(x) = 0$$

$$\Rightarrow \frac{d}{dx}\left\{p(x)u_1'(x)\right\}u_2(x) - \frac{d}{dx}\left\{p(x)u_2'(x)\right\}u_1(x) + (\lambda_1 - \lambda_2) r_1(x)u_1(x)u_2(x) = 0$$

$$\Rightarrow (\lambda_2 - \lambda_1) r(x)u_1(x)u_2(x) = \frac{d}{dx}\left\{p(x)\left[u_1'(x)u_2(x) - u_2'(x)u_1(x)\right]\right\}$$

$$\Rightarrow \int_a^b (\lambda_2 - \lambda_1)u_1(x)u_2(x)r(x)\,dx = p(x)\left[u_1'(x)u_2(x) - u_2'(x)u_1(x)\right]\Big|_a^b$$

$$\Rightarrow (\lambda_2 - \lambda_1)\int_a^b u_1(x)u_2(x)r(x)\,dx = p(b)\left[u_1'(b)u_2(b) - u_2'(b)u_1(b)\right] - p(a)\left[u_1'(a)u_2(a) - u_2'(a)u_1(a)\right]$$

$$\Rightarrow p(a)\left[u_1'(a)u_2(a) - u_2'(a)u_1(a)\right] - p(a)\left[u_1'(a)u_2(a) - u_2'(a)u_1(a)\right] = 0$$

$$\Rightarrow (\lambda_2 - \lambda_1)\int_a^b u_1(x)u_2(x)r(x)\,dx = 0$$

As $\lambda_1 \neq \lambda_2$ so $\lambda_2 - \lambda_1 \neq 0$

$$\Rightarrow \int_a^b u_1(x)u_2(x)r(x)\,dx = 0$$

$\Rightarrow u_1$ and u_2 are orthogonal w.r.t weight function $r(x)$

⇒ Square Integrable Function:-

A function $f(x)$ is said to be square integrable function w.r.t weight function $w(x) > 0$ over $[a, b]$ if

$$\int_a^b w(x)|f(x)|^2 dx < \infty$$

⇒ Lagrange Identity:-

Suppose $u(x)$ and $v(x)$ are solutions of an SL system then the following identity always hold.

$$u\mathcal{L}(v) - v\mathcal{L}(u) = \frac{d}{dx}\left\{p(x)\left[u(x)v'(x) - v(x)u'(x)\right]\right\}$$

which is called differentiable form of Lagrange identity its integral form is given by

$$\int_a^b \left[u\mathcal{L}(v) - v\mathcal{L}(u)\right]dx = p(x)\left[u(x)v'(x) - v(x)u'(x)\right]\Big|_a^b$$

Derivation:- L.H.S

$$u\mathcal{L}(v) - v\mathcal{L}(u) = u\left[\frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)\right](v) - v\left[\frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)\right](u)$$

$$= u(x)\frac{d}{dx}\left\{p(x)v'(x)\right\} + \cancel{q(x)v(x)u(x)} - v(x)\frac{d}{dx}\left\{p(x)u'(x)\right\} - \cancel{q(x)u(x)v(x)}$$

$$= u(x)\frac{d}{dx}\left\{p(x)v'(x)\right\} - v(x)\frac{d}{dx}\left\{p(x)u'(x)\right\}$$

$$= \frac{d}{dx}\left\{p(x)\left[u(x)v'(x) - v(x)u'(x)\right]\right\}$$

$$= R.H.S$$

Question – Show by Lagrang's Identity that eigen values of a regular periodic SL system are real.

Solution –

Let λ be eigen value of regular SL system $\mathcal{L}(u) + \lambda r(x) u = 0$ and $u(x)$ be the corresponding eigen function.

Let $\alpha u(a) + \alpha' u'(a) = 0$

$\beta u(b) + \beta' u'(b) = 0$ be end point conditions

Now $\mathcal{L}(u) + \lambda r(x) u = 0$

$$\Rightarrow \mathcal{L}(u) = -\lambda r(x) u \longrightarrow ①$$

To prove λ is real. Suppose λ is complex then from ① $\mathcal{L}(\bar{u}) = -\bar{\lambda} r(x) \bar{u}$ $\because r$ is real

Now consider Lagranges identity

$$\int_a^b [u\mathcal{L}(v) - v\mathcal{L}(u)]dx = p(x)[uv' - vu']\Big|_a^b$$

Put $v = \bar{u}$ then

$$\int_a^b [u\mathcal{L}(\bar{u}) - \bar{u}\mathcal{L}(u)]dx = p(x)[u\bar{u}' - \bar{u}u']\Big|_a^b$$

$$\Rightarrow \int_a^b \{u(-\bar{\lambda} r(x)\bar{u}) - \bar{u}(-\lambda r(x) u)\}dx = p(b)[u(b)\bar{u}'(b) - \bar{u}(b)u'(b)] - p(a)[u(a)\bar{u}'(a) - \bar{u}(a)u'(a)] = 0$$

$$\Rightarrow \int_a^b (\lambda - \bar{\lambda}) r(x) u(x) \bar{u}(x)\, dx = 0$$

$$\Rightarrow (\lambda - \bar{\lambda}) \int_a^b r(x)|u(x)|^2 dx = 0$$

$\Rightarrow \lambda - \bar{\lambda} = 0 \Rightarrow \lambda = \bar{\lambda}$ $\quad$ [$\because r(x)$ & $u(x)$ are not zero

$\Rightarrow \lambda$ is real.

⇒ **Wronskian:-** Wronskian of two functions f & g is denoted & defined by

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

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Theorem:- Let f and g be defined over ~~linearly~~ closed interval $[a,b]$ and for some $x \in [a,b]$. $W(f(x), g(x)) \neq 0$. then f and g are linearly independent.

Proof Suppose on the contrary f and g are linearly dependent then $\exists$ some constants A and B such that atleast one of them is non-zero & $Af + Bg = 0$

i.e $Af' + Bg' = 0$

Now

$$Af(x) + Bg(x) = 0 \quad \&$$
$$Af'(x) + Bg'(x) = 0 \quad \forall x$$

Since atleast one of A and B are non-zero, so $\begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = 0 \Rightarrow W(f(x), g(x)) = 0$

which is contradiction

So our supposition is wrong. Hence f and g are linearly independent.

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Theorem:- Prove that eigen values of a regular SL system are simple. (OR) To each eigen value there corresponds only one linearly independent function OR If $u(x)$ and $v(x)$ are eigen functions of a regular SL system corresponds to same eigen value then they must differ by a multiplication constant.

Proof

Let λ be an eigen value and $u(x)$, $v(x)$ be corresponding eigen functions of a regular SL system then

$$\mathcal{L}(u) = -\lambda r(x)\, u(x) \quad \text{and}$$

$$\mathcal{L}(v) = -\lambda r(x)\, v(x)$$

Now $\quad u\mathcal{L}(v) - v\mathcal{L}(u) = 0$

$$\Rightarrow \frac{d}{dx}\left\{p(x)\left[u(x)v'(x) - v(x)u'(x)\right]\right\} = 0$$

$$\Rightarrow p(x)\left[uv' - vu'\right] = 0 \Rightarrow p(x)\, W(u,v) = 0$$

$$\Rightarrow W(u,v) = 0 \Rightarrow u \;\&\; v \text{ are lin. Indep.}$$

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Theorem An eigen value λ can be related to its eigen function $u(x)$ by Rayleigh Quotent

$$\lambda = \frac{-puu'\Big|_a^b + \int_a^b (pu'^2 - qu^2)\,dx}{\int_a^b r(x)\, u^2\, dx}$$

Proof

SL equation is given by

$$\frac{d}{dx}\left\{p(x)\, u'\right\} + q(x)u + \lambda r(x) u = 0$$

i.e. $\quad \left\{p(x)u'\right\}' + q(x)u + \lambda r(x) u = 0$

Xing both sides by u and integrating

from a to b

$$\int_a^b \left[u(P(x)u')' + q(x)u^2 + \lambda r(x)u^2\right] dx = 0$$

$$\Rightarrow \int_a^b u(P(x)u')' dx + \int_a^b q(x)u^2 dx + \lambda \int_a^b r(x)u^2 dx = 0$$

$$\Rightarrow uu'p\Big|_a^b - \int_a^b u'Pu' dx + \int_a^b qu^2 dx + \lambda \int_a^b ru^2 dx = 0$$

$$\Rightarrow \lambda \int_a^b ru^2 dx = -puu'\Big|_a^b + \int_a^b pu'^2 dx - \int_a^b qu^2 dx$$

$$\Rightarrow \lambda = \frac{-puu'\Big|_a^b + \int_a^b [pu'^2 - qu^2] dx}{\int_a^b r(x)\, u^2 dx}$$

As requ

It follows from this result that $\lambda \geq 0$ if $-puu'\Big|_a^b \geq 0$ and $q \leq 0$

This result can not be used to determine the eigen values this is so because to evaluate λ we need to know the corresponding eigen function which in term can not be found without first knowing the eigen values. However interesting & important results can be obtained from the Rayleigh Quotent without solving the differential equation.

———★★———

Example:- Using the Rayleigh Quotent discuss the sign of eigen values of the SL system

$$u'' + \lambda u = 0, \quad u(0) = 0 \quad \& \quad u(l) = 0$$

Solution

Here $p(x) = 1$, $q(x) = 0$, $r(x) = 1$

Now by Rayleigh Quotent

$$\lambda = \frac{-puu'\big|_0^l + \int_0^l \{pu'^2 - qu^2\}dx}{\int_a^l r u^2 dx}$$

$$= \frac{-uu'\big|_0^l + \int_0^l u'^2 dx}{\int_0^l u^2 dx}$$

$$= \frac{-u(l)u'(l) + u(0)u'(0) + \int_0^l u'^2 dx}{\int_0^l u^2 dx}$$

$$= \frac{\int_0^l u'^2 dx}{\int_0^l u^2 dx}$$

$\Rightarrow \lambda \geq 0 \quad \because u^2 \,\&\, u'^2 \geq 0$

If $F \geq 0$
$\Rightarrow \int_a^b F \geq 0$

However note that if $\lambda = 0$ then we have $u'' = 0 \Rightarrow u(x) = Ax + B$

Now $u(0) = 0 \Rightarrow B = 0$

$u(l) = 0 \Rightarrow A = 0$

$\Rightarrow u = 0$, which is not possible so $\lambda \neq 0$. Hence $\lambda > 0$

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⇒ Self Adjoint Operator:-

An operator A defined over a linear space of functions is said to be self adjoint if for all u and v,

$$\langle u, Av \rangle = \langle Au, v \rangle$$ or equivalently

$$\int_a^b u(Av)\,dx = \int_a^b (Au)v\,dx$$

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Theorem:- Prove that SL operator is Self Adjoint.

Proof:-

$$\int_a^b [u\mathcal{L}(v) - v\mathcal{L}(u)]\,dx = p(x)[uv' - vu']\Big|_a^b = 0$$

$$\Rightarrow \int_a^b u\mathcal{L}(v)\,dx = \int_a^b v\mathcal{L}(u)\,dx$$

$\Rightarrow \mathcal{L}$ is self Adjoint.

**

⇒ Eigen Function Expansion:-

Consider the series $f(x) = \sum_{n=0}^{\infty} a_n u_n(x) \longrightarrow (1)$

where we have assumed that the series is uniformly convergent. such a series is called generalized fourier series. Where the co-efficient "a_n" can be determined by using the orthogonality condition of the eigen function as follows

Multiply both sides of equ (1) by $u_m(x) r(x)$ and then integrating from a to b and then using orthogonality condition we have

$$\int_a^b r(x) f(x) u_m(x)\, dx = \int_a^b \left(\sum_{n=0}^{\infty} a_n u_m(x) u_n(x) r(x) \right) dx$$

$$= a_m \int_a^b u_m(x) u_m(x) r(x)\, dx$$

$$\Rightarrow a_m = \frac{\int_a^b r(x) f(x) u_m(x)\, dx}{\int_a^b r(x) u_m^2(x)\, dx}$$

$$\Rightarrow a_n = \frac{\int_a^b r(x) f(x) u_n(x)\, dx}{\int_a^b r(x) u_n^2(x)\, dx}$$

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Example:- Varify that for the SL system
$u'' + \lambda u = 0$, $u'(0) = 0$, $u'(l) = 0$

(i) There are an infinite number of eigen values with smallest but no largest

(ii) The nth eigen function has exactly $n-1$ zeros in $]0, l[$

(iii) The eigen functions are orthogonal and form a complete set.

Solution Auxialary equation is
$D^2 + \lambda = 0$

$\Rightarrow D = \pm i\sqrt{\lambda}$ $\Rightarrow$ $u(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$

$u'(x) = -c_1\sqrt{\lambda} \sin\sqrt{\lambda}\, x + c_2\sqrt{\lambda} \cos\sqrt{\lambda}\, x$

Now $u'(0) = 0 \Rightarrow 0 + c_2\sqrt{\lambda} = 0 \Rightarrow c_2 = 0$

$\Rightarrow u'(x) = -c_1\sqrt{\lambda} \sin\sqrt{\lambda}\, x$

$u'(l) = 0 \Rightarrow -c_1\sqrt{\lambda} \sin\sqrt{\lambda}\, l = 0$

$\Rightarrow \sin\sqrt{\lambda}\, l = 0 \Rightarrow \sqrt{\lambda}\, l = n\pi$

$$\Rightarrow \sqrt{\lambda} = \frac{n\pi}{\ell} \Rightarrow \lambda = \frac{n^2\pi^2}{\ell^2}, n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow \lambda_n = \frac{n^2\pi^2}{\ell^2}, \quad n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow u_n(x) = C_n \cos\left[\frac{n\pi}{\ell} x\right]$$

1) $\lambda_n = \frac{n^2\pi^2}{\ell^2}$, $n = 0, 1, 2, 3, \ldots$ shows that there are infinite numbers of eigen values with smallest but no largest.

2) Consider $u_0(x) = C_0 \cos(0x) = C_0$ which has no zero in $]0, \ell[$

Consider $u_1(x) = C_1 \cos(\pi/\ell \, x)$

Now $u_1(x) = 0 \Rightarrow C_1 \cos\left(\frac{\pi}{\ell} x\right) = 0$

$$\Rightarrow \cos\left(\frac{\pi}{\ell} x\right) = 0 \Rightarrow \frac{\pi}{\ell} x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

$$\Rightarrow \frac{x}{\ell} = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$$

$$\Rightarrow x = \pm\frac{\ell}{2}, \pm\frac{3\ell}{2}, \pm\frac{5\ell}{2}, \ldots$$

$\Rightarrow x = \frac{\ell}{2}$ is the only value in $]0, \ell[$ s.t $u_1(x) = 0$

Next consider $u_2(x) = 0 \Rightarrow C_2 \cos\left(\frac{2\pi}{\ell} x\right) = 0$

$$\Rightarrow \cos\left(\frac{2\pi}{\ell} x\right) = 0 \Rightarrow \frac{2\pi}{\ell} x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

$$\Rightarrow \frac{2}{\ell} x = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$$

$\Rightarrow x = \frac{\ell}{4}, \frac{3\ell}{4}, \frac{5\ell}{4}, \ldots$ are the only values in interval $]0, \ell[$ s.t $u_2(x) = 0$

Hence we conclude that

$u_0(x)$ i.e 1st eigen function has 0 i.e 1−1 zeros in $]0,\ell[$
$u_1(x)$ " 2nd " " " 1 i.e 2−1 " " $]0,\ell[$ interval
$u_2(x)$ " " " " " 2 i.e 3−1 " " " "

And so on nth eigen function has $n-1$ zeros in $]0,\ell[$ interval.

3) Since every function $f(x)$ defined in $[0,\ell]$ and satisfying some conditions can be written as

$$f(x) = \sum_{n=0}^{\infty} a_n u_n(x)$$

So eigen functions form a complete set and also as

$$\int_0^{\ell} r(x) u_m(x) u_n(x)\, dx = 0 \quad \text{for } m \neq n$$

implies $\{u_n(x)\}$ are orthogonal.

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Example:- Show that if $u(x)$ and $v(x)$ are the periodic solutions of the Mathieu's difference equation $u'' + \lambda u + 16d \cos 2x\, u = 0$ with period π having distinct eigen values then $\int_a^b u(x) v(x)\, dx = 0$

Solution:-

Here the end point conditions are $u(0) = u(\pi)$ and $u'(0) = u'(\pi)$

Let λ_1, λ_2 be the two distinct eigen values corresponding to the eigen functions $u(x)$ and $v(x)$ respectively.

Then $u'' + 16d \cos 2x\, u + \lambda_1 u = 0$ —— (1)

with $u(0)=u(\pi)$ & $u'(0)=u'(\pi)$

and $v''+16d\cos 2x\, v+\lambda_2 v=0 \longrightarrow$ ②

with $v(0)=v(\pi)$ & $v'(0)=v'(\pi)$

Now Multiplying equ ① by v and equ ② by u and then subtracting.

$$u''v+16d\cancel{\cos 2x}\,uv+\lambda_1 uv-uv''-16d\cancel{\cos 2x}\,uv-\lambda_2 uv=0$$

$$u''v-uv''=(\lambda_2-\lambda_1)uv$$

$$\Rightarrow (\lambda_2-\lambda_1)uv=\frac{d}{dx}(u'v-uv')$$

Integrating both sides from 0 to π

$$(\lambda_2-\lambda_1)\int_0^{\pi}u(x)v(x)dx=u'(x)v(x)-u(x)v'(x)\Big|_0^{\pi}$$

$$\Rightarrow (\lambda_2-\lambda_1)\int_0^{\pi}u(x)v(x)\,dx=u'(\pi)v(\pi)-u(\pi)v'(\pi)-u'(0)v(0)+u(0)v'(0)$$

$$=u'(\pi)v'(\pi)-u'(\pi)v'(\pi)-u'(0)v'(0)+u'(0)v'(0)$$

By using end point conditions

$$=0$$

$$\Rightarrow \int_0^{\pi}u(x)v(x)\,dx=0 \quad \because \lambda_2-\lambda_1\neq 0 \quad \text{As } \lambda_1\neq\lambda_2$$

**

$\Rightarrow$ **Abel's Formula**:- If $u(x)$ & $v(x)$ are any two solutions of a regular or periodic SL ~~s~~ System then

$$p(x)\,W(u,v)=\text{constant}\quad, x\in[a,b]$$

Proof As we know that for a regular or periodic SL system.

$u\mathcal{L}(v)-v\mathcal{L}(u)=0$ for any pair of solutions $u(x)$ and $v(x)$.

So by differential form of Lagrang's identity $\frac{d}{dx}\{p(x)\,W(u,v)(x)\}=0$

$$\Rightarrow p(x)\,W(u,v)=\text{constant}\quad \forall\, x\in[a,b]$$

Question:- Verify that $\lambda_n^2=\left(\frac{n\pi}{\ln b}\right)^2$ and $u_n = C_n \sin(\lambda_n \ln x)$ are eigen values and eigen functions of $(xu')' + \lambda^2\left(\frac{1}{x}\right)u = 0$
$1 < x < b$ with $u(1) = 0$ and $u(b) = 0$

Solution:

Given $x^2u'' + xu' + \lambda^2 u = 0$
which is cauchy's Euler's equation [104 Method BSc]

$$\Rightarrow u = Ax^{i\lambda} + Bx^{-i\lambda}$$

$$= Ae^{i\lambda \ln x} + Be^{-i\lambda \ln x}$$

$$= C_1 \cos(\lambda \ln x) + C_2 \sin(\lambda \ln x)$$

$u(1) = 0 \quad \Rightarrow C_1 = 0$

$u(b) = 0 \quad \Rightarrow C_2 \sin(\lambda \ln b) = 0$

$\Rightarrow \sin(\lambda \ln b) = 0 \quad \because C_2 \neq 0$

$\Rightarrow \lambda \ln b = n\pi \Rightarrow \lambda_n^2 = \left(\frac{n\pi}{\ln b}\right)^2 \quad n \in N$

and $u_n(x) = C_n \sin(\lambda_n \ln x)$

Example:- Verify that for the SL system
$u'' + \lambda u = 0$ with $u'(0), u(l) = 0$

a) There are infinite number of eigen values with smallest but no largest.

b) The nth eigen function has exactly $n-1$ zeros.

c) The eigen functions are orthogonal and form a complete set.

Solution:

Given $u'' + \lambda u = 0$

$$\Rightarrow u = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$$

$$u' = -\sqrt{\lambda}\, c_1 \sin\sqrt{\lambda}\, x + \sqrt{\lambda}\, c_2 \cos\sqrt{\lambda}\, x$$

$u'(0) = 0 \Rightarrow \sqrt{\lambda}\, c_2 = 0$

$\Rightarrow c_2 = 0 \quad \because \sqrt{\lambda} \neq 0$

$\because$ If $\sqrt{\lambda} = 0$, then solution is trivial.

So $u = c_1 \cos\sqrt{\lambda}\, x$

$u(\ell) = 0 \Rightarrow c_1 \cos\sqrt{\lambda}\, \ell = 0$

$\Rightarrow \cos\sqrt{\lambda}\, \ell = 0 \quad \because$ for non trivial sol. $c_1 \neq 0$

$$\Rightarrow \sqrt{\lambda}\, \ell = (2n+1)\frac{\pi}{2} \qquad n \in \mathbb{Z}$$

$$\Rightarrow \lambda_n = (2n+1)^2 \frac{\pi^2}{4\ell^2} \qquad n = 0, 1, 2, 3, \cdots$$

which are required eigen values.

The corresponding eigenfunctions are

$$u_n = C_n \cos\left(\frac{(2n+1)\pi}{2\ell}\right) x \qquad n = 0, 1, 2, 3, \cdots$$

Now

a) Obviously there are infinite number of eigenvalues with $\lambda_0 = \pi^2/4\ell^2$ as the smallest eigenvalue. But there is no largest eigenvalue.

b)

Eigen function corresponding to nth eigen value is given by

$$u_{n-1} = C_{n-1} \cos[(2n-1)\pi/2\ell]\, x, \quad n = 1, 2, 3, \cdots$$

or equivalently

$$u_n = C_n \cos\left[\frac{(2n+1)\pi}{2\ell}\right] x, \quad n = 0, 1, 2, 3, \cdots$$

$n = 0 \Rightarrow u_0 = C_0 \cos\left(\frac{\pi}{2\ell}\right) x = 0$

$= \cos\left(\frac{\pi}{2\ell} x\right) = 0 \quad \because C_0 \neq 0$

$\Rightarrow \frac{\pi}{2\ell} x = \frac{\pi}{2}, \frac{3\pi}{2}, \cdots \Rightarrow x = \ell, x = 3\ell$

As $\ell, 3\ell, \cdots \notin \,]0, \ell[$

$\Rightarrow u_0$ has no zero in $]0,l[$

$n=1 \Rightarrow u_1 = C_1 \cos\left(\frac{3\pi}{2l}x\right)$

$u_1 = 0 \Rightarrow \frac{3\pi}{2l}x = \pi/2, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\Rightarrow x = l/3, l, \ldots$

As $l/3 \in (0,l)$ & $l, 3l, \ldots \notin (0,l)$

$\Rightarrow u_1$ has only one zero in $(0,l)$

$n=2 \Rightarrow u_2 = C_2 \cos\left(\frac{5\pi}{2l}\right)x$

$u_2 = 0 \Rightarrow \frac{5\pi}{2l}x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\Rightarrow x = l/5, 3l/5, l, \frac{7l}{5}, \ldots$

only $l/5, 3l/5 \in]0,l[$ & $l, 7l/5, \ldots \notin]0,l[$

$\Rightarrow u_2$ has exactly 2 zeros in $(0,l)$

$\Rightarrow u_0$ i-e 1st e.f has no zero in $(0,l)$

u_1 " 2nd " " 1 " " "

u_2 " 3rd " " 2 " " "

& so on

nth function has exactly $n-1$ zeros in $(0,l)$

c) Eigen functions are orthogonal and form a complete set for half interval $[0,l]$. Every function $f(x)$ in this interval and satisfying some conditions can be expressed as

$$f(x) = \sum_{n=0}^{\infty} a_n u_n(x)$$

Example:- Show that the following B.Cs yield self-adjoint problems.

(a) $u(0)=0$ $\quad u(L)=0$

(b) $u'(0)=0$ $\quad u(L)=0$

(c) $u(a)=u(b)$, $p(a)u'(a)=p(b)u'(b)$

Solution:-

(a) Here $a=0$ $\quad b=L$

Let u and v be the two eigen functions

then $u(0)=0$ $\quad u(L)=0$

& $v(0)=0$ $\quad v(L)=0$

Now

$$\int_0^{L}(u\mathcal{L}(v)-v\mathcal{L}(u))\,dx=P(x)[uv'-vu']\Big|_0^{L}$$

$$=P(L)[u(L)v'(L)-v(L)u'(L)]$$

$$-P(0)[u(0)v'(0)-v(0)u'(0)]$$

$$=P(L)[0-0]-P(0)[0-0]=0$$

$$\Rightarrow \int_0^L u\mathcal{L}(v)\,dx=\int_0^L v\mathcal{L}(u)\,dx$$

$\Rightarrow \mathcal{L}$ is self adjoint.

(b)

$$\int_0^L(u\mathcal{L}(v)-v\mathcal{L}(u))\,dx=P(L)[(0)v'(L)-(0)u'(L)]$$

$$-P(0)[u(0)(0)-v(0)(0)]$$

$$=0$$

$$\Rightarrow \int_0^L u\mathcal{L}(v)\,dx=\int_0^L v\mathcal{L}(u)\,dx$$

$\Rightarrow \mathcal{L}$ is self adjoint.

(c)

$$\int_a^b(u\mathcal{L}(v)-v\mathcal{L}(u))\,dx=P(b)[u(b)v'(b)-u'(b)v(b)]$$

$$-P(a)[u(a)v'(a)-u'(a)v(a)]$$

$$=P(b)u(b)v'(b)-P(b)u'(b)v(b)$$

$$-P(a)u(a)v'(a)+P(a)u'(a)v(a)$$

$= u(a)\cancel{p(a)}v'(a) - \cancel{p(a)}u'(a)v(a) - \cancel{p(a)}u(a)v'(a)$
$+ p(a)\cancel{u'(a)}v(a)$

$= 0$ $\quad \mathcal{L}$ is self adjoint.

———————

Example: Using the Rayleigh quotent, discuss the sign of the eigen values of the S.L system.

$u'' + \lambda u = 0 \quad , u(0) = 0 \quad u(\ell) = 0$
$p(x) = 1 \quad q(x) = 0 \quad r(x) = 1$

Solution:

Rayleigh quotent is given by

$$\lambda = \frac{-u(1)u'\big|_0^{\ell} + \int_0^{\ell}((1)u'^2 - (0)u^2)\,dx}{\int_0^{\ell}(1)u^2\,dx}$$

$$= \frac{-(u(\ell)u'(\ell) - u(0)u'(0)) + \int_0^{\ell}u'^2\,dx}{\int_0^{\ell}u^2\,dx}$$

$$\lambda = \frac{\int_0^{\ell}u'^2\,dx}{\int_0^{\ell}u^2\,dx} \longrightarrow (1)$$

Since u is unknown so λ can not be determined from (1). Now as u is an eigen function. So $u \neq 0$

$\Rightarrow u'$ is constant $\Rightarrow u' = c$

Now $u(0) = 0 \Rightarrow c = 0$ & $u(\ell) = 0 \Rightarrow c = 0$

$\Rightarrow c = 0 \Rightarrow u = 0$ which is trivial sol.

So $\lambda \neq 0 \Rightarrow \lambda > 0$

Hence all the eigen values are +ve.

———— ★★★ ————

Ordinary Differential Equations...

⇒ Bessel's Differential Equation:-

A 2nd order linear differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

is known as Bessel's differential equation. Solution of this differential equation is oftenly denoted by $J_n(x)$ and is known as Bessel's function. This equation is also sometimes called Bessel's differential equation of order 'n'.

* Solution of Bessel's Differential Equation:-

Bessel's differential equation is given by

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \longrightarrow (1)$$

Put $y=\frac{u(x)}{\sqrt{x}}$ then $\frac{dy}{dx}=\frac{\sqrt{x}\frac{du}{dx}-u(x)\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{x}\frac{du}{dx}-\frac{u(x)}{2\sqrt{x}}}{x} \Rightarrow x\frac{dy}{dx}=\sqrt{x}\frac{du}{dx}-\frac{u(x)}{2\sqrt{x}}$$

$$\Rightarrow x\frac{dy}{dx}=\frac{2x\frac{du}{dx}-u(x)}{2\sqrt{x}}$$

Again differentiate w.r.t x

$$x\frac{d^2y}{dx^2}+1\cdot\frac{dy}{dx}=\frac{(2\sqrt{x})\{2xu''(x)+2u'(x)-u'(x)\}-[2xu'(x)-u(x)]\left[\frac{1}{\sqrt{x}}\right]}{(2\sqrt{x})^2}$$

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{4x\sqrt{x}\,u''(x) + \cancel{2\sqrt{x}\,u'(x)} - \cancel{2\sqrt{x}\,u'(x)} + \frac{u(x)}{\sqrt{x}}}{4x}$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = x\sqrt{x}\,u''(x) + \cancel{\frac{1}{2}\sqrt{x}\,u'(x)} + \frac{u(x)}{4\sqrt{x}}$$

Now from eqn ①

$$x\sqrt{x}\,u''(x) + \frac{u(x)}{4\sqrt{x}} + (x^2-n^2)\frac{u(x)}{\sqrt{x}} = 0$$

Dividing both sides by $x\sqrt{x}$

$$\Rightarrow u''(x) + \frac{u(x)}{4x^2} + (x^2-n^2)\cdot\frac{u(x)}{x^2} = 0$$

$$u''(x) + \left[\frac{1}{4x^2} + \frac{(x^2-n^2)}{x^2}\right]u(x) = 0$$

$$u''(x) + \left[1 - \frac{n^2 - 1/4}{x^2}\right]u(x) = 0$$

As a special case when $n = \pm\frac{1}{2}$ then

$$n^2 - 1/4 = 0$$

$$\Rightarrow u''(x) + u(x) = 0 \qquad \Rightarrow (D^2+1)\,u(x) = 0$$

characteristic equation is

$$D^2 + 1 = 0 \qquad \Rightarrow D = \pm i$$

$$\Rightarrow u(x) = C_1\cos x + C_2\sin x$$

For $n = \pm\frac{1}{2}$ $\quad y(x) = \dfrac{C_1\cos x + C_2\sin x}{\sqrt{x}}$ is the general solution of Bessel's differential equation.

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* Series Solution of Bessel's Differential Equation:-

Bessel's differential equation is given by $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \longrightarrow (1)$

here we assume n is non negative integer

Let $y(x) = \sum_{s=0}^{\infty} C_s x^{r+s}$ then

$$\frac{dy}{dx} = \sum_{s=0}^{\infty} C_s (r+s) x^{r+s-1},$$

$$\frac{d^2y}{dx^2} = \sum_{s=0}^{\infty} C_s (r+s)(r+s-1) x^{r+s-2}$$

Putting in (1)

$$x^2\left[\sum_{s=0}^{\infty}(r+s)(r+s-1)C_s x^{r+s-2}\right] + x\left[\sum_{s=0}^{\infty} C_s (r+s) x^{r+s-1}\right]$$

$$+ (x^2 - n^2)\sum_{s=0}^{\infty} C_s x^{r+s} = 0$$

$$\Rightarrow \sum_{s=0}^{\infty} C_s (r+s)(r+s-1) x^{r+s} + \sum_{s=0}^{\infty} C_s (r+s) x^{r+s} +$$

$$\sum_{s=0}^{\infty} C_s x^{r+s+2} - n^2 \sum_{s=0}^{\infty} C_s x^{r+s} = 0$$

$$\Rightarrow \sum_{s=0}^{\infty} C_s \left[(r+s)(r+s-1) + (r+s) - n^2\right] x^{r+s} + \sum_{s=0}^{\infty} C_s x^{r+s+2} = 0$$

$$\Rightarrow \sum_{s=0}^{\infty} C_s \left[(r+s)^2 - n^2\right] x^{r+s} + \sum_{s=0}^{\infty} C_s x^{r+s+2} = 0$$

Now equating like powers of coefficients of x

سب سے چھوٹی پاور | x^r

Coefficient of $x^r \Rightarrow C_0[r^2 - n^2] = 0 \longrightarrow (2)$

— " — $x^{r+1} \Rightarrow C_1[(r+1)^2 - n^2] = 0 \longrightarrow (3)$

— " — $x^{r+2} \Rightarrow C_2[(r+2)^2 - n^2] + C_0 = 0 \longrightarrow (4)$

Coefficient of $x^{r+3} \Rightarrow C_3[(r+3)^2 - n^2] + C_1 = 0 \longrightarrow (5)$

—— " —— $x^{r+4} \Rightarrow C_4[(r+4)^2 - n^2] + C_2 = 0 \longrightarrow (6)$

⋮

—— " —— $x^{r+2m} \Rightarrow C_{2m}[(r+2m)^2 - n^2] + C_{2m-2} = 0 \longrightarrow (7)$

—— " —— $x^{r+2m+1} \Rightarrow C_{2m+1}[(r+2m+1)^2 - n^2] + C_{2m-1} = 0 \longrightarrow (8)$

from (2) $C_0[r^2 - n^2] = 0$ As $C_0 \neq 0$ So $r^2 - n^2 = 0$

$\Rightarrow r = \pm n$

As here $n \geq 0$ so we take $r = n$

from (3) $C_1[(n+1)^2 - n^2] = 0 \Rightarrow C_1(n^2 + 2n + 1 - n^2) = 0$

$\Rightarrow C_1[2n+1] = 0$ As $2n+1 \neq 0$ $\because n \geq 0$

So $C_1 = 0$

Now from (8) $C_{2m+1} = -\dfrac{C_{2m-1}}{(n+2m+1)^2 - n^2}$

$m = 1 \Rightarrow C_3 = \dfrac{-C_1}{(n+3)^2 - n^2} = \dfrac{0}{(n+3)^2 - n^2} = 0$

$m = 2 \Rightarrow C_5 = \dfrac{-C_3}{?} = \dfrac{0}{?} = 0$

$\Rightarrow C_1 = C_3 = C_5 = C_7 = \cdots = 0$

Now from (7) $C_{2m} = -\dfrac{C_{2m-2}}{(n+2m)^2 - n^2}$

$= -\dfrac{C_{2m-2}}{4m(n+m)}$

$m = 1 \Rightarrow C_2 = \dfrac{-C_0}{4(n+1)} = \left[\dfrac{-1}{4(1)(n+1)}\right] C_0$

$m = 2 \Rightarrow C_4 = \dfrac{-C_2}{4(2)(n+2)} = \left[-\dfrac{1}{4(2)(n+2)}\right]\left[\dfrac{-1}{4(1)(n+1)}\right] C_0$

$m = 3 \Rightarrow C_6 = \dfrac{-C_4}{4(3)(n+3)} = \left[\dfrac{-1}{4(3)(n+3)}\right]\left[\dfrac{-1}{4(2)(n+2)}\right]\left[\dfrac{-1}{4(1)(n+1)}\right] C_0$

If we choose $C_0 = \frac{1}{2^n n!}$ then

$$C_2 = \frac{-1}{4(1)(n+1)} \cdot \frac{1}{2^n n!} = (-1)^1 \cdot \frac{1}{2^{n+2}(n+1)!\,1!}$$

$$C_4 = \frac{-1}{4(2)(n+2)} \cdot \frac{-1}{4(1)(n+1)} \cdot \frac{1}{2^n n!} = (-1)^2 \frac{1}{2^{n+4}(n+2)!\,2!}$$

$$C_6 = \frac{-1}{4(3)(n+3)} \cdot \frac{-1}{4(2)(n+2)} \cdot \frac{-1}{4(1)(n+1)} \cdot \frac{1}{2^n n!}$$

$$= (-1)^3 \cdot \frac{1}{2^{n+6}(n+3)!\,3!}$$

$$\vdots$$

$$C_{2m} = (-1)^m \cdot \frac{1}{2^{n+2m}(n+m)!\,m!}$$

Now

$$y(x) = \sum_{s=0}^{\infty} C_s x^{n+s} = \sum_{s=0}^{\infty} C_s x^{n+s}$$

$$\Rightarrow y(x) = C_0 x^n + C_1 x^{n+1} + C_2 x^{n+2} + C_3 x^{n+3} + C_4 x^{n+4} + C_5 x^{n+5}$$

$$+ C_6 x^{n+6} + \cdots\cdots$$

$$= C_0 x^n + 0 + C_2 x^{n+2} + 0 + C_4 x^{n+4} + 0 + C_6 x^{n+6} + \cdots,$$

$$= \sum_{s=0}^{\infty} C_{2s} x^{n+2s}$$

$$\Rightarrow y(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{2^{n+2s}\,s!\,(n+s)!} x^{n+2s}$$

$$\Rightarrow y(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s}$$

B. Diff. Eq has the solution

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s}$$

Question:- Prove the following Recurrance relations

(i) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

(ii) $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ [Differential Recurrane formula]

(iii) $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$ [Pure Rec formula]

(iv) $x J_n'(x) + n J_n(x) = x J_{n-1}(x)$

(v) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

(vi) $J_0'(x) = -J_1(x)$

(vii) $J_{-n}(x) = (-1)^n J_n(x)$

(viii) $\frac{d}{dx}\left[x^{n} J_n(x)\right] = x^{n} J_{n-1}(x)$

Solution:- (i) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

As $$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$$

$$J_n'(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\cdot(n+2s)\left(\frac{x}{2}\right)^{n+2s-1}\cdot\frac{1}{2}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\cdot n\left(\frac{x}{2}\right)^{n+2s}\cdot\left(\frac{2}{x}\right)\cdot\frac{1}{2} + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\cdot \cancel{2}s\left(\frac{x}{2}\right)^{n+2s-1}\cdot\frac{1}{\cancel{2}}$$

$\because \left(\frac{x}{2}\right)^{-1} = \frac{2}{x}$

$n+2s$ کو علیحدہ علیحدہ کر دیا

$$= \frac{n}{x}\sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\cdot\left(\frac{x}{2}\right)^{n+2s} + \sum_{s=1}^{\infty} \frac{(-1)^s}{(s-1)!(n+s)!}\left(\frac{x}{2}\right)^{n+2s-1}$$

$$= \frac{n}{x} J_n(x) + \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!(n+s+1)!}\left(\frac{x}{2}\right)^{n+2(s+1)-1}$$

put $s=0$ کرنے سے s والی Term 0 ہو جاتی ہے اور $s!$ کی جگہ $(s-1)!$ اس لیے آیا کہ s سے کٹ گیا ... 2

$$J_n'(x) = \frac{n}{x} J_n(x) + \sum_{s=0}^{\infty} \frac{(-1)^s(-1)}{s!\,[(n+1)+s]!}\left(\frac{x}{2}\right)^{n+1+2s}$$

$$= \frac{n}{x} J_n(x) - J_{n+1}(x)$$

$$\Rightarrow x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

(ii) $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

As $J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$

Differentiate both sides w.r.t x

$$\Rightarrow J_n'(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!}(n+2s)\left(\frac{x}{2}\right)^{n+2s-1}\cdot\left(\frac{1}{2}\right)$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!}(n+s+s)\left(\frac{x}{2}\right)^{n+2s-1}\cdot\frac{1}{2}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!}(n+s)\left(\frac{x}{2}\right)^{(n-1)+2s}\cdot\frac{1}{2}$$

$$+ \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!}\cdot s\left(\frac{x}{2}\right)^{n+2s-1}\cdot\frac{1}{2}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s-1)!}\left(\frac{x}{2}\right)^{n-1+2s}\cdot\frac{1}{2} +$$

$$\sum_{s=1}^{\infty} \frac{(-1)^s}{(s-1)!\,(n+s)!}\left(\frac{x}{2}\right)^{n-1+2s}\cdot\frac{1}{2}$$

$$= \frac{1}{2}\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,[(n-1)+s]!}\left(\frac{x}{2}\right)^{n-1+2s} + \frac{1}{2}\sum_{s=1}^{\infty} \frac{(-1)^s}{(s-1)!\,(n+s)!}\left(\frac{x}{2}\right)^{n+2s-1}$$

$$= \frac{1}{2} J_{n-1}(x) + \frac{1}{2}\sum_{s=1}^{\infty} \frac{(-1)^s}{(s-1)!\,(n+s)!}\left(\frac{x}{2}\right)^{n-1+2s}$$

Put $s = p+1$

$$\Rightarrow J_n'(x) = \frac{1}{2} J_{n-1}(x) + \frac{1}{2} \sum_{p=0}^{\infty} \frac{(-1)^{p+1}}{p!\,[(n+1)+p]!} \left(\frac{x}{2}\right)^{n+1+2p}$$

$$= \frac{1}{2} J_{n-1}(x) - \frac{1}{2} \sum_{p=0}^{\infty} \frac{(-1)^{p}}{p!\,[(n+1)+p]!} \left(\frac{x}{2}\right)^{n+1+2p}$$

$$J_n'(x) = \frac{1}{2} J_{n-1}(x) - \frac{1}{2} J_{n+1}(x)$$

$$\Rightarrow 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \qquad \text{Proved.}$$

(iii) $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

Since we know that

$$x J_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

$$\Rightarrow J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x) \longrightarrow (1)$$

Also $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

$$\Rightarrow J_n'(x) = \frac{1}{2} J_{n-1}(x) - \frac{1}{2} J_{n+1}(x) \longrightarrow (2)$$

from (1) & (2)

$$\frac{n}{x} J_n(x) - J_{n+1}(x) = \frac{1}{2} J_{n-1}(x) - \frac{1}{2} J_{n+1}(x)$$

$$\Rightarrow \frac{n}{x} J_n(x) = \frac{1}{2} J_{n-1}(x) + \frac{1}{2} J_{n+1}(x)$$

$$\Rightarrow \frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x) \qquad \text{Proved.}$$

(iv) $x J_n'(x) + nJ_n(x) = xJ_{n-1}(x)$

As $J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \cdot \left(\frac{x}{2}\right)^{n+2s}$

$$J_n'(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \cdot (n+2s) \left(\frac{x}{2}\right)^{n+2s-1} \cdot \frac{1}{2}$$

$$\Rightarrow J_n'(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{-n}{2} + \frac{2n+2s}{2}\right)\cdot\left(\frac{x}{2}\right)^{n+2s-1}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{-n}{2}\right)\cdot\left(\frac{x}{2}\right)^{n+2s-1} + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{2n+2s}{2}\right)\cdot\left(\frac{x}{2}\right)^{n+2s-1}$$

$$= \left(\frac{-n}{x}\right)\sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\cdot\left(\frac{x}{2}\right)^{n+2s-1}\cdot\left(\frac{x}{2}\right) + \sum_{s=0}^{\infty} \frac{(-1)^s}{s![(n-1)+s]!}\left(\frac{x}{2}\right)^{n-1+2s}$$

$$= \left(\frac{-n}{x}\right)\sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} + J_{n-1}(x) \quad \text{(iii)}$$

$$J_n'(x) = \left(\frac{-n}{x}\right)J_n(x) + J_{n-1}(x)$$

$$\Rightarrow x J_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$$

$$\Rightarrow x J_n'(x) + nJ_n(x) = xJ_{n-1}(x)$$

(V) $\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$

$$\text{L.H.S} = \frac{d}{dx}\left[x^{-n}J_n(x)\right]$$

$$= x^{-n}J_n'(x) + (-n)x^{-n-1}J_n(x) \longrightarrow (1)$$

But $J_n'(x) = \left(\frac{n}{x}\right)J_n(x) - J_{n+1}(x)$ Put in (1)

$$\Rightarrow \frac{d}{dx}\left[x^{-n}J_n(x)\right] = x^{-n}\left[\left(\frac{n}{x}\right)J_n(x) - J_{n+1}(x)\right] - nx^{-n-1}J_n(x)$$

$$= \cancel{nx^{-n-1}J_n(x)} - x^{-n}J_{n+1}(x) - \cancel{nx^{-n-1}J_n(x)}$$

$$\Rightarrow \frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$$

(vi) $J_0'(x) = -J_1(x)$

As we know that

$$xJ_n'(x) + nJ_n(x) = xJ_{n-1}(x)$$

Put $n=0$ we get

$$xJ_0'(x) + 0 = xJ_{-1}(x)$$

$$\Rightarrow J_0'(x) = J_{-1}(x)$$

$$\Rightarrow J_0'(x) = -J_1(x) \qquad \because J_{-n}(x) = (-1)^n J_n(x)$$

(vii) $J_{-n}(x) = (-1)^n J_n(x)$

As $J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(-n+s)!} \cdot \left(\frac{x}{2}\right)^{-n+2s}$

Put $s = p+n \Rightarrow p = s-n$

$$\Rightarrow J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{p+n}}{(p+n)!\,p!} \left(\frac{x}{2}\right)^{-n+2(p+n)}$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^p(-1)^n}{p!(n+p)!} \cdot \left(\frac{x}{2}\right)^{n+2p}$$

$$= (-1)^n \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(n+p)!} \left(\frac{x}{2}\right)^{n+2p}$$

$$\Rightarrow J_{-n}(x) = (-1)^n \cdot J_n(x)$$

(viii) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

L.H.S $\frac{d}{d(x)}\left[x^n J_n(x)\right]$

$$= x^n J_n'(x) + n x^{n-1} J_n(x) \longrightarrow (1)$$

As we know that

$$J_n'(x) = \left(-\frac{n}{x}\right) J_n(x) + J_{n-1}(x)$$

$$\Rightarrow \frac{d}{dx}\left[x^n J_n(x)\right] = x^n\left[\left(\frac{-n}{x}\right)J_n(x) + J_{n-1}(x)\right] + n x^{n-1} J_n(x)$$

$$= -n x^{n-1} \cancel{J_n(x)} + x^n J_{n-1}(x) + n x^{n-1} \cancel{J_n(x)}$$

$$= x^n J_{n-1}(x)$$

***

Question:- Show that

$$e^{\frac{x}{2}\left[t-\frac{1}{t}\right]} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

Proof:- L.H.S

$$e^{\frac{x}{2}\left[t-\frac{1}{t}\right]} = e^{\frac{x}{2}t - \frac{x}{2}\cdot\frac{1}{t}} = e^{\frac{xt}{2} - \frac{xt^{-1}}{2}}$$

$$\Rightarrow e^{\frac{x}{2}\left[t-\frac{1}{t}\right]} = \sum_{r=0}^{\infty} \frac{\left(\frac{xt}{2}\right)^r}{r!} \sum_{s=0}^{\infty} \frac{\left(-\frac{xt^{-1}}{2}\right)^s}{s!}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{r+s}}{r!\,s!} t^{r-s}$$

$$\boxed{\begin{aligned} &\because e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots \\ &= \frac{\theta^0}{0!} + \frac{\theta^1}{1!} + \frac{\theta^2}{2!} + \cdots \\ &= \sum_{p=0}^{\infty} \frac{\theta^p}{p!} \end{aligned}}$$

Put $r - s = n$ i.e $r = n+s$ then

$$\text{L.H.S} = \sum_{n=-\infty}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \cdot \left(\frac{x}{2}\right)^{n+2s} \cdot t^n$$

$$= \sum_{n=-\infty}^{\infty} J_n(x) t^n = \text{R.H.S}$$

*

Question:- Prove that

(i) $\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots\cdots$

(ii) $\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \cdots\cdots$

(iii) $\cos(x\cos\theta) = J_0(x) - 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta - \cdots\cdots$

(iv) $\sin(x\cos\theta) = 2J_1(x)\cos\theta - 2J_3(x)\cos 3\theta + \cdots\cdots$

Solution we know that

$$e^{\frac{x}{2}\left[t-\frac{1}{t}\right]} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

$$\Rightarrow e^{\frac{x}{2}[t-t^{-1}]} = J_0(x)t^0 + J_1(x)t + J_{-1}(x)t^{-1} + J_2(x)t^2 + J_{-2}(x)t^{-2} + J_3(x)t^3 + J_{-3}(x)t^{-3} + \cdots$$

Now using $J_{-n}(x) = (-1)^n J_n(x)$ we have

$$e^{\frac{x}{2}[t-t^{-1}]} = J_0(x) + J_1(x)t - J_1(x)t^{-1} + J_2(x)t^2 + J_2(x)t^{-2} + J_3(x)t^3 - J_3(x)t^{-3} + \cdots$$

$$= J_0(x) + [t - t^{-1}]J_1(x) + [t^2 + t^{-2}]J_2(x) + [t^3 - t^{-3}]J_3(x) + \cdots$$

Putting $t = e^{i\theta}$ we have

$$e^{\frac{x}{2}[e^{i\theta} - e^{-i\theta}]} = J_0(x) + [e^{i\theta} - e^{-i\theta}]J_1(x) + [e^{2i\theta} + e^{-2i\theta}]J_2(x) + [e^{3i\theta} - e^{-3i\theta}]J_3(x) + \cdots$$

$$\Rightarrow \boxed{\cancel{e^{2x\sin\theta} = J_0(x) \times 2}}$$

$$e^{ix\left[\frac{e^{i\theta} - e^{-i\theta}}{2i}\right]} = J_0(x) + 2i\left[\frac{e^{i\theta} - e^{-i\theta}}{2i}\right]J_1(x) + 2\left[\frac{e^{2i\theta} + e^{-2i\theta}}{2}\right]J_2(x) + 2i\left[\frac{e^{3i\theta} - e^{-3i\theta}}{2i}\right]J_3(x) + \cdots$$

$$\Rightarrow e^{ix\sin\theta} = J_0(x) + 2i\sin\theta J_1(x) + 2\cos 2\theta J_2(x) + 2i\sin 3\theta J_3(x) + 2\cos 4\theta J_4(x) + \cdots$$

$$\Rightarrow \cos(x\sin\theta) + i\sin(x\sin\theta) = [J_0(x) + 2\cos 2\theta J_2(x) + 2\cos 4\theta J_4(x) + \cdots] + i[2\sin\theta J_1(x) + 2\sin 3\theta J_3(x) + \cdots]$$

Equating real & imaginary parts

$$\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots$$

$$\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \cdots$$

(iii) & (iv) As $\cos(x\sin\theta) + i\sin(x\sin\theta) = J_0(x) + 2J_1(x)\sin\theta\cdot i$
$+ 2J_2(x)\cos 2\theta + 2J_3(x)\sin 3\theta\cdot i + 2J_4(x)\cos 4\theta$
$+ 2J_5(x)\sin 5\theta\cdot i + \dots$

Replacing θ by $\pi/2 - \theta$

$\Rightarrow \cos(x\sin(\frac{\pi}{2}-\theta)) + i\sin(x\sin(\pi/2-\theta)) = J_0(x) + 2J_1(x)\sin(\pi/2-\theta)\cdot i$
$+ 2J_2(x)\cos 2(\pi/2-\theta) + 2J_3(x)\sin 3(\pi/2-\theta)\cdot i + 2J_4(x)$
$\cos 4(\pi/2-\theta) + 2J_5(x)\sin 5(\pi/2-\theta) + \dots$

$\Rightarrow \cos(x\cos\theta) + i\sin(x\cos\theta) = J_0(x) + 2J_1(x)\cos\theta\cdot i - 2J_2(x)\cos 2\theta$
$- 2J_3(x)\cos 3\theta\cdot i + 2J_4(x)\cos 4\theta + \dots$

Compairing real & imaginary parts

$\cos(x\cos\theta) = J_0(x) - 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \dots$
$\sin(x\cos\theta) = 2J_1(x)\cos\theta - 2J_3(x)\cos 3\theta + \dots$

*** ——————— ***

*** Expression For $J_n(x)$ when n is a half of an odd integer:-**

Question:- Prove that $J_{\frac{1}{2}}x = \sqrt{\frac{2}{\pi x}}\sin x$

Solution we know that

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$$

Now define $\pi(x) = \int_0^\infty e^{-t}\, t^x\, dt$. Then $\pi(x) = x!$ where x is an non-ve odd integer

Then
$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\pi(s)\cdot\pi(n+s)}\cdot\left(\frac{x}{2}\right)^{n+2s}$$

Now consider $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{1/2}\, dt$

Put $t = \theta^2$ then

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-\theta^2} . \theta . 2\theta\, d\theta = 2\int_0^\infty \theta^2 e^{-\theta^2}\, d\theta$$

$$= \int_0^\infty \underbrace{e^{-\theta^2}(-2\theta)}_{II}\, \underbrace{(-\theta)}_{I}\, d\theta$$

$$= -\theta e^{-\theta^2}\Big|_0^\infty - \int_0^\infty (-1)(e^{-\theta^2})\, d\theta$$

$$= 0 + \int_0^\infty e^{-\theta^2}\, d\theta = \cancel{\sqrt{\pi}/2} \quad \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2}$$

Now $J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s)\,\Gamma(n+s)} \cdot \left(\frac{x}{2}\right)^{n+2s}$

$$\Rightarrow J_{\frac{1}{2}}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s)\,\Gamma(\frac{1}{2}+s)} \cdot \left(\frac{x}{2}\right)^{\frac{1}{2}+s}$$

$$= \frac{(-1)^0}{\Gamma(0)\,\Gamma(\frac{1}{2})}\left(\frac{x}{2}\right)^{1/2} + \frac{(-1)^1}{\Gamma(1)\,\Gamma(3/2)} \cdot \left(\frac{x}{2}\right)^{5/2} + \frac{(-1)^2}{\Gamma(2)\,\Gamma(5/2)}\left(\frac{x}{2}\right)^{9/2}$$

$$+ \frac{(-1)^3}{\Gamma(3)\,\Gamma(7/2)} \cdot \left(\frac{x}{2}\right)^{11/2} + \cdots$$

$$= \frac{1}{0!\,\Gamma(1/2)}\left(\frac{x}{2}\right)^{1/2} - \frac{1}{1!\,\frac{3}{2}\,\Gamma(1/2)} \cdot \left(\frac{x}{2}\right)^{5/2} +$$

$$\frac{1}{2!\cdot\frac{5}{2}\cdot\frac{3}{2}\,\Gamma(1/2)}\left(\frac{x}{2}\right)^{9/2} - \frac{1}{3!\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\,\Gamma(\frac{1}{2})}\left(\frac{x}{2}\right)^{\frac{11}{2}}$$

$$+ \cdots$$

$$\Rightarrow J_{\frac{1}{2}}(x) = \frac{1}{\frac{\sqrt{\pi}}{2}} \cdot \left(\frac{x}{2}\right)^{-\frac{1}{2}} \cdot \left(\frac{x}{2}\right)^{1} - \frac{1}{\frac{3}{2}\cdot\frac{\sqrt{\pi}}{2}} \left(\frac{x}{2}\right)^{-\frac{1}{2}} \left(\frac{x}{2}\right)^{3} +$$

$$\frac{1}{2\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{\sqrt{\pi}}{2}} \left(\frac{x}{2}\right)^{-\frac{1}{2}} \left(\frac{x}{2}\right)^{5} + \dots$$

$$= \left(\frac{x}{2}\right)^{-\frac{1}{2}} \cdot \frac{1}{\sqrt{\pi}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right]$$

$$= \sqrt{\frac{2}{\pi x}} \sin x$$

* *

Question:- Show that $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos x$

Solution Let us define

$$\Gamma(x) = \int_0^\infty e^{-t} \cdot t^{x} \, dt \quad \text{then}$$

$$\Gamma(x) = x\,\Gamma(x-1) \quad \text{and}$$

$$\Gamma\left(-\tfrac{1}{2}\right) = \int_0^\infty e^{-t}\, t^{-\frac{1}{2}} \, dt$$

Put $t = \theta^2 \Rightarrow dt = 2\theta\, d\theta$

$$\Gamma\left(-\tfrac{1}{2}\right) = \int_0^\infty e^{-\theta^2} \cdot (\theta^2)^{-\frac{1}{2}} \cdot 2\theta\, d\theta = 2\int_0^\infty e^{-\theta^2}\, d\theta$$

$$= \frac{2\sqrt{\pi}}{2} = \sqrt{\pi}$$

Now

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \cdot \left(\frac{x}{2}\right)^{n+2s}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s)\,\Gamma(n+s)} \cdot \left(\frac{x}{2}\right)^{n+2s}$$

$$J_{-\frac{1}{2}}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s)\,\Gamma(-\frac{1}{2}+s)} \cdot \left(\frac{x}{2}\right)^{-\frac{1}{2}+2s}$$

$$= \frac{(-1)^0}{\Gamma(0)\,\Gamma(-\frac{1}{2})} \left(\frac{x}{2}\right)^{-\frac{1}{2}} + \frac{(-1)^1}{\Gamma(1)\,\Gamma(-\frac{1}{2}+1)} \left(\frac{x}{2}\right)^{\frac{3}{2}} +$$

$$\frac{(-1)^2}{\Gamma(2)\,\Gamma(3/2)}\left(\frac{x}{2}\right)^{7/2}+\frac{(-1)^3}{\Gamma(3)\,\Gamma(5/2)}\left(\frac{x}{2}\right)^{\frac{11}{2}}+\ldots$$

$$=\frac{1}{1\sqrt{\pi}}\cdot\left(\frac{x}{2}\right)^{-1/2}-\frac{1}{\frac{1}{2}\sqrt{\pi}}\cdot\left(\frac{x}{2}\right)^{2}\left(\frac{x}{2}\right)^{-\frac{1}{2}}+$$

$$\frac{1}{2!\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\left(\frac{x}{2}\right)^{4}\cdot\left(\frac{x}{2}\right)^{-\frac{1}{2}}-\frac{1}{3!\,\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\left(\frac{x}{2}\right)^{6}\left(\frac{x}{2}\right)^{-\frac{1}{2}}+\ldots$$

$$=\sqrt{\frac{2}{\pi x}}\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right]$$

$$=\sqrt{\frac{2}{\pi x}}\cos x$$

* *

Question:- Calculate $J_{3/2}(x)$, $J_{-3/2}(x)$, $J_{5/2}(x)$, $J_{-5/2}(x)$, $J_{7/2}(x)$, $J_{-7/2}(x)$

Solution $J_{3/2}(x)$

using formula

$$\frac{2n}{x}J_n(x)=J_{n-1}(x)+J_{n+1}(x)$$

Put $n=1/2$

$$\Rightarrow \frac{1}{x}J_{1/2}(x)=J_{-1/2}(x)+J_{3/2}(x)\longrightarrow (*)$$

Since $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$ &

$$J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$$

So $(*)$ becomes

$$\frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x=\sqrt{\frac{2}{\pi x}}\cos x+J_{3/2}(x)$$

$$\Rightarrow J_{3/2}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x}-\cos x\right]$$

$J_{-3/2}(x) = ?$ using formula

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

Put $n = -1/2$

$$-\frac{1}{x} J_{-\frac{1}{2}}(x) = J_{-\frac{3}{2}}(x) + J_{1/2}(x) \longrightarrow (1)$$

Since $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ & $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

So equ (1) becomes

$$-\frac{1}{x}\sqrt{\frac{2}{\pi x}} \cos x = J_{-3/2}(x) + \sqrt{\frac{2}{\pi x}} \sin x$$

$$\Rightarrow J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left[-\sin x - \frac{\cos x}{x}\right]$$

$J_{5/2}(x) = ?$

By Pure Recurrance relation

$$\frac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x)$$

Put $n = 3/2$

$$\frac{3}{x} J_{3/2}(x) = J_{5/2}(x) + J_{1/2}(x) \longrightarrow (1)$$

Since $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right)$

& $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

So equ (1) becomes

$$\frac{3}{x}\sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right) = J_{5/2}(x) + \sqrt{\frac{2}{\pi x}} \sin x$$

$$\Rightarrow J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{3}{x^2}\sin x - \frac{3}{x}\cos x - \sin x\right]$$

$$= \sqrt{\frac{2}{\pi x}} \left[\frac{3-x^2}{x^2} \sin x - \frac{3}{x} \cos x\right]$$

$J_{-5/2}(x) = ?$

By Pure Recurrane relation

$$\frac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x)$$

Put $n = -3/2$

$$\Rightarrow \frac{-3}{x} J_{-3/2}(x) = J_{-1/2}(x) + J_{-5/2}(x)$$

Since $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(-\sin x - \frac{\cos x}{x}\right)$

& $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

$$\Rightarrow J_{-5/2}(x) = \frac{-3}{x}\sqrt{\frac{2}{\pi x}}\left(-\sin x - \frac{\cos x}{x}\right) - \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x + \frac{3-x^2}{x^2}\cos x\right]$$

$J_{7/2}(x) = ?$

By Pure Recurrane relation

$$\frac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x)$$

Put $n = 5/2$

$$\Rightarrow \frac{5}{x} J_{5/2}(x) = J_{7/2}(x) + J_{3/2}(x) \longrightarrow (1)$$

Since $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right]$

& $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$

Therefore ① becomes

$$\frac{5}{x}\sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right] = J_{7/2}(x) + \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$$

$$\Rightarrow J_{7/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{15-5x^2}{x^3}\sin x - \frac{15}{x^2}\cos x - \frac{\sin x}{x} + \cos x\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{15-5x^2-x^2}{x^3}\sin x + \frac{x^2\cos x - 15\cos x}{x^2}\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{15-6x^2}{x^3}\sin x + \frac{x^2-15}{x^2}\cos x\right]$$

$$J_{-7/2}(x) = ?$$

By Pure recurrance relation

$$\frac{2n}{x}J_n(x) = J_{n+1}(x) + J_{n-1}(x)$$

Put $n = -5/2$

$$\Rightarrow \frac{-5}{x}J_{-5/2}(x) = J_{-3/2}(x) + J_{-7/2}(x) \longrightarrow ①$$

Since $J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x - \frac{3-x^2}{x^2}\cos x\right]$

& $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(-\sin x - \frac{\cos x}{x}\right)$

Therefore ① becomes

$$\frac{-5}{x}\sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x - \frac{3-x^2}{x^2}\cos x\right] = \sqrt{\frac{2}{\pi x}}\left[-\sin x - \frac{\cos x}{x}\right] + J_{-7/2}(x)$$

$$\Rightarrow J_{-7/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{-15}{x^2}\sin x + \frac{15-5x^2}{x^3}\cos x + \sin x + \frac{\cos x}{x}\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{-15+x^2}{x^2}\sin x + \frac{15-5x^2+x^2}{x^3}\cos x\right]$$

$$\Rightarrow J_{-7/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{x^2-15}{x^2}\sin x + \frac{15-4x^2}{x^3}\cos x\right]$$

★———★★★———★

⇒ Legender's Differential Equation :-

The 2nd order linear differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

is called 2nd order legender differential equation of degree 'n'.

