20-2-14

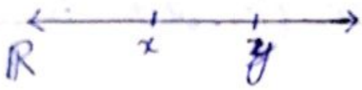
What is Functional Analysis?

Functional Analysis is the branch of mathematics that deals with abstract spaces and the linear operators (linear function) between them.

By an abstract space we mean a set whose elements are unspecified together with some axioms.

operation
$f: X \longrightarrow Y$
V. space   V. Space

"Metric Spaces"



1):- $|x-y| \geq 0$

2):- $|x-y| = 0$ iff $x=y$

3):- $|x-y| = |y-x|$ (Symmetry)

4):- $|x-y| \leq |x-z| + |z-y|$ (Triangle Inequality)

↳ Because in a triangle if we add two sides, then the sum can't be less than the third side. To generalized the concept of distance to arbitrary spaces we need the ~~con~~ concept of metric.

Definition: Let $X \neq \phi$, then the function $d: X \times X \longrightarrow \mathbb{R}$ is called a metric if it is satisfied the following Conditions:

M1): $d(x, y) \geq 0$

M2): $d(x, y) = 0 \Leftrightarrow x = y$

M3): $d(x,y) = d(y,x)$ (Symetery)

M4): $d(x,y) \leq d(x,z) + d(z,y)$

(Triangle Inequality)

The triangle inequality can be generalized for n-points between x and y as

$$d(x,y) \leq d(x,x_1) + d(x_1, x_2) + \cdots + d(x_n, y)$$

which is called generalized triangle inequality. The space (X,d) is called a metric space.

Example 1: Let $X = \mathbb{R}$ defined by $d(x,y) = |x-y|$ then d is metric because:

M1: $d(x,y) = |x-y| \geq 0$

M2: $d(x,y) = |x-y| = 0 \iff x = y$

M3: $d(x,y) = |x-y| = |y-x| = d(y,x)$

M4: $d(x,y) = |x-y| = |(x-z)+(z-y)|$

$$\leq |x-z| + |z-y| = d(x,z) + d(z,y)$$

i.e $d(x,y) \leq d(x,z) + d(z,y)$

Hence d is a metric and (X,d) is a metric space.

Example 2: Let $X = \mathbb{R}^2$ and defined on X by

$$d(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}$$

Where $x = (\xi_1, \xi_2)$, $y = (\eta_1, \eta_2) \in \mathbb{R}^2$

Then d is metric on X, since

M1: $d(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2} \geq 0$

M2: $d(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2} = 0 \Leftrightarrow \xi_1 = \eta_1$ and $\xi_2 = \eta_2 \Leftrightarrow x = y$

M3: $d(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}$

$$= \sqrt{(\eta_1 - \xi_1)^2 + (\eta_2 - \xi_2)^2} = d(y,x)$$

M4: Let $z = (\zeta_1, \zeta_2) \in \mathbb{R}^2$ if we identify $\mathbb{R}^2$ with $\mathbb{C}$, then

$d(x,y) = |x - y|$, the modulus of the complex number $x - y$.

then

$x = \xi_1 + i\xi_2$

$y = \eta_1 + i\eta_2$

$x - y = (\xi_1 - \eta_1) + i(\xi_2 - \eta_2)$

$|x - y| = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}$

$= d(x,y)$

$$\begin{aligned} d(x,y) &= |x - y| \\ &= |(x - z) + (z - y)| \\ &\leq |x - z| + |z - y| \\ &= d(x,z) + d(z,y) \end{aligned}$$

i.e. $d(x,y) \leq d(x,z) + d(z,y)$

So, d is metric and (X,d) is a metric space.

$$d(x,y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2|$$

Example-3- Let X be the space of continuous function defined on $[a,b]$. This space is denoted by $C[a,b]$ 21-02-2014

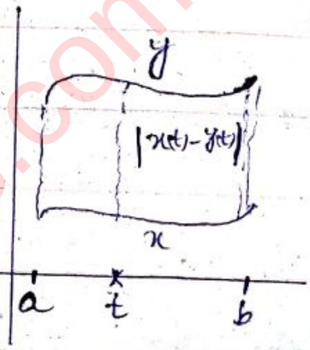
i.e $C[a,b] = \{x \mid x \text{ is continuous on } [a,b]\}$

Defined:

d on $C[a,b]$ by

$$d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|$$

This function is welldefined since difference of two continuous functions is continuous and a continuous defined on a closed interval is bounded.



We can show that d is a metric as follows.

M1) $d(x,y) = \max\limits_{t\in[a,b]} |x(t) - y(t)| \geq 0$

M2) $d(x,y) = \max\limits_{t\in[a,b]} |x(t) - y(t)| = 0 \Leftrightarrow x(t) = y(t) \Leftrightarrow x = y$
$\forall t \in [a,b]$

M3) $d(x,y) = \max\limits_{t\in[a,b]} |x(t) - y(t)|$

$$= \max_{t\in[a,b]} |y(t) - x(t)| = d(y,x)$$

M4) Let $z \in C[a,b]$ then for $t \in [a,b]$ we have

$$|x(t)-y(t)| = |(x(t)-z(t)) + (z(t)-y(t))|$$
$$\leq |x(t)-z(t)| + |z(t)-y(t)|$$
$$\leq \max_{t\in[a,b]} |x(t)-z(t)| + \max_{t\in[a,b]} |z(t)-y(t)|$$
$$= d(x,z) + d(z,y)$$

i.e

$$|x(t)-y(t)| \leq d(x,z) + d(z,y)$$

Since this is true for an arbitrary $t \in [a,b]$ therefore

$$\max_{t\in[a,b]} |x(t)-y(t)| \leq d(x,z)+d(z,y)$$

i.e $d(x,y) \leq d(x,z) + d(z,y)$

Hence d is metric and (X,d) is a metric space.

Example 4: Consider $C[a,b]$ with

$$d(x,y) = \int_a^b |x(t)-y(t)|\,dt$$

then d is metric. Since

M1) $d(x,y) = \int_a^b |x(t)-y(t)|\,dt \geq 0$

M2) $d(x,y) = \int_a^b |x(t)-y(t)|\,dt = 0$ iff either

$a=b$ or $x(t)-y(t)=0 \;\forall\, t\in[a,b]$

Since $a \neq b$, therefore $x=y$.

Note:- yahan per agar $a=b$ ho to $t\in[a,b]$ interval nahi rehta.

M3) $d(x,y) = \int_a^b |x(t) - y(t)|\, dt$
$$= \int_a^b |y(t) - x(t)|\, dt = d(y,x)$$

M4) Let $z \in C[a,b]$ then for $t \in [a,b]$,

we have
$$d(x,y) = \int_a^b |x(t) - y(t)|\, dt$$
$$= \int_a^b \left|(x(t) - z(t)) + (z(t) - y(t))\right| dt$$
$$\leq \int_a^b |x(t) - z(t)|\, dt + \int_a^b |z(t) - y(t)|\, dt$$
$$= d(x,z) + d(z,y)$$

i.e
$$d(x,y) \leq d(x,z) + d(z,y)$$

Hence d is metric and (X,d) is a metric space.

Example. 5:- Consider the space of all bounded or unbounded sequance of complex no. This space is denoted by S. define d on S by
$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}$$

Where $x = \{\xi_i\}$ & $y = \{\eta_i\} \in S$ this form is well defined

Since
$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \quad \& \quad 0 \leq \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|} < 1$$

Then d is metric.

M1) $d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|} \geq 0$

M2) $d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|} = 0 \Leftrightarrow \xi_i = \eta_i \ \forall i \Leftrightarrow x = y$

M3) $d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|}$

$$= \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|\eta_i - \xi_i|}{1+|\eta_i - \xi_i|} = d(y,x)$$

M4) Consider

$$f(t) = \frac{t}{1+t}$$

$$\Rightarrow f'(t) = \frac{(1+t)(1) - t(1)}{(1+t)^2} = \frac{1}{(1+t)^2} > 0$$

$\Rightarrow f$ is monotonically increasing.

So, $|a+b| \leq |a| + |b|$

$$\Rightarrow f(|a+b|) \leq f(|a| + |b|)$$

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|+|b|}{1+|a|+|b|} = \frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|}$$

$$\leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

i.e

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|} \longrightarrow (1)$$

Let $z = \zeta_i \in S$ then

Put $a = \xi_i - \zeta_i$, $b = \zeta_i - \eta_i$ in (1).

$$d(x,y) = \sup |x(t) - y(t)|$$

$$\frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|} \leq \frac{|\xi_i - \delta_i|}{1+|\xi_i - \delta_i|} + \frac{|\delta_i - \eta_i|}{1+|\delta_i - \eta_i|}$$

$$\frac{1}{2^i} \cdot \frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|} \leq \frac{1}{2^i} \frac{|\xi_i - \delta_i|}{1+|\xi_i - \delta_i|} + \frac{1}{2^i} \cdot \frac{|\delta_i - \eta_i|}{1+|\delta_i - \eta_i|}$$

Summing over i, we get,

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|\xi_i - \delta_i|}{1+|\xi_i - \delta_i|} + \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\delta_i - \eta_i|}{1+|\delta_i - \eta_i|}$$

$$\Rightarrow d(x,y) \leq d(x,z) + d(z+y)$$

Hence "d" is metric and (X,d) is a metric space.

Example: 6: Let X be any set. define d on X by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

M1) $d(x,y) \geq 0$ (by definition)

M2) $d(x,y) = 0 \Leftrightarrow x = y$ "

M3) $d(x,y) = d(y,x)$

M4) Let $x, y, z \in X$

Case 1: if x, y, z are distinct.

$$d(x,y) = 1 \quad , \quad d(x,z) + d(z+y) = 2$$

i.e $d(x,y) \leq d(x,z) + d(z,y)$

Case II: if x, y, z are not distinct.

$d(x,y) = 0$, $d(x,z) + d(z+y) = 0$

i.e $d(x,y) \leq d(x,z) + d(z+y)$

Similarly for other cases.

ℓ^p – space, $p \geq 1$ Lec # 3 24-02-14

Let $p \geq 1$ be a fixed real no. the ℓ^p is the space of all sequences of the type $x = (\xi_i)_1^\infty = (\xi_1, \xi_2, \cdots)$ such that $|\xi_1|^p + |\xi_2|^p + \cdots$ converges. i.e

$$\ell^p = \left\{ x = (\xi_i)_1^\infty \;\middle|\; \sum_1^\infty |\xi_i|^p < \infty \right\}$$

and the metric is defined by

$$d(x,y) = \left(\sum_{i=1}^\infty |\xi_i - \eta_i|^p \right)^{1/p}$$

where $y = \eta_i$ & $\sum_{i=1}^\infty |\eta_i|^p < \infty$.

If ξ_i's are real, the ℓ^p space is said to be real ℓ^p-space. If ξ_i's are complex, the ℓ^p-space is called complex ℓ^p-space.

Note: In the case $p=2$ we have the famous **Hilbert sequence Space** ℓ^2 with metric defined by $d(x,y) = \left(\sqrt{\sum_{i=1}^\infty |\xi_i - \eta_i|^2} \right)$.

We prove some inequalities which are helpful in proving triangle inequalities for different spaces.

(I) Auxiliary Inequality

Let α and β be any two positive real numbers and $p \geq 1$ define q

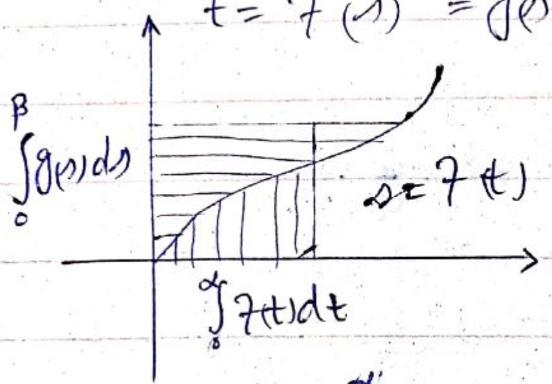
such that

$$\frac{1}{p} + \frac{1}{q} = 1 \text{ (i.e } p \text{ \& } q \text{ are conjugates exponents)}$$

then prove that

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q} \longrightarrow (1)$$

Youngs Inequality



$$\alpha\beta \leq \int_0^\alpha f(t)\,dt + \int_0^\beta g(s)\,ds$$

Proof:- From $\frac{1}{p} + \frac{1}{q} = 1$ we have $p+q = pq$

So, that

$$1 + \frac{p}{q} = p \quad \& \quad 1 + \frac{q}{p} = q$$

i.e $\quad p-1 = \frac{p}{q} \quad \& \quad q-1 = \frac{q}{p}$

Then $(p-1)(q-1) = 1 \Rightarrow q-1 = \frac{1}{p-1}$

The inequality (1) is true if either

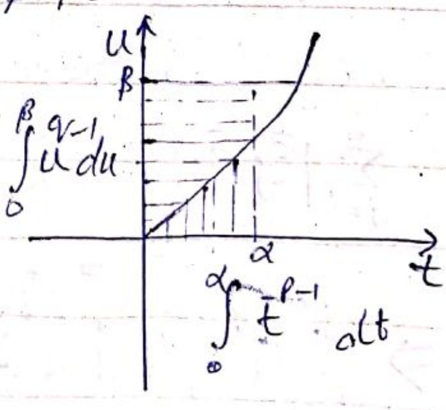
$\alpha = 0$ or $\beta = 0$

So, suppose $\alpha \neq 0$ & $\beta \neq 0$

consider a function

$$u = t^{p-1}$$

$$\Rightarrow t = u^{\frac{1}{p-1}} = u^{q-1}$$



From figure it is clear that

$$\alpha\beta \le \int_0^\alpha t^{p-1}\,dt + \int_0^\beta u^{q-1}\,du$$

$$\Rightarrow \alpha\beta \le \frac{t^p}{p}\Big|_0^\alpha + \frac{u^q}{q}\Big|_0^\beta$$

$$\Rightarrow \alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

Holder's Inequality:-

Let $x = (\xi_j) \in l^p$ & $y = (\eta_j) \in l^q$ where p, q are conjugates (i.e $\frac{1}{p} + \frac{1}{q} = 1$)

Then

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \le \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p} \left(\sum_{m=1}^{\infty} |\eta_m|^q\right)^{1/q}$$

Proof:- Let $(\bar{\xi}_j)$ & $(\bar{\eta}_j)$ be two spaces such that

$$\sum_{j=1}^{\infty} |\bar{\xi}_j|^p = 1 \quad \& \quad \sum_{j=1}^{\infty} |\bar{\eta}_j|^q = 1$$

If we said $\alpha = |\bar{\xi}_j|$ & $\beta = |\bar{\eta}_j|$ then, by Auxiliary inequality we obtain

$$|\bar{\xi}_j \bar{\eta}_j| \le \frac{|\bar{\xi}_j|^p}{p} + \frac{|\bar{\eta}_j|^q}{q}$$

Taking sumation over index j we get,

$$\sum_{j=1}^{\infty} |\bar{\xi}_j \bar{\eta}_j| \le \frac{1}{p}\sum_{j=1}^{\infty} |\bar{\xi}_j|^p + \frac{1}{q}\sum_{j=1}^{\infty} |\bar{\eta}_j|^q$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \sum_{j=1}^{\infty} |\bar{\xi}_j \bar{\eta}_j| \le 1 \longrightarrow (1)$$

∴ Suppose $x = (\xi_j) \in l^p$ & $y = (\eta_j) \in l^q$ ∴ then if we define

$$\bar{\xi}_j = \frac{\xi_j}{\left(\sum_{k=1}^{\infty}|\xi_k|^p\right)^{1/p}} \to (i) \quad \& \quad \bar{\eta}_j = \frac{\eta_j}{\left(\sum_{m=1}^{\infty}|\eta_m|^q\right)^{1/q}} \to (ii)$$

So, that

$$\sum_{j=1}^{\infty} |\bar{\xi}_j|^p = 1 \quad \& \quad \sum_{j=1}^{\infty} |\bar{\eta}_j|^q = 1$$

~~So, by (1) we get~~

(a)

$$\sum_{j=1}^{\infty} |\bar{\xi}_j|^p = \frac{\sum |\xi_j|^p}{\sum_{k=1}^{\infty} |\xi_k|^p} = 1$$

So by (1) we get (from (i) & (ii))

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| * \frac{1}{\left(\sum_{k=1}^{\infty}|\xi_k|^p\right)^{1/p} \cdot \left(\sum_{m=1}^{\infty}|\eta_m|^q\right)^{1/q}} \leq 1$$

$$\Rightarrow \sum_{j=1}^{\infty} |\xi_j \eta_j| \leq \left(\sum_{k=1}^{\infty}|\xi_k|^p\right)^{1/p} \cdot \left(\sum_{m=1}|\eta_m|^q\right)^{1/q}$$

If $p = q = 2$ then it called **Cauchy-Schwarz Inequality**

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \leq \sqrt{\sum_{k=1} |\xi_k|^2} \sqrt{\sum_{m=1}^{\infty} |\eta_m|^2}$$

Minkowski Inequality

Let $x = (\xi_j) \in l^p$, $y = (\eta_j) \in l^p$ & $p \geq 1$,

then

$$\left(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p\right)^{1/p} \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p} + \left(\sum_{m=1}^{\infty} |\eta_m|^p\right)^{1/p} \to (1)$$

Proof:— For $p=1$, the inequality ① is true. Since 27-02-2014

$$|\xi_j+\eta_j| \leq |\xi_j| + |\eta_j|$$

Such that

$$\sum_1^\infty |\xi_j+\eta_j| \leq \sum_1^\infty |\xi_j| + \sum_1^\infty |\eta_j|$$

Let $p>1$ and suppose

$$\omega_j = \xi_j + \eta_j \quad \text{then}$$

$$\begin{aligned} |\omega_j|^p &= |\omega_j|\,|\omega_j|^{p-1} \\ &= |\xi_j+\eta_j|\,|\omega_j|^{p-1} \\ &\leq (|\xi_j|+|\eta_j|)\,|\omega_j|^{p-1} \\ &= |\xi_j||\omega_j|^{p-1} + |\eta_j|\,|\omega_j|^{p-1} \end{aligned}$$

Summing from $J=1$ to $J=n$, we obtain

$$\sum_1^\infty |\omega_j|^p \leq \sum_1^\infty |\xi_j|\,|\omega_j|^{p-1} + \sum_1^\infty |\eta_j|\,|\omega_j|^{p-1} \longrightarrow ②$$