* Series Solution of Legender Differential Equation :-

Consider $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \longrightarrow (1)$

Let

$$y = \sum_{r=0}^{\infty} a_r x^{m+r}$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1} , \quad \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2}$$

Putting in (1)

$$\left[\sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2}\right](1-x^2) - 2x\left[\sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1}\right] + n(n+1)\sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (m+r)(m+r-1)\cdot x^{m+r-2} - \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r} - 2\sum_{r=0}^{\infty} a_r (m+r) x^{m+r} + n(n+1)\sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2} + \sum_{r=0}^{\infty} a_r \left[n(n+1) - 2(m+r) - (m+r)(m+r-1)\right] x^{m+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2} + \sum_{r=0}^{\infty} a_r \left[n(n+1) - (m+r)(2+m+r-1)\right] x^{m+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2} + \sum_{r=0}^{\infty} a_r [n(n+1) - (m+r)(m+r+1)] x^{m+r} = 0$$

Equating co-efficients of like powers of x

Co-efficient of $x^{m-2} \Rightarrow a_0 (m)(m-1) = 0 \longrightarrow$ ②

— " — $x^{m-1} \Rightarrow a_1 (m+1)(m) = 0 \longrightarrow$ ③

— " — $x^{m} \Rightarrow a_2 (m+2)(m+1) + a_0 [n(n+1) - m(m+1)] = 0 \longrightarrow$ ④

— " — $x^{m+1} \Rightarrow a_3 (m+3)(m+2) + a_1 [n(n+1) - (m+1)(m+2)] = 0 \longrightarrow$ ⑤

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

— " — $x^{m+r-2} \Rightarrow a_r (m+r)(m+r-1) + a_{r-2} [n(n+1) - (m+r-2)(m+r-1)] = 0 \longrightarrow$ ⑥

from ①

$$a_0 (m)(m-1) = 0$$

As $a_0 \neq 0$ so $m(m-1) = 0 \Rightarrow m = 0$ & $m = 1$

when $m = 0$ then from ⑥

$$a_r = \frac{(0+r-2)(0+r-1) - n(n+1)}{(0+r)(0+r-1)} a_{r-2}$$

$$= \frac{(r-2)(r-1) - n(n+1)}{r(r-1)} a_{r-2}$$

$$\Rightarrow a_r = -\frac{(n-r+2)(n+r-1)}{r(r-1)} a_{r-2} \longrightarrow (*)$$

Now $r = 2 \Rightarrow$
$$a_2 = \frac{-n(n+1)}{2(2-1)} a_0$$

$r = 4 \Rightarrow$
$$a_4 = -\frac{(n-2)(n+3)}{4(3)} a_2$$

$$= \left[-\frac{(n-2)(n+3)}{4(3)}\right]\left[-\frac{n(n+1)}{2(1)}\right] a_0$$

$$r=6 \Rightarrow a_6 = \frac{-(n-4)(n+5)}{6(5)} a_4$$

$$= \left[-\frac{(n-4)(n+5)}{6(5)}\right]\left[-\frac{(n-2)(n+3)}{4(3)}\right]\left[\frac{-n(n+1)}{2(1)}\right] a_0$$

$$\Rightarrow a_2 = \frac{(-1)^1 \, n(n+1)}{2!} a_0$$

$$a_4 = \frac{(-1)^2 \, n(n+1)(n-2)(n+3)}{4!} a_0$$

$$a_6 = \frac{(-1)^3 \, n(n+1)(n-2)(n+3)(n-4)(n+5)}{6!} a_0$$

$\vdots \qquad \vdots \qquad \vdots$

Again from (*)

when $r=3$

$$a_3 = -\frac{(n-1)(n+2)}{3!} a_1$$

$$r=5 \Rightarrow a_5 = \frac{(-1)^2 (n-1)(n+2)(n-3)(n+4)}{5!} a_1$$

$$r=7 \Rightarrow a_7 = \frac{(-1)^3 (n-1)(n+2)(n-3)(n+4)(n-5)(n+6)}{7!} a_1$$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

Hence $y = \sum_{r=0}^{\infty} a_r x^{m+r} = \sum_{r=0}^{\infty} a_r x^r$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= [a_0 + a_2 x^2 + a_4 x^4 + \cdots] + [a_1 x + a_3 x^3 + a_5 x^5 + \cdots]$$

$$\Rightarrow y = \left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n+3)}{4!}x^4 + \cdots\right]a_0$$
$$+\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!}x^5 - \frac{(n-1)(n+2)(n-3)(n+4)(n-5)(n+6)}{7!}x^7 + \cdots\right]a_1$$

It can be easily seen that each of the above series converges in $]-1,1[$ by ratio test.

Now by taking $m=1$, we get nothing new but second series of eqn ⊛.

Since a_0 and a_1 are arbitrary, Therefore this is the general solution of Legender's differential equation. We further note that first series reduces to a polynomial when 'n' is even and second series reduces to a polynomial when 'n' is odd.

Now, if we give such numerical values to a_0 and a_1 so that the polynomial becomes equal to unity when x is unity, then we obtain a system of polynomials.

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}[5x^3 - 3x]$, $P_5(x) = \frac{1}{8}[63x^5 - 70x^3 + 15x]$

$P_6(x) = \frac{1}{16}[231x^6 - 315x^4 + 105x^2 - 5]$

Now

$$P_4(x) = \left[1 - 10x^2 + \frac{35}{3}x^4\right]a_0$$

$$P_4(1) = 1 \Rightarrow 1 = \left[1 - 10 + \frac{35}{3}\right]a_0 \Rightarrow 1 = \left[\frac{3-30+35}{3}\right]a_0$$

$$\Rightarrow a_0 = 3/8$$

So $P_4(x) = \frac{3}{8}\left[1 - 10x^2 + \frac{35}{3}x^4\right] = \frac{1}{8}\left[35x^4 - 30x^2 + 3\right]$

These polynomials are called Legender's polynomial and each satisfies Legender's differential equation, such that the polynomial has the degree equals to the degree of linear differential equation.

The general form of $P_n(x)$ is given by

$$P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n r! (n-r)! (n-2r)!} \cdot x^{n-2r}$$

where

$$N = \begin{cases} n/2 & , \text{if } n \text{ is even} \\ \frac{n-1}{2} & , \text{if } n \text{ is odd} \end{cases}$$

Further note that $P_n(x)$ is even or odd according to n is even or odd.

** ——————— **

⇒ Rodrigue's Formula for Legender's formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

Proof consider Legender's differential eqn.

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \longrightarrow (1)$$

Now Put $v = (x^2-1)^n$ then

$$\frac{dv}{dx} = n(x^2-1)^{n-1}(2x)$$

$$= \frac{n(x^2-1)^n}{x^2-1} \cdot 2x$$

$$\Rightarrow (x^2-1)\frac{dv}{dx} = 2nxv$$

$$\Rightarrow (1-x^2)\frac{dv}{dx} + 2nxv = 0$$

Again differentiate w.r.t x

$$(1-x^2)\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + 2nx\frac{dv}{dx} + 2nv = 0$$

$$(1-x^2)\frac{d^2v}{dx^2} + (2n-2)x\frac{dv}{dx} + 2nv = 0 \longrightarrow (2)$$

Again differentiate w.r.t x

$$(1-x^2)\frac{d^3v}{dx^3} + 2x(n-2)\frac{d^2v}{dx^2} + 2(2n-1)\frac{dv}{dx} = 0$$

This can be written as

$$(1-x^2)\frac{d^2v_1}{dx^2} + 2x(n-2)\frac{dv_1}{dx} + 2(2n-1)v_1 = 0$$

here $v_1 = \frac{dv}{dx}$

This can be written as

$$(1-x^2)\frac{d^2v_1}{dx^2} + 2x(n-1-1)\frac{dv_1}{dx} + (2n-1)(1+1)v_1 = 0 \longrightarrow **$$

Now (2) can be written as

$$(1-x^2)\frac{d^2v}{dx^2} + 2x(n-0-1)\frac{dv}{dx} + (2n-0)(1+0)v = 0 \longrightarrow *$$

Keeping * & ** in view, if we differentiate equ (2) n times

$$(1-x^2)\frac{d^2v_n}{dx^2} + 2x(n-n-1)\frac{dv_n}{dx} + (2n-n)(n+1)v_n = 0$$

$$(1-x^2)\frac{d^2v_n}{dx^2} - 2x\frac{dv_n}{dx} + n(n+1)v_n = 0$$

This shows that v_n is the solution of Legender's differential equation.

But $v_n = \frac{d^nv}{dx^n} = \frac{d^n}{dx^n}(x^2-1)^n$

But solution of Legender's differential equation is $P_n(x)$ and is unique. So, then for some c

$$P_n(x) = c\cdot\frac{d^n}{dx^n}(x^2-1)^n$$

$$c\frac{d^n}{dx^n}(x^2-1)^n = \sum_{r=0}^{N} \frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r}$$

Equating co-efficients of x^n on both side

$$\Rightarrow c(2n)(2n-1)\cdots(n+1) = \frac{(-1)^0(2n-0)!}{2^n\cdot 0!(n-0)!(n-0)!}$$

$$\Rightarrow \frac{c(2n)(2n-1)\cdots(n+1)n!}{n!} = \frac{2n!}{2^n n!n!}$$

$$\Rightarrow c\frac{\cancel{2n!}}{\cancel{n!}} = \frac{\cancel{2n!}}{2^n n!\cancel{n!}} \quad \Rightarrow c = \frac{1}{2^n n!}$$

$$\Rightarrow P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

* * *

* Generating Function of $P_n(x)$:-

Theorem Prove that the coefficient of z^n in $(1-2xz+z^2)^{-1/2}$ is $P_n(x)$

Proof:- Consider $(1-2xz+z^2)^{-1/2}$

$$= [1+(-2xz+z^2)]^{-1/2}$$

$$= 1+\left(-\frac{1}{2}\right)(-2xz+z^2)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-2xz+z^2)^2+$$

$$\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}(-2xz+z^2)^3+\cdots$$

$$= 1-\frac{1}{2}(2xz+z^2)+\frac{3}{8}(4x^2z^2+z^4-4xz^3)-$$

$$\frac{5}{6}(-8x^3z^3+z^6-6xz^5+12x^2z^4)+\cdots$$

$$= 1 + (x)z + \left[\tfrac{1}{2}(3x^2-1)\right]z^2 + \left[\tfrac{1}{2}(5x^3-3x)\right]z^3$$
$$+ \left[\tfrac{1}{8}(35x^4 - 30x^2 + 3)\right]z^4 + \cdots$$

$$\Rightarrow (1-2xz+z^2)^{-1/2} = P_0(x)z^0 + P_1(x)z^1 + P_2(x)z^2 + P_3(x)z^3 + \cdots$$
$$\cdots + P_n(x)z^n + \cdots$$

Hence coefficient of z^n in the expression of $(1-2xz+z^2)^{-1/2}$ is $P_n(x)$

***

Theorem:- Prove that the Legender's Polynomials are orthogonal and

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

Proof:- Let us consider Legender's differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \longrightarrow (1)$$

Let $P_m(x)$ and $P_n(x)$ be the legender's polynomials

then $$(1-x^2)\frac{d^2}{dx^2}(P_m(x)) - 2x\frac{d}{dx}(P_m(x)) + m(m+1)P_m(x) = 0 \longrightarrow (2)$$

$$(1-x^2)\frac{d^2}{dx^2}(P_n(x)) - 2x\frac{d}{dx}(P_n(x)) + n(n+1)P_n(x) = 0 \longrightarrow (3)$$

Multiply equ (2) by $P_n(x)$ & equ (3) by $P_m(x)$ & subtract

$$(1-x^2)P_m''(x)P_n(x) - (1-x^2)P_n''(x)P_m(x) - 2xP_m'(x)P_n(x) + 2xP_n'(x)P_m(x) + [m(m+1) - n(n+1)]P_m(x)P_n(x) = 0$$

$$\Rightarrow (1-x^2)[P_m''(x)P_n(x) - P_n''(x)P_m(x)] - 2x[P_m'(x)P_n(x) - P_n'(x)P_m(x)]$$
$$= [n(n+1) - m(m+1)]P_m(x)P_n(x)$$

$$\Rightarrow [n(n+1) - m(m+1)]P_m(x)P_n(x) = [(1-x^2)P_m''(x)P_n(x) - 2xP_m'(x)P_n(x)] - [(1-x^2)P_n''(x)P_m(x) - 2xP_n'(x)P_m(x)]$$

$$\Rightarrow [n(n+1)-m(m+1)]P_m(x)P_n(x) = P_n(x)[(1-x^2)P_m''(x)-2xP_m'(x)] - P_m(x)[(1-x^2)P_n''(x)-2xP_n'(x)]$$

$$= P_n(x)\frac{d}{dx}\{(1-x^2)P_m'(x)\} - P_m(x)\frac{d}{dx}\{(1-x^2)P_n'(x)\}$$

$$\Rightarrow [n(n+1)-m(m+1)]\int_{-1}^{1} P_m(x)P_n(x)\,dx = \int_{-1}^{1}[\underset{I}{P_n(x)}\underset{II}{\{(1-x^2)P_m'(x)\}'}]\,dx - \int_{-1}^{1}[P_m(x)\{(1-x^2)P_n'(x)\}']\,dx$$

$$= [P_n(x)][(1-x^2)P_m'(x)]\Big|_{-1}^{1} - \int_{-1}^{1} P_n'(x)[(1-x^2)P_m'(x)]\,dx - [P_m(x)][(1-x^2)P_n'(x)]\Big|_{-1}^{1} + \int_{-1}^{1} P_m'(x)[(1-x^2)P_n'(x)]\,dx = 0$$

$$\Rightarrow \int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \quad \text{for} \quad m \neq n$$

Now we prove for $m = n$

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$$

for this consider $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$

& $(1-2xt+t^2)^{-1/2} = \sum_{m=0}^{\infty} P_m(x)t^m$

$$\Rightarrow \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_m(x)P_n(x)t^{m+n} = (1-2xt+t^2)^{-1}$$

$$\Rightarrow \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_{-1}^{1} P_m(x)P_n(x)t^{m+n}\,dx = \int_{-1}^{1}\frac{1}{1-2xt+t^2}\,dx$$

$$= \frac{-1}{2t}\int_{-1}^{1}\frac{-2t}{1-2xt+t^2}\,dx$$

$$= \frac{-1}{2t} \ln(1-2xt+t^2)\Big|_{-1}^{1}$$

$$= \frac{-1}{2t}\left[\ln(1-2t+t^2) - \ln(1+2t+t^2)\right]$$

$$= \frac{-1}{2t}\left[\ln(1-t)^2 - \ln(1+t)^2\right]$$

$$= \frac{-1}{t}\left[\left(-t-\frac{t^2}{2}-\frac{t^3}{3}-\frac{t^4}{4}-\cdots\right)-\left(t-\frac{t^2}{2}+\right.\right.$$

$$\left.\left.\frac{t^3}{3}-\frac{t^4}{4}+\cdots\right)\right]$$

$$= \frac{-1}{t}\left[-2t-\frac{2t^3}{3}-\frac{2t^5}{5}-\cdots\right]$$

$$= 2+\frac{2t^2}{3}+\frac{2t^4}{5}+\cdots$$

when $m=n$

$$\sum_{n=0}^{\infty}\int_{-1}^{1} P_n(x)P_n(x)t^{2n}\,dx = \sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n}$$

$$\Rightarrow \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$$

***

* Recurrance Relation For $P_n(x)$:-

Prove that (1) $P_{n-1}(x) = P_n'(x) - 2xP_{n-1}'(x) + P_{n-2}'(x)$

(2) $nP_{n+1}(x) = (2n+1)xP_n(x) + (n-1)P_{n-1}(x)$

Proof:-

(1) $P_{n-1}(x) = P_n'(x) - 2xP_{n-1}'(x) + P_{n-2}'(x)$

Consider the generating function

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Differentiating both sides w.r.t 'x' then we have

$$-\frac{1}{2}(1-2xt+t^2)^{\frac{-1}{2}-1}(-2t) = \sum_{n=0}^{\infty} P_n'(x)t^n$$

$$\Rightarrow t(1-2xt+t^2)^{-\frac{1}{2}}\cdot(1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty} P_n'(x)t^n$$

$$\Rightarrow t(1-2xt+t^2)^{-1/2} = (1-2x+t^2)\sum_{n=0}^{\infty} P_n'(x)t^n$$

$$\Rightarrow t\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} P_n'(x)t^n - 2xt\sum_{n=0}^{\infty} P_n'(x)t^n + t^2\sum_{n=0}^{\infty} P_n'(x)t^n$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x)t^{n+1} = \sum_{n=0}^{\infty} P_n'(x)t^n - 2x\sum_{n=0}^{\infty} P_n'(x)t^{n+1} + \sum_{n=0}^{\infty} P_n'(x)t^{n+2}$$

Now equating the coefficient of t^n then we have

$$P_{n-1}(x) = P_n'(x) - 2xP_{n-1}'(x) + P_{n-2}'(x)$$

(2) $nP_{n+1}(x) = (2n+1)xP_n(x) + (n-1)P_{n-1}(x)$

Consider the generating function

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Now differentiating both sides w.r.t t

then $-\frac{1}{2}(1-2xt+t^2)^{-1/2-1}(-2x+2t) = \sum_{n=0}^{\infty} P_n(x)\,n\,t^{n-1}$

$$\Rightarrow (1-2xt+t^2)^{-1/2}(1-2xt+t^2)^{-1}(x-t) = \sum_{n=0}^{\infty} P_n(x).nt^{n-1}$$

$$\Rightarrow (1-2xt+t^2)^{-1/2}(x-t) = (1-2xt+t^2)\sum_{n=0}^{\infty} P_n(x).nt^{n-1}$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x)t^n(x-t) = \sum_{n=0}^{\infty} P_n(x).nt^{n-1} - 2xt\sum_{n=0}^{\infty} P_n(x)nt^{n-1} + t^2\sum_{n=0}^{\infty} P_n(x).nt^{n-1}$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x)t^n(x) - \sum_{n=0}^{\infty} P_n(x)t^{n+1} = \sum_{n=0}^{\infty} P_n(x)nt^{n-1} - 2x\sum_{n=0}^{\infty} P_n(x).nt^n + \sum_{n=0}^{\infty} P_n(x)nt^{n+1}$$

Now, equating the coefficient of t^n

$$x P_n(x) - P_{n-1}(x) = n P_{n+1}(x) - 2n x P_n(x) + P_{n-1}(x) \cdot n$$

$$\Rightarrow P_n(x)(2n+1) \cdot x = n P_{n+1}(x) + P_{n-1}(x)(n+1)$$

$$\star\ \underline{n P_{n+1}(x) = P_n(x)(2n+1) \cdot x - P_{n-1}(x)(n+1)}\ \star$$

* Power Series :-

Legender's Method to find the solutions of ~~Linear~~ Differential Equations. ~~is given by~~ L.O.Eq is given by

$$(1-x^2) y''(x) - 2x y'(x) + n(n+1) y = 0 \longrightarrow ①$$

Let $y = \sum_{m=0}^{\infty} C_m x^m$

$$y' = \sum_{m=0}^{\infty} C_m m x^{m-1}$$

$$y'' = \sum_{m=0}^{\infty} C_m m(m-1) x^{m-2}$$

Putting in ① & simplyfying

$$\sum_{m=0}^{\infty} C_m m(m-1) x^{m-2} - \sum_{m=0}^{\infty} C_m m(m-1) x^m -$$

$$2 \sum_{m=0}^{\infty} C_m m x^m + n(n+1) \sum_{m=0}^{\infty} C_m x^m = 0$$

Now equating coefficients of x^m

$$C_{m+2}(m+2)(m+1) - C_m m(m-1) - 2m C_m +$$

$$n(n+1) C_m = 0$$

$$\Rightarrow C_{m+2}(m+1)(m+2) - C_m[(m-n)(m+n+1)] = 0$$

$$\Rightarrow C_{m+2} = \frac{(m-n)(m+n+1)}{(m+1)(m+2)} C_m \longrightarrow ②$$

$m = 0$ $$\Rightarrow C_2 = \left[\frac{(-n)(n+1)}{(1)(2)}\right] C_0 = \frac{(-1)^1 n(n+1)}{2!} C_0$$

$$m=1 \Rightarrow C_3 = \frac{(1-n)(n+2)}{(2)(3)} C_1 = \frac{(-1)^1 (n-1)(n+2)}{3!} C_1$$

$$m=2 \Rightarrow C_4 = \frac{(2-n)(n+3)}{(3)(4)} C_2 = \frac{(-1)^2 n(n+1)(n-2)(n+3)}{4!} C_0$$

$$m=3 \Rightarrow C_5 = \frac{(3-n)(n+4)}{(4)(5)} C_3 = \frac{(-1)^2 (n-1)(n+2)(n-3)(n+4)}{5!} C_1$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y = \sum_{m=0}^{\infty} C_m x^m = C_0 \left\{ 1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n-2)(n+3)}{4!} x^4 + \cdots \right.$$

$$\left. \cdots + C_1 \left\{ x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!} x^5 + \cdots \right\} \right.$$

$$\Rightarrow y(x) = C_0 y_1(x) + C_1 y_2(x)$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n-2)(n+3)}{4!} x^4 + \cdots$$

&

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!} + \cdots$$

Since $y_1(x)$ & $y_2(x)$ are linearly independent. Therefore it gives two solutions of Legendeng's differential Equation.

***

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* Convergence of Series Solution of Legender's Differential Equation:-

Consider the mth term of series solution, then

$$\left|\frac{U_{m+2}}{U_m}\right| = \left|\frac{C_{m+2}\, x^{m+2}}{C_m\, x^m}\right| = \left|\frac{C_{m+2}}{C_m}\right| x^2$$

$$= \left|\frac{(m-n)(m+n+1)}{(m+1)(m+2)}\right| x^2$$

$$\lim_{m\to\infty} \left|\frac{U_{m+2}}{U_m}\right| = \lim_{m\to\infty} \left|\frac{\left(1-\frac{n}{m}\right)\left(1+\frac{n+1}{m}\right)}{\left(1+\frac{1}{m}\right)\left(1+\frac{2}{m}\right)}\right| x^2$$

$$= \left|\frac{(1-0)(1+0)}{(1+0)(1+0)}\right| x^2 = x^2$$

Series solution will converge if $x^2 < 1$.

$$\Rightarrow -1 < x < 1$$

$\rightarrow$ Series solution of Legender Differential converge for $-1 < x < 1$

Question Show that $P_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x-1}{2}\right)^{n-k} \left(\frac{x+1}{2}\right)^k$

Solution By Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x-1)^n (x+1)^n\right]$$

Now by using Leibniz's formula

$$D^n[fg] = \sum_{k=0}^{n} \binom{n}{k} D^k f \cdot D^{n-k} g \text{ , we get}$$

$$P_n(x) = \frac{1}{2^n n!} \sum_{k=0}^{n} \binom{n}{k} D^k (x-1)^n D^{n-k} (x+1)^n$$

$$\Rightarrow P_n(x) = \frac{1}{2^n n!}\sum_{k=0}^{n}\binom{n}{k}\left[n(n-1)(n-2)\cdots(n-k+1)(x-1)^{n-k}\right]\left[n(n-1)(n-2)\cdots(k+1)(x+1)^k\right]$$

$\Rightarrow$ x & ÷ by $k!$ & $(n-k)!$

$n-(n-k-1) = k+1$

$$= \frac{1}{2^n \cancel{n!}}\sum_{k=0}^{n}\binom{n}{k}\cancel{n!}\frac{n!}{k!(n-k)!}(x-1)^{n-k}(x+1)^k$$

$$= \sum_{k=0}^{n}\binom{n}{k}^2\left(\frac{x-1}{2}\right)^{n-k}\left(\frac{x+1}{2}\right)^k \quad \because \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Question– If $f(x) = \sum_{m=0}^{\infty} b_m P_m(x)$, then show that

$$b_m = \frac{2m+1}{2}\int_{-1}^{1} P_m(x)\,dx$$

Solution Given $f(x) = \sum_{m=0}^{\infty} b_m P_m(x)$

$$f(x)P_n(x) = \sum_{m=0}^{\infty} b_m P_m(x)P_n(x)$$

$$\int_{-1}^{1} f(x)P_n(x) = \sum_{m=0}^{\infty} b_m \int_{-1}^{1} P_m(x)P_n(x)\,dx$$

$$= b_n\int_{-1}^{1} P_n(x)P_n(x)\,dx$$

$\because P_m(x)$ & $P_n(x)$ are orthogonal

$$= b_n\cdot\frac{2}{2n+1}$$

$$\Rightarrow b_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx$$

Replacing n by m

$$\Leftrightarrow b_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)P_m(x)\,dx$$

**

Theorem:- Prove that if n is +ve integer, then

$$\int_{-1}^{1} P_n(x)(1-2xt+t^2)^{-1/2}\,dx = \frac{2t^n}{2n+1}$$ & hence

using Rodrigues formula show that

$$\int_{-1}^{1} (1-x^2)^n (1-2xt+t^2)^{-\frac{1}{2}-n}\,dx = \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

Proof:- We know that

$$(1-2xt+t^2)^{-1/2} = \sum_{m=0}^{\infty} P_m(x)\,t^m$$

Multiplying both sides by $P_n(x)$ & then integrating from -1 to 1

$$\int_{-1}^{1} P_n(x)(1-2xt+t^2)^{-\frac{1}{2}}\,dx = \sum_{m=0}^{\infty} t^m \int_{-1}^{1} P_m(x)P_n(x)\,dx$$

$$= t^n \int_{-1}^{1} P_n(x)P_n(x)\,dx$$

$\because P_m(x)$ & $P_n(x)$ are orthogonal

$$= t^n \frac{2}{2n+1}$$

Now by Rodrigues formula

$$\frac{2t^n}{2n+1} = \int_{-1}^{1} \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]\cdot(1-2xt+t^2)^{-1/2}\,dx$$

$$= \frac{1}{2^n n!}\int_{-1}^{1}(1-2xt+t^2)^{-\frac{1}{2}}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]dx$$

$$= \frac{1}{2^n n!}\left[(1-2xt+t^2)^{-\frac{1}{2}}\frac{d^{n-1}}{dx^{n-1}}\left[(x^2-1)^n\right]\Big|_{-1}^{1} - \int_{-1}^{1}\left(-\frac{1}{2}\right)(1-2xt+t^2)^{-\frac{1}{2}-1}\right.$$

$$\left.(-2t)\frac{d^{n-1}}{dx^{n-1}}\left[(x^2-1)^n\right]\right]dx$$

$$= \frac{1}{2^n n!}(-1)^1 \cdot t^1 \cdot 1 \int_{-1}^{1} (1-2xt+t^2)^{\frac{-1}{2}-1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx$$

using upto n-times

$$\frac{2t^n}{2n+1} = \frac{1}{2^n n!}(-1)^n \int_{-1}^{1} \left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-(n-1)\right)$$

$$(1-2xt+t^2)^{-\frac{1}{2}-n} (-2t)^n (x^2-1)^n \, dx$$

$$= \frac{1}{n!}\int_{-1}^{1} (-1)^n \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\cdots\left(\frac{2n-1}{2}\right)(1-2x+t^2)^{-\frac{1}{2}-n}$$

$$t^n (x^2-1)^n dx$$

$$= \frac{1}{n!}\int_{-1}^{1} \frac{(1)(3)(5)\cdots(2n-1)}{2^n} (1-2xt+t^2)^{-\frac{1}{2}-n} \cdot t^n (1-x^2)^n dx$$

$$\Rightarrow \frac{2t^n}{2n+1} = \frac{1}{n!\,2^n} \frac{t^n (2n)!}{n!\,2^n} \int_{-1}^{1} (1-2xt+t^2)^{-\frac{1}{2}-n} (1-x^2)^n dx$$

$$\Rightarrow \int_{-1}^{1} (1-2xt+t^2)^{-\frac{1}{2}-n} (1-x^2)^n dx = \frac{2t^n}{2n+1} \cdot \frac{n!\,2^n}{1} \cdot \frac{n!\,2^n}{t^n (2n)!}$$

$$= \frac{2^{2n+1} (n!)^2}{(2n+1)!}$$

—★★—

Question Show that if m & n are +ve integers then $$\int_{-1}^{1} (1+x)^{m+n} P_n(x) dx = \frac{2^{m+n+1}}{m!} \frac{[(m+n)!]^2}{(m+2n+1)!}$$

Solution By Rodrigues Formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

Now multiplying both sides by $(1+x)^{m+n}$ & then integrating w.r.t 'x'

Then $\int_{-1}^{1}(1+x)^{m+n} P_n(x)\,dx = \int_{-1}^{1}(1+x)^{m+n} \cdot \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n\,dx \longrightarrow *$

$$= \frac{1}{2^n n!}\left[(1+x)^{m+n} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \Big|_{-1}^{1} - \int_{-1}^{1}(m+n)(1+x)^{m+n-1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx\right]$$

$$= \frac{1}{2^n n!}\left[0-0-\int_{-1}^{1}(m+n)(1+x)^{m+n-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx\right]$$

$$\Rightarrow \int_{-1}^{1}(1+x)^{m+n} P_n(x)\,dx = \frac{-1\cdot(m+n)}{2^n \cdot n!}\int_{-1}^{1}(1+x)^{m+n-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx$$

Now integrating R.H.S w.r.t 'x' upto $n-1$ times

then $\int_{-1}^{1}(1+x)^{m+n} P_n(x)\,dx = \frac{(-1)^n (m+n)(m+n-1)(m+n-2)\cdots(m+1)}{2^n \cdot n!}\int_{-1}^{1}(1+x)^m \cdot (x^2-1)^n\,dx$

$$\int_{-1}^{1}(1+x)^{m+n} P_n(x)\,dx = \frac{(-1)^n (m+n)(m+n-1)\cdots(m+1)}{2^n \cdot n!}\int_{-1}^{1}(1+x)^m (x+1)^n (x-1)^n\,dx \longrightarrow (1)$$

Consider $\int_{-1}^{1}(1+x)^{m+n}(x-1)^n\,dx$

$$= \left|(x-1)^n \frac{(1+x)^{m+n+1}}{m+n+1}\right|_{-1}^{1} - \int_{-1}^{1}\frac{n(x-1)^{n-1}(1+x)^{m+n+1}}{m+n+1}\,dx$$

$$= [0-0] - \int_{-1}^{1}\frac{n(x-1)^{n-1}(x+1)^{m+n+1}}{m+n+1}\,dx$$

$$= \frac{(-1)\cdot n}{m+n+1}\int_{-1}^{1}(x-1)^{n-1}(x+1)^{m+n+1}\,dx$$

Now integrating n-times

$$\Rightarrow \int_{-1}^{1}(1+x)^{m+n}(x-1)^n\,dx = \frac{(-1)^n n!}{(m+n+1)(m+n+2)\cdots(m+2n)}\int_{-1}^{1}(x-1)^0(x+1)^{m+2n}dx$$

$$= \frac{(-1)^n n!}{(m+n+1)(m+n+2)\cdots\cdots(m+2n)}\int_{-1}^{1}(x+1)^{m+2n}\,dx$$

$$= \frac{(-1)^n n!}{(m+n+1)(m+n+2)\cdots(m+2n)}\left|\frac{(x+1)^{m+2n+1}}{m+2n+1}\right|_{-1}^{1}$$

$$= \frac{(-1)^n n!}{(m+n+1)(m+n+2)\cdots(m+2n)}\left[\frac{2^{m+2n+1}}{m+2n+1}-0\right]$$

$$= \frac{(-1)^n n!\cdot 2^{m+2n+1}}{(m+n+1)(m+n+2)\cdots(m+2n)(m+2n+1)}$$

Now equ ① becomes

$$\int_{-1}^{1}(1+x)^{m+n}P_n(x)\,dx = \frac{(-1)^n(m+n)(m+n-1)\cdots(m+1)(-1)^n\cdot \cancel{n!}\,2^{m+2n+1}}{2^n\cancel{n!}(m+n+1)\cdots(m+2n)(m+2n+1)}$$

$$= \frac{(-1)^{2n}\cdot 2^{m+n+1}\cdot \cancel{2^n}(m+n)(m+n-1)\cdots(m+1)}{\cancel{2^n}\cdot(m+n+1)\cdots(m+2n)(m+2n+1)}$$

$$= \frac{(-1)^{2n}\cdot 2^{m+n+1}(m+n)(m+n-1)\cdots(m+1)}{(m+n+1)\cdots(m+2n)(m+2n+1)}$$

$$= \frac{(-1)^{2n}\cdot 2^{m+n+1}(m+n)(m+n-1)\cdots(m+1)}{(m+2n+1)(m+2n)\cdots(m+n+1)}\times\frac{(m+n)!}{(m+n)!}$$

$$\times\frac{m(m-1)\cdots 3\cdot 2\cdot 1}{m(m-1)\cdots 3\cdot 2\cdot 1}$$

$$= \frac{(-1)^{2n}\cdot 2^{m+n+1}\cdot(m+n)(m+n-1)\cdots(m+1)\,m(m-1)}{(m+2n+1)(m+2n)\cdots(m+n+2)(m+n)!\,m(m-1)}$$

$$\frac{\cdots\cdot 3\cdot 2\cdot 1(m+n)!}{\cdots\cdot 3\cdot 2\cdot 1}$$

$$= \frac{(-1)^{2n} \cdot 2^{m+n+1} [(m+n)(m+n-1) \cdots (m+1)\, m(m-1) \cdots 3 \cdot 2 \cdot 1]}{[(m+2n+1)(m+2n) \cdots (m+n+2)(m+n)(m+n-1) \cdots m(m-1) \cdots 3 \cdot 2 \cdot 1]\, m(m-1) \cdots 3 \cdot 2 \cdot 1}$$

$$= \frac{(-1)^{2n}\, 2^{m+n+1}\, (m+n)!\, (m+n)!}{(m+2n+1)! \cdot m!}$$

$$\Rightarrow \int_{-1}^{1} (1+x)^{m+n} P_n(x)\, dx = \frac{(-1)^{2n}\, 2^{m+n+1} [(m+n)!]^2}{m!\,(m+2n+1)!}$$

$$\Rightarrow \int_{-1}^{1} (1+x)^{m+n} P_n(x)\, dx = \frac{2^{m+n+1} [(m+n)!]^2}{m!\,(m+2n+1)!} \quad \because (-1)^{2n} = 1$$

* * * *

Question. If $q = x + \sqrt{x^2-1}$, then show that

$$(1-tq)^{-\frac{1}{2}} (1 - t/q)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

Solution L.H.S $= (1-tq)^{-\frac{1}{2}} (1 - t/q)^{-1/2}$

$$= (1-tq)^{-\frac{1}{2}} (q-t)^{-\frac{1}{2}} \cdot (1/q)^{-1/2}$$

$$= \sqrt{q}\, [(1-tq)(q-t)]^{-1/2}$$

$$= \sqrt{q}\, [q - t + t^2 q - t q^2]^{-1/2}$$

$$= \sqrt{q}\, [x + \sqrt{x^2-1} - t + t^2 (x + \sqrt{x^2-1}) - t(x + \sqrt{x^2-1})^2]$$

$$= \sqrt{q}\, [x + \sqrt{x^2-1} - t - t[x^2 + (x^2-1) + 2x\sqrt{x^2-1}) + t^2 (x + \sqrt{x^2-1})]^{-1/2}$$

$$= \sqrt{q}\, [x + \sqrt{x^2-1} - t - 2tx^2 + t - 2xt\sqrt{x^2-1} + t^2 x + t^2 \sqrt{x^2-1}]^{-1/2}$$

$$= \sqrt{q}\left[x+\sqrt{x^2-1} - 2tx\left(x+\sqrt{x^2-1}\right) + t^2\left(x+\sqrt{x^2-1}\right)\right]^{-\frac{1}{2}}$$

$$= \sqrt{q}\left[q - 2txq + t^2 q\right]^{-1/2}$$

$$= (q)^{1/2}(q)^{-\frac{1}{2}}\left[1-2tx+t^2\right]^{-1/2}$$

$$= (1-2tx+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

$$= R.H.S$$

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⇒ Hypergeometric Differential Equations:-

The differential equation

$$z(1-z)y''(z) + [\gamma - (\alpha+\beta+1)z]y'(z) - \alpha\beta y(z) = 0$$

is called hypergeometric differential equation.

* Hypergeometric Function:-

The function denoted & defined by

$$F_{2,1}(\alpha,\beta,\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n\, n!} z^n$$

is called hypergeometric function, where α, β, γ are real parameters & z is a variable and

$$(\alpha)_n = (\alpha)(\alpha+1)(\alpha+2)\dots(\alpha+n-1)$$ and

$(\alpha)_n$, $(\beta)_n$ & $(\gamma)_n$ are called Christopher Symbols. Further $(\alpha)_0 = 1$

Moreover, γ is neither zero nor any -ve integer and the suffix 2 and

1 means that two parameters are in the numerator & one parameter in the denominator.

(VI) The number of parameters in the numerator & in the denominator, and further more the number of variables can vary. But here we will discuss the case only described above.

The hypergeometric function is also denoted by

$$F\begin{bmatrix}\alpha, \beta & \\ \gamma & z\end{bmatrix}$$

★ Series Solution of Hypergeometric Differential Equations:- Consider Hypergeometric Differential Equation

$$z(1-z)y''(z)+[\gamma-(\alpha+\beta+1)z]y'(z)-\alpha\beta y(z)=0 \longrightarrow ①$$

Let $y=\sum_{m=0}^{\infty} C_m z^{m+n}$ Then ① becomes

$$\sum_{m=0}^{\infty} C_m[(m+n)(m+n-1)+\gamma(m+n)]z^{m+n-1}-\sum_{m=0}^{\infty} C_m[(m+n)(m+n+\alpha+\beta)+\alpha\beta]z^{m+n}=0 \longrightarrow \star$$

Now equating co-efficients of like powers

Coefficient of $z^{n-1} \Rightarrow C_0[n(n-1)+\gamma n]=0 \longrightarrow (i)$

" of $z^{n} \Rightarrow C_1[(1+n)n+\gamma(1+n)]-[n(n+\alpha+\beta)+\alpha\beta]C_0=0 \longrightarrow (ii)$

" $z^{n+1} \Rightarrow C_2[(2+n)(1+n)+\gamma(2+n)]-[C_1[(1+n)(1+n+\alpha+\beta)+\alpha\beta]=0 \longrightarrow (iii)$

⋮

Coefficient of z^{m+n-1} $\Rightarrow C_m[(m+n)(m+n-1)+\gamma(m+n)] - C_{m-1}[$
$(m-1+n)(m-1+n+\alpha+\beta)+\alpha\beta] = 0$ $\longrightarrow$ (iv)

from (i)

$$C_0[n(n-1)+\gamma n] = 0$$

As $C_0 \neq 0 \Rightarrow n(n-1)+\gamma n = 0$

$$\Rightarrow n(n-1+\gamma) = 0 \quad \Rightarrow n=0, \; n-1+\gamma = 0$$

$$\Rightarrow n = 0 \;, \quad n = 1-\gamma$$

<u>When $n=0$</u>

from (iv)

$$C_m = \frac{(m+n-1)(m+n+\alpha+\beta-1)+\alpha\beta}{(m+n)(m+n-1)+\gamma(m+n)} C_{m-1}$$

Since $n=0$ so

$$C_m = \frac{(m-1)(m+\alpha+\beta-1)+\alpha\beta}{m(m-1)+\gamma(m)} C_{m-1}$$

$$\Rightarrow C_m = \frac{(\alpha+m-1)(\beta+m-1)}{m(\gamma+m-1)} C_{m-1} \longrightarrow (2)$$

Put $m=1$

$$\Rightarrow C_1 = \frac{(\alpha)(\beta)}{1.\gamma} C_0$$

Put $m=2 \Rightarrow C_2 = \frac{(\alpha+1)(\beta+1)}{2(\gamma+1)} C_1 = \frac{(\alpha+1)(\beta+1)}{2(\gamma+1)} \cdot \frac{\alpha.\beta}{\gamma} C_0$

$$= \frac{\alpha(\alpha+1).\beta(\beta+1)}{2.1\;\gamma(\gamma+1)} C_0$$

Put $m=3 \Rightarrow$

$$C_3 = \frac{(\alpha+2)(\beta+2)}{3(\gamma+2)} \cdot C_2 = \frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{3.2.1\;\gamma(\gamma+1)(\gamma+2)} C_0$$

Continuing in this way

$$C_m = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)\,\beta(\beta+1)(\beta+2)\cdots(\beta+m-1)}{m(m-1)(m-2)\cdots 3.2.1 \cdot \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+m-1)} C_0$$

$$\Rightarrow C_m = \frac{(\alpha)_m (\beta)_m}{m!\,(\gamma)_m} c_0$$

Thus $y(z) = \sum_{m=0}^{\infty} C_m z^{m+n} = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{m!\,(\gamma)_m} z^{m+0} c_0 \quad \because n=0$

Choose $y(z) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{m!\,(\gamma)_m} z^m = F_{2,1}[\alpha, \beta, \gamma; z]$

which is required solution of Hypergeometric defferential equation.

When $n = 1-\gamma$: then let $y(z) = z^{1-\gamma} v(z)$

where $v(z)$ is a solution of ① for $n=0$

then $y'(z) = z^{1-\gamma} v'(z) + (1-\gamma) z^{-\gamma} v(z)$

& $y''(z) = z^{1-\gamma} v''(z) + (1-\gamma) z^{-\gamma} v'(z) + (1-\gamma)(-\gamma) z^{-\gamma-1} v(z) + (1-\gamma) z^{-\gamma} v'(z)$

Putting these values in equ ① and after some simplification we get

$(-\gamma)(1-\gamma) z^{-\gamma} (1-z) v(z) + z(1-\gamma) z^{1-\gamma} (1-z) v'(z) + z^{2-\gamma} v''(z) + \gamma(1-\gamma) z^{-\gamma} v(z) + \gamma z^{1-\gamma} v'(z) - (\alpha+\beta+1)(1-\gamma) z^{1-\gamma} v(z) - (\alpha+\beta+1) z^{2-\gamma} v'(z) - \alpha\beta z^{1-\gamma} v(z) = 0$

Dividing by $z^{-\gamma}$ and after some simplification we get

$$[z^2(1-z)] v''(z) + [z(1-\gamma) z(1-z) + \gamma z - (\alpha+\beta+1) z^2] v'(z) + [(-\gamma)(1-\gamma)(1-z) + \gamma(1-\gamma) - (\alpha+\beta+1)(1-\gamma) z - \alpha\beta z] v(z) = 0$$

$$\Rightarrow [z^2(1-z)] v''(z) + [z(1-\gamma) z(1-z) + \gamma z - (\alpha+\beta+1) z^2] v'(z) + [\gamma(1-\gamma) z - (\alpha+\beta+1)(1-\gamma) z - \alpha\beta z] v(z) = 0$$

Dividing by z

$$\Rightarrow z(1-z) v''(z) + [z(1-\gamma)(1-z) + \gamma - (\alpha+\beta+1) z] v'(z) + [\gamma(1-\gamma) - (\alpha+\beta+1)(1-\gamma) - \alpha\beta] v(z) = 0$$

$$\Rightarrow z(1-z)v''(z)+[(2-\gamma)-\{(\alpha-\gamma+1)+(\beta-\gamma+1)+1\}v'(z)-$$
$$(\alpha-\gamma+1)(\beta-\gamma+1)v(z)=0 \longrightarrow (3)$$

(3) is now hypergeometric differential equation.

For $n=0$

$$v(z)=\sum_{m=0}^{\infty}\frac{(\alpha-\gamma+1)_m(\beta-\gamma+1)_m}{(2-\gamma)_m\ m!}z^m$$

$$\Rightarrow y(z)=z^{1-\gamma}v(z)=z^{1-\gamma}\sum_{m=0}^{\infty}\frac{(\alpha-\gamma+1)_m(\beta-\gamma+1)_m}{(2-\gamma)_m\cdot m!}z^m$$

$$=z^{1-\gamma}F_{2,1}[\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z]$$

which is required solution of hyper-geometric differential equation for $n=1-\gamma$

***

* Convergence of Series Solution of Hypergeometric Differential Equation:

By Ratio test

$$\left|\frac{U_{n+1}}{U_n}\right|=\left|\frac{(\alpha)_{n+1}(\beta)_{n+1}}{(\gamma)_{n+1}\cdot(n+1)!}z^{n+1}\cdot\frac{(\gamma)_n\cdot n!}{(\alpha)_n(\beta)_n}\cdot\frac{1}{z^n}\right|$$

$$=\left|\frac{(\alpha+n)(\beta+n)}{(\gamma+n)(n+1)}\cdot z\right|$$

$$=\frac{(\alpha/n+1)(\beta/n+1)}{(\gamma/n+1)(1+1/n)}|z|$$

when $n\longrightarrow\infty$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right|=\frac{(0+1)(0+1)}{(0+1)(1+0)}|z|=|z|$$

$\Rightarrow$ Series solution converges if $|z| < 1$

$$\Rightarrow -1 < z < 1$$

***

$\Rightarrow$ Gamma Function:-

$$\Gamma(x) = \int_0^\infty e^{-t} \cdot t^{x-1} \cdot dt$$

&

$$\Gamma x = (x-1)\,\Gamma(x-1)$$

and if x is non-negative integer, then $\Gamma x = (x-1)!$

$\Rightarrow$ Beta Function:-

$$\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}\, dx$$

Further $\beta(l,m) = \dfrac{\Gamma l\,\Gamma m}{\Gamma l+m}$

**

Question:- Show that $(1-x)^{-\alpha} = \sum_{m=0}^{\infty} \frac{(\alpha)_m x^m}{m!}$

Solution

$$(1-x)^{-\alpha} = 1 + (-\alpha)(-x) + \frac{(-\alpha)(-\alpha-1)}{2!}(-x)^2 + \frac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!}(-x)^3 + \cdots\cdots$$

$$\Rightarrow (1-x)^{-\alpha} = 1 + \alpha x + \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{3!}x^3 + \cdots\cdots$$

$$= \frac{(\alpha)_0 x^0}{0!} + \frac{(\alpha)_1 x^1}{1!} + \frac{(\alpha)_2 x^2}{2!} + \frac{(\alpha)_3 x^3}{3!} + \cdots\cdots$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m x^m}{m!}$$

***

Question:- Show that $\dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = (\alpha)_n$

Solution:

$$L.H.S = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

$$= \frac{(\alpha+n-1)\Gamma(\alpha+n-1)}{\Gamma(\alpha)}$$

$$= \frac{(\alpha+n-1)(\alpha+n-2)(\alpha+n-3)\cdots(\alpha+2)(\alpha+1)\alpha\,\Gamma(\alpha)}{\Gamma(\alpha)}$$

$$= (\alpha+n-1)(\alpha+n-2)(\alpha+n-3)\cdots(\alpha+2)(\alpha+1)\alpha$$

$$= \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-2)(\alpha+n-1)$$

$$= (\alpha)_n = R.H.S$$

**

* Integral Representation of Hypergeometric Function:-

Consider $\beta(\alpha+n, \gamma-\alpha) = \dfrac{\Gamma(\alpha+n)\,\Gamma(\gamma-\alpha)}{\Gamma(\alpha+n+\gamma-\alpha)}$

$$\because \beta(l,m) = \frac{\Gamma l\,\Gamma m}{\Gamma l+m}$$

$$\Rightarrow \beta(\alpha+n, \gamma-\alpha) = \frac{\Gamma(\alpha+n)\,\Gamma(\gamma-\alpha)}{\Gamma(\gamma+n)} \cdot \frac{\Gamma(\gamma)}{\Gamma(\gamma)} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha)}$$

$$= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} \cdot \Gamma\alpha \cdot \frac{\Gamma(\gamma)}{\Gamma(\gamma+n)}$$

$$= (\alpha)_n \frac{\Gamma(\gamma-\alpha)\,\Gamma(\alpha)}{\Gamma(\gamma)} \cdot \frac{1}{(\gamma)_n}$$

$$\Rightarrow \frac{(\alpha)_n}{(\gamma)_n} \frac{\Gamma(\gamma-\alpha)\Gamma(\alpha)}{\Gamma(\gamma)} = \beta(\alpha+n, \gamma-\alpha)$$

$$= \int_0^1 t^{\alpha+n-1} (1-t)^{\gamma-\alpha-1} dt \longrightarrow (1)$$

$\because$ By definition of $\beta(l,m)$

Now,

$$F_{2,1}[\alpha,\beta,\gamma;z] = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \, n!} \cdot z^n$$

$$= \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} z^n \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)} \int_0^1 t^{\alpha+n-1} (1-t)^{\gamma-\alpha-1} dt$$

By equ (1)

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)} \int_0^1 t^{\alpha-1} \cdot t^n (1-t)^{\gamma-\alpha-1} \frac{(\beta)_n}{n!} z^n dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)} \int_0^1 t^{\alpha-1} (1-t)^{\gamma-\alpha-1} \cancel{(1-tz)} \left(\sum_{n=0}^{\infty} \frac{(\beta)_n (tz)^n}{n!}\right) dt$$

$$\Rightarrow F_{2,1}[\alpha,\beta,\gamma;z] = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)} \int_0^1 t^{\alpha-1} (1-t)^{\gamma-\alpha-1} (1-tz)^{-\beta} dt$$

$$\because (1-t)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n t^n}{n!}$$

which is required integral representation of Hypergeometric function.

Question Show that $F_{2,1}[\alpha,\beta,\gamma;1] = \dfrac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}$

Solution

By integral representation of Hypergeometric function.

$$F_{2,1}[\alpha,\beta,\gamma;z] = \frac{\Gamma\gamma}{\Gamma(\gamma-\alpha)\,\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tz)^{-\beta}dt$$

$$\Rightarrow F_{2,1}[\alpha,\beta,\gamma;1] = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\,\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-t)^{-\beta}dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\,\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{(\gamma-\alpha-\beta)-1}dt$$

$$= \frac{\Gamma\gamma}{\Gamma(\gamma-\alpha)\,\Gamma\alpha}\cdot\frac{\Gamma\alpha\;\Gamma(\gamma-\alpha-\beta)}{\Gamma(\alpha+\gamma-\alpha-\beta)}$$

$$= \frac{\Gamma(\gamma)\;\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\;\Gamma(\gamma-\beta)}$$

* ____________ *

Question:- Show that

(1) $F(-n,\alpha+n,\gamma;1) = \dfrac{(\gamma-\alpha-n)_n}{(\gamma)_n}$

(2) $F(-n,1-\beta-n,\alpha;1) = \dfrac{(\alpha+\beta-1)_{2n}}{(\alpha)_n(\alpha+\beta-1)_n}$

(3) $\dfrac{d}{dz}[F(\alpha,\beta,\gamma;z)] = \dfrac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1;z)$

Solution:- (1) $F(-n,\alpha+n,\gamma;1) = \dfrac{(\gamma-\alpha-n)_n}{(\gamma)_n}$

Since $(\alpha,\beta,\gamma;1) = \dfrac{\Gamma\gamma\;\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\;\Gamma(\gamma-\beta)} \longrightarrow ①$

Now Put $\alpha=-n$, $\beta=\alpha+n$ in ①

$$\Rightarrow F(-n,\alpha+n,\gamma;1) = \frac{\Gamma\gamma\;\Gamma(\gamma-(-n)-(\alpha+n))}{\Gamma(\gamma-(-n))\;\Gamma(\gamma-(\alpha+n))}$$

$$= \frac{\Gamma\gamma \, \Gamma(\gamma+n-\alpha-n)}{\Gamma(\gamma+n)\, \Gamma(\gamma-\alpha-n)}$$

$$= \frac{1}{\frac{\Gamma(\gamma+n)}{\Gamma\gamma}} \cdot \frac{\Gamma((\gamma-\alpha-n)+n)}{\Gamma(\gamma-\alpha-n)}$$

$$= \frac{1}{(\gamma)_n} \cdot (\gamma-\alpha-n)_n$$

$$\Rightarrow F[-n, \alpha+n, \gamma; 1] = \frac{(\gamma-\alpha-n)_n}{(\gamma)_n}$$

2) $F(-n, 1-\beta-n, \alpha; 1) = \dfrac{(\alpha+\beta-1)_{2n}}{(\alpha)_n (\alpha+\beta-1)_n} \longrightarrow ④$

Since $F(\alpha, \beta, \gamma; 1) = \dfrac{\Gamma\gamma \, \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\, \Gamma(\gamma-\beta)} \longrightarrow ①$

Put $\alpha = -n$, $\beta = 1-\beta-n$, $\gamma = \alpha$ in ①

$$\Rightarrow F(-n, 1-\beta-n, \alpha; 1) = \frac{\Gamma\alpha \, \Gamma(\alpha-(-n)-(1-\beta-n))}{\Gamma(\alpha-(-n))\, \Gamma(\alpha-(1-\beta-n))}$$

$$= \frac{\Gamma\alpha}{\Gamma(\alpha+n)} \frac{\Gamma(\alpha+n-1+\beta+n)}{\Gamma(\alpha-1+\beta+n)}$$

$$= \frac{1}{\frac{\Gamma(\alpha+n)}{\Gamma\alpha}} \cdot \frac{\Gamma(\alpha+\beta-1+2n)}{\Gamma((\alpha+\beta-1)+n)} \times \frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha+\beta-1)}$$

$$= \frac{1}{(\alpha)_n} \frac{\Gamma(\alpha+\beta-1+2n)}{\Gamma(\alpha+\beta-1)} \cdot \frac{1}{\frac{\Gamma((\alpha+\beta-1)+n)}{\Gamma(\alpha+\beta-1)}}$$

$$\Rightarrow F(-n, 1-\beta-n, \alpha; 1) = \frac{1}{(\alpha)_n} \cdot \frac{(\alpha+\beta-1)_{2n}}{(\alpha+\beta-1)_n}$$

3) $\frac{d}{dz}\left[F(\alpha,\beta,\gamma;z)\right] = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+1; z)$

Since $F(\alpha,\beta,\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \cdot n!} \cdot z^n$

$$\Rightarrow \frac{d}{dz}\left[F(\alpha,\beta,\gamma;z)\right] = \frac{d}{dz}\left[\sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n\, n!} \cdot z^n\right]$$

$$= \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \cdot n!} n z^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n (n-1)!} \cdot z^{n-1}$$

$$\Rightarrow \frac{d}{dz}\left[F(\alpha,\beta,\gamma;z)\right] = \frac{(\alpha)_1 (\beta)_1}{(\gamma)_1 (1-1)!} z^{1-1} + \frac{(\alpha)_2 (\beta)_2}{(\gamma)_2 (2-1)!} z^{2-1} +$$

$$\frac{(\alpha)_3 (\beta)_3}{(\gamma)_3 (3-1)!} z^{3-1} + \frac{(\alpha_4) (\beta)_4}{(\gamma)_4 (4-1)!} z^{4-1} + \cdots$$

$$= \frac{\alpha\beta}{\gamma} + \frac{\alpha(\alpha+1) \beta(\beta+1)}{\gamma(\gamma+1)} z +$$

$$\frac{\alpha(\alpha+1)(\alpha+2) \beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)\, 2!} z^2 + \cdots$$

$$= \frac{\alpha\beta}{\gamma}\left[1 + \frac{(\alpha+1)_1 (\beta+1)_1}{(\gamma+1)_1} z + \frac{(\alpha+1)_2 (\beta+1)_2}{(\gamma+1)_2\, 2!} z^2 + \cdots\right]$$

$$= \frac{\alpha\beta}{\gamma} \sum_{n=0}^{\infty} \frac{(\alpha+1)_n (\beta+1)_n}{(\gamma+1)_n \cdot n!} \cdot z^n$$

$$\Rightarrow \frac{d}{dz}\left[F(\alpha,\beta,\gamma;1)\right] = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+1; z)$$

★———★★★

Question:- Show that $(\alpha)_{n-k} = \dfrac{(\alpha)_n}{(-1)^k(1-\alpha-n)_k}$ and hence deduce the case $(1)_{n-k} = \dfrac{n!}{(-1)^k(-n)_k}$

Solution:-

Since $(\alpha)_{n-k} = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+(n-k)-1)$

$$\Rightarrow (\alpha)_{n-k} = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-k-1) \times \frac{(\alpha+n-k)(\alpha+n-k+1)\ldots\ldots(\alpha+n-1)}{(\alpha+n-k)(\alpha+n-k+1)\ldots\ldots(\alpha+n-1)}$$

$$= \frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-k-1)(\alpha+n-k)(\alpha+n-k+1)\ldots(\alpha+n-1)}{(\alpha+n-k)(\alpha+n-k+1)\ldots(\alpha+n-1)}$$

$$= \frac{(\alpha)_n}{(-1)^k(1-\alpha-n)(2-\alpha-n)\ldots(k-\alpha-n)}$$

$$= \frac{(\alpha)_n}{(-1)^k(1-\alpha-n)(1-\alpha-n+1)\ldots(1-\alpha-n+(k-1))}$$

$$= \frac{(\alpha)_n}{(-1)^k(1-\alpha-n)_k}$$

Now put $\alpha = 1$

$$\Rightarrow (1)_{n-k} = \frac{(1)_n}{(-1)^k(-n)_k}$$

$$= \frac{(1)(1+1)(1+2)\ldots(1+(n-1))}{(-1)^k(-n)_k}$$

$$\Rightarrow (1)_{n-k} = \frac{n!}{(-1)^k(-n)_k}$$

** ——— **

* Hypergeometric form of Legender's Polynomial:-

we know that $\sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2)^{-\frac{1}{2}}$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x)t^n = [(1+t^2-2t)-2xt+2t]^{-\frac{1}{2}}$$

$$= [(1-t)^2 - 2t(x-1)]^{-\frac{1}{2}}$$

$$= [(1-t)^2]^{-\frac{1}{2}} \left[1 - \frac{2t(x-1)}{(1-t)^2}\right]^{-\frac{1}{2}}$$

$$= (1-t)^{-1} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k}{k!} \left[\frac{2t(x-1)}{(1-t)^2}\right]^k$$

$$\because (1-x)^{-\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k x^k}{k!} \longrightarrow *$$

$$= \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k 2^k t^k (x-1)^k}{k!} (1-t)^{-(2k+1)}$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_k 2^k t^k}{k!} (x-1)^k \frac{(2k+1)_n t^n}{n!}$$

$\because$ Again by *

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_k 2^k t^{n+k} (x-1)^k (2k+n)!}{k!\, n!\, (2k)!}$$

$$\left(\because (2k+1)_n = \frac{\Gamma(2k+1+n)}{\Gamma(2k+1)} = \frac{\Gamma(2k+n+1)}{\Gamma(2k+1)} = \frac{(2k+n)!}{(2k)!}\right.$$

Now $\frac{(\frac{1}{2})_k 2^k}{(2k)!} = \frac{(\frac{1}{2})(\frac{1}{2}+1)(\frac{1}{2}+2)\cdots(\frac{1}{2}+k-1)\cdot 2^k}{(2k)(2k-1)(2k-2)\cdots 4\cdot 3\cdot 2\cdot 1}$

$$= \frac{(1)(3)(5)\cdots(2k-1)\cdot \cancel{\frac{1}{2^k}}\cdot \cancel{2^k}}{(2k)(2k-2)(2k-4)\cdots 4\cdot 2\,(2k-1)(2k-3)(2k-5)\cdots 5\cdot 3\cdot 1}$$

$$\Rightarrow \frac{(\frac{1}{2})_K \cdot 2^K}{(2K)!} = \frac{1}{2^K K!}$$

Hence by ①

$$\sum_{n=0}^{\infty} P_n(x) t^n = \sum_{K=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^{n+K}(x-1)^K}{K!\,(n)!} \cdot \frac{(2K+n)!}{2^K \cdot K!}$$

Now replacing n by $n-K$ on R.H.S

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x) t^n = \sum_{K=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^n (x-1)^K}{K!\,(n-K)!} \cdot \frac{(K+n)!}{2^K \cdot K!}$$

$$= \sum_{K=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^n (x-1)^K}{(K!)^2\, 2^K} \cdot \frac{(n+K)!}{(n-K)!}$$

$$= \sum_{K=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^n (x-1)^K}{(K!)^2 \cdot 2^K} \cdot (-1)^K (-n)_K (n+1)_K$$

(by Previous result)

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} \left[\sum_{K=0}^{\infty} \frac{(-n)_K (n+1)_K}{(1)_K \cdot K!} \left(\frac{1-x}{2}\right)^K \right] t^n$$

$\because (1)_K = K!$

$$= \sum_{n=0}^{\infty} \left[F_{2,1}\left(-n, n+1, 1; \frac{1-x}{2}\right) \right] t^n$$

which is required Hypergeometric form of Legender's polynomial i.e

$$P_n(x) = F_{2,1}\left[-n, n+1, 1; \frac{1-x}{2}\right]$$

Now replace x by $-x$ on both sides

$$\Rightarrow P_n(-x) = F_{2,1}\left[-n, n+1, 1; \frac{1+x}{2}\right]$$

$$(-1)^n P_n(x) = F_{2,1}\left[-n, n+1, 1; \frac{1+x}{2}\right]$$

$$\Rightarrow P_n(x) = (-1)^n F_{2,1}\left[-n, n+1, 1; \frac{1+x}{2}\right]$$

which is another hypergeometric form.

* ——— * * * ——— *

Green's Functions AND Associated Boundry Value Problems.

⇒ Kronecker Delta:-

δ_{ij} is defined as

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

It is a tensors of rank 2 and it has the following well known property

$$\sum_i \delta_{ij} a_i = a_j$$

In δ_{ij}, the indices i and j have the integral values as $1, 2, 3, \ldots$

⇒ Dirac Delta Function:-

Dirac delta function is actually the generalization of the Kronecker delta function δ_{ij}.

It is denoted as defined by

$$\delta(x-t) = \begin{cases} 0 & x \neq t \\ \infty & x = t \end{cases}$$

Contrary to δ_{ij}, in $\delta(x-t)$ the values of x and t are real as continuous.

Properties:- It has the following properties

(i) $\int_{-\infty}^{\infty} \delta(x-t)\,dx = 1$, (ii) $\int_{-\infty}^{\infty} \delta(x-t) f(x)\,dx = f(t)$

* * ———————— * *

⇒ Motivation for Green's Function:-

We have to solve the problem associated with non-homogeneous differential equation

$$\mathcal{L}\{u(x)\} + \lambda r(x)\,u(x) = f(x) \longrightarrow (1)$$

where $\mathcal{L} = \frac{d}{dx}\left\{P(x)\frac{d}{dx}\right\} + q(x)$ and u satisfies some suitable end point conditions.

The solution of non-homogeneous differential equation subject to boundry conditions is closely related to the existence of green's function associated with homogeneous differential equation

$$\mathcal{L}(u) + \lambda r(x)\,u = 0.$$

If a function $G(x,t,\lambda)$ which does not depend on the source function $f(x)$ exist, Then the solution of (1) can be written as

$$u(x) = \int_a^b G(x,t,\lambda)\,f(t)\,dt$$

In this case $G(x,t,\lambda)$ is called green's function & satisfies the equation

$$\mathcal{L}\{G\} + \lambda r(x)\,G = \delta(x-t)$$

⇒ Green's Function Associated with Regular SL System**

Let

$$\mathcal{L}(u) + \lambda r(x)\, u = 0 \longrightarrow ①$$ be the

SL equation with end point conditions

$$\alpha u(a) + \alpha' u'(a) = 0 \longrightarrow ②$$

and $$\beta u(b) + \beta' u'(b) = 0 \longrightarrow ③$$

equation ② & ③ can also be written as

$$B_1(u) = 0 \text{ and } B_2(u) = 0$$

Under the assumption that $\lambda = 0$ is not an eigen value of this system (Regular SL system) i.e it gives trivial solution, and there associates a function with it, called green's function which has the following properties.

1) Green's function $G(x,t)$ considered a function of x satisfies the differential equation $\mathcal{L}\{G(x,t)\} = 0$ in each of sub interval $[a,t)$ & $(t,b]$.
2) $G(x,t)$ is continuous in $[a,b]$
3) $G(x,t)$ as a function of x satisfies the end point conditions $B_1(G) = 0$ & $B_2(G) = 0$
4) $\frac{dG}{dx}$ is discontinuous as $x \to t$ & moreover

$$\lim_{x\to t-0} G'(x,t) - \lim_{x\to t+0} G'(x,t) = \frac{1}{P(t)}$$

*———***

Examples**

Example 1:- Construct a Green's function associated with the problem
$$u''+\lambda u=0 \quad \text{with } u(0)=0 \ \& \ u(1)=0$$

Solution

1) First we check whether $\lambda=0$ is eigen value or not.

$\lambda=0 \Rightarrow u''=0 \Rightarrow u'=A$
$$\Rightarrow u=Ax+B$$
$u(0)=0 \Rightarrow A(0)+B=0 \Rightarrow \boxed{B=0}$
$u(1)=0 \Rightarrow A(1)+B=0 \Rightarrow \boxed{A=0} \quad \because B=0$
$$\Rightarrow u(0)=0$$
$\Rightarrow \lambda=0$ is not the eigen value.

2) $G(x,t)$ satisfies the differential equation
$\Rightarrow G''(x,t)=0$ in each of the sub-interval $[0,t[\,,\ ]t,1]$

Hence
$$G(x,t)=\begin{cases} Ax+B & 0\le x<t \\ A'x+B' & t<x\le 1 \end{cases}$$

3) As $G(x,t)$ satisfies end point conditions
$G(0,t)=0 \quad \& \quad G(1,t)=0$
$\Rightarrow A(0)+B=0 \quad \& \quad A'(1)+B'=0$
$B=0 \quad \& \quad B'=-A'$
$$\Rightarrow G(x,t)=\begin{cases} Ax & 0\le x<t \\ A'x-A' & t<x\le 1 \end{cases}$$

4) As $G(x,t)$ is continuous on $[0,t[$ so is continuous on $x=t$
$$\Rightarrow G(t-0,t)=G(t+0,t)$$

$$\Rightarrow At = A't - A'$$

$$\Rightarrow At = A'(t-1) \quad \Rightarrow A = \frac{A'(t-1)}{t}$$

$$\Rightarrow G(x,t) = \begin{cases} A'\dfrac{(t-1)}{t}x & 0 \le x < t \\ A'(x-1) & t < x \le 1 \end{cases}$$

5) Now $G'(x,t)$ satisfies discontinuous condition which is

$$G'(t-0,t) - G'(t+0,t) = \frac{1}{p(t)}$$

$$\Rightarrow A'\left[\frac{t-1}{t}\right] - A' = \frac{1}{1} \qquad \because p(x)=1 \Rightarrow p(t)=1$$

$$\Rightarrow A'\left[1-\frac{1}{t}\right] - A' = 1$$

$$\Rightarrow \cancel{A'} - \frac{A'}{t} - \cancel{A'} = 1 \Rightarrow A' = -t$$

$$\Rightarrow G(x,t) = \begin{cases} (1-t)x & 0 \le x < t \\ (1-x)t & t < x \le 1 \end{cases}$$

* ——————— *

Example 2:- Construct Green's function associated with the problem

$$xu'' + u' + \lambda r(x)u = 0$$

$u(0)$ is finite & $u(1) = 0$

Solution

1) 1st we check whether $\lambda = 0$ is eigen value or not

$$\lambda = 0 \Rightarrow xu'' + u' = 0$$

$$\Rightarrow \frac{d}{dx}(xu') = 0 \Rightarrow xu' = A$$

$$\Rightarrow u' = \frac{A}{x} \Rightarrow u(x) = A\ln x + B$$

$u(0)$ is finite $\Rightarrow A = 0$

$\Rightarrow u(x) = B$, $u(1) = 0 \Rightarrow B = 0$

$\Rightarrow u = 0 \Rightarrow \lambda = 0$ is not the eigen value of the system

2) $G(x,t)$ satisfies the differential equation $x G''(x,t) + G'(x,t) = 0$ in each of the subinterval $[0,t[$ & $]t,1]$. So

$$G(x,t) = \begin{cases} A \ln x + B & 0 \le x < t \\ A' \ln x + B' & t < x \le 1 \end{cases}$$

3) Sin $G(x,t)$ is continuous in $[0,1]$ so is continuous at $x = t$

$\Rightarrow G(t-0,t) = G(t+0,t)$

$\Rightarrow A \ln t + B = A' \ln t + B'$

$\Rightarrow -B' = (A' - A) \ln t - B$

$\Rightarrow B' = (A - A') \ln t + B$

$$\Rightarrow G(x,t) = \begin{cases} A \ln x + B & 0 \le x < t \\ A' \ln x + (A - A') \ln t + B & t < x \le 1 \end{cases}$$

4) $G(x,t)$ satisfies the end point conditions

$G(0,t)$ is finite $\Rightarrow \boxed{A = 0}$

$G(1,t) = 0 \Rightarrow A' \ln(1) + (A - A') \ln t + B = 0$

$\Rightarrow B = (A' - A) \ln t$

$$\Rightarrow G(x,t) = \begin{cases} (A' - 0) \ln t & 0 \le x < t \\ A' \ln x & t < x \le 1 \end{cases}$$

5) By the discontinuity condition

$$G'(t-0,t) - G'(t+0,t) = \frac{1}{P(t)}$$

$$\Rightarrow 0 - \frac{A'}{t} = \frac{1}{t} \Rightarrow A' = -1$$

$$\Rightarrow G(x,t) = \begin{cases} -\ln t & 0 \leq x < t \\ -\ln x & t < x \leq 1 \end{cases}$$

* *

Example 3:- Construct Green's function associated with the problem

$$x u'' + u' - \frac{n^2}{x} u + \lambda r(x) u = 0$$

with $u(0)$ is finite & $u(1) = 0$

Solution

Given equation can be written as

$$\frac{d}{dx}\{x u'\} + \left(\frac{-n^2}{x}\right) u + \lambda r(x) u = 0$$

1) $\Rightarrow P(x) = x$ & $q(x) = \frac{-n^2}{x}$

Now $\lambda = 0 \Rightarrow \frac{d}{dx}\{x u'\} - \frac{n^2}{x} u = 0$

$$\Rightarrow x u'' + u' - \frac{n^2}{x} u = 0$$

$$\Rightarrow x^2 u'' + x u' - n^2 u = 0$$

which is cauchy's euler's equation

& $\Rightarrow u(x) = A x^n + B x^{-n}$

Now $u(0)$ is finite $\Rightarrow B = 0$

$$\Rightarrow u(x) = A x^n$$

Now $u(1) = 0 \Rightarrow A(1)^n = 0 \Rightarrow A = 0$

$\Rightarrow \lambda = 0$ is not an eigen value.

2) As $G(x,t)$ satisfies the differential equation in each of sub interval

$[0,t[$ & $]t,1]$ so

$$G(x,t) = \begin{cases} Ax^n + Bx^{-n} & 0 \le x < t \\ A'x^n + B'x^{-n} & t < x \le 1 \end{cases}$$

3) $G(x,t)$ satisfies the end point conditions

$G(0,t)$ is finite $\Rightarrow B = 0$

& $G(1,t) = 0 \Rightarrow A' + B' = 0 \Rightarrow B' = -A'$

$$\Rightarrow G(x,t) = \begin{cases} Ax^n & 0 \le x < t \\ A'(x^n - x^{-n}) & t < x \le 1 \end{cases}$$

4) $G(x,t)$ is continuous in $[0,1]$ so is continuous at $x = t$

$\Rightarrow G(t-0,t) = G(t+0,t)$

$\Rightarrow At^n = A'(t^n - t^{-n}) \Rightarrow A = A'(1 - t^{-2n})$

$$\Rightarrow G(x,t) = \begin{cases} A'(1-t^{-2n})x^n & 0 \le x < t \\ A'(x^n - x^{-n}) & t < x \le 1 \end{cases}$$

5) By Discontinuous Condition

$G'(t-0,t) - G'(t+0,t) = \frac{1}{P(t)}$

$\Rightarrow A'(1-t^{-2n})nt^{n-1} - A'(nt^{n-1} + nt^{-n-1}) = \frac{1}{t}$

$\Rightarrow A'[\cancel{nt^{n-1}} - nt^{-n-1} - \cancel{nt^{n-1}} - nt^{-n-1}] = \frac{1}{t}$

$A'[-2nt^{-n+1}] = \frac{1}{t} \Rightarrow A' = \frac{-1}{2n}t^n$

$$\Rightarrow G(x,t) = \begin{cases} \frac{-1}{2n}[t^n - t^{-n}]x^n & 0 \le x < t \\ \frac{-1}{2n}[x^n - x^{-n}]t^n & t < x \le 1 \end{cases}$$

Example 4:- Construct the Green's function for $\frac{d}{dx}\left\{(1-x^2)u'\right\} - \frac{k^2}{1-x^2}u + \lambda r(x)u = 0$

with $u(\pm 1)$ are finite.

Solution Here $p(x) = 1-x^2$ & $q(x) = \frac{-k^2}{1-x^2}$

$a=-1$, $b=1$

1) First we check whether $\lambda = 0$ is eigen value or not

$$\lambda = 0 \Rightarrow \frac{d}{dx}\left\{(1-x^2)u'\right\} - \frac{k^2}{1-x^2}u = 0$$

$$\Rightarrow (1-x^2)u'' - 2xu' - \frac{k^2}{1-x^2}u = 0 \longrightarrow (1)$$

Put $t = \ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$

$$\frac{dt}{dx} = \frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2}$$

$$\frac{du}{dx} = \frac{du}{dt}\cdot\frac{dt}{dx} = \frac{2}{1-x^2}\cdot\frac{du}{dt}$$

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left[\frac{2}{1-x^2}\frac{du}{dt}\right]$$

$$= -2(1-x^2)^{-2}(-2x)\frac{du}{dt} + \frac{2}{1-x^2}\frac{d}{dt}\left(\frac{du}{dt}\right)\frac{dt}{dx}$$

$$= \frac{4x}{(1-x^2)^2}\frac{du}{dt} + \frac{2}{1-x^2}\cdot\frac{2}{1-x^2}\frac{d^2u}{dt^2}$$

$$= \frac{4}{(1-x^2)^2}\left[\frac{d^2u}{dt^2} + x\frac{du}{dt}\right]$$

Putting these values in equ (1) we get.

$$(1-x^2)\frac{4}{(1-x^2)^2}\left[\frac{d^2u}{dt^2}+x\frac{du}{dt}\right]-2x\cdot\frac{2}{1-x^2}\frac{du}{dt}-\frac{k^2}{1-x^2}u=0$$

$$\Rightarrow 4\left[\frac{d^2u}{dt^2}+x\frac{du}{dt}\right]-4x\frac{du}{dt}-\frac{k^2}{1}u=0$$

$$\Rightarrow 4\frac{d^2u}{dt^2}-k^2u=0 \quad\Rightarrow \frac{d^2u}{dt^2}-\frac{k^2}{4}u=0$$

$$\Rightarrow u=Ae^{kt/2}+Be^{-kt/2}$$

$$\Rightarrow u=A\left[\frac{1+x}{1-x}\right]^{\frac{1}{2}k}+B\left[\frac{1+x}{1-x}\right]^{-\frac{1}{2}k}$$

$u(1)$ is finite $\Rightarrow A=0$

$u(-1)$ is finte $\Rightarrow B=0$

$\Rightarrow u=0$

$\Rightarrow \lambda=0$ is not the eigen value.

2) $G(x,t)$ satisfies the given differential equation $(1-x^2)G''-2xG'-\frac{k^2}{1-x^2}G=0$ in each of the sub interval $[0,t[$ & $]t,1]$

$$\Rightarrow G(x,t)=\begin{cases} A\left[\frac{1+x}{1-x}\right]^{\frac{1}{2}k}+B\left[\frac{1+x}{1-x}\right]^{-\frac{1}{2}k} & -1\leq x<t \\ A'\left[\frac{1+x}{1-x}\right]^{\frac{1}{2}k}+B'\left[\frac{1+x}{1-x}\right]^{-\frac{1}{2}k} & t<x\leq 1\end{cases}$$

3) $G(x,t)$ satisfies the end point conditions

$G(-1,t)$ is finite $\Rightarrow B=0$

$G(1,t)$ " " $\Rightarrow A'=0$

$$\Rightarrow G(x,t)=\begin{cases} A\left[\frac{1+x}{1-x}\right]^{\frac{1}{2}k} & -1\leq x<t \\ B'\left[\frac{1+x}{1-x}\right]^{-\frac{1}{2}k} & t<x\leq 1\end{cases}$$

4) $G(x,t)$ is continuous at $x=t$

$$\Rightarrow G(t-0,t) = G(t+0,t)$$

$$\Rightarrow A\left[\frac{1+t}{1-t}\right]^{\frac{1}{2}k} = B'\left[\frac{1+t}{1-t}\right]^{-\frac{1}{2}k} = C$$

$$\Rightarrow A = \left[\frac{1+t}{1-t}\right]^{-\frac{1}{2}k} C \quad , \quad B' = \left[\frac{1+t}{1-t}\right]^{\frac{1}{2}k} C$$

$$\Rightarrow G(x,t) = \begin{cases} \left(\frac{1+t}{1-t}\right)^{-\frac{1}{2}k}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}k} C & -1 \leq x < t \\ \left(\frac{1+t}{1-t}\right)^{\frac{1}{2}k}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}k} C & t < x \leq 1 \end{cases}$$

5) By the discontinuity condition.

$$G'(t-0,t) - G'(t+0,t) = \frac{1}{p(t)}$$

$$\left(\frac{1+t}{1-t}\right)^{-\frac{1}{2}k} \cdot \frac{1}{\cancel{2}}k\left(\frac{1+t}{1-t}\right)^{\frac{1}{2}k-1} C \cdot \left(\frac{\cancel{2}}{(1-t)^2}\right) - \left(\frac{1+t}{1-t}\right)^{\frac{1}{2}k}\left(\frac{-1}{\cancel{2}}k\right)$$
$$\left(\frac{1+t}{1-t}\right)^{-\frac{1}{2}k-1}\left(\frac{\cancel{2}}{(1-t)^2}\right) \cdot C = \frac{1}{1-t^2}$$

$$\frac{kc}{1-t^2} + \frac{kc}{1-t^2} = \frac{1}{1-t^2} \quad \Rightarrow \quad c = \frac{1}{2k}$$

$$\Rightarrow G(x,t) = \begin{cases} \frac{1}{2k}\left(\frac{1+t}{1-t}\right)^{-\frac{1}{2}k}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}k} & -1 \leq x < t \\ \frac{1}{2k}\left(\frac{1+t}{1-t}\right)^{\frac{1}{2}k}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}k} & t < x \leq 1 \end{cases}$$

* ——————— *

Example 5:- Construct Green's function Associated with the problem.

$u'' + \lambda u = 0$ with

$u(0) + u'(1) = 0$, $u(1) + 2u'(0) = 0$

Solution

1) First we check whether $\lambda = 0$ is eigen value or not

$\lambda = 0 \Rightarrow u'' = 0 \Rightarrow u' = A$

$\Rightarrow u = Ax + B$

$u(0) + u'(1) = 0 \Rightarrow B + A = 0 \longrightarrow ①$

$u(1) + 2u'(0) = 0 \Rightarrow A + B + 2A = 0$

$\Rightarrow 3A + B = 0 \longrightarrow ②$

from ① & ② $A = 0$ & $B = 0$

$\Rightarrow u = 0 \Rightarrow \lambda = 0$ is not eigen value.

2) $G(x,t)$ satisfies the differential equation.

$G''(x,t) = 0$ in each of the subinterval $[0, t[$ & $]t, 1]$

$$\Rightarrow G(x,t) = \begin{cases} Ax + B & 0 \le x < t \\ A'x + B' & t < x \le 1 \end{cases}$$

3) $G(x,t)$ is continuous at $x = t$

$\Rightarrow G(t-0, t) = G(t+0, t)$

$At + B = A't + B' \Rightarrow B = (A' - A)t + B'$

$$\Rightarrow G(x,t) = \begin{cases} Ax + (A' - A)t + B' & 0 \le x < t \\ A'x + B' & t < x \le 1 \end{cases}$$

4) $G(x,t)$ satisfies the end point condition,

so $(A' - A)t + B' + A' = 0$

& $A' + B' + 2A = 0 \longrightarrow \star$

$\Rightarrow (A' - A)t - 2A = 0$

$$\Rightarrow A' = (2A + At)/t = A\left(\frac{t+2}{t}\right)$$

$$\& \quad A\left(\frac{t+2}{2}\right) + B' + 2A = 0 \qquad \text{by } (*)$$

$$\frac{2}{t}A + B' + 3A = 0$$

$$\Rightarrow B' = -\left(3 + \frac{2}{t}\right)A$$

$$\Rightarrow G(x,t) = \begin{cases} Ax - A\left(\frac{t+2}{t}\right) & 0 \le x < t \\ A\left(\frac{t+2}{t}\right)x - A\left(\frac{3t+2}{t}\right) & t < x \le 1 \end{cases}$$

5) By The discontinuity condition.

$$G'(t-0,t) - G'(t+0,t) = \frac{1}{P(t)}$$

$$\Rightarrow A - A\left(1 + \frac{2}{t}\right) = \frac{1}{1} \Rightarrow \frac{-2A}{t} = 1$$

$$\Rightarrow A = -\frac{1}{2}t$$

$$\Rightarrow G(x,t) = \begin{cases} -\frac{1}{2}tx + \frac{1}{2}\cancel{t}\left(\frac{t+2}{\cancel{t}}\right) & 0 \le x < t \\ -\frac{1}{2}\cancel{t}\left(\frac{t+2}{\cancel{t}}\right)x + \frac{1}{2}\cancel{t}\left(\frac{3t+2}{\cancel{t}}\right) & t < x \le 1 \end{cases}$$

$$= \begin{cases} -\frac{1}{2}tx + \frac{1}{2}t + 1 & 0 \le x < t \\ \left(-\frac{1}{2}t - 1\right)x + \frac{3}{2}t + 1 & t < x \le 1 \end{cases}$$

$$\Rightarrow G(x,t) = \begin{cases} \frac{1}{2}(2 + t - tx) & 0 \le x < t \\ \frac{1}{2}(2 + 3t - tx - 2x) & t < x \le 1 \end{cases}$$

Example 6- Construct Green's function associated with the system.

$$u'' + \lambda u = 0 \quad \text{with} \quad u'(0) = 0 \ \ \& \ \ u(1) = 0.$$

Solution

Here $p(x) = 1$, $a = 0$ & $b = 1$

1) 1st we check whether $\lambda = 0$ is eigen value or not.

put $\lambda = 0 \Rightarrow u'' = 0 \Rightarrow u = Ax + B$

$\cancel{u'(0) = 0 \Rightarrow}$ $u(1) = 0 \Rightarrow A + B = 0$

& $u'(0) = 0 \Rightarrow A = 0 \Rightarrow B = 0$

$\Rightarrow u = 0 \quad \Rightarrow \lambda = 0$ is not eigen value.

2) $G(x,t)$ satisfies the differential equation

$G'' = 0$ in $[0, t[$ & $]t, 1]$

$$\Rightarrow G(x,t) = \begin{cases} Ax + B & 0 \leq x < t \\ A'x + B' & t < x \leq 1 \end{cases}$$

3) As $G(x,t)$ is continuous at $x = t$

So $G(t-0, t) = G(t+0, t)$

$\Rightarrow At + B = A't + B'$

$\Rightarrow B = (A' - A)t + B'$

$$\Rightarrow G(x,t) = \begin{cases} Ax + (A' - A)t + B' & 0 \leq x < t \\ A'x + B' & t < x \leq 1 \end{cases}$$

4) $G(x,t)$ satisfies the end point conditions

$G(1,t) = 0 \quad \Rightarrow A' + B' = 0 \longrightarrow (1)$

$G'(0,t) = 0 \quad \Rightarrow A = 0$

So By (1) $\quad B' = -A'$

So

$$G(x,t) = \begin{cases} A't - A' & 0 \leq x < t \\ A'x - A' & t < x \leq 1 \end{cases}$$

5) By the discontinuity condition

$$G'(t-0,t) - G'(t+0,t) = \frac{1}{P(t)}$$

$$\Rightarrow 0 - A' = \frac{1}{1} \quad \Rightarrow A' = -1$$

So $G(x,t) = \begin{cases} 1-t & 0 \le x < t \\ 1-x & t < x \le 1. \end{cases}$

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⇒ Modified Green's Function:-

We know that If $\lambda = 0$ is not the eighen value of the S.L system $\mathcal{L}(u) + \lambda r(x) u = 0$ with $B_1(u) = 0$ & $B_2(u) = 0$ then we can associate a function with it having certain properties. and is called Green's function.

If $\lambda = 0$ is eigen value of the S.L system defined above then the associated Green's function is called Modified Green's Function. It is denoted by $G_M(x,t)$ and can be constructed as follows.

1) Let $u_0(x)$ be the normalized eigen function corresponding to $\lambda = 0$. i.e

$$\langle u_0, u_0 \rangle = \int_a^b u_0(x) u_0(x)\, dx = 1$$

2) $G_M(x,t)$ Satisfies the equation

$$\mathcal{L}[G_M(x,t)] = u_0(x) u_0(t)$$

in each of the sub interval $[a,t)$ & $(t,b]$.

3) $G_M(x,t)$ satisfied the end point condi-

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tions. $B_1(G_M) = 0$, $B_2(G_M) = 0$

4) $G_M(x,t)$ is continuous everywhere in $[a,b]$, and is continuous particularly at $x=t$

5) $G_M(x,t)$ satisfies the discontinuity condition
$$G_M'(t-0,t) - G_M'(t+0,t) = \frac{1}{P(t)}$$

6) $G_M(x,t)$ satisfies the orthogonality condition
$$\int_a^b G_M(x,t)\, u_0(x)\, dx = 0$$

The above conditions uniquely determine the Modified Green's function.

** **

Examples**

Example 1:- Construct Green's function associated with the system
$$u'' + \lambda r(x) u = 0 \quad \text{with } u'(0) = 0 = u'(1)$$

Solution

Here $P(x) = 1$

1) 1st we check whether $\lambda = 0$ is eigen value or not

$\lambda = 0 \Rightarrow u'' = 0 \Rightarrow u' = A$

$\Rightarrow u = Ax + B$

$u'(x) = A$, $u'(0) = 0 \Rightarrow A = 0$

$u'(1) = 0 \Rightarrow A = 0$

$\Rightarrow u = B \neq 0$

$\Rightarrow \lambda = 0$ is eigen value of the system so associated Green's function is the Modified Green's function

2) Let $u_0(x)$ be the normalized eigen function so

$$\int_0^1 B\cdot B\,dx = 1 \Rightarrow B^2 x\Big|_0^1 = 1 \Rightarrow B^2 = 1$$

$$\Rightarrow B = 1$$

So $u_0(x) = 1$

3) $G_M(x,t)$ satisfies the Differential equation

$G''_M(x,t) = u_0(x)u_0(t)$ in $[0,t[$ & $]t,1]$

$G''_M(x,t) = (1)(1) = 1$

$G'_M(x,t) = x + A \rightarrow G_M(x,t) = \frac{1}{2}x^2 + Ax + B$

So $G_M(x,t) = \begin{cases} \frac{1}{2}x^2 + Ax + B & 0 \leq x < t \\ \frac{1}{2}x^2 + A'x + B' & t < x \leq 1 \end{cases}$

4) $G_M(x,t)$ is continuous at $x = t$

$\Rightarrow G_M(t-0,t) = G_M(t+0,t)$

$\Rightarrow \frac{t^2}{2} + t + B = t^2/2 + A't + B'$

$\Rightarrow B = (A' - A)t + B'$

$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} + Ax + (A' - A)t + B' & 0 \leq x < t \\ \frac{x^2}{2} + A'(x) + B' & t < x \leq 1 \end{cases}$

5) $G_M(x,t)$ satisfies the end point conditions

$G'_M(0,t) = 0 \Rightarrow 0 + A = 0 \Rightarrow A = 0$

$G'_M(1,t) = 0 \Rightarrow 1 + A' = 0 \Rightarrow A' = -1$

$\Rightarrow G_M(x,t) = \begin{cases} \frac{x^2}{2} - t + B' & 0 \leq x < t \\ \frac{x^2}{2} - x + B' & t < x \leq 1 \end{cases}$

6) The discontinuity condition does not help in determining the un-

known constants.

7) By orthogonality condition

$$\int_0^1 G_M(x,t)\, u_0(x)\, dx = 0$$

$$\Rightarrow \int_0^t \left(\frac{x^2}{2} - t + B'\right) dx + \int_t^1 \left(\frac{x^2}{2} - x + B'\right) dx = 0$$

$$\Rightarrow \left.\frac{x^3}{6} - tx + B'x\right|_0^t + \left.\frac{x^3}{6} - \frac{x^2}{2} + B'x\right|_t^1 = 0$$

$$\cancel{\frac{t^3}{6}} - t^2 + \cancel{B't} + \frac{1}{6} - \frac{1}{2} + B' - \cancel{\frac{t^3}{6}} + \frac{t^2}{2} - \cancel{B't} = 0$$

$$\Rightarrow \frac{-t^2}{2} - \frac{1}{3} + B' = 0 \qquad \Rightarrow B' = \frac{t^2}{2} + \frac{1}{3}$$

$$\Rightarrow G_M(x,t) = \begin{cases} \dfrac{x^2}{2} - t + \dfrac{t^2}{2} + \dfrac{1}{3} & 0 \le x < t \\ \dfrac{x^2}{2} - x + \dfrac{t^2}{2} + \dfrac{1}{3} & t < x \le 1 \end{cases}$$

Example 2:- Construct Green's Function associated with the problem $u'' + \lambda u = 0$ with $u(0) = u(1)$ & $u'(0) = u'(1)$

Solution

Here $a = 0$, $b = 1$ & $P(x) = 1$

1) First we check whether $\lambda = 0$ is eigen value or not

$\lambda = 0 \Rightarrow u'' = 0 \Rightarrow u(x) = Ax + B$

$u(0) = u(1) \Rightarrow A(0) + B = A(1) + B \Rightarrow A = 0$

$u'(0) = u'(1) \Rightarrow A = A = 0 \Rightarrow A = 0$

$\Rightarrow u(x) = B$

$\Rightarrow \lambda = 0$ is eigen value so corresponding Green's function is Modified Green's function

2) Let $u_0(x)$ be the normalized eigen function then $\int_0^1 B.B\,dx = 1 \Rightarrow u_0(x) = 1$

3) $G_m(x,t)$ satisfies $G_m''(x,t) = u_0(x)u_0(t) = 1$

$$\Rightarrow G_m(x,t) = \frac{x^2}{2} + Ax + B$$

So for each subinterval $[0,t[$ & $]t,1]$ we can write

$$G_m(x,t) = \begin{cases} \frac{x^2}{2} + Ax + B & 0 \leq x < t \\ \frac{x^2}{2} + A'x + B' & t < x \leq 1 \end{cases}$$

4) As $G_m(x,t)$ is continuous at $x = t$

So $G_m(t-0,t) = G_m(t+0,t)$

$$\frac{t^2}{2} + At + B = \frac{t^2}{2} + A't + B'$$

$$\Rightarrow B = (A' - A)t + B'$$

$$\Rightarrow G_m(x,t) = \begin{cases} \frac{x^2}{2} + Ax + (A' - A)t + B' & 0 \leq x < t \\ \frac{x^2}{2} + A'x + B' & t < x \leq 1 \end{cases}$$

5) $G_m(x,t)$ satisfies the end point conditions

$$G_m'(0,t) = G_m'(1,t)$$

$$A = 1 + A'$$

& $G_m(0,t) = G_m(1,t)$

$$\Rightarrow (A' - A)t + B' = \frac{1}{2} + A' + B'$$

$$\Rightarrow (\cancel{A'} - 1 - \cancel{A'})t = \frac{1}{2} + A'$$

$$\Rightarrow A' = -t - \frac{1}{2}$$

$$\Rightarrow G_m(x,t) = \begin{cases} \frac{x^2}{2} + \left(\frac{1}{2} - t\right)x - t + B' & 0 \le x < t \\ \frac{x^2}{2} - \left(t + \frac{1}{2}\right)x + B' & t < x \le 1 \end{cases}$$

6) The discontinuity does not help us to determine B'.

7) By the orthogonality condition

$$\int_0^1 G_m(x,t)\, u_0(x)\, dx = 0$$

$$\Rightarrow \int_0^t \left(\frac{x^2}{2} + \left(\frac{1}{2} - t\right)x - t + B'\right)dx + \int_t^1 \left[\frac{x^2}{2} - \left(t + \frac{1}{2}\right)x + B'\right]dx = 0$$

$$\Rightarrow \frac{t^3}{6} + \left(\frac{1}{2} - t\right)\frac{t^2}{2} - t^2 + B't + \frac{1}{6} - \left(t + \frac{1}{2}\right)\cdot\frac{1}{2} + B' - \frac{t^3}{6} + \left(t + \frac{1}{2}\right)\cdot\frac{t^2}{2} - B't = 0$$

$$\Rightarrow \frac{t^2}{4} - \frac{t^3}{2} - t^2 + \frac{1}{6} - \frac{t}{2} - \frac{1}{4} + B' + \frac{t^3}{2} + \frac{t^2}{4} = 0$$

$$\Rightarrow -\frac{1}{2}t^2 - \frac{1}{12} - \frac{t}{2} + B' = 0$$

$$\Rightarrow B' = \frac{1}{4}t^2 + \frac{1}{2}t + \frac{1}{12}$$

$$\Rightarrow G_m(x,t) = \begin{cases} \frac{x^2}{2} + \frac{x}{2} - tx - \frac{t}{2} + \frac{1}{2}t^2 + \frac{1}{12} & 0 \le x < t \\ \frac{x^2}{2} - tx - \frac{x}{2} + \frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{12} & t < x \le 1 \end{cases}$$

$$\Rightarrow G_m(x,t) = \begin{cases} \frac{1}{2}x^2 - tx + \frac{1}{2}x + \frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{12} & 0 \le x < 1 \\ \frac{1}{2}x^2 - tx - \frac{1}{2}x + \frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{12} & t < x \le 1 \end{cases}$$

Example 3 Construct Green's function associated with the problem
$u'' + \lambda u = 0$ with
$u(-1) = u(1)$ and $u'(-1) = u'(1)$

Solution

1) First we check whether $\lambda = 0$ is eigen value or not

$\lambda = 0 \Rightarrow u = Ax + B$

$u(-1) = u(1) \Rightarrow -A + B = A + B \Rightarrow A = 0$

$u'(-1) = u'(1) \Rightarrow A = A \Rightarrow A = 0$

$\Rightarrow u \neq 0 \Rightarrow \lambda = 0$ is eigen value.

$\Rightarrow$ Associated G.F is M.G.F

2) Let $u_0(x)$ be the normalized eigen function then

$$\int_{-1}^{1} u_0(x)\, u_0(x)\, dx = 1$$

$$\Rightarrow B^2 x \Big|_{-1}^{1} = 1 \Rightarrow B = 1/\sqrt{2}$$

$$\Rightarrow u_0(x) = 1/\sqrt{2}$$

3) $G_M(x,t)$ satisfies the Diff Eq—

$$G_M''(x,t) = u_0(x)u_0(t) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$$

$$\Rightarrow G_M'(x,t) = \frac{x}{2} + A$$

$$\Rightarrow G_M(x,t) = \frac{x^2}{4} + Ax + B$$

So for $[-1, t[$ and $]t, 1]$ we can write

$$G_M(x,t) = \begin{cases} \frac{x^2}{4} + Ax + B & -1 \leq x < t \\ \frac{x^2}{4} + A'x + B' & t < x \leq 1 \end{cases}$$

4) As $G_M(x,t)$ is continuous at $x=t$ so

$$G_M(t-0,t) = G_M(t+0,t)$$

$$\Rightarrow \frac{t^2}{4} + At + B = \frac{t^2}{4} + A't + B'$$

$$\Rightarrow B = (A'-A)t + B'$$

$$\Rightarrow \quad G_M(x,t) = \begin{cases} \frac{x^2}{4} + Ax + (A'-A)t + B' & -1 \leq x < t \\ \frac{x^2}{4} + A'x + B' & t < x \leq 1 \end{cases}$$

5) By the end point conditions

$$G_M(-1,t) = G_M(1,t)$$

$$\Rightarrow \cancel{\tfrac{1}{4}} - A + (A'-A)t + \cancel{B'} = \cancel{\tfrac{1}{4}} + A' + \cancel{B'}$$

$$\Rightarrow -A + A't - At = A' \qquad \Rightarrow A' = A\left[\frac{t+1}{t-1}\right]$$

and $G_M'(-1,t) = G_M'(1,t)$

$$\Rightarrow \frac{-1}{2} + A = \frac{1}{2} + A' = \frac{1}{2} + A\left[\frac{t+1}{t-1}\right]$$

$$\Rightarrow \left[1 - \frac{t+1}{t-1}\right]A = \frac{1}{2} + \frac{1}{2} = 1$$

$$\Rightarrow \frac{-2}{t-1}A = 1 \qquad \Rightarrow A = -\frac{t-1}{2}$$

So $A' = -\dfrac{t-1}{2} \times \dfrac{t+1}{t-1} = -\dfrac{t+1}{2}$

$$\Rightarrow \quad G_M(x,t) = \begin{cases} \frac{x^2}{4} - \frac{t-1}{2}x - t + B' & -1 \leq x < t \\ \frac{x^2}{4} - \frac{t+1}{2}x + B' & t < x \leq 1 \end{cases}$$

6) discontinuity condition does not help us to determine B'

7) By the orthogonality condition

$$\int_{-1}^{t}\left(\frac{x^2}{4} - \frac{t-1}{2}x - t + B'\right)\left(\frac{1}{\sqrt{2}}\right)dx + \int_{t}^{1}\left(\frac{x^2}{4} - \frac{t+1}{2}x + B'\right)\frac{1}{\sqrt{2}}\,dx = 0$$

$$\Rightarrow \frac{1}{\sqrt{2}}\left[\frac{t^3}{12} - \frac{t-1}{2}\cdot\frac{t^2}{2} - t^2 + \cancel{B'}t + \frac{1}{12} + \frac{t-1}{4} - t + B' + \frac{1}{12}\right.$$

$$\left. - \frac{t+1}{4} + B' - \frac{t^3}{12} + \frac{t+1}{2}\cdot\frac{t^2}{2} - \cancel{B'}t\right] = 0$$

$$\Rightarrow B' = \frac{t^2}{4} + \frac{1}{6} + \frac{t}{2}$$

$$\Rightarrow G_m(x,t) = \begin{cases} \frac{x^2}{4} - \frac{t-1}{2}x - \frac{t}{2} + \frac{t^2}{4} + \frac{1}{6}, & -1 \leq x < t \\ \frac{x^2}{4} - \frac{t+1}{2}x + \frac{t^2}{4} + \frac{1}{6} + \frac{t}{2}, & t < x \leq 1 \end{cases}$$

____

Example 4:- Construct Green's Function Associated with the system

$u'' + \lambda u = 0$ with $u'(0) = 0$ & $u(2) = 0$

Solution

1) First we check whether $\lambda = 0$ is eigen value or not

$\lambda = 0 \Rightarrow u'' = 0 \Rightarrow u = Ax + B$

$u(2) = 0 \Rightarrow 2A + B = 0 \longrightarrow \circledast$

$u'(0) = 0 \Rightarrow A = 0$

So $\circledast \Rightarrow 2(0) + B = 0 \Rightarrow B = 0$

$\Rightarrow u = 0 \Rightarrow \lambda = 0$ is not eigen value.

So Green's Function can be associated.

2) G.F satisfies

$G'' = 0 \Rightarrow G = Ax + B$

So

$$G(x,t) = \begin{cases} Ax + B & 0 \leq x < t \\ A'x + B' & t < x \leq 2 \end{cases}$$

3) As $G(x,t)$ is continuous on $[0,2]$ so is also continuous at $x = t \in [0,2]$

$\Rightarrow G(t-0,t) = G(t+0,t)$

$$\Rightarrow At+B = A't+B'$$
$$\Rightarrow B = (A'-A)t+B'$$
$$\Rightarrow G(x,t) = \begin{cases} Ax+(A'-A)t+B' \\ A'x+B' \end{cases}$$

4) $G(x,t)$ as a function of x satisfies the end point conditions. So

$G'(0,t) = 0 \quad \Rightarrow \quad A = 0$

$G(2,t) = 0 \quad \Rightarrow \quad A'(2)+B' = 0$

$\Rightarrow B' = -2A'$

$$\Rightarrow G(x,t) = \begin{cases} A't-2A' & 0 \leq x < t \\ A'x-2A' & t < x \leq 2 \end{cases}$$

5) By the discontinuity conditions.

$$G'(t-0,t) - G'(t+0,t) = \frac{1}{p(t)}$$

$0 - A' = \frac{1}{1} \quad \Rightarrow A' = -1$

$$\Rightarrow G(x,t) = \begin{cases} 2-t & 0 \leq x < t \\ 2-x & t < x \leq 2 \end{cases}$$

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* Solution of Boundry Value Problems Using Green's Function.

Example: Solve the problem $u'' = f(x) \longrightarrow (1)$
with $u(0) = \alpha, \; u(\ell) = \beta \,] \longrightarrow (2)$

Solution Let $G(x,t)$ be the

Corresponding Green's function, then

$$G(x,t) = \begin{cases} -\dfrac{x(l-t)}{l} & 0 \le x < t \\ -\dfrac{t(l-x)}{l} & t < x \le l \end{cases}$$

Further then

$$\frac{d^2G}{dx^2} = \delta(x-t) \quad \& \quad G(0,t) = 0$$

$$\& \quad G(l,t) = 0$$

Now considering Lagrange identity

$$\int_0^l [u\mathcal{L}(v) - v\mathcal{L}(u)]\,dx = (uv' - vu')\Big|_0^l$$

put $v = G(x,t)$

then we have

$$\int_0^l [u\mathcal{L}(G) - G\mathcal{L}(u)]\,dx = (uG' - Gu')\Big|_0^l$$

$$= [u(l)G'(l,t) - G(l,t)u'(l)]$$

$$- [u(0)G'(0,t) - G(0,t)u'(0)]$$

$$= [\beta G'(l,t) - 0\,u'(l)] - [\alpha G'(0,t) - 0\,u'(0)]$$

$$= \beta G'(l,t) - \alpha G'(0,t)$$

$$= \beta(t/l) - \alpha\left(\frac{1}{l}\right)(t-l)$$

$$= \left(\frac{\beta-\alpha}{l}\right)t + \alpha$$

$$\Rightarrow \int_0^l [u(x)\delta(x-t) - G(x,t)f(x)]\,dx = \frac{\beta-\alpha}{l}t + \alpha$$

$$\Rightarrow u(t) - \int_0^l G(x,t)f(x)\,dx = \frac{\beta-\alpha}{l}t + \alpha$$

$$\Rightarrow u(t) = \int_0^{\ell} G(x,t)\, f(x)\, dx + \frac{\beta-\alpha}{\ell} t + \alpha$$

$$\Rightarrow u(x) = \int_0^{\ell} G(t,x)\, f(t)\, dt + \frac{\beta-\alpha}{\ell} x + \alpha$$

$$\Rightarrow u(x) = \int_0^{\ell} G(x,t)\, f(t)\, dt + \frac{\beta-\alpha}{\ell} x + \alpha$$

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Laplace Transform And Its Applications*

⇒ Transformation:-

A transformation is a law or rule which changes, sometimes does not change, some variable or function into another variable or function.

* ~~Linear~~ Integral Transformation:-

There are several types transformations but we are concern only with integral transformation i.e the transformation involving integral

* Types of Integral Transformations

The important (& well known) Integral Transformations are.

1) Laplace Transformation.
2) Fourier Transform.
3) Mellin Transform.
4) Hankel Transform.
5) Kontrovich-Lebedev Transform.
6) Mehler - Fock Transform.
7) Chebyshev Tranform.

***Linear Transformation:-** A transformation T is said to be linear if

$$T(\alpha_1 f_1(x) + \alpha_2 f_2(x)) = \alpha_1 T(f_1(x)) + \alpha_2 T(f_2(x))$$

⇒ Definition:-

Let $f(t)$ be a continuous or sectionally continuous function of t defined over $[0, \infty)$, then the Laplace transform of $f(t)$ is a function $F(s)$ of another variable 's' defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Remarks:- 1) The existance of Laplace transform of a function depends upon the existance of the defining integral.

2) Every function may or may not has the Laplace transform.

The laplace transform of a function $f(t)$, if it exists, is denoted by any one of the following notations.

$F(s)$, $\bar{f}(s)$, $\mathcal{L}\{f(t)\}$, $\mathcal{L}[f(t)]$ or $\mathcal{L}\{f(t); s\}$

** **

Theorem:- The Laplace transformation operator $\mathcal{L}$ is a linear operator.

Proof:

$$\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} = \int_0^\infty e^{-st}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} dt$$

$$\Rightarrow \mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} = \int_0^\infty e^{-st}\{\alpha_1 f_1(t)\} dt + \int_0^\infty e^{-st}\{\alpha_2 f_2(t)\} dt$$

$$= \alpha_1 \int_0^\infty e^{-st} f_1(t)\, dt + \alpha_2 \int_0^\infty e^{-st} f_2(t)\, dt$$

$$= \alpha_1 \mathcal{L}\{f_1(t)\} + \alpha_2 \mathcal{L}\{f_2(t)\}$$

$\Rightarrow \mathcal{L}$ is linear operator.

⇒ Definition:-

Let $f(t)$ be a continuous or sectionally continuous function of t defined over $[0,\infty)$, then the Laplace transform of $f(t)$ is a function $F(s)$ of another variable 's' defined by

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** **

Theorem:- The Laplace transformation operator $\mathcal{L}$ is a linear operator.

Proof

$$\mathcal{L}\{\alpha_1 f_1(t)+\alpha_2 f_2(t)\} = \int_0^\infty e^{-st}\{\alpha_1 f_1(t)+\alpha_2 f_2(t)\}\,dt$$

$$\Rightarrow \mathcal{L}\{\alpha_1 f_1(t)+\alpha_2 f_2(t)\} = \int_0^\infty e^{-st}\{\alpha_1 f_1(t)\}\,dt + \int_0^\infty e^{-st}\{\alpha_2 f_2(t)\}\,dt$$

$$= \alpha_1\int_0^\infty e^{-st} f_1(t)\,dt + \alpha_2\int_0^\infty e^{-st} f_2(t)\,dt$$

$$= \alpha_1 \mathcal{L}\{f_1(t)\} + \alpha_2 \mathcal{L}\{f_2(t)\}$$

$\Rightarrow \mathcal{L}$ is linear operator.

⇒ **Definition:-** A function $f(t)$ is said to be of exponential order c if

$$|f(t)| \leq Me^{ct} \quad \text{for } t > t_0$$

where M & t_0 are +ve constants and c is any constant.

⇒ **Function of Class A:-**

A function which is piece wise continuous and of some exponential order is said to be a function of class A.

Theorem:- Sufficient condition for the existance of Laplace transformation is that it should be function of class A i.e it should be piece wise continuous and of exponential order.

Proof: Since f is of exponential order, so then by definition

$$|f(t)| \leq Me^{ct} \quad \text{for } t > t_0$$

where M & t_0 are +ve constants while c is any constant.

Now $|f(t)| \leq Me^{ct}$

$$\Rightarrow |e^{-st} f(t)| \leq Me^{-st} e^{ct} \quad \text{for } t > t_0$$

$$= Me^{-(s-c)t} \quad \text{for } t > t_0$$

$$|\mathcal{L}\{f(t)\}| = \left|\int_0^\infty e^{-st} f(t)\,dt\right| \leq M \int_0^\infty e^{-(s-c)t}\,dt$$

$$\Rightarrow \mathcal{L}\{f(t)\} \leq \frac{M}{|s-c|}$$

Thus it is clear that $\mathcal{L}\{f(t)\}$ exists.

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Remark:- 1) The conditions described in the above theorem are sufficient but not necessary. e.g the function $t^{-1/2}$ has Laplace transform but is not piecewise continuous in any interval $[0,T]$ where $T>0$

2) From above if

$$\lim_{s\to\infty} F(s) \neq 0$$

Then $F(s)$ can never be the L.T of any function $f(t)$.

So we prove $\lim\limits_{s\to\infty} F(s) = 0$

Since $f(t)$ is piecewise continuous and is of exponantial order, so then

$$|f(t)| \leq Me^{ct} \quad \forall\ t > t_0$$

$$\text{Now } |F(s)| = \left|\int_0^\infty e^{-st} f(t)\,dt\right|$$

$$\leq \left|\int_0^\infty e^{-st} Me^{ct}\,dt\right|$$

$$= \left|\int_0^\infty e^{-(s-c)t} M\,dt\right|$$

$$= \frac{M}{s-c}$$

$$\left|\lim_{s\to\infty} F(s)\right| \leq \lim_{s\to\infty} \frac{M}{s-c} = 0$$

$$\Rightarrow \lim_{s\to\infty} F(s) = 0$$

————————

* Laplace Transformation of Some Functions *

1) L.T of a Constant c :-

$$\mathcal{L}\{c\} = \int_0^\infty e^{-st} \cdot c \, dt$$

$$= c \cdot \frac{e^{-st}}{-s}\Big|_0^\infty = \frac{c}{-s}[0-1]$$

$$\mathcal{L}\{c\} = \frac{c}{s}$$

2) L.T of t :-

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} \cdot t \, dt$$

$$= t \cdot \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty (1) \frac{e^{-st}}{-s} \, dt$$

$$= (0-0) + \frac{1}{s}\int_0^\infty 1 \cdot e^{-st} \, dt$$

$$= 0 + \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

$$\Rightarrow \mathcal{L}\{t\} = \frac{1}{s^2}$$

3) L.T of t^2 :-

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st} t^2 \, dt$$

$$= t^2 \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty 2t \cdot \frac{e^{-st}}{-s} \, dt$$

$$= (0-0) + \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s} \cdot \frac{1}{s^2}$$

$$= \frac{2!}{s^3}$$

4) L.T of t^3 :-

$$\mathcal{L}\{t^3\} = \int_0^\infty e^{-st} t^3 \, dt$$

$$= t^3 \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty 3t^2 \frac{e^{-st}}{-s} dt$$

$$= 0 + \frac{3}{s}\mathcal{L}\{t^2\} = \frac{3}{s}\cdot\frac{2!}{s^3}$$

$$\mathcal{L}\{t^3\} = \frac{3!}{s^4}$$

5) L.T of t^n :-

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n \, dt$$

$$= t^n \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty (nt^{n-1}) \frac{e^{-st}}{-s} dt$$

$$= \frac{n}{s}\int_0^\infty e^{-st} t^{n-1} dt = \frac{n}{s}\mathcal{L}\{t^{n-1}\}$$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\mathcal{L}\{t^{n-2}\}$$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdot\mathcal{L}\{t^{n-3}\}$$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdots\cdots\frac{1}{s}\mathcal{L}\{t^0\}$$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdots\cdots\frac{3}{s}\cdot\frac{2}{s}\cdot\frac{1}{s}\cdot\frac{1}{s}$$

$$= \frac{n!}{s^{n+1}}$$

6) L.T of e^{kt} :-

$$\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-st}\cdot e^{kt} dt$$

$$= \int_0^\infty e^{-(s-k)t} dt$$

Let $s-k = s'$ then

$$\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-s't}\,dt$$

$$= \int_0^\infty 1 \cdot e^{-s't}\,dt = \frac{1}{s'}$$

$$\Rightarrow \mathcal{L}\{e^{kt}\} = \frac{1}{s-k} \quad \text{for } s > k$$

7) L.T of Sin kt:-

$$\mathcal{L}\{\sin kt\} = \mathcal{L}\left\{\frac{e^{ikt} - e^{-ikt}}{2i}\right\}$$

$$= \frac{1}{2i}\left[\frac{1}{s-ik} - \frac{1}{s+ik}\right]$$

$$= \frac{1}{2i}\left[\frac{s+ikt - \cancel{s} + ikt}{s^2 - (ik)^2}\right]$$

$$= \frac{k}{s^2+k^2}$$

$$\Rightarrow \mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

8) L.T of Cos kt:-

$$\mathcal{L}\{\cos kt\} = \mathcal{L}\left\{\frac{e^{ikt} + e^{-ikt}}{2}\right\}$$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{ikt}\} + \mathcal{L}\{e^{-ikt}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-ik} + \frac{1}{s+ik}\right]$$

$$= \frac{1}{2}\left[\frac{s+ik+s-ik}{s^2+k^2}\right] = \frac{1}{2}\left[\frac{2s}{s^2+k^2}\right]$$

$$\Rightarrow \mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$$

9) L.T of $\sinh kt$:-

$$\mathcal{L}\{\sinh kt\} = \mathcal{L}\left\{\frac{e^{kt}-e^{-kt}}{2}\right\}$$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{kt}\} - \mathcal{L}\{e^{-kt}\}\right]$$

$$= \frac{1}{2}\left\{\frac{1}{s-k} - \frac{1}{s+k}\right\} = \frac{1}{2}\left[\frac{s+k-s+k}{s^2-k^2}\right]$$

$$\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$$

10) L.T of $\cosh kt$:-

$$\mathcal{L}\{\cosh kt\} = \mathcal{L}\left\{\frac{e^{kt}+e^{-kt}}{2}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s-k}+\frac{1}{s+k}\right] = \frac{1}{2}\left[\frac{s+k+s-k}{s^2-k^2}\right]$$

$$\Rightarrow \mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$$

⇒ 1st Shifting Theorem:-

$$\mathcal{L}\{e^{kt}f(t)\} = \mathcal{L}\{f(t)\}_{s\to s-k}$$

$$= F(s)\Big|_{s\to s-k} = F(s-k)$$

It is also known as 1st Translation Theorem

Proof:

$$\mathcal{L}\{e^{kt}f(t)\} = \int_0^\infty e^{-st}\cdot e^{kt} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-k)t} f(t)\,dt$$

$$= \int_0^\infty e^{-s't} f(t)\,dt \quad \because s' = s-k$$

$$= F(s') = F(s-k)$$

$$= F(s)\Big|_{s\to s-k}$$

Exercise ایکسرسائز

Question Find L.T of $\sin^2 \omega t$

Solution

$$\mathcal{L}\{\sin^2\omega t\} = \mathcal{L}\left\{\frac{1-\cos 2\omega t}{2}\right\}$$

$$\Rightarrow \mathcal{L}\{\sin^2\omega t\} = \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos 2\omega t\}$$

$$= \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s}{s^2+4\omega^2}$$

$$= \frac{1}{2s} - \frac{s}{2(s^2+4\omega^2)}$$

$$= \frac{\cancel{s^2}+4\omega^2-\cancel{s^2}}{2s(s^2+4\omega^2)} = \frac{2\omega^2}{s(s^2+4\omega^2)}$$

Question Find L.T of $\cos^2 \omega t$.

Solution

$$\mathcal{L}\{\cos^2\omega t\} = \mathcal{L}\left\{\frac{1+\cos 2\omega t}{2}\right\}$$

$$\Rightarrow \mathcal{L}\{\cos^2\omega t\} = \frac{1}{2s} \cdot \frac{s}{2(s^2+4\omega^2)}$$

$$= \frac{s^2+4\omega^2+s^2}{2s(s^2+4\omega^2)} = \frac{2(s^2+2\omega^2)}{2s(s^2+4\omega^2)}$$

$$\Rightarrow \mathcal{L}\{\cos^2\omega t\} = \frac{s^2+2\omega^2}{s(s^2+4\omega^2)}$$

Question Find $\mathcal{L}\{\sin(\omega t-\phi)\}$

Solution

$$\mathcal{L}\{\sin(\omega t-\phi)\} = \mathcal{L}\left\{\frac{e^{i(\omega t-\phi)} - e^{-i(\omega t-\phi)}}{2i}\right\}$$

$$= \frac{1}{2i}\left[\mathcal{L}\{e^{i\omega t}\cdot e^{-i\phi}\} - \mathcal{L}\{e^{-i\omega t}\cdot e^{i\phi}\}\right]$$

$$= \frac{1}{2i}\left[\frac{e^{-i\phi}}{s}\Big|_{s\to s-i\omega} - \frac{e^{i\phi}}{s}\Big|_{s\to s+i\omega}\right]$$

$$= \frac{1}{2i}\left[\frac{e^{-i\phi}}{s-i\omega} - \frac{e^{i\phi}}{s+i\omega}\right]$$

$$= \frac{1}{2i}\left[\frac{\cos\phi - i\sin\phi}{s-i\omega} - \frac{\cos\phi + i\sin\phi}{s+i\phi}\right]$$

$$= \frac{1}{2i}\left[\frac{\cancel{s\cos\phi} - is\sin\phi + i\omega\cos\phi + \cancel{\omega\sin\phi} - \cancel{s\cos\phi} - is\sin\phi + i\omega\cos\phi - \cancel{\omega\sin\phi}}{s^2+\omega^2}\right]$$

$$= \frac{\omega\cos\phi - s\sin\phi}{s^2+\omega^2}$$

Question Find $\mathcal{L}\{\cos(\omega t - \phi)\}$

Solution

$$\mathcal{L}\{\cos(\omega t-\phi)\} = \mathcal{L}\left\{\frac{e^{i(\omega t-\phi)} + e^{-i(\omega t-\phi)}}{2}\right\}$$

$$= \frac{1}{2}\left[\frac{\cos\phi - i\sin\phi}{s-i\omega} + \frac{\cos\phi + i\sin\phi}{s+i\omega}\right]$$

$$= \frac{1}{2}\left[\frac{s\cos\phi - is\cancel{\sin\phi} + i\omega\cancel{\cos\phi} + \omega\sin\phi + s\cos\phi + is\cancel{\sin\phi} - i\omega\cancel{\cos\phi} + \omega\sin\phi}{(s-i\omega)(s+i\omega)}\right]$$

$$= \frac{s\cos\phi + \omega\sin\phi}{s^2+\omega^2}$$

Question L.T of $e^{2(t+1)}$

Solution

$$\mathcal{L}\{e^{2(t+1)}\} = \mathcal{L}\{e^{2t}\cdot e^{2}\}$$

$$= \mathcal{L}\{e^{2}\}_{s\to s-2}$$

$$= \frac{e^2}{s}\Big|_{s\to s-2} = \frac{e^2}{s-2}$$

Question $\mathcal{L}\{\sin \omega t\, e^{-2t}\}$

Solution

$$\mathcal{L}\{\sin \omega t\, e^{-2t}\}$$

$$= \mathcal{L}\{\sin \omega t\}\Big|_{s \to s+2}$$

$$= \frac{\omega}{s^2+\omega^2}\Big|_{s \to s+2}$$

$$= \frac{\omega}{(s+2)^2+\omega^2}$$

$$\Rightarrow \mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+4s+4+\omega^2}$$

Question L.T of $\cos 3\omega t\, e^{4t}$

Solution

$$\mathcal{L}\{\cos 3\omega t\, e^{4t}\} = \frac{s-4}{(s-4)^2+9\omega^2}$$

Question L.T of $e^{3t}t^4$

Solution $$\mathcal{L}\{e^{3t}t^4\} = \frac{4!}{s^5}\Big|_{s \to s-3}$$

$$= \frac{24}{(s-3)^5}$$

Question Calculate $\mathcal{L}\{t^\alpha\}$, where $\alpha \in \mathbb{R}$

Solution $$\mathcal{L}\{t^\alpha\} = \int_0^\infty e^{-st} t^\alpha \, dt$$

Put $st = u$, $s\,dt = du \Rightarrow dt = \frac{1}{s}du$

$$\mathcal{L}\{t^\alpha\} = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^\alpha \cdot \frac{1}{s}du$$

$$= \frac{1}{s^{\alpha+1}}\int_0^\infty e^{-u} u^\alpha \, du$$

$$= \frac{1}{s^{\alpha+1}} \int_0^\infty e^{-u} u^{\overline{\alpha+1}-1} du$$

$$= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

Here we have used the definition of Gamma function defined by

$$\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du$$

Question L.T of $t^{1/2}$

Solution

$$\mathcal{L}\{t^{1/2}\} = \frac{\Gamma(\frac{1}{2}+1)}{s^{1/2+1}} = \frac{\Gamma(3/2)}{s^{3/2}}$$

$$= \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\frac{1}{2}\Gamma(1/2)}{s^{3/2}} \quad \because \Gamma n = (n-1)\Gamma{n-1}$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} \quad \because \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

Question L.T of $t^{-1/2}$

Solution

$$\mathcal{L}\{t^{-1/2}\} = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-1/2+1}} = \frac{\Gamma(1/2)}{\sqrt{s}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\pi/s}$$

*** ___

Question:- Prove that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

Solution Let

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$\Rightarrow I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\, dy$$

put $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = J\,dr\,d\theta = r\,dr\,d\theta$

and $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$

$$\Rightarrow I^2 = \int_0^{\infty}\int_0^{2\pi} e^{-r^2} r\,dr\,d\theta$$

$$= \left[\frac{-1}{2}\int_0^{\infty} e^{-r^2}(-2r)\,dr\right]\left[\int_0^{2\pi} d\theta\right]$$

$$= \frac{-1}{2}\left(e^{-r^2}\right)\Big|_0^{\infty} \cdot \theta\Big|_0^{2\pi}$$

$$= \frac{-1}{2}(0-1)(2\pi - 0) = \pi$$

$$\Rightarrow I = \sqrt{\pi}$$

***

Question:- Use the result

$$\mathcal{L}\{t^{\alpha}\} = \frac{\alpha}{s}\mathcal{L}\{t^{\alpha-1}\} \;;\; s>0 \;;\; \alpha > -1$$

Calculate $\mathcal{L}\{t^{k/2}\}$; where k is an odd +ve integer.