Let

$$I = \sum_1^\infty |\xi_j|\,|\omega_j|^{p-1}$$

note that $|\xi_j| \in l^p$ we claim that $|\omega_j|^{p-1} \in l^q$ since

$$\begin{aligned} \sum_1^\infty \left(|\omega_j|^{p-1}\right)^q &= \sum_1^\infty |\omega_j|^{(p-1)q} \\ &= \sum_1^\infty |\omega_j|^p < \infty \quad (\text{B/c } (\omega_j) \in l^p) \end{aligned}$$

So, from Holder's inequality, we obtain.

$$I = \sum_1^n |\xi_j|\,|\omega_j|^{p-1} \leq \left(\sum_1^n |\xi_j|^p\right)^{1/p} \left(\sum_1^n \left(|\omega_j|^{p-1}\right)^q\right)^{1/q}$$

i.e

$$\sum_1^n |\xi_j||w_j|^{p-1} \leq \left(\sum_1^n |\xi_j|^p\right)^{1/p} \left(\sum_1^n |w_j|^p\right)^{1/q} \longrightarrow (3)$$

By similarly argument, we have

$$\sum_1^n |\eta_j||w_j|^{p-1} \leq \left(\sum_1^n |\eta_j|^p\right)^{1/p} \left(\sum_1^n |w_j|^p\right)^{1/q} \longrightarrow (4)$$

Put eq. (3) & eq. (4) in eq. (2). we get,

$$\sum_1^n |w_j|^p \leq \left[\left(\sum_1^n |\xi_j|^p\right)^{1/p} + \left(\sum_1^n |\eta_j|^p\right)^{1/p}\right] \left(\sum_1^n |w_j|^p\right)^{1/q}$$

$$\Rightarrow \left(\sum_1^n |w_j|^p\right)^{1-1/q} \leq \left(\sum_1^n |\xi_j|^p\right)^{1/p} + \left(\sum_1^n |\eta_j|^p\right)^{1/p}$$

$$\Rightarrow \left(\sum_1^n |w_j|^p\right)^{1/p} \leq \left(\sum_1^n |\xi_j|^p\right)^{1/p} + \left(\sum_1^n |\eta_j|^p\right)^{1/p}$$

$$\frac{1}{p} + \frac{1}{q} = 1, \quad \frac{1}{p} = 1 - \frac{1}{q}$$

Letting $n \to \infty$ we get,

$$\left(\sum_1^\infty |\xi_j + \eta_j|^p\right)^{1/p} \leq \left(\sum_1^\infty |\xi_k|^p\right)^{1/p} + \left(\sum_1^\infty |\eta_m|^p\right)^{1/p}$$

Example:- Show that

$$l^p = \left\{x = (\xi_i) \mid \sum_1^\infty |\xi_i|^p < \infty\right\}$$

with d defined by

$$d(x,y) = \left(\sum_1^\infty |\xi_i - \eta_i|^p\right)^{1/p}$$

is a metric space. Where $x = (\xi_i) \in l^p$, $y = (\eta_i) \in l^p$.

Solution:- **M1)** $d(x,y) = \left(\sum_1^\infty |\xi_i - \eta_i|^p\right)^{1/p} \geq 0$

since $|\xi_i - \eta_i| \geq 0 \quad \forall i$

M2):- $d(x,y) = \left(\sum_1^\infty |\xi_i - \eta_i|^p\right)^{1/p} = 0$

$\Leftrightarrow |\xi_i - \eta_i| = 0 \ \forall i \Leftrightarrow \xi_i = \eta_i \ \forall i \Leftrightarrow x = y$

M3):- $d(x,y) = \left(\sum_1^\infty |\xi_i - \eta_i|^p\right)^{1/p}$

$= \left(\sum_1^\infty |\eta_i - \xi_i|^p\right)^{1/p} = d(y,x)$

M4): Let $z = (\zeta_i) \in \ell^p$ then

$d(x,y) = \left(\sum_1^\infty |\xi_i - \eta_i|^p\right)^{1/p}$

$= \left(\sum_1^\infty |(\xi_i - \zeta_i) + (\zeta_i - \eta_i)|^p\right)^{1/p}$

$\underset{M\cdot I}{\leq} \left(\sum_1^\infty |\xi_i - \zeta_i|^p\right)^{1/p} + \left(\sum_1^\infty |\zeta_i - \eta_i|^p\right)^{1/p}$

$= d(x,z) + d(z,y)$

i.e

$d(x,y) \leq d(x,z) + d(z,y)$

Hence (X,d) is a metric space.

Example: Let $X = \mathbb{R}^n$. Define "d" on $\mathbb{R}^n$ by $d(x,y) = \sqrt{(\xi_1-\eta_1)^2 + (\xi_2-\eta_2)^2 + \cdots + (\xi_n-\eta_n)^2} = \left(\sum_1^n |\xi_i - \eta_i|^2\right)^{1/2}$

Show that (X,d) is a metric space.

Solution:- M1):- $d(x,y) = \left(\sum_1^n |\xi_i - \eta_i|^2\right)^{1/2} \geq 0$

since $|\xi_i - \eta_i| \geq 0 \ \forall i$.

M2):- $d(x,y) = \left(\sum_1^n |\xi_i - \eta_i|^2\right)^{1/2} = 0$

$\Leftrightarrow |\xi_i - \eta_i| = 0 \ \forall i \Leftrightarrow \xi_i = \eta_i \ \forall i \Leftrightarrow x = y$

M3):- $d(x,y) = \left(\sum_1^n |\xi_i - \eta_i|^2\right)^{1/2}$

$$= \left(\sum_1^n |\eta_i - \xi_i|^2\right)^{1/2} = d(y,x)$$

M4:) Let $z = (\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n$

$$d(x,y) = \left(\sum_1^n |\xi_i - \eta_i|^2\right)^{1/2}$$

$$= \left(\sum_1^n |(\xi_i - \zeta_i) + (\zeta_i - \eta_i)|^2\right)^{1/2}$$

$$\underset{M.I}{\leq} \left(\sum_1^n |\xi_i - \zeta_i|^2\right)^{1/2} + \left(\sum_1^n |\zeta_i - \eta_i|^2\right)^{1/2}$$

$$= d(x,z) + d(z,y)$$

i.e $d(x,y) \leq d(x,z) + d(z,y)$

Hence $(\mathbb{R}^n, d)$ is a metric space.

Note:- Similar arguments can be used to prove that $\mathbb{C}^n$ with

$$d(x,y) = \left(\sum_1^n |\xi_i - \eta_i|^2\right)^{1/2}$$

is a metric space.

Some Topological Concepts in Metric Spaces

28-02-14

Definitions:- Let (X,d) be a metric space $x_0 \in X$ & $r > 0$ then

1:) $B(x_0; r) = \{x \in X \mid d(x_0, x) < r\}$

is called an **open ball** with centre at x_0 and radius r.

2:) $\overline{B}(x_0;r)=\{x\in X \mid d(x_0,x)\le r\}$ is called a **closed ball** with centre at x_0 and radius r.

3): $S(x_0;r)=\{x\in X \mid d(x_0,x)=r\}$ is called a **sphere** with centre at x_0 and radius r.

Note: $S(x_0;r)=\overline{B}(x_0;r)-B(x_0;r)$

Example: Consider the **discrete metric** space where d is

$$d(x,y)=\begin{cases}0 & \text{if } x=y\\ 1 & \text{if } x\ne y\end{cases}$$

Then

$$B(x_0;1)=\{x\in X \mid d(x_0,x)<1\}=\{x_0\}$$

$$S(x_0;1)=\{x\in X \mid d(x_0,x)=1\}=X\setminus\{x_0\}$$

$$S(x_0;r)=\{x\in X \mid d(x_0,x)=r\}=\phi \text{ if } r\ne 1$$

Example:- Let $X=\mathbb{C}$ with

$$d(x,y)=|x-y| \quad x,y\in\mathbb{C}$$

then

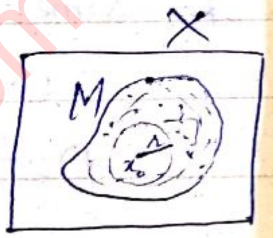
$$B(0;1)=\{x\in\mathbb{C} \mid d(0,x)<1\}$$
$$=\{x\in\mathbb{C} \mid |x|<1\}$$

$$\overline{B}(0;1)=\{x\in\mathbb{C} \mid |x|\le 1\}$$

$$S(0;1)=\{x\in\mathbb{C} \mid |x|=1\}$$

Open Set

Let (X, d) be a metric space & $M \subseteq X$. We say that M is open set in X if it contains an open ball of suitable radius about each of it's points. i.e $\underbrace{B(x_0; r)}_{\text{open ball}} \subseteq M$



Closed Set:-

A subset K of X is said to be closed set if it's complement (in X) is open.

i.e $K^c = X | K$ is open.

ε-neighborhood:-

An open Ball $B(x_0; \varepsilon)$ is called an ε-neighborhood of x_0.

Neighborhood:-

By a neighborhood of a point x_0 we mean a subset N of X which contains an ε-neighborhood of x_0

$\because$ N contain ε-neighborhood of x_0

Interior Point

A point $x_0 \in X$ is called an interior point of $M \subseteq X$ if M is nieghborhood of x_0.

Interior of a Set

The set of all interior points of a set $M \subseteq X$ is called interior of M and is denoted by M° or $Int(M)$

Result:- Every Open Ball is an Open set.

Proof:- Let $B(x_0; r)$ be an open ball in some metric space (X, d).

Let $y \in B(x_0; r)$

Define $r_1 = r - d(x_0, y)$.

then $B(y; r_1) \subseteq B(x_0; r)$

if $z \in B(y; r_1)$ then

$d(x_0, z) \leq d(x_0, y) + d(y, z) < r_1 + d(x_0, y) = r$

i.e $d(x_0, z) < r$

So that $z \in B(x_0, r)$ which proves that

$B(y, r_1) \subseteq B(x_0, r)$

Hence

$B(x_0, r)$ is open set.

(self) **Result:-** Let (X, d) be a metric space. Then

(I) ϕ and X are open.

(II) The union of any collection (countable or uncountable) of open sets is open.

III) Finite intersection of open sets is open.

Proof:- I) Let X and ϕ are closed sets.

$$X^c = X \setminus X = \phi \quad \because \phi \text{ is closed}$$

$\Rightarrow X^c$ is closed

$\Rightarrow X$ is open.

Similarly $\phi^c = X \setminus \phi = X \quad \because X$ is closed

$\Rightarrow \phi^c$ is closed.

$\Rightarrow \phi$ is open.

II):- Let $\{A_K : K \in I\}$ be an indexed family of open sets in X and let

$$A = \bigcup_{K \in I} A_K$$

If $x \in A$ then there is an index i such that $x \in A_i$. Since A_i is open there is an $r > 0$ such that

$$B(x; r) \subseteq A_i \subseteq \bigcup_{K \in I} A_K$$

Therefore A is open.

III):- $A_1, A_2, \ldots, A_n$ be open subsets of X and let $A = \bigcap_{K=1}^{n} A_K$.

If $x \in A$ then $x \in A_K$, $K = 1, 2, \ldots, n$

It follows that there are $r_1, r_2, \ldots, r_n$ such that

$$B(x, r_k) \subseteq A_k \;,\; k = 1, 2, \cdots, n$$

Let $\quad r = \min\{r_1, r_2, \cdots, r_n\}$

Then, obviously,

$$B(x; r) \neq \phi$$

and $\quad B(x; r) \subseteq A_k \subseteq A \quad K = 1, 2, \cdots, n$

Thus "A" is open.

Remarks:- By above results we conclude that every metric space is a topological space.

Note:- Countable intersection of open sets need not to be open.

For example if $X = \mathbb{R}$, then the collection

$$G_i = \left(\frac{-1}{i}, \frac{1}{i}\right) \;\forall\, i \in N \text{ is a}$$

collection of open sets in $\mathbb{R}$. Then

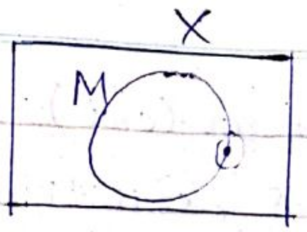
$$\bigcap_{1}^{\infty} G_i = \{0\} \text{ is not open.}$$

3-3-2014

Limit point / Accomulation Point.

Let (X, d) be a metric space and $M \subset X$. A point $x \in X$ (may or may not ~~not~~ be in M) is called a limit point or accomulative point if every neighborhood of x contains at least one point of M.

$B(x; r) \cap \{M - \{x\}\} \neq \phi$



Closure of a Set

A set M together with all it's limit points is called closure of set M and is denoted by $\overline{M}$.

So,

$$\overline{M} = M \cup \{\text{Limit points of } M\}$$

Dense Set

A subset M of a metric space X is called dense in X if $\overline{M} = X$

Example: Let $X = \mathbb{R}$ the $\mathbb{Q}$ is dense in $\mathbb{R}$.

Remarks:- If M is dense in X, then every ball in X no matter how small will contain points of M. In other words, there is no point $x \in X$ which has a neighborhood that does not contain points of M.

Continuity

The concept of continuity plays very important role in calculus. We also have the concept of continuity in metric space. There are three different (But equivalent) ways of defining continuity $\varepsilon - \delta$ definition, the Sequence criterian and the topological definition.

Each of them is intresting in it's own right.

Definition:-

Let (X,d) & $(Y,\bar{d})$ be two metric spaces.

A mapping $T:(X,d)\to(Y,\bar{d})$ is said to be continuous at a point $x_0 \in X$ if $\forall\, \varepsilon>0$, $\exists\, \delta>0$ such that

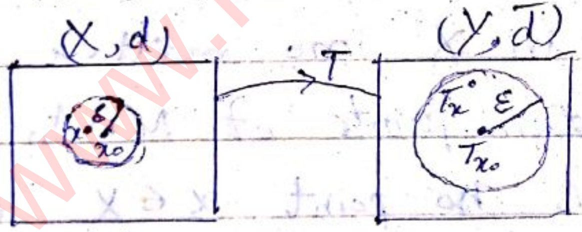
$\bar{d}(Tx, Tx_0) < \varepsilon \;\forall\, x$ satisfying

$d(x,x_0)<\delta \longrightarrow$ (A)

$\forall\, \varepsilon>0\; \exists\, \delta>0$ s.t. $|f(x)-f(x_0)|<\varepsilon$ whenever $|x-x_0|<\delta$

$T:(X,d)\to(Y,\bar{d})$

$\bar{d}(T_x, T_{x_0})<\varepsilon$ whenever $d(x,x_0)<\delta$



If "T" is continuous at each point of X then it is continuous over X.

Remarks:- Condition (A) equivalent to $T(B(x,\delta)) \subseteq B(T_x, \varepsilon)$

Image of radius $<\delta$ is contain in image of ball centre at T_x.

Continuous mapping are

characterized in terms of open sets as follows:

Theorem:- A mapping $T:(X,d)\to(Y,\bar{d})$ is continuous iff inverse image of any open subset of Y is an open subset of X.

Proof:- $\Rightarrow$ Suppose "T" is continuous & Let "O" be open subset of Y. We have to prove that

$$T^{-1}(O) \text{ is open in } X.$$

Let $x\in T^{-1}(O)$ then $Tx\in O$

(By def.) Since O is open there must exists $\varepsilon>0$ such that $B(Tx,\varepsilon)\subseteq O$.

Since T is continuous $\exists\ \delta>0$ such that $T(B(x,\delta))\subseteq B(Tx,\varepsilon)\subseteq O$

i.e $B(x,\delta)\subseteq T^{-1}(O)$

So, that $T^{-1}(O)$ is open.

$\Leftarrow$ Suppose that $T^{-1}(O)$ is open for every open subset O of Y.

We have to show that T is Continuous. Let $x\in X$ & $\varepsilon>0$ be given.

Then $B(Tx,\varepsilon)$ is open subset of Y.

By our supposition $T^{-1}(B(Tx,\varepsilon))$ is open in X.

Then $\exists$ $\delta > 0$ such that

$$B(x, \delta) \subseteq T^{-1}(B(Tx, \varepsilon))$$

$$\Rightarrow T(B(x, \delta)) \subseteq B(Tx, \varepsilon)$$

which proves that T is continuous.

Separable:-

A metric space (X, d) is said to be seperable if it contains a countable subset which is dense in X.

Example 1: Real line $\mathbb{R}$. 6-03-14

Let $X = \mathbb{R}$, then $\mathbb{Q} \subseteq \mathbb{R}$ is countable & $\overline{\mathbb{Q}} = \mathbb{R}$. So that $\mathbb{R}$ is seperable.

Example 2: Complex plane $\mathbb{C}$.

Let $X = \mathbb{C}$, then

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$$

Let

$$M = \{p + iq \mid p, q \in \mathbb{Q}\}$$

then M is countable subset of $\mathbb{C}$ (because $\mathbb{Q}$ is countable)

$$\overline{M} = \mathbb{C} \quad (\because \overline{\mathbb{Q}} = \mathbb{R})$$

$\Rightarrow \mathbb{C}$ is seperable.

Result:- A discrete metric space (X, d) is seperable iff X is countable.

Proof:- Given that (X, d) is a

discrete metric space. Then

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Suppose X is seperable. Let $M \subset X$ we claim that M can not be dense in X.

i.e $\overline{M} \neq X$

Let $x \in X \setminus M$ then for all $m \in M$, $d(x,m) = 1$. Then ~~then~~ open ball $B(x, \frac{1}{2})$ will not contain any point of M. So, that M is not dense in X. So no proper subset of X is dense in X.

Since X is seperable, is only dense set in X is X itself. So that X is countable.

Conversely:- Suppose X is countable. Since $\overline{X} = X$, therefore X is seperable.

Example:- The space ℓ^∞ is not seperable.

Solution:- We have

$$\ell^\infty = \{x = (\xi_i) \mid \sup_i |\xi_i| < \infty\}$$

with $d(x,y) = \sup_i |\xi_i - \eta_i|$, where

$$x = (\xi_i),\ y = \eta_i \in \ell^\infty$$

Let $y = (\eta_1, \eta_2, \eta_3, \ldots)$ such that $\eta_i = 0$ or 1

then, since

$\sup_i |\eta_i| = 1 < \infty$, therefore $y \in \ell^\infty$

with this y we can associate a real no.

$$\hat{y} = \frac{\eta_1}{2} + \frac{\eta_2}{2^2} + \frac{\eta_3}{2^3} + \cdots \in [0,1]$$

Note that $\hat{y}$ is binary representation of a real number in $[0,1]$. So, there is one-one correspondingce between $[0,1]$ & the sequance of 0's and 1's in ℓ^∞.