Solution

Since k is an odd +ve integer.

so it can be written as $k = 2m+1$, $m \in W$

$$\Rightarrow t^{k/2} = t^{\frac{2m+1}{2}} = t^{m+\frac{1}{2}}$$

$$\mathcal{L}\{t^{m+\frac{1}{2}}\} = \frac{m+\frac{1}{2}}{s}\,\mathcal{L}\{t^{m-\frac{1}{2}}\}$$

$$= \frac{m+\frac{1}{2}}{s}\cdot\frac{m-\frac{1}{2}}{s}\,\mathcal{L}\{t^{m-\frac{3}{2}}\}$$

$$= \frac{m+\frac{1}{2}}{s}\cdot\frac{m-\frac{1}{2}}{s}\cdot\frac{m-\frac{3}{2}}{s}\cdots\frac{3/2}{s}\cdot\frac{\frac{1}{2}}{s}\cdot\mathcal{L}\{t^{-1/2}\}$$

$$= \frac{2m+1}{2s}\cdot\frac{2m-1}{2s}\cdot\frac{2m-3}{2s}\cdots\frac{3}{2s}\cdot\frac{1}{2s}\cdot\sqrt{\frac{\pi}{s}}$$

$$= \frac{k(k-2)(k-4)\cdots 3\cdot 1}{(2s)^{(k+1)/2}}\cdot\frac{\sqrt{\pi}}{\sqrt{s}} \qquad \boxed{\begin{array}{l}2m+1 = k\\ 2m-1 = k-1-1\\ m = \frac{k-1}{2}\end{array}}$$

$$= \frac{k(k-2)(k-4)\cdots 3\cdot 1}{2^{k/2+\frac{1}{2}}}\cdot\sqrt{\frac{\pi}{s^{k+2}}}$$

***

Question:- Derive L.T of $t^{1/2}$ and $t^{-1/2}$ from definition.

Solution:

$$\mathcal{L}\{t^{1/2}\} = \int_0^\infty e^{-st}\,t^{1/2}\,dt$$

Put $st = u$, $dt = \frac{1}{s}du$ and $t = u/s$

$$\Rightarrow \mathcal{L}\{t^{\frac{1}{2}}\} = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^{\frac{1}{2}}\cdot\frac{1}{s}\,du$$

$$= \frac{1}{s^{3/2}}\int_0^\infty e^{-u}\,u^{1/2}\,du$$

Put $u^{1/2} = x \Rightarrow u = x^2$

$$\Rightarrow du = 2x\,dx$$

$$\Rightarrow \mathcal{L}\{t^{1/2}\} = \frac{1}{s^{3/2}}\int_0^\infty e^{-x^2}x\cdot 2x\,dx$$

$$= \frac{1}{s^{3/2}}(-1)\int_0^\infty x\left(e^{-x^2}(-2x)\right)dx$$

$$\Rightarrow \mathcal{L}\{t^{1/2}\} = \frac{1}{s^{3/2}}(-1)\left[x \cdot e^{-x^2}\Big|_0^\infty - \int_0^\infty 1 \cdot e^{-x^2} dx\right]$$

$$= \frac{1}{s^{3/2}}\int_0^\infty e^{-x^2} dx$$

$$= \frac{1}{s^{3/2}} \cdot \frac{1}{2}\int_{-\infty}^\infty e^{-x^2} dx = \frac{1}{2 \cdot s^{3/2}}\sqrt{\pi}$$

$$\mathcal{L}\{t^{1/2}\} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

Now

$$\mathcal{L}\{t^{-1/2}\} = \int_0^\infty e^{-st} t^{-1/2} dt$$

Put $st = u$

$$\Rightarrow \mathcal{L}\{t^{-\frac{1}{2}}\} = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^{-\frac{1}{2}} \cdot \frac{1}{s} du$$

$$= \frac{1}{s^{\frac{1}{2}}}\int_0^\infty e^{-u} u^{-\frac{1}{2}} du$$

Put $u = x^2$, $du = 2x\,dx$

$$\Rightarrow \mathcal{L}\{t^{-\frac{1}{2}}\} = \frac{1}{s^{1/2}}\int_0^\infty e^{-x^2} \cdot x^{-1} \cdot 2x\,dx$$

$$= \frac{1}{s^{1/2}}\int_{-\infty}^\infty e^{-x^2} dx = \frac{1}{s^{1/2}} \cdot \sqrt{\pi}$$

$$\Rightarrow \mathcal{L}\{t^{-\frac{1}{2}}\} = \sqrt{\frac{\pi}{s}}$$

***

Question Prove that

1) $\Gamma(x+1) = x\Gamma x$ (2) $\Gamma(1) = 1$

Solution

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^{x+1-1} dt$$

$$= t^{x}\frac{e^{-t}}{-1}\Big|_0^\infty - \int_0^\infty x t^{x-1}\frac{e^{-t}}{-1}\,dt$$

$$= 0 + x\int_0^\infty e^{-t} t^{x-1}\,dt = x\,\Gamma(x)$$

$$\Rightarrow \Gamma(x+1) = x\,\Gamma(x)$$

Further $\Gamma(x+1) = x\,\Gamma(x) = x(x-1)\,\Gamma(x-1)$

$$= x(x-1)(x-2)\,\Gamma(x-2)$$

$$= x(x-1)(x-2)\cdots 3\cdot 2\cdot 1\cdot 1 = x! \quad \text{if } x\in Z^+$$

$$\Gamma(1) = \int_0^\infty e^{-t} t^{1-1}\,dt = \int_0^\infty e^{-t}\,dt = e^{-t}\Big|_0^\infty$$

$$= -(0-1) = 1$$

**

⇒ Laplace Transforms of Derivatives And Integrals***

Theorem:- If $f(t), f'(t), f''(t) \ldots\ldots f^{(n-1)}(t)$ are continuous and $f^{(n)}(t)$ is piecewise continuous on the interval $[0,\infty)$ and all are of exponential order, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0)$$
$$\ldots\ldots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

Find the case when $n=2$.

Proof:-

We prove the theorem by the principle of mathematical Induction

For $n=1$:-

$$\text{L.H.S} = \mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st}\cdot f(t)\Big|_0^\infty - \int_0^\infty (-s)e^{-st}\cdot f(t)\,dt$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s^1 F(s)$$

R.H.S $= s^1 F(s) - s^{1-1} f(0)$

$= sF(s) - f(0)$

$\Rightarrow$ L.H.S $=$ R.H.S

$\Rightarrow$ Theorem is true for $n=1$

So case I is satisfied.

Now suppose that theorem is true for $n=k$ i.e

$$\mathcal{L}\{f^{(k)}(t)\} = s^k F(s) - s^{k-1} f(0) - s^{k-2} f'(0) - s^{k-3} f''(0) - \cdots - s f^{(k-2)}(0) - f^{(k-1)}(0) \quad (1)$$

Now we prove the theorem for $n = k+1$.

for this

$$\mathcal{L}\{f^{(k+1)}(t)\} = \int_0^\infty e^{-st} f^{(k+1)}(t)\,dt$$

$$= e^{-st} f^{(k)}(t)\Big|_0^\infty - \int_0^\infty (-s) e^{-st} f^{(k)}(t)\,dt$$

$$= -f^{(k)}(0) + s\mathcal{L}\{f^{(k)}(t)\}$$

$$= -f^{(k)}(0) + s\{s^k F(s) - s^{k-1} f(0) - s^{k-2} f'(0) - s^{k-3} f''(0) - \cdots - s f^{k-2}(0) - f^{k-1}(0)\} \text{ by (1)}$$

$$= s^{k+1} F(s) - s^{\overline{k+1}-1} f(0) - s^{\overline{k+1}-2} f'(0) - s^{\overline{k+1}-3} f''(0) - \cdots - s^2 f^{(\overline{k+1}-3)}(0) - s f^{\overline{k+1}-2}(0) - f^{\overline{k+1}-1}(0)$$

$\Rightarrow$ Theorem is true for $n=K+1$

$\Rightarrow$ Case 2 is satisfied

$\Rightarrow$ Theorem is true upto the laplace of nth derivative of $f(t)$

Now if $n=2$, then

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

****

Theorem: Prove that $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$

Proof:

Let $g(t) = \int_0^t f(\tau)\,d\tau$

Then by the fundamental theorem of integral calculas

$$g'(t) = f(t)$$

Then $\mathcal{L}\{g'(t)\} = \mathcal{L}\{f(t)\}$

$s\mathcal{L}\{g(t)\} - g(0) = \mathcal{L}\{f(t)\}$

$\Rightarrow s\mathcal{L}\{g(t)\} - \int_0^0 f(\tau)\,d\tau = \mathcal{L}\{f(t)\}$

$\Rightarrow s\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} - 0 = \mathcal{L}\{f(t)\}$

$= \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$

Question: Calculate $\mathcal{L}\left\{\int_0^t e^{kt}\cos\omega t\,dt\right\}$

Solution: $\mathcal{L}\left\{\int_0^t e^{kt}\cos\omega t\right\}$

$= \frac{1}{s}\mathcal{L}\{e^{kt}\cos\omega t\}$

$$= \frac{1}{s}\left[\left(\frac{s}{s^2+\omega^2}\right)\Big|_{s \to s-K}\right]$$ By 1st Shifting Theorem

$$= \frac{1}{s} \cdot \frac{s-K}{(s-K)^2+\omega^2} = \frac{s-K}{s((s-K)^2+\omega^2)}$$

***

⇒ Inverse Laplace Transform:-

If $F(s)$ is the Laplace Transform of $f(t)$, Then $f(t)$ is called the inverse Laplace Transform of $F(s)$.

The invese $\mathcal{L}.T$ of $F(s)$ is denoted by $\mathcal{L}^{-1}\{F(s)\} = f(t)$

The following table shows the relation b/w $F(s)$ and $\mathcal{L}^{-1}\{F(s)\}$

	$F(s)$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
(i)	c/s	$c \quad , s>0$
(ii)	$n!/s^{n+1} \quad , n \in Z^+$	$t^n \quad , s>0$
(iii)	$\frac{1}{s-K}$	$e^{Kt} \quad , s>K$
(iv)	$\frac{K}{s^2+K^2}$	$\sin Kt \quad , s>0$
(v)	$\frac{s}{s^2+K^2}$	$\cos Kt \quad , s>0$
(vi)	$\frac{K}{s^2-K^2}$	$\sinh Kt \quad , s>\lvert K\rvert$
(vii)	$\frac{s}{s^2-K^2}$	$\cosh Kt \quad , s>\lvert K\rvert$
(viii)	$F(s-K)$	$e^{Kt}\mathcal{L}^{-1}\{F(s)\} \quad , s>K$

(ix)	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$	$f^{(n)}(t)$
(x)	$\frac{1}{s} F(s)$	$\int_0^t f(\xi) d\xi$
(xi)	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$	$\mathcal{L}\{t^{\alpha}\}$, $\alpha \in \mathbb{R}$
(xii)	$\frac{1}{s} \sqrt{\frac{\pi}{s}}$	$2t^{\frac{1}{2}}$
(xiii)	$\sqrt{\frac{\pi}{s}}$	$t^{-1/2}$

⇒ First Shifting theorem for Inverse L.T:-

$$\mathcal{L}^{-1}\{F(s-k)\} = e^{kt} \mathcal{L}^{-1}\{F(s)\}$$

Example Calculate $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s}\right\}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+1-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2-1}\right\} = e^{-1 \cdot t} \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\}$$

∴ by 1st shifting theorem.

$$= e^{-t} \sinh t$$

2nd Method:-

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s+2}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{1}{2}(1) - \frac{1}{2} e^{-2t}$$

Examples Calculate $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s}\right\}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s}\right\} = \mathcal{L}^{-1}\left\{\frac{\cancel{s}}{\cancel{s}(s+2)}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$$

★ ★

Use of Partial Fraction in Calculating Inverse Laplace Transform.

Example: calculat $\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\}$

Solution Consider

$$\frac{s+4}{s^2+3s+2} = \frac{A}{s+1} + \frac{B}{s+2}$$

xing both sides by $(s+1)(s+2)$

$$s+4 = A(s+2) + B(s+1)$$

Put $s=-1$

$$\Rightarrow 3 = A(-1+2) + 0 \Rightarrow \boxed{A=3}$$

Put $s=-2$

$$\Rightarrow -2+4 = A(0) + B(-2+1)$$

$$\Rightarrow \boxed{B=-2}$$

$$\Rightarrow \frac{s+4}{s^2+3s+2} = \frac{3}{s+1} + \frac{-2}{s+2}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-2}{s+2}\right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= 3e^{-t} - 2e^{-2t}$$

★★ ★★

* Applications of Laplace Transform to Initial Value Problem.

Before applying L.T to initial value problems the following results must be kept in view.

$$\mathcal{L}\{u(t)\} = U(s)$$

$$\mathcal{L}\{u'(t)\} = sU(s) - u(0)$$

$$\mathcal{L}\{u''(t)\} = s^2U(s) - su(0) - u'(0)$$

$$\mathcal{L}\{u'''(t)\} = s^3U(s) - s^2u(0) - su'(0) - u''(0)$$

Example:- Apply L.T to solve

$$u' - 2u = 0 \quad , \quad u(0) = 1$$

Solution

Given $u' - 2u = 0$

$$\mathcal{L}\{u' - 2u\} = \mathcal{L}\{0\} \Rightarrow \mathcal{L}\{u'\} - 2\mathcal{L}\{u\} = 0$$

$$\Rightarrow sU(s) - u(0) - 2U(s) = 0$$

$$\Rightarrow sU(s) - 1 - 2U(s) = 0$$

$$\Rightarrow (s-2)U(s) = 1 \quad \Rightarrow \quad U(s) = \frac{1}{s-2}$$

$$\Rightarrow \mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$\Rightarrow u(t) = e^{2t}$$

Example 2:- Solve the I.V.P

$$u' + 2u = 0 \quad , \quad u(0) = 1$$

Solution

$$\mathcal{L}\{u'\} + 2\mathcal{L}\{u\} = \mathcal{L}\{0\}$$

$$sU(s) - u(0) + 2U(s) = 0$$

$$\Rightarrow (s+2)U(s) = 1$$

$$\Rightarrow U(s) = \frac{1}{s+2}$$

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\Rightarrow u(t) = e^{-2t}$$

* *

Example 3:- Apply L.T to solve

$$u'' + 4u' + 3u = 0 \quad \text{with } u(0) = 1 \ \& \ u'(0) = 0$$

Solution

$$\mathcal{L}\{u'' + 4u' + 3u\} = \mathcal{L}\{0\}$$

$$\Rightarrow \mathcal{L}\{u''\} + 4\mathcal{L}\{u'\} + 3\mathcal{L}\{u\} = \frac{0}{s}$$

$$\Rightarrow s^2U(s) - su(0) - u'(0) + 4\{sU(s) - u(0)\} + 3U(s) = 0$$

$$\Rightarrow s^2U(s) - s(1) - 0 + 4\{sU(s) - 1\} + 3U(s) = 0$$

$$\Rightarrow \{s^2 + 4s + 3\}U(s) - s - 4 = 0$$

$$\Rightarrow U(s) = \frac{s+4}{s^2+4s+3}$$

$$\Rightarrow u(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4s+3}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+4}{(s+1)(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{3/2}{s+1} + \frac{-1/2}{s+3}\right\}$$

$$= \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}$$

* *

* Unit Step Function:-

Unit step function or Heaviside step function is denoted & defined by

$$H(t - t_0) = \begin{cases} 0 & \text{if } t < t_0 \\ 1 & \text{if } t \geq t_0 \end{cases}$$

⇒ **Convolution:-** If $f(t)$ and $g(t)$ are piecewise continuous functions over the interval $[0,\infty[$ then convolution of f and g is denoted & defined by

$$f*g = \int_0^t f(y)g(t-y)\,dy$$

⇒ **Theorem:-** If f, g and h are piece wise continuous functions over $[0,\infty)$, then

(i) $f*g = g*f$

(ii) $f*(g+h) = f*g + f*h$

(iii) $f*(g*h) = (f*g)*h$

Proof:- (i) $f*g = \int_0^t f(y)\,g(t-y)\,dy$

Put $t-y = T \Rightarrow -dy = dT$

$y = 0 \Rightarrow T = t$

$y = t \Rightarrow T = 0$

So $f*g = \int_t^0 f(t-T)\,g(T)\,(-dT)$

$= \int_0^t g(T)\,f(t-T)\,dT$

$= g*f$

$\Rightarrow f*g = g*f$

(ii) $f*(g+h) = \int_0^t f(y)[g(t-y)+h(t-y)]\,dy$

$= \int_0^t f(y)g(t-y)\,dy + \int_0^t f(y)h(t-y)\,dy$

$= f*g + f*h$

(iii) = ?

⇒ Convolution Theorem:-

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{G(s)\} = g(t)$ then

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = f(t) * g(t)$$

Proof:-

We prove $\mathcal{L}\{f(t) * g(t)\} = F(s)\,G(s)$
before proving this we prove

$$f * g = \int_0^\infty H(t-\tau) f(\tau) g(t-\tau)\, d\tau \qquad (i)$$

where $H(t-\tau)$ is unit step function denoted and defined by

$$H(t-\tau) = \begin{cases} 0 & \text{if } t < \tau \\ 1 & \text{if } t \geq \tau \end{cases} \qquad (ii)$$

Consider

$$\int_0^\infty H(t-\tau) f(\tau) g(t-\tau)\, d\tau$$

$$= \int_0^t H(t-\tau) f(\tau) g(t-\tau)\, d\tau + \int_t^\infty H(t-\tau) f(\tau) g(t-\tau)\, d\tau$$

$$= \int_0^t (1) f(\tau) g(t-\tau)\, d\tau + \int_t^\infty (0) f(\tau) g(t-\tau)\, d\tau$$

$$= f * g$$

Now consider

$$\mathcal{L}\{f(t) * g(t)\} = \int_0^\infty e^{-st} [f(t) * g(t)]\, dt$$

$$= \int_0^\infty e^{-st} \int_0^\infty H(t-\tau) f(\tau) g(t-\tau)\, d\tau\, dt$$

$$= \int_0^\infty \int_0^\infty e^{-st} H(t-\tau) f(\tau) g(t-\tau)\, d\tau\, dt$$

$$= \int_0^\infty \int_0^\infty e^{-st} H(t-\xi) f(\xi) g(t-\xi)\, dt\, d\xi$$

$\because$ by changing the order of integral

$$= \int_0^\infty f(\xi) \left[\int_0^\infty e^{-st} H(t-\xi) g(t-\xi)\, dt \right] d\xi \longrightarrow \circledast$$

Now consider

$$\int_0^\infty e^{-st} H(t-\xi) g(t-\xi)\, dt$$

Put $(t-\xi) = T$ then $dt = dT$

$t = 0 \Rightarrow T = -\xi$ & $t = \infty \Rightarrow T = \infty$

Then $\displaystyle\int_0^\infty e^{-st} H(t-\xi) g(t-\xi) dt = \int_{-\xi}^\infty e^{-s(T-\xi)} H(T) g(T)\, dT$

$$= \int_{-\xi}^0 e^{-sT} e^{-s\xi} H(T) g(T)\, dT + \int_0^\infty e^{-sT} e^{-s\xi} H(T) g(T)\, dT$$

$$= 0 + \int_0^\infty e^{-sT} e^{-s\xi} g(T)\, dT$$

$\because \int_{-\xi}^0 H(T) = 0$

& $\int_0^\infty H(T) = 1$

unit step funct

So from $\circledast$

$$\mathcal{L}\{f(t) * g(t)\} = \int_0^\infty f(\xi) \left[\int_0^\infty e^{-sT} e^{-s\xi} g(T)\, dT \right] d\xi$$

$$= \left[\int_0^\infty e^{-s\xi} f(\xi)\, d\xi \right] \left[\int_0^\infty e^{-sT} g(T)\, dT \right]$$

$$= \mathcal{L}\{f(t)\}\, \mathcal{L}\{g(t)\}$$

$\Rightarrow \mathcal{L}\{f(t) * g(t)\} = F(s)\, G(s)$

$$\Rightarrow f(t) * g(t) = \mathcal{L}^{-1}\{F(s)\, G(s)\}$$

** **

Example: Solve $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+k^2)}\right\}$ by convolution theorem.

Solution

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+k^2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+k^2}\right\}$$

$$= \frac{1}{k}\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\cdot\frac{1}{s}\right\}$$

$$= \frac{1}{k}\left[\sin kt * 1\right] \quad \text{by convolution theorem}$$

$$= \frac{1}{k}\int_0^t \sin ky\cdot(t-y)^0\,dy \quad \because 1 = t^0$$

$$= \frac{1}{k}\int_0^t \sin ky\,dy$$

$$= \frac{1}{k}\left[\frac{-\cos ky}{k}\right]\Big|_0^t$$

$$= \frac{1}{k^2}\left[-\cos kt + \cos(0)\right]$$

$$= \frac{1}{k^2}\left[1 - \cos kt\right]$$

Example: Calculat L.T of $\int_0^t (t-\beta)\sin 3\beta\,d\beta$

Solution

$$\mathcal{L}\left\{\int_0^t \sin 3\beta\,(t-\beta)\,d\beta\right\} = \mathcal{L}\left\{\sin 3t * t\right\}$$

$$= \mathcal{L}\{\sin 3t\}\,\mathcal{L}\{t\}$$

$$= \frac{3}{s^2+9}\cdot\frac{1}{s^2} = \frac{3}{s^2(s^2+9)}$$

EXERCISE.

Q1:- Find the Laplace Transform of

a) $\int_0^t e^{-(t-\beta)} \sin\beta \, d\beta$, b) $\int_0^t (t-\beta)^3 \sin\beta \, d\beta$

Solution

a) $\mathcal{L}\left\{\int_0^t \sin\beta \, e^{-(t-\beta)} d\beta\right\} = \mathcal{L}\{\sin t * e^{-t}\}$

$= \mathcal{L}\{\sin t\} \, \mathcal{L}\{e^{-t}\}$

$= \frac{1}{s^2+1^2} \cdot \frac{1}{s+1} = \frac{1}{(s+1)(s^2+1)}$

b) $\mathcal{L}\left\{\int_0^t (t-\beta)^3 \sin\beta \, d\beta\right\} = \mathcal{L}\{\sin t * t^3\}$

$= \frac{1}{s^2+1} \cdot \frac{3!}{s^4} = \frac{6}{s^4(s^2+1)}$

* * *

Q2:- Find Inverse Laplace Transform by Convolution theorem

a) $\frac{4}{s^2(s-2)}$, b) $\frac{1}{(s^2+1)^2}$

Solution

a) $\mathcal{L}^{-1}\left\{\frac{4}{s^2(s-2)}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s^2} \cdot \frac{1}{s-2}\right\}$

$= 4 \cdot [t * e^{2t}] = 4\int_0^t y \, e^{2(t-y)} dy$

$= 4e^{2t}\left[y \frac{e^{-2y}}{-2}\Big|_0^t - \int_0^t \frac{e^{-2y}}{-2} dy\right]$

$= 4e^{2t}\left[-\frac{1}{2} t e^{-2t} - 0 + \frac{1}{2} \cdot \frac{e^{-2y}}{-2}\Big|_0^t\right]$

$= 4e^{2t}\left[-\frac{1}{2} t e^{-2t} - \frac{1}{4}\left[e^{-2t} - e^0\right]\right]$

$= 4e^{2t}\left[-\frac{1}{2} t e^{-2t} - \frac{1}{4} e^{-2t} + \frac{1}{4}\right]$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{4}{s^2(s-2)}\right\} = -2t - 1 + e^{2t}$

b) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{1}{s^2+1}\right\} = \sin t * \sin t$$

$$= \int_0^t \sin y \, \sin(t-y)\, dy$$

$$= \frac{-1}{2}\int_0^t (-2\sin y \sin(t-y))\, dy$$

$$= -\frac{1}{2}\int_0^t [\cos(\cancel{y}+t-\cancel{y}) - \cos(y-t+y)]\, dy$$

$$= \frac{-1}{2}\left[\int_0^t \cos t \, dy - \int_0^t \cos(2y-t)\, dy\right]$$

$$= -\frac{1}{2}\left[t\sin t - \left.\frac{\sin(2y-t)}{2}\right|_0^t\right]$$

$$= -\frac{1}{2}\left[t\sin t - \frac{1}{2}(\sin t + \sin t)\right]$$

$$= -\frac{1}{2}[t\sin t - \sin t]$$

$$= \frac{\sin t}{2} - \frac{t\sin t}{2}$$

***

Q3:- Show that $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(y)\, dy$

Solution

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \mathcal{L}^{-1}\left\{F(s)\cdot\frac{1}{s}\right\}$$

$$= f(t) * 1 = f(t) * t^0$$

$$= \int_0^t f(y)(t-y)^0\, dy$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(y)\, dy \qquad \because (t-y)^0 = 1$$

Q4:- Show that $\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \left\{\int_0^y f(\lambda)\,d\lambda\right\} dy$

Solution

$$L.H.S = \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\cdot\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} * 1$$

$$= \mathcal{L}^{-1}\left\{F(s)\cdot\frac{1}{s}\right\} * 1 = [f(t) * 1] * 1$$

$$= \left(\int_0^t f(\lambda)\,d\lambda\right) * 1$$

$$= \int_0^t \left\{\int_0^y f(\lambda)\,d\lambda\right\}(t-y)^0\,dy$$

$$= \int_0^t \left\{\int_0^y f(\lambda)\,d\lambda\right\} dy = R.H.S$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \left\{\int_0^y f(\lambda)\,d\lambda\right\} dy$$

———————

Q5:- a) Show that $\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} = t\int_0^t f(\lambda)\,d\lambda - \int_0^t t f(t)\,dt$

b) Calculate $\mathcal{L}\{\ln t\}$

Solution

a) $\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \left\{\int_0^y f(\lambda)\,d\lambda\right\} dy$

$$= \int_0^t \left\{\int_0^y f(\lambda)\,d\lambda\right\}\cdot 1\,dy$$

$$= \left(\int_0^y f(\lambda)\,d\lambda\right)\cdot y\Big|_0^t - \int_0^t f(y)\cdot y\,dy \qquad \because \text{Integrating By Parts}$$

$$= t\int_0^t f(\lambda)\,d\lambda - \int_0^t y f(y)\,dy$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} = t\int_0^t f(\lambda)\,d\lambda - \int_0^t y f(y)\,dy$$

b) To calculate $\mathcal{L}\{\ln t\}$

$$\mathcal{L}\{\ln t\} = \int_0^\infty e^{-st} \ln t \, dt$$

Put $st = u \Rightarrow \mathcal{L}\{\ln t\} = \int_0^\infty e^{-u}(-\ln s + \ln u)\frac{du}{s}$

$$\Rightarrow \mathcal{L}\{\ln t\} = -\frac{\ln s}{s}\int_0^\infty e^{-u} du + \frac{1}{s}\int_0^\infty e^{-u} \ln u \, du$$

$$= -\frac{\ln s}{s} \cdot 1 + \frac{1}{s}\int_0^\infty e^{-u} \ln u \, du$$

Now $\Gamma(x+1) = \int_0^\infty e^{-u} u^x \, du$

$$\Gamma'(x+1) = \int_0^\infty e^{-u} u^x \ln u \, du$$

$$\Rightarrow \Gamma'(1) = \int_0^\infty e^{-u} \ln u \, du$$

$$\Rightarrow \mathcal{L}\{\ln t\} = -\frac{\ln s}{s} + \frac{1}{s}\Gamma'(1)$$

$$\Rightarrow \mathcal{L}\{\ln t\} = \frac{1}{s}(\Gamma'(1) - \ln s)$$

***

Q6:- Solve the integral equation

$$y(t) = at + \int_0^t y(z) \sin(t-z) \, dz$$

Solution

Let $u(t) = \int_0^t y(z) \sin(t-z) \, dz$

$$= y(t) * \sin t$$

$$\mathcal{L}\{u(t)\} = \mathcal{L}\{y(t) * \sin t\}$$

$$= \mathcal{L}\{y(t)\}\,\mathcal{L}\{\sin t\}$$

$$\Rightarrow U(s) = Y(s) \cdot \frac{1}{s^2+1}$$

Now $y(t) = at + u(t)$

$$\Rightarrow \mathcal{L}\{y(t)\} = \mathcal{L}\{at\} + \mathcal{L}\{u(t)\}$$

$$\Rightarrow Y(s) = a/s^2 + U(s)$$

$$= \frac{a}{s^2} + Y(s) \cdot \frac{1}{s^2+1}$$

$$\Rightarrow Y(s)\left[1 - \frac{1}{s^2+1}\right] = a/s^2$$

$$\Rightarrow Y(s)\left[\frac{s^2+1-1}{s^2+1}\right] = a/s^2 \Rightarrow Y(s) = a/s^2 \cdot \frac{s^2+1}{s^2}$$

$$\Rightarrow Y(s) = a\left(\frac{s^2+1}{s^4}\right) = a\left[\frac{1}{s^2} + \frac{1}{s^4}\right]$$

$$= \frac{a}{s^2} + \frac{a}{3!} \cdot \frac{3!}{s^4}$$

$$\Rightarrow y(t) = at + \frac{a}{3!} t^3$$

***

Q7:- Solve $y(t) = f(t) + \int_0^t g(t-\tau)\, y(\tau)\, d\tau$

Solution $\mathcal{L}\{y(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\left\{\int_0^t y(\tau) g(t-\tau)\, d\tau\right\}$

$$\Rightarrow Y(s) = F(s) + Y(s)\, G(s) \qquad \text{By convolution theorem}$$

$$\Rightarrow Y(s)[1 - G(s)] = F(s)$$

$$\Rightarrow Y(s) = \frac{F(s)}{1-G(s)} \Rightarrow y(t) = \mathcal{L}^{-1}\left[\frac{F(s)}{1-G(s)}\right]$$

****

Q8:- Solve the differential equations by L.T Method

a) $y''(t) + k^2 y(t) = f(t)$

b) $y''(t) - 2k\, y'(t) + k^2 y(t) = f(t)$

c) $y''(t) + \lambda\, y'(t) + k^2 y(t) = f(t)$

{In each case check the physical significance.

Solution

a) $y''(t) + k^2 y(t) = f(t)$

$$\mathcal{L}\{y''(t)\} + k^2 \mathcal{L}\{y(t)\} = \mathcal{L}\{f(t)\}$$

$$\Rightarrow s^2 Y(s) - s y(0) - y'(0) + k^2 Y(s) = F(s)$$

$$Y(s)[s^2+k^2] = F(s) - s y(0) - y'(0)$$

$$Y(s) = \frac{F(s) - s y(0) - y'(0)}{s^2+k^2}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left[\frac{F(s) - s y(0) - y'(0)}{s^2+k^2}\right]$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left[\frac{F(s)}{s^2+k^2}\right] - c_1 \mathcal{L}^{-1}\left[\frac{s}{s^2+k^2}\right] - \frac{c_2}{k}\mathcal{L}^{-1}\left[\frac{k}{s^2+k^2}\right]$$

where $y(0) = c_1$ & $y'(0) = c_2$

$$y(t) = \frac{1}{k} f(t) * \sin kt - c_1 \cos kt - \frac{c_2}{k} \sin kt$$

b) Given

$$y''(t) - 2k y'(t) + k^2 y(t) = f(t)$$

Applying Laplace transform

$$s^2 Y(s) - s y(0) - y'(0) - 2k s Y(s) + 2k y(0) + k^2 Y(s) = F(s)$$

$$\Rightarrow (s^2 - 2ks + k^2) Y(s) = F(s) + s y(0) + y'(0) - 2k y(0)$$

$$\Rightarrow Y(s) = \frac{F(s) + s y(0) + y'(0) - 2k y(0)}{(s-k)^2}$$

Applying I.L.T

$$y(t) = \mathcal{L}^{-1}\left[\frac{F(s) + s y(0) + y'(0) - 2k y(0)}{(s-k)^2}\right]$$

$$\Rightarrow y(t) = f(t) * t e^{kt} + c_1(e^{kt} + kt e^{kt}) + c_2 t e^{kt} - 2k c_1 t e^{kt}$$

c) Given $y''(t) + \lambda y'(t) + k^2 y(t) = f(t)$

Applying L.T on both sides

$$s^2 Y(s) - s y(0) - y'(0) + \lambda s Y(s) - \lambda y(0) + k^2 Y(s) = F(s)$$

$$\Rightarrow (s^2+\lambda s+k^2)Y(s) = F(s)+sy(0)+y'(0)+\lambda y(0)$$

$$\Rightarrow Y(s) = \left[\frac{F(s)+sy(0)+y'(0)+\lambda y(0)}{s^2+\lambda s+k^2}\right]$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left[\frac{F(s)+sy(0)+y'(0)+\lambda y(0)}{s^2+\lambda s+k^2}\right]$$

————————

Q9:- Solve the problem

$$y''+\omega^2 y = f(t) \quad ; y(0)=y_0 ,\ y'(0)=y_1$$

and discuss the case when

$$f(t) = \begin{cases} f_0 & t_0<t<t_1 \\ 0 & \text{for other values of } 't' \end{cases}$$

Solution

Given $y''+\omega^2 y = f(t)$

$$\mathcal{L}\{y''\}+\omega^2\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$\Rightarrow s^2Y(s)-sy(0)-y'(0)+\omega^2 Y(s) = F(s)$$

$$\Rightarrow (s^2+\omega^2)Y(s) = F(s)+sy_0+y_1$$

$$\Rightarrow Y(s) = \frac{F(s)}{s^2+\omega^2}+\frac{s}{s^2+\omega^2}y_0+\frac{1}{s^2+\omega^2}y_1$$

$$\Rightarrow y(t) = y_0\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\}+\frac{y_1}{\omega}\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\}$$

$$+\frac{1}{\omega}\mathcal{L}^{-1}\left\{F(s)\cdot\frac{\omega}{s^2+\omega^2}\right\}$$

$$\Rightarrow y(t) = y_0\cos\omega t+\frac{y_1}{\omega}\sin\omega t+\frac{1}{\omega}f(t)*\sin\omega t$$

Now $f(t)*\sin\omega t = \int_0^t f(\tau)\sin\omega(t-\tau)\,d\tau$

$$= \int_0^{t_0} f(\tau)\sin\omega(t-\tau)\,d\tau+\int_{t_0}^{t_1} f(\tau)\sin\omega(t-\tau)\,d\tau$$

$$+\int_{t_1}^{t} f(\tau)\sin\omega(t-\tau)\,d\tau$$

$$= 0 + \int_{t_0}^{t_1} f(\tau)\sin\omega(t-\tau)\,d\tau + 0$$

$$= \int_{t_0}^{t_1} f(\tau)\sin\omega(t-\tau)\,d\tau$$

$$\Rightarrow y(t) = y_0 \cos\omega t + \frac{y_1}{\omega}\sin\omega t + \frac{1}{\omega}\int_{t_0}^{t_1} f(\tau)\sin\omega(t-\tau)\,d\tau$$

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Q 10:- Solve the inhomogenous problems with zero initial conditions. i.e

$u(0) = 0$ & $u'(0) = 0$

a) $u'' + au = 1$, b) $u'' + u = t$

c) $u'' + 2u' = 1 - e^{-t}$

d) $u'' - u = 1$, e) $u'' + 4u = \sin t$

Solution a) $\mathcal{L}\{u''\} + \mathcal{L}\{au\} = \mathcal{L}\{1\}$

$$\Rightarrow s^2 U(s) - s u(0) - u'(0) + aU(s) = 1/s$$

$$(s^2 + a) U(s) = 1/s$$

$$\Rightarrow U(s) = \frac{1}{s(s^2+a)}$$

$$\Rightarrow u(t) = \frac{1}{a}\left[1 - \cos\sqrt{a}\,t\right] \quad \because \text{by exp } 1$$

b) $\mathcal{L}\{u''\} + \mathcal{L}\{u\} = \mathcal{L}\{t\}$

$$\Rightarrow s^2 U(s) - s u(0) - u'(0) + U(s) = 1/s^2$$

$$(s^2+1) U(s) = 1/s^2$$

$$\Rightarrow U(s) = \frac{1}{s^2(s^2+1)}$$

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} \cdot \frac{1}{s^2+1}\right\}$$

$$= t * \sin t = \sin t * t$$

$$\Rightarrow u(t) = \int_0^t \sin\tau\,(t-\tau)\,d\tau$$

$$\Rightarrow u(t) = (t-\tau)\cdot[-\cos\tau]\Big|_0^t - \int_0^t (0-1)(-\cos\tau)\,d\tau$$

$$= (t-t)[-\cos t] - (t-0)(-\cos(0)) - \sin\tau\Big|_0^t$$

$$= t - \sin t$$

$$\Rightarrow u(t) = t - \sin t$$

c) $\mathcal{L}\{u''\} + 2\mathcal{L}\{u'\} = \mathcal{L}\{1-e^{-t}\}$

$$s^2U(s) - s\,u(0) - u'(0) + 2sU(s) - 2u(0) = \frac{1}{s} - \frac{1}{s+1}$$

$$\Rightarrow (s^2+2)U(s) = \frac{1}{s} - \frac{1}{s+1}$$

$$\Rightarrow U(s) = \frac{1}{s(s^2+2)} - \frac{1}{(s+1)(s^2+2)}$$

$$\Rightarrow u(t) = \frac{1}{2}(1-\cos\sqrt{2}t) - \frac{1}{\sqrt{2}}e^{-t} * \sin\sqrt{2}t$$

$$= \frac{1}{2}[1-\cos\sqrt{2}t] - \frac{1}{\sqrt{2}}\left[\int_0^t e^{-\tau}\sin\sqrt{2}(t-\tau)\,d\tau\right]$$

Let $I = \int_0^t e^{-\tau}\sin\sqrt{2}(t-\tau)\,d\tau$

$$= e^{-\tau}\frac{(-\cos\sqrt{2}(t-\tau))}{-\sqrt{2}}\Big|_0^t - \int(-e^{-\tau})\left[\frac{-\cos\sqrt{2}(t-\tau)}{-\sqrt{2}}\right]d\tau$$

$$= e^{-t} - \cos\sqrt{2}t + \frac{1}{\sqrt{2}}\left[e^{-\tau}\frac{(\sin\sqrt{2}(t-\tau))}{-\sqrt{2}}\Big|_0^t - \int_0^t(-e^{-\tau})\left(\frac{\sin\sqrt{2}(t-\tau)}{-\sqrt{2}}\right)d\tau\right]$$

$$= e^{-t} - \cos\sqrt{2}t + \frac{1}{2}\sin\sqrt{2}t - \frac{1}{2}[I]$$

$$\Rightarrow I = \frac{2}{3}\left[e^{-t} - \cos\sqrt{2}t\right] + \frac{1}{3}\sin\sqrt{2}t$$

$$\Rightarrow u(t) = \frac{1}{2}(1-\cos\sqrt{2}t) - \frac{\sqrt{2}}{3}(e^{-t} - \cos\sqrt{2}t) - \frac{1}{2\sqrt{3}}\sin\sqrt{2}t$$

d) $u'' - u = 1$

$$\Rightarrow s^2U(s) - su(0) - u'(0) - U(s) = 1/s$$

$$\Rightarrow U(s) = \frac{1}{s(s^2-1)}$$

$$\Rightarrow u(t) = 1 * \sinh t = \sinh t * 1$$

$$= \int_0^t \sinh \tau (t-\tau)^0 \, d\tau$$

$$= \cosh \tau \Big|_0^t = \cosh \tau - 1$$

$$\Rightarrow u(t) = \cosh \tau - 1$$

e) $u'' + 4u = \sin t$

$$s^2U(s) - su(0) - u'(0) + 4U(s) = \frac{1}{s^2+1}$$

$$\Rightarrow U(s) = \frac{1}{(s^2+4)(s^2+1)}$$

$$\Rightarrow u(t) = \frac{1}{2}\sin 2t * \sin t$$

$$= \frac{1}{2}\int_0^t \sin 2\tau \sin(t-\tau)\, d\tau$$

$$= -\frac{1}{4}\int_0^t (-2\sin 2\tau \sin(t-\tau))\, d\tau$$

$$= -\frac{1}{4}\int_0^t (\cos(\tau+t) - \cos(3\tau-t))\, d\tau$$

$$= -\frac{1}{4}\sin(\tau+t)\Big|_0^t + \frac{1}{12}\sin(3\tau-t)\Big|_0^t$$

$$= -\frac{1}{4}(\sin 2t - \sin t) + \frac{1}{12}(\sin 2t + \sin t)$$

$$= -\frac{1}{6}\sin 2t + \frac{1}{3}\sin t$$

Q 11:- Find Inverse L.T by Partial fraction

a) $\frac{1}{s^2-4}$ b) $\frac{s+3}{s(s^2+2)}$ c) $\frac{1}{s(s+1)}$

Solution a)

Given $\frac{1}{s^2-4}$

$$\Rightarrow \frac{1}{s^2-4} = \frac{A}{s-2} + \frac{B}{s+2} \Rightarrow 1 = A(s+2) + B(s-2)$$

$s = 2 \Rightarrow 1 = 4A \Rightarrow \boxed{A = 1/4}$

$s = -2 \Rightarrow 1 = -4B \Rightarrow \boxed{B = -1/4}$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^2-4}\right\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{4}L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}$$

b) Let

$$\frac{s+3}{s(s^2+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2}$$

$$\Rightarrow s+3 = A(s^2+2) + s(Bs+C)$$

$$= (A+B)s^2 + Cs + 2A$$

Comparing co-efficients

$2A = 3 \Rightarrow \boxed{A = 3/2}$

$\boxed{C = 1}$ & $A+B=0 \Rightarrow \boxed{B = -3/2}$

$$\text{So } L^{-1}\left\{\frac{s+3}{s(s^2+2)}\right\} = L^{-1}\left\{\frac{3/2}{s}\right\} + L^{-1}\left\{\frac{-\frac{3}{2}(s+1)}{s^2+2}\right\}$$

$$= \frac{3}{2}L^{-1}\left\{\frac{1}{s}\right\} - \frac{3}{2}L^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{1}{\sqrt{2}}L^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\}$$

$$= \frac{3}{2}(1) - \frac{3}{2}\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s+3}{s(s^2+2)}\right\} = \frac{3}{2} - \frac{3}{2}\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t$$

c)
$$\frac{4}{s(s+1)} = 4\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

$$\mathcal{L}^{-1}\left\{\frac{4}{s(s+1)}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$
$$= 4(1) - 4e^{-t}$$
$$= 4(1-e^{-t})$$

————————

Q 12:- Solve
$$y''(t) + t\,y'(t) - y(t) = 0$$
with $y(0) = 0$ & $y'(0) = 1$

Solution
$$\mathcal{L}\{y''(t)\} + \mathcal{L}\{t\,y'(t)\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{0\}$$
$$s^2Y(s) - sy(0) - y'(0) + (-1)^1\frac{d}{ds}\{sY(s) - y(0)\} - Y(s) = 0$$

[Here we have used the formula $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$ which will discussed later]

Now
$$s^2Y(s) - 0 - 1 - sY'(s) - (1)Y(s) + 0 - Y(s) = 0$$
$$(s^2-2)Y(s) - sY'(s) - 1 = 0$$
$$sY'(s) - (s^2-2)Y(s) = -1$$
$$Y'(s) - \frac{s^2-2}{s}Y(s) = -1/s$$
$$I.F = e^{-\int\frac{s^2-2}{s}ds} = e^{-\int(s-\frac{2}{s})ds}$$
$$= e^{-s^2/2 + \ln s^2} = e^{-s^2/2}\cdot e^{\ln s^2} = s^2 e^{-s^2/2}$$

Multiplying both sides by $s^2 e^{-s^2/2}$ and integrating

$$\int d\left(s^2 e^{-s^2/2} Y(s)\right) = \int \left(-s e^{-s^2/2}\right) ds$$

$$s^2 e^{-s^2/2} Y(s) = e^{-s^2/2} + c$$

Now $y(0) = 0$

$$\mathcal{L}\{y(0)\} = \mathcal{L}\{0\} \Rightarrow Y(0) = 0$$

$$0 = e^0 + c \Rightarrow \boxed{c = -1}$$

$$\Rightarrow s^2 e^{-s^2/2} Y(s) = e^{-s^2/2} - 1$$

$$\Rightarrow Y(s) = \frac{1}{s^2} - \frac{e^{s^2/2}}{s^2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{s^2/2}}{s^2}\right\}$$

$$\Rightarrow y(t) = t - \mathcal{L}^{-1}\left\{\frac{e^{s^2/2}}{s^2}\right\}$$

*** ***

⇒ Laplace Transform of Unit Step Function:-

$$\mathcal{L}\{H(t-a)\} = \int_0^\infty e^{-st} H(t-a)\, dt$$

$$\Rightarrow \mathcal{L}\{H(t-a)\} = \int_0^a e^{-st} H(t-a)\, dt + \int_a^\infty e^{-st} H(t-a)\, dt$$

$$= \int_0^a e^{-st}(0) + \int_a^\infty e^{-st}(1)\, dt$$

$$= 0 + \left.\frac{e^{-st}}{-s}\right|_a^\infty = -\frac{1}{s}\left[e^{-\infty} - e^{-sa}\right]$$

$$= -\frac{1}{s}\left[0 - e^{-sa}\right]$$

$$\Rightarrow \mathcal{L}\{H(t-a)\} = \frac{1}{s} e^{-sa}$$

Theorem:- Prove that:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$$

Proof:- For $n=1$

$$\mathcal{L}\{t f(t)\} = \int_0^\infty e^{-st}(t f(t))\,dt$$

Now $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$F'(s) = \int_0^\infty (-t)e^{-st} f(t)\,dt = -\int_0^\infty e^{-st}(t f(t))\,dt$$

$$\Rightarrow \mathcal{L}\{t f(t)\} = (-1)^1 \frac{d}{ds}\{F(s)\}$$

$\Rightarrow$ C-I is satisfied

Suppose it is true for $n=K$ i.e

$$\mathcal{L}\{t^K f(t)\} = (-1)^K \frac{d^K}{ds^K}\{F(s)\}$$

Now we prove it is true for $n=K+1$

$$(-1)^K \frac{d^K}{ds^K}\{F(s)\} = \mathcal{L}\{t^K f(t)\}$$

$$= \int_0^\infty e^{-st}(t^K f(t))\,dt$$

Differentiate w.r.t s

$$(-1)^K \frac{d^{K+1}}{ds^{K+1}}\{F(s)\} = \int_0^\infty (-t e^{-st})(t^K f(t))\,dt$$

$$\Rightarrow (-1)^K \frac{d^{K+1}}{ds^{K+1}}\{F(s)\} = -\int_0^\infty e^{-st}(t^{K+1} f(t))\,dt$$

$$\Rightarrow (-1)^{K+1} \frac{d^{K+1}}{ds^{K+1}}\{F(s)\} = \mathcal{L}\{t^{K+1} f(t)\}$$

Therefore this is true for $n=K+1$ $\Rightarrow$ C-2 is satis

So theorem is true for all derivatives upto nth order.

→ The Second Shifting Theorem:-

It states that $\mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)f(t-a)$, $a>0$

Proof:- To prove $\mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)f(t-a)$

we prove $\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s)$

$$\mathcal{L}\{H(t-a)f(t-a)\} = \int_0^\infty H(t-a)f(t-a)e^{-st}\,dt$$

$$= \int_0^a H(t-a)f(t-a)e^{-st}\,dt + \int_a^\infty H(t-a)f(t-a)e^{-st}\,dt$$

$$= \int_0^a e^{-st}(0)f(t-a)\,dt + \int_a^\infty e^{-st}(1)f(t-a)\,dt$$

$$= 0 + \int_a^\infty e^{-st}f(t-a)\,dt$$

Put $t-a = y$

$t = a \Rightarrow y = 0$

$t = \infty \Rightarrow y = \infty$, $dt = dy$

So $$\mathcal{L}\{H(t-a)f(t-a)\} = \int_0^\infty e^{-s(y+a)}f(y)\,dy$$

$$= e^{-as}\int_0^\infty e^{-sy}f(y)\,dy$$

$$= e^{-as}\int_0^\infty e^{-st}f(t)\,dt$$

$$= e^{-as}\mathcal{L}\{f(t)\} = e^{-as}F(s)$$

$$\Rightarrow \mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)f(t-a)$$

** ——————————— **

Corollary:- Prove that

$$\mathcal{L}\{P_n(t)f(t)\} = P_n(-D)F(S)$$

Proof:-

Let $P_n(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$

$$\mathcal{L}\{P_n(t)f(t)\} = \mathcal{L}[\{a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0\}f(t)]$$

$$= a_n\mathcal{L}\{t^n f(t)\} + a_{n-1}\mathcal{L}\{t^{n-1}f(t)\} + \cdots$$

$$\cdots + a_1\mathcal{L}\{t f(t)\} + a_0\mathcal{L}\{f(t)\}$$

$$= a_n\{(-1)^n D^n F(S)\} + a_{n-1}\{(-1)^{n-1}D^{n-1}F(S)\} +$$

$$\cdots + a_1\{(-1)^1 D F(S)\} + a_0 F(S)$$

$$= [a_n(-D)^n + a_{n-1}(-D)^{n-1} + \cdots + a_1(-D)^1 + a_0]F(S)$$

$$= P_n(-D)F(S)$$

$$\Rightarrow \mathcal{L}\{P_n(t)f(t)\} = P_n(-D)F(S)$$

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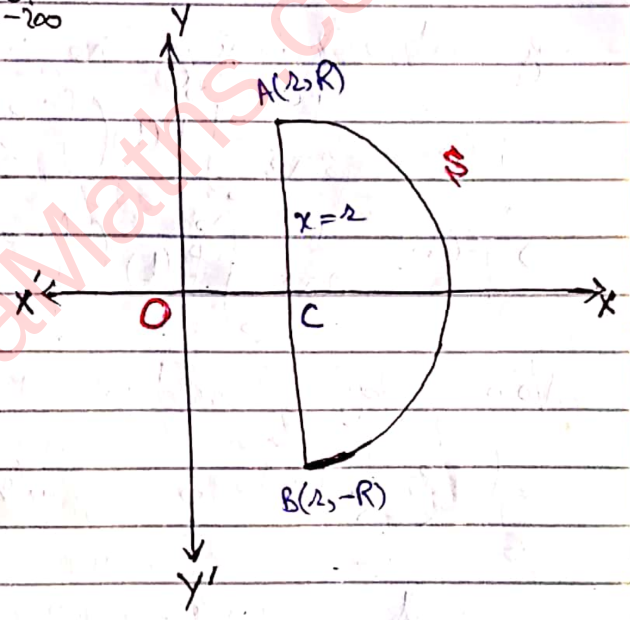
⇒ The Laplace Inversion Integral OR Fourier Mellin Integral:-

Statement:- If $f(t)$ is the inverse Laplace Transform of $F(s)$, and all the singularities of $F(s)$ in the complex plane lie to the left of the line $x=r$, then

$$f(t) = \frac{1}{2\pi i} \lim_{r\to\infty} \int_{r-i\infty}^{r+i\infty} e^{st} F(s)\, ds$$

Proof:-

Draw the line $x=r$ in the XY-plane and mark the point $A(r,R)$ and $B(r,-R)$ on the line and draw a semi-circle S of radius R to right of the line $x=r$ as shown in the figure.



Consider the closed contour C consisting of the line segment $\overline{AB}$ and the semicircle S i.e $C = \overline{AB} \cup S$. Now consider the function $F(z)$, then $F(z)$ is analytic on and with in the closed contour C. because all the singularities of $F(z)$ lie to the left of the line $x=r$. If s is any point inside the C then by cauchy's integral theorem

$$F(s) = \frac{1}{2\pi i}\int_C \frac{F(z)}{z-s}\,dz \longrightarrow ①$$

where $F(z)$ is L.T of $f(t)$ i.e

$$F(z) = \int_0^\infty e^{-zt} f(t)\,dt \longrightarrow ②$$

from ① and ②

$$F(s) = \frac{1}{2\pi i}\int_C \frac{1}{z-s}\left(\int_0^\infty e^{-zt} f(t)\,dt\right) dz$$

Since z and t are independent so by interchanging the integrals we get

$$F(s) = \frac{1}{2\pi i}\int_0^\infty f(t)\left(\int_C \frac{e^{-zt}}{z-s}\,dz\right) dt$$

$$\Rightarrow F(s) = \frac{1}{2\pi i}\int_0^\infty f(t)\left(\int_S \frac{e^{-zt}}{z-s}\,dz + \int_A^B \frac{e^{-zt}}{z-s}\,dz\right) dt \longrightarrow ③$$

Now by Jordan's Lemma

$$\int_S \frac{e^{-zt}}{z-s}\,dz = 0 \quad \text{when } R \to \infty$$

and hence

$$\int_A^B \frac{e^{-zt}}{z-s}\,dz = \int_{r+i\infty}^{r-i\infty} \frac{e^{-zt}}{z-s}\,dz$$

So by ③

$$F(s) = \frac{1}{2\pi i}\int_0^\infty f(t)\int_{r-i\infty}^{r+i\infty} \frac{e^{-zt}}{s-z}\,dz\,dt$$

$$= \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{1}{s-z}\left(\int_0^\infty f(t)\cdot e^{-zt}\,dt\right) dz$$

$$\Rightarrow F(s) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} F(z)\frac{1}{s-z}\,dz$$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} F(z)\,\mathcal{L}^{-1}\left\{\frac{1}{s-z}\right\}dz$$

$$= \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} F(z)\,e^{zt}\,dz$$

$$\Rightarrow f(t) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{st}F(s)\,ds$$

***

Remark:- By L.I.T formula

$$f(t) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{st}F(s)\,ds$$

$$\Rightarrow f(t) = \frac{1}{2\pi i}\int_C e^{st}F(s)\,ds$$

$$= \frac{1}{2\pi i}\left(\sum_j 2\pi i R_j\right)$$

$$\Rightarrow f(t) = \sum_j R_j$$

where R_j = residue of $e^{st}F(s)$ at the pole $s = s_j$

***

Example Use Inversion Integral to evaluate

$$\mathcal{L}^{-1}\left\{\frac{s^3+2s^2+1}{s^2(s^2+1)}\right\}$$

Solution Here $F(s) = \dfrac{s^3+2s^2+1}{s^2(s^2+1)}$

$F(s)$ is simple pole at $s = \pm i$ and double pole at $s = 0$

So $R_1 = \lim_{s\to i} \cancel{(s-i)} \dfrac{s^3+2s^2+1}{s^2(s+i)\cancel{(s-i)}} e^{st}$

$= \dfrac{-i-2+1}{-1(2i)} e^{it} = \dfrac{i+1}{2i} e^{it} = \dfrac{-i(i+1)}{2} e^{it}$

$= \dfrac{1-i}{2} e^{it}$

$R_2 = \lim_{s\to -i} \cancel{(s+i)} \dfrac{(s^3+2s^2+1)}{s^2\cancel{(s+i)}(s-i)} e^{st}$

$= \dfrac{i-2+1}{-1(-2i)} e^{-it} = \dfrac{i-1}{2i} e^{-it} = -i\left(\dfrac{i-1}{2}\right) e^{-it}$

$= \dfrac{1}{2}(1+i) e^{-it}$

$R_3 = \lim_{s\to 0} \dfrac{d}{ds}\left\{\cancel{(s-0)^2} \dfrac{s^3+2s^2+1}{\cancel{s^2}(s^2+1)} e^{st}\right\}$

$= \lim_{s\to 0}\left\{t e^{st} \dfrac{s^3+2s^2+1}{s^2+1} + e^{st} \cdot \dfrac{(s^2+1)(3s^2+4s)-(s^3+2s^2+1)(\ldots)}{(s^2+1)^2}\right\}$

$= t(1) + 0 = t$

So by inversion integral formula

$f(t) = R_1 + R_2 + R_3 = \frac{1}{2}(1-i)e^{it} + \frac{1}{2}(1+i)e^{-it} + t$

$= \frac{1}{2}[(1-i)(\cos t + i\sin t)] + \frac{1}{2}[(1+i)(\cos t - i\sin t)] + t$

$= \frac{1}{2}[\cos t + \cancel{i\sin t} - \cancel{i\cos t} + \sin t + \cos t - \cancel{i\sin t} + \cancel{i\cos t}] + t$

$= \cos t + \sin t + t$

* * *

Example:- Calculate ~~the~~ by Inversion Integral Method

a) $\mathcal{L}^{-1}\left\{\frac{2(s+1)}{s(s^2+1)}\right\}$, b) $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$

Solution

a) $\frac{2(s+1)}{s(s^2+1)}$ has all simple poles which are $0, -i$ and i. Residues are there be respectively R_0, R_1, R_2

$$R_0 = \lim_{s\to 0}\left\{(s-0)\frac{2(s+1)}{s(s^2+1)}\cdot e^{st}\right\} = \frac{2(0+1)}{0+1}e^0 = 2$$

$$\Rightarrow R_0 = 2$$

$$R_2 = \lim_{s\to i}\left\{(s-i)\frac{2(s+1)}{s(s-i)(s+i)}e^{st}\right\} = \frac{2(i+1)}{i(2i)}e^{+it}$$

$$= -(1+i)(\cos t + i\sin t)$$

$$= -\cos t - i\cos t - i\sin t + \sin t$$

$$R_1 = \lim_{s\to -i}\left\{(s+i)\frac{2(s+1)}{s(s+i)(s-i)}e^{st}\right\} = \frac{2(1-i)}{-i(-2i)}e^{-it}$$

$$= (i-1)e^{-it} = (i-1)(\cos t - i\sin t)$$

$$= i\cos t - \cos t + \sin t + i\sin t$$

Now by inversion integral formula

$$f(t) = R_0 + R_1 + R_2$$

$$= 2 + 2\sin t - 2\cos t$$

$$= 2(\sin t - \cos t + 1)$$

b) $\frac{1}{s^2(s+1)}$ has simple pole at $s=-1$ and double pole at $s=0$

$$R_0 = \text{Residue at } s=0 = \lim_{s\to 0}\frac{d}{ds}\left\{(s-0)^2\cdot\frac{1}{s^2(s+1)}\cdot e^{st}\right\}$$

$$= \lim_{s\to 0}\left[t\,e^{st}/_{s+1} - e^{st}\frac{1}{(s+1)^2}\right]$$

$$= t - 1$$

$$R_1 = \text{Residue at } s=-1 = \lim_{s\to -1}\left\{(s+1)\cdot\frac{e^{st}}{s^2(s+1)}\right\}$$

$$\Rightarrow R_1 = e^{-t}/1 = e^{-t}$$

So $f(t) = R_0 + R_1 = t - 1 + e^{-t}$

$$\Rightarrow f(t) = e^{-t} + t - 1$$

* Applications to Partial Differential Equations *

$$\mathcal{L}\{u(x,t)\} = U(x,s)$$

$$\mathcal{L}\{u_t(x,t)\} = sU(x,s) - u(x,0) \quad \boxed{\mathcal{L}\{f'(t)\} = sF(s) - f(0)}$$

$$\mathcal{L}\{u_x(x,t)\} = \frac{\partial}{\partial x}\{U(x,s)\}$$

$$\mathcal{L}\{u_{tt}(x,t)\} = s^2 U(x,s) - s u(x,0) - u_t(x,0)$$

$$\mathcal{L}\{u_{xx}(x,t)\} = \frac{\partial^2}{\partial x^2}\{U(x,s)\}$$

Example: Solve the problem by L.T method

$$u_{tt}(x,t) = a^2 u_{xx}(x,t) \quad , t > 0 \ \& \ x > 0$$

$$u(x,0) = 0, \ u_t(x,0) = 0, \ u(0,t) = f(t) \ \&$$

$$\lim_{x\to\infty} u(x,t) = 0$$

Solution: $u_{tt}(x,t) = a^2 u_{xx}(x,t)$

$$\mathcal{L}\{u_{tt}(x,t)\} = \mathcal{L}\{a^2 u_{xx}(x,t)\}$$

$$s^2 U(x,s) - s u(x,0) - u_t(x,0) = a^2 \frac{\partial^2}{\partial x^2}[U(x,s)]$$

$$\Rightarrow s^2 U(x,s) - 0 - 0 = a^2 \frac{\partial^2}{\partial x^2} U(x,s)$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} - \frac{s^2}{a^2}\right) U(x,s) = 0$$

Auxiliary Equation is

$$D^2 - s^2/a^2 = 0$$

$\Rightarrow D = \pm s/a$

$$\Rightarrow U(x,s) = c_1 e^{sx/a} + c_2 e^{-sx/a}$$

where c_1 and c_2 may or may not depend upon s but it is independent of x.

$$U(0,s) = c_1 + c_2$$

As $u(0,t) = f(t) \Rightarrow U(0,s) = F(s)$

So $c_1 + c_2 = F(s) \longrightarrow (1)$

Also as

$$\lim_{x\to\infty} u(x,t) = 0$$

$$\Rightarrow \lim_{x\to\infty} U(x,s) = 0$$

$$\Rightarrow \lim_{x\to\infty} \left[c_1 e^{sx/a} + c_2 e^{-\frac{sx}{a}}\right] = 0$$

$$\Rightarrow c_1 = 0$$

Then by (1) $c_2 = F(s)$

$$\Rightarrow U(x,s) = 0 + F(s) e^{-\frac{sx}{a}}$$

$$\Rightarrow U(x,s) = F(s) e^{-sx/a}$$

Applying $\mathcal{L}^{-1}$ on both sides

$$\mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\{e^{-\frac{sx}{a}} F(s)\}$$

$$\Rightarrow u(x,t) = H\left(t-\frac{x}{a}\right) f\left(t-\frac{x}{a}\right)$$

$$\Rightarrow u(x,t) = \begin{cases} f\left(t-\frac{x}{a}\right) & t > \frac{x}{a} \\ 0 & \end{cases}$$

————

Example:- Use L.T method to solve the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad 0 < x < a, \; 0 \leq t < \infty$$

with $u(0,t) = 0$, $u(1,t) = 0$, $t > a$

$u(x,0) = 1 + \sin \pi x \qquad 0 < x < a$

Solution It is a one dimensional heat problem describing conduction of heat through a rod of unit length, whose end points are maintained at zero temprature, and whose initial temprature profile is prescribed.

Given that $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

$$\Rightarrow \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \mathcal{L}\left\{\frac{\partial u}{\partial t}\right\}$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) = sU(x,s) - u(x,0)$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) = sU(x,s) - 1 - \sin \pi x$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} U(x,s) - sU(x,s) = -(1+\sin\pi x)$$

Auxiliary Equation is $D^2 - s = 0$

$$\Rightarrow D = \pm\sqrt{s}$$

$$\Rightarrow U_c = C_1 e^{\sqrt{s}x} + C_2 e^{-\sqrt{s}x}$$

Now $U_p = \frac{1}{D^2-s}\left[-(1+\sin\pi x)\right]$

$$= -\frac{1}{D^2-s} e^{0x} - \mathrm{Im}\, \frac{1}{D^2-s} e^{i\pi x}$$

$$= \frac{-1}{0^2-s} - \mathrm{Im}\left(\frac{e^{i\pi s}}{-\pi^2-s}\right)$$

$$= \frac{1}{s} + \frac{\sin\pi x}{\pi^2+s^2}$$

Therefore $U(x,s) = U_c + U_p$

$$\Rightarrow U(x,s) = C_1 e^{\sqrt{s}x} + C_2 e^{-\sqrt{s}x} + \frac{1}{s} + \frac{\sin\pi x}{\pi^2+s}$$

$U(0,s) = C_1 + C_2 + \frac{1}{s}$, $U(1,s) = C_1 e^{\sqrt{x}} + C_2 e^{-\sqrt{x}} + \frac{1}{s}$

Also $u(0,t) = 0$ | $u(1,t) = 0$

$\Rightarrow U(0,s) = 0$ | $U(1,s) = 0$

$$\Rightarrow c_1 + c_2 + \frac{1}{s} = 0 \qquad \Big\} \Rightarrow c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} + \frac{1}{s} = 0$$

$$\Rightarrow c_1 = 0 \quad , \quad c_2 = 0$$

So $U(x,s) = \frac{1}{s} + \frac{\sin \pi x}{s+\pi^2}$

$$\mathcal{L}^{-1}\{U(x,s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{\sin \pi x}{s+\pi^2}\right\}$$

$$\cancel{\Rightarrow u(x,t) = 1 + e^{-\pi^2 t} \sin \pi x}$$

$$\Rightarrow u(x,t) = 1 + e^{-\pi^2 t} \sin \pi x$$

** **

Example 3:- Use L.T method to solve

$$u_{tt}(x,t) = a^2 u_{xx}(x,t) - g$$

$$u(x,0) = u_t(x,0) = 0 \quad , \quad u(0,t) = 0$$

$$\lim_{x \to \infty} u_x(x,t) = 0$$

Solution

$$\mathcal{L}\{u_{tt}(x,t)\} = a^2 \mathcal{L}\{u_{xx}(x,t)\} - \mathcal{L}\{g\}$$

$$\Rightarrow s^2 U(x,s) - s\,u(x,0) - u_t(x,0) = a^2 \frac{\partial^2}{\partial x^2} U(x,s) - g/s$$

$$\Rightarrow s^2 U(x,s) - 0 + 0 = a^2 \frac{\partial^2}{\partial x^2} U(x,s) - g/s$$

$$\left(a^2 \frac{\partial^2}{\partial x^2} - s^2\right) U(x,s) = g/s$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} - \frac{s^2}{a^2}\right) U(x,s) = \frac{g}{sa^2}$$

Auxiliary Equation is

$$D^2 - s^2/a^2 = 0 \qquad \Rightarrow D = \pm \frac{s}{a}$$

So $U_c = c_1 e^{\frac{sx}{a}} + c_2 e^{-\frac{sx}{a}}$

$$U_p = \frac{1}{D^2 - s^2/a^2} \cdot g/sa^2 = \frac{g}{sa^2} \cdot \frac{1}{D^2 - s^2/a^2} e^{0x}$$

$$= \frac{g}{sa^2} \cdot \frac{1}{0 - s^2/a^2} = \frac{g}{sa^2} \cdot \left(\frac{-a^2}{s^2}\right) = \frac{-g}{s^3}$$

So $U(x,s) = U_c + U_p$

$$\Rightarrow U(x,s) = c_1 e^{\frac{sx}{a}} + c_2 e^{-\frac{sx}{a}} - g s^{-3}$$

$$U(0,s) = c_1 + c_2 - g s^{-3}$$

$$\Rightarrow c_1 + c_2 - g s^{-3} = 0 \longrightarrow (1) \qquad \because u(0,t) = 0 \Rightarrow U(0,s) = 0$$

Now $U_x(x,s) = \frac{s}{a} c_1 e^{\frac{sx}{a}} - \frac{s}{a} c_2 e^{-\frac{sx}{a}}$

$$\lim_{x\to\infty} U_x(x,s) = 0$$

$$\Rightarrow c_1 = 0$$

So by (1)

$$0 + c_2 - g s^{-3} = 0$$

$$\Rightarrow c_2 = g s^{-3}$$

$$\Rightarrow U(x,s) = \frac{g}{s^3} e^{-\frac{sx}{a}} - \frac{g}{s^3}$$

$$\Rightarrow u(x,t) = g\mathcal{L}^{-1}\left\{e^{-\frac{sx}{a}} \cdot \frac{1}{s^3}\right\} - g\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= \frac{g}{2!}\mathcal{L}^{-1}\left\{e^{-\frac{sx}{a}} \cdot \frac{2!}{s^3}\right\} - \frac{g}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\}$$

$$\Rightarrow u(x,t) = \frac{g}{2!} H\left(t-\frac{x}{a}\right)\left(t-\frac{x}{a}\right)^2 - \frac{g}{2!} t^2$$

$$\Rightarrow u(x,t) = \frac{g}{2} H\left(t-\frac{x}{a}\right)\left(t-\frac{x}{a}\right)^2 - \frac{g}{2} t^2$$

* *

Example 4:- Solve by L.T method

$$u_{xx} = u_{tt} \qquad 0<x<1, \; t>0$$

$$u(0,t) = 0, \quad u(1,t) = 0$$

$$u(x,0) = \sin \pi x, \quad u_t(x,0) = -\sin \pi x$$

Solution

$$\mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_{tt}\}$$

$$\frac{\partial^2}{\partial x^2} U(x,s) = s^2 U(x,s) - s u(x,0) - u_t(x,0)$$

$$= s^2 U(x,s) - s\sin \pi x + \sin \pi x$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} - s^2\right) U(x,s) = \sin \pi x - s \sin \pi x$$

A.E is $D^2 - s^2 = 0 \Rightarrow D = \pm s$

$$\Rightarrow U_c = c_1 e^{sx} + c_2 e^{-sx}$$

$$U_p = \frac{(1-s)}{D^2 - s^2} \sin \pi x = \text{Im} \frac{(1-s)}{D^2 - s^2} e^{i\pi x}$$

$$= \text{Im} \frac{(1-s)}{(i\pi)^2 - s^2} e^{i\pi x} = \frac{-(1-s) \sin \pi x}{\pi^2 + s^2}$$

$$\Rightarrow U(x,s) = U_c + U_p$$

$$= c_1 e^{sx} + c_2 e^{-sx} - \frac{(1-s) \sin \pi x}{s^2 + \pi^2}$$

$U(1,s) = c_1 e^{s} + c_2 e^{-s} = 0 \quad \because u(1,t) = 0$

$$\Rightarrow c_1 e^{s} + c_2 e^{-s} = 0 \longrightarrow ①$$

Also $u(0,t) = 0 \Rightarrow U(0,s) = 0$

$$\Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$$

So by ① $\quad -c_2 e^{s} + c_2 e^{-s} = 0$

$$\Rightarrow c_2 (e^{-s} - e^{s}) = 0 \Rightarrow c_2 = 0$$

$$\Rightarrow c_1 = 0$$

$$\Rightarrow U(x,s) = 0 + 0 - \frac{(1-s) \sin \pi x}{s^2 + \pi^2}$$

$$\Rightarrow u(x,t) = \sin \pi x \left[-\frac{1}{\pi} \mathcal{L}^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} \right]$$

$$\Rightarrow u(x,t) = \sin \pi x \left[\cos \pi t - \frac{1}{\pi} \sin \pi t \right]$$

***

Theorem:- $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\,ds \quad , F(s) = \mathcal{L}\{f(t)\}$

Proof:-

$$R.H.S = \int_s^{\infty} F(s)\,ds = \int_s^{\infty}\left\{\int_0^{\infty} e^{-st} f(t)\,dt\right\} ds$$

$$= \int_0^{\infty} f(t)\left[\int_s^{\infty} e^{-st}\,ds\right] dt$$

$$= \int_0^{\infty} f(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty} dt$$

$$= \int_0^{\infty} f(t)\left[\frac{e^{-\infty}}{-t} - \frac{e^{-st}}{-t}\right] dt$$

$$= \int_0^{\infty} e^{-st}\frac{f(t)}{t}\,dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

Proved.

Examples:-

Example 1:- Prove that

$$\mathcal{L}\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\ln\frac{s^2+b^2}{s^2+a^2}$$

Solution

$$\mathcal{L}\left\{\frac{\cos at - \cos bt}{t}\right\} = \mathcal{L}\left\{\frac{\cos at}{t}\right\} - \mathcal{L}\left\{\frac{\cos bt}{t}\right\}$$

$$\Rightarrow \mathcal{L}\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^{\infty}\frac{s}{s^2+a^2}\,ds - \int_s^{\infty}\frac{s}{s^2+b^2}\,ds$$

$$= \frac{1}{2}\left[\ln|s^2+a^2| - \ln|s^2+b^2|\right]_s^{\infty}$$

$$= \frac{1}{2}\ln\frac{s^2+a^2}{s^2+b^2}\Big|_s^{\infty} = \frac{1}{2}\left|\ln\frac{1+a^2/s^2}{1+b^2/s^2}\right|_s^{\infty}$$

$$= \frac{1}{2}\ln(1) - \frac{1}{2}\ln\frac{1+a^2/s^2}{1+b^2/s^2}$$

$$= -\frac{1}{2}\ln(s^2+a^2) + \frac{1}{2}\ln(s^2+b^2)$$

$$= \frac{1}{2}\ln\frac{s^2+b^2}{s^2+a^2}$$

Example 2:- Evaluate $\mathcal{L}\left\{\frac{1-\cos t}{t^2}\right\}$

Solution Let $f(t) = \frac{1-\cos t}{t}$

Then $\mathcal{L}\{f(t)\} = F(s) = \mathcal{L}\left\{\frac{1-\cos t}{t}\right\}$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds$$

$$= \frac{1}{2}\ln s^2 - \frac{1}{2}\ln|s^2+1|\Big|_s^\infty$$

$$= \frac{1}{2}\ln(s^2+1) - \frac{1}{2}\ln(s^2)$$

$$= \frac{1}{2}\ln\left\{\frac{s^2+1}{s^2}\right\}$$

Now $\mathcal{L}\left\{\frac{1-\cos t}{t^2}\right\} = \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{1}{2}\ln\left(\frac{s^2+1}{s^2}\right) ds$

$$= \frac{1}{2}\left[\int_s^\infty 1\cdot\ln(s^2+1)\,ds - 2\int_s^\infty 1\cdot\ln s\,ds\right]$$

$$= \frac{1}{2}\left[\ln(s^2+1)\cdot s\Big|_s^\infty - \int_s^\infty \frac{2s}{s^2+1}\cdot s\,ds - 2\left\{\ln s\cdot s\Big|_s^\infty - \int_s^\infty \frac{1}{s}\cdot s\,ds\right.\right.$$

$$= \frac{1}{2}\left[s\ln(s^2+1)\Big|_s^\infty - 2\int_s^\infty\left(1-\frac{1}{s^2+1}\right)ds - 2s\ln s\Big|_s^\infty - 2s\Big|_s^\infty\right.$$

$$= \frac{1}{2}\left[s\ln(s^2+1)\Big|_s^\infty - \cancel{2s}\Big|_s^\infty + 2\tan^{-1}s\Big|_s^\infty - 2s\ln s\Big|_s^\infty + \cancel{2s}\Big|_s^\infty\right]$$

$$= \frac{1}{2}\left[s(\ln(s^2+1) - \ln s^2)\Big|_s^\infty + 2\left[\tan^{-1}(\infty) - \tan^{-1}(s)\right]\right]$$

$$= \frac{1}{2}\left[s\left(\ln s^2 - \ln(s^2+1)\right) + 2\left(\frac{\pi}{2} - \tan^{-1}s\right)\right]$$

$$= \frac{s}{2}\ln\frac{s^2}{s^2+1} + \frac{\pi}{2} - \tan^{-1}s$$

$$= \frac{s}{2}\ln\frac{s^2}{s^2+1} + \cot^{-1}s$$

$$\Rightarrow \mathcal{L}\left\{\frac{1-\cos t}{t^2}\right\} = \frac{s}{2}\ln\frac{s^2}{s^2+1} + \cot^{-1}s$$

***

Formula:-

$$F(s) = -\int_s^\infty F'(s)\,ds$$

$$= -\mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

where $f(t) = \mathcal{L}^{-1}\{F'(s)\}$

$$F(s) = -\int_s^\infty F'(s)\,ds$$

$$= -\mathcal{L}\left\{\frac{\mathcal{L}^{-1}\{F'(s)\}}{t}\right\}$$

$$= -\mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

* *

Example:- Calculat

$$\mathcal{L}^{-1}\left\{\tan^{-1}\frac{a}{s}\right\}$$

Solution

$$\tan^{-1}\left(\frac{a}{s}\right) = -\int_s^\infty \left(\tan^{-1}\frac{a}{s}\right)' ds$$

$$= -\int_s^\infty \frac{1}{\frac{a^2+s^2}{s^2}}\cdot\left(-\frac{a}{s^2}\right) ds$$

$$= \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$$= \mathcal{L}\left\{\frac{\sin at}{t}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right)\right\} = \frac{1}{t}\sin at$$

** **

Example 2:- Calculate $\mathcal{L}^{-1}\left\{\ln \frac{s^2+1}{(s-1)^2}\right\}$

Solution

$$\ln \frac{s^2+1}{(s-1)^2} = -\int_s^\infty \left(\ln \frac{s^2+1}{(s-1)^2}\right)' ds$$

$$\Rightarrow \ln \frac{s^2+1}{(s-1)^2} = -\int_s^\infty \frac{(s-1)^2}{s^2+1} \cdot \left(\frac{s^2+1}{(s-1)^2}\right)' ds$$

$$= -\int_s^\infty \frac{(s-1)^2}{s^2+1} \cdot \frac{(s-1)^2(2s)-(s^2+1)(2)(s-1)}{(s-1)^4} ds$$

$$= -\int_s^\infty \frac{\cancel{(s-1)^2}}{s^2+1} \cdot \frac{\cancel{2s^3}-4s^2+\cancel{2s}-\cancel{2s^3}+2s^2-\cancel{2s}+2}{\cancel{(s-1)^2}(s-1)^2} ds$$

$$= -\int_s^\infty \frac{-2(s^2-1)}{(s^2+1)(s-1)^2} ds \quad = 2\int_s^\infty \frac{1}{(s-1)^2} ds$$

$$= -\int_s^\infty \frac{-2(s^2-1)}{(s^2+1)(s-1)^2} ds = 2\int_s^\infty \frac{(s-1)(s+1)}{(s^2+1)(s-1)^2} ds$$

$$= 2\int_s^\infty \frac{s+1}{(s^2+1)(s-1)} ds$$

$$= 2\int_s^\infty \left[\frac{1}{s-1} - \frac{s}{s^2+1}\right] ds = \mathcal{L}\left\{\frac{(2e^t - 2\cos t)}{t}\right\}$$

$$= \mathcal{L}\left\{\frac{2}{t}\{e^t - \cos t\}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\ln \frac{s^2+1}{(s-1)^2}\right\} = \frac{2e^t}{t} - \frac{2\cos t}{t}$$

***

Example 3:- Calculate $\mathcal{L}^{-1}\left\{\ln \frac{s^2+a^2}{s^2+b^2}\right\}$

Solution

$$\ln \frac{s^2+a^2}{s^2+b^2} = -\int_s^\infty \left[\ln \frac{s^2+a^2}{s^2+b^2}\right]' ds$$

$$\Rightarrow \ln\frac{s^2+a^2}{s^2+b^2} = -\int_s^\infty \left[\ln(s^2+a^2)-\ln(s^2+b^2)\right]' ds$$

$$= -\int_s^\infty \left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right] ds$$

$$= 2\mathcal{L}\left\{\frac{\cos bt}{t}\right\} - 2\mathcal{L}\left\{\frac{\cos at}{t}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\ln\frac{s^2+a^2}{s^2+b^2}\right\} = \frac{2\cos bt - 2\cos at}{t}$$

***

⇒ Laplace Transform of Periodic Function:- (Theorem)

If $f(t)$ is a periodic function of period $T>0$ then calculate $\mathcal{L}\{f(t)\}$

Calculation:-

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \cdots \longrightarrow (1)$$

Now Consider

$$\int_T^{2T} e^{-st} f(t)\,dt \quad \text{put } t = \lambda + T \text{ then}$$

$$dt = d\lambda$$

$$t = T \Rightarrow \lambda = 0, \quad t = 2T \Rightarrow \lambda = T$$

So
$$\int_T^{2T} e^{-st} f(t)\,dt = \int_0^T e^{-s(\lambda+T)} f(\lambda+T)\,d\lambda$$

$$= \int_0^T e^{-s\lambda}\cdot e^{-sT} f\lambda\, d\lambda \quad \because f(\lambda+T) = f(\lambda)$$

$$\Rightarrow \int_T^{2T} e^{-st} f(t)\,dt = e^{-sT}\int_0^T e^{-s\lambda} f(\lambda)\,d\lambda$$

Now Consider $\int_{2T}^{3T} e^{-st} f(t)\,dt$

Put $t = 2T + \lambda$ Then

$$\int_{2T}^{3T} e^{-st} f(t)\,dt = \int_0^T e^{-s(2T+\lambda)} f(\lambda+2T)\,d\lambda = e^{-2sT}\int_0^T e^{-\lambda s} f(\lambda)\,d\lambda$$

So
$$\mathcal{L}\{f(t)\} = \int_0^T e^{-s\lambda} f(\lambda)\,d\lambda + e^{-sT}\int_0^T e^{-s\lambda} f(\lambda)\,d\lambda + e^{-2sT}\int_0^T e^{-s\lambda} f(\lambda)\,d\lambda + \cdots\cdots$$

$$= \left[1 + e^{-sT} + e^{-2sT} + \cdots\right]\int_0^T e^{-\lambda s} f(\lambda)\,d\lambda$$

$$= \left\{1 - e^{-sT}\right\}^{-1}\int_0^T e^{-\lambda s} f(\lambda)\,d\lambda$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^T \frac{e^{-st} f(t)}{1 - e^{-sT}}\,dt$$

Question:- Prove that
$$\mathcal{L}^{-1}\left\{\frac{A}{(s-k)^n}\right\} = \frac{At^{n-1}}{(n-1)!} e^{kt}$$

Solution

$$\mathcal{L}^{-1}\left\{\frac{A}{(s-k)^n}\right\} = \mathcal{L}^{-1}\{F(s-k)\}$$

$$= e^{kt}\mathcal{L}^{-1}\{F(s)\} \quad \text{by 1st shifting theorem}$$

$$= e^{kt}\mathcal{L}^{-1}\left\{\frac{A}{s^n}\right\} = \frac{Ae^{kt}}{(n-1)!}\mathcal{L}^{-1}\left\{\frac{(n-1)!}{s^n}\right\}$$

$$= \frac{Ae^{kt}}{(n-1)!} t^{n-1}$$

Example: Solve $\frac{dx}{dt} + \frac{dy}{dt} - 4y = 1$, $x(0) = 0$

& $x + \frac{dy}{dt} - 3y = t^2$ $\quad y(0) = 0$

Solution Given

$$\dot{x} + \dot{y} - 4y = 1 \longrightarrow (1)$$

& $$x + \dot{y} - 3y = t^2 \longrightarrow (2)$$

from (1) $\mathcal{L}\{\dot{x}\} + \mathcal{L}\{\dot{y}\} - 4\mathcal{L}\{y\} = \mathcal{L}\{1\}$

$$sX(s) - x(0) + sY(s) - y(0) - 4Y(s) = 1/s$$

$$sX(s) - 0 + sY(s) - 0 - 4Y(s) = 1/s$$

$$sX(s) + (s-4)Y(s) = 1/s \longrightarrow (3)$$

From (2) $\mathcal{L}\{x\} + \mathcal{L}\{\dot{y}\} - 3\mathcal{L}\{y\} = \mathcal{L}\{t^2\}$

$$X(s) + sY(s) - y(0) - 3Y(s) = 2!/s^3$$

$$X(s) + (s-3)Y(s) = 2/s^3 \longrightarrow (4)$$

(3) − s(4) gives

$$sX(s) + (s-4)Y(s) = 1/s$$

$$\underline{-}\; sX(s) \;\underline{+}\; s(s-3)Y(s) = 2/s^2$$

$$(s-4-s^2+3s)Y(s) = \frac{1}{s} - \frac{2}{s^2}$$

$$\Rightarrow Y(s) = \frac{2}{s^2(s-2)^2} - \frac{1}{s(s-2)^2} = \frac{2-s}{s^2(s-2)^2}$$

$$= \frac{1/4}{s} + \frac{1/2}{s^2} + \frac{-1/4}{s-2}$$

$$\Rightarrow y(t) = \frac{1}{4} + \frac{1}{2}t - \frac{1}{4}e^{2t} \longrightarrow *$$

Now $(s-3)(3) - (s-4)(4)$ gives

$$[s(s-3) - (s-4)]X(s) = (s-3)\cdot\frac{1}{s} - (s-4)\cdot\frac{2}{s^3}$$

$$(s^2 - 4s + 4)X(s) = \frac{s^2(s-3) - 2(s-4)}{s^3}$$

$$(s-2)^2 X(s) = \frac{s^3-3s^2-2s+8}{s^3}$$

$$X(s) = \frac{s^3-3s^2-2s+8}{s^3(s-2)^2}$$

$$= \frac{(s-2)(s^2-s-4)}{s^3(s-2)(s-2)}$$

$$\Rightarrow X(s) = \frac{s^2-s-4}{s^3(s-2)}$$

$$= \frac{1/4}{s} + \frac{3/2}{s^2} + \frac{2}{s^3} + \frac{-1/4}{s-2}$$

$$x(t) = \frac{1}{4} + \frac{3}{2}t + t^2 - \frac{1}{4}e^{2t} \longrightarrow **$$

* & ** are required solution.

***

⇒ The Error Function:-

The error function of 'x' is denoted & defined by

$$\text{erf}\, x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\lambda^2} d\lambda$$

Inversion Formula:- (Related to Error funct.)

We have the following inversion formula related to error function

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = 1 - \text{erf}\left(\frac{2a}{\sqrt{t}}\right)$$

which is equivalent to

$$\mathcal{L}\left\{\text{erf}\, t^{1/2}\right\} = \frac{1}{s}\left\{1 - e^{-\sqrt{s}/2}\right\}$$

Question: Calculate $\mathcal{L}\{\operatorname{erf} t\}$

Solution

$$\operatorname{erf} t = \frac{2}{\sqrt{\pi}}\int_0^t e^{-\lambda^2}\,d\lambda$$

$$= \frac{2}{\sqrt{\pi}}\int_0^t \left[1+(-\lambda^2)+\frac{(-\lambda^2)^2}{2!}+\frac{(-\lambda^2)^3}{3!}+\cdots\right]d\lambda$$

$$= \frac{2}{\sqrt{\pi}}\int_0^t \left[1-\lambda^2+\frac{\lambda^4}{2}-\frac{\lambda^6}{6}+\cdots\right]d\lambda$$

$$= \frac{2}{\sqrt{\pi}}\left[\lambda-\lambda^3/3+\frac{\lambda^5}{10}-\frac{\lambda^7}{42}+\cdots\right]_0^t$$

$$= \frac{2}{\sqrt{\pi}}\left[t-t^3/3+\frac{t^5}{10}-\frac{t^7}{42}+\cdots\right]$$

$$\mathcal{L}\{\operatorname{erf} t\} = \frac{2}{\sqrt{\pi}}\left\{\frac{1}{s^2}-\frac{1}{3}\left(\frac{3!}{s^4}\right)+\frac{1}{10}\left(\frac{5!}{s^6}\right)-\frac{1}{42}\left(\frac{7!}{s^8}\right)+\cdots\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{1}{s^2}-\frac{2}{s^4}+\frac{12}{s^6}-\frac{120}{s^8}+\cdots\right]$$

Question: Calculate

$$\mathcal{L}\{e^{3t}\operatorname{erf}\sqrt{t}\}$$

Solution

$$\operatorname{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-\lambda^2}\,d\lambda$$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left[1-\lambda^2+\frac{\lambda^4}{2}-\frac{\lambda^6}{6}+\frac{\lambda^8}{24}-\cdots\right]d\lambda$$

$$= \frac{2}{\sqrt{\pi}}\left[\lambda-\frac{\lambda^3}{3}+\frac{\lambda^5}{10}-\frac{\lambda^7}{42}+\cdots\right]_0^{\sqrt{t}}d\lambda$$

$$= \frac{2}{\sqrt{\pi}}\left[t^{1/2} - \frac{1}{3}t^{3/2} + \frac{1}{10}t^{5/2} - \frac{1}{42}t^{7/2} + \cdots\right]$$

$$\mathcal{L}\{e^{3t}\operatorname{erf}\sqrt{t}\} = \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma\frac{3}{2}}{s^{3/2}} - \frac{1}{3}\frac{\Gamma\frac{5}{2}}{s^{5/2}} + \frac{1}{10}\frac{\Gamma\frac{7}{2}}{s^{7/2}} - \frac{1}{42}\frac{\Gamma\frac{9}{2}}{s^{9/2}} + \cdots\right]_{s \to s-3}$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}} - \frac{1}{3}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{5/2}} + \frac{1}{10}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{7/2}} - \frac{1}{42}\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{9/2}} + \cdots\right]_{s \to s-3}$$

$$= \frac{2}{\sqrt{\pi}}\cdot\frac{1}{2}\Gamma\frac{1}{2}\left[\frac{1}{s^{3/2}} - \frac{1}{3}\cdot\frac{3}{2}\frac{1}{s^{5/2}} + \frac{1}{10}\cdot\frac{15}{4}\cdot\frac{1}{s^{7/2}} - \frac{1}{42}\cdot\frac{105}{8}\cdot\frac{1}{s^{9/2}}\right]_{s \to s-3}$$

$$= \frac{1}{s^{3/2}}\left[1 - \frac{1}{2s} + \frac{3}{8s^2} - \frac{5}{16s^3} + \cdots\right]_{s \to s-3}$$

$$= \left[\frac{1}{s^{3/2}}\left[1 + \frac{1}{s}\right]^{-\frac{1}{2}}\right]_{s \to s-3}$$

$$= \left[\frac{1}{s^{3/2}}\left\{\frac{(s+1)^{-1/2}}{s^{-1/2}}\right\}\right]_{s \to s-3}$$

$$\Rightarrow \mathcal{L}\{e^{3t}\operatorname{erf}\sqrt{t}\} = \left.\frac{1}{s\sqrt{s+1}}\right|_{s \to s-3}$$

$$\Rightarrow \mathcal{L}\{e^{3t}\operatorname{erf}\sqrt{t}\} = \frac{1}{(s-3)\sqrt{s-2}}$$

★★ ——— ★★★ ——— ★★

M. TAHIR M.S. MATH COMSATS

THE Fourier Transforms And Its Applications

⇒ Definition:-

Fourier Transform of a function $f(x)$, if it exists, is denoted and defined by

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx$$

In this case $f(x)$ is called the inverse Fourier Transform of $F(k)$ and it is defined as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F(k)\, dk$$

⇒ Definition:-

If $F(k)$ is the F.T of a function $f(x)$ and $f(x)$ is the Inverse Fourier Transform of $F(k)$ with

$$F(k) = C_1 \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx$$

and

$$f(x) = C_2 \int_{-\infty}^{\infty} e^{-ikx} F(k)\, dk$$

Then C_1 and C_2 be related as follows for each case

(i) $C_1 = \frac{1}{\sqrt{2\pi}}$ $\qquad$ $C_2 = \frac{1}{\sqrt{2\pi}}$

(ii) $C_1 = 1$ $\qquad$ $C_2 = 1/2\pi$

(iii) $C_1 = \frac{1}{2\pi}$ $\qquad$ $C_2 = 1$

Fourier and its inverse transforms can also be related by the following definitions

$$F(K) = \int_{-\infty}^{\infty} e^{2\pi i k x} f(x)\, dx$$

and

$$f(x) = \int_{-\infty}^{\infty} e^{-2\pi i k x} f(K)\, dK$$

<u>Notations</u>:- The pairs (x, K), (x, P), (x, ξ) are frequently also used by various authors. F.T of $f(x)$ is also denoted by $F\{f(x)\}$ & $\bar{f}(K)$

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★ Properties :-

1) Linearity Property:- Fourier Transform is Linear

Proof

$$F\{\alpha_1 f_1(x) + \alpha_2 f_2(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} (\alpha_1 f_1(x) + \alpha_2 f_2(x)) dx$$

$$\Rightarrow F\{\alpha_1 f_1(x) + \alpha_2 f_2(x)\} = \alpha_1 \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f_1(x) dx\right] + \alpha_2 \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f_2(x) dx\right]$$

$$= \alpha_1 F\{f_1(x)\} + \alpha_2 F\{f_2(x)\}$$

2) Conjugation Property:-

If $f(x)$ is real then $F(-K) = \bar{F}(K)$

Proof

$$F(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx$$

$$\Rightarrow \bar{F}(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx \qquad \because f \text{ is real}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(-K)x} f(x)\, dx$$

$$\Rightarrow \bar{F}(K) = F(-K)$$

3) Real & Complex Values of F.T

a) If $f(x)$ is real ~~then~~ and even then $F(k)$ is real.

b) If $f(x)$ is real and odd then $F(k)$ is Pure imaginary.

c) If $f(x)$ is complex then

$$F\{\bar{f}(-x)\} = \bar{F}(k)$$

Proof: a)

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

$$\bar{F}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx \quad \because f \text{ is real}$$

Put $x = -x' \Rightarrow dx = -dx'$

$$\Rightarrow \bar{F}(k) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-ik(-x')} f(-x')(-dx')$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'} f(x')\,dx' \quad \because f \text{ is even}$$

$$= F(k) \qquad \Rightarrow \bar{F}(k) = F(k)$$

$\Rightarrow F(k)$ is real

b)

$$\bar{F}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{e^{ikx}} f(x)\,dx \quad \because f \text{ is real}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$$

Put $x = -x' \Rightarrow dx = -dx'$

$$\text{So } \bar{F}(k) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-ik(-x')} f(-x')(-dx')$$

$$\bar{F}(k) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx'} [-f(x')](-dx') \quad \because f \text{ is odd}$$

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx'} f(x') dx' = -F(k)$$

$$\Rightarrow \bar{F}(k) = -F(k)$$

$\Rightarrow F(k)$ is Pure imaginary

c) $$\bar{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \bar{f}(x) dx$$

Put $x = -x' \Rightarrow dx = -dx'$

So $$\bar{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{-ik(-x')} \bar{f}(-x')(-dx')$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx'} \bar{f}(-x') dx'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \bar{f}(-x) dx$$

$$\Rightarrow \bar{F}(k) = F\{\bar{f}(-x)\}$$

4) Attenuation Property:-

$$F\{e^{ax} f(x)\} = F(k - ia)$$

Proof:-

$$F\{e^{ax} f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} e^{ax} f(x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx + ax} f(x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(k - ia)x} f(x) dx$$

$$\Rightarrow F\{e^{ax} f(x)\} = F(k - ia)$$

5) Shifting Property:-

$$F\{f(x-a)\} = e^{ika} F(k)$$

Proof:-

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x-a)\, dx$$

Put $x - a = t \Rightarrow dx = dt$

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(t+a)} f(t)\, dt$$

$$= e^{ika} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikt} f(t)\, dt$$

$$= e^{ika} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx$$

$$\Rightarrow F\{f(x-a)\} = e^{ika} F(k)$$

6) Scaling Property:-

If c is non zero constant then

$$F\{f(cx)\} = \frac{1}{|c|} F(k/c)$$

Proof:-

Case I:- If $c > 0$

$$F\{f(cx)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(cx)\, dx$$

Put $cx = x' \Rightarrow c\,dx = dx'$

$x = -\infty \Rightarrow x' = -\infty$

$x = \infty \Rightarrow x' = \infty$

$$\Rightarrow F\{f(cx)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x'/c)} f(x') \cdot \frac{1}{c} dx'$$

$$= \frac{1}{c} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(k/c)x'} f(x')\, dx' \right.$$

$$= \frac{1}{c} F(k/c)$$

Case II :- If $c < 0$

$$F\{f(cx)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(cx)\,dx$$

put $cx = x'$ $\Rightarrow x = \frac{1}{c} x'$ $\Rightarrow dx = \frac{1}{c} dx'$

$x = \infty \Rightarrow x' = -\infty$ $\because c < 0$

$x = -\infty \Rightarrow x' = \infty$ $\because c < 0$

$$\Rightarrow F\{f(cx)\} = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{ik(x'/c)} f(x') \frac{1}{c} dx'$$

$$= \frac{1}{-c} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(k/c)x'} f(x')\,dx' \right]$$

$$= \frac{1}{-c} F(k/c)$$

~~bo~~ combining both cases $F\{f(cx)\} = \frac{1}{|c|} F(k/c)$

7) Modulation Property :-

a) $F\{\cos\alpha x\, f(x)\} = \frac{1}{2}[F(k+\alpha) + F(k-\alpha)]$

b) $F\{\sin\alpha x\, f(x)\} = \frac{1}{2i}[F(k+\alpha) - F(k-\alpha)]$

proof)

a) $F\{\cos\alpha x\, f(x)\} = F\left\{\left(\frac{e^{i\alpha x} + e^{-i\alpha x}}{2}\right) f(x)\right\}$

$$= \frac{1}{2}\left[F\{e^{i\alpha x} f(x)\} + F\{e^{-i\alpha x} f(x)\}\right]$$

$$= \frac{1}{2}[F(k+\alpha) + F(k-\alpha)]$$

b) $F\{\sin\alpha x\, f(x)\} = F\left\{\left(\frac{e^{i\alpha x} - e^{-i\alpha x}}{2i}\right) f(x)\right\}$

$$= \frac{1}{2i}\left[F\{e^{i\alpha x} f(x)\} - F\{e^{-i\alpha x} f(x)\}\right]$$

$$= \frac{1}{2i}[F(k+\alpha) - F(k-\alpha)]$$

***

Theorem:- The Fourier Transform of $f(x)$ exists if f is absolutely integrable over $]-\infty, \infty[$

Proof:- If f is absolutely integrable on $]-\infty, \infty[$, Then $\int_{-\infty}^{\infty} |f(x)|\, dx$ exist

Now $F\{f(x)\} = \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx$

Now $\int_{-\infty}^{\infty} |e^{ikx} f(x)|\, dx = \int_{-\infty}^{\infty} |e^{ikx}||f(x)|\, dx$

$$= \int_{-\infty}^{\infty} (1)|f(x)|\, dx \quad \because |e^{ikx}| = 1$$

$$= \int_{-\infty}^{\infty} |f(x)|\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} |e^{ikx} f(x)|\, dx = \int_{-\infty}^{\infty} |f(x)|\, dx$$

since integral on R.H.S exists. So integral on L.H.S also exists. So $e^{ikx} f(x)$ is absolutely integrable on $]-\infty, \infty[$ and is therefore integrable on $]-\infty, \infty[$. i.e

$$\int_{-\infty}^{\infty} e^{ikx} f(x)\, dx \quad \text{exists}$$

$$\Rightarrow F\{f(x)\} \text{ exists.}$$

***

* Fourier Transform of Some Simple Functions **

Example 1:- Evaluate F.T of Gaussian function defined by $g(x) = Ne^{-\alpha x^2}$ where N and α are constants and $\alpha > 0$

Solution:-

$$F\{g(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, g(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} . Ne^{-\alpha x^2} dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx - \alpha x^2} dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[x^2 - \frac{ik}{\alpha} x\right]} dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[x^2 - 2\left(\frac{ik}{2\alpha}\right)x + \left(\frac{ik}{2\alpha}\right)^2 - \left(\frac{ik}{2\alpha}\right)^2\right]} dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[\left(x - \frac{ik}{2\alpha}\right)^2 + k^2/4\alpha^2\right]} dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left[x - \frac{ik}{2\alpha}\right]^2} . e^{-\frac{k^2}{4\alpha}} dx$$

$$= \frac{N}{\sqrt{2\pi}} e^{-k^2/4\alpha} \int_{-\infty}^{\infty} e^{-\left[\sqrt{\alpha}\left(x - \frac{ik}{2\alpha}\right)\right]^2} dx$$

Put $\sqrt{\alpha}\left(x - \frac{ik}{2\alpha}\right) = y$

$\Rightarrow \sqrt{\alpha}\, dx = dy \Rightarrow dx = \frac{1}{\sqrt{\alpha}} dy$

So $F\{g(x)\} = \frac{N}{\sqrt{2\pi}} e^{-k^2/4\alpha} \int_{-\infty}^{\infty} e^{-y^2} \frac{1}{\sqrt{\alpha}} dy$

$$= \frac{N}{\sqrt{2\alpha\cancel{\pi}}} e^{-k^2/4\alpha} \sqrt{\cancel{\pi}} \qquad \because \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$$

$$\Rightarrow F\{g(x)\} = \frac{N}{\sqrt{2\alpha}} e^{-k^2/4\alpha}$$

***

Example 2:- Calculate F.T of

$$g(x) = \frac{a}{x^2+a^2}, \quad a>0$$

Solution

$$F\{g(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} g(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \cdot \frac{a}{x^2+a^2} dx$$

clearly the integral on the R.H.S leads to the contour Integration. So we use Cauchy's Residues Theorem to evaluate the integral.

Here arises three cases.

Case I:- If $k=0$

Then $F\{g(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{0} \frac{a}{x^2+a^2} dx$

$$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{x^2+a^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \tan^{-1}\left(\frac{x}{a}\right)\Big|_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \sqrt{\frac{\pi}{2}}$$

Case II:- If $K > 0$

If $K > 0$ then singularities lies on the upper half plane.

The singularities are at where $x^2 + a^2 = 0$

$\Rightarrow x^2 = -a^2 \quad \Rightarrow \quad x = \pm ia$

Since $a > 0$, so the only singularity which lie on the upper half plane is ai

Residue at $x = ai$

$$= \lim_{x \to ai} (x - ai) \cdot \frac{a e^{ikx}}{(x - ai)(x + ai)}$$

$$= \frac{e^{-ka}}{2i}$$

So $F\{g(x)\} = \frac{1}{\sqrt{2\pi}} \cdot 2\pi i \sum R$ (Query ?)

$$= \sqrt{2\pi} \cdot i \cdot \frac{e^{-ka}}{2i} = \sqrt{\frac{\pi}{2}}\, e^{-ka}$$

Case III:- If $K < 0$

Then singularities lie in lower half plane. But the only singularity which lies on lower half plane is $x = -ai$

Residue at $x = -ai$

$$= \lim_{x \to -ai} (x + ai) \cdot \frac{a e^{ikx}}{(x - ai)(x + ai)}$$

$$= \frac{e^{ka}}{-2i}$$

So $F\{g(x)\} = \frac{1}{\sqrt{2\pi}} \cdot -2\pi i \cdot \frac{e^{ka}}{-2i} = \sqrt{\frac{\pi}{2}}\, e^{ka}$

Combining all the above cases we have

$$F\{g(x)\} = \sqrt{\frac{\pi}{2}}\, e^{-|K| a} \quad \text{for all } K.$$

* *

$\because \sqrt{\frac{\pi}{2}}\, e^{-|k| a} = \sqrt{\frac{\pi}{2}}$ if $k = 0$; $\sqrt{\frac{\pi}{2}}\, e^{-ka} = \frac{\pi}{2} e^{-|k|a} \quad \because k > 0$; $\sqrt{\frac{\pi}{2}}\, e^{ka} = \frac{\pi}{2} e^{-|k|a} \quad \because k < 0$

Example 3:- Find F.T of box function

$$f(x) = \begin{cases} 0 & |x| > a \quad , a > 0 \\ 1 & |x| \leq a \end{cases}$$

Solution Given

$$f(x) = \begin{cases} 1 & -a \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

$$F(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} e^{ikx} f(x) dx + \int_{-a}^{a} e^{ikx} f(x) dx + \int_{a}^{\infty} e^{ikx} f(x)\, dx \right]$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{ikx} \cdot 1 \cdot dx + 0$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{ikx}}{iK} \Big|_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{iK} \left[e^{iKa} - e^{-iKa} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{K} \left[\frac{e^{iKa} - e^{-iKa}}{2i} \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\sin Ka}{K}$$

Example 4:- Show that $F\{x^n f(x)\} = (-i)^n \frac{d^n}{dK^n}(F(K))$ ★★

Solution

$$F(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx$$

$$\frac{d}{dK}(F(K)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ix) e^{ikx} f(x)\, dx = i F\{x f(x)\}$$

$$\Rightarrow \frac{1}{i}\frac{d}{dK}(F(K)) = F\{x f(x)\}$$

$$\Rightarrow (-i)\frac{1}{}\frac{d}{dK}(F(K)) = F\{x f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} x f(x)\,dx$$

$$\Rightarrow (-i)\frac{1}{}\frac{d^2}{dK^2}(F(K)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (ix)\, e^{ikx} x f(x)\,dx$$

$$\Rightarrow (-i)^2\frac{d^2}{dK^2}(F(K)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}(x^2 f(x))\,dx$$

$$\Rightarrow (-i)^2\frac{d^2}{dK^2}(F(K)) = F\{x^2 f(x)\}$$

Continuing this process and differentiating upto n times we get

$$F\{x^n f(x)\} = (-i)^n \frac{d^n}{dK^n}(F(K))$$

Example 5:- Find Fourier Transform of

$$f(x) = e^{-\lambda x^2}\cos\beta x \quad (\lambda > 0)$$

Solution

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, e^{-\lambda x^2}\cos\beta x\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\lambda\left[x^2 - \frac{ik}{\lambda}x\right]}\cos\beta x\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\, Re\int_{-\infty}^{\infty} e^{-\lambda\left[x^2 - \frac{ik}{\lambda}x\right]}\cdot e^{i\beta x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\, Re\int_{-\infty}^{\infty} e^{-\lambda\left[x^2 - i\left(\frac{\beta+K}{\lambda}\right)x\right]}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\, Re\int_{-\infty}^{\infty} e^{-\lambda\left[x^2 + 2\left(\frac{\beta+K}{2i\lambda}\right)x\right]}\,dx$$

$$\Rightarrow F\{f(x)\} = \frac{1}{\sqrt{2\pi}} Re \int_{-\infty}^{\infty} e^{-\lambda\left[x+\frac{\beta+K}{2i\lambda}\right]^2 + \lambda\left[\frac{\beta+K}{2i\lambda}\right]^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} Re\, e^{-\lambda\left(\frac{\beta+K}{2\lambda}\right)^2} \int_{-\infty}^{\infty} e^{-\lambda\left[x+\frac{\beta+K}{2i\lambda}\right]^2} dx$$

Put $\sqrt{\lambda}\left[x+\frac{\beta+K}{2i\lambda}\right] = y$

$$\Rightarrow dx = \frac{1}{\sqrt{\lambda}} dy$$

So $$F\{f(x)\} = \frac{e^{-\lambda\left[\frac{\beta+K}{2\lambda}\right]^2}}{\sqrt{2\pi\lambda}} Re \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{e^{-\frac{(\beta+K)^2}{4\lambda}}}{\sqrt{2\lambda\pi}} Re(\sqrt{\pi})$$

Hence $$F\{f(x)\} = \frac{1}{\sqrt{2\lambda}} e^{-\frac{(\beta+K)^2}{4\lambda}}$$

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⇒ Boundedness And Continuity**

Theorem:- If $f(x)$ is piecewise smooth and absolutely integrable on the interval $]-\infty, \infty[$, Then its F.T $F(K)$ is bounded and continuous.

Proof:- Since $f(x)$ is absolutely integrable on $]-\infty, \infty[$ so

$$\int_{-\infty}^{\infty} |f(x)|\, dx \text{ exists}$$

Let $$\int_{-\infty}^{\infty} |f(x)|\, dx = \lambda$$

Now by definition

$$F(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx$$

$$|F(K)| = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx \right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{ikx}| |f(x)|\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \lambda$$

$$\Rightarrow |F(K)| \leq \frac{\lambda}{\sqrt{2\pi}}$$

$\Rightarrow F(K)$ is bounded

Now

$$\lim_{h \to 0} F(K+h) = \lim_{h \to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(K+h)x} f(x)\, dx$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} e^{ihx} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \left(\lim_{h \to 0} e^{ihx} \right) f(x)\, dx$$

[$\because h$ & x are indipendent]

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} (1) f(x)\, dx$$

$$\Rightarrow \lim_{h \to 0} F(K+h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx$$

$$\Rightarrow \lim_{h \to 0} F(K+h) = F(K)$$

$\Rightarrow F(K)$ is continuous.

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⇒ Fourier Transforms of Derivatives.

Theorem:- If $f(x), f'(x), f''(x), \dots, f^{(n-1)}(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$, Then

$$F\{f^{(n)}(x)\} = (-ik)^n F(k)$$

Proof:-

We prove the theorem by principle of Mathematical Induction

For $n=1$

$$F\{f'(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f'(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[e^{ikx} f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(ik) e^{ikx} f(x)\,dx \right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 - 0 - \int_{-\infty}^{\infty}(ik) e^{ikx} f(x)\,dx \right]$$

$$= (-ik)^1 \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

$$\Rightarrow F\{f'(x)\} = (-ik)^1 F(k)$$

So Case-I is satisfied

Now assume theorem is true for $n=m$ i.e

$$F\{f^{(m)}(x)\} = (-ik)^m F(k) \longrightarrow (1)$$

Now we prove the theorem for $n=k+1$

$$F\{f^{m+1}(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f^{(m+1)}(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[e^{ikx} f^{(m)}(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(ik) e^{ikx} f^{(m)}(x)\,dx \right]$$

$$= (-ik) F\{f^{m}(x)\} = (-ik)(-ik)^m F(k) \quad \text{by (1)}$$

$\Rightarrow F\{f^{m+1}(x)\} = (-ik)^{m+1} F(k)$

$\Rightarrow$ C-2 is satisfied.

So theorem is true for all n.

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⇒ Convolution:-

The convolution of two functions $f(x)$ and $g(x)$, defined over $]-\infty,\infty[$ is denoted and defined by

$$f*g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\eta)\,g(x-\eta)\,d\eta$$

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Convolution Theorem:- If $F(k)$ and $G(k)$ are Fourier Transforms of $f(x)$ and $g(x)$, then

$$F\{f*g\} = F(k)\,G(k)$$

or $$F^{-1}\{F(k)G(k)\} = f*g$$

Proof:

We prove $F^{-1}\{F(k)G(k)\} = f*g$

$$F^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)G(k)\,dk$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'} g(x')\,dx'\right)dk$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ik(x-x')} F(k)\,dk\right\} g(x')\,dx'$$

(Interchanging order of integral)

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-x')\,g(x')\,dx' = g*f$$

$\Rightarrow F^{-1}\{F(k)G(k)\} = f*g$

———

⇒ Parseval's Theorem:-

(French Mathematician Chenes Parseval 1755 — 1836)

Parseval's 1st and 2nd theorems are given by

$$\int_{-\infty}^{\infty} |F(K)|^2 dK = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

and

$$\int_{-\infty}^{\infty} F(K)\, G(K)\, dK = \int_{-\infty}^{\infty} f(x)\, g(-x)\, dx$$

Proof)

First we prove Parsvel's 2nd theorem. 1st theorem is a special case of 2nd theorem.

By Definition of Convolution theorem

$$F^{-1}\{F(K)G(K)\} = f(x) * g(x)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F(K)G(K)\, dK = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\eta)\cdot g(x-\eta)\, d\eta$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-ikx} F(K)\, G(K)\, dK = \int_{-\infty}^{\infty} f(\eta)\, g(x-\eta)\, d\eta$$

put $x = 0$

$$\Rightarrow \int_{-\infty}^{\infty} e^{0} F(K)\, G(K)\, dK = \int_{-\infty}^{\infty} f(\eta)\, g(0-\eta)\, d\eta$$

$$\Rightarrow \int_{-\infty}^{\infty} F(K)\, G(K)\, dK = \int_{-\infty}^{\infty} f(\eta)\, g(-\eta)\, d\eta$$

$$\Rightarrow \int_{-\infty}^{\infty} F(K)\, G(K)\, dK = \int_{-\infty}^{\infty} f(x)\, g(-x)\, dx$$

which is required Parseval's 2nd Theorem

Now Take $g(x) = \bar{f}(-x)$

$$\Rightarrow g(-x) = \bar{f}(x)$$

$$F\{g(x)\} = F\{\bar{f}(-x)\}$$

$$\Rightarrow G(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} \bar{f}(-x)\, dx$$

put $x = -x'$, $dx = -dx'$

$x = \infty \Rightarrow x' = -\infty$

$x = -\infty \Rightarrow x' = \infty$

So
$$G(K) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{-iKx'} \bar{f}(x')(-dx')$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iKx'} \bar{f}(x')\, dx'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iKx} \bar{f}(x)\, dx$$

$$= \bar{F}(K)$$

So
$$\int_{-\infty}^{\infty} F(K)\, G(K)\, dK = \int_{-\infty}^{\infty} f(x)\, g(-x)\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} F(K)\, \bar{F}(K)\, dK = \int_{-\infty}^{\infty} f(x)\, \bar{f}(x)\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} |F(K)|^2\, dK = \int_{-\infty}^{\infty} |f(x)|^2\, dx \qquad \because z\bar{z} = |z|^2$$

Proved.

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⇒ Rieman Lebesgue Theorem:-

If $f(x)$ is piecewise smooth and absolutely integrable function, then

$$\lim_{|K|\to\infty} F(K) = 0$$

Proof

By Definition

$$F(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[f(x) \cdot \frac{e^{iKx}}{iK} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) \frac{e^{iKx}}{iK} dx \right]$$

$$|F(K)| \leq \frac{1}{\sqrt{2\pi}} \left[\lim_{x\to\infty} \frac{|f(x)|}{|iK|} |e^{iKx}| - \lim_{x\to-\infty} \frac{|f(x)|}{|iK|} |e^{iKx}| - \int_{-\infty}^{\infty} |f'(x)| \frac{|e^{iKx}|}{|iK|} dx \right]$$

Since $f(x)$ is absolutely integrable

So $\lim_{|x|\to\infty} |f(x)| = 0$

$$\Rightarrow |F(K)| \leq \frac{1}{\sqrt{2\pi}} \frac{1}{|K|} \int_{-\infty}^{\infty} |f'(x)| dx$$

$$\Rightarrow \lim_{|K|\to\infty} |F(K)| \leq 0$$

$$\Rightarrow \lim_{|K|\to\infty} |f(K)| = 0$$

★★ ★★

⇒ Fourier Integral Theorem:-

If $f(x)$ is a real function defined over $]-\infty,\infty[$ and the integral $\int_{-\infty}^{\infty} f(x)dx$ is absolutely convergent, Then

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(x')\cos k(x-x')dx'dk$$

Proof

Since $\int_{-\infty}^{\infty} f(x)dx$ is absolutely convergent so F.T of $f(x)$ and correspondind Inverse F.T exists. So by Definition of Inverse F.T

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}F(k)dk$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{-ikx}F(k)dk + \int_0^{\infty} e^{-ikx}F(k)dk\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{\infty}^{0} e^{ik'x}F(-k')(-dk') + \int_0^{\infty} e^{-ikx}F(k)dk\right]$$ Since by Putting $k=-k'$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{ik'x}F(-k')dk' + \int_0^{\infty} e^{-ikx}F(k)dk\right]$$

Since $f(x)$ is real, so by conjugation property

$$F(-k') = \bar{F}(k') = \bar{F}(k)$$

So

$$f(x) = \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{ik'x}\bar{F}(k')dk' + \int_0^{\infty} e^{-ikx}F(k)dk\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{ikx}\bar{F}(k)dk + \int_0^{\infty} e^{-ikx}F(k)dk\right]$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\left[e^{ikx}\bar{F}(k) + e^{-ikx}F(k)\right]dk \longrightarrow (1)$$

Now $$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'}f(x')dx'$$

$$\Rightarrow e^{-ikx}F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\cdot e^{ikx'}f(x')dx'$$

$$e^{-ikx} F(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(x-x')} f(x')\,dx'$$

$$\Rightarrow e^{ikx} \bar{F}(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x-x')} f(x')\,dx' \quad \because f \text{ is real}$$

So by eqn ①

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x-x')} f(x')\,dx' + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(x-x')} f(x')\,dx' \right] dK$$

$$= \frac{1}{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \left[e^{ik(x-x')} + e^{-ik(x-x')} \right] f(x')\,dx'\,dK$$

$$= \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \left[\frac{e^{ik(x-x')} + e^{-ik(x-x')}}{2} \right] f(x')\,dx'\,dK$$

$$= \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \cos K(x-x')\, f(x')\,dx'\,dK$$

$$\Rightarrow f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(x') \cos K(x-x')\,dx'\,dK$$

***

⇒ Fourier Sine & Cosine Transforms:-

If a function $f(x)$ is defined over the interval $[0, \infty[$, Then we can define Fourier Sine and Fourier Cosine Transforms

$$F_s(K) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin Kx\,dx$$

and $F_c(K) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos Kx\,dx$ respectively

The corresponding Inverse Fourier sine and Inverse Fourier Cosine Transforms are

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(K) \sin Kx \, dK$$

and $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(K) \cos Kx \, dK .$$

* Justification For The Definition:-

The above definitions are directly followed from the definition of complex or exponential Fourier Transform. Let a real valued function $f(x)$ be defined over $[0, \infty[$.

Now define a function

$$f_c(x) = \begin{cases} f(x) & \text{for } 0 \leq x < \infty \\ f(-x) & \text{for } -\infty < x \leq 0 \end{cases}$$

Then $f_c(x)$ is the extension of $f(x)$. Now

$$F\{f_c(x)\} = F_c(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iKx} f_c(x) \, dx$$

$$\Rightarrow F_c(K) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{iKx} f_c(x) \, dx + \int_0^\infty e^{iKx} f_c(x) \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{iKx} f(-x) \, dx + \int_0^\infty e^{iKx} f(x) \, dx \right]$$

Put $x = -x'$ in 1st integral

$$\Rightarrow F_c(K) = \frac{1}{\sqrt{2\pi}} \left[\int_{\infty}^{0} e^{-iKx'} f(x') \, d(-x') + \int_0^\infty e^{iKx} f(x) \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^\infty e^{-iKx} f(x) \, dx + \int_0^\infty e^{iKx} f(x) \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2 \int_0^\infty \left(\frac{e^{iKx} + e^{-iKx}}{2} \right) f(x) \, dx \right]$$

$$\Rightarrow F_c(K) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos Kx \, dx$$

⇒ Fourier Sine & Cosine Transforms of Derivatives:-

1) If $f(x)$ is real valued and $|f(x)| \to 0$ as $x \to \infty$, Then

$$F_c\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\cos kx \, f(x) \Big|_0^\infty - \int_0^\infty f(x)(-k \sin kx) dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-f(0) + k \int_0^\infty \sin kx \, f(x) \, dx \right]$$

$$= -\sqrt{\frac{2}{\pi}} f(0) + k F_s(k)$$

2) $$F_s\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\sin kx \, f(x) \Big|_0^\infty - \int_0^\infty k \cos kx \, f(x) dx \right]$$

$$= -k \sqrt{\frac{2}{\pi}} \int_0^\infty \cos kx \, f(x) \, dx$$

$$\Rightarrow F_s\{f'(x)\} = -k\sqrt{\frac{2}{\pi}} F_c(k)$$

3) If $f(x), f'(x) \to 0$ as $x \to \infty$, Then

$$F_s\{f''(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx \, f''(x) dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \sin kx \, f'(x) \Big|_0^\infty - \int_0^\infty k \cos kx \, f'(x) dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - k \left\{ \cos kx \, f(x) \Big|_0^\infty - \int_0^\infty (-k \sin kx) f(x) dx \right\} \right]$$

$$= k\sqrt{\frac{2}{\pi}} f(0) - k^2 \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx \, f(x) \, dx$$

$$\Rightarrow F_s\{f''(x)\} = k\sqrt{\frac{2}{\pi}}\, f(0) - k^2 F_s(k)$$

4) $$F_c\{f''(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f''(x)\cos kx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\cos kx\, f'(x)\Big|_0^\infty - \int_0^\infty (-k\sin kx)\, f'(x)\, dx\right]$$

$$= -\sqrt{\frac{2}{\pi}}\, f'(0) + \sqrt{\frac{2}{\pi}}\, k\left[\sin kx\, f(x)\Big|_0^\infty - \int_0^\infty k\cos kx\, \cancel{\frac{f(x)}{dx}}\right]$$

$$= -\sqrt{\frac{2}{\pi}}\, f'(0) - \sqrt{\frac{2}{\pi}}\, k^2\int_0^\infty \cos kx\, f(x)\, dx$$

$$= -\sqrt{\frac{2}{\pi}}\, f'(0) - k^2 F_c(k)$$

***

Examples

Example 1:- Calculate $F_s(k)$ s.t $f(x) = e^{-x}\cos x$

Solution

$$F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin kx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos x\sin kx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{1}{2}\int_0^\infty e^{-x}(2\sin kx\cos x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x}\{\sin(k+1)x + \sin(k-1)x\}\, dx$$

Using $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\{a\sin bx - b\cos bx\}$

$a = -1,\ b = k+1$

$$F_s(k) = \frac{1}{\sqrt{2\pi}}\left[e^{-x}\left\{\frac{1}{(k+1)^2+1}\left(-\sin(k+1)x - (k+1)\cos(k+1)x\right) + \frac{1}{(k-1)^2+1}\left(-\sin(k-1)x - (k-1)\cos(k-1)x\right)\right\}\right]_0^\infty$$

$$\Rightarrow F_s(K) = \frac{1}{\sqrt{2\pi}}\left[0 - \left\{\frac{1}{(K+1)^2+1}(0-(K+1)) + \frac{1}{(K-1)^2+1}(0-(K-1))\right\}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{K+1}{(K+1)^2+1} + \frac{K-1}{(K-1)^2+1}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{K+1}{K^2+2K+2} + \frac{K-1}{K^2-2K+2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2K^3}{K^4+4}$$

$$\Rightarrow F_s\{e^{-x}\cos x\} = \sqrt{\frac{2}{\pi}} \cdot \frac{K^3}{K^4+4}$$

***

Example 2:- If

$$f(x) = \begin{cases} \sin x & 0 \leq x \leq \pi \\ 0 & x > \pi \end{cases}$$

Then calculate $F_s(K)$.

Proof

$$F_s(K) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin Kx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\pi \sin x \sin Kx\,dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{-2}\int_0^\pi (-2\sin Kx \sin x)\,dx$$

$$= -\sqrt{\frac{2}{\pi}} \cdot \frac{1}{2}\int_0^\pi \cos(K+1)x - \cos(K-1)x\,dx$$

$$= -\frac{1}{\sqrt{2\pi}}\left.\frac{\sin(K+1)x}{K+1} - \frac{\sin(K-1)x}{K-1}\right|_0^\pi$$

$$= -\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(K+1)\pi}{K+1} - \frac{\sin(K-1)\pi}{K-1} - 0 + 0\right]$$

$$= -\frac{1}{\sqrt{2\pi}}\left[\frac{(K-1)[\sin(K+1)\pi] - (K+1)\sin(K-1)\pi}{K^2-1}\right]$$

$$\Rightarrow F_s(K) = -\frac{1}{\sqrt{2\pi}}\left[\frac{-(K-1)\sin K\pi - (K+1)\sin K\pi}{K^2-1}\right]$$

$$= \frac{-1}{\sqrt{2\pi}}\left[\frac{-K\sin K\pi + \cancel{\sin K\pi} - K\sin K\pi - \cancel{\sin K\pi}}{K^2-1}\right]$$

$$= -\frac{1}{\sqrt{2\pi}} \cdot \frac{-2K\sin K\pi}{K^2-1}$$

$$= \frac{2K\sin K\pi}{\sqrt{2\pi}\,(K^2-1)}$$

**

Example 3 :- Calculate Fourier Sine Transform of $f(x) = x\,e^{-ax}$

Solution

$$F_s(K) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin Kx \cdot x\,e^{-ax}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[x\left\{\frac{e^{-ax}}{a^2+K^2}(-a\sin Kx - K\cos Kx)\right\}\Big|_0^\infty - \int_0^\infty 1\cdot\frac{e^{-ax}}{a^2+K^2}(-a\sin Kx - K\cos Kx)\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0 + \frac{1}{a^2+K^2}\left\{a\int_0^\infty e^{-ax}\sin Kx\,dx + K\int_0^\infty e^{-ax}\cos Kx\,dx\right\}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+K^2}\left\{\frac{e^{-ax}}{a^2+K^2}(-a\sin Kx - K\cos Kx)\right\} + \frac{K}{a^2+K^2}\left\{\frac{e^{-ax}}{a^2+K^2}(-a\cos Kx + K\sin Kx)\right\}\right]\Big|_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{a}{a^2+K^2}\left\{\frac{1}{a^2+K^2}(0-K)\right\} - \frac{K}{a^2+K^2}\left\{\frac{1}{a^2+K^2}(-a+0)\right\}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{aK}{(a^2+K^2)^2} + \frac{aK}{(a^2+K^2)^2}\right]$$

$$\Rightarrow F_s(K) = \sqrt{\frac{2}{\pi}} \cdot \frac{2aK}{(a^2+K^2)^2}$$

Example 4:- Calculate $F_c(K)$, $f(x) = x^{\alpha-1}$

Solution

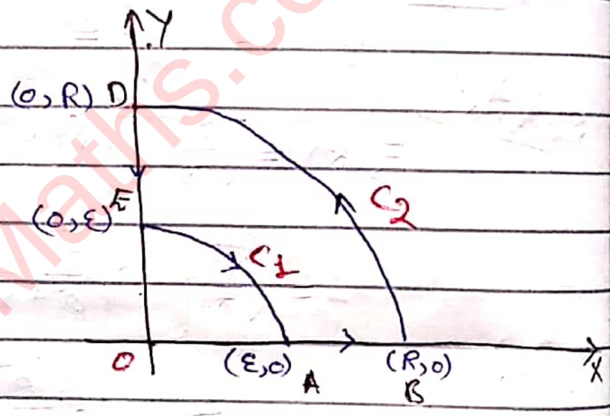
$$F_c(K) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{\alpha-1} \cos Kx \, dx \longrightarrow (1)$$

$$F_s(K) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{\alpha-1} \sin Kx \, dx \longrightarrow (2)$$

Define a function

$$f(z) = z^{\alpha-1} e^{-Kz}, \quad 0 < \alpha < 1$$

and choose the contour as shown in the figure.