Since $[0,1]$ is uncountable therefore there are uncountable many sequances of 0's & 1's. If we let each of these sequances be centre of small balls with radius ~~tel~~ less than 1 (say $\frac{1}{3}$) then these balls are non-intersecting and we have uncountable many of these. If M is dense in ℓ^∞, each of these non-intersecting ball must contains an element of M. Hence M can't be countable & ℓ^∞ is not a seperable.

Example:- The space ℓ^p with $1 \leq p < \infty$ is seperable.

Proof:- We have

$$\ell^p = \{x = (\xi_i) \mid \sum_1^\infty |\xi_i|^p < \infty\}$$

with $d(x,y) = \left(\sum_1^\infty |\xi_i - \eta_i|^p\right)^{1/p}$

where $x = (\xi_i)_1^\infty$, $y = (\eta_i)_1^\infty \in \ell^p$

To prove the ℓ^p is seperable, we have to show that existance of a set in ℓ^p which is countable & dense.

Let $x = (\xi_i)_1^\infty \in l^p$, then $\sum_1^\infty |\xi_i|^p < \infty$

this mean's that $\forall\ \varepsilon > 0$, we can find n (depending upon ε) such that

$$\sum_{i=n+1}^{\infty} |\xi_i|^p < \frac{\varepsilon^p}{2} \longrightarrow (1)$$

Construct $M = \{y = \{\eta_1, \eta_2, \dots, \eta_n, 0, 0 \cdots\} \mid \eta_i \in \mathbb{Q}\} \subseteq l^p$

Since $\sum_1^\infty |\eta_i|^p < \infty$

M is countable. Since $\mathbb{Q}$ is countable.

Also, since rationals are dense in $\mathbb{R}$ for each $\xi_i \in \mathbb{R}$, there is a rational close to it. Then we can find a $y \in M$ such that

$$\sum_{i=1}^{n} |\xi_i - \eta_i|^p < \frac{\varepsilon^p}{2} \longrightarrow (ii)$$

If we chose $x \in l^p$ & $y \in M$

$$d^p(x,y) = \sum_{i=1}^{\infty} |\xi_i - \eta_i|^p$$

$$= \sum_{i=1}^{n} |\xi_i - \eta_i|^p + \sum_{i=n+1}^{\infty} |\xi_i|^p < \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} = \varepsilon^p$$

i.e $d(x,y) < \varepsilon$

which shows that M is dense in l^p. Since M is countable therefore l^p is seperable.

Q: Show that 7-03-2014

$$B([a,b]) = \{x \mid x \text{ is bounded on } [a,b]\}$$

with $d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|$

is not seperable.

Convergence of a Sequance in Metric Space.

A sequance (x_n) in a metric space (X,d) is said to be convergence if $\exists\ x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0$$

i.e $d(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$

x is called limit point of (x_n) & we write

$$\lim_{x\to\infty} x_n = x$$

or $x_n \longrightarrow x$ as $n \longrightarrow \infty$

Remark:- The limit x of a convergent sequance (x_n) must be a point of X. For example

$X = (0,1]$ not converges

$X = (0,1]$ & $(x_n) = \frac{1}{n}$ is a sequance in $(0,1]$, but $\lim_{n\to\infty} x_n = 0 \notin X$

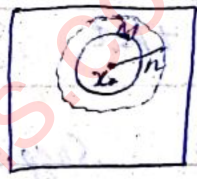
So $x_n \not\to 0$ in X.

A non-empty subset $M \subseteq X$ is said to bounded it's diameter

$\sup_{x,y \in M} d(x,y)$ is finite.

So, a sequance (x_n) is bounded in a metric space (X,d) if corresponding point set $\{x_1, x_2, \cdots\}$ is bounded.

Note:- If M is bounded, $M \subset B(x_0, r)$ where x_0 is any point of M and r is sufficiently large real number.



Theorem:-

Let (X,d) be a metric space.

Then,

(a) Convergence sequance in X is bounded and it's limit is unique.

b) If $x_n \to x$ & $y_n \to y$ in X then $d(x_n, y_n) \longrightarrow d(x,y)$

Proof: a):- Suppose (x_n) is a convergent sequance in the metric space (X,d).

Then $\exists$ $x \in X$ such that

$x_n \to x$ as $n \to \infty$

In order to prove that (x_n) is bounded, we need to find a number $b \in \mathbb{R}$ such that

$d(x_n, x) < b \quad \forall \; n \geq N$

Since (x_n) is converges, $\forall \ \varepsilon > 0$

$\exists \ N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon \quad \forall \ n \geq N$$

In particular, if $\varepsilon = 1$, $\exists \ N_1 \in \mathbb{N}$

such that $d(x_n, x) < 1 \quad \forall \ n > N_1$

If we said

$$k = \max\left[d(x_1, x), d(x_2, x), \dots, d(x_{N_1}, x), 1\right]$$

then $\quad d(x_n, x) < k \quad \forall \ n \in \mathbb{N}$

So, that (x_n) is bounded.

Now suppose $x_n \to x$ & $x_n \to y$

as $n \to \infty$ then $\qquad d(x_n, x) \to 0$ as $n \to \infty$

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \to 0$$

as $n \to \infty$

i.e $\quad d(x, y) \leq 0 \quad$ But $d(x, y) \geq 0$

So that $d(x, y) = 0$ & $x = y$.

(b). Consider $d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$

$$\Rightarrow d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y) \to (1)$$

Interchanging the roles of x_n & x

and y_n & y.

$$d(x, y) - d(x_n, y_n) \leq d(x_n, x) + d(y_n, y)$$

i.e $\quad -\left[d(x_n, y_n) - d(x, y)\right] \leq d(x_n, x) + d(y_n, y) \to (II)$

(I) & (II) $\Rightarrow |d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y) \to 0$

as $n \to \infty$

i.e $\quad d(x_n, y_n) \to d(x, y)$

Remark:— Converse of part **(a)** is not true in generally. i.e a bounded sequence may or may not be convergent.

For example: $(x_n) = (-1)^n = -1, 1, -1, 1, \cdots$ is bounded but not convergent.

Cauchy Sequence

10-03-2014

A sequence (x_n) in metric space (X, d) is said to be cauchy or (Fundamental) if $\forall\ \varepsilon > 0$, $\exists\ N(\varepsilon) \in \mathbb{N}$ such that

$$d(x_m, x_n) < \varepsilon \quad \forall\ m, n \in \mathbb{N}$$

Complete

A metric space (X, d) is said to be complete if every cauchy sequence in X converges in X. i.e has a limit point which is an element of X.

Example:— $\mathbb{R}$ and $\mathbb{C}$ are complete by cauchy convergence criterian.

Example:— Let $X = (0, 1]$ with $d(x,y) = |x-y|$ & let $(x_n) = \frac{1}{n}$ $\forall\ n \in \mathbb{N}$, then (x_n) is cauchy sequence because we can find an $N \in \mathbb{N}$ such that

$$|x_m - x_n| = \left|\frac{1}{m} - \frac{1}{n}\right| < \varepsilon \quad \forall\ m, n < N$$

$$x_n = \frac{1}{n} \Longrightarrow [0,1]$$

$$x_n = (-1)^n \Longrightarrow \{-1, 1\}$$

But

$$x_n \longrightarrow 0 \notin X$$

Hence (X,d) is not complete.

Example:- Let $X = R - \{a\}$, with $d(x,y) = |x-y|$ and let $(x_n) = a + \frac{1}{n}$, then (x_n) is cauchy sequence but

$x_n \longrightarrow a \notin X$ so that (X,d) is not complete.

Example: Let $X = (a,b)$ with $d(x,y) = |x-y|$ then the sequences $x_n = a + \frac{1}{n}$, $y_n = b - \frac{1}{n}$ are cauchy and

$x_n \longrightarrow a \notin X$ & $y_n \longrightarrow b \notin X$ So, (X,d) is not complete.

Theorem:- Every convergent sequence in a metric space (X,d) is a cauchy sequence

Proof:- Let (x_n) be a convergent sequence in (X,d). Then $\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that

$$d(x_n, x) < \frac{\varepsilon}{2} \quad \forall n > N \longrightarrow (1)$$

Now for $m, n > N$,

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) \underset{\text{by (1)}}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

i.e $d(x_m, x_n) < \varepsilon \quad \forall\ m, n > N$

$\Rightarrow (x_n)$ is a cauchy sequence.

Theorem:- Let M be a non-empty subset of a metric space & $\overline{M}$ it's closure. Then

(a):- $x \in \overline{M}$ iff there is a sequence (x_n) in M such that $x_n \to x$.

(b):- M is closed iff the ~~statement~~ situation $x_n \in M$ & $x_n \to x$ implies $x \in M$.

Proof:- (a) $\Rightarrow$ Let $x \in \overline{M} = M \cup \{\text{limit points of } M\}$

Case I:- if $x \in M$, then $(x, x, x, \ldots)$ is in M and converges to x.

Case II:- If $x \notin M$ then it is limit point of M. Then for each $n = 1, 2, \cdots$ the ball $B(x, \frac{1}{n})$ contains a point $x_n \in M$ then $x_n \to x$ because $\frac{1}{n} \to 0$ as $n \to \infty$.

$\Leftarrow$ Suppose that (x_n) is a sequence in M such that $x_n \to x$. Then either $x \in M$ or every neighborhood of x contains point $x_n \neq x$. This means that x is limit point of M so, that $x \in \overline{M}$.

$B(x_1, 1)$	x_1
$B(x_2, \frac{1}{2})$	x_2
$B(x_3, \frac{1}{3})$	x_3
$\vdots$	$\vdots$
$B(x, \frac{1}{n})$	x_n
	$\downarrow$
	x

b):- M is closed iff $\overline{M} = M$ so that part (b) follows directly from part (a).

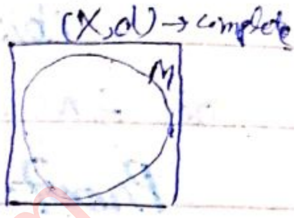
Theorem: (Complete Subspace)

A subspace M of a complete metric space X is itself complete iff the set M is closed in X.

Proof:- $\Rightarrow$ Suppose M is complete.

we have to prove that

M is closed i.e $M = \bar{M}$



clearly $M \subseteq \bar{M}$

We only need to show that $\bar{M} \subseteq M$

Let $x \in \bar{M}$ then part (a) of previous theorem $\exists$ (x_n) in M such that $x_n \longrightarrow x$. Since every convergent sequence is cauchy, therefore (x_n) is cauchy sequance in M. Since M is complete. (x_n) is converges to a point of M.

Since limit is unique, $x \in M \Rightarrow \bar{M} \subseteq M$.

So $M = \bar{M}$. i.e. M is closed.

$\Leftarrow$ Suppose that M is closed in X.

Let (x_n) is cauchy sequence in X. Since X is complete.

$\exists$ $x \in X$ such that $x_n \longrightarrow x$.

By part (a) of previous theorem, $x \in \bar{M} = M$ this means that M is complete.

$x \in \bar{M} \Leftrightarrow \exists$ (x_n) in M s.t $x_n \longrightarrow x$

Theorem:- 13-3-14

A mapping $T:(X,d)\to(Y,\bar{d})$ is continuous at a point $x_0\in X$ iff

$x_n\to x_0 \Rightarrow Tx_n\to Tx_0$

Proof: $\Rightarrow$ Suppose $T:(X,d)\to(Y,\bar{d})$ is continuous at $x_0\in X$. Then $\forall\ \varepsilon>0, \exists\ \delta>0$ such that

$$\bar{d}(Tx,Tx_0)<\varepsilon \quad \forall\ x \text{ satisfying } d(x,x_0)<\delta$$

i.e $d(x,x_0)<\delta \Rightarrow \bar{d}(Tx,Tx_0)<\varepsilon \longrightarrow (a)$

Suppose $x_n\to x_0$ then $\exists\ N\in\mathbb{N}$ such that

$d(x_n,x_0)<\delta \quad \forall\ n>N$ By def. of conver.

Then (a) $\Rightarrow \bar{d}(Tx_n,Tx_0)<\varepsilon \quad \forall\ n>N$

$\Rightarrow Tx_n\to Tx_0$

$\Leftarrow$ Suppose that $x_n\to x_0 \Rightarrow Tx_n\to Tx_0$ to prove that T is continuous.

Suppose on the contrary that T is not continuous at $x_0\in X$. Then $\exists$ $\varepsilon>0$ such that for $\delta>0$ we have

$$d(x,x_0)<\delta \Rightarrow \bar{d}(Tx,Tx_0)\geq\varepsilon$$

In particular if $\delta=\frac{1}{n}$ then $\exists\ x_n$ such that $d(x_n,x_0)<\delta=\frac{1}{n}$ but $\bar{d}(Tx_n,Tx_0)\geq\varepsilon$

$\Rightarrow x_n\to x_0$ but $Tx_n\not\to Tx_0$, which is a contradiction.

$\Rightarrow T_p$ is continuous at $x_0 \in X$.

Examples of complete metric space.

To prove that a space X is complete we take a cauchy sequence in X & show that it converges in X.

For this we generally use the following three steps:

1):- Construct an element x (to be used as limit)

2):- Prove that x is in the space considered.

3):- Prove that $x_n \to x$ in the space of metric.

Example 1:- Show that $\mathbb{R}^n$ with

$$d(x,y) = \left(\sum_{1}^{n} |\xi_i - \eta_i|^2\right)^{1/2}$$

where $x = (\xi_1, \xi_2, \dots, \xi_n)$, $y = (\eta_1, \eta_2, \dots, \eta_n)$ is complete.

Solution/proof:- Let (x_n) be cauchy sequence in $\mathbb{R}^n$ then

$$x_m = \left(\xi_1^{(m)}, \xi_2^{(m)}, \dots, \xi_n^{(m)}\right)$$

$\Rightarrow \forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ such that

$d(x_m, x_n) < \varepsilon \quad \forall\ m, n > N$

$$\Rightarrow \left(\sum_{i=1}^{n} |\xi_i^{(m)} - \xi_i^{(n)}|^2\right)^{1/2} < \varepsilon \quad \forall\ m,n > N \longrightarrow ①$$

$$\Rightarrow \sum_{i=1}^{n} |\xi_i^{(m)} - \xi_i^{(n)}|^2 < \varepsilon^2 \quad \forall\ m,n > N$$

$\Rightarrow$ For each fixed i $(1 \le i \le n)$ we have

$$|\xi_i^{(m)} - \xi_i^{(n)}|^2 < \varepsilon^2 \quad \forall\ m,n > N$$

$$\Rightarrow |\xi_i^{(m)} - \xi_i^{(n)}| < \varepsilon \quad \forall\ m,n > N$$

$\Rightarrow$ $\xi_i^{(m)}$ is a cauchy sequence in $\mathbb{R}$ $\forall\ i\ (1 \le i \le n)$

Since $\mathbb{R}$ is complete $\therefore$ $\xi_i^{(m)} \to \xi_i \in \mathbb{R}$

Using these n limits, construct

$$x = (\xi_1, \xi_2, \xi_3, \xi_4, \dots, \xi_n)$$

Condition II):- Clearly $x \in \mathbb{R}^n$

III): We now need to prove that $x_m \to x$. Letting $n \to \infty$ in (i) we get $\left(\sum_{i=1}^{\infty} |\xi_i^{(m)} - \xi_i|^2\right)^{1/2} \in \varepsilon \quad \forall\ m > N$

$$\Rightarrow d(x_m, x) < \varepsilon \quad \forall\ m > N$$

$$\Rightarrow x_m \to x \text{ as } m \to \infty$$

$\Rightarrow (\mathbb{R}^m, d)$ is complete metric space

Example 2:– Show that the space 14-03-2014

$$\ell^{\infty} = \{x = (\xi_i)_{i=1}^{\infty} \mid \sup_i |\xi_i| < \infty\}$$

with $d(x,y) = \sup_i |\xi_i - \eta_i|$ is a complete metric space.

Solution:– Let (x_m) be a C.S in ℓ^{∞}

then $x_m = (\xi_i^{(m)})_{i=1}^{\infty}$

$\Rightarrow \forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N}$ s.t

$d(x_m, x_n) < \varepsilon \quad \forall\ m, n > N$

$\Rightarrow \sup_i |\xi_i^{(m)} - \xi_i^{(n)}| < \varepsilon \quad \forall\ m, n > N \longrightarrow \text{(I)}$

$\Rightarrow$ For fixed i $(1 \leq i < \infty)$, we have

$|\xi_i^{(m)} - \xi_i^{(n)}| < \varepsilon \quad \forall\ m, n > N \longrightarrow \text{(I)}$

$\Rightarrow$ For each fixed i, the sequence $(\xi_i^{(1)}, \xi_i^{(2)}, \ldots)$ is a cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete. The cauchy sequence converges in $\mathbb{R}$ i.e

$$\xi_i^{(m)} \longrightarrow \xi_i \in \mathbb{R}$$

Using these infinitly many limits construct

$$x = (\xi_1, \xi_2, \ldots)$$

In order to prove that ℓ^{∞} is complete, ~~we~~ need to show that $x \in \ell^{\infty}$ and $x_n \longrightarrow x$.

Letting $r \to \infty$ in (I), we get

$$|\xi_i^{(m)} - \xi_i| < \varepsilon \quad \forall m > N \longrightarrow \text{(II)}$$

Since $x_m = (\xi_i^{(m)})_{i=1}^{\infty} \in l^{\infty}$, therefore $\exists K$ such that

$$|\xi_i^{(m)}| < K_m \quad \forall m > N$$

because $\sup_i |\xi_i^{m}| < \infty$

then $|\xi_i| = |\xi_i - \xi_i^{(m)} + \xi_i^{(m)}|$

$$\leq |\xi_i - \xi_i^{(m)}| + |\xi_i^{(m)}| < \varepsilon + K_m = K \quad \forall i$$

$$\Rightarrow \quad x = (\xi_i)_1^{\infty} \in l^{\infty}$$

From (II) we have $\sup_i |\xi_i^{(m)} - \xi_i| < \varepsilon \quad \forall m > N$

$\Rightarrow d(x_m, x) < \varepsilon \quad \forall m > N$

$\Rightarrow x_m \longrightarrow x$ as $m \to \infty$

$\Rightarrow l^{\infty}$ is complete.

Sometimes, we use a different line of proof to show the completeness and that gives a simple method as it is shown in the following examples.

Example:- Show that

c-convergent

$$c = \{x = (\xi_j)_{j=1}^{\infty} \mid x \text{ is convergent}\}$$

is a complete metric space.

Sol:- Note that $c \subseteq l^{\infty}$.

So, to prove that c is compleste we only need to show that c is closed

*(Subspace M is complete iff M is closed.)

We can define d on C as

$$d(x,y) = \sup_j |\xi_j - \eta_j|$$

prove $C = \bar{C}$ as $C \subseteq \bar{C}$ & $\bar{C} \subseteq C$

Obviously $C \subseteq \bar{C}$.

Now let $x \in \bar{C}$ then (By theorem) $\exists$ a seq. (x_n) in C such that $x_n \to x$ where

$$x_n = \left(\xi_j^{(n)}\right)_{j=1}^{\infty}$$

then $\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ Such that

$$|\xi_j^{(n)} - \xi_j| \leq d(x_n, x) < \frac{\varepsilon}{3} \quad \forall\ n > N$$

In particular if $n = N$ then

$$|\xi_j^{(N)} - \xi_j| < \frac{\varepsilon}{3}$$

$\Rightarrow x_N = (\xi_1^{(N)}, \xi_2^{(N)}, \cdots)$ is a convergent sequence. Since every convergent sequence is a cauchy sequence. Therefore $\forall\ \varepsilon > 0$, $\exists\ N_1 \in \mathbb{N}$ such that

$$|\xi_j^{(N)} - \xi_k^{(N)}| < \frac{\varepsilon}{3} \quad \forall\ j, k \geq N_1$$

$\Rightarrow \forall\ j, k \geq N_1$ we have,

$$|\xi_j - \xi_k| \leq |\xi_j - \xi_j^{(N)}| + |\xi_j^{(N)} - \xi_k^{(N)}| + |\xi_k^{(N)} - \xi_k|$$

$\xi_j \quad \xi_j^{(N)} \quad \xi_k^{(N)} \quad \xi_k$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$\Rightarrow x = (\xi_j)_1^\infty$ is a cauchy sequence & hence convergent. So, that $x \in C$ & $\Rightarrow \bar{C} \subseteq C$.

$$C = \bar{C}$$

Hence C is complete.

Note: A metric space X may be complete under one metric but by changing the metric on X, it may happen that X is no longer a complete metric space. This is illustrated in the following examples.

Example:- Consider **17-3-2014**

$C[a,b] = \{x \mid x \text{ is continuous on } [a,b]\}$

introduce two metrics on $C[a,b]$ by,

$$d_1(x,y) = \max_{t \in [a,b]} |x(t) - y(t)|$$

&

$$d_2(x,y) = \int_a^b |x(t) - y(t)|\, dt$$

Show that $(C[a,b], d_1)$ is complete but $(C[a,b], d_2)$ is not complete.

Sol:(a):- To show that $(C[a,b], d_1)$ is complete. Let (x_m) be a cauchy sequence in $C[a,b]$ then $\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that

$$d_1(x_m, x_n) < \varepsilon \quad \forall\, m, n \geq N \longrightarrow ①$$

$$\Rightarrow \max_{t\in[a,b]} |x_m(t) - x_n(t)| < \varepsilon \quad \forall\, m, n \geq N$$

$\Rightarrow \exists\ t_0 \in [a,b]$ such that maximum is acheived at t_0 then,

$$|x_m(t_0) - x_n(t_0)| < \varepsilon \quad \forall\, m, n \geq N$$

$\Rightarrow (x_m(t_0))_{m=1}^{\infty}$ is a cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, $(x_m(t_0))$ converges in $\mathbb{R}$.

So, $x_m(t_0) \longrightarrow x(t_0)$.

In this way we can assocaite with each $t \in [a,b]$ a unique real number $x(t)$.

This defines a function x on $[a,b]$.

To prove that $[C[a,b], d_1]$ is complete we need to show that $x \in C[a,b]$ & $x_m \xrightarrow{d_1} x$

Letting $n \to \infty$ in ① we get.

$$\max_{t\in[a,b]} |x_m(t) - x(t)| < \varepsilon \quad \forall\, m \geq N \longrightarrow ②$$

$\Rightarrow \forall\, t \in [a,b]$, we have

$$|x_m(t) - x(t)| < \varepsilon \quad \forall\, m \geq N$$

$\Rightarrow x_m \longrightarrow x$ uniformaly on $[a,b]$

Since (x_m) is a sequence of continuous function on $[a,b]$ and $x_n \to x$ uniformaly, therefore x is continuous.

($\because$ if a sequence (x_m) of continuous function as converges uniformally to x on $[a,b]$ then x is continuous).

$$\Rightarrow x \in C[a,b]$$

$$② \Rightarrow d_1(x_m, x) < \varepsilon \quad \forall\, m \geq N$$

$$\Rightarrow x_m \xrightarrow{d_1} x$$

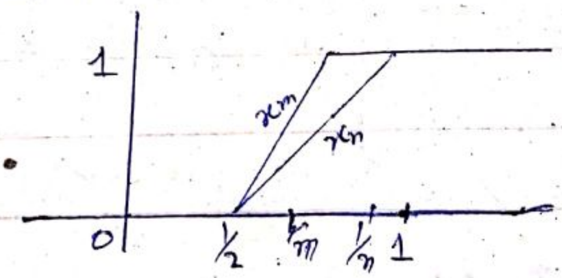
$$\Rightarrow (C[a,b], d_1) \text{ is complete.}$$

(b):— To show that $(C[a,b], d_2)$ is not complete.

We only need to construct a cauchy sequence in $C[a,b]$ which donot converge under the metric d_2.

Let $[a,b] = [0,1]$

Consider the function, x_m in figure below,



$$d_2(x_m, x_n) < \varepsilon \quad \forall\, m,n \geq N$$

or $d_2(x_m, x_n) \to 0$ as $m,n \to \infty$

Consider

$$d_2(x_m, x_n) = \int_0^1 |x_m(t) - x_n(t)|\,dt$$

$$= \int_0^{1/2} |x_m(t) - \cancel{x_n(t)}|\,dt + \int_{1/2}^{a_m} |x_m(t) - x_n(t)|\,dt$$

$$+ \int_{a_m}^{a_n} |x_m(t) - x_n(t)|\,dt + \int_{a_n}^{1} |x_m(t) - \cancel{x_n(t)}|\,dt$$

$x_m(t) \to 1$

$x_n(t) \to 1$

$$= \int_{1/2}^{\frac{1}{2}+\frac{1}{m}} |x_m(t) - x_n(t)|\,dt + \int_{\frac{1}{2}+\frac{1}{m}}^{\frac{1}{2}+\frac{1}{n}} |x_m(t) - x_n(t)|\,dt \to 0$$

as $m, n \to \infty$

$\Rightarrow (x_m)$ is a cauchy sequence in $C[a,b]$, we have to prove (x_m) is not convergent in $C[a,b]$.

Suppose on the contrary that (x_m) is convergent in $C[a,b]$

$\Rightarrow \exists\ x \in C[a,b]$ such that

$x_m \xrightarrow{d_2} x$ ie $d_2(x_m, x) \to 0$ as $m \to \infty$

Consider

$$d_2(x_m, x) = \int_0^1 |x_m(t) - x(t)|\,dt$$

$$= \int_0^{1/2} |\cancel{x_m(t)} - x(t)|\,dt + \int_{1/2}^{a_m} |x_m(t) - x(t)|\,dt$$

$$+ \int_{a_m}^{1} |x_m(t) - x(t)|\,dt$$

$$= \int_0^{1/2} |x(t)|\,dt + \int_{1/2}^{\frac{1}{2}+\frac{1}{m}} |x_m(t) - x(t)|\,dt + \int_{\frac{1}{2}+\frac{1}{m}}^{1} |1 - x(t)|\,dt$$

$$\rightarrow \int_0^{1/2} x(t)\,dt + \int_{1/2}^{1/2} |x_m(t) - x(t)|\,dt + \int_{1/2}^{1} |1-x(t)|\,dt$$

$$\Rightarrow 0 \text{ as } m \to \infty$$

which is possible only if

$$x(t) = \begin{cases} 0 & 0 \le t \le 1/2 \\ 1 & \frac{1}{2} < t \le 1 \end{cases}$$

discontinuous at $t = \frac{1}{2}$

$\Rightarrow x \notin C[a,b]$, a contradiction

$$\Rightarrow x_m \xrightarrow{d_2}\!\!\!\!\!\!/\;\; x$$

$(C[a,b], d_2)$ is not complete.

Example:- Show that l^p is complete. 20-03-2014

where $1 \le p < \infty$

Proof:- We have

$$l^p = \{x \mid \sum_1^\infty |\xi_i|^p < \infty\}$$

& $d(x,y) = \left(\sum_1^\infty |\xi_i - \eta_i|^p\right)^{1/p}$

Let (x_m) be a Cauchy sequence in l^p, where

$$x_m = \left(\xi_i^{(m)}\right)_{i=1}^{\infty}$$

Then $\forall \ \varepsilon > 0 \ \exists \ N \in \mathbb{N}$ such that

$d(x_m, x_n) < \varepsilon \quad \forall m, n \geq N$

$$\Rightarrow \left(\sum_1^{\infty} |\xi_i^{(m)} - \eta_i^{(n)}|^p \right)^{1/p} < \varepsilon \quad \forall m, n \geq N \longrightarrow (1)$$

$$\Rightarrow \forall i \quad |\xi_i^{(m)} - \eta_i^{(n)}| < \varepsilon \quad \forall m, n \geq N$$

$\Rightarrow (\xi_i^{(m)})_{m=1}^{\infty}$ is a cauchy sequence in $\mathbb{R}$ (or $\mathbb{C}$)

For each fixed i -

Since $\mathbb{R}$ (or $\mathbb{C}$) is complete.

This cauchy sequence is converges in $\mathbb{R}$ (or $\mathbb{C}$)

$\Rightarrow \quad \xi_i^{(m)} \longrightarrow \xi_i \in \mathbb{R}$ (or $\mathbb{C}$) as $m \longrightarrow \infty$

Using these infinitely many limits we construct $x = (\xi_1, \xi_2, \cdots)$,

We need to prove that

$x \in l^p$ and $x_m \longrightarrow x$

Letting $n \longrightarrow \infty$ in (1) we obtained

$$\left(\sum_{i=1}^{\infty} |\xi_i^{(m)} - \xi_i|^p \right)^{1/p} < \varepsilon, \ \forall m \geq N \longrightarrow (2)$$

$d(x_m, x) < \varepsilon, \ \forall m \geq N$

$\Rightarrow x_m \longrightarrow x$

From (2) we have

$$\sum_{i=1}^{\infty} |\xi_i^{(m)} - \xi_i|^p < \varepsilon^p, \ \forall m \geq N$$

$\Rightarrow x_m - x \in l^p$ or $x - x_m \in l^p$

Now: $x = (x - x_m) + x_m$

$$\Rightarrow \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p} = \left(\sum_{i=1}^{\infty} \left|(\xi_i - \xi_i^{(m)}) + (\xi_i^{m})\right|^p\right)^{1/p}$$

$$\underset{M.I}{\leq} \underbrace{\left(\sum_{1}^{\infty} |\xi_i - \xi_i^{(m)}|^p\right)^{1/p}}_{<\infty} + \underbrace{\left(\sum_{i=1}^{\infty} |\xi_i^{m}|^p\right)^{1/p}}_{<\infty}$$

$(\because x - x_m \in l^p)$ $\quad (\because x_m \in l^p)$

$$\Rightarrow \sum_{1}^{\infty} |\xi_i|^p < \infty$$

$$\Rightarrow x \in l^p$$

Hence (l^p, d) is complete.

Chap#2 Normed Spaces

Self
Page. 49-55

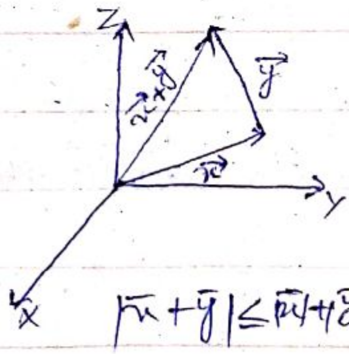
In $\mathbb{R}^3$

$$\vec{x} = a\hat{i} + b\hat{j} + c\hat{k}$$

$$|\vec{x}| = \sqrt{a^2 + b^2 + c^2} \geq 0$$

$$|\vec{x}| = 0 \Leftrightarrow \vec{x} = 0$$

$$|\alpha \vec{x}| = |\alpha| \, |\vec{x}|$$



$|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$

$\|x\|$ norm of x

Definition:- Let X be vector space. A real valued function $\|\cdot\| : X \longrightarrow \mathbb{R}$ is said to be a norm on X if

N1):- $\|x\| \geq 0$

N2):- $\|x\| = 0 \iff x = 0$

N3):- $\|\alpha x\| = |\alpha| \|x\|$ where $\alpha \in \mathbb{R}$ (or $\mathbb{C}$)

N4):- $\|x+y\| \leq \|x\| + \|y\|$

The pair $(X, \|\cdot\|)$ is called normed space.

A Banach space is a complete normed space (complete under the metric induced by the norm)

Remark:- The norm $\|\cdot\|$ generalizes the concept of lenth of a vector in $\mathbb{R}^3$ to a general vector space X. In this case $(X = \mathbb{R}^3)$ $|x| = \|x\|$.

*(Every N.S is a M.S but every M.S may or may not be a N.S)

Metric Induced by Norm

A metric d on X can be defined by using the norm $\|\cdot\|$ on X as follows,

$$d(x,y) = \|x-y\|$$

this metric is called the metric induced by norm.

M1):- $d(x,y) = \|x-y\| \geq 0$

M2):- $d(x,y) = \|x-y\| = 0 \Leftrightarrow x-y=0 \Leftrightarrow x=y$

M3):- $d(x,y) = \|x-y\| = \|(-1)(y-x)\|$

$= |-1| \quad \|y-x\|$

$= \|y-x\| = d(y, x)$

Let $z \in X$

M4):- $d(x,y) = \|x-y\|$

$= \|x-z+z-y\|$

$\leq \|x-z\| + \|z-y\| = d(x,z) + d(z,y)$

i.e $d(x,y) \leq d(x,z) + d(z,y)$

Hence this is metric induced by norm.

Remark:- 21.03.2014

Every norm space is a metric space but converse is not true in general.

For Example:- Consider

$S = \{x \mid x \text{ is bounded or unbounded}\}$

with

$$d(x,y) = \sum_{1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|}$$

Suppose $\|\cdot\|$ is define on S & d is induced by $\|\cdot\|$ i.e $d(x,y) = \|x-y\|$

then $\|x\| = d(x,0) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i|}{1+|\xi_i|}$

$$\Rightarrow \|\alpha x\| = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|\alpha \xi_i|}{1+|\alpha \xi_i|}$$

$$\neq |\alpha| \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i|}{1+|\xi_i|}$$

i.e $\|\alpha x\| \neq |\alpha|\,\|x\|$

$\Rightarrow$ S can't be converted into a normed space.

Examples of Normed Space:-

Example 1:- Let $X = \mathbb{R}^n$ with $\|\cdot\|$ defined by

$$\|x\| = \sqrt{\sum_{i=1}^{n} |\xi_i|^2} \quad \text{where } x = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$$

Show that $(X, \|\cdot\|)$ is a normed space.

Sol:- N1):- $\|x\| = \sqrt{\sum_{i=1}^{n} |\xi_i|^2} \geq 0$

N2): $\|x\| = \left(\sum_{i=1}^{n} |\xi_i|^2\right)^{1/2} = 0 \Leftrightarrow \xi_i = 0 \ \forall i \Leftrightarrow x = 0$

N3):- $\|\alpha x\| = \left(\sum_{i=1}^{n} |\alpha \xi_i|^2\right)^{1/2}$

$$= |\alpha| \left(\sum_{i=1}^{n} |\xi_i|^2\right)^{1/2}$$

$$= |\alpha|\,\|x\|$$

N4):- Let $y = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{R}^n$

then $\|x+y\| = \left(\sum_{i=1}^{n} |\xi_i + \eta_i|^2\right)^{1/2}$

$$\underset{M.I}{\leq} \left(\sum_{i=1}^{n} |\xi_i|^2\right)^{1/2} + \left(\sum_{i=1}^{n} |\eta_i|^2\right)^{1/2}$$

$$= \|x\| + \|y\|$$

i.e $\|x+y\| \leq \|x\| + \|y\| \Rightarrow (\mathbb{R}^n, \|\cdot\|)$ is a normed space.

The metric induced by $\|\cdot\|$ on R^n is

$$d(x,y) = \|x-y\|$$
$$= \left(\sum_{i=1}^{n} |\xi_i - \eta_i|^2\right)^{1/2}$$

which is usual metric on R^n. Since R^n is complete under this induced metric, therefore $(R^n, \|\cdot\|)$ is a Banach space.

Note:- By similar argument we can show that C^n with

$$\|x\| = \left(\sum_{i=1}^{n} |\xi_i|^2\right)^{1/2}$$

$|\xi_i|$ is modulus of a complex no.

where $x = (\xi_1, \xi_2, \ldots \xi_n) \in C^n$

is a Banach space.

(self) Example:- Consider

$$l^p = \{x = (\xi_i)_1^\infty \mid \sum_1^\infty |\xi_i|^p < \infty\}$$

with $\|x\| = \left(\sum_1^\infty |\xi_i|^p\right)^{1/p}$

Then $(l^p, \|\cdot\|)$ is a normed space.

Also the metric induced by $\|\cdot\|$ in l^p is

$$d(x,y) = \left(\sum_{i=1}^{\infty} |\xi_i - \eta_i|^p\right)^{1/p}$$

Under this metric l^p is complete.

$\Rightarrow (l^p, \|\cdot\|)$ is a Banach Space.