The $f(z)$ is analytic on & with in the contour C, so then by Cauchy's Theorem.

$$\int_C f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz + \int_\varepsilon^R f(z)\,dz + \int_{C_2} f(z)\,dz + \int_R^\varepsilon f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz + \int_\varepsilon^R x^{\alpha-1} e^{-Kx}\,dx + \int_{C_2} f(z)\,dz + \int_R^\varepsilon (iy)^{\alpha-1} e^{-iKy}(i\,dy) = 0$$

When $\varepsilon \longrightarrow 0$, $R \to \infty$, Then

$$\int_{C_1} f(z)\,dz = 0 \quad \& \quad \int_{C_2} f(z)\,dz = 0$$

Then $$\int_0^\infty x^{\alpha-1} e^{-Kx}\,dx = i \int_0^\infty (iy)^{\alpha-1} e^{-iKy}\,dy$$

$$\Rightarrow \int_0^\infty x^{\alpha-1} e^{-Kx}\,dx = \int_0^\infty i^\alpha y^{\alpha-1} (e^{-iKy})\,dy$$

$$i^{\alpha} = (0+1i)^{\alpha} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{\alpha}$$

Now $(i)^{\alpha} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{\alpha} = e^{i\alpha\frac{\pi}{2}}$

$$\Rightarrow \int_0^\infty x^{\alpha-1} e^{-kx}\,dx = \int_0^\infty e^{i\alpha\frac{\pi}{2}} y^{\alpha-1} e^{-iky}\,dy$$

$$\Rightarrow e^{-\frac{i\alpha\pi}{2}} \int_0^\infty x^{\alpha-1} e^{-kx}\,dx = \int_0^\infty y^{\alpha-1} e^{-iky}\,dy$$

Equating real and imaginary parts

$$\cos\frac{\alpha\pi}{2} \int_0^\infty x^{\alpha-1} e^{-kx}\,dx = \int_0^\infty y^{\alpha-1} \cos ky\,dy \longrightarrow (3)$$

$$\& \sin\frac{\alpha\pi}{2} \int_0^\infty x^{\alpha-1} e^{-kx}\,dx = \int_0^\infty y^{\alpha-1} \sin ky\,dy \longrightarrow (4)$$

From (1) & (3)

$$F_c(k) = \sqrt{\frac{2}{\pi}} \cos\frac{\alpha\pi}{2} \int_0^\infty x^{\alpha-1} e^{-kx}\,dx$$

Put $kx = t$

$$F_c(k) = \sqrt{\frac{2}{\pi}} \cos\frac{\alpha\pi}{2} \int_0^\infty (t/k)^{\alpha-1} e^{-t} \frac{1}{k}\,dt$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\cos\frac{\alpha\pi}{2}}{k^{\alpha}} \int_0^\infty t^{\alpha-1} e^{-t}\,dt$$

$$= \sqrt{\frac{2}{\pi}} \frac{\cos\frac{\alpha\pi}{2}}{k^{\alpha}} \Gamma\alpha$$

Now from (2) & (4)

$$F_s(k) = \sqrt{\frac{2}{\pi}} \sin\frac{\alpha\pi}{2} \int_0^\infty x^{\alpha-1} e^{-kx}\,dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin\frac{\alpha\pi}{2}}{k^{\alpha}} \Gamma\alpha$$

**

Example 5:- Show that $F_c\{x e^{-ax}\} = \sqrt{\frac{2}{\pi}} \cdot \frac{a^2-k^2}{(a^2+k^2)^2}$

Solution $F_c\{x e^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-ax} \cos kx\,dx$

$$\Rightarrow F_c\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\Bigg[x\cdot\frac{e^{-ax}}{a^2+k^2}(-a\cos kx + k\sin kx)\Big|_0^\infty$$

$$-\int_0^\infty \frac{e^{-ax}}{a^2+k^2}(-a\cos kx + k\sin kx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\Bigg[\frac{-a}{a^2+k^2}\int_0^\infty \frac{e^{-ax}}{1}\cos kx\,dx +$$

$$\frac{k}{a^2+k^2}\int_0^\infty \frac{e^{-ax}}{1}\sin kx\,dx\Bigg]$$

$$= \sqrt{\frac{2}{\pi}}\Bigg[\frac{-a}{a^2+k^2}\left\{\frac{e^{-ax}}{a^2+k^2}(-a\cos kx + k\sin kx)\right\}\Big|_0^\infty$$

$$+\frac{k}{a^2+k^2}\left\{\frac{e^{-ax}}{a^2+k^2}(-a\sin kx - k\cos kx)\right\}\Big|_0^\infty\Bigg].$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-a^2}{(a^2+k^2)^2} + \frac{k^2}{(a^2+k^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{a^2-k^2}{(a^2+k^2)^2}$$

——— ★★★ ———

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* Use of $F_s(K)$ & $F_c(K)$ in Boundry and Initial Value Problems **

Example 1:- Solve the potentional equation in the semi-infinite strip $0<x<c$, $y>0$ that satisfies the following conditions.

$$u(0,y)=0, \quad u_y(x,0)=0 \quad u_x(c,y)=f(y)$$

Solution

The potentional equation is given by

$$U_{xx}+U_{yy}=0$$

Now Apply Fourier Cosine Transform w.r.t y

$$F_c\{U_{xx}\}+F_c\{U_{yy}\}=0$$

$$\frac{d^2}{dx^2}U_c(x,K)-\sqrt{\frac{2}{\pi}}\,u_y(x,0)-K^2U_c(x,K)=0$$

$$\Rightarrow\left(\frac{d^2}{dx^2}-K^2\right)U_c(x,K)-0=0$$

A.E is $\frac{d^2}{dx^2}-K^2=0 \quad \Rightarrow D^2=\pm K$

$$\Rightarrow U_c(x,K)=C_1e^{Kx}+C_2e^{-Kx} \longrightarrow ①$$

Now $u(0,y)=0 \quad \Rightarrow U_c(0,K)=0$

So $C_1+C_2=0 \longrightarrow ②$ by ①

Also $u_x(c,y)=f(y)$

$$\Rightarrow \frac{d}{dx}U_c(c,K)=F_c(K) \longrightarrow ③$$

Now by ① $\frac{d}{dx}U_c(x,K)=C_1Ke^{Kx}-C_2Ke^{-Kx}$

$$\Rightarrow \frac{d}{dx}U_c(c,K)=C_1Ke^{cK}-C_2Ke^{-cK}$$

$$\Rightarrow C_1Ke^{cK}-C_2Ke^{-cK}=F_c(K) \longrightarrow ④$$

by ③

Solving ③ & ④

$$C_1 = \frac{2F_c(K)}{K\cosh cK} \quad , \quad C_2 = -\frac{2F_c(K)}{K\cosh cK}$$

Hence

$$U_c(x,K) = \frac{2F_c(K)e^{Kx}}{K\cosh cK} - \frac{2F_c(K)e^{-Kx}}{K\cosh cK}$$

$$= \frac{2F_c(K)}{K\cosh cK}\left[\frac{e^{Kx}-e^{-Kx}}{2}\right]\times 2$$

$$= \frac{4F_c(K)\sinh Kx}{K\cosh cK}$$

Taking I.F.C.T

$$u(x,y) = \sqrt{\frac{2}{\pi}}\cdot 4\int_0^\infty \frac{F_c(K)\sinh Kx}{K\cosh cK}\cdot \cos Ky \, dK$$

$$= \sqrt{\frac{2}{\pi}}\cdot 4\int_0^\infty \frac{\sinh Kx \cos Ky}{K\cosh cK}\cdot\left(\sqrt{\frac{2}{\pi}}\int_0 \cos Ky' f(y')dy'\right)dK$$

$$\Rightarrow u(x,y) = \frac{8}{\pi}\int_0^\infty\int_0^\infty \frac{\sinh Kx\cos Ky\cos Ky'}{K\cosh cK} f(y')\,dy'\,dK$$

If $f(y')$ is known then on further integrating $u(x,y)$ can be obtained.

————————

Example 2:- Solve $u_t = u_{xx}$ $\quad x>0, t>0$

$u(0,t) = u_0 \ , \ u(x,0) = 0 \ , \ u_0 > 0$

Solution

B.C suggests that we should apply Fourier Sine Transform w.r.t x

$$F_s\{u_t\} = F_s\{u_{xx}\}$$

$$\frac{d}{dt}\{U_s(K,t)\} = \sqrt{\frac{2}{\pi}}\,K\,u(0,t) - K^2 U_s(K,t)$$

$$= \sqrt{\frac{2}{\pi}}\,K\,u_0 - K^2 U_s(K,t)$$

$$\Rightarrow \frac{d}{dt} U_s(k,t) + k^2 U_s(k,t) = \sqrt{\frac{2}{\pi}}\, k u_0$$

$$I.F = e^{\int k^2 dt} = e^{k^2 t}$$

So $$d\left(U_s(k,t)\, e^{k^2 t}\right) = \sqrt{\frac{2}{\pi}}\, k u_0 e^{k^2 t}\, dt$$

Integrating both sides

$$U_s(k,t)\, e^{k^2 t} = \sqrt{\frac{2}{\pi}} (1/k) u_0 e^{k^2 t} + c$$

Now $u(x,0) = 0 \Rightarrow U_s(k,0) = 0$

So $$U_s(k,0) = \sqrt{\frac{2}{\pi}} \cdot \frac{u_0}{k} + c$$

$$\Rightarrow c = -\sqrt{\frac{2}{\pi}} \cdot \frac{u_0}{k}$$

$$\Rightarrow e^{k^2 t} U_s(k,t) = \sqrt{\frac{2}{\pi}} \frac{u_0}{k} e^{k^2 t} - \sqrt{\frac{2}{\pi}} \frac{u_0}{k}$$

$$\Rightarrow U_s(k,t) = \sqrt{\frac{2}{\pi}} \cdot \frac{u_0}{k} - \sqrt{\frac{2}{\pi}} u_0 \frac{1}{k} e^{-k^2 t}$$

Taking Inverse Fourier sine Transform on both sides

$$u(x,t) = \sqrt{\frac{2}{\pi}} u_0 \left[\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin kx}{k} dk - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-k^2 t}}{k} \sin kx\, dk \right]$$

$$= \sqrt{\frac{2}{\pi}} u_0 \left[\int_0^\infty \frac{\sin kx}{k} dk - \int_0^\infty \frac{1}{k} e^{-k^2 t} \sin kx\, dk \right]$$

$$= \sqrt{\frac{2}{\pi}} u_0 [I_1 - I_2] \longrightarrow ⓐ$$

$$I_1 = \int_0^\infty \frac{\sin kx}{k} dk = \pi/2$$

$$I_2 = \int_0^\infty e^{-k^2 t} \frac{\sin kx}{k} dk$$

$$= \int_0^\infty e^{-k^2 t} \left(\int_0^{x} \cos kx' dx' \right) dk \qquad \because \frac{\sin kx}{k} = \int_0^x \cos kx' dx'$$

$$\Rightarrow I_2 = \int_0^{x} \left[\int_0^\infty e^{-k^2 t} \cos kx'\, dk \right] dx' \longrightarrow ①$$

Consider $\int_0^\infty e^{-k^2 t} \cos kx' \, dk'$

$$= Re \int_0^\infty e^{-k^2 t} e^{ikx'} \, dk'$$

$$= Re \int_0^\infty e^{-k^2 t} e^{ikx'} \, dk$$

$$= Re \int_0^\infty e^{-t\left[k^2 - \frac{ikx'}{t}\right]} \, dk$$

$$= Re \int_0^\infty e^{-t\left(k - \frac{ix'}{2t}\right)^2} \cdot e^{-x'^2/4t} \, dk$$

$$= e^{\frac{-x'^2}{4t}} \cdot Re \int_0^\infty e^{-t\left(k - \frac{ix'}{2t}\right)^2} \, dk$$

Put $k - ix'/2t = k' \Rightarrow dk = dk'$

$$= e^{\frac{-x'^2}{4t}} Re \int_0^\infty e^{-tk'^2} \, dk$$

$$= e^{-x'^2/4t} \cdot \sqrt{\pi/t}$$

Thus by ① $I_2 = \int_0^x e^{-x'^2/4t} \cdot \sqrt{\pi/t} \, dx'$

$$= \frac{\sqrt{\pi}}{t} \int_0^x e^{-x'^2/4t} \, dx'$$

$$= \frac{\sqrt{\pi}}{t} \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) \qquad \because \operatorname{erf}(x) = \int_0^x e^{-t^2} dt$$

So by ⓐ

$$u(x,t) = \frac{2}{\pi} u_0 \left(\frac{\pi}{2} - \frac{\sqrt{\pi}}{t} \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)$$

$$= u_0 \left(1 - \frac{2}{\sqrt{\pi} t} \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)$$

* Use of Complex Fourier Transform In Solving Boundry Value And Initial Value Problems**

Example 1:- Solve the problem by using F.T method $u_{xx}(x,t) = u_t(x,t)$

$$u(x,0) = e^{-ax^2}$$

where $u(x,t),\ u_x(x,t) \longrightarrow 0$ as $x \longrightarrow \pm\infty$

Solution

$$F\{u_{xx}(x,t)\} = F\{u_t(x,t)\}$$

$$(-ik)^2\, U(k,t) = \frac{d}{dt} U(k,t)$$

$$\Rightarrow -k^2\, U(k,t) = \frac{d}{dt} U(k,t)$$

$$\Rightarrow \frac{U'(k,t)}{U(k,t)} = -k^2$$

$$\Rightarrow \ln U(k,t) = -k^2 t + \ln A = \ln e^{-k^2 t} + \ln A$$

$$\Rightarrow \ln U(k,t) = \ln A e^{-k^2 t}$$

$$\Rightarrow U(k,t) = A e^{-k^2 t} \longrightarrow (1)$$

Now $u(x,0) = e^{-\alpha x^2}$ using Gaussian function

$$\Rightarrow U(k,0) = \frac{1}{\sqrt{2\alpha}} e^{-k^2/4\alpha}$$

$$\Rightarrow A = \frac{1}{\sqrt{2\alpha}} e^{-k^2/4\alpha} \quad \text{by } (1)$$

$$\Rightarrow U(k,t) = \frac{1}{\sqrt{2\alpha}} e^{-k^2 t} \cdot e^{-k^2/4\alpha}$$

$$= \frac{1}{\sqrt{2\alpha}} e^{-k^2\left(1/4\alpha + t\right)}$$

$$= \frac{1}{\sqrt{2\alpha}} e^{-\beta k^2}, \quad \beta = \frac{1}{4\alpha} + t$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\alpha}} F^{-1}\{e^{-\beta k^2}\}$$

$$= \frac{1}{\sqrt{2\alpha}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \cdot e^{-\beta k^2} dk$$

$$= \frac{1}{2\sqrt{\pi\alpha}} \int_{-\infty}^{\infty} e^{-\beta\left(k^2 + \frac{ikx}{\beta}\right)} dk$$

$$= \frac{1}{2\sqrt{\pi\alpha}} \int_{-\infty}^{\infty} e^{-\beta\left(k + \frac{ix}{2\beta}\right)^2} \cdot e^{-x^2/4\beta} dk$$

$$= \frac{1}{2\sqrt{\cancel{\pi}\alpha}} e^{-x^2/4\beta} \cdot \sqrt{\cancel{\pi}/\beta}$$

$$= \frac{1}{2\sqrt{\alpha\beta}} e^{-x^2/4\beta}$$

$$\Rightarrow u(x,t) = \frac{1}{2\sqrt{\alpha\left(\frac{1}{4\alpha}+t\right)}} \cdot e^{\frac{-x^2}{4\left(\frac{1}{4\alpha}+t\right)}}$$

$$= \frac{1}{\sqrt{1+4\alpha t}} e^{\frac{-\alpha x^2}{1+4\alpha t}}$$

Example 2:- Solve by using F.T method

$$u_t = u_{xx} \qquad x > 0, t > 0$$

$$u_x(0,t) = f(t) \qquad t > 0$$

$$u(x,0) = 0 \qquad x > 0$$

Solution

The 2nd B.C suggests that we should apply Fourier Cosine Transform

So $F_c\{u_t\} = F_c\{u_{xx}\}$

$$\Rightarrow \frac{d}{dt} U_c(k,t) = -\sqrt{\frac{2}{\pi}}\, u_x(0,t) - k^2 U_c(k,t)$$

$$= -\sqrt{\frac{2}{\pi}}\, f(t) - k^2 U_c(k,t)$$

$$\Rightarrow \frac{d}{dt} U_c(k,t) + k^2 U_c(k,t) = -\sqrt{\frac{2}{\pi}}\, f(t)$$

$$I.F = e^{k^2 t}$$

$$\Rightarrow e^{k^2 t} U_c(k,t) = -\sqrt{\frac{2}{\pi}} \int e^{k^2 t} f(t)\, dt + c \longrightarrow ①$$

Now $u(x,0) = 0$

$$\Rightarrow U_c(k,0) = 0$$

By ① $c = 0 \Rightarrow e^{k^2 t} U_c(k,t) = -\sqrt{\frac{2}{\pi}} \int e^{k^2 t} f(t)\, dt$

$$\Rightarrow U_c(k,t) = -\sqrt{\frac{2}{\pi}}\, e^{-k^2 t} \int f(t)\, e^{k^2 t}\, dt$$

$$= -\sqrt{\frac{2}{\pi}}\, e^{-k^2 t} \int f(t')\, e^{k^2 t'}\, dt'$$

$$= -\sqrt{\frac{2}{\pi}} \int f(t')\, e^{k^2 (t'-t)}\, dt'$$

$$\Rightarrow u(x,t) = -\sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\int f(t')\, e^{k^2(t'-t)}\, dt' \right) \cos kx\, dk$$

$$= -\frac{2}{\pi} \int f(t') \int_0^\infty e^{k^2(t'-t)} \cos kx\, dk\, dt'$$

$$= -\sqrt{\frac{2}{\pi}} \int f(t') \left[\sqrt{\frac{2}{\pi}} \int_0^\infty e^{k^2(t'-t)} \cos kx\, dk \right] dt'$$

$$\Rightarrow u(x,t) = -\sqrt{\frac{2}{\pi}} \int f(t')\, \frac{1}{t'-t}\, e^{\left(-x^2/4(t'-t)\right)}\, dt'$$

As required.

* *

Example 3:- Solve $u_t = \alpha^2 u_{xx}$ $\quad -\infty < x < \infty,\ t > 0$

with $u(x,0) = f(x)$ $\quad -\infty < x < \infty$

$|u(x,0)| < \infty$ $\quad$ for all x, t

Solution

Given $u_t = \alpha^2 u_{xx}$

$F\{u_t\} = \alpha^2 F\{u_{xx}\}$

$\Rightarrow U'(k,t) = \alpha^2 (-ik)^2 U(k,t)$

$$\Rightarrow \frac{U'(k,t)}{U(k,t)} = -\alpha^2 k^2$$

$\Rightarrow \ln U(k,t) = -\alpha^2 k^2 t + \ln A$

$= \ln A e^{-\alpha^2 k^2 t}$

$\Rightarrow U(k,t) = A e^{-\alpha^2 k^2 t}$

$U(k,0) = A$

Also $u(x,0) = f(x)$

$\Rightarrow U(k,0) = F(k)$ $\quad \Rightarrow A = F(k)$

$\Rightarrow U(k,t) = F(k) e^{-\alpha^2 k^2 t}$

Applying F^{-1}

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} e^{-\alpha^2 k^2 t} F(k)\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} e^{-\alpha^2 k^2 t} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx'} f(x')\, dx' \right) dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x') \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} e^{-\alpha^2 k^2 t} e^{ikx'}\, dk \right) dx'$$

Consider

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2 t \left[k^2 + \frac{ikx}{\alpha^2 t} - \frac{ikx'}{\alpha^2 t} \right]}\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2 t \left[k^2 + 2\left(\frac{ix - ix'}{2\alpha^2 t}\right)k + \left\{\frac{i(x-x')}{2\alpha^2 t}\right\}^2 - \left\{\frac{i(x-x')}{2\alpha^2 t}\right\}^2\right]} dk$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-x')^2}{4\alpha^2 t}} \int_{-\infty}^{\infty} e^{-\alpha^2 t\left(k + \frac{i(x-x')}{2\alpha^2 t}\right)^2} dk$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-x')^2}{4\alpha^2 t}} \int_{-\infty}^{\infty} e^{-\alpha^2 t k'^2} dk'$$

$$= \frac{1}{\sqrt{2\pi}} e^{-(x-x')^2/4\alpha^2 t} \cdot \sqrt{\frac{\pi}{\alpha^2 t}}$$

$$= \frac{1}{\sqrt{2\alpha^2 t}} e^{-\frac{(x-x')^2}{4\alpha^2 t}}$$

So
$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x') \frac{1}{\sqrt{2\alpha^2 t}} e^{-\frac{(x-x')^2}{4\alpha^2 t}} dx'$$

$$= \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} f(x') e^{-\frac{(x-x')^2}{4\alpha^2 t}} dx'$$

**

Example 4:- Solve $u_t = u_{xx}$, $0 < x < \infty$, $t > 0$

$u_x(0,t) = 0$ $\quad t > 0$

$u(x,0) = f(t)$ $\quad 0 < x < \infty$

Solution

As $0 < x < \infty$, so we have to use Fourier Sine or Fourier Cosine Transform. But 2nd B.C suggests that we should use Fourier Cosine Transform

$$F_c\{u_t\} = F_c\{u_{xx}\}$$

$$U_c'(k,t) = -\sqrt{\frac{2}{\pi}}\, u_x(0,t) - k^2 U_c(k,t)$$

$$= 0 - k^2 U_c(k,t)$$

$$\Rightarrow \ln U_c(k,t) = \ln A e^{-k^2 t}$$

$$\Rightarrow U_c(k,t) = Ae^{-k^2t}$$

$$U_c(k,0) = A$$

Now

$$u(x,0) = f(x)$$

$$\Rightarrow U_c(k,0) = F_c(k)$$

$$\Rightarrow A = F_c(k)$$

$$\Rightarrow U_c(k,t) = F_c(k)e^{-k^2t}$$

$$\Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos kx\, F_c(k)\, e^{-k^2t}\, dk$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \cos kx \left[\sqrt{\frac{2}{\pi}}\int_0^\infty \cos kx' f(x')\, dx'\right] e^{-k^2t}\, dk$$

$$= \frac{2}{\pi}\int_0^\infty\int_0^\infty \cos kx \cos kx' f(x') e^{-k^2t}\, dx'\, dk$$

Example 5:- Solve

$$u_{xxxx} = \frac{1}{a^2}u_{tt}$$

with

$$u(x,0) = f(x)$$

$$u_t(x,0) = ag'(x)$$

and $g, u, u_{xx}, u_{xxx} \longrightarrow 0$ as $x \longrightarrow \pm\infty$

Solution:

As $x \to \pm\infty$

So we have to Apply Fourier Transform

$$F\{u_{xxxx}\} = \frac{1}{a^2}F\{u_{tt}\}$$

$$\Rightarrow (-ik)^4 U(k,t) = \frac{1}{a^2}\frac{d^2}{dt^2}U(k,t)$$

$$\Rightarrow \frac{d^2}{dt^2}U(k,t) - a^2k^4U(k,t) = 0$$

A.E is $D^2 - a^2k^4 = 0$

$$\Rightarrow \quad D = \pm a k^2$$

$$\Rightarrow U(k,t) = A e^{ak^2 t} + B e^{-ak^2 t} \longrightarrow ①$$

$$U(k,0) = A + B \longrightarrow ②$$

$$U'(k,t) = ak^2 (A e^{ak^2 t} - B e^{-ak^2 t})$$

$$U'(k,0) = ak^2 (A - B) \longrightarrow ③$$

Now $u(x,0) = f(x) \Rightarrow U(k,0) = F(k)$

$$u_t(x,0) = a g'(x)$$

$$\Rightarrow \frac{d}{dt} U(k,0) = a(-ik) G(k)$$

By ② $\quad A + B = F(k)$

By ③ $\quad ak^2(A-B) = -iak\,G(k) \Rightarrow A - B = \frac{-i\alpha}{\alpha k} G(k)$

$$\Rightarrow A = \frac{1}{2}\left\{F(k) - \frac{i}{k} G(k)\right\} \longrightarrow ④$$

$$B = \frac{1}{2}\left\{F(k) + \frac{i}{k} G(k)\right\} \longrightarrow ⑤$$

Hence

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left(A e^{ak^2 t} + B e^{-ak^2 t}\right) dk$$

where A and B are given in ④ & ⑤

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Example 6:- Solve

$$u_{xx}(x,t) = \frac{1}{c^2} u_{tt}(u,t)$$

$$u(x,0) = P(x) \qquad u_t(x,0) = q(x)$$

$$u, u_x \to 0 \qquad \text{as } x \to \pm\infty$$

Solution

As $x \to \pm\infty$ so we use fourier complex Transform.

$$F\{u_{xx}\} = F\{u_{tt}\}$$

$$-k^2 U(k,t) = \frac{1}{c^2}\frac{d^2}{dt^2} U(k,t)$$

$$\Rightarrow \frac{d^2}{dt^2} U(k,t) + c^2k^2 U(k,t) = 0$$

A.E is $D^2 + c^2k^2 = 0$

$$\Rightarrow D = \pm i c k$$

$$\Rightarrow U(k,t) = A\cos ckt + B\sin kct$$

$$U(k,0) = A$$

$$U_t(k,t) = -ckA\sin ckt + ckB\cos ckt$$

$$U_t(k,0) = ckB$$

Now $u(x,0) = P(x) \Rightarrow U(k,0) = P(k)$

$u_t(x,0) = q(x) \Rightarrow U_t(k,0) = Q(k)$

So $A = P(k)$

$$B = \frac{1}{ck} Q(k)$$

So $U(k,t) = P(k)\cos ckt + \frac{1}{ck} Q(k)\sin ckt$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[P(k)\cos ckt + \frac{1}{ck} Q(k)\sin ckt\right] dk$$

where $P(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} p(x)\, dx$

& $Q(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} q(x)\, dx.$

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⇒ Multidimensional Fourier Transform:-

Let $f(x,y)$ be a function defined over whole XY plane then fourier transform of $f(x,y)$ is denoted & defined by

$$F(k_1, k_2) = \frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_1x+k_2y)} f(x,y)\,dx\,dy$$

In this case corresponding Inverse Fourier transform is given by

$$f(x,y) = \frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_1x+k_2y)} F(k_1,k_2)\,dk_1\,dk_2$$

Next let $f(x,y,z)$ be a function defined over whole space $\mathbb{R}^3$ then the Fourier Transform of $f(x,y,z)$ is denoted and defined by

$$F(k_1,k_2,k_3) = \frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_1x+k_2y+k_3z)} f(x,y,z)\,dx\,dy\,dz$$

And in this case Inverse fourier transform is given by

$$f(x,y,z) = \frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_1x+k_2y+k_3z)} F(k_1,k_2,k_3)\,dk_1\,dk_2\,dk_3$$

———★★★———

Examples

Example 1:- Find Fourier Transform of

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$$

Solution

$$F\{\nabla^2 u\} = F\{u_{xx} + u_{yy} + u_{zz}\}$$

$$= F\{u_{xx}\} + \text{Two similar Terms}$$

$$\Rightarrow F\{\nabla^2 u\} = \frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_1x+k_2y+k_3z)} u_{xx}(x,y,z)\,dx\,dy\,dz$$

$$+\text{Two similar Terms}$$

$$= \frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[e^{i(k_1x+k_2y+k_3z)}\cdot u_x(x,y,z)\Big|_{-\infty}^{\infty}$$

$$- \int_{-\infty}^{\infty}(ik_1)\, e^{i(k_1x+k_2y+k_3z)} u_x(x,y,z)\,dx\Big]dy\,dz$$

$$+T.S.T$$

$$= \frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[0 + (-ik_1)\int_{-\infty}^{\infty} e^{i(k_1x+k_2y+k_3z)} u_x(x,y,z)\,dx\Big]dy\,dz$$

$$+\text{Two similar Terms}$$

$$= (-ik_1)\frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[e^{i(k_1x+k_2y+k_3z)} u(x,y,z)\Big|_{-\infty}^{\infty}$$

$$- \int_{-\infty}^{\infty}(ik_1)\, e^{i(k_1x+k_2y+k_3z)} u(x,y,z)\,dx\Big]dy\,dz + T.S.T$$

$$= (-ik_1)^2\frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_1x+k_2y+k_3z)} u(x,y,z)\,dx\,dy\,dz + T.S.T$$

$$= (-k_1)^2\; F\{u(x,y,z)\} + T.S.T$$

$$\Rightarrow F\{\nabla^2 u\} = -k_1^2\, U(k_1, k_2, k_3) + T.S.T$$
$$= (-k_1^2 - k_2^2 - k_3^2)\, U(k_1, k_2, k_3)$$
$$= -(k_1^2 + k_2^2 + k_3^2)\, U(k_1, k_2, k_3)$$

Example 2:- Use Fourier Transform to solve the poisson Equation $\nabla^2 u = g(x, y, z)$

Solution

Given $\nabla^2 u = g(x, y, z)$

$$\Rightarrow F\{\nabla^2 u\} = F\{g(x, y, z)\}$$
$$\Rightarrow -(k_1^2 + k_2^2 + k_3^2)\, U(k_1, k_2, k_3) = G(k_1, k_2, k_3)$$

$$\boxed{\begin{array}{l} x, y, z = r \\ k_1, k_2, k_3 = k \end{array}}$$

$$\Rightarrow -k^2 U(k) = G(k)$$
$$\Rightarrow U(k) = -\frac{G(k)}{k^2}$$

Now applying Inverse Fourier Transform on both sides

$$\Rightarrow u(r) = -\frac{1}{(\sqrt{2\pi})^3} \int_{\text{all space}} e^{-ik\cdot r} \frac{G(k)}{k^2}\, d^3k$$

$$\Rightarrow u(r) = -\frac{1}{(\sqrt{2\pi})^3} \int_{\text{all space}} \frac{e^{-ik\cdot r}}{k^2} \left[\frac{1}{(\sqrt{2\pi})^3} \int_{\text{all space}} e^{ik\cdot r'} g(r')\, d^3r'\right] d^3k$$

$\boxed{\text{F.T of } g(r') = G(k)}$

$$= -\frac{1}{(\sqrt{2\pi})^6} \iint_{\text{all space}} \frac{e^{-ik\cdot(r-r')}\, g(r')}{k^2}\, d^3r'\, d^3k$$

$$= -\frac{1}{(\sqrt{2\pi})^6} \iint_{\text{all space}} \left[\frac{e^{-ik\cdot(r-r')}}{k^2}\, d^3k\right] g(r')\, d^3r'$$

$\because$ by interchanging order of the integral

$$\Rightarrow \quad u(r) = -\frac{1}{(\sqrt{2\pi})^6} \int I \, g'(r) \, d^3r' \longrightarrow \textcircled{*}$$

where

$$I = \int \frac{e^{-i k\cdot(r - r')}}{k^2} \, d^3k$$

Now using spherical polar coordinates (k, θ, ϕ) and choosing k_3 - axis in the direction of $R = r - r'$, Then

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{e^{-ikR\cos\theta}}{\cancel{k^2}} (\cancel{k^2}\sin\theta \, dk \, d\theta \, d\phi$$

$$= \left[\int_0^{2\pi} d\phi\right]\left[\int_0^{\infty} \frac{1}{ikR}\left[\int_0^{\pi} e^{-iRk\cos\theta} (ikR\sin\theta) d\theta\right] dk\right]$$

$$= 2\pi \int_0^{\infty} \frac{1}{ikR} \cdot e^{ikR\cos\theta}\Big|_0^{\pi} \, dk$$

$$= 2\pi \int_0^{\infty} \frac{1}{ikR} \cdot \left[e^{ikR} - e^{-ikR}\right] dk$$

$$= 4\pi \int_0^{\infty} \frac{1}{kR} \cdot \frac{e^{ikR} - e^{-ikR}}{2i} \, dk$$

$$= 4\pi \int_0^{\infty} \frac{1}{kR} \sin kR \, dk$$

Put $kR = \eta$ then

$$I = 4\pi \int_0^{\infty} \frac{\sin\eta}{\eta} \cdot \frac{1}{R} \, d\eta$$

$$= \frac{4\pi}{R} \int_0^{\infty} \frac{\sin\eta}{\eta} \, d\eta = \frac{4\pi}{R}\left(\frac{\pi}{2}\right)$$

$$\Rightarrow I = \frac{2\pi^2}{R}$$

Hence $u(r) = -\frac{1}{8\pi^3}\int \left(\frac{2\pi^2}{R}\right) g(r')\, d^3r'$

$$= -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{g(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\, dx'dy'dz'$$

As required.

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Variational Methods

OR Calculus of Variation.

⇒ **Functional:-**

A function whose domain is a set C_1 of functions and whose co-domain (Range) C_2 consists of functions or real numbers.

⇒ **Stationary Value:-**

The maximum or minimum value of a function or functional is called stationary value.

⇒ **Extremal:-**

The curve $y = f(x)$ along which a functional takes the stationary value is called the Extremal.

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⇒ **Fundamental Theorem of Variational Calculus:-** (One Independent Variable)

If $F(x)$ is continuous in the interval (x_1, x_2) and the integral

$$\int_{x_1}^{x_2} F(x)\, G(x)\, dx = 0$$

where $G(x)$ satisfies the conditions

1) It is an arbitrary function with continuous derivatives in the interval (x_1, x_2)
2) $G(x_1) = G(x_2) = 0$

Then $F(x) = 0 \quad \forall\, x \in [x_1, x_2]$

Proof) Suppose $F(x) \neq 0$ in $[x_1, x_2]$. Then there is a point $x_0 \in [x_1, x_2]$ s.t $F(x_0) \neq 0$. Without any loss of generality suppose $F(x_0) > 0$. Then by the continuity of F, there is an internal $(x_0-\varepsilon, x_0+\varepsilon)$, $\varepsilon > 0$ s.t $F(x) > 0$

$\forall\, x \in]x_0-\varepsilon, x_0+\varepsilon[$

Since $G(x)$ is arbitrary so we define

$$G(x) = \begin{cases} (x-x_0-\varepsilon)^2(x-x_0+\varepsilon)^2 & \text{if } x_0-\varepsilon \leq x \leq x_0+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Then $$0 = \int_{x_1}^{x_2} F(x)\,G(x)\,dx$$

$$\Rightarrow \quad 0 = \int_{x_1}^{x_0-\varepsilon} F(x)\,G(x)\,dx + \int_{x_0-\varepsilon}^{x_0+\varepsilon} F(x)\,G(x)\,dx + \int_{x_0+\varepsilon}^{x_2} F(x)\,G(x)\,dx$$

$$= 0 + \int_{x_0-\varepsilon}^{x_0+\varepsilon} F(x)\,G(x)\,dx + 0$$

$$= \int_{x_0-\varepsilon}^{x_0+\varepsilon} F(x)\,G(x)\,dx > 0$$

$$\Rightarrow \quad 0 > 0$$

which is a contradiction. So our supposition is wrong. Hence $F(x) = 0$ for all $x \in [x_1, x_2]$

——— ★★★ ———

⇒ Fundamental Theorem of Variational Calculus:- [Two Independent Variables]

Let a function $F(x,y)$ be continuous in a region D of the xy plane and $G(x,y)$ be an arbitrary function with continuous partial derivatives in D and let $G(x,y)$ vanishes on the boundry curve C of the domain D

If
$$\iint_D F(x,y)\,G(x,y)\,dx\,dy = 0$$

Then $F(x,y) = 0$ for all (x,y) in the domain D.

Proof

Suppose given is not true i.e $F(x,y) \neq 0$ in D. Then there is atleast one point (x_0, y_0) in D s.t $F(x_0, y_0) \neq 0$. Without any loss of generality suppose $F(x_0, y_0) > 0$. Since F is continuous in D, So there exists a circular domain centered at (x_0, y_0) and with radius $\varepsilon > 0$ i.e $C_0 : (x-x_0)^2 + (y-y_0)^2 \leq \varepsilon^2$ s.t $F(x,y) > 0$ in this domain. Now as $G(x,y)$ is arbitrary so we can choose

$$G(x,y) = \begin{cases} k[(x-x_0)^2 + (y-y_0)^2] & (x,y) \in C_0 ,\ k > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\iint_D F(x,y)\,G(x,y)\,dx\,dy = \iint_D F(x,y)\cdot k\left[(x-x_0)^2 + (y-y_0)^2\right] dx\,dy$$
$$> 0$$

A contradiction. So our supposition is wrong

Hence $F(x,y) = 0 \quad \forall\ (x,y) \in D$

Integrable ← Continuous ← Differentiable

⇒ Euler's-Lagrange Equations:-

Theorem Let $I = \int_{x_1}^{x_2} F(x, y, y')\, dx$

where $y = y(x)$ is a continuous function having continuous 1st and 2nd order derivatives satisfying the following end point conditions

$$y(x_1) = y_1 \quad , \quad y(x_2) = y_2$$

If F is supposed to have continuous 1st and 2nd order derivatives w.r.t its arguments, then the function $y(x)$ will extremise the given integral if it satisfies the differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Proof) Clearly between two fixed points $A(x_1, y_1)$ and $B(x_2, y_2)$ infinite no. of curves can be drawn. Let the family of such curves be defined as

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x) \longrightarrow ①$$

where $\eta(x)$ denotes the derivation from the curve $y = y(x) = y(x, 0)$ i.e $\eta(x_1) = 0 = \eta(x_2)$. where α is the parameter labelling different paths and is independent of x. We have to find the curve which gives the stationary value. We suppose that the extremal curve corresponds to the value $\alpha = 0$

Now $$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \frac{\partial F}{\partial \alpha}\, dx$$

$$\Rightarrow \frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \alpha}\right] dx$$

Now $\dfrac{\partial y}{\partial \alpha} = \eta(x)$ by ①

$\dfrac{\partial y'}{\partial \alpha} = \eta'(x)$ $\because y'(x,\alpha) = y'(x,0) + \alpha\eta'(x)$

So $\dfrac{\partial I}{\partial \alpha} = \displaystyle\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \cdot \eta(x) + \frac{\partial F}{\partial y'} \eta'(x) \right] dx$

$= \displaystyle\int_{x_1}^{x_2} \frac{\partial F}{\partial y} \eta(x)\, dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \eta'(x)\, dx \longrightarrow ②$

Consider $\displaystyle\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \eta'(x)\, dx = \frac{\partial F}{\partial y'} \eta(x) \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \eta(x)\, dx$

$= 0 - \displaystyle\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \eta(x)\, dx$

So $\dfrac{\partial I}{\partial \alpha} = \displaystyle\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) \eta(x)\, dx$ by ②

Since $\eta(x)$ satisfies all the conditions of the fundamental theorem of variational calculus so for extreme value

$\dfrac{\partial I}{\partial \alpha} = 0 \Rightarrow \displaystyle\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) \eta(x)\, dx = 0$

gives

$\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ Proved

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★ Special Cases:-

Case I:- If F is independent of y then

$\frac{\partial F}{\partial y} = 0$ Then Euler's Lagrange equations takes the form

$$0 - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial y'} = \text{constant.}$$

Case II:- If F is independent of x explicitly

then $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \Rightarrow \frac{\partial F}{\partial y} = \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx}$$

$$\Rightarrow \frac{\partial F}{\partial y}dy = y' d\left(\frac{\partial F}{\partial y'}\right) \longrightarrow (*)$$

Now $F = F(y, y')$

$$\Rightarrow dF = \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial y'}dy'$$

$$= y' d\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'}dy' \quad \because \text{by } (*)$$

$$= d\left(y'\frac{\partial F}{\partial y'}\right)$$

$$\Rightarrow d\left[F - y'\frac{\partial F}{\partial y'}\right] = 0$$

$$\Rightarrow F - y'\frac{\partial F}{\partial y'} = c$$

★ ——————— ★

Example 1:- Find the Euler Lagrange Equation for the following

(i) $F = x^2 y^2 - y'^2$ (ii) $F = \sqrt{xy} + y'^2$

(iii) $F = \sin(xy')$ (iv) $F = \dfrac{x^2 y'}{\sqrt{1+y'^2}}$

Solution

(i) $F = x^2 y^2 - y'^2$

$$\frac{\partial F}{\partial y} = 2x^2 y \quad , \quad \frac{\partial F}{\partial y'} = -2y'$$

Euler Lagrange Equation is given by

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow 2x^2 y - \frac{d}{dx}(-2y') = 0$$

$$\Rightarrow 2x^2 y + 2y'' = 0$$

$$\Rightarrow x^2 y + y'' = 0 \quad \Rightarrow y'' + x^2 y = 0$$

(ii)

Here $F = \sqrt{xy} + y'^2$

$$\frac{\partial F}{\partial y} = \frac{1}{2\sqrt{xy}} \cdot x = \frac{1}{2}\sqrt{\frac{x}{y}}$$

& $\dfrac{\partial F}{\partial y'} = 2y'$

So $\dfrac{1}{2}\sqrt{\dfrac{x}{y}} - \dfrac{d}{dx}(2y') = 0$

$$\Rightarrow \frac{1}{2}\sqrt{\frac{x}{y}} - 2y'' = 0 \Rightarrow \sqrt{\frac{x}{y}} - 4y'' = 0$$

$$\Rightarrow \sqrt{x} - 4\sqrt{y}\, y'' = 0$$

(iii) Here $F = \sin(xy')$

As F is independent of y so we have

$$\frac{\partial F}{\partial y'} = \text{constant}$$

Now $\dfrac{\partial F}{\partial y'} = \cos(xy') \cdot x$

So Euler Lagrange Equation is

$$x \cos(xy') = c$$

(iv) Here $F = \dfrac{x^2 y'}{\sqrt{1+y'^2}}$

$$\frac{\partial F}{\partial y'} = \frac{\sqrt{1+y'^2}\cdot x^2 - x^2 y'\left(\frac{y'}{\sqrt{1+y'^2}}\right)}{1+y'^2}$$

$$= \frac{x^2 + \cancel{x^2 y'^2} - \cancel{x^2 y'^2}}{(1+y'^2)^{3/2}} = \frac{x^2}{(1+y'^2)^{3/2}}$$

So Euler Lagrange Equation is

$$\frac{\partial F}{\partial y'} = c \quad \Rightarrow \frac{x^2}{(1+y'^2)^{3/2}} = c$$

———————————————

Example 2:- [Brachistochrone/Bernoulli Problem] (The problem solved by Newton, Bernoulli and Euler in 1696)

Find the equation of the path in space down which a particle will fall from one point to another in the shortest possible time.

Solution: Let $A(x_1, y_1)$ & $B(x_2, y_2)$ be the two points. Then the time taken by the particle is given by

$$\int_A^B dt = \int_A^B \frac{dt}{ds}\cdot ds$$

$$= \int_A^B \frac{1}{v}\cdot\sqrt{(dx)^2+(dy)^2} \qquad \left[\because \frac{ds}{dt} = v \ \ \& \ ds = \sqrt{(dx)^2+(dy)^2}\right.$$

$$= \int_A^B \frac{1}{v}\cdot\sqrt{1+y'^2}\, dx \qquad \because \text{taking } \sqrt{(dx)^2} \text{ as common}$$

$$\Rightarrow \int_A^B dt = \int_A^B \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\, dx$$

$\because v = \sqrt{2gy}$

$v_f^2 - v_i^2 = 2gy \Rightarrow v^2 - 0 = 2gy$

$\Rightarrow v = \sqrt{2gy}$

$$= \frac{1}{\sqrt{2g}} \int_A^B \frac{\sqrt{1+y'^2}}{\sqrt{y}}\, dx$$

Here $F = \dfrac{\sqrt{1+y'^2}}{\sqrt{y}}$

Here F is independent of x explicitly, So Euler's Lagrange Equation is

$$F - y'\frac{\partial F}{\partial y'} = c$$

Now $\dfrac{\partial F}{\partial y'} = \dfrac{y'}{\sqrt{1+y'^2}\sqrt{y}}$

So $\dfrac{\sqrt{1+y'^2}}{\sqrt{y}} - y'\dfrac{y'}{\sqrt{1+y'^2}\sqrt{y}} = c$

$\Rightarrow \dfrac{1}{y(1+y'^2)} = c^2 \qquad \Rightarrow y' = \sqrt{\dfrac{1-c^2y}{c^2y}}$

$\Rightarrow \sqrt{\dfrac{c^2y}{1-c^2y}}\, dy = dx \longrightarrow (*)$

Put $c^2y = \sin^2\frac{\theta}{2}$

$c^2 dy = \cancel{2}\cdot\frac{1}{\cancel{2}}\sin\frac{\theta}{2}\cos\theta/2\, d\theta$

$\Rightarrow dy = \frac{1}{c^2}\sin\theta/2 \cos\theta/2\, d\theta$

$I = \int \sqrt{\dfrac{\sin^2\theta/2}{\cos^2\theta/2}}\cdot\dfrac{1}{c^2}\sin\theta/2\cos\frac{\theta}{2}\, d\theta$

where $I = \int\sqrt{\dfrac{c^2y}{1-c^2y}}\, dy$

$$\Rightarrow I = \frac{1}{c^2}\int \sin^2 \theta/2 \, d\theta \quad = \frac{1}{c^2}\int \frac{1-\cos\theta}{2} d\theta$$

$$= \frac{1}{c^2}\left[\frac{1}{2}\theta - \frac{1}{2}\sin\theta\right] + a$$

From ®

$$x = \frac{1}{2c^2}[\theta - \sin\theta] + a$$

&

$$y = \frac{1}{c^2}\sin^2\frac{\theta}{2} = \frac{1}{c^2}\left[\frac{1-\cos\theta}{2}\right]$$

$$= \frac{1}{2c^2}[1 - \cos\theta]$$

So Required path is

$$x = \frac{1}{2c^2}[\theta - \sin\theta] + a \quad \& \quad y = \frac{1}{2c^2}[1 - \cos\theta]$$

***

Example 3:- Find the curve joining the ~~path~~ Points (x_1, y_1) and (x_2, y_2) which gives minimum area of surface of revolution generated around (i) y-axis (ii) x-axis

Solution

Case I:- About X-axis

$$\text{Area} = \int_A^B 2\pi y \, ds = \int_A^B 2\pi y\sqrt{1+y'^2}\, dx$$

Here $F = y\sqrt{1+y'^2}$

Euler's Lagrange Equation is

$$F - y'\frac{\partial F}{\partial y'} = c$$

$$\Rightarrow y\sqrt{1+y'^2} - y'\cdot\frac{yy'}{\sqrt{1+y'^2}} = c$$

$$\Rightarrow y(1+y'^2) - yy'^2 = c\sqrt{1+y'^2}$$

$$\Rightarrow y = c\sqrt{1+y'^2} \quad \Rightarrow y^2 = c^2(1+y'^2)$$

$$y'^2 = \frac{y^2 - c^2}{c^2}$$

$$\Rightarrow y' = \frac{\sqrt{y^2-c^2}}{c} \quad \Rightarrow \frac{1}{\sqrt{y^2-c^2}}\,dy = \frac{1}{c}\,dx$$

$$\Rightarrow \frac{1}{c}x = \cosh^{-1}\left(\frac{y}{c}\right) + a$$

$$\Rightarrow x = c\cosh^{-1}\left(\frac{y}{c}\right) + b \qquad , b = ac$$

Case II :- About Y- axis.

$$A = \int_A^B 2\pi x\, ds = \int_A^B 2\pi x\sqrt{1+y'^2}\,dx$$

Here $F = x\sqrt{1+y'^2}$

Euler's Lagrange Equation is

$$\frac{\partial F}{\partial y'} = c \Rightarrow \frac{xy'}{\sqrt{1+y'^2}} = c$$

$$\Rightarrow \frac{x^2y'^2}{1+y'^2} = c^2 \qquad \Rightarrow y' = \frac{c}{\sqrt{x^2-c^2}}$$

$$\Rightarrow dy = \frac{c}{\sqrt{x^2-c^2}}\,dx$$

$$\Rightarrow y = c\cosh\left(\frac{x}{c}\right) + a$$

Example 4:- Give the geomatrical interpretation of the variational calculas

$$\int_0^1 \sqrt{1+y'^2}\,dx$$

with the boundry conditions $y(0)=0$, $y(1) = 1$.

Solve the problem for the external

Find the stationary values of the integral and compare it with the values of the integral which are obtained for the curves that joint the same end points but are different from the external.

Solution:-

Geometrical Interpretation:-

Given that
$$I = \int_0^1 \sqrt{1+y'^2}\, dx$$
$$\Rightarrow I = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{(dx)^2+(dy)^2} = \int_0^1 ds$$

which gives the distance b/w two points on a surface. As the end points are $A=(0,0)$ and $B=(1,1)$. So this shows that the surface is a plane, xy-plane. We have to find the volume of y which minimize this integral.

Now we find y

Here
$$F = \sqrt{1+y'^2}$$

Since F is independent of y. So Euler's Lagrange Equation is
$$\frac{\partial F}{\partial y'} = c \Rightarrow \frac{y'}{\sqrt{1+y'^2}} = c$$
$$\Rightarrow y' = \sqrt{\frac{c^2}{1-c^2}} = a \Rightarrow y = ax+b$$
$$y(0) = 0 \Rightarrow b = 0$$
$$y(1) = 1 \Rightarrow a(1) = 1 \Rightarrow a = 1$$
$$\text{So } y(x) = x$$

Stationary Value:-

$$S.V = \int_0^1 \sqrt{1+y'^2}\,dx$$

$\Rightarrow$ Stationary Value $= \int_0^1 \sqrt{1+1^2}\,dx = \sqrt{2}(x)\Big|_0^1$

$$= \sqrt{2}(1-0)$$

$$= \sqrt{2}$$

Comparison:-

Choose $y = x^2$ then

$$I = \int_0^1 \sqrt{1+y'^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx$$

$$= \int_0^1 \sqrt{1+(2x)^2}\,dx$$

$$= \frac{1}{2}\left[\frac{2x(\sqrt{1+4x^2})}{2} + \frac{1}{2}\ln\left(\frac{2x+\sqrt{1+4x^2}}{1}\right)\right]_0^1$$

$$= \frac{1}{2}\left[\sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5})\right]$$

$$= \frac{1}{2}[2.24 + .72] = 1.48 > \sqrt{2}$$

So $\sqrt{2}$ is the minimum i.e Stationary value.

———

Theorem.- State suitable boundry conditions for the functional

$$I = \int F(x, y, y', y'', \dots y^{(n)})\, dx$$

and find Euler's Lagrange Equation for the corresponding exteremal

Solution

Let $A(x_1, y_1)$ & $B(x_2, y_2)$ be the two fixed points in plane through which the exteremal curve is situated. then the boundry conditions will be

$$y(x_1) = y'(x_1) = y''(x_1) = \dots = y^{(n)}(x_1) = \text{constant}$$

$$y(x_2) = y'(x_2) = y''(x_2) = \dots = y^{(n)}(x_2) = \text{constant}$$

Let δy be the variation in y and Let δF and δI be the corresponding variations in F & I respectively.

Then

$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y''}\delta y'' + \dots + \frac{\partial F}{\partial y^{(n)}}\delta y^{(n)} \right) dx \quad \text{--- (1)}$$

Consider

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y'\, dx = \frac{\partial F}{\partial y'}\delta y \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\, dx$$

$$= 0 - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\, dx$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta y''\, dx = \frac{\partial F}{\partial y''}\delta y' \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\delta y'\, dx$$

$$= 0 - \left[\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\delta y \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\delta y\, dx\right.$$

$$= (-1)^2 \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\delta y\, dx$$

& So on

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y^n}\, \delta y^n\, dx = (-1)^n \int_{x_1}^{x_2} \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right) \delta y\, dx$$

So by ①

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\,\delta y + (-1)^1 \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\delta y + \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right)\delta y \right] dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right) \right] \delta y\, dx$$

For the extremal curve $\delta I = 0$

$$\Rightarrow \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right) \right] \delta y\, dx = 0$$

From fundamental Theorem

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right) + \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right) = 0$$

$\because \delta y$ is orbitrary

So required Euler's Lagrange equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right) = 0$$

★★★ ———————— ★★★

Theorem:- Find the external of the functional

$$I = \int_{x_1}^{x_2} F(x, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')\, dx$$

with boundry conditions

$y_k(x_1) =$ constant , $y_k(x_2) =$ constant

$K = 1, 2, 3, \ldots, n$

Proof

Let δI be the variation in I corresponding to the variation δy in y. Then

$$\delta I = \int_{x_1}^{x_2} \left[\left\{\frac{\partial F}{\partial y_1}\delta y_1 + \frac{\partial F}{\partial y_2}\delta y_2 + \cdots + \frac{\partial F}{\partial y_n}\delta y_n\right\} + \left\{\frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y_2'}\delta y_2' + \cdots + \frac{\partial F}{\partial y_n'}\delta y_n'\right\}\right] dx \longrightarrow (1)$$

Now Consider $\displaystyle\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y_k'}\delta y_k'\right) dx$

$$= \frac{\partial F}{\partial y_k'}\delta y_k \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)\delta y_k\, dx$$

$$= 0 - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)\delta y_k\, dx \qquad k = 1, 2, 3, \ldots, n$$

$$\Rightarrow \delta I = \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1'}\right)\right)\delta y_1 + \left(\frac{\partial F}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2'}\right)\right)\delta y_2 + \cdots + \left(\frac{\partial F}{\partial y_n} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_n'}\right)\right)\delta y_n\right] dx$$

For external $\delta I = 0$

$$\Rightarrow \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right)\right]\delta y_k\, dx = 0, \qquad k = 1, 2, 3, \ldots, n$$

As δy_k is orbitrary so

$$\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'}\right) = 0 \quad , k = 1, 2, 3, \dots, n$$

which are required E.L Equs.

* * *

Example 1:- Find the external for

$$I = \int_0^{\pi/2} (y'^2 + z'^2 + 2yz)\, dx$$

with $y(0) = 0$, $y(\pi/2) = 1$

$z(0) = 0$, $z(\pi/2) = -1$

Solution

Here two variables y and z are functions of x. So corresponding Euler's Lagrange Equations are

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \longrightarrow (1)$$

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial z}{\partial z'}\right) = 0 \longrightarrow (2)$$

Here $F = y'^2 + z'^2 + 2yz$

$$\frac{\partial F}{\partial y} = 2z \quad , \quad \frac{\partial F}{\partial z} = 2y$$

$$\frac{\partial F}{\partial y'} = 2y' \quad , \quad \frac{\partial F}{\partial z'} = 2z'$$

By (1) $2z - \frac{d}{dx}(2y') = 0$

$$\Rightarrow z - y'' = 0 \longrightarrow (3)$$

By (2) $2y - \frac{d}{dx}(2z') = 0 \Rightarrow y - z'' = 0 \longrightarrow (4)$

From (3) & (4) $z - z'''' = 0 \Rightarrow z'''' - z = 0$

$$(D^4 - 1)z = 0$$

A.E is $D^4 - 1 = 0$

$\Rightarrow D = \pm 1, \pm i$

$\Rightarrow z = Ae^{x} + Be^{-x} + C\cos x + D\sin x$

$z(0) = 0 \Rightarrow A + B + C = 0 \longrightarrow (5)$

$z(\pi/2) = -1 \Rightarrow Ae^{\pi/2} + Be^{-\pi/2} + D = -1 \longrightarrow (6)$

Also $z' = Ae^{x} - Be^{-x} - C\sin x + D\cos x$

$z'' = Ae^{x} + Be^{-x} - C\cos x - D\sin x$

As $y = z'' \Rightarrow y = Ae^{x} + Be^{-x} - C\cos x - D\sin x$

$y(0) = 0 \Rightarrow A + B - C = 0 \longrightarrow (7)$

$y(\pi/2) = 1 \Rightarrow Ae^{\pi/2} + Be^{-\pi/2} - D = 1 \longrightarrow (8)$

Solving (5), (6), (7) and (8) we have

$\boxed{A = 0}$, $\boxed{B = 0}$, $\boxed{C = 0}$, $\boxed{D = -1}$

So $y = \sin x$, $z = \cos x$

————

Example 2:- Find the extrimal

$$I = \int_0^{\pi/2} (y''^2 - y^2 + x^2)\, dx$$

with $y(0) = 1$, $y(\pi/2) = 0$, $y'(0) = 0$, $y'(\pi/2) = 1$

Solution

Here $F = x^2 - y^2 + y''^2$

and corresponding E.L equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \longrightarrow (1)$$

Now $\frac{\partial F}{\partial y} = -2y$, $\frac{\partial F}{\partial y'} = 0$, $\frac{\partial F}{\partial y''} = 2y''$

So by ① $-2y - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(2y'') = 0$

$\Rightarrow -2y + 2y'''' = 0 \quad \Rightarrow y'''' - y = 0$

A.E equ is $D^4 - 1 = 0$

$\Rightarrow D = \pm 1, \pm i$

$\Rightarrow y(x) = Ae^{x} + Be^{-x} + C\cos x + D\sin x$

$y'(x) = Ae^{x} - Be^{-x} - C\sin x + D\cos x$

$y(0) = 0 \Rightarrow A + B + C = 0 \longrightarrow ②$

$y(\pi/2) = 0 \Rightarrow Ae^{\pi/2} + Be^{-\pi/2} + D = 0 \longrightarrow ③$

$y'(0) = 0 \Rightarrow A - B + D = 0 \longrightarrow ④$

$y'(\pi/2) = 1 \Rightarrow Ae^{\pi/2} - Be^{-\pi/2} - C = 0 \longrightarrow ⑥$

$A = \frac{1}{2}\left[1 + e^{-\pi/2}\right], \quad B = \frac{1}{2}\left[1 - e^{\pi/2}\right]$

$C = \frac{1}{2}\left[e^{\pi/2} - e^{-\pi/2}\right] = \sinh\frac{\pi}{2}$

$D = -\frac{1}{2}\left[e^{\pi/2} + e^{-\pi/2}\right] = -\cosh\frac{\pi}{2}$

Hence $y = \frac{1}{2}\left(1 + e^{-\pi/2}\right)e^{x} + \frac{1}{2}\left(1 - e^{\pi/2}\right)e^{-x} +$

$\sinh\frac{\pi}{2}\cos x - \cosh\frac{\pi}{2}\sin x$

Example 3:- Use C.V to prove that a straight line is the shortest distance b/w the two points in the plane.

Solution

Let $A(x_1, y_1)$ & $B(x_2, y_2)$ be two points in xy-plane. Then we have to minimize $I = \int_A^B \sqrt{1+y'^2}\, dx$

with $y(x_1) = y_1$ & $y(x_2) = y_2$

Here $F = \sqrt{1+y'^2}$

Solving $\frac{\partial F}{\partial y'} = c$ we get

$y = ax + b$ which is straight line.