Example:- Consider $C[a,b] = \{x \mid x$ is continuous on $[a,b]\}$

Define
$$\|x\|_1 = \max_{t\in[a,b]} |x(t)|$$
& $$\|x\|_2 = \int_a^b |x(t)|\, dt$$
then $(C[a,b], \|\cdot\|_1)$ & $(C[a,b], \|\cdot\|_2)$ are normed spaces.

$(C[a,b], d_1)$ where $d_1(x,y) = \|x-y\|_1$
$$= \max_{t\in[a,b]} |x(t)-y(t)|$$
is complete So that $(C[a,b], \|\cdot\|_1)$ is a Banach space.

& $(C[a,b], d_2)$ where $d_2(x,y) = \|x-y\|_2$
$$= \int_a^b |x(t)-y(t)|\, dt$$

is not complete. So that $(C[a,b], \|\cdot\|_2)$ is not a Banach space.

Lemma:- A metric d induced by a norm $\|\cdot\|$ on a normed space $(X, \|\cdot\|)$ satisfies the following properties.

1):- $d(x+z, y+z) = d(x,y)$

2):- $d(\alpha x, \alpha y) = |\alpha|\, d(x,y)$

Proof:- Since d is induced by $\|\cdot\|$ on X. Therefore $d(x,y) = \|x-y\|$

1):- $d(x+z, y+z) = \|(x+z) - (y+z)\|$

$$= \|x+z-y-z\|$$
$$= \|x-y\|$$
$$= d(x,y)$$

2):- $d(\alpha x, \alpha y) = \|\alpha x - \alpha y\|$
$$= \|\alpha(x-y)\|$$
$$= |\alpha|\,\|x-y\| \quad \text{by N3}$$
$$= |\alpha|\, d(x,y)$$

Remark: The above lemma gives a criterion to judge that whether the given metric can be induced by a norm or not.

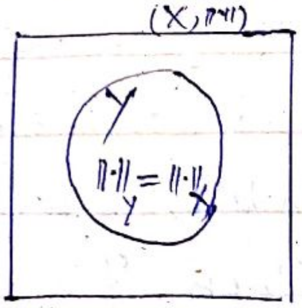
$d(x,y) = \sum_{1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1+|\xi_i-\eta_i|}$ Not satisfies these two conditions.

Subspace

24-03-2014

A subspace Y of a normed space $(X, \|\cdot\|)$ is subspace of X considered as a vector space, with the norm obtained by restricting the norm on X to the subset Y. *This norm on Y is said to be induced by the norm on X. If Y is closed in X, then Y is called a closed subspace of X.

$$\|\cdot\|_X : X \to \mathbb{R}$$
$$\|\cdot\|_X\big|_Y = \|\cdot\|_Y : Y \to \mathbb{R}$$



A subspace Y of a Banach space $(X, \|\cdot\|)$ is a subspace of X considered as a normed space, Hence we don't require Y to be complete.

Theorem:-

A subspace Y of a Banach Space $(X, \|\cdot\|)$ is complete iff the set Y is closed in X.

Proof: $\Rightarrow$ Let Y is complete normed space. To prove $Y = \overline{Y}$

Obviously $Y \subseteq \overline{Y}$

Now Let $y \in \overline{Y}$ then $\exists$ a seq (y_n) in Y such that $y_n \longrightarrow y$.

Since every convergent sequence is a cauchy sequence, therefore (y_n) is a cauchy sequence in Y.

Since Y is complete. (y_n) converges in Y. Since limit is unique $y \in Y$ So, that $\overline{Y} \subseteq Y$

then $\overline{Y} = Y$

$\Leftarrow$ Suppose that $Y = \overline{Y}$. To prove Y is complete. Let (y_n) be an arbitrary cauchy sequence in Y. Then (y_n) is a cauchy sequence in X.

Since X is Banach space, therefore $y \in X$ such that $y_n \longrightarrow y$ then

$$y \in \overline{Y} \quad (\because y \text{ is limit point})$$

Thus Y is complete.

Convergence in Normed Spaces

A sequence (x_n) in a norm space X is convergent if $\exists$ $x \in X$ such that

$$\lim_{n \to \infty} \|x_n - x\| = 0$$

In this case we write $x_n \xrightarrow{\|\cdot\|} x$ and call x the limit of (x_n).

Def: Cauchy Sequence

A sequence (x_n) in a norm space $(X, \|\cdot\|)$ is said to be cauchy if $\forall$ $\epsilon > 0$, $\exists$ $N \in \mathbb{N}$ such that

$$\|x_m - x_n\| < \epsilon \quad \forall \; m, n \geq N$$

Definition:-

If (x_K) is a sequence in a norm space $(X, \|\cdot\|)$, then $\sum_1^{\infty} x_K$ is a series in X. We associate with (x_K) a sequence (S_n) of partial sums defined by

$$S_n = x_1 + \cdots + x_n \qquad n = 1, 2, \ldots$$

If (S_n) is convergent, say

$S_n \longrightarrow S$

i.e $\|S_n - S\| \longrightarrow 0$ as $n \longrightarrow \infty$

Then the series $\sum_1^\infty x_k$ is said to converges & S is called the sum of the series and we write

$$S = \sum_1^\infty x_k = x_1 + x_2 + \cdots$$

If $\sum_1^\infty \|x_k\|$ ~~the~~ converges then we say that

$\sum_1^\infty x_k$ converges absolutely in $(X, \|\cdot\|)$

Note: In case of $\mathbb{R}$ or $\mathbb{C}$, we have absolute converges $\Rightarrow$ converges

i.e $\sum_1^\infty |x_k| < \infty \Rightarrow \sum_1^\infty x_k < \infty$

But converse may not be true. For Examples:- $\sum_1^\infty (-1)^n$ is convergent but not absolutly convergent

In a norm space $(X, \|\cdot\|)$ absolute convergence $\not\Rightarrow$ convergence.

i.e

$$\sum_1^\infty \|x_k\| < \infty \not\Rightarrow \sum_1^\infty x_k < \infty$$

For Example: Consider

$$\ell^\infty = \{x = (\xi_i)_1^\infty \mid \xi_i \in \mathbb{R} \text{ or } \mathbb{C} \ \& \ \sup_i |\xi_i| < \infty\}$$

with $\|x\| = \sup_i |\xi_i|$, $x = (\xi_i)_1^\infty \in \ell^\infty$

Then $(\ell^\infty, \|\cdot\|)$ is a Banach space.

Sol:- Let Y be set of sequence will only finitely non-zero terms.

i-e

$$Y = \{y = (y_i)_1^\infty \mid y_i = 0 \ \forall \ i > n, n \in \mathbb{N}\}$$

Then $Y \subseteq \ell^\infty$ $(\sup_i |y_i| < \infty)$

Note that Y is not closed in ℓ^∞

Since,

$$y_n = (1, \tfrac{1}{2}, \cdots, \tfrac{1}{n}, 0, 0, 0, \cdots) \in Y$$

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But

$$\lim_{n \to \infty} y_n = (1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots, \tfrac{1}{n}, \tfrac{1}{n+1}, \cdots) \notin Y$$

Consider (y_n) defined by $y_n = (\eta_j^{(n)})$

where

$$\eta_j^{(n)} = \begin{cases} 1/n^2 & \text{if } j = n \\ 0 & \text{if } j \neq n \end{cases}$$

then $y_1 = \eta_j^{(1)} = (\frac{1}{1^2}, 0, 0, \ldots)$

& $y_2 = \eta_j^{(2)} = (0, \frac{1}{2^2}, 0, 0, \cdots)$

$y_3 = \eta_j^{(3)} = (0, 0, \frac{1}{3^2}, 0, 0, \ldots)$

then

$y_n \in Y \ \forall \ n \in \mathbb{N}$ $\qquad \|x\| = \sup_i |\xi_i|$

Now

$\|y_1\| = 1$, $\|y_2\| = \frac{1}{2^2}$, $\|y_3\| = \frac{1}{3^2}$, ---

therefore

$$\sum_1^\infty \|y_n\| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$= \sum_1^\infty \frac{1}{n^2} < \infty$$

by p-series test $\Big|$ $\sum_1^n \frac{1}{n^p} < \infty$ if $p > 1$ or div if $p \leq 1$

So, $\sum_1^\infty y_n$ is absolutly convergent

Now consider

$$\sum_1^\infty y_n = \left(1, \frac{1}{2^2}, \frac{1}{3^2}, \cdots, \frac{1}{n^2}, \frac{1}{(n+1)^2}, \cdots\right) = y \notin Y$$

$\Rightarrow \sum_1^\infty y_n$ is not convergent.

Hence, in an arbitrary normed space $(X, \|\cdot\|)$ absolute convergence $\not\Rightarrow$ Convergence

Basis:-

In a vector space the set of Linearly independent vectors that span the whole space is called a basis.

This definition does not use the

concept of a norm & the vector operations "+" & "·".

We can use these concept to define a basis called schauder basis defined as follows:

Schauder Basis

If a normed space $(X, \|\cdot\|)$ contains a sequence (e_n) with the property that for every $x \in X$, there is a unique sequence of scalars (α_n) such that

$$\|x - (\alpha_1 e_1 + \cdots + \alpha_n e_n)\| \to 0 \text{ as } n \to \infty$$

then (e_n) is called a schauder basis for X.

$\boxed{\text{Book}}$ The series $\sum_{k=1}^{\infty} \alpha_k e_k$ which has the sum x is then called the expansion of x w.r.t (e_n), and we write $x = \sum_{k=1}^{\infty} \alpha_k e_k$.

Example:- l^p has Shauder basis (e_n)

where $e_1 = (1, 0, 0, \ldots)$

$e_2 = (0, 1, 0, \ldots)$

$e_3 = (0, 0, 1, \ldots)$

$\vdots$

$e_n = (0, 0, \ldots, 1, 0, 0, \ldots)$

These are schauder basis for l^p.

Theorem:- If a normed space X has a schauder basis then it is separable.

Proof: Suppose $(X, \|\cdot\|)$ has a schauder basis (e_n).

Case(I):- Suppose $(X, \|\cdot\|)$ is real normed space.

To prove X is separable.

Let

$$M = \{x \mid x = \alpha_1 e_1 + \dots + \alpha_n e_n \,,\ \alpha_i \in \mathbb{Q}\}$$

Since $\mathbb{Q}$ is countable therefore M is countable.

Suppose $z \in X$.

Since (e_n) is schauder basis, $\exists$ a sequence (α_i') such that

$$\|z - \sum_{i=1}^{n} \underbrace{\alpha_i' e_i}_{\text{element of } M}\| \to 0 \quad \text{as } n \to \infty$$

$\forall\ \varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ such that

$$\|z - \sum_{1}^{n} \alpha_i' e_i\| < \varepsilon \quad \forall\ n \geq N \longrightarrow (1)$$

If $\alpha_i' \in \mathbb{Q}$ then $\sum_{1}^{n} \alpha_i' e_i \in M$

If $\alpha_i' \notin \mathbb{Q}$, then we can approximate

α_i' by rational β_i such that

$\beta_1 e_1 + \cdots + \beta_n e_n$ approximates z,

where

$$\sum_1^n \beta_i e_i \in M$$

Since z and ε are arbitrary.

we conclude from (1) that every nbd of z contain an element of M.

$\forall z \in X$ Hence,

$$\bar{M} = X$$

So, X is separable.

Case (II): If X is complex normed space, then consider

$$M = \{x \mid x = \alpha_1 e_1 + \cdots + \alpha_n e_n\}$$

where

$$\text{Real}\,\alpha_i \in \mathbb{Q} \;\&\; \text{Img}\,\alpha_i \in \mathbb{Q}$$

So, M is countable by the following the same procedure as above we can show that M is dense

or $\bar{M} = X$.

So, that X is separable in this case as well.

Remarks:

Converse of the above theorem is not true in general.

i.e

A separable space need not have schauder basis.

This was proved by Enflo in 1973 by giving a counter example.

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Q:- Show that $B([a,b])$ is not separable.

Sol:- We have

$$B([a,b]) = \{x \mid x \text{ is bounded on } [a,b]\}$$

with

$$d(x,y) = \sup_{t \in [a,b]} |x(t) - y(t)|$$

For each $c \in [a,b]$, define

$$x_c(t) = \begin{cases} 1 & \text{if } t = c \\ 0 & \text{if } t \neq c \end{cases}$$

then

$$x_c \in B([a,b])$$

So,

$$A = \{x_c \mid c \in [a,b]\} \subseteq B([a,b])$$

Since interval $[a,b]$ is uncountable therefore A is uncountable.

For x_c & $x_d \in A$, we have

$$d(x_c, x_d) = \sup_{t \in [a,b]} |x_c(t) - x_d(t)| = 1$$

If we let each x_c as centre of a ball with radius $\frac{1}{3}$ then these balls are non-intersecting and we have uncountably many of them. If M is dense in $B([a,b])$, then each of these non-intersecting balls must contains an element of M.

Hence M can not be countable. This shows that $B([a,b])$ can not be have a dense subset which is countable. So $B([a,b])$ is not separable.

Finite Dimensional Normed Spaces

R, C, R^n, C^n etc are finite dimensional.

l^p, l^∞ etc are infinite dimensional.

In order to develope a fruitful theory on finite demensional normed spaces, we need a Lemma which is backbone of the whole study.

Lemma:- (Linear Combination)

Let $\{x_1, x_2, \dots, x_n\}$ be a set of linearly independent vectors in a normed space $(X, \|\cdot\|)$ of any dimension. Then $\exists$ a number $c > 0$ such that for every choice of scalars $\alpha_1, \dots, \alpha_n$, we have

$$\|\alpha_1 x_1 + \dots + \alpha_n x_n\| \geq c\,(|\alpha_1| + \dots + |\alpha_n|) \longrightarrow (1)$$

(i.e norm of linear combination of $\{x_1, x_2, \dots, x_n\}$ is at least $c\,(|\alpha_1| + \dots + |\alpha_n|)$).

Proof:- Let $s = |\alpha_1| + \dots + |\alpha_n|$

If $s = 0$

Then all α_i's are zero and (1) hold trivially for any c.

If $s \neq 0$, then (1) is equivalent to

$$\frac{\|\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n\|}{s} \geq c$$

i.e

$$\|\beta_1 x_1 + \dots + \beta_n x_n\| \geq c \longrightarrow (2)$$

where $|\beta_i| = \dfrac{|\alpha_i|}{s}$ so that

$$\sum_{i=1}^{n} |\beta_i| = \sum_{i=1}^{n} \frac{|\alpha_i|}{s} = 1$$

We required to prove the existance of a number $c > 0$ such that (2) holds with the condition that $\sum_{1}^{n} |\beta_i| = 1$

Suppose that this is false.
Then $\exists$ a sequence (y_m) of vectors where

$$y_m = \beta_1^{(m)} x_1 + \cdots + \beta_n^{(m)} x_n$$

s.t

$$\|y_m\| = \|\beta_1^{(m)} x_1 + \cdots + \beta_n^{(m)} x_n\| \to 0$$

as $m \to \infty$

Since $\sum_{1}^{n} |\beta_i^{(m)}| = 1$ we have $|\beta_i^{(m)}| \leq 1$

So that for each fixed i, $\forall i$
the sequence $(\beta_i^{(m)})$ defined by $\beta_i^{(m)} = (\beta_i^1, \beta_i^2, \cdots)$ is a bounded sequence of scalars.

By Bolzano-Weierstrass theorem $(\beta_i^{(m)})$ must have a convergent subsequence $(\gamma_i^{(m)})$ which is convergent to β_i (say)

Consequently the sequence (y_m) has a convergent subsequence $(y_{1,m})$,

$$y_{1,m} = \gamma_1^{(m)} x_1 + \beta_2^{(m)} x_2 + \cdots + \beta_n^{(m)} x_n$$

By the same argument, $(y_{1,m})$ has a subsequence $(y_{2,m})$ for which the corresponding subsequence $(\gamma_2^{(m)})$ of scalars converges to β_2 (say).

where,

$$y_{2,m} = \gamma_1^{(m)} x_1 + \gamma_2^{(m)} x_2 + \beta_3^{(m)} x_3 + \cdots + \beta_n^{(m)} x_n$$

Continuing in this way, after n-steps we obtain a subsequence $(y_{n,m})$ of (y_m) such that

$$(y_{n,m}) = \gamma_1^{(m)} x_1 + \cdots + \gamma_n^{(m)} x_n$$

with $\gamma_i^{(m)} \longrightarrow \beta_i$ as $m \longrightarrow \infty$

$$y_{n,m} \longrightarrow \sum_1^n \beta_i x_i = y \quad \text{with } \sum_1^n |\beta_i| = 1$$

Since all $\beta_i \neq 0$ and $\{x_1, \cdots, x_n\}$ are linearly independent therefore $y \neq 0$ also

$$y_{n,m} \longrightarrow y \Rightarrow \|y_{n,m}\| \longrightarrow \|y\|$$

Since

$\|y_m\| \longrightarrow 0$ (by assumption) and

$(y_{n,m})$ is a subsequence of (y_m) we must have $\|y_{n,m}\| \longrightarrow 0$

So that $\|y\| = 0$ i.e $y = 0$, a contradiction. Hence ② Holds.

Theorem:- (Completeness) 03-04-14

Every finite dimensional subspace Y of a normed space $(X, \|\cdot\|)$ is complete. In particular, if X is finite dimensional, then X itself is complete.

Proof:- Let $\dim Y = n$ and Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of Y. Let (y_m) be a cauchy sequence in Y.

Then each y_m can be expressed as

$$y_m = \alpha_1^{(m)} e_1 + \cdots + \alpha_n^{(m)} e_n$$

Since (y_m) is a cauchy Sequence. Therefore $\forall \varepsilon > 0 \; \exists \; N \in \mathbb{N}$ such that

$$\|y_m - y_r\| < \varepsilon \quad \forall m, r \geq N$$

$$\Rightarrow \varepsilon > \|y_m - y_r\| = \|(\alpha_1^{(m)} - \alpha_1^{(r)})e_1 + \cdots + (\alpha_n^{(m)} - \alpha_n^{(r)})e_n\| \quad \forall m, r \geq N$$

$$\geq C\left(\sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(r)}|\right)$$

(by previous Lemma)

$$\Longrightarrow \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(r)}| < \frac{\varepsilon}{c} \quad \forall m, r \geq N$$

$$\Rightarrow \alpha_i^{(m)} - \alpha_i^{(r)} < \frac{\varepsilon}{c}$$

$\Rightarrow$ For each fixed i, $(\alpha_i^{(m)})$ is a cauchy sequence of real or complex numbers. Since $\mathbb{R}$ or $\mathbb{C}$ are complete.

$$\alpha_i^{(m)} \longrightarrow \alpha_i \in \mathbb{R} \text{ or } \mathbb{C} \text{ as } m \to \infty$$

Then $y_m \longrightarrow \alpha_1 e_1 + \cdots + \alpha_n e_n = y$

clearly $y \in Y$

Now we prove that this convergence is under the norm $\|\cdot\|$.

For this consider,

$$\|y_m - y\| = \left\| \sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i) e_i \right\| \leq K \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i|$$

where $K = \max\limits_{1 \leq i \leq n} \|e_i\|$

Note

$$\|(\alpha_1^{m} - \alpha_1) e_1 + \cdots + (\alpha_n^{(m)} - \alpha_n) e_n\| \leq |\alpha_1^{(m)} - \alpha_1| \|e_1\| + \cdots + |\alpha_n^{(m)} - \alpha_n| \|e_n\|$$

If $m \to \infty$ then $\|y_m - y\| \longrightarrow 0$

Hence Y is complete.

From above theorem & from "A subspace M of a complete metric space X is complete iff it is closed".

$\bar{M} = M$ We have the following theorem,

$\overline{Y} \neq Y$

$Y \cup Y^d \neq Y$

Theorem:-

Every finite dimensional subspace Y of a normed space $(X, \|\cdot\|)$ is closed in X.

Remark:- In case of infinite dimensional subspace, the above result need not be true. i.e an infinite dimensional subspace need not be closed.

For Example:- Let $X = C[a,b]$ with $\|x\| = \max_{t \in [a,b]} |x(t)|$

Let $Y = \text{Span}\{1, t, t^2, \dots\} =$

$= \text{Set of all polynomial} \subseteq C[a,b]$

Let (y_n) be a sequence in Y defined by

$$y_n = 1 + t + \frac{t^2}{2!} + \dots + \frac{t^n}{n!}$$

$$\lim_{n \to \infty} y_n = 1 + t + \frac{t^2}{2!} + \dots + \frac{t^n}{n!} + \frac{t^{n+1}}{(n+1)!} + \dots = e^t \notin Y$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$\Rightarrow Y$ is not closed ($\because$ a limit point does not belongs to Y)

Hence Y is not closed then Y is not complete.

Definition:- (Equivalent norms)

A norm $\|\cdot\|$ on a vector space X is said to be equivalent to a norm $\|\cdot\|_0$ on X if $\exists$ scalars α & β such that $\forall x \in X$

$$\alpha \|x\|_0 \leq \|x\| \leq \beta \|x\|_0$$

Example:-

Let $X = R^2$ define two norms on X by

$\|x\|_1 = |\xi_1| + |\xi_2|$, $x = (\xi_1, \xi_2) \in R^2$

$\|x\|_2 = \sqrt{|\xi_1|^2 + |\xi_2|^2}$, $x = (\xi_1, \xi_2) \in R^2$

$= \left(\sum_{i=1}^{2} |\xi_i|^2\right)^{1/2}$

Show that $\|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent.

Sol: $\|x\|_1 = |\xi_1| + |\xi_2| = \sum_{i=1}^{2} |\xi_i|$

$= \sum_{i=1}^{2} |1 \cdot \xi_i| \underset{H \cdot I}{\leq} \left(\sum_{i=1}^{2} |1|^2\right)^{1/2} \underbrace{\left(\sum_{i=1}^{2} |\xi_i|^2\right)^{1/2}}_{\|x\|_2}$

i.e $\|x\|_1 \leq \sqrt{2}\, \|x\|_2 \longrightarrow (1)$

Now $\|x\|_2 = \sqrt{|\xi_1|^2 + |\xi_2|^2} \leq |\xi_1| + |\xi_2|$

$\left(\because |\xi_1|^2 + |\xi_2|^2 \leq (|\xi_1| + |\xi_2|)^2\right)$

i.e $\|x\|_2 \leq 1 \cdot \|x\|_1 \longrightarrow (2)$

From (1) & (2)

$\frac{1}{\sqrt{2}} \|x\|_1 \leq \|x\|_2 \leq 1 \cdot \|x\|_1$

$\Rightarrow \|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent.

Theorem:- 04-04-14

On a finit dimensional normed space X any norm $\|\cdot\|$ is equivalent to any other norm $\|\cdot\|_0$.

Proof: Let $\dim X = n$ & $\{e_1, e_2, \dots, e_n\}$ be a basis for X.

Then each $x \in X$ has a unique representation:

$$x = \alpha_1 e_1 + \dots + \alpha_n e_n$$

So, that $\|x\| = \|\sum_{i=1}^{n} \alpha_i e_i\|$

$$= \|\alpha_1 e_1 + \dots + \alpha_n e_n\|$$

Since

$\{e_1, e_2, \dots, e_n\}$ are linearly independent by perivious Lemma $\exists$ $c > 0$ such that

$$\|x\| \geq c\left(\sum_{i=1}^{n} |\alpha_i|\right) \longrightarrow (1)$$

Now $\|x\|_0 = \|\alpha_1 e_1 + \dots + \alpha_n e_n\|_0$

$$\leq |\alpha_1| \, \|e_1\| + \dots + |\alpha_n| \, \|e_n\|$$

$$\leq K\left(\sum_{i=1}^{n} |\alpha_i|\right) \quad \text{where } K = \max_{1 \leq i \leq n} \|e_i\|$$

i.e. $\|x\|_0 \leq K\left(\sum_{i=1}^{n} |\alpha_i|\right) \leq \frac{K}{c} \|x\|$ (by (1))

i.e $\|x\|_0 \leq B \|x\| \longrightarrow (2)$, where $B = \frac{K}{c}$

Interchanging roles of $\|\cdot\|_0$ & $\|\cdot\|$ we obtain $\|x\| \leq d \|x\|_0 \longrightarrow (3)$

from (2) & (3).

$$\frac{1}{B} \|x\|_0 \leq \|x\| \leq d \|x\|_0$$

Hence $\|\cdot\|$ & $\|\cdot\|_0$ are equivalent.

Remark:

In case of finite dimensional space, we need not worry about the norm because all norms are equivalent.

Compactness in Normed Spaces

A metric space X is said to be a **compact** (or Sequentially compact) if every sequence in X has a convergent subsequence.

A subset M of X is said to be **compact** if M is compact considered as a subspace of X.

i.e if every space in M has a convergent subsequence whose limit is an element of M.

Remark:— There are 3 ways of defining the concept of compactness in a topological space.

1):— Every open cover has a finite subcover.

2):— Every countable cover has a finit subcover.

3):— Every sequence has a convergent subsequence.

In case of metric space, the 3 concepts become identical so that the distinction does not matter in our work.

Lemma:

A compact subset M of a metric space (X, d) (finite or infinite dimensional) is closed & bounded.

Proof:- Suppose M is compact.

$\Rightarrow$ To prove M is closed

clearly $M \subseteq \bar{M}$

Let $x \in \bar{M}$, then $\exists$ a sequence (x_n) in M. Such that $x_n \xrightarrow{d} x$

Since M is compact (by 3)

(x_n) has a convergent subsequence which converges to x ($\because x_n \longrightarrow x$).

Then $x \in M$. So that $\bar{M} \subseteq M$

$\Rightarrow \bar{M} = M$ and M is closed.

$\Rightarrow$ To prove M is bounded.

Suppose on the contrary that M is not bounded. Then $\exists$ an unbounded sequence (y_n) in M such that $d(y_n, b) > n$ where $n \in \mathbb{N}$ & b is any point of M. Thus (y_n) cannot have a convergent subsequence which contradicts the fact that M is compact.

Hence M is bounded.

Remark:- 10-4-14

The converse of above lemma is not true in general. i.e A closed and bounded set need not be compact.

Example:- Let $X = l^\infty$, then $\dim X = \infty$ and $\|x\| = \sup_i |\xi_i|$,

where $x = (\xi_i)_1^\infty \in l^\infty$

Define

$M = \{e_1 = (1,0,0,\ldots), e_2 = (0,1,0,0,\ldots), \ldots\}$

Since $\|e_1\| = 1 = \|e_2\| = \cdots$ therefore M is bounded.

M is closed.

($\because$ M has no limit point say that all limit points are in M. Actually M is a point set)

$\{1,2,3\}$ Points sets are closed.

M is not compact.

Since the subsequence.

$e_5, e_{10}, e_{15}, \ldots \not\to$ limit point

$\times$ A seq which is not convergent it will not be compact.

$M = \overline{M} = M \cup \phi$

Example:- Let $X = \mathbb{Z}$ with

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Indiscrete space every subset of X is open as well as closed. In particular X is closed. It is bounded because each point lies within a distance 1 for a fixed point x_0 of X.

$X = \mathbb{Z}$ is not compact.

Since $\{\{x\} \mid x \in \mathbb{Z}\}$ is an open cover of X but it can not have a finite subcover.

$\{\{x\} \mid x \in \mathbb{Z}\}$ open cover of $\mathbb{Z}$. It has no finite subcover.

Theorem:- In a finite dimensional normed space $(X, \|\cdot\|)$ any subset $M \subseteq X$ is compact iff M is closed and bounded.

Proof: ⟹ Suppose M is compact to prove M is closed and bounded.

*{Same as before given in Lemma page#75

Clearly $M \subseteq \overline{M}$

Let $x \in \overline{M}$. Then $\exists$ a subsequence (x_n) in M. Such that $x_n \xrightarrow{d\|\cdot\|} x$.

Since M is compact (by 3). (x_n) has a convergent subsequence which converges to x ($\because x_n \to x$).

Then $x \in M$. So that $\bar{M} \subseteq M$

$\Rightarrow \bar{M} = M$ and M is closed.

To prove M is bounded.

Suppose on the contrary that M is not bounded then $\exists$ an unbounded sequence (y_n) in M such that $\|y_n - b\| > n$ where $n \in \mathbb{N}$ & b is any point of M.

Thus (y_n) can not have a convergent subsequence which contradicts the fact that M is compact.

Hence M is bounded.

$\Leftarrow$ Suppose that M is closed and bounded. To prove M is compact.

Let (x_m) be a sequence in M. Since X is finite dimensional. Let $\dim X = n$ & $\{e_1, e_2, \cdots, e_n\}$ be a basis for X. Then

$$x_m = \alpha_1^{(m)} e_1 + \alpha_2^{(m)} e_2 + \cdots + \alpha_n^{(m)} e_n \quad \forall m \in \mathbb{N}$$

Since M is bounded. Therefore x_m is bounded. So, $\exists$ $K > 0$ such that

$$\|x_m\| \leq K \quad \forall m$$

$$\Rightarrow K \geq \|\alpha_1^{(m)} e_1 + \cdots + \alpha_n^{(m)} e_n\| \geq C \left(\sum_{i=1}^{n} |\alpha_i^{(m)}|\right)$$

where $c>0$

$$\Rightarrow \sum_{i=1}^{n} |\alpha_i^{(m)}| \leq \frac{K}{c}$$

$\Rightarrow \forall$ fixed i $(1 \leq i \leq n)$ $(\alpha_i^{(m)})_{m=1}^{\infty}$ is a bounded sequence so that by Bolzano-Weierstrass theorem it has a convergent subsequence which convergent to α (say)

Then (x_m) has a convergent sub seq. (z_m) which converges to

$$\alpha_1 e_1 + \cdots + \alpha_n e_n = z$$

Since M is closed, $z \in M$.

Hence M is compact.

F. Riesz's Lemma 11-04-2014.

Let Y & Z be subspaces of a normed space X (of any dimension) and suppose that Y is closed and proper subset of Z. then for every real number $\theta \in (0,1)$ $\exists$ $z \in Z$ such that

$$\|z\| = 1 \quad \& \quad \|z-y\| \geq \theta \quad \forall y \in Y$$

Proof:- Choose $v \in Z - Y$.

Let "a" be distance of v from Y.

$$a = \inf_{y \in Y} \|v - y\|$$

Clearly $a > 0$ because Y is closed.

Now choose $\theta \in (0,1)$ such that for $y \in Y$.

$$\|v - y_0\| \leq \frac{a}{\theta}$$

$$\Rightarrow \quad a \leq \|v - y_0\| \leq \frac{a}{\theta} \longrightarrow (1) \quad \text{(By def. of inf.)}$$

note that $\frac{a}{\theta} > a$.

Since $0 < \theta < 1$. Define $z = c(v - y_0)$

where $c = \frac{1}{\|v - y_0\|}$

Then $\|z\| = 1$

Now consider $\|z - y\| = \|c(v - y_0) - y\|$

$$= c\|(v - y_0) - c^{-1} y\| \quad (\because c \text{ is positive})$$

$$= c\|v - (y_0 + \tfrac{1}{c} y)\|$$

$$= c\|v - y_1\|$$

where $y_1 = y_0 + \frac{1}{c} y \in Y$

$$\geq ca \quad (\because a \text{ is } \inf_{y \in Y} \|v - y\|)$$

$$= \frac{1}{\|v - y_0\|} \cdot a$$

$$\geq \frac{\theta}{a} \cdot a \quad \text{from (1)}. \quad a \neq 0$$

i.e. $\|z - y\| \geq \theta \quad \forall \; y \in Y$.

Linear Operator

A linear operator T is an operator such that

(i):- the domain $\mathcal{D}(T)$ of T is a vector space and the range $\mathcal{R}(T)$ lies in a vector space over the same field.

$$T : X \longrightarrow Y$$

(ii):- $\forall\, x, y \in \mathcal{D}(T)$ & scalars α.

We have

$$T(x+y) = T(x) + T(y)$$

&

$$T(\alpha x) = \alpha\, T(x)$$

<u>OR</u> $\quad T(\alpha x + \beta y) = \alpha T x + \beta T y$

(Kernal) **Null Space:–**

Let $T : X \longrightarrow Y$ be an operator then the null space of T denoted by $N(T)$ is defined by

$$N(T) = \{x \in \mathcal{D}(T) \mid Tx = 0\}$$

Note:– Let $T : X \longrightarrow Y$ be a linear operator. Then if we choose $\alpha = 0$. Then

$$T(\alpha x) = T(0 \cdot x) = 0 \cdot Tx = 0$$

i.e

$$T(\underset{\text{v.s}}{0}) = 0$$

Example. 1:— Identity Operator

The identity operator $I_X : X \longrightarrow X$ is defined by

$$I_X x = x \quad \forall \; x \in X$$

is a linear operator because

$$I_X(\alpha x + \beta y) = \alpha x + \beta y$$
$$= \alpha I_X x + \beta I_X y$$

Example. 2:— Zero Operator

The zero operator $O : X \longrightarrow Y$ is defined by $Ox = 0 \quad \forall \; x \in X$ is a linear operator because

$$O(\alpha x + \beta y) = 0 = \alpha Ox + \beta Oy$$

Example. 3:— Differentiation

Let X be the vector space of all polynomial on $[a, b]$. Then $T : X \to X$ is defined by

$$Tx = x' \quad \forall \; x \in X$$

i.e $\quad Tx(t) = x'(t) \quad , \; t \in [a, b]$.

Then T is linear because,

$$T(\alpha x(t) + \beta y(t)) = (\alpha x(t) + \beta y(t))'$$
$$= \alpha x'(t) + \beta y'(t)$$
$$= \alpha Tx(t) + \beta Ty(t)$$

Example. 4:- Integration

Consider $T : C[a,b] \longrightarrow \mathbb{R}$

defined by

$$Tx(t) = \int_a^b x(t)\,dt \qquad t \in [a,b]$$

Then T is linear since

$$T(\alpha x(t) + \beta y(t)) = \int_a^b (\alpha x(t) + \beta\, y(t))\,dt$$
$$= \alpha \int_a^b x(t)\,dt + \beta \int_a^b y(t)\,dt$$
$$= \alpha\, Tx(t) + \beta\, Ty(t)$$

Example 5:- Both Elementary vetor Algebra 14-4-14

Consider $T_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

defined by

$T_1 x = x \times a$, where $a = (a_1, a_2, a_3)$ is a fixed vector in $\mathbb{R}^3$. T_1 is linear because

$$T_1(\alpha x + \beta y) = (\alpha x + \beta y) \times a$$
$$= \alpha (x \times a) + \beta (y \times a)$$
$$= \alpha\, T_1 x + \beta\, T_1 y$$

Now, Consider,

$T_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}$: defined by

$$T_2\, x = x \cdot a$$

T_2 is linea becuse,

$$T_2(\alpha x + \beta y) = (\alpha x + \beta y) \cdot a$$
$$= \alpha (x \cdot a) + \beta (y \cdot a) = \alpha T_2 x + \beta T_2 y$$

Example. 6:- Matrices:

Consider

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \quad \& \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

Then

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$A: \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad (\text{or } A: \mathbb{C}^n \longrightarrow \mathbb{C}^m)$$

The matrix $A = (a_{ij})_{m\times n}$ can be used as an operator from $\mathbb{R}^n \to \mathbb{R}^m$ i.e $A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by

$$Ax = y$$

This operator is linear because matrix multiplication is linear.

★ If A is complex then it will be a linear operator from $\mathbb{C}^n \to \mathbb{C}^m$.

Theorem:- Let T be a linear operator. then

a):- The range $R(T)$ is a vector space.

b):- If $\dim \mathcal{D}(T) = n < \infty$, then $\dim R(T) \leq n$

c):- The null space $N(T)$ is a V. Space.

PROOF: a):- Let $y_1, y_2 \in R(T)$

Then $\exists$ $x_1, x_2 \in D(T)$ such that

$$Tx_1 = y_1 \quad \& \quad Tx_2 = y_2$$

$$\Rightarrow \alpha y_1 + \beta y_2 = \alpha Tx_1 + \beta Tx_2$$

$$= T(\alpha x_1 + \beta x_2) \in R(T) \quad (\because T \text{ is linear})$$

$$\Rightarrow \alpha y_1 + \beta y_2 \in R(T) \quad (\because \alpha x_1 + \beta x_2 \in D(T))$$

$\Rightarrow R(T)$ is a V.space.

b):- We choose $n+1$ elements $y_1, y_2, \cdots, y_{n+1} \in R(T)$

Then $y_1 = Tx_1$, $y_2 = Tx_2$, $\cdots$, $y_{n+1} = Tx_{n+1}$.

for some

$$x_1, x_2, \cdots, x_n, x_{n+1} \in D(T)$$

Since $\dim D(T) = n$ (given),

The set $\{x_1, x_2, \cdots, x_{n+1}\}$ is L.D. Then

$$\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1} = 0$$

for some scalars $\alpha_1, \cdots, \alpha_{n+1}$ not all zeros.

$$T(\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1}) = T0$$

Since T is linear then

$$\Rightarrow \alpha_1 Tx_1 + \cdots + \alpha_{n+1} Tx_{n+1} = T0 \qquad \because T0 = 0$$

$$\Rightarrow \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_{n+1} y_{n+1} = 0$$

where all scalars are not zero.