**

⇒ Euler's Lagrange Equation for Two Independent Variables in the Integral:-

Theorem:- The extremal of the functional

$$I = \iint_R F(x, y, u, u_x, u_y)\, dx\, dy$$

where u, has different values inside the region R but is prescribed on the boundry of the region R. Moreover u is supposed to possess continuous partial derivatives upto 2nd order, is given by

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0$$

Proof)

$$\delta I = \iint_R \left[\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y\right] dx\, dy$$

Now consider $\frac{\partial}{\partial x}\left[\frac{\partial F}{\partial u_x}\delta u\right] = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial F}{\partial u_x}\delta u_x$

$$\Rightarrow \frac{\partial F}{\partial u_x}\delta u_x = \frac{\partial}{\partial x}\left[\frac{\partial F}{\partial u_x}\delta u\right] - \frac{\partial}{\partial x}\left[\frac{\partial F}{\partial u_x}\right]\delta u$$

Similarly $\frac{\partial F}{\partial u_y}\delta u_y = \frac{\partial}{\partial y}\left[\frac{\partial F}{\partial u_y}\delta u\right] - \frac{\partial}{\partial y}\left[\frac{\partial F}{\partial u_y}\right]\delta u$

G.T.:- $\iint_R (Q_x - P_y)\,dx\,dy = \int_C (P\,dx + Q\,dy)$

$$\Rightarrow \delta I = \iint_R \Big[\frac{\partial F}{\partial u}\delta u + \frac{\partial}{\partial x}\Big(\frac{\partial F}{\partial u_x}\delta u\Big) - \frac{\partial}{\partial x}\Big(\frac{\partial F}{\partial u_x}\Big)\delta u + \frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial u_y}\delta u\Big) - \frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial u_y}\Big)\delta u\Big]\,dx\,dy$$

$$= \iint_R \Big[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\Big(\frac{\partial F}{\partial u_x}\Big) - \frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial u_y}\Big)\Big]\delta u\,dx\,dy + \iint_R \Big[\frac{\partial}{\partial x}\Big(\frac{\partial F}{\partial u_x}\delta u\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial u_y}\delta u\Big)\Big]\,dx\,dy$$

$$\Rightarrow \delta I = I_1 + I_2$$

where $I_1 = \iint_R \Big[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\Big(\frac{\partial F}{\partial u_x}\Big) - \frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial u_y}\Big)\Big]\delta u\,dx\,dy$

$$I_2 = \iint_R \Big[\frac{\partial}{\partial x}\Big(\frac{\partial F}{\partial u_x}\delta u\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial u_y}\delta u\Big)\Big]\,dx\,dy$$

Applying Green's Theorem on I_2

$$\iint_R \Big[\frac{\partial}{\partial x}\Big(\frac{\partial F}{\partial u_x}\delta u\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial u_y}\delta u\Big)\Big]\,dx\,dy = \int_C \Big(\frac{\partial F}{\partial u_x}dy - \frac{\partial F}{\partial u_y}dx\Big)\delta u$$

$$\Big[\iint_R (Q_x - P_y)\,dx\,dy = \int_C (P\,dx + Q\,dy)\Big]$$

Since u is prescribed on the boundry so $\delta u = 0$

$$\Rightarrow I_2 = 0$$

$$\Rightarrow \delta I = \iint_R \Big[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\Big(\frac{\partial F}{\partial u_x}\Big) - \frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial u_y}\Big)\Big]\delta u\,dx\,dy$$

For extremal $\delta I = 0$

$$\Rightarrow \iint_R \Big[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\Big(\frac{\partial F}{\partial u_x}\Big) - \frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial u_y}\Big)\Big]\delta u\,dx\,dy = 0$$

As δu is arbitrary, so for Fundamental theorem of calculas of variation.

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0$$

***

⇒ Euler's Lagrange Equation for Three Independent Variables in The Integral:-

Theorem- Given the Functional

$$I = \iiint_R F(x, y, z, u, u_x, u_y, u_z)\, dx\,dy\,dz$$

where u has differential values in a three dimentional region R but is prescribed on the boundry surface S of the region; it is assumed that u has continuous partial derivatives upto 2nd order in the region R. Then the necessary condition for this function to have an extremum is that $u(x,y,z)$ must satisfy the PDE

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) = 0$$

Proof) Given integral implies

$$\delta I = \iiint_V \left[\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial u_z}\delta u_z\right] dx\,dy\,dz \longrightarrow (1)$$

Consider $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial F}{\partial u_x}\delta u_x$

$$\Rightarrow \frac{\partial F}{\partial u_x}\delta u_x = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u$$

Similarly $\frac{\partial F}{\partial u_y}\delta u_y = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u$

$$\frac{\partial F}{\partial u_z}\delta u_z = \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\delta u$$

So by ①

$$\delta I = \iiint_V \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\right]\delta u \, dx\,dy\,dz$$

$$- \iiint_V \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right)\right] dx\,dy\,dz$$

Consider

$$I_1 = \iiint_V \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right)\right] dx\,dy\,dz$$

$$= \iiint_V \operatorname{div}\left(\frac{\partial F}{\partial u_x}\delta u\,\hat{i} + \frac{\partial F}{\partial u_y}\delta u\,\hat{j} + \frac{\partial F}{\partial u_z}\delta u\,\hat{k}\right) dx\,dy\,dz$$

$$= \iiint_V \operatorname{div} G \cdot dv$$

where $G = \left(\frac{\partial F}{\partial u_x}\hat{i} + \frac{\partial F}{\partial u_y}\hat{j} + \frac{\partial F}{\partial u_z}\hat{k}\right)\delta u$

& $dv = dx\,dy\,dz$

Now by Gauss's Divergence Theorem

$$\iiint_V \operatorname{div} G\, dv = \iint_S (G \cdot n)\, ds$$

where n is outward normal to the region.

But since $\delta u = 0$ on the boundry surface S.

So ~~δI~~ $I_1 = 0$

$$\Rightarrow \delta I = \iiint_V \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\right]\delta u\, dx\,dy\,dz = 0$$

For extremal $\delta I = 0$

But as δu is arbitrary. So by F.T.C.V

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) = 0$$

⇒ Costrained Extrema:-

These problems are also called variational problems with constraints or variational problems with side conditions or isoperimetrical problems.

In such problems we have to find a curve $y = y(x)$ which extremize the integral

$$I = \int F \, dx$$

subject to the certain condition.

* Working Rule:-

The curve $y = y(x)$ extremize the integral $I = \int F \, dx$

subject to the condition $G = \text{constant}$

Then y satisfies the Euler Lagrange Equation

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$

where

$$H = F + \lambda G \quad , \lambda \text{ is any parameter.}$$

Example 1:- Find the curve joining the points $A(0,0)$ & $B(1,0)$ with the given length s.t the y coordinate of its centroid is minimum.

Solution: Let $y = y(x)$ be the length of the curve. then y co-ordinate of its centroid is given by

$$\bar{y} = \frac{\int_0^1 m y \, ds}{\int_0^1 m \, ds} = \frac{\int_0^1 y \, ds}{\int_0^1 ds}$$

$$\Rightarrow \bar{y} = \frac{\int_0^1 y\sqrt{1+y'^2}}{l} \, dx$$

where $l = \int_0^1 \sqrt{1+y'^2}\,dx$ is the length of the curve througth the given points.

Thus we have to find the curve $y=y(x)$ which minimize $\bar{y}$ subject to the condition

$$l = \int_0^1 \sqrt{1+y'^2}\,dx$$

This is an isoperimetric problem with

$$F = y\sqrt{1+y'^2}, \quad G = \sqrt{1+y'^2}$$

Let $H = F + \lambda G$

$$= y\sqrt{1+y'^2} + \lambda\sqrt{1+y'^2}$$

Eq corresponding Euler's Lagrange Equation be

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$

Now

$$\frac{\partial H}{\partial y} = \sqrt{1+y'^2}, \quad \frac{\partial H}{\partial y'} = \frac{yy'}{\sqrt{1+y'^2}} + \frac{\lambda y'}{\sqrt{1+y'^2}}$$

So $\sqrt{1+y'^2} - \dfrac{d}{dx}\left[\dfrac{(\lambda+y)y'}{\sqrt{1+y'^2}}\right] = 0$

But This will lead to the complication.

As H is independent of x, so we use

$$H - y'\frac{\partial H}{\partial y'} = c$$

$$\Rightarrow (y+\lambda)\sqrt{1+y'^2} - y'\left[\frac{(\lambda+y)y'}{\sqrt{1+y'^2}}\right] = c$$

$$\Rightarrow (y+\lambda)(1+y'^2) - (\lambda+y)y'^2 = c\sqrt{1+y'^2}$$

$$\Rightarrow y + \cancel{yy'^2} + \lambda + \cancel{\lambda y'^2} - \cancel{\lambda y'^2} - \cancel{yy'^2} = c\sqrt{1+y'^2}$$

$$\Rightarrow (y+\lambda)^2 = c^2(1+y'^2)$$

$$\Rightarrow \frac{(y+\lambda)^2 - c^2}{c^2} = y'^2$$

$$\Rightarrow y' = \frac{\sqrt{(y+\lambda)^2 - c^2}}{c} \qquad \Rightarrow \frac{c}{\sqrt{(y+\lambda)^2 - c^2}}\, dy = dx$$

$$\Rightarrow c \cosh^{-1}\left(\frac{y+\lambda}{c}\right) = x + a$$

$y(0) = 0 \quad \Rightarrow c \cosh^{-1}\left[\frac{0+\lambda}{c}\right] = 0 + a$

$$\Rightarrow \cosh^{-1}\left[\frac{\lambda}{c}\right] = a/c$$

$y(1) = 0 \quad \Rightarrow c \cosh^{-1}\left[\frac{0+\lambda}{c}\right] = 1 + a$

$$\Rightarrow \cosh^{-1}\left(\frac{\lambda}{c}\right) = \frac{1}{c} + \frac{a}{c}$$

Now $\frac{\lambda}{c} = \cosh\left(\frac{a}{c}\right)$ & $\frac{\lambda}{c} = \cosh\left[\frac{a}{c} + \frac{1}{c}\right]$

$$\Rightarrow \cosh\left(\frac{a}{c}\right) = \cosh\left(\frac{a}{c} + \frac{1}{c}\right)$$

$$\Rightarrow -\frac{a}{c} = \frac{a}{c} + \frac{1}{c} \Rightarrow \frac{-2a}{c} = \frac{1}{c}$$

$$\boxed{\begin{array}{l} \cos\theta_1 = \cos\theta_2 \\ \Rightarrow \theta_1 = \pm\theta_2 \end{array}}$$

$$\Rightarrow a = -\frac{1}{2}$$

$$\Rightarrow c \cosh^{-1}\left(\frac{y+\lambda}{c}\right) = x - \frac{1}{2} = \frac{2x-1}{2}$$

$$\Rightarrow \frac{y+\lambda}{c} = \cosh\left[\frac{2x-1}{2c}\right]$$

$$\Rightarrow y = c\left[\cosh\left(\frac{2x-1}{2c}\right) - \frac{\lambda}{c}\right]$$

$$\Rightarrow y = c\left[\cosh\left(\frac{2x-1}{2c}\right) - \cosh\left(\frac{a}{c}\right)\right]$$

$$\Rightarrow y = c\left[\cos\left(\frac{2x-1}{2c}\right) - \cosh\left(\frac{1}{2c}\right)\right]$$

***

Example 2:- Show that a solid of revolution which for a given surface area has maximum volume is a sphere

OR find a curve which generates a surface of revolution of a given area which encloses maximum volume.

Solution Let $y = y(x)$ with $y(0) = 0$, $y(a) = 0$ be routed about X-axis so as to generate a surface of ~~ro~~ revolution. An element of surface area is $2\pi y\,ds$, where $ds = \sqrt{(dx)^2 + (dy)^2}$. Then the total area will be

$$A = 2\pi \int_0^a y\sqrt{1+y'^2}\,dx$$

Also in this case

$$V = \pi \int_0^a y^2\,dx$$

Here we have to maximize V subject to A

$$\Rightarrow F = y^2 \quad , \quad G = y\sqrt{1+y'^2}$$

$$\Rightarrow H = F + \lambda G \quad \Rightarrow H = y^2 + \lambda y\sqrt{1+y'^2}$$

Since H is independent of x, so

Euler Lagrange equation is

$$H - y'\left(\frac{\partial H}{\partial y'}\right) = C$$

$$\Rightarrow y^2 + \lambda y\sqrt{1+y'^2} - y'\left[\frac{\lambda y y'}{\sqrt{1+y'^2}}\right] = C$$

$$y^2\sqrt{1+y'^2} + \lambda y + \cancel{\lambda y y'^2} - \cancel{\lambda y y'^2} = C\sqrt{1+y'^2}$$

$$\Rightarrow \frac{\lambda y}{C - y^2} = \sqrt{1+y'^2}$$

$$\Rightarrow \frac{\lambda^2 y^2}{(c-y^2)^2} = 1+y'^2 \Rightarrow y'^2 = \frac{\lambda^2 y^2 - (c-y^2)^2}{(c-y^2)^2}$$

$$\Rightarrow y' = \frac{\sqrt{\lambda^2 y^2 - (c-y^2)^2}}{c-y^2}$$

Now using $y(0) = 0$

$$\Rightarrow y'(0) = \frac{\sqrt{0-c^2}}{c-0}$$

$$\Rightarrow y'(0) = \frac{\sqrt{-c^2}}{c} \Rightarrow c\,y'(0) = \sqrt{-c^2}$$

$\Rightarrow c = 0$ $\quad \because y'(0)$ is finite & y is real

So $y' = \dfrac{\sqrt{\lambda^2 y^2 - y^4}}{-y^2} \Rightarrow y' = -\dfrac{\sqrt{\lambda^2 - y^2}}{y}$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{\lambda^2 - y^2}}{y}$$

$$\Rightarrow \frac{y}{\sqrt{\lambda^2 - y^2}}\,dy = -dx$$

$$\Rightarrow -\frac{1}{2}\int (\lambda^2 - y^2)^{-1/2}(-2y)\,dy = -\int dx$$

$$\Rightarrow \sqrt{\lambda^2 - y^2} = x + b$$

$$\Rightarrow \lambda^2 - y^2 = (x+b)^2 \Rightarrow (x+b)^2 + y^2 = \lambda^2$$

which is equation of a sphere with centre $(-b, 0)$ and radius λ. Which is required proof.

***

Example 3:- [Dido's Problem] Find the closed curve of given length which encloses maximum area.

Solution

clearly the required curve is convex

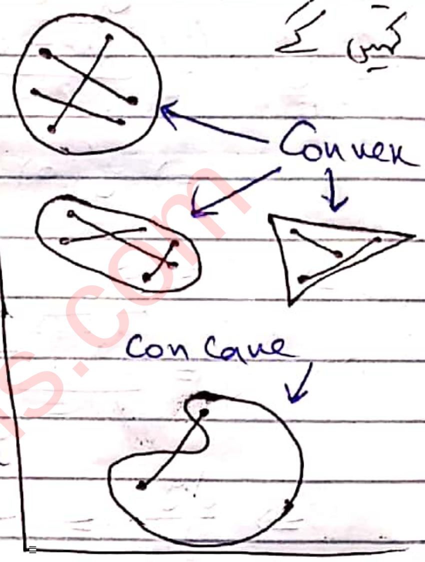
Also we note that a straight line that bisects a closed curve bounding a maximum area will divide the area in equal two halves.

We suppose that x-axis is the line which divides the curve in two equal halves and it meets the x-axis at $A(x_1, 0)$ and $B(x_2, 0)$ (as)

Area enclosed $= A_1 = I = \int_A^B y\,dx$

$$\Rightarrow I = \int_A^B y\,dx$$

ایسی Closed Curve جس میں سے 2 Points کو ملا کر اگر ملایا جائے تو ملانے والی Line Segment بھی اسی Curve میں آ جائے تو اسے Convex کہیں گے



Also length of the curve $= l = \int_A^B \sqrt{1+y'^2}\,dx$

We have to maximize I subject to l

Here $F = y$, $G = \sqrt{1+y'^2}$

$$\Rightarrow H = y + \lambda\sqrt{1+y'^2}$$

Euler Lagrange Equation is

$$H - y'\frac{\partial H}{\partial y'} = c$$

$$\Rightarrow y + \lambda\sqrt{1+y'^2} - y'\left[\frac{\lambda y'}{\sqrt{1+y'^2}}\right] = c$$

$$y\sqrt{1+y'^2} + \lambda + \cancel{\lambda y'^2} - \cancel{\lambda y'^2} = c\sqrt{1+y'^2}$$

$$\Rightarrow (c-y)\sqrt{1+y'^2} = \lambda$$

$$\Rightarrow \sqrt{1+y'^2} = \frac{\lambda}{c-y} \quad \Rightarrow y'^2 = \frac{\lambda^2}{(c-y)^2} - 1$$

$$\Rightarrow y'^2 = \frac{\lambda^2 - (c-y)^2}{(c-y)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{\lambda^2 - (c-y)^2}}{c-y}$$

$$\Rightarrow \int \frac{c-y}{\sqrt{\lambda^2 - (c-y)^2}}\, dy = \int dx$$

$$\Rightarrow \int [\lambda^2 - (c-y)^2]^{-1/2} (c-y)\, dy = \int dx$$

$$\Rightarrow \frac{1}{2}\int [\lambda^2 - (c-y)^2]^{-1/2}\, 2(c-y)\, dy = \int dx$$

$$\Rightarrow [\lambda^2 - (c-y)^2]^{1/2} = x + d$$

$$\Rightarrow \lambda^2 - (c-y)^2 = (x+d)^2$$

$$\Rightarrow (x+d)^2 + (y-c)^2 = \lambda^2$$

$$\Rightarrow (x-\lambda_1)^2 + (y-\lambda_2)^2 = \lambda^2$$

which is the equation of a circular arc. So Required closed curve is a circle.

***

⇒ Geodesic Problems:-

A Geodesic is a curve of shortest length joining two points in space.

***

Example:- Find the arc of shortest length b/w two given points in a plane, using polar coordinates (r, θ)

Solution:-

Let A and B be two points in plane then arc length between A and B is given by

$$I = \int_A^B ds = \int_A^B \sqrt{(dr)^2 + (r\,d\theta)^2}$$

$$= \int_A^B \sqrt{r^2 + r'^2}\; d\theta$$

Subject to $r(\theta_1) = \text{constant}$
and $r(\theta_2) = \text{constant}$

Here $F(\theta, r, r') = \sqrt{r^2 + r'^2}$

Since F is independent of θ explicitly, so Euler's Lagrange Equation is

$$F - r' \frac{\partial F}{\partial r'} = c$$

$$\Rightarrow \sqrt{r^2 + r'^2} - r' \frac{r'}{\sqrt{r^2 + r'^2}} = c \quad \Rightarrow r^2 + \cancel{r'^2} - \cancel{r'^2} = c\sqrt{r^2 + r'^2}$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{r}{c}\sqrt{r^2 - c}$$

$$\Rightarrow \int \frac{c\,dr}{r\sqrt{(r^2 - c^2)}} = \int d\theta \quad \Rightarrow \sec^{-1}\left(\frac{r}{c}\right) = \theta + d$$

$$\Rightarrow \frac{r}{c} = \sec(\theta+\alpha) \quad \Rightarrow c = r\cos(\theta+\alpha)$$

$$\Rightarrow c = r[\cos\theta\cos\alpha - \sin\theta\sin\alpha]$$

$$= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$$

$$\Rightarrow c = x\cos\alpha - y\sin\alpha$$

$$\Rightarrow ax + by + c = 0 \quad \text{where } a = -\cos\alpha,\ b = \sin\alpha$$

Which is equation of line

Hence curve of shortest length in a plane is a straight line.

———————————

Example 2:- Find a curve of shortest length on the surface of a sphere.

Solution:-

Let A and B be two points on the surface of a sphere of radius "a" then arc length between A and B is given by

$$I = \int_A^B ds = \int_A^B \sqrt{(dr)^2 + (r\,d\theta)^2 + (r\sin\theta\, d\phi)^2}$$

As $r = a$ so $dr = 0$

$$\Rightarrow I = \int_A^B \sqrt{a^2 d\theta^2 + a^2\sin^2\theta\, d\phi^2}$$

$$= \int_A^B a\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\, d\theta$$

Here we have to minimize it

Now

$$F(\theta,\phi) = a\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}$$

$$= a\sqrt{1+\sin^2\theta(\phi')^2}$$

Eulers Lagrange Equation is

$$\frac{\partial F}{\partial \phi} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \phi'}\right) = 0 - \frac{d}{d\theta}\left(\frac{\sin^2\theta\, \phi'}{\sqrt{1+\sin^2\theta\, \phi'^2}}\right) = 0$$

$$\Rightarrow \frac{\sin^2\theta\, \phi'}{\sqrt{1+\sin^2\theta\,(\phi')^2}} = C_1$$

$$\Rightarrow \frac{\sin^4\theta\, \phi'^2}{1+\sin^2\theta\, \phi'^2} = C_1^2 \qquad \Rightarrow \phi' = \frac{C_1}{\sin\theta\sqrt{\sin^2\theta - C_1^2}}$$

$$\Rightarrow \int d\phi = \int \frac{C_1}{\sin\theta\sqrt{\sin^2\theta - C_1^2}}\, d\theta$$

$$\Rightarrow \phi = \int \frac{C_1}{\sin\theta\sqrt{\sin^2\theta - C_1^2}}\, d\theta$$

$$= \int \frac{C_1}{\frac{1}{\operatorname{cosec}\theta}\sqrt{\frac{1}{\operatorname{cosec}^2\theta} - C_1^2}}\, d\theta = \int \frac{C_1 \operatorname{cosec}^2\theta}{\sqrt{1 - C_1^2\operatorname{cosec}^2\theta}}\, d\theta$$

$$= \int \frac{C_1 \operatorname{cosec}^2\theta}{\sqrt{1 - C_1^2(1+\cot^2\theta)}}\, d\theta$$

Put $\cot\theta = \psi$

$$-\operatorname{cosec}^2\theta\, d\theta = d\psi \Rightarrow \operatorname{cosec}^2\theta\, d\theta = -d\psi$$

$$\Rightarrow \phi = C_1 \int \frac{-d\psi}{\sqrt{1 - C_1^2(1+\psi^2)}}$$

$$= -\int \frac{d\psi}{\sqrt{1 - C_1^2 - C_1^2\psi^2}} = \frac{-C_1}{C_1}\int \frac{d\psi}{\sqrt{\frac{1-C_1^2}{C_1^2} - \psi^2}}$$

$$= \frac{-C_1}{C_1}\int \frac{d\psi}{\sqrt{C_2^2 - \psi^2}}, \quad C_2 = \frac{1-C_1^2}{C_1^2}$$

$$\Rightarrow \phi = \frac{c_1}{c_1}\int \frac{-d\psi}{\sqrt{c_2^2-\psi^2}} \quad \Rightarrow \phi = \cos^{-1}\left(\frac{\psi}{c_2}\right)+c_3$$

$$\Rightarrow \phi = \cos^{-1}\left(\frac{\cot\theta}{c_2}\right)+c_3$$

$$\Rightarrow \cos(\phi - c_3) = \frac{\cot\theta}{c_2}$$

$$\Rightarrow c_2[\cos\phi\cos c_3 + \sin\phi\sin c_3] = \cot\theta$$

$$\Rightarrow (c_2\cos c_3)\cos\phi + (c_2\sin c_3)\sin\phi = \frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow \alpha\sin\theta\cos\phi + \beta\sin\theta\sin\phi = \cos\theta$$

$$\Rightarrow \alpha(a\sin\theta\cos\phi)+\beta(a\sin\theta\sin\phi) = a\cos\theta$$

$$\Rightarrow \alpha x + \beta y = z$$

Which is the equation of the plane through the centre of the sphere.

Hence the curve of shortest length joining A and B is the arc of the great circle through A and B.

**

Example 3:- Find the geodesic curve for the cylinder $x^2+y^2=a^2$

Solution

$$I = \int_A^B \sqrt{dr^2 + r^2 d\theta^2 + dz^2}$$

As $r = a \Rightarrow dr = 0$

$$\Rightarrow I = \int_A^B \sqrt{a^2 d\theta^2 + dz^2} = \int_A^B \sqrt{a^2+z'^2}\, d\theta$$

Euler's Lagrange Equation is

$$\frac{\partial F}{\partial z} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial z'}\right) = 0$$

$$\Rightarrow 0 - \frac{d}{d\theta}\left[\frac{z'}{\sqrt{a^2+z'^2}}\right] = 0$$

$$\Rightarrow z'/\sqrt{a^2+z'^2} = c_1$$

$$\Rightarrow z' = c_1\sqrt{a^2+z'^2} \quad \Rightarrow z'^2 = c_1^2 a^2 + c_1^2 z'^2$$

$$\Rightarrow (1-c_1^2)z'^2 = c_1^2 a^2$$

$$\Rightarrow z'^2 = \frac{c_1^2 a^2}{1-c_1^2} = \alpha^2$$

$$\Rightarrow z' = \alpha \quad \Rightarrow \frac{dz}{d\theta} = \alpha$$

$$\Rightarrow z = \alpha\theta + \beta \quad \Rightarrow z = \alpha \operatorname{Tan}^{-1}\left(\frac{y}{x}\right) + \beta$$

$$\Rightarrow \frac{z-\beta}{\alpha} = \operatorname{Tan}^{-1}\frac{y}{x}$$

$$\Rightarrow y = x \operatorname{Tan}\left(\frac{z-\beta}{\alpha}\right)$$

Which is required curve.

* ——— * ——— *

Example 4:- Find the shortest distance b/w the points $A(1,-1,0)$ and $B(2,1,-1)$ in the plane $15x-7y+z-22=0$

Solution

Here $I = \int_A^B \sqrt{(dx)^2+(dy)^2+(dz)^2}$

$$= \int_A^B \sqrt{1+(y')^2+(z')^2}\, dx$$

Here $F = \sqrt{1+y'^2+z'^2}$ and $G = 15x-7y+z-22$

And Euler's Lagrange Equations are

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \longrightarrow (1)$$

& $\frac{\partial H}{\partial z} - \frac{d}{dx}\left(\frac{\partial H}{\partial z'}\right) = 0 \longrightarrow (2)$

where $H = \sqrt{1+y'^2+z'^2} + \lambda(15x - 7y + z - 22)$

By (1) $-7\lambda - \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2+z'^2}}\right] = 0$

$$\Rightarrow 7\lambda + \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2+z'^2}}\right] = 0 \longrightarrow (3)$$

By (2) $\lambda - \frac{d}{dx}\left[\frac{z'}{\sqrt{1+y'^2+z'^2}}\right] = 0 \longrightarrow (4)$

Eliminating λ from (3) & (4)

$$-\frac{d}{dx}\left[\frac{y' + 7z'}{\sqrt{1+y'^2+z'^2}}\right] = 0$$

$$\Rightarrow \frac{y'+7z'}{\sqrt{1+y'^2+z'^2}} = c_1 \longrightarrow (*)$$

Now given that

$$15x - 7y + z - 22 = 0$$

$$\Rightarrow 15 - 7y' + z' = 0 \quad \Rightarrow z' = 7y' - 15$$

So $(*) \Rightarrow \frac{y' + 7(7y'-15)}{\sqrt{1+y'^2+(7y'-15)^2}} = c_1$

$$\Rightarrow 50(250 - c_1^2)y'^2 + 210(c_1^2 - 50)y' + (11025 - 226) = 0$$

$$\Rightarrow \alpha_1 y'^2 + \beta_1 y' + \gamma_1 = 0$$

Which is quadratic in y'. Let α be the one root of this equation. Then

$y' = \alpha \qquad \Rightarrow y = \alpha x + \beta$

Now $y(1) = -1 \Rightarrow \alpha + \beta = -1$

& $y(2) = 1 \Rightarrow 2\alpha + \beta = 1$

$$\alpha = 2$$

$$\Rightarrow \beta = -3$$

Hence $y = 2x - 3$

So $z = 7y - 15x + 22 = -x + 1$

So Shortest Distance $= \int_1^2 \sqrt{1 + (2)^2 + (-1)^2}\, dx$

$$= \sqrt{6}$$

⇒ Applications To Mechanics:-

* Principle of Least Action:-

Let a particle move in an external field of force which is conservative. If the motion takes place in the interval of time t_1 to t_2 where $t_2 > t_1$. Then the actual path traced by the particle is the one along which

$$I = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

is minimum where $\mathcal{L}$ is Lagrangian and for the conservative system

$$\mathcal{L} = K.E - P.E = T - V$$

⇒ Hamilton's Principle:-

According to this principle, the path of motion of a rigid body in time interval $t_2 - t_1$ is such that the integral

$$I = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

has stationary value.

Where $\mathcal{L}$ is Lagrangian function.

**

Example 1:- Find the equation of Motion of particle moving in a conservative field of force described by the function $V(x, y, z)$

Solution

Here $T = \frac{1}{2} m v^2 = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$\mathcal{L} = T - V = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$$

By the principle of least action the equation of motion are given by

$$I = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

where I has the minimum value which implies that

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \longrightarrow (1)$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \longrightarrow (2)$$

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0 \longrightarrow (3)$$

$(1) \Rightarrow$ $$-\frac{\partial V}{\partial x} - \frac{d}{dt}(m\dot{x}) = 0$$

$$\Rightarrow \frac{\partial V}{\partial x} + m\ddot{x} = 0 \longrightarrow (4)$$

② implies that $\frac{\partial V}{\partial y} + m\ddot{y} = 0 \longrightarrow ⑤$

③ implies that $\frac{\partial V}{\partial z} + m\ddot{z} = 0 \longrightarrow ⑥$

Equation ④, ⑤ and ⑥ are required equation of motions.

***

Example 2:- Use the principle of Least Action of obtain... describing the vibration of Simple Harmonic oscilation.

Solution

By definition for a particle executing S.H. Motion along X-axis. The force F is given by

$$F = -Kx$$

where x denotes displacement from centre of vibration, the constant K which is +ve is called spring constant.

* The K.E is given by

$$T = \frac{1}{2} m\dot{x}^2$$

As $F = dV/dx$ so $V = -\int Kx\,dx$

$$= -\frac{1}{2}Kx^2$$

So $\mathcal{L} = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2$

So E-L equation is

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$$\Rightarrow Kx - \frac{d}{dt}(m\dot{x}) = 0$$

$$\Rightarrow Kx - m\ddot{x} = 0$$

$$\Rightarrow m\ddot{x} - Kx = 0$$

** ——————— **

Example 3:- Obtain the equation of motion of a stretched string by invoking Hamilton's principle of least action.

Solution

Let us consider a string of uniform density ρ and length l. s.t initially it is streched along X-axis, and when plucked. Let it vibrate in XY-plane. Consider the K.E of an element ds of the string. Then

$$dT = \frac{1}{2} dm v^2 = \frac{1}{2}\rho y_t^2 \, ds$$

$$\Rightarrow T = \frac{1}{2}\rho \int_0^l y_t^2 \, ds$$

Here we assume that deflection of the string is very small so

$$T = \frac{1}{2}\rho \int_0^l y_t^2 \, dx$$

and P.E = Increase in length due to tension of the string = Work done by the tension in producing extension $ds - dx$

$$= \int_0^l (ds - dx)\tau$$

$$= \tau\left[\int_0^l \left[1 + \frac{1}{2} y_x^2\right] dx - l\right]$$

$$= \tau\left[\cancel{l} + \frac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx - \cancel{l}\right]$$

$$= \frac{\tau}{2}\int_0^l y_x^2 \, dx$$

So $\mathcal{L} = T - V = \frac{\rho}{2}\int_0^l y_t^2\,dx - \frac{T}{2}\int_0^l y_x^2\,dx$

$$= \int_0^l (\rho y_t^2 - T y_x^2)\,dx$$

By Hamilton's principle the equation of motion of the string will correspond to stationary value of

$$\int_{t_1}^{t_2} \mathcal{L}\,dt \quad \text{i.e.} \quad \frac{1}{2}\int_{t_1}^{t_2}\int_0^l [\rho y_t^2 - T y_x^2]\,dx\,dt$$

Let $F = \rho y_t^2 - T y_x^2$

For the extremal

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial y_t}\right) = 0 \longrightarrow (1)$$

$$\frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial y_x} = -2T y_x, \quad \frac{\partial F}{\partial y_t} = 2\rho y_t$$

So by (1)

$$0 - \frac{\partial}{\partial x}(-2T y_x) - \frac{\partial}{\partial t}(2\rho y_t) = 0$$

$$\Rightarrow 2T\frac{\partial^2 y}{\partial x^2} - 2\rho\frac{\partial^2 y}{\partial t^2} = 0$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} - \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} = 0$$

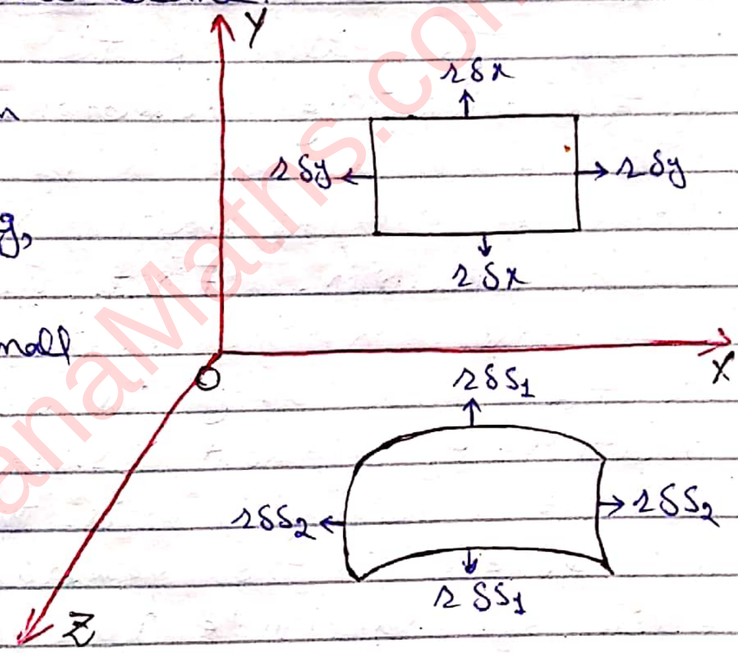
***

Example 4:* Discuss the vibrations of a stretched membrane and obtain its equ. of motion by using Hamilton's Principle.

Solution

We consider the transverse ~~motion~~ vibration of a membrane of arbitrary shape which is supposed to be perfectly flexible but inextensible. First we calculate K.E and P.E of the vibrating membrane.

Let the membrane lies in XY-plane. As in case of the string, we suppose that displacements are small and are $\perp$ to XY-plane i.e in the direction of Z-axis.



When the membrane is vibrating a rectangular element of area ABCD originally lying in XY-plane is brought to the position of curved element ds in space centred around the point (x,y,z) at time t. Here the corresponding z coordinate denotes the displacement of the membrane at time 't' i.e $z = z(x,y,t)$.

To calculate K.E and P.E we analyse the motion of an element of the membrane.

$$T = \frac{1}{2}\int z_t^2\, dm$$

$$= \frac{1}{2}\int z_t^2 \rho\, ds = \frac{\rho}{2}\int z_t^2\, dx\, dy$$

where we have taken $ds = dx\,dy$ in the 1st approximation

$$\because dx = \sqrt{1+z_x^2+z_y^2}\; dx\, dy$$

and z_x and z_y are neglecting because vibrations are very small.

To calculate the potential energy V of the vibrating membrane, we have to calculate work done in beginning membrane from a original position XY- plane to its curved position. First we calculate the work done on the rectangular element $ABCD$ of the area $dx\,dy$ in bringing it to the position of element δs.

Now if T is the tension per unit length in the membrane then two forces each of magnitude $T\,dx$ are pulling the sides AD and BC, where as two forces each of magnitude $T\,dy$ are pulling the sides AB and CD of the element $ABCD$.

Now remembering that the work done on these sides are respectively $(ds_1 - dx)T\,dy$ and $(ds_2 - dy)T\,dx$.

Now work done for the element of area ds will be

$$dV = T(ds_1 - dx)\,dy + T(ds_2 - dy)\,dx$$

$$= T(ds_1\,dy + ds_2\,dx - 2dx\,dy)$$

Now
$$(ds_1 dy) = \sqrt{(dx)^2+(dz)^2}\, dy = \sqrt{1+z_x^2}\, dx\, dy$$
$$= \left(1+\tfrac{1}{2}z_x^2\right) dx\, dy$$
$$ds_2\, dx = \sqrt{(dy)^2+(dz)^2}\, dx = \left(1+\tfrac{1}{2}z_y^2\right) dx\, dy$$

Where we have used the fact that z_x and z_y are small because of displacements being small. Therefore we can write
$$dV = T\left[\left(1+\tfrac{1}{2}z_x^2\right)dx\,dy + \left(1+\tfrac{1}{2}z_y^2\right)dx\,dy - 2\,dx\,dy\right]$$
$$= \tfrac{T}{2}\left(z_x^2+z_y^2\right) dx\, dy$$

Hence $\quad V = \tfrac{T}{2}\int_S \left(z_x^2+z_y^2\right) dx\, dy$

Hence $\quad L = T - V = \frac{1}{2}\int_S \left(\rho z_t^2 - T\left(z_x^2+z_y^2\right)\right) dx\, dy$

By Hamilton's principle
$$\frac{1}{2}\int_{t_1}^{t_2}\int_S \left(\rho z_t^2 - T z_x^2 - T z_y^2\right) dx\, dy\, dt = 0$$

Here $\quad F = F(t, x, y, z, z_t, z_x, z_y) = \rho z_t^2 - T z_x^2 - T z_y^2$

And corresponding Euler Lagrang equation is
$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial z_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = 0$$
$$\Rightarrow 0 - \frac{\partial}{\partial t}(2\rho z_t) - \frac{\partial}{\partial x}(-2T z_x) - \frac{\partial}{\partial y}(-2T z_y) = 0$$
$$\Rightarrow -\rho z_{tt} + T z_{xx} + T z_{yy} = 0$$
$$\Rightarrow z_{xx} + z_{yy} = \frac{\rho}{T} z_{tt}$$
$$\Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad c = \sqrt{T/\rho}$$

is required

Sum More Example

Example 1:- If ϕ and ψ are two function s.t

$$\delta\phi(x_1) = \delta\phi(x_2) = 0$$

$$\delta\psi(x_1) = \delta\psi(x_2) = 0$$

Then show that the necessary condition for

$$I = \int_{x_1}^{x_2} F(x, \phi, \psi, \phi', \psi')\,dx$$

to have a stationary value is that

$$\frac{\partial F}{\partial \phi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \phi'}\right) = 0 \quad \& \quad \frac{\partial F}{\partial \psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right) = 0$$

Solution

Given $I = \int_{x_1}^{x_2} F(x, \phi, \psi, \phi', \psi')\,dx$

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \phi}\delta\phi + \frac{\partial F}{\partial \psi}\delta\psi + \frac{\partial F}{\partial \phi'}\delta\phi' + \frac{\partial F}{\partial \psi'}\delta\psi'\right] dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \phi}\delta\phi - \frac{d}{dx}\left(\frac{\partial F}{\partial \phi'}\right)\delta\phi' + \frac{\partial F}{\partial \psi}\delta\psi - \frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right)\delta\psi'\right] dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \phi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \phi'}\right)\right]\delta\phi\,dx + \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right)\right]\delta\psi\,dx$$

For extremal $\delta I = 0$

$$\Rightarrow \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \phi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \phi'}\right)\right]\delta\phi\,dx + \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right)\right]\delta\psi\,dx = 0$$

Since $\delta\phi$ and $\delta\psi$ are arbitrary so

$$\frac{\partial F}{\partial \phi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \phi'}\right) = 0$$

and $$\frac{\partial F}{\partial \psi} - \frac{d}{dx}\left(\frac{\partial F}{\partial \psi'}\right) = 0$$

———

Example 2:- Solve that the Euler's Lagrange Equs. for the functional

$$I = \int_A^B F(x, y, y', z, z')\,dx$$

Admit that

(i) $\frac{\partial F}{\partial y'} = c$ if F is independent of y

(ii) $F - y'\frac{\partial F}{\partial y'} - z'\frac{\partial F}{\partial z'} = c$ if F is independent of x.

Solution Corresponding Euler Lagrange Equations are

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \longrightarrow ①$$

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0 \longrightarrow ②$$

(i) If F is independent of "y" then

$$\frac{\partial F}{\partial y} = 0 \quad \text{so then} \quad 0 - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow \int \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)dx = \int 0 \cdot dx$$

$$\Rightarrow \frac{\partial F}{\partial y'} = c$$

(ii) Here $F = F(y, y', z, z')$

$$\Rightarrow dF = \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial y'}dy' + \frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial z'}dz'$$

Also by ① $\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx}$

$$\Rightarrow \frac{\partial F}{\partial y}dy = y'\, d\left(\partial F/\partial y'\right)$$

Similarly $\frac{\partial F}{\partial z}dz = z'\, d\left(\partial F/\partial z'\right)$

Hence

$$dF = y' d\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'}dy' + z' d\left(\partial F/\partial z'\right) + \frac{\partial F}{\partial z'}dz'$$

$$= d\left[y'\frac{\partial F}{\partial y'} + z'\frac{\partial F}{\partial z'}\right]$$

$$\Rightarrow F - y'\frac{\partial F}{\partial y'} - z'\frac{\partial F}{\partial z'} = c$$

Example 3:- Find the eigen values and eigen vectors of the functional

$$I = \int_0^3 \left[(2x+3)^2 y'^2 - y^2\right] dx$$

subject to conditions $y(0) = 0 = y(3)$ and

$$\int_0^3 y^2(x)\, dx = 1$$

Solution

Here $F = (2x+3)^2 y'^2 - y^2$ & $G = y^2$

So $H = F + \lambda G = (2x+3)^2 y'^2 - y^2 + \lambda y^2$

The corresponding Lagrange equation is

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$

$$\Rightarrow -2y + 2\lambda y - \frac{d}{dx}\{2(2x+3)^2 y'\} = 0$$

$$\Rightarrow \frac{d}{dx}\{2(2x+3)^2 y'\} + 2y - 2\lambda y = 0$$

$$\Rightarrow \frac{d}{dx}\{(2x+3)^2 y'\} + (1-\lambda) y = 0$$

$$\Rightarrow \frac{d}{dx}\left\{(2x+3)^2 \frac{dy}{dx}\right\} + (1)y + (-1)\lambda y = 0$$

Which is an S.L equation with

$P(x) = (2x+3)^2$, $q(x) = 1$, $r(x) = -1$

The above equation can be written as

$$(2x+3)^2 \frac{d^2y}{dx^2} + 4(2x+3)\frac{dy}{dx} + (1-\lambda)y = 0 \longrightarrow (1)$$

Which is Cauchy Euler's equation

So put $2x+3 = e^t \Rightarrow \ln(2x+3) = t$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{2}e^t$$

$$\Rightarrow \frac{dy}{dx} = 2e^{-t}\frac{dy}{dt}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 4e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

So equ ① becomes

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + (1-\lambda)y = 0$$

Auxiliary equation is

$$4D^2 + 4D + (1-\lambda) = 0$$

$$\Rightarrow D = \frac{-4 \pm \sqrt{16 - 4(1-\lambda)(4)}}{2(4)} = \frac{-1 \pm \sqrt{\lambda}}{2}$$

If $\lambda \geq 0$ then solution is trivial. If $\lambda < 0$ then we can write $\lambda = -\lambda_0$, $\lambda_0 > 0$

Then general solution is given by

$$y = e^{-t/2}\left[c_1 \cos\sqrt{\lambda_0}\, t/2 + c_2 \sin\sqrt{\lambda_0}\, t/2\right]$$

$$\Rightarrow y(x) = (2x+3)^{-1/2}\left[c_1 \cos\left(\tfrac{1}{2}\sqrt{\lambda_0}\, \ln(2x+3)\right) + c_2 \sin\left(\tfrac{1}{2}\sqrt{\lambda_0}\, \ln(2x+3)\right)\right]$$

Applying $y(0) = 0$ & $y(3) = 0$

$$\frac{c_1}{\sqrt{3}} \cos\left[\tfrac{1}{2}\sqrt{\lambda_0}\ln 3\right] + \frac{c_2}{\sqrt{3}} \sin\left[\tfrac{1}{2}\sqrt{\lambda_0}\ln 3\right] = 0 \longrightarrow ②$$

and $$\frac{c_1}{3}\cos\left[\sqrt{\lambda_0}\ln 3\right] + \frac{c_2}{3}\sin\left[\sqrt{\lambda_0}\ln 3\right] = 0$$

For non trivial solution

$$\begin{vmatrix} \cos(\tfrac{1}{2}\sqrt{\lambda_0}\ln 3) & \sin(\tfrac{1}{2}\sqrt{\lambda_0}\ln 3) \\ \cos(\sqrt{\lambda_0}\ln 3) & \sin(\sqrt{\lambda_0}\ln 3) \end{vmatrix} = 0$$

$$\Rightarrow \cos(\tfrac{1}{2}\sqrt{\lambda_0}\ln 3)\sin(\sqrt{\lambda_0}\ln 3) - \sin(\tfrac{1}{2}\sqrt{\lambda_0}\ln 3)\cos(\sqrt{\lambda_0}\ln 3) = 0$$

$$\Rightarrow \sin\left[\sqrt{\lambda_0}\ln 3 - \tfrac{1}{2}\sqrt{\lambda_0}\ln 3\right] = 0$$

$$\Rightarrow \sin\left[\tfrac{1}{2}\sqrt{\lambda_0}\ln 3\right] = 0$$

$$\Rightarrow \tfrac{1}{2}\sqrt{\lambda_0}\ln 3 = n\pi \qquad n = \pm 1, \pm 2, \pm 3, \cdots$$

$$\Rightarrow \sqrt{\lambda_0} = \frac{2n\pi}{\ln 3} \quad \Rightarrow \lambda_{0n} = \frac{4n^2\pi^2}{(\ln 3)^2}, \; n = 1,2,3,\ldots$$

Which are required eigen values.
Now by ②

$$C_1 \cos n\pi + C_2 \sin n\pi = 0$$

Hence the corresponding eigen functions are

$$y_n(x) = \frac{A_n}{\sqrt{2x+3}} \sin\left[\frac{n\pi \ln(2x+3)}{\ln 3}\right], \; n = 1,2,3,\ldots$$

Where the constants D_n can be determined from the condition $\int_0^3 y^2(x)\,dx = 1$

Which gives $D_n = \pm 2/\sqrt{\ln 3}$

***

Example 4:- On what curves can the functional

$$I = \int_0^{\pi/2} (y'^2 - y^2)\,dx$$

$y(0) = 0$, $y(\pi/2) = 1$ be extremised.

Solution

Here $F = y'^2 - y^2$

Euler Lagrange Equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$-2y - \frac{d}{dx}(2y') = 0 \quad \Rightarrow y'' + y = 0$$

A.E is $D^2 + 1 = 0 \quad \Rightarrow D = \pm i$

$$\Rightarrow y = A\cos x + B\sin x$$

using given conditions $A = 0$, $B = 1$

so $y = \sin x$

Example 5:- The Lagrangian $\mathcal{L}$ for a system of n particles is a function of generalized coordinates q_i'. Show that the minimization of the integral $I = \int_{t_1}^{t_2} \mathcal{L}\, dt$ leads to the following eqn of motion

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right) = 0$$

Solution Here $\mathcal{L} = \mathcal{L}(q_i, q_i')$

$$\Rightarrow \delta I = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial q_i'}\delta q_i'\right) dt$$

Now $$\int_{t_1}^{t_2} \frac{\partial L}{\partial q_i'}\delta q_i'\, dt = \frac{\partial L}{\partial q_i'}\delta q_i \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right)\delta q_i\, dt$$

$$= 0 - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right)\delta q_i\, dt$$

$\because q_i$ are fixed at t_1 & t_2

$$\Rightarrow \delta I = \int_{t_1}^{t_2} \sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right)\right)\delta q_i\, dt$$

For the extremal curve $\delta I = 0$

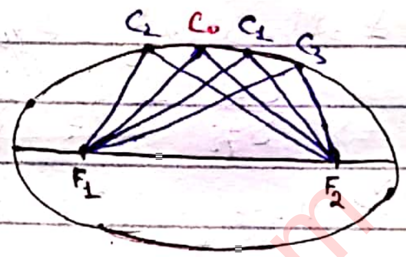
$$\Rightarrow \int_{t_1}^{t_2} \sum_{i=1}^{n}\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right)\right]\delta q_i\, dt = 0$$

Sin δq_i is arbitrary so by fundamental theorem of V.M

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right) = 0, \quad i = 1, 2, 3, \ldots, n$$

Example 6:- Use law of reciprocity to prove the isoscelles triangle has the smallest perimeter for a given area and a given base.

Solution



First we show using a well known property of an ellipse that of all triangles with a given base and a given perimeter, The isoscelles triangle has maximum area.

Let F_1, F_2 be foci of an ellipse. We take F_1F_2 as the base for the triangles $C_1F_1F_2$, $C_2F_1F_2$, $C_3F_1F_2$ etc.

Since the sum of distances of any point on an ellipse from its foci is a constant. it follows that the triangles $C_1F_1F_2$, $C_2F_1F_2$, $C_3F_1F_2$ etc. have the same perimeter. However the triangle $C_0F_1F_2$ which has the largest altitude and which is an isoscelles triangle has the maximum area.

By Law of Resiprocity, it follows that all triangles with given base and given area the isoscelles triangle have the smallest perimeter.

—— ** ——

Question: Find the shortest distance b/w $A(1, 0, -1)$ and $B(0, -1, 1)$ lying on the surface $x+y+z=0$

Solution

Here we have to minimize

$$I = \int_A^B \sqrt{(dx)^2+(dy)^2+(dz)^2}$$

$$\Rightarrow I = \int_A^B \sqrt{1+(y')^2+(z')^2}\, dx$$

subject to the condition $x+y+z=0$

So Here $F = \sqrt{1+(y')^2+(z')^2}$

& $G = x+y+z$

with $y(0) = -1$, $z(0) = 1$

$y(1) = 0$ & $z(1) = -1$

Let $H = F + \lambda G$

$$= \sqrt{1+(y')^2+(z')^2} + \lambda(x+y+z)$$

Corresponding Euler's Lagrange equations are

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \longrightarrow (1)$$

$$\& \quad \frac{\partial H}{\partial z} - \frac{d}{dx}\left(\frac{\partial H}{\partial z'}\right) = 0 \longrightarrow (2)$$

From (1) and (2) respectively we get

$$\lambda - \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2+z'^2}}\right] = 0 \longrightarrow (3)$$

$$\text{and} \quad \lambda - \frac{d}{dx}\left[\frac{z'}{\sqrt{1+y'^2+z'^2}}\right] = 0 \longrightarrow (4)$$

subtracting (3) from (4)

$$\frac{d}{dx}\left[\frac{y'-z'}{\sqrt{1+y'^2+z'^2}}\right] = 0$$

$$\Rightarrow \frac{y'-z'}{\sqrt{1+y'^2+z'^2}} = c_1$$

Now $x+y+z = 0$

$$\Rightarrow 1+y'+z' = 0$$

$$\Rightarrow z' = -1-y' \quad \text{so}$$

$$\frac{y'-(-1-y')}{\sqrt{1+y'^2+(-1-y')^2}} = c_1$$

$$\Rightarrow \frac{1+2y'}{\sqrt{1+y'^2+1+y'^2+2y'}} = c_1$$

$$\Rightarrow \frac{1+2y'}{\sqrt{2+2y'^2+2y'}} = c_1$$

$$\Rightarrow 1+4y'^2+4y' = c_1^2(2+2y'+2y'^2)$$

$$\Rightarrow 1+4y'^2+4y'-2c_1^2-2c_1^2y'-2c_1^2y'^2 = 0$$

$$ay'^2+by'+c = 0 \longrightarrow ⑤$$

where $a = 4-2c_1^2$, $b = 4-2c_1^2$

$$c = 1-2c_1^2$$

is quadratic in y' and can be made to have real roots. Let its one ~~one~~ of real roots be α. then

$$y' = \alpha$$

$$\Rightarrow y = \alpha x + \beta$$

using Boundry Conditions, we have

$$y = x-1 \quad \text{q}$$

$$z = -2x+1$$

So $I = \int_A^B \sqrt{1 + y'^2 + z'^2}\, dx$

$= \int_0^1 \sqrt{1 + 1^2 + 4}\, dx = \int_0^1 \sqrt{6}\, dx$

$= \sqrt{6}\, x \Big|_0^1 = \sqrt{6}(1 - 0)$

$= \sqrt{6}$

** ———— * * * * ———— **

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Initial Value And Boundry Value Problems**

Associated with Heat + Wave Equs.

⇒ **Ordinary Differential Equation:-** An equation involving derivatives or differentials i.e derivative w.r.t one variable is called an ordinary differential equation. e.g

$$\frac{dy}{dx} + 7y = 0 \; ;$$

$$x\frac{d^2y}{dx^2} + 27\frac{dy}{dx} - 14xy = 17$$

are ordinary differential equations.

⇒ **Partial Differential Equation:-** A equation involving partial derivatives i.e derivative w.r.t more than one variable is called partial differential equation. eg

$$u_{yy} + u_{xx} = \alpha^2 u_t \quad \& \quad u_{xx} = \frac{1}{c^2} u_{tt}$$

are partial differential equations.

⇒ **Order of Differential Equation:-** The highest derivative involved in the differential equation is called order of the differential equation. e.g

$$\frac{dy}{dx} - 3y = 0 \quad \&$$

$$\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^3 + 8y = 0$$

are differential equations of order 1 and 2 respectively.

⇒ Degree of Differential Equation:-

The highest exponent of the highest derivative involved in the differential equation is called ~~gre~~ degree of the differential equation.

e.g

$$\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^3 - 8y = 0$$

and

$$\frac{dy}{dx} + 3xy = 0$$

are differential equation of degree 1.

⇒ General Solution:-

A linear combination of all possible solutions of a differential equation is called general solution of the equation.

Example:- Suppose $\frac{d^2y}{dx^2} - 4y = 0$

Then it can easily be seen that $y = e^{2x}$ & $y = e^{-2x}$ are two its solutions. So its general solution is given by

$$y = Ae^{2x} + Be^{-2x}$$

⇒ Linear Differential Equation:-

A differential equation is said to be linear if

1) The dependent variable y and all of its derivatives are of degree 1.
2) No product of y and/or any of its derivative occures.
3) No transcedental function of y and/or any of its derivatives occures.

A differential equation which is not linear is said to be non-linear.

Example:- 1) $U_{xx} + U_{yy} + U_{zz} = 0$

2) $U_{xx} + U_{yy} = \alpha^2 U_t$

3) $U_{xx} = \frac{1}{c^2} U_t$

are all linear differential equations of order 2 and degree 1.

4) $U U_{xx} + 2 U_{xy} = 0$

5) $U_{xx} + U_{yy} + U^2 = 0$

6) $U_{xx} + U_{xy} + \tan U = 0$

are all non-linear differential equations of order 2 and degree 1.

***

⇒ Initial Value & Boundry Value Problems:-

A differential equation with conditions at one initial points e.g

$u(x_0) = u_0, \quad u'(x_0) = u_0$

is called initial value problem

A differential equation with conditions at two different points e.g

$u(x_0) = u_0 \quad , \quad u(x_1) = u_1$

or $u'(x_0) = u_0 \quad , \quad u'(x_1) = u_1$

are called boundry value problems.

⇒ Operator:-

If A and B are two classes of functions. A rule which assigns each function of A to a unique function of B is called operator.

Examples:-

1) The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplace operator.

2) The operator $\nabla^2 - \alpha^2 \frac{\partial}{\partial t}$ is called heat operator.

3) The operator $\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is called wave operator.

⇒ Linear Operator:-

An operator A is said to be linear if

$$A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$$

otherwise it is said to be non-linear.

Examples:-

1) Laplace operator is linear
2) operator $\left(\frac{d}{dx}\right)^2$ is non-linear.

⇒ Sum of Linear Operators:-

If A & B are

two linear operators. then their sum is defined as

$$(A+B)(f) = A(f)+B(f)$$

⇒ Definition:-

If "$\mathcal{L}$" is a linear operator and "f" is is function then the equation $\mathcal{L}(u) = f$ is called a linear Homogeneous equation, otherwise it is said to be non-homogeneous.

***

⇒ Principal of Super Position:-

Let $f_1, f_2, \ldots, f_n$ be any functions and $c_1, c_2, \ldots, c_n$ be any constants. Let $u_1, u_2, \ldots, u_n$ be the functions s.t.

$$\mathcal{L}(u_i) = f_i \quad , i = 1, 2, \ldots, n$$

then $\mathcal{L}(u) = f$

where $u = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$

$f = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$

Proof)

$$\mathcal{L}(u) = \mathcal{L}(c_1 u_1 + c_2 u_2 + \cdots + c_n u_n)$$
$$= \mathcal{L}(c_1 u_1) + \mathcal{L}(c_2 u_2) + \cdots + \mathcal{L}(c_n u_n)$$
$$= c_1 \mathcal{L}(u_1) + c_2 \mathcal{L}(u_2) + \cdots + c_n \mathcal{L}(u_n)$$
$$= c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$
$$\Rightarrow \mathcal{L}(u) = f$$

***

⇒ Derivation of Laplace Equation:-

According to Guass's flux theorem in electrostatic "Total flux of electro static intensity $\bar{E}$ across a closed surface S is equal to 4π times the total charge enclosed within the surface S"

Mathematically

$$\iint_S \bar{E}\cdot ds = \iiint_V 4\pi \rho \, dv \longrightarrow ①$$

where ρ is volume density.

But By Guass's divergence theorem

$$\iint_S \bar{E}\cdot ds = \iiint_V \operatorname{div} \bar{E} \, dv \longrightarrow ②$$

from ① & ②

$$\iiint_V \operatorname{div} \bar{E} \, dv = \iiint_V 4\pi\rho \, dv$$

$$\Rightarrow \iiint_V (\operatorname{div} \bar{E} - 4\pi\rho) \, dv = 0$$

$$\Rightarrow \operatorname{div} \bar{E} - 4\pi\rho = 0 \qquad \because dv \neq 0$$

$$\Rightarrow \operatorname{div} \bar{E} = 4\pi\rho$$

$$\Rightarrow \nabla\cdot \bar{E} = 4\pi\rho$$

But By Fourier's Law $\bar{E} = -\nabla\phi$ So

$$\nabla\cdot(-\nabla\phi) = 4\pi\rho \quad \Rightarrow \nabla^2\phi = -4\pi\rho \text{ (Poisson equ)}$$

Now for the region where there is no charge, $\rho = 0$ then $\nabla^2\phi = 0$

This equ is called Laplace equation.

Mass = Vol × density

$\Rightarrow \rho = \frac{Mass}{Vol}$

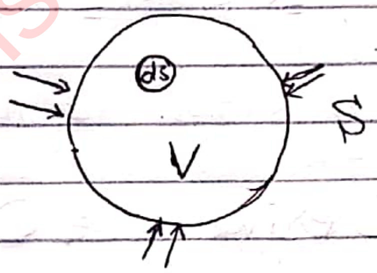
⇒ Derivation of Heat Equation:-

The Fourier Law states that $E = -K \nabla T$

where E is intensity i.e flow of heat, T is temperature and K is the constant of conductivity. The negative sign shows the direction of flow of heat.

Consider a closed surface enclosing a volume V, then the total out flow of heat energy across the surface S per unit time is equal to

$$-\iint_S E \cdot ds = -\iiint_V \operatorname{div} \bar{E} \, dV$$



Here we have applied Guass's divergence theorem.

$$\Rightarrow -\iint_S \bar{E} \cdot ds = -\iiint_V \operatorname{div}(-K\nabla T) \, dV \qquad \because \bar{E} = -K\nabla T$$

$$\Rightarrow \iint_S \bar{E} \cdot ds = K \iiint_V \nabla^2 T \, dV$$

Now according to Law of conservation of Energy

Out flow of Heat = Inflow of Heat

If C_p is the specific heat of the body (i.e C_p is heat required to raise the temperature of unit mass of a body through 1 kelvin), then the total heat energy required to raise the temperature of an element of mass $\rho \, dV$ to the temperature $T°C$ is $\rho \, dV \, C_p T$, where ρ is dencity of element thus rate of increase of temperature per unit time is $\frac{\partial}{\partial t}\iiint_V C_p T \rho \, dV$ which is also total inflow of heat.

Now by ①

$$K\iiint_V \nabla^2 T \, dV = \frac{\partial}{\partial t}\iiint_V C_p T \rho \, dV$$

$$\Rightarrow \iiint_V \left(K\nabla^2 T - \rho C_p \frac{\partial T}{\partial t}\right) dV = 0$$

$$\Rightarrow K\nabla^2 T - \rho C_p \frac{\partial T}{\partial t} = 0 \qquad \because dV \neq 0$$

$$\Rightarrow \nabla^2 T = \alpha^2 T_t \qquad \text{where } \alpha^2 = \frac{\rho C_p}{K}$$

which is required heat conduction equation.

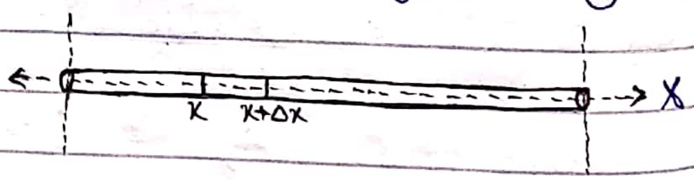
***

⇒ Derive One Dimentional Heat Equation:-

Question:- Derive heat equation for the flow of heat through a rod of uniform thickness taking into consideration source of heat.

Solution

Consider one dimentional flow of heat in a rod of uniform cross sectional area A. Let



A be the area of cross section of the rod let X-axis be choosen along the length of the rod.

We use law of conservation

جب Mass distribution تمام راڈ پر ہو تو Mass
کو Density کے ذریعے ظاہر کرتے ہیں

of energy to a small portion of rod. Thus law may be stated as

"Amount of Heat energy that enters a region + Amount of energy generated inside the region in a given time = Amount of energy that leaves the region + Amount of energy that observed or stored" $\longrightarrow$ ①

Let $q(x,t)$ be the flow of heat at point 'x' at time 't' then the rate at which heat is stored in small volume element = Specific Heat × Mass of material × rise in temperature in one sec

$$= C_p \times A \Delta x \, \rho \times \frac{\partial T}{\partial t} \quad (\because \text{Mass} = \text{Vol.} \times \text{Density})$$

Now the rate at which heat enters the volume element $= A\, q(x,t)$

& rate at which heat leaves the volume element $= A\, q(x+\Delta x, t)$.