$\Rightarrow \{y_1, y_2, \cdots, y_{n+1}\}$ is L.D.

Hence $\dim R(T) \leq n$.

Pf (C):- Let $x_1, x_2 \in N(T)$, then

$$Tx_1 = 0 \quad , \quad Tx_2 = 0$$

Consider

$$T(\alpha x_1 + \beta x_2) = \alpha \underbrace{Tx_1}_{0} + \beta \underbrace{Tx_2}_{0} \quad (\because T \text{ is linear})$$

$$= 0$$

$\Rightarrow \alpha x_1 + \beta x_2 \in N(T)$

$\Rightarrow N(T)$ is a vector space.

Remarks:- From proof of Part (b), we conclude that "Linear operator preserve linear dependence".

Def: Injective or One-to-One

An operator $T : D(T) \longrightarrow Y$ is said to be injective or one-to-one if different points in domain have different images. i.e

$$x_1 \neq x_2 \Longrightarrow Tx_1 \neq Tx_2$$

OR

$$Tx_1 = Tx_2 \Rightarrow x_1 = x_2$$

In this case $\exists$ a mapping

$T^{-1} : R(T) \longrightarrow D(T)$ which mapps every $y \in R(T)$ onto $x \in D(T)$ for which $Tx = y$

Remarks:- 17-04-14

$$T^{-1}Tx = x \quad \forall \; x \in D(T)$$
$$T T^{-1}y = y \quad \forall \; y \in R(T)$$

Theorem:- Let X and Y be vector spaces both real (or both complex).

Let $T: D(T) \longrightarrow Y$ be a linear operator with $D(T) \subseteq X$ & $R(T) \subseteq Y$. Then

If $T(\alpha x) = \alpha Tx$
complex Real
It has no meaning.

a):- The inverse $T^{-1}: R(T) \longrightarrow D(T)$ exists iff $Tx = 0 \Rightarrow x = 0$

b):- If T^{-1} exists, then it is linear.

c):- If $\dim D(T) = n < \infty$ & T^{-1} exists, then

$$\dim D(T) = \dim R(T)$$

17-4-14

PROOF:- a):- If T^{-1} exists. Then T is 1-1.

$$\Rightarrow \forall \; x_1, x_2 \in D(T) :$$
$$Tx_1 = Tx_2 \Rightarrow x_1 = x_2 \longrightarrow (1)$$

If we choose $x_1 = x$ & $x_2 = 0$ in (1).

then $Tx = T_0 \Rightarrow x = 0$

i.e $Tx = 0 \Rightarrow x = 0$

$\because T_0 = 0$ if T is linear

Conversely:- Suppose $Tx = 0 \Rightarrow x = 0$

To prove that T is 1-1.

Let $x_1, x_2 \in D(T)$ is so that

$Tx_1 = Tx_2$

$\Rightarrow Tx_1 - Tx_2 = 0$

$\Rightarrow T(x_1 - x_2) = 0$ ($\because$ T is linear)

$\Rightarrow x_1 - x_2 = 0$ (by assumption)

$\Rightarrow x_1 = x_2$

b):- Let $y_1, y_2 \in D(T^{-1}) = R(T)$

then $y_1 = Tx_1$ & $y_2 = Tx_2$

where $x_1, x_2 \in D(T)$

Now

$$\alpha y_1 + \beta y_2 = \alpha Tx_1 + \beta Tx_2$$

$$= T(\alpha x_1 + \beta x_2) \quad (\because T \text{ is linear})$$

$$\Rightarrow T^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2$$

$$= \alpha T^{-1} y_1 + \beta T^{-1} y_2$$

$\Rightarrow T^{-1}$ is linear.

C):- We know that if T is linear

then $\dim R(T) \leq \dim D(T) \longrightarrow (1)$

Since T^{-1} is also linear. (part b)

Therefore $\dim R(T^{-1}) \leq \dim D(T^{-1})$

$\Rightarrow \dim D(T) \leq \dim R(T) \longrightarrow (2)$

From (1) and (2) we get

$\dim D(T) = \dim R(T)$.

Bounded and Continuous Operators

In defining linear operators, we have not used the norm, we can take norm into acount in the following definition.

Def: Bounded linear operator.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and $T: \mathcal{D}(T) \subset X \longrightarrow Y$ be a linear operator.

The operator T is said to be bounded if there is a real number C such that $\forall x \in \mathcal{D}(T)$,

$$\|Tx\|_Y \leq C\|x\|_X \longrightarrow (1)$$

Question:- What is the minimum value of C such that (1) hold $\forall x$ $\mathcal{D}(T)$ with $x \neq 0$?

Ans:- $(1) \Rightarrow \dfrac{\|Tx\|}{\|x\|} \leq C \quad \forall x \in \mathcal{D}(T), \; x \neq 0$

$$\Rightarrow \sup_{\substack{x \in \mathcal{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \leq C$$

Hence ~~maximum~~ minimum value of C is

$\displaystyle\sup_{\substack{x \in \mathcal{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$. This quantity is denoted by $\|T\|$. So that

$$\|T\| = \sup_{\substack{x \in \mathcal{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

Then ① can be written as

$$\|Tx\| \leq \|T\| \cdot \|x\| \longrightarrow ②$$

(Boundedness)

This is very important result that will be used frequently.

Note:- If $\mathcal{D}(T) = \{0\}$ then we defined $\|T\| = 0$. In this case $T = 0$ Since $T0 = 0$.

Lemma:- (Norm).

Let T be a bounded linear operator then

a):- An alternate formula for the norm of T is

$$\|T\| = \sup_{\substack{x \in \mathcal{D}(T) \\ \|x\| = 1}} \|Tx\|$$

b):- The norm defined by

$$\|T\| = \sup_{\substack{x \in \mathcal{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

satisfies all the conditions of a norm.

Proof: a):- Let $x \neq 0$ So that $\|x\| = a$. Define $y = \frac{1}{a}x$, $\forall x \neq 0$

Then $\|y\| = 1$

Consider

$$\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

$$= \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|}{a} \qquad \|\alpha x\| = |\alpha| \, \|x\|$$

$$= \sup_{\substack{x \in D(T) \\ x \neq 0}} \left\| \frac{1}{a} Tx \right\|$$

$$= \sup_{\substack{x \in D(T) \\ x \neq 0}} \left\| T\left(x \frac{1}{a}\right) \right\| \qquad \because T \text{ is Linear.}$$

$$= \sup_{\substack{y \in D(T) \\ \|y\| = 1}} \|Ty\|$$

y is a dummy variable. So;

$$\|T\| = \sup_{\substack{x \in D(T) \\ \|x\| = 1}} \|Tx\|$$

b):- We have to show that,

$$\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \text{ is a normed space.}$$

N1:-

$$\|T\| = \sup_{\substack{x \in \mathcal{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \geq 0$$

N2:-

$$\|T\| = \sup_{\substack{x \in \mathcal{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = 0$$

$$\Leftrightarrow \|Tx\| = 0 \Leftrightarrow Tx = 0 \quad \forall x \in \mathcal{D}$$

$$\Leftrightarrow T = 0.$$

N3):- $$\|T\| = \sup_{\substack{x \in \mathcal{D}(T) \\ \|x\| = 1}} \|Tx\|$$

$$\|\alpha T\| = \sup_{\substack{x \in \mathcal{D}(T) \\ \|x\| = 1}} \|\alpha Tx\|$$

$$= \sup_{\substack{x \in \mathcal{D}(T) \\ \|x\| = 1}} |\alpha| \, \|Tx\|$$

$$= |\alpha| \sup_{\substack{x \in \mathcal{D}(T) \\ \|x\| = 1}} \|Tx\|$$

i.e $\|\alpha T\| = |\alpha| \, \|T\|$

N4):- $$\|T_1 + T_2\| = \sup_{\substack{x \in \mathcal{D}(T) \\ \|x\| = 1}} \|(T_1 + T_2)x\|$$

$$= \sup_{\substack{x \in \mathcal{D}(T) \\ \|x\| = 1}} \|T_1x + T_2x\|$$

$$\leq \sup_{\substack{x \in D(T) \\ \|x\|=1}} \|T_1 x\| + \sup_{\substack{x \in D(T) \\ \|x\|=1}} \|T_2 x\|$$

i.e

$$\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$$

Hence this is a normed space.

Example:- (Identity Operator) 21-04-14

The identity operator $I : X \to X$ defined by $Ix = x$ is bounded.

Since

$$\|Ix\| = \|x\| \leq 1 \cdot \|x\|$$

where $\|I\| = \sup_{\substack{x \in D(I) \\ x \neq 0}} \frac{\|Ix\|}{\|x\|} = 1$

Example:- (Zero Operator)

The zero operator $O : X \to Y$ defined by $Ox = 0$, $\forall x \in X$ is bounded. since

$$\|Ox\| = \|0\| = \|O\| \, \|x\|$$

i.e $\|Ox\| \leq \|O\| \, \|x\|$ $\quad \because \|Tx\| \leq \|T\| \, \|x\|$

Where

$$\|O\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Ox\|}{\|x\|} = 0$$

Example:

Let X be the normed space of all polynomials on $[0,1]$ will be norm defined by

$$\|x\| = \max_{t\in[0,1]} |x(t)|$$

Define $T: X \longrightarrow Y$ by

$Tx = x'$ we claim that T is unbounded.

To see this define (x_n) in X by $x_n(t) = t^n$, $t \in [0,1]$

$$Tx_n = n t^{n-1}$$

$$\|Tx_n\| = \max_{t\in[0,1]} |n t^{n-1}| = n$$

($t=1$ is max.)

$$\|x_n\| = \max_{t\in[0,1]} |t^n| = 1$$

We have

$$\frac{\|Tx_n\|}{\|x_n\|} = \frac{n}{1} = n$$

Since $n \in N$ is arbitrary, we are unable to find a fixed number C s.t

$$\|Tx_n\| \leq C\|x_n\|$$

$\Rightarrow$ T is not bounded.

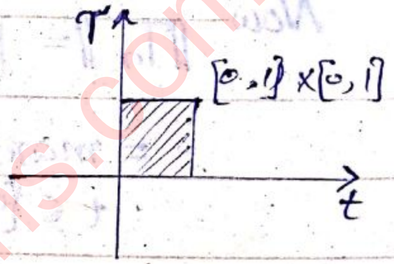
Example:-

Define $T: C[0,1] \longrightarrow C[0,1]$

by $Tx = \int_0^1 K(t,\tau)\, x(\tau)\, d\tau = y(t)$

Where $K(t;\tau)$ is a given function which is called **Kernal** of T and is assumed to be continuous on the closed square $[0,1]\times[0,1]$ in the $t\tau$-plane.

Obviously T is linear.



Since,

$$T(\alpha x+\beta y) = \int_0^1 K(t,\tau)\,(\alpha x+\beta y)(\tau)\,d\tau$$

$$= \int_0^1 K(t,\tau)\big(\alpha x(\tau)+\beta y(\tau)\big)\,d\tau$$

$$= \alpha \int_0^1 K(t,\tau)\,x(\tau)\,d\tau + \beta \int_0^1 K(t,\tau)\,y(\tau)\,d\tau$$

$$= \alpha Tx + \beta Ty$$

We claim that T is bounded.

First note that K being continuous on closed square is bounded. (Because every continuous function defined on closed interval is bounded). So;

$$|K(t,\tau)| \leq K_0, \quad \forall\, (t,\tau) \in [0,1]\times[0,1]$$

where K_0 is a real number.

$$\left|\int f(x)\,dx\right| \le \int |f(x)|\,dx$$

Furthermore,

$$\|x\| = \max_{t\in[0,1]} |x(t)|$$

$$\Rightarrow |x(t)| \le \max_{t\in[0,1]} |x(t)|$$

i.e $|x(t)| \le \|x\|$

Now $\|Tx\| = \|y\|$

$$= \max_{t\in[0,1]} |y(t)|$$

$$= \max_{t\in[0,1]} \left|\int_0^1 K(t,\tau)\, x(\tau)\, d\tau\right|$$

$\because \int |f(x)|dx \ge \left|\int f(x)dx\right|$

$$\le \max_{t\in[0,1]} \int_0^1 |K(t,\tau)|\,|x(\tau)|\, d\tau$$

$$\le K_0 \|x\| \max_{t\in[0,1]} \underbrace{\int_0^1 d\tau}_{1}$$

$$= K_0 \|x\| \qquad \because |K(t,\tau)| \le K_0$$

$|x(\tau)| \le \|x\|$

i.e

$$\|Tx\| \le K_0 \|x\|$$

$\Rightarrow T$ is bounded.

Example:- The matrix $A = (a_{ij})_{m\times n}$ defines a linear operator

$$A : R^n \longrightarrow R^m \quad \text{by} \quad Ax = y$$

where

$$x = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}, \quad y = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_m \end{bmatrix}$$

From $Ax = y$ we have

$$\eta_i = \sum_{j=1}^{n} a_{ij}\, \xi_j$$

We show that A is bounded.

$$\|Ax\|^2 = \|y\|^2$$

$$= \sum_{i=1}^{m} \eta_i^2$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij}\, \xi_j \right)^2$$

$$\underset{H.I}{\leq} \sum_{i=1}^{m} \left[\left(\sum_{j=1}^{n} a_{ij}^2 \right)^{1/2} \cdot \left(\sum_{j=1}^{n} \xi_j^2 \right)^{1/2} \right]^2$$

$$= \sum_{i=1}^{m} \left[\sum_{j=1}^{n} a_{ij}^2 \cdot \underbrace{\sum_{j=1}^{n} \xi_j^2}_{\|x\|^2} \right]$$

$$= \sum_{i=1}^{m} c_i^2 \, \|x\|^2 \qquad \text{where } c_i^2 = \sum_{j=1}^{n} a_{ij}^2$$

$$= c^2 \, \|x\|^2 \qquad \text{where } c^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2$$

i.e

$$\|Ax\| \leq c\|x\|$$

$\Rightarrow A$ is bounded.

Theorem:- (Finite dimensional) 24-04-14

If a normed space X is finite dimensional, then every linear operator on X is bounded.

Proof:- Let $\dim X = n$ & $\{e_1, e_2, \ldots, e_n\}$ be a basis for X. Then each $x \in X$ can be expressed as;

$$x = \sum_1^n \alpha_i e_i$$

$$Tx = T\left(\sum_1^n \alpha_i e_i\right)$$

$$= \sum_1^n \alpha_i T(e_i) \quad \because T \text{ is linear}$$

$$\Rightarrow \|Tx\| = \left\|\sum_1^n \alpha_i T(e_i)\right\|$$

$$\leq \sum_1^n \|\alpha_i T(e_i)\| \quad \text{Triangle inequality for } n\text{-terms.}$$

$$= \sum_1^n |\alpha_i| \, \|T(e_i)\|$$

$$\leq M \sum_1^n |\alpha_i| \longrightarrow (1)$$

where $M = \max\limits_{1 \leq i \leq n} \|Te_i\|$

Also since $\{e_1, e_2, \ldots, e_n\}$ are linearly independent therefore,

$$\|x\| = \left\|\sum_1^n \alpha_i e_i\right\| \geq C\left(\sum_1^n |\alpha_i|\right)$$

(By previous lemma)

$$\Rightarrow \sum_1^n |\alpha_i| \leq \frac{1}{C}\|x\| \longrightarrow (2)$$

using (2) and (1).

$$\|Tx\| \leq \frac{M}{C} \|x\|$$

i.e $\|Tx\| \leq K \|x\|$ where $K = \frac{M}{C}$

$\Rightarrow$ T is bounded.

Continuity :-

Let $T: \mathcal{D}(T) \longrightarrow Y$ be any operator, not necessarily linear. where $\mathcal{D}(T) \subset X$ and X, Y are normed spaces. The operator T is said to be continuous at a point $x_0 \in \mathcal{D}(T)$ if $\forall \epsilon > 0$, $\exists$ a $\delta > 0$ such that $\|Tx - Tx_0\| < \epsilon$ $\forall$ $x \in \mathcal{D}(T)$ satisfying $\|x - x_0\| < \delta$

Theorem:- Let $T: \mathcal{D}(T) \longrightarrow Y$ be a linear operator, where $\mathcal{D}(T) \subset X$ and X, Y are normed spaces. Then,

a):- T is continuous $\Leftrightarrow$ T is bounded.

b):- If T is continuous at a single point, it is continuous.

Proof: a):- If $T = 0$. Then the result is trivially true.

Let $T \neq 0$ then $\|T\| \neq 0$

Assume that T is bounded.

Let $x_0 \in D(T)$ & $\epsilon > 0$ be given.

Let $x \in D(T)$ such that $\|x - x_0\| < \delta$

where $\delta = \frac{\epsilon}{\|T\|}$, we have

$\|Tx - Tx_0\| = \|T(x - x_0)\|$ $\because T$ is linear.

$\leq \|T\| \|x - x_0\|$ $\because T$ is bounded.

$< \|T\| \cdot \delta = \|T\| \cdot \frac{\epsilon}{\|T\|} = \epsilon$

i.e $\|Tx - Tx_0\| < \epsilon$

$\Rightarrow T$ is continuous at $x_0 \in D(T)$.

Since x_0 was arbitrary, therefore T is continuous.

Conversely:- Suppose that T is continuous at an arbitrary point $x_0 \in D(T)$. Then $\forall \epsilon > 0 \ \exists \ \delta > 0$ such that $\|Tx - Tx_0\| < \epsilon \ \ \forall x \in D(T)$

satisfying $\|x - x_0\| \leq \delta \longrightarrow (1)$

Choose any $y \neq 0 \in D(T)$ | $\because \delta, \epsilon$ are very very small.

★ & set $x = x_0 + \frac{\delta}{\|y\|} \cdot y$

$\boxed{x - x_0 = \frac{\delta}{\|y\|} y}$

then $\|x - x_0\| = \delta$

Then $(1) \Rightarrow \|Tx - Tx_0\| < \epsilon$

$\Rightarrow \|T(x - x_0)\| < \epsilon$ ($\because T$ is linear)

$$\Rightarrow \left\|T\left(\frac{\delta}{\|y\|}y\right)\right\| < \epsilon$$

$$\Rightarrow \frac{\delta}{\|y\|}\|Ty\| < \epsilon$$

$$\Rightarrow \|Ty\| < \frac{\epsilon}{\delta}\|y\|$$

i.e $\|Ty\| < C\|y\|$, where $C = \frac{\epsilon}{\delta}$

$\Rightarrow$ T is bounded.

b):- Suppose T is continuous at a single point in $D(T)$, then by converse of part (a), it is bounded and Hence continuous by part (a).

Corollary:- Let T be a bounded linear operator, Then

a):- $x_n \to x \Rightarrow Tx_n \to Tx$ where $x_n, x \in D(T)$

b):- The null space $N(T)$ is closed.

Proof:- a):-

$\|Tx_n - Tx\| = \|T(x_n - x)\|$ ($\because$ T is linear)

$\leq \|T\|\,\|x_n - x\| \to 0$ as $n \to \infty$

$\because$ T is bounded

$\Rightarrow Tx_n \to Tx$

b):- Clearly $N(T) \subseteq \overline{N(T)} \to (1)$

Let $x \in \overline{N(T)}$ then $\exists$ a seq. (x_n)

in $N(T)$ such that $x_n \to x$ then $Tx_n \to Tx$ by part (a).

Since $x_n \in \mathcal{D}(T)$, $Tx_n = 0 \;\forall\, n \in N$

So, $Tx = 0$

i.e $x \in N(T)$.

So that

$$\overline{N(T)} \subseteq N(T) \longrightarrow ②$$

from ① and ② we have

$$\overline{N(T)} = N(T)$$

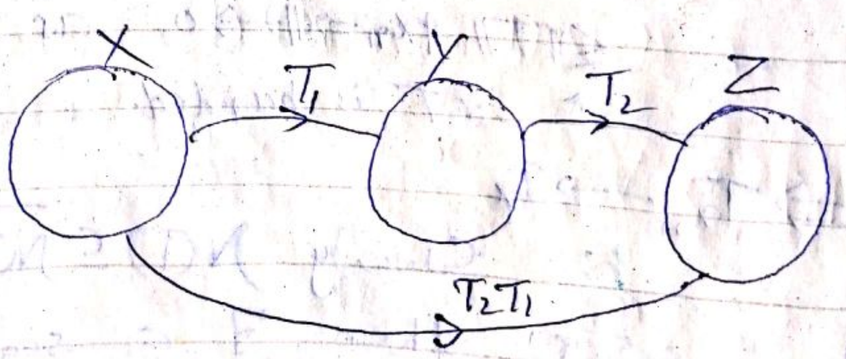
$\Rightarrow N(T)$ is closed.

Def: Composition 28-04-14

Let $T_1 : X \to Y$ & $T_2 : Y \to Z$ be operators. Then the composition

$T_2 T_1 : X \to Z$ is defined as

$$T_2 T_1 x = T_2 (T_1 x)$$



Results:-

If $T_1 : X \to Y$ and $T_2 : Y \to Z$ are bounded linear operators. Then $T_2 T_1$ is also bounded and linear.

More over $\boxed{\|T_2 T_1\| \leq \|T_2\| \, \|T_1\|}$

Proof:- To prove $T_2 T_1$ is linear.

$T_2 T_1 (\alpha x + \beta y) = T_2 (T_1 (\alpha x + \beta y))$ By defn of composition.

$= T_2 (\alpha T_1 x + \beta T_1 y)$ $\because T_1$ is linear.

$= \alpha T_2 (T_1 x) + \beta T_2 (T_1 y)$ $\because T_2$ is linear

$= \alpha T_2 T_1 x + \beta T_2 T_1 y$

$\Rightarrow T_2 T_1$ is linear.

Now we have to prove that $T_2 T_1$ is bounded.

$\|T_2 T_1 x\| = \|T_2 (T_1 x)\|$

$\leq C_1 \|T_1 x\|$ $\because T_2$ is bounded.

$\leq C_1 C_2 \|x\|$ $\because T_1$ is bounded.

i.e $\|T_2 T_1 x\| \leq C\|x\|$ where $C = C_1 C_2$

$\Rightarrow T_2 T_1$ is bounded.

Now for $\|T_2 T_1\| \leq \|T_2\| \|T_1\|$

$$\|T_2 T_1\| = \sup_{\substack{x \in \mathcal{D}(T_2 T_1) \\ x \neq 0}} \frac{\|T_2 T_1 x\|}{\|x\|}$$

$$= \sup_{\substack{x \in \mathcal{D}(T_2T_1) \\ x \neq 0}} \frac{\|T_2(T_1x)\|}{\|x\|}$$

$$\leq \sup_{\substack{x \in \mathcal{D}(T_2T_1) \\ x \neq 0}} \frac{\|T_2\| \, \|T_1 x\|}{\|x\|} \quad (\because T_2 \text{ is bdd})$$

$$\leq \sup_{\substack{x \in \mathcal{D}(T_2T_1) \\ x \neq 0}} \frac{\|T_2\| \|T_1\| \cancel{\|x\|}}{\cancel{\|x\|}} \quad (\because T_1 \text{ is bdd})$$

$$= \|T_2\| \, \|T_1\| \quad \text{i.e. } \|T_2T_1\| \leq \|T_2\| \|T_1\|$$

Note:- If $T : X \to Y$ then by repeated use of $\|T_2T_1\| \leq \|T_2\| \|T_1\|$ We can obtained.

$$\|T^n\| \leq \|T\|^n$$

Def: Equal Operators

The operator T_1, T_2 are said to be equal if

$$\mathcal{D}(T_1) = \mathcal{D}(T_2)$$

and

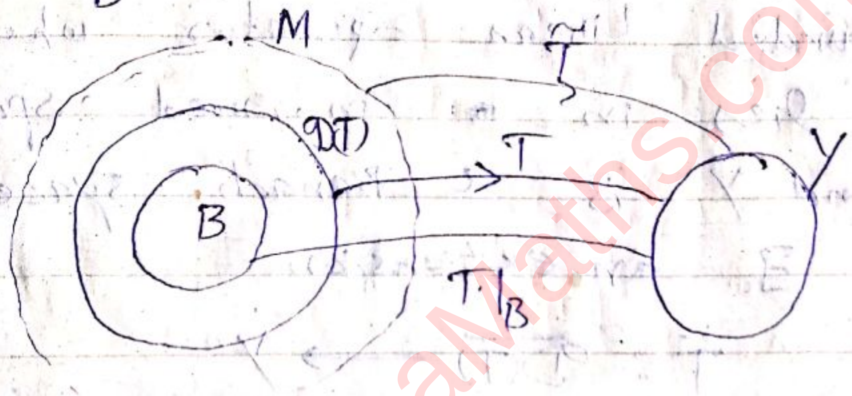
$$T_1x = T_2x \quad , \forall x \in \mathcal{D}(T_1) = \mathcal{D}(T_2)$$

Definitions:-

The restriction of an operator $T: D(T) \to Y$ is to a subset $B \subset D(T)$ is denoted by $T|_B$ and is the operator defined by

$$T|_B : B \longrightarrow Y \quad \text{such that}$$

$$T|_B(x) = Tx, \ \forall x \in B$$



An extension of T to a set $M \supset D(T)$ is an operator

$$\tilde{T} : M \longrightarrow Y \quad \text{such that}$$

$$\tilde{T}|_{D(T)} = T$$

i.e $\tilde{T}|_{D(T)}(x) = Tx \ \forall x \in D(T)$

If $D(T)$ is a proper subset of M, then a given T has many extensions of particular interest linearity (if T happens to be linear) or boundedness (if T is bounded)

The following theorem describes a way of getting bounded linear extensions.

★ **Theorem:-** $\mathcal{X}$ not included

Let $T : \mathcal{D}(T) \longrightarrow Y$ be a bounded linear operator where $\mathcal{D}(T)$ lies in a normed space X, and Y is a Banach space. Then $\exists$ an extension

$$\tilde{T} : \overline{\mathcal{D}(T)} \longrightarrow Y$$

where $\boxed{\tilde{T}}$ is $\boxed{\text{bounded}}$ and $\boxed{\text{linear}}$

also $\boxed{\|\tilde{T}\| = \|T\|}$

Proof:- First we show the $\boxed{\text{existence}}$ of $\tilde{T}$ i.e

We justify that such a $\tilde{T}$ exists. Let $x \in \overline{\mathcal{D}(T)}$, then $\exists$ a sequence (x_n) in $\mathcal{D}(T)$ such that $x_n \longrightarrow x$. Since T is linear and bounded therefore

$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\|$ $\because$ T is linear.

$\leq \|T\| \, \|x_n - x_m\| \longrightarrow 0$ (∵ T is bounded)
as $m, n \longrightarrow \infty$
∵ (x_n) being a convergent
is cauchy.

$\Rightarrow \{Tx_n\}^{\infty}$ is a cauchy sequence in Y. Since Y is complete, $\exists$ $y \in Y$ such that

$Tx_n \longrightarrow y$. So, we define $\tilde{T}$ by

$$\tilde{T}x = y = \lim_{n \to \infty} Tx_n$$

Clearly, $\tilde{T}x = Tx$, $\forall x \in D(T)$.

So that $\tilde{T}$ is an extension of T.

We now show that this definition of $\tilde{T}$ is independent of the choice of sequence in $D(T)$ converging to x.

Suppose that $x_n \longrightarrow x$ & $z_n \longrightarrow x$ then $v_m \longrightarrow x$ where (v_m) is the sequence $(x_1, z_1, x_2, z_2, \ldots)$

Then Tv_m converges (∵ T is linear & bdd $x_n \to x \Rightarrow Tx_n \to Tx$)

And the two subsequences

(Tx_n) and (Tz_n) must have the same limit.

This proves that $\tilde{T}$ is uniquely define at every $x \in \overline{D(T)}$.

☆ Now we prove $\tilde{T}$ is linear.

Consider

$$T(\alpha x_n + \beta x_n') = \alpha Tx_n + \beta Tx_n' \quad (\because T \text{ is linear})$$

Where
$$x_n \to x \;\; \& \;\; x_n' \to x'$$

$$\Rightarrow \lim_{n\to\infty} T(\alpha x_n + \beta x_n') = \alpha \lim_{n\to\infty} Tx_n + \beta \lim_{n\to\infty} Tx_n'$$

$$\tilde{T}(\alpha x + \beta x') = \alpha \tilde{T}x + \beta \tilde{T}x'$$

$\Rightarrow \tilde{T}$ is linear.

2-5-14

☆ Now we have to show that $\tilde{T}$ is bounded.

Since T is bounded, therefore

$$\|Tx_n\| \leq \|T\| \, \|x_n\| \quad , \text{where } x_n \to x$$

$$\lim_{n\to\infty} \|Tx_n\| \leq \|T\| \lim_{n\to\infty} \|x_n\|$$

$$\Rightarrow \|\lim_{n\to\infty} Tx_n\| \leq \|T\| \, \|\lim_{n\to\infty} x_n\|$$

$$\|\tilde{T}x\| \leq \|T\| \, \|x\| \Rightarrow \tilde{T} \text{ is bounded.}$$

★ Now, for $\|\tilde{T}\| = \|T\|$

Since

$$\|\tilde{T}x\| \leq \|T\|\,\|x\|$$

$$\Rightarrow \frac{\|\tilde{T}x\|}{\|x\|} \leq \|T\| \quad \forall\ x \neq 0 \in \overline{D(T)}$$

$$\Rightarrow \sup_{\substack{x \in \overline{D(T)} \\ x \neq 0}} \frac{\|\tilde{T}x\|}{\|x\|} \leq \|T\|$$

$$\Rightarrow \|\tilde{T}\| \leq \|T\| \longrightarrow (1)$$

Since $D(\tilde{T}) \supseteq D(T)$ therefore

$$\|\tilde{T}\| \geq \|T\| \longrightarrow (2)$$

(1) & (2) $\Rightarrow$

$$\|\tilde{T}\| = \|T\|$$

Def: Linear Functional.

A linear functional is a linear operator with domain vector space X and range in the scalar field K of X,

Thus $f : D(f) \longrightarrow K$

where K is real if X is real and K is complex if X is complex vector space.

Bounded linear Functional

A bounded linear functional 'f' is a bounded linear operator with range in

the scalar field K of the normed space X in which the $D(f)$ lies.
Thus $\exists$ a real number C such that
$\forall x \in D(f)$

$$|f(x)| \leq C\|x\|$$

Further more the norm of 'f' is

$$\|f\| = \sup_{\substack{x \in D(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|}$$

OR

$$\|f\| = \sup_{\substack{x \in D(f) \\ \|x\| = 1}} |f(x)|$$

Also, $|f(x)| \leq \|f\| \, \|x\|$

Note:- The results that we prove for bound linear operators continue to hold true for bounded linear functionals.

Example:- The norm

$\|\cdot\| : X \longrightarrow \mathbb{R}$ on a normed space is a functional which is not linear, because

$$\|x+y\| \neq \|x\| + \|y\|$$

$$T(x+y) \neq Tx + Ty$$

Example:- $f : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$f(x) = x \cdot a$$

where $x = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$ & $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3$

is a functional. Obviously f is linear. Now we check it is bounded or not?

i.e $|f(x)| \leq c \, \|x\|$

Sol:- $|f(x)| = |x \cdot a| = \|x\| \|a\| |\cos\theta| \leq \|a\| \|x\|$

i.e $|f(x)| \leq \underset{\uparrow \, (\text{constant})}{\|a\|} \, \|x\| \quad (\because |\cos\theta| \leq 1)$

$\Rightarrow f$ is bounded.

Now $|f(x)| \leq \|a\| \|x\| \quad \forall \; x \neq 0$

$$\Rightarrow \frac{|f(x)|}{\|x\|} \leq \|a\|$$

$$\Rightarrow \sup_{\substack{x \in \mathcal{D}(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|} \leq \|a\|$$

$$\Rightarrow \|f\| \leq \|a\| \longrightarrow (1)$$

$$\|f\| = \sup_{\substack{x \in D(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|} \geq \frac{|f(a)|}{\|a\|} \quad \text{at } x=a$$

$$= \frac{|a \cdot a|}{\|a\|} \quad (\text{by def.})$$

i.e $\|f\| \geq \dfrac{\|a\|^2}{\|a\|}$

$\Rightarrow \|f\| \geq \|a\| \longrightarrow (2)$

From (1) & (2).

$$\|f\| = \|a\|$$

Example:- Define $f: C[a,b] \to \mathbb{R}$

by
$$f(x) = \int_a^b x(t)\,dt$$

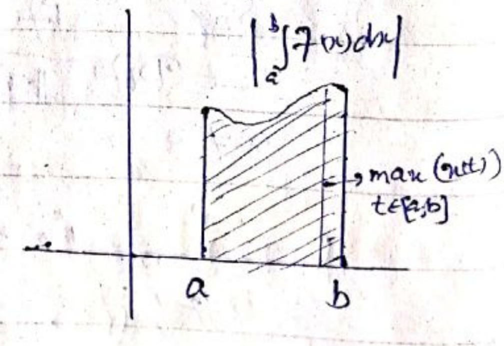
then f is a linear functional.

Show that f is bounded & $\|f\| = b-a$

Proof:- We have

$$\|x\| = \max_{t \in [a,b]} |x(t)| \quad , \quad x \in C[a,b]$$

$$|f(x)| = \left| \int_a^b x(t)\,dt \right|$$



$$\leq (b-a) \max_{t \in [a,b]} |x(t)|$$

$$= (b-a)\,\|x\|$$

i.e $|f(x)| \leq (b-a)\,\|x\| \longrightarrow (1)$

$\Rightarrow f$ is bounded.

From (I)

$$|f(x)| \leq (b-a)\|x\| \quad \forall x \neq 0$$

$$\Rightarrow \frac{|f(x)|}{\|x\|} \leq (b-a)$$

$$\Rightarrow \sup_{\substack{x \in D(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|} \leq (b-a)$$

$$\Rightarrow \|f\| \leq (b-a) \longrightarrow \text{(ii)}$$

We have

$$\|f\| = \sup_{\substack{x \in D(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|} \geq \frac{|f(x_0)|}{\|x_0\|}$$

where $x_0(t) = 1 \quad \forall t \in [a,b]$

$$\geq \int_a^b 1\, dt = b-a$$

i.e

$$\|f\| \geq (b-a) \longrightarrow \text{(iii)}$$

From (ii) & (iii)

$$\|f\| = (b-a)$$

Example:- 5-5-14

Consider $C[a,b]$. If we choose $t_0 \in [a,b]$ then we can define a functional $f: C[a,b] \longrightarrow \mathbb{R}$ by

$$f(x) = x(t_0)$$

Sol:- f is linear (obvious).

f is also bounded. Since

$$|f(x)| = |x(t)| \le \max_{t\in[a,b]} |x(t)| \rightarrow \|x\|$$

i.e $|f(x)| \le 1 \cdot \|x\|$

$\Rightarrow f$ is bounded.

$\Rightarrow \dfrac{|f(x)|}{\|x\|} \le 1$ for all $x \in D(f)$, $x \ne 0$

$$\Rightarrow \sup_{\substack{x\in D(f)\\ x\neq 0}} \frac{|f(x)|}{\|x\|} \le 1$$

$\Rightarrow \|f\| \le 1 \longrightarrow$ (i)

On the other hand, for $x_0 = 1 \in C[a,b]$, we have $\|x_0\| = 1$ &

$$\|f\| \ge \frac{|f(x_0)|}{\|x_0\|} = |x_0(t)| = 1$$

i.e $\|f\| \ge 1 \longrightarrow$ (ii)

(i) & (ii) $\Rightarrow \|f\| = 1$.

Example: Consider a space $X = l^2$. Choose a fixed $a = (\alpha_i)_1^\infty \in l^2$ and define a functional on l^2 by

$$f(x) = \sum_1^\infty \xi_i \alpha_i \quad , \quad x = (\xi_i)_1^\infty \in l^2$$

Sol:- f is linear (obvious