Now let r be the rate of generation of heat ~~generation~~ energy per unit volume then Heat generation inside volume element

$$= r A \Delta x$$

Putting these values in equ ①

$$A\, q(x,t) + r A \Delta x = A\, q(x+\Delta x, t) + A \Delta x \, C_p \rho \frac{\partial T}{\partial t}$$

Dividing both sides by A

$$q(x,t) + r \Delta x = q(x+\Delta x, t) + \Delta x \, C_p \rho \frac{\partial T}{\partial t}$$

$$\Rightarrow \frac{q(x+\Delta x, t) - q(x,t)}{\Delta x} = r - C_p \rho \frac{\partial T}{\partial t}$$

taking limit as $\Delta x \to 0$

$$\frac{\partial q}{\partial x} = r - C_p \rho \frac{\partial T}{\partial t}$$

Heat is time dependent & Point dependent

Flow of heat is directly proportional to –ve of gradient of temprature $q(x,t) \propto -\nabla T$

$$\Rightarrow \frac{\partial}{\partial x}\left(-K\frac{\partial T}{\partial x}\right) = r - C_p \rho \frac{\partial T}{\partial t}$$

$\because$ by Fourier Law $q = -K\frac{\partial T}{\partial x}$

$$\Rightarrow -K\frac{\partial^2 T}{\partial x^2} = r - C_p \rho \frac{\partial T}{\partial t}$$

$$\Rightarrow \frac{\partial^2 T}{\partial x^2} - \frac{\rho C_p}{K}\frac{\partial T}{\partial t} = \frac{-r}{K} \qquad \div \text{ by } -K$$

which is one dimentional form of heat equation in the presence of source of heat. If there is no source of heat then $r = 0$

$$\Rightarrow \frac{\partial^2 T}{\partial x^2} = \frac{\rho C_p}{K}\cdot\frac{\partial T}{\partial t}$$

$$\Rightarrow \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t} \qquad \text{where } \alpha^2 = \frac{K}{\rho C_p}$$

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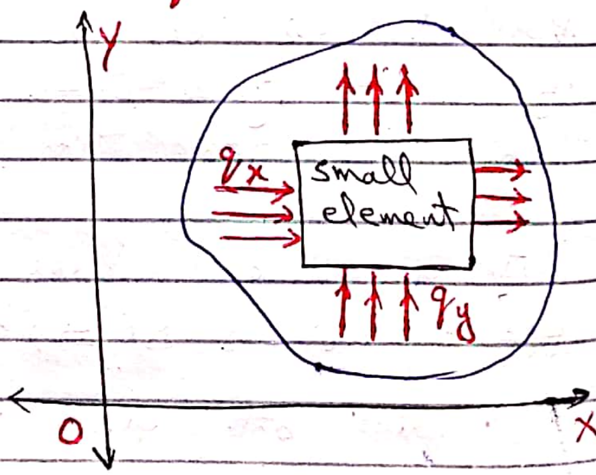
⇒ Two Dimential Heat Equation:-

Consider a plate in XY- plane as shown in figure.



In order to derive the heat equation we apply the law of conservation of energy to small element of plate in the form of rectangle with dimentions Δx and Δy. According to this law

"Total energy that enters the body + Total energy that generates inside the body = Total energy that leaves the body

+ Total energy stored in body"

Let $q_x(x,y,t)$ and $q_y(x,y,t)$ denotes the flow of heat in the directions of x-axis and y-axis respectively.

Let θ be the thickness of the plate, r be rate of generation per unit volume, ρ is density and C be heat capacity then

Rate of generation $= r\theta \Delta x \Delta y$

" " Storage $= \rho C \theta \Delta x \Delta y u_t$

Rate at which heat enters the body

$= q_x \theta \Delta y + q_y \theta \Delta x$

Rate at which heat leaves the body

$= q_x(x+\Delta x, y, t)\Delta y \theta + q_y(x, y+\Delta y, t)\theta \Delta x$

So by Law of conservation of energy

$q_x(x,y,t)\theta\Delta y + q_y(x,y,t)\theta\Delta x + r\theta\Delta x\Delta y = q_x(x+\Delta x, y,t)\theta\Delta y + q_y(x,y+\Delta y,t)\theta\Delta x + \rho\theta C\Delta x\Delta y u_t$

$$\Rightarrow \frac{q_x(x+\Delta x,y,t)-q_x(x,y,t)}{\Delta x} + \frac{q_y(x,y+\Delta y,t)-q_y(x,y,t)}{\Delta y}$$

$$= r - \rho C u_t$$

Applying limit as $\Delta x, \Delta y \longrightarrow 0$

$$\Rightarrow \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = r - \rho C \frac{\partial u}{\partial t} \longrightarrow (1)$$

By Fourier Law

$$q_x = -K_x \frac{\partial u}{\partial x}, \quad q_y = -K_y \frac{\partial u}{\partial y}$$

Then by (1)

$$-K_x u_{xx} - K_y u_{yy} = r - \rho C u_t$$

$$\Rightarrow K_x u_{xx} + K_y u_{yy} = \rho C u_t - r$$

in general K_x and K_y are different.

If $K_x = K_y = K$ say

Then $u_{xx} + u_{yy} = \frac{\delta c}{K} u_t - \frac{r}{K}$

In absence of source of heat

$u_{xx} + u_{yy} = \frac{1}{\alpha^2} u_t \quad \because \alpha^2 = K/\delta c \ \& \ r = 0$

is required.

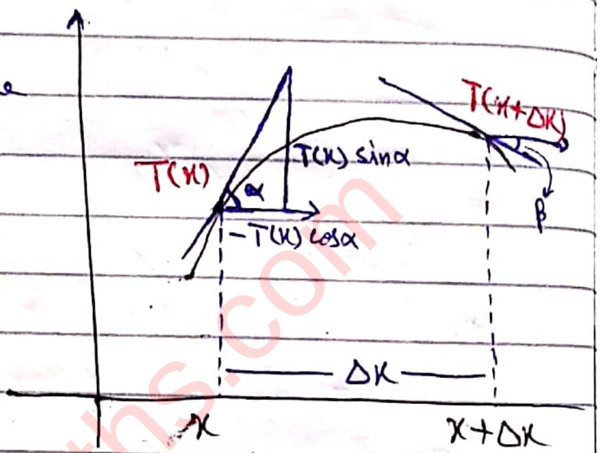
⇒ Three Dimentional Heat Equation:-

⇒ Derivation of Wave Equation:-

We shall derive the wave equation by applying the law of motion to a small element of the string b/w x and $x+\delta x$.

Also $u(x,t)$ will denote the displacement at the point x and at time t.

Let $T(x)$ denote the tension in the string and $T(x+\delta x)$ denotes the tension at other end at the same time.



Now as weight is acting downward and no external force is supposed to be present. Then the total force in the x-direction must be zero.

$$\Rightarrow -T(x)\cos\alpha + T(x+\delta x)\cos\beta = 0$$

$$\Rightarrow T(x)\cos\alpha = T(x+\delta x)\cos\beta = T \text{ (say)}$$

$$\Rightarrow T(x) = \frac{T}{\cos\alpha} \quad \& \quad T(x+\delta x) = \frac{T}{\cos\beta}$$

Similarly considering the force on the string in the vertical direction

$$-T(x)\sin\alpha + T(x+\delta x)\sin\beta - mg = m\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{-T}{\cos\alpha}\sin\alpha + \frac{T}{\cos\beta}\sin\beta - mg = m\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow -T\tan\alpha + T\tan\beta - mg = m\frac{\partial^2 u}{\partial t^2} \longrightarrow (1)$$

But $\tan\alpha$ = slope of string at $x = \frac{\partial u}{\partial x}(x,t)$

$\tan\beta$ = slop of string at $x+\delta x = \frac{\partial u(x+\delta x,t)}{\partial x}$

So by (1)

$$-T\,u_x(x,t) + T\,u_x(x+\delta x,t) - mg = m u_{tt}$$

Now as $m = \rho\,\delta x$

$$\Rightarrow -T\,u_x(x,t) + T\,u_x(x+\delta x,t) - \rho\,\delta x\,g = \rho\,\delta x\,u_{tt}$$

$$\Rightarrow \frac{u_x(x+\delta x,t) - u_x(x,t)}{\delta x} = \frac{\rho g}{T} + \frac{\rho}{T}u_{tt}$$

Applying limit as $\delta x \to 0$

$$u_{xx}(x,t) = \frac{\rho g}{T} + \frac{\rho}{T}u_{tt}$$

Now $T/\rho = c^2$

$$\Rightarrow u_{xx} = \frac{g}{c^2} + \frac{1}{c^2}u_{tt}$$

$$\Rightarrow u_{xx} = \frac{1}{c^2}u_{tt} \qquad \because g \lll c^2$$

which is required wave equation in one dimention.

*** —— ** ——

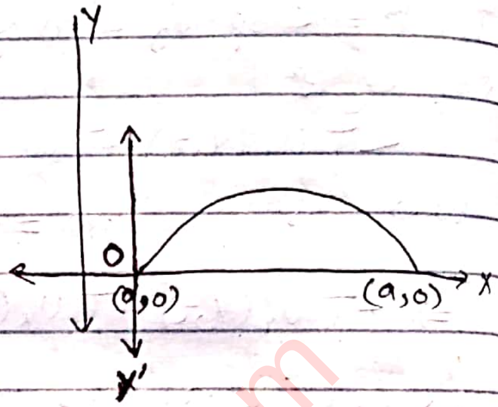
⇒ Modified Form of Wave Equation:-

Suppose a distributed vertical force $F(x,t)$ (+ve upward) acts on the string. Then the equation of motion is

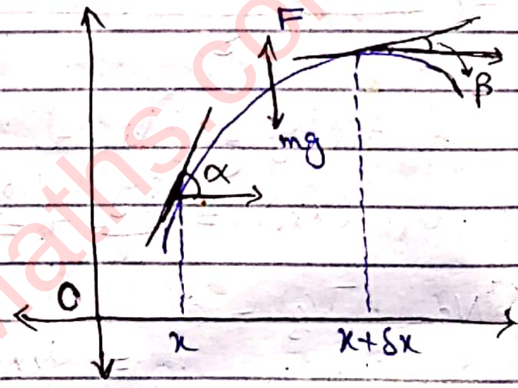
$$u_{xx} = \frac{1}{c^2}u_{tt} - \frac{F(x,t)}{T}$$

Derivation:- Let us consider the vibration of a stretched string of length "a". suppose that the string is completely flexible and has uniform density ρ, and is fixed at $x=0$ and $x=a$

We shall derive the equation by applying the law of motion to a small element of the string lying b/w x and $(x+\delta x)$. Also $u(x,t)$ denotes the displacement at the point x at at time



Let $T(x)$ and $T(x+\delta x)$ be the tensions in the string at the points x and $x+\delta x$.



Then for the force acting in horizontal direction

$$-T(x)\cos\alpha + T(x+\delta x)\cos\beta = 0$$

$$\Rightarrow T(x)\cos\alpha = T(x+\delta x)\cos\beta = T \text{ (say)}$$

$$\Rightarrow T(x) = \frac{T}{\cos\alpha},\quad T(x+\delta x) = \frac{T}{\cos\beta} \Big\} \longrightarrow (1)$$

For vertical forces

$$-T(x)\sin\alpha + T(x+\delta x)\sin\beta - mg + F\delta x = m u_{tt}$$

using (1)

$$-T\tan\alpha + T\tan\beta - mg + F\delta x = m u_{tt}$$

$$\Rightarrow -T u_x(x,t) + T u_x(x+\delta x,t) - \rho\delta x g + F(x,t)\delta x = \rho\delta x\, u_{tt} \longrightarrow (2)$$

$\because$ $\tan\alpha =$ slope of string at $x = u_x(x,t)$

$\tan\beta =$ " " " " $x+\delta x = u_x(x+\delta x,t)$

& $m = \rho\,\delta x$

$$\Rightarrow \lim_{\delta x\to 0} \frac{u_x(x+\delta x,t)-u_x(x,t)}{\delta x} = \frac{\rho g}{T} - \frac{F}{T} + \frac{\rho}{T}u_{tt}$$

$$\Rightarrow u_{xx} = \frac{1}{c^2}u_{tt} - \frac{F(x,t)}{T}$$

Which is required wave equation in the presence of external force $F(x,t)$. In the Absence of external force

$$u_{xx} = \frac{1}{c^2}u_{tt}$$

Remark:- If the distributed vertical force is the load $W(x,t)$, acting downward. Then replacing $F(x,t)$ by $-W(x,t)$ we get the equation $u_{xx} = \frac{1}{c^2}u_{tt} + \frac{W(x,t)}{T}$

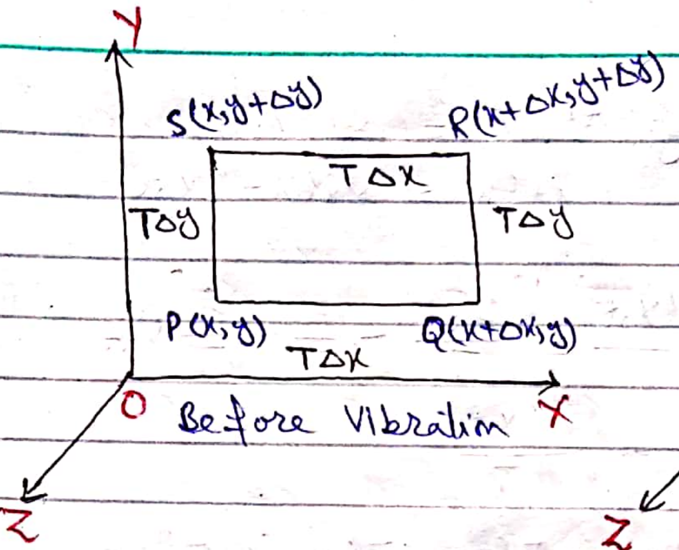
*** —— ** [91, 93, 94]

⇒ Two Dimensional Wave Equation:-

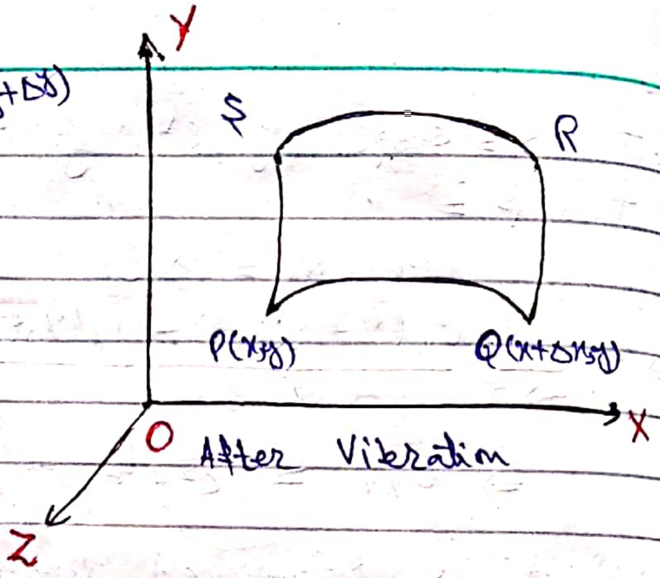
To derive the two dimensional wave equation we consider the vibration of a stretched membrane which is completly flexible, has negligible mass and has uniform density ρ.

Further suppose that the displacements of the membrane are small and are always ⊥ to the plane in which it lies initially.

Let the membrane initially lies in XY-plane and T denotes



Before Vibration



After Vibration

the surface tension. To derive the equation we consider a small element PQRS of membrane with sides $\Delta x, \Delta y$. Let the forces act along tangent planes to the sides.

Let α, β be the angles b/w XY-plane and tangent planes and PS and QR respectively and γ, δ be the angles between XY-plane and tangent planes at PQ and SR respectively.

Now for the forces acting along Z-axis

$$\Delta y T.(\sin\beta - \sin\alpha) + \Delta x T.(\sin\delta - \sin\gamma)$$
$$= \Delta x \Delta y \, \rho \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\sin\beta - \sin\alpha}{\Delta x} + \frac{\sin\delta - \sin\gamma}{\Delta y} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} \longrightarrow (1)$$

But as $\alpha, \beta, \gamma, \delta$ are very small so

$$\sin\alpha \approx \tan\alpha = \frac{\partial u}{\partial x}(x,y,t) \text{ etc}$$

$$\Rightarrow \frac{u_x(x+\Delta x, y, t) - u_x(x,y,t)}{\Delta x} + \frac{u_y(x, y+\Delta y, t) - u_y)}{\Delta y}$$

$$= \frac{\delta}{T} u_{tt}$$

$$\Rightarrow u_{xx} + u_{yy} = \frac{\delta}{T} u_{tt} \quad \because \text{As } \Delta x, \Delta y \to 0$$

$$\Rightarrow u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt} \quad \because \frac{1}{c^2} = \frac{\delta}{T}$$

***

⇒ Solution of Heat Equation:-

The conduction of heat is described by the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial u}{\partial t} \longrightarrow ①$$

with initial condition

$$u(x,0) = f(x) \longrightarrow ②$$

and boundry conditions

$$u(0,t) = T_0 \longrightarrow ③ \quad \text{and} \quad u(a,t) = T_1 \longrightarrow ④$$

Here boundry conditions are non-homogeneous.

If $T_0 = 0$ and $T_1 = 0$

Then the boundry conditions are said to be homogeneous.

To solve the partial differential equation by using the standard method of seperation of variable.

1st the boundry conditions are made homogeneous if they are not homogeneous.

** **

Question:- Solve by using method of seperation of variables $u_{xx} = \alpha^2 u_t \longrightarrow ①$

Solution

Let $u = X(x)\,T(t)$

$$\rightarrow u_{xx} = X_{xx} T \quad , \quad u_t = X T_t$$

Hence from ①

$$X_{xx}T = \alpha^2 X T_t$$

$$\Rightarrow \frac{X_{xx}}{X} = \alpha^2 \frac{T_t}{T} = \lambda^2$$

$$\Rightarrow \frac{X_{xx}}{X} = \lambda^2 \quad \Rightarrow X_{xx} - \lambda^2 X = 0$$

$$\Rightarrow (D^2 - \lambda^2) X = 0$$

Auxiliary Equation is

$$D^2 - \lambda^2 = 0 \quad \Rightarrow D = \pm\lambda$$

So $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$

Now $\alpha^2 \frac{T_t}{T} = \lambda^2 \quad \Rightarrow T_t - \frac{\lambda^2}{\alpha^2} T = 0$

$$\Rightarrow \int \frac{T_t}{T} dt = \int \frac{\lambda^2}{\alpha^2} dt + \ln c$$

$$\Rightarrow \ln T = \lambda^2/\alpha^2\, t + \ln c$$

$$\Rightarrow \ln \frac{T}{c} = \lambda^2/\alpha^2\, t$$

$$\Rightarrow T = c\left(e^{(\lambda^2/\alpha^2)t}\right)$$

Hence

$$u(x,t) = [Ae^{\lambda x} + Be^{-\lambda x}][ce^{(\lambda^2/\alpha^2)t}]$$

Which is required solution.

***

⇒ Steady State OR Equilibrium State:

* Temperature Distribution:-

If u depends upon t also then the flow of heat is time dependent or non-steady.

And when $t \to \infty$ i.e after a long time the non-steady temperature tends to become steady. If $V(x)$ represents steady flow of temperature then

$$u(x,t) \to V(x) \quad \text{when } t \to \infty$$

Now if we put

$$\lim_{t\to\infty} u(x,t) = V(x) \quad \text{then}$$

from ① $\quad \dfrac{d^2V}{dx^2} = 0 \longrightarrow ⑤ \quad \because \dfrac{\partial u}{\partial t} = 0$ as $t \to \infty$

Solution of Heat Equation

From ③ $\quad V(0) = T_0 \longrightarrow ⑥$

From ④ $\quad V(a) = T_1 \longrightarrow ⑦$

Now from ⑤ $\quad V(x) = Ax + B$

$V(0) = T_0 \Rightarrow B = T_0$

$V(a) = T_1 \Rightarrow Aa + B = T_1 \Rightarrow A = \dfrac{T_1 - T_0}{a}$

Hence

$$V(x) = \left[\frac{T_1 - T_0}{a}\right]x + T_0 \longrightarrow ⑧$$

which is required solution of steady state problem.

⇒ * Transient Temperature Distribution:-

If we define

$$w(x,t) = u(x,t) - V(x) \longrightarrow ⑨$$

then $\displaystyle\lim_{t\to\infty} w(x,t) = V(x) - V(x)$

$$= 0$$

i.e $w(x,t)$ is non-zero only for that value of t which are not very large. $w(x,t)$ is called transient temperature distribution.

Now from ⑨

$$u(x,t) = w(x,t) + v(x)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} + 0 \qquad \because \frac{d^2 v}{dx^2} = 0 \text{ by } ⑤$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial x^2}$$

Also

$$\frac{\partial u}{\partial t} = \frac{\partial w}{\partial t}$$

using these values equ ① takes the form $\frac{\partial^2 w}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial w}{\partial t} \longrightarrow ⑩$

Now by ⓐ

$$w(x,0) = u(x,0) - v(x) = f(x) - v(x)$$
$$= g(x)$$

$$\Rightarrow w(x,0) = g(x)$$

$$\Rightarrow w(0,t) = u(0,t) - v(0) = T_0 - T_0 = 0$$

$$w(a,t) = u(a,t) - v(a) = T_1 - T_1 = 0$$

Hence by ⓐ the given problem reduces to

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial w}{\partial t}$$

$$w(x,0) = f(x) - v(x)$$

$$w(0,t) = 0 \qquad w(a,t) = 0$$

Now the boundry conditions are homogeneous, so the system can be solved by standard method of seperation of variables (S.O.V)

** ———— **

Question:- Find the steady solution of the equation

$$u_{xx} = \frac{1}{k} u_t \qquad 0<x<a \; , \; t>0$$

subject to $u_x(0,t) = 0$, $u_x(a,t) = 0$

$u(x,0) = f(x) \qquad 0<x<a$

Also discuss the uniqueness of the solution.

Solution

When $t \to \infty$ then non-steady solution tends to the steady solution

$\therefore$ Put $\lim\limits_{t\to\infty} u(x,t) = v(x)$ where $v(x)$ is the steady solution. Using this

$u_{xx} = \frac{1}{k} u_t$ becomes $\frac{d^2v}{dx^2} = 0$

$\Rightarrow v(x) = Ax + B \qquad , v_x(x) = A$

Also $u_x(0,t) = 0 \Rightarrow v_x(0) = 0 \quad \Rightarrow A = 0$

$u_x(a,t) = 0 \Rightarrow v_x(a) = 0 \quad \Rightarrow A = 0$

So $v(x) = B$

Since for different values of B $v(x)$ is different and so is not unique. However by physical consideration we can determine the constant (i.e its unique value) and therefore the unique solution.

The problem describes the flow of heat in a rod of length 'a' at the end points $x = 0$, $x = a$. It means that the heat contained in the rod is constant for all the time. e.g at $t = 0$ and at $t = \infty$

Now heat stored in time t is $\int_0^a \rho C A\, u(x,t)\, dx = \rho C A \int_0^a u(x,t)\, dx$

The quantity is same at $t=0, t=\infty$

$$\cancel{\delta A}c\int_0^a u(x,0)\,dx = \cancel{\delta A}c \lim_{t\to\infty}\int_0^a u(x,t)\,dx$$

$$\Rightarrow \int_0^a f(x)\,dx = \int_0^a V(x)\,dx = \int_0^A B\,dx = Ba$$

$$\Rightarrow B = \frac{1}{a}\int_0^a f(x)\,dx$$

So unique solution is $V(x) = \frac{1}{a}\int_0^a f(x)\,dx$

***

Question- Solve $u_{xx} = \frac{1}{c^2}u_{tt}$ with

$$u(0,t)=0, \quad u(l,t)=0$$

$$u(x,0)=\phi(x), \quad \dot{u}(x,0)=0 \quad [87, 89, 90, 91, 92]$$

Solution

Since boundry conditions are homogeneous so we use method of seperation of variable. So let

$$u = XT = X(x)T(t)$$

Then given equation becomes

$$X_{xx}T = \frac{1}{c^2}XT_{tt}$$

$$\Rightarrow \frac{X_{xx}}{X} = \frac{1}{c^2}\frac{T_{tt}}{T} = \lambda^2$$

$$\Rightarrow X_{xx} - \lambda^2 X = 0$$

$$\Rightarrow \text{A.E is } D^2 - \lambda^2 = 0 \quad \Rightarrow D = \pm\lambda$$

$$\Rightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x} \longrightarrow (1)$$

Also $\frac{1}{c^2}\frac{T_{tt}}{T} = \lambda^2$

$$\Rightarrow (D^2 - c^2\lambda^2)T = 0$$

A.E is $D^2 - c^2\lambda^2 = 0 \Rightarrow D = \pm c\lambda$

So $T(t) = De^{c\lambda t} + Ee^{-c\lambda t} \longrightarrow ②$

From ① and ②

$u(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})(De^{c\lambda t} + Ee^{-c\lambda t}) \longrightarrow ③$

$u(0,t) = 0 \Rightarrow A + B = 0 \Rightarrow A = -B$

Hence by ① $X(x) = A[e^{\lambda x} - e^{-\lambda x}]$

Also $u(l,t) = 0 \Rightarrow X(l) = 0$

$\Rightarrow A[e^{\lambda l} - e^{-\lambda l}] = 0 \Rightarrow A = 0$

$\Rightarrow B = 0$

$\Rightarrow u = 0$ which is trivial solution

For non-trivial solution we take

$$\frac{X_{xx}}{X} = \frac{1}{c^2}\frac{T_{tt}}{T} = -\lambda^2$$

then $X = A\cos\lambda x + B\sin\lambda x$

$T = D\cos(\lambda c)t + E\sin(\lambda c)t$

Now $u(0,t) = 0 \Rightarrow X(0) = 0 \Rightarrow A = 0$

$u(l,t) = 0 \Rightarrow X(l) = 0 \Rightarrow B\sin\lambda l = 0$

$\Rightarrow \sin\lambda l = 0 \quad \because B \neq 0$

$\Rightarrow \lambda = \frac{m\pi}{l}, \; m = \pm 1, \pm 2, \pm 3, \ldots$

So $X_m(x) = B\sin\left(\frac{m\pi}{l}\right)x$

Now $\dot{u}(x,0) = 0 \Rightarrow \dot{T}(0) = 0$

Now $T(t) = D\cos(c\lambda)t + E\sin(c\lambda)t$

$\dot{T}(t) = -c\lambda D\sin(c\lambda)t + Ec\lambda\cos(c\lambda)t$

$\dot{T}(0) = 0 \Rightarrow E = 0$

So $T(t) = D\cos(c\lambda)t$

$\cancel{\phi(t)}$ ≠ $T_m(t) = D\cos\left(\frac{Cm\pi}{\ell}\right)t$

Hence $u_m(x,t) = \left(B\sin\frac{m\pi}{\ell}x\right)\left(D\cos\left(\frac{Cm\pi}{\ell}\right)t\right)$

$= BD\sin\left(\frac{m\pi}{\ell}x\right)\cos\left(\frac{Cm\pi}{\ell}t\right)$

So $u_n(x,t) = BD\sin\left(\frac{m\pi}{\ell}x\right)\cos\left(\frac{Cm\pi}{\ell}t\right)$

$\Rightarrow u_n(x,t) = d_n\sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{Cn\pi t}{\ell}\right)$

$n = 1, 2, 3, \ldots$

Now by principle of superposition

$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

$= \sum_{n=1}^{\infty} d_n\sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi ct}{\ell}\right)$

$u(x,0) = \phi(x)$

$\Rightarrow \phi(x) = \sum_{n=1}^{\infty} d_n\sin\left(\frac{n\pi x}{\ell}\right)$

$\Rightarrow \int_0^\ell \phi(x)\sin\left(\frac{m\pi x}{\ell}\right)dx = \sum_{n=1}^{\infty} d_n\int_0^\ell \sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)dx$

$= d_n\left(\frac{\ell}{2}\right)$ for $m = n$

$\because \int_0^\ell \sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)dx = \begin{cases} 0 & m \neq n \\ \ell/2 & m = n \end{cases}$

Hense $d_n = \frac{2}{\ell}\int_0^\ell \phi(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$

So general solution is given by

$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{\ell}\int_0^\ell \phi(x)\sin\left(\frac{n\pi x}{\ell}\right)dx\right]\sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi ct}{\ell}\right)$

***

⇒ D'Alembert's Solution of Wave Equation. [91, 94]

The wave equation is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \longrightarrow (1)$$

with initial conditions

$$u(x,0) = f(x) \quad , \quad u_t(x,0) = g(x)$$

and with boundry conditions $u(0,t) = 0$, $u(a,t) = 0$

Let the variables w and z be defined as

$$w = x + ct \quad , \quad z = x - ct$$

Then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dx}$$

$$= \frac{\partial u}{\partial w}(1) + \frac{\partial u}{\partial z}(1)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial w} + \frac{\partial u}{\partial z} \Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial w} + \frac{\partial}{\partial z}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial w} + \frac{\partial u}{\partial z}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial w}\right] + \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial z}\right]$$

$$= \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial w}\right)\cdot\frac{\partial w}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial w}\right)\cdot\frac{\partial z}{\partial x}$$

$$+ \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial z}\right)\frac{\partial w}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)\cdot\frac{\partial z}{\partial x}$$

$$= \frac{\partial^2 u}{\partial w^2}(1) + \frac{\partial^2 u}{\partial z \partial w}(1) + \frac{\partial^2 u}{\partial w \partial z}(1) + \frac{\partial^2 u}{\partial z^2}(1)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial w^2} + 2\frac{\partial^2 u}{\partial w \partial z} + \frac{\partial^2 u}{\partial z^2} \longrightarrow (*)$$

Now $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial t} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}$

$$= c\frac{\partial u}{\partial w} - c\frac{\partial u}{\partial z}$$

$$\Rightarrow \frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial}{\partial w} - \frac{\partial}{\partial z}$$

So $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial}{\partial w} - \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial w} - \frac{\partial u}{\partial z}\right)$

$$= \frac{\partial^2 u}{\partial w^2} - 2\frac{\partial^2 u}{\partial w\partial z} + \frac{\partial^2 u}{\partial z^2} \longrightarrow **$$

using * and ** equ (1) becomes

$$\frac{\partial^2 u}{\partial w\partial z} = 0 \Rightarrow \frac{\partial u}{\partial z} = \phi(z)$$

$$\Rightarrow u = \int\phi(z)\,dz + \phi(w)$$

$$\Rightarrow u = \psi(z) + \phi(w)$$

$$\Rightarrow \boxed{u = \phi(w) + \psi(z)}$$

Which is the most general form of the solution of wave equation called D' Alembert's solution.

——— *** ———

Question:- Let $u(x,t)$ be the solution of the equation

$$u_{xx} = \frac{1}{c^2}u_{tt} \quad 0<x<a, \; t>0$$

$u(x,0) = f(x)$, $u_t(x,0) = 0$

$u(0,t) = 0$, $u(a,t) = 0$

where $f(x)$ is a function whose graph is a isosceles triangle, of width 'a' and height 'h' find $u(x,t)$ for

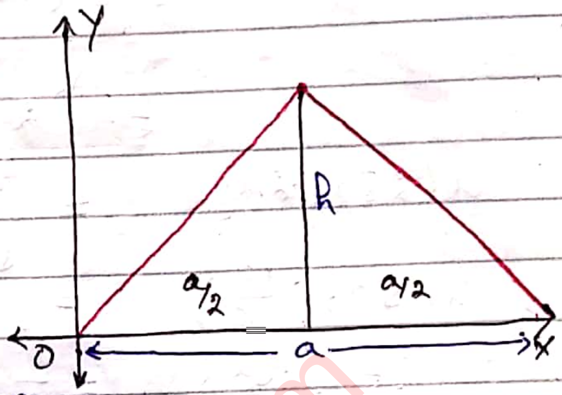
$x = 0.25a$, $0.5a$ and for
$t=0$, $0.2\frac{a}{c}$, $0.4\, a/c$, $0.8\, a/c$

Solution

The wave equation is given by

$$u_{xx} = \frac{1}{c^2} u_{tt}$$

with initial conditions



$$u(x,0) = f(x) = \begin{cases} \frac{2hx}{a} & 0 \leq x \leq \frac{a}{2} \\ \frac{-2h(x-a)}{a} & \frac{a}{2} < x < a \end{cases}$$

By Two point formula. Equation of OA

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

and

$$u_t(x,0) = g(x) = 0$$

and with boundry conditions

$$u(0,t) = 0 \qquad , \qquad u(a,t) = 0$$

Now D' Alembert's general solution is given by

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

Now $u(x,0) = f(x)$

$$\Rightarrow \phi(x) + \psi(x) = f(x) \longrightarrow ①$$

$$u_t(x,0) = 0 \Rightarrow \phi'(x) - \psi'(x) = 0$$

$$\Rightarrow \phi(x) - \psi(x) = A \longrightarrow ②$$

Adding equation ① and ②

$$\phi(x) = \frac{1}{2}\{f(x) + A\}$$

subtracting equ ② from ①

$$\rightarrow \psi(x) = \frac{1}{2}\{f(x) - A\}$$

Hence $u(x,t) = \frac{1}{2}\{f(x+ct) + f(x-ct)\}$

Now

$$u(.25a, 0) = \frac{1}{2}\{f(.25a) + f(.25a)\}$$

$$= f(.25a) = \frac{2h(.25\cancel{a})}{\cancel{a}}$$

$$= h/2$$

$$u(.25a, .2a/c) = h/2$$

$$u(.5a, \frac{.4a}{c}) = h/5$$

$$u(.5a, \frac{.8a}{c}) = \frac{1}{2}[f(.5a + .8a) + f(.5a - .8a)]$$

$$= \frac{1}{2}[f(1.3a) + f(-0.3a)]$$

solution does not exist.

$$u(.25a, \frac{.4a}{c}) = \frac{1}{2}[f(.25a + .4a) + f(.25a - .4a)]$$

$$= \frac{1}{2}[f(.65a) + f(-.15a)]$$

solution does not exist

$$u(.25a, \frac{.8a}{c}) = \frac{1}{2}[f(.25a + .8a) + f(.25a - .8a)]$$

$$= \frac{1}{2}[f(1.5a) + f(-0.55a)]$$

solution does not exist

$$u(0.5a, 0) = \frac{1}{2}[f(0.5a) + f(0.5a)]$$

$$= f(0.5a) = \frac{2h(0.5\cancel{a})}{\cancel{a}}$$

$$= h$$

$$u(0.5a, \frac{.2a}{c}) = \frac{1}{2}[f(0.5a + .2a) + f(0.5a - 0.2a)]$$

$$= \frac{1}{2}[f(0.7a) + f(0.3a)]$$

$$= \frac{1}{2}\left[\frac{-2h(0.7a - a)}{a} + \frac{2h(0.3a)}{a}\right]$$

$$= \frac{1}{2}\left[\frac{-2h(-.3\cancel{a})}{\cancel{a}} + 2h(0.3)\right]$$

$$= 0.8h$$

Question:- Solve $u_{xx} - \frac{1}{c^2} u_{tt} = -\frac{1}{T} F(x,t)$
if $F(x,t) = T\cos t$. Find general solution of the equation.

Solution

By D' Alembert's method

$$4\frac{\partial^2 u}{\partial w \partial z} = \frac{-1}{T} F(x,t)$$

where $w = x + ct$, $z = x - ct$

$$\Rightarrow \frac{\partial^2 u}{\partial w \partial z} = -\frac{1}{4T} T\cos t$$

$$\Rightarrow \frac{\partial^2 u}{\partial w \partial z} = -\frac{1}{4}\cos\left(\frac{w-z}{2c}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial z \partial w} = -\frac{1}{4}\cos\left(\frac{w-z}{2c}\right)$$

$$\Rightarrow \frac{\partial u}{\partial w} = -\frac{1}{4}\sin\left(\frac{w-z}{2c}\right)(-2c) + \phi(w)$$

$$= \frac{1}{2} c \sin\left[\frac{w-z}{2c}\right] + \phi(w)$$

$$\Rightarrow u(x,t) = \frac{1}{2} c\left[-\cos\left(\frac{w-z}{2c}\right)(2c)\right] + \int \phi(w)\, dw + \theta(z)$$

$$\Rightarrow u(x,t) = -c^2 \cos t + \psi(x+ct) + \theta(x-ct)$$

***

Question:- $u_{xx} = \alpha^2 u_t \longrightarrow (1)$
with $u(0,t) = T_1 \longrightarrow (2)$
$u(l,t) = T_2 \longrightarrow (3)$ & $u(x,0) = \phi(x) \longrightarrow (4)$

Solution Define a function

$$V(x,t) = u(x,t) - \left(1 - \frac{x}{l}\right)T_1 - \frac{x}{l}T_2$$

substitution learn by Heart

Then $V(0,t) = u(0,t) - \left(1 - \frac{0}{l}\right)T_1 - \frac{0}{l}T_2$

$$\Rightarrow V(0,t) = T_1 - T_1 = 0$$

$$V(l,t) = u(l,t) - (1-1)T_1 - T_2$$

$$= T_2 - T_2 = 0$$

Also $V(x,0) = u(x,0) - (1-\frac{x}{l})T_1 - \frac{x}{l}T_2$

$$= \phi(x) - (1-\frac{x}{l})T_1 - \frac{x}{l}T_2$$

$$\Rightarrow V(x,0) = \psi(x)$$

So given problem reduces to

$V_{xx} = \alpha^2 V_t$ with

$V(0,t) = 0$, $V(l,t) = 0$, $V(x,0) = \psi(x)$

Let $V(x,t) = X(x)T(t)$

$$\Rightarrow X_{xx}T = \alpha^2 X T_t$$

$$\Rightarrow \frac{X_{xx}}{X} = \alpha^2 \frac{T_t}{T} = -\lambda^2$$

$$\Rightarrow \frac{X_{xx}}{X} = -\lambda^2 \Rightarrow X_{xx} + \lambda^2 X = 0$$

Auxiliary Equation is

$$D^2 + \lambda^2 = 0 \Rightarrow D = \pm i\lambda$$

So $X(x) = A\cos\lambda x + B\sin\lambda x$

Also $\frac{\alpha^2 T_t}{T} = -\lambda^2 \Rightarrow \ln T = -\lambda^2/\alpha^2\, t + \ln c$

$$\Rightarrow T(t) = c\, e^{-\lambda^2/\alpha^2\, t}$$

$V(0,t) = 0 \Rightarrow X(0) = 0 \Rightarrow A = 0$

$V(l,t) = 0 \Rightarrow X(l) = 0 \Rightarrow \lambda = \frac{-n\pi}{l}$

So

$$X(x) = B\sin\frac{n\pi x}{l}, \quad T(t) = Ce^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$$

$$\Rightarrow V(x,t) = BC\sin\frac{n\pi x}{l}\, e^{-(n^2\pi^2/\alpha^2 l^2)t}$$

$$\Rightarrow V_n(x,t) = d_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n^2\pi^2}{\alpha^2 l^2}\right)t}$$

$$V(x,t) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{l}\right)\cdot e^{-\left(\frac{n^2\pi^2}{\alpha^2 l^2}\right)t}$$

$$V(x,0) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow \psi(x) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx = \sum_{n=1}^{\infty} d_n \int_0^l \sin\frac{m\pi x}{l}\cdot\sin\frac{n\pi x}{l}\,dx$$

$$\Rightarrow \int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx = d_n\left(\frac{l}{2}\right) \quad \text{for } m=n$$

$$\Rightarrow d_n = \frac{2}{l}\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

So $$V(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx\right] \times \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-n^2\pi^2}{\alpha^2 l^2}t}$$

Thus

$$u(x,t) = \left(1-\frac{x}{l}\right)T_1 - \frac{x}{l}T_2 + \sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx\right]\sin\frac{n\pi x}{l}\, e^{\frac{-n^2\pi^2}{\alpha^2 l^2}t}$$

where $\psi(x) = \phi(x) - \left(1-\frac{x}{l}\right)T_1 - \frac{x}{l}T_2$

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Question: $u_{xx} = \alpha^2 u_t$ with $u_x(0,t) = 0$
$u_x(l,t) = 0$ & $u(x,0) = \phi(x)$

Solution Let
$$u(x,t) = X(x)\,T(t)$$
$$X(x) = A\cos\lambda x + B\sin\lambda x$$
$$T(t) = Ce^{-\lambda^2/\alpha^2 t}$$
$$X_x(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$$
$$X_x(0) = 0 \qquad \Rightarrow B = 0$$
$$X_x(l) = 0 \qquad \Rightarrow -A\lambda\sin\lambda l = 0$$
$$\Rightarrow \lambda = \frac{n\pi}{l}$$
$$\Rightarrow u(x,t) = AC\cos\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$$
$$\Rightarrow u_n(x,t) = \lambda_n\cos\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$$

So by principle of super position
$$u(x,t) = \sum_{n=0}^{\infty}\lambda_n\cos\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$$
$$= \lambda_0 + \sum_{n=1}^{\infty}\lambda_n\cos\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$$
$$u(x,0) = \phi(x)$$
$$\Rightarrow \phi(x) = \lambda_0 + \sum_{n=1}^{\infty}\lambda_n\cos\frac{n\pi x}{l}$$
$$(\phi(x) - \lambda_0) = \sum_{n=1}^{\infty}\lambda_n\cos\left(\frac{n\pi x}{l}\right)$$
$$\int_0^l(\phi(x) - \lambda_0)\cos\frac{n\pi x}{l}\,dx = \lambda_n\left(\frac{l}{2}\right)$$

$$\Rightarrow a_n = \frac{2}{\ell}\int_0^{\ell}(\phi(x) - a_0)\cos\left(\frac{n\pi x}{\ell}\right)dx$$

Hence

$$u(x,t) = a_0 + \sum_{n=1}^{\infty}\left[\frac{2}{\ell}\int_0^{\ell}(\phi(x) - a_0)\cos\left(\frac{n\pi x}{\ell}\right)dx\right] \times \cos\left(\frac{n\pi x}{\ell}\right)dx$$

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Question:- $u_{xx} = \alpha^2 u_t$ with

$u(0,t) = T_0$, $u_x(\ell,t) = 0$ & $u(x,0) = \phi(x)$

Solution

Define a function

$$V(x,t) = u(x,t) - T_0$$

Substitution
Learn by Heart
∵ One B.C is non Homogeneous

Then given problem becomes

$V_{xx} = \alpha^2 V_t$ with

$V(0,t) = 0$, $V_x(\ell,t) = 0$

$$V(x,0) = u(x,0) - T_0$$
$$= \phi(x) - T_0 = \psi(x)$$

Let $V(x,t) = X(x)\,T(t)$

Then $X(x) = A\cos\lambda x + B\sin\lambda x$

& $T(t) = e^{(-\lambda^2/\alpha^2)t}$

$X(0) = 0 \Rightarrow A = 0$

So $X(x) = B\sin\lambda x \Rightarrow X_x(x) = B\lambda\cos\lambda x$

$X_x(\ell) = 0 \Rightarrow B\lambda\cos\lambda\ell = 0$

$\Rightarrow \lambda\ell = (2n+1)\pi/2$

$\Rightarrow \lambda = (2n+1)\frac{\pi}{2\ell}$

So $V_n(x,t) = a_n\sin\left[\frac{(2n+1)\pi x}{2\ell}\right]e^{-\frac{(2n+1)^2\pi^2}{4\ell^2\alpha^2}t}$

So by principle of superposition

$$V(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{(2n+1)\pi x}{2\ell}\right) e^{-\frac{(2n+1)^2\pi^2}{4\ell^2\alpha^2}t}$$

where $\lambda_n = \frac{2}{\ell}\int_0^{\ell} (\phi(x) - T_0)\sin\left[\frac{2n+1}{2\ell}\pi x\right] dx$

So $u(x,t) = T_0 + \sum_{n=1}^{\infty} \lambda_n \sin\left[\frac{(2n+1)\pi x}{2\ell}\right] e^{-\frac{(2n+1)^2\pi^2}{4\alpha^2\ell^2}t}$

where $\lambda_n = \frac{2}{\ell}\int_0^{\ell}(\phi(x) - T_0)\sin\left[\frac{(2n+1)\pi x}{2\ell}\right] dx$

* ——— ** ——— *

Question – $u_{xx} = \alpha^2 u_t$ with

$u(0,t) = 0$, $u(\ell,t) = T_0$

& $u(x,0) = \phi(x)$

Solution

Define a function

$V(x,t) = u(x,t) - \frac{x}{\ell}T_0$

Then give problem reduces to

$V_{xx} = \alpha^2 V_t$ with $V(0,t) = 0$

$V(\ell,t) = 0$, $V(x,0) = \phi(x) - \frac{x}{\ell}T_0 = \psi(x)$

Let $V(x,t) = X(x)\,T(t)$

$\Rightarrow X(x) = A\cos\lambda x + B\sin\lambda(x)$

& $T(t) = e^{-\lambda^2/\alpha^2 t}$.

$V(0,t) = 0 \Rightarrow X(0) = 0 \Rightarrow A = 0$

$V(\ell,t) = 0 \Rightarrow X(\ell) = 0 \Rightarrow B\sin\lambda\ell = 0$

$$\Rightarrow \quad \lambda = \frac{n\pi}{l}$$

So
$$V(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$$

where
$$\lambda_n = \frac{2}{l}\int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

So
$$u(x,t) = \frac{x}{l}T_0 + \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2}{\alpha^2 l^2}t}$$

where
$$\lambda_n = \frac{2}{l}\int_0^l \left(\phi(x) - \frac{x}{l}T_0\right) \sin\left(\frac{n\pi x}{l}\right) dx$$

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Question:- Solve $u_{xx} = \alpha^2 u_t$ with
$u(0,t) = T_1$, $u(l,t) = T_2$ and
$u(x,0) = 0$

Solution

Define a function
$$V(x,t) = u(x,t) - \left(1 - \frac{x}{l}\right)T_1 - \frac{x}{l}T_2$$

Then
$$V(0,t) = u(0,t) - T_1 - 0$$
$$= T_1 - T_1 = 0$$

$$\Rightarrow V(0,t) = 0$$

~~Then the~~
$$V(l,t) = u(l,t) - 0 - T_2$$
$$= T_2 - T_2 = 0$$

$$\Rightarrow V(l,t) = 0$$

Then the given problem reduces to
$$V_{xx} = \alpha^2 V_t$$

with $V(0,t) = 0$, $V(l,t) = 0$
& $V(x,0) = -\left(1 - \frac{x}{l}\right)T_1 - \frac{x}{l}T_2$

Now as

Put $V = XT$

$\Rightarrow V = X(x)\,T(t)$

$\Rightarrow X_{xx}T = \alpha^2 X T_t$

$\Rightarrow \dfrac{X_{xx}}{X} = \alpha^2 \dfrac{T_t}{T} = -\lambda^2$

$\Rightarrow X_{xx} + \lambda^2 X = 0$

A.E is $D^2 + \lambda^2 = 0 \Rightarrow D^2 = -\lambda^2$

$\Rightarrow D = \pm i\lambda$

So $X(x) = A\cos\lambda x + B\sin\lambda x$

Also $\dfrac{T_t}{T} = \dfrac{-\lambda^2}{\alpha^2}$

$\Rightarrow T(t) = c\,e^{-\lambda^2/\alpha^2\, t}$

So $V(x,t) = [A\cos\lambda x + B\sin\lambda x]\,c\,e^{-\lambda^2/\alpha^2\, t}$

$V(0,t) = 0 \Rightarrow X(0) = 0$

$\Rightarrow A\cos(0) + B\sin(0) = 0$

$\Rightarrow A = 0$

So $X(x) = B\sin\lambda x$

$X(\ell) = 0 \Rightarrow B\sin\lambda\ell = 0$

$\Rightarrow \sin\lambda\ell = 0 \Rightarrow \lambda\ell = n\pi$

$\Rightarrow \lambda = \dfrac{n\pi}{\ell}$

$\Rightarrow V(x,t) = B\sin\left(\dfrac{n\pi x}{\ell}\right)\cdot c\,e^{-\frac{n^2\pi^2}{\alpha^2\ell^2}t}$

$\Rightarrow V_n(x,t) = d_n \sin\left(\dfrac{n\pi x}{\ell}\right) e^{-\frac{n^2\pi^2}{\alpha^2\ell^2}t}$

Now by Principle of superposition

$$V(x,t) = \sum_{n=1}^{\infty} V_n(x,t)$$

$$= \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\frac{n^2\pi^2}{a^2\ell^2}t}$$

$$V(x,0) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{\ell}\right)(1)$$

$$\Rightarrow \left(\frac{x}{\ell}T_1 - \frac{x}{\ell}T_2 - T_1\right) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow \left(\frac{x}{\ell}T_1 - \frac{x}{\ell}T_2 - T_1\right)\sin\frac{m\pi x}{\ell} = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{m\pi x}{\ell}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} d_n \int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{m\pi x}{\ell}\right)dx = \int_0^{\ell}\left(\frac{x}{\ell}T_1 - \frac{x}{\ell}T_2 - T_1\right)\sin\frac{m\pi x}{\ell}\,dx$$

For $m = n$

$$d_n(\text{———}) = \int_0^{\ell}\left(\frac{x}{\ell}T_1 - \frac{x}{\ell}T_2 - T_1\right)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$= \left(\frac{T_1 - T_2}{\ell}\right)\int_0^{\ell} x\sin\left(\frac{n\pi x}{\ell}\right)dx - T_1\int_0^{\ell}\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$= \left(\frac{T_1 - T_2}{\ell}\right)\left\{\frac{-x\cos\left(\frac{n\pi x}{\ell}\right)}{n\pi/\ell}\Bigg|_0^{1} - \int_0^{\ell}(1)\left[\frac{-\cos\frac{n\pi x}{\ell}}{n\pi/\ell}\right]dx\right\}$$

$$- T_1\left[\frac{-\cos\frac{n\pi x}{\ell}}{n\pi/\ell}\right]\Bigg|_0^{\ell}$$

$$= \left[\frac{T_1 - T_2}{\ell}\right]\left\{\frac{-\ell\cos n\pi}{n\pi/\ell} - 0 + \int_0^{\ell}\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\,dx\right\}$$

$$+ \frac{\ell T_1}{n\pi}\left[\cos n\pi - \cos 0\right]$$

$$= \left[\frac{T_1 - T_2}{\ell}\right]\left[-\frac{\ell^2}{n\pi}(-1)^n + \frac{\ell}{n\pi}\cdot\frac{\sin\frac{n\pi x}{\ell}}{n\pi/\ell}\Bigg|_0^{\ell}\right.$$

$$+ \frac{\ell T_1}{n\pi}\left[(-1)^n - 1\right]$$

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$$\Rightarrow \lambda_n \left(\frac{\quad}{\quad}\right) = \left(\frac{T_1 - T_2}{\ell}\right)\left[\frac{-\ell^2}{n\pi}(-1)^n + \frac{\ell^2}{n^2\pi^2}(0-0)\right]$$

$$+ \frac{\ell T_1}{n\pi}\left[(-1)^n - 1\right]$$

$$= -\frac{(T_1 - T_2)\ell}{n\pi}(-1)^n + \frac{\ell T_1}{n\pi}(-1)^n - \frac{\ell T_1}{n\pi}$$

$$= \cancel{\frac{-\ell T_1 (-1)^n}{n\pi}} + \frac{\ell T_2}{n\pi}(-1)^n + \cancel{\frac{\ell T_1}{n\pi}(-1)^n}$$

$$- \frac{\ell T_1}{n\pi}$$

$$= \frac{\ell T_2}{n\pi}(-1)^n - \frac{\ell T_1}{n\pi}$$

$$\Rightarrow \lambda_n = \frac{2}{\cancel{\ell}}\left[\frac{\cancel{\ell}}{n\pi}\left(T_2(-1)^n - T_1\right)\right]$$

$$\lambda_n = \frac{2}{n\pi}\left[(-1)^n T_2 - T_1\right]$$

Hence

$$V(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{n\pi x}{\ell}\right) e^{\frac{-n^2\pi^2}{\alpha^2\ell^2}t}$$

$$\Rightarrow V(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[(-1)^n T_2 - T_1\right] \sin\left(\frac{n\pi x}{\ell}\right) e^{\frac{-n^2\pi^2}{\alpha^2\ell^2}t}$$

v. Imp

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Question:- Discuss the Dirichlet, Neumann, Robin and Mixed boundry conditions Associated with heat equation as well as the initial condition [91, 93, 94, 95, 96]

Solution

There must be one initial and two boundry conditions to solve the problem uniquely. Such conditions are usually of the form $u(x,0) = f(x)$ which gives the initial temperature distribution. Now the Associated boundry conditions can be divided into four different types

1:- Dirichlet Conditions OR Boundry Conditions of 1st Kind:-

These condition can be described by $u(0,t) = T_0$ and $u(a,t) = T_1$, $t>0$ where T_0 and T_1 are the temperatures and $x=0$, $x=a$ are the end points of rod. Here T_0 and T_1 may be different or same.

2:- Neumann's Conditions OR Boundry Conditions of 2nd Kind:-

An other possibility is that the rate of flow of heat is specified at one or more boundry points. Since rate of flow is related to gradient of temperature by Fourier's Law (i.e $E = -K \nabla T$) such a condition may be written as

$$u_x(0,t) = \gamma(t) \quad \& \quad u_x(a,t) = \delta(t),$$ where in general γ and δ are function of 't' and in particular γ and δ may be constants or zero. If $\gamma = 0$ Then there is no flow at $x=0$

3:- Robin's Conditions OR Boundry Conditions of 3rd Kind:-

The boundry conditions of the form

$$a_1 u(0,t) + a_2 u_x(0,t) = \text{constant}$$
$$b_1 u(a,t) + b_2 u_x(a,t) = \text{constant}$$

are called Robin's conditions or Boundry conditions of 3rd kind.

* Physical Meanings of Robin's Conditions:-

According to the newton's law of cooling

"If a hot body is in contact with a less hot body (sourrounding body), it loses heat by convention. Rate of transfer of heat is proportional to the difference of temperature between the two bodies i.e

$$q(x,t) \propto u - T$$

$$\Rightarrow q(x,t) = h(u-T)$$

Applying this law at end point $x = x_0$

Then $q(x_0,t) = h[u(x_0,t) - T]$

$$\Rightarrow -K\frac{\partial u}{\partial x}(x_0,t) = h[u(x_0,t) - T]$$

$\because$ by Fourier Law $q = -K\frac{\partial u}{\partial x}$

$$\Rightarrow hu(x_0,t) + K\frac{\partial u(x_0,t)}{\partial x} = hT$$

Which is Robin's condition at $x = x_0$

4. Mixed Boundry Conditions:-

All the above boundry conditions involve the function u and its derivatives at one point. If more than one point is involved then the Basic condition is called Mixed Boundry Conditions

e.g if a uniform rod is bent s.t the ends $x = 0$ and $x = a$

jointed together. Then appropriate mixed Boundry conditions are

$$u(0,t) = u(a,t),$$

$$\frac{\partial}{\partial x} u(0,t) = \partial/_{\partial x}\, u(a,t) \quad , t>0$$

***

Question: Find steady state solution of the problem

$$u_{xx} = \frac{1}{k} u_t \longrightarrow ① \quad 0<x<a,\ t>0$$

$$u(0,t) = T_0 \longrightarrow ②$$

$$-K\, u_x(a,t) = h[u(a,t) - T_1] \longrightarrow ③$$

$$u(x,0) = f(x) \longrightarrow ④$$

Solution

Put $\lim\limits_{t\to\infty} u(x,t) = V(x)$

where $V(x)$ is the steady state solution

Now by ① $\frac{d^2V}{dx^2} = 0 \quad \Rightarrow V(x) = Ax + B$

$$u(0,t) = T_0 \quad \Rightarrow V(0) = T_0 \quad \Rightarrow B = T_0$$

$$-K\, u_x(a,t) = h[u(a,t) - T_1]$$

$$-KA = h[Aa + T_0 - T_1]$$

$$\Rightarrow A = \frac{T_1 - T_0}{ha + K}$$

So required steady state solution is

$$V(x) = \left[\frac{T_1 - T_0}{ha+K}\right] x + T_0$$

———

Question Interpret and solve the problem

$$u_{xx} = \frac{1}{k} u_t \qquad 0<x<a, \; t>0$$

$$u(0,t) = T_0 \;, \; u(a,t) = T_1$$

$$u(x,0) = f'(x)$$

Solution

Physical Interpretation:-

Obviously given problem describes the flow of heat in a rod of length 'a' which is included along the sides and whose end points are maintained at temperature T_0 and T_1. The original temperature of the distribution is $f'(x)$

1st we obtain steady state solution by using $\lim_{t\to\infty} u(x,t) = v(x)$

Then given equation becomes

$$\frac{d^2v}{dx^2} = 0 \qquad \Rightarrow v(x) = Ax + B$$

$$v(0) = T_0 \qquad \Rightarrow B = T_0$$

$$v(a) = T_1 \qquad \Rightarrow A = \frac{T_1 - T_0}{a}$$

$$\Rightarrow v(x) = \left(\frac{T_1 - T_0}{a}\right)x + T_0$$

Now we obtain transient temperature distribution by substitution

$$u(x,t) = \omega(x,t) + v(x)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial \omega}{\partial x} + \frac{\partial v}{\partial x} \;\; \& \;\; \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \omega}{\partial x^2} + 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} \quad \text{and also} \quad \frac{\partial u}{\partial t} = \frac{\partial w}{\partial t}$$

Hence we have

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}$$

Then $w(x,t) = u(x,t) - v(x)$

$$w(0,t) = u(0,t) - v(0) = T_0 - T_0 = 0$$

$$w(a,t) = u(a,t) - v(a) = T_1 - T_0 + T_0 - T_1 = 0$$

$$w(x,0) = u(x,0) - v(x) = f'(x) - v(x) = g(x)$$

Hence given problem becomes

$$w_{xx} = \frac{1}{k} w_t \quad \text{with}$$

$$w(0,t) = 0, \quad w(a,t) = 0, \quad w(x,t) = g(x)$$

and can be solved by method of seperation of variable.

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Question Find the steady state solution of the problem

$$\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) = c\rho\left(\frac{\partial u}{\partial t}\right) \quad 0<x<a,\ t>0$$

$u(0,t) = T_0$, $u(a,t) = T_1$. If $k(x) = b + dx$, b and d are constant.

Solution Put $\lim_{t\to\infty} u(x,t) = v(x)$

then $\frac{d}{dx}\left(k\frac{dv}{dx}\right) = 0 \longrightarrow (1)$

$$\Rightarrow k\frac{dv}{dx} = A \Rightarrow \frac{dv}{dx} = \frac{A}{k} = \frac{A}{b+dx}$$

$$\Rightarrow v = \frac{A}{d}\ln|b+dx| + B \longrightarrow *$$

Now $u(0,t) = T_0 \Rightarrow v(0) = T_0$

$$\Rightarrow \frac{A}{d}\ln|b| + B = T_0$$

$$\Rightarrow B = T_0 - \frac{A}{d}\ln|b| \longrightarrow (2)$$

$u(a,t) = T_1 \Rightarrow V(a) = T_1$

$$\Rightarrow \frac{A}{d}\ln|b+ad| + B = T_1$$

$$\Rightarrow A = \frac{d(T_1 - T_0)}{\ln\left|1+\frac{ad}{b}\right|} \quad \text{eq}$$

$$B = T_0 - \frac{\ln|b|(T_1 - T_0)}{\ln\left|1+\frac{ad}{b}\right|}$$

Hence by * required steady state solution is

$$V(x) = \frac{T_1 - T_0}{\ln\left|1+\frac{ad}{b}\right|}\ln|b+dx| + T_0 - \frac{\ln|b|(T_1 - T_0)}{\ln\left|1+\frac{ad}{b}\right|}$$

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⇒ Wronskian:-

Wronskian of two functions f and g is denoted and defined by

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

$$= fg' - f'g$$

——

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Theorem Let the functions $f_1, f_2, f_3, \dots, f_n$ have derivatives up to '$n-1$' order and

$$\begin{vmatrix} f_1 & f_2 & f_3 & \cdots & f_{n-1} & f_n \\ f_1' & f_2' & f_3' & \cdots & f_{n-1}' & f_n' \\ f_1'' & f_2'' & f_3'' & \cdots & f_{n-1}'' & f_n'' \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_{n-1}^{(n-1)} & f_n^{(n-1)} \end{vmatrix} \neq 0$$

for at least one value of $x \in [a,b]$. then functions are linearly independent.

Proof We prove the theorem for $n=2$

Given $W(f_1, f_2) \neq 0$

Suppose f_1 and f_2 are linearly dependent then $\exists$ c_1 and c_2 (not both zero) s.t

$$c_1 f_1 + c_2 f_2 = 0 \longrightarrow (1)$$

$$\& \quad c_1 f_1' + c_2 f_2' = 0 \longrightarrow (2)$$

Since c_1 and c_2 are not zero simulta-

so $\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = 0 \quad \Rightarrow W(f_1, f_2) = 0$

A contradiction. so our supposition is wrong. Hence f_1 and f_2 are lin. Ind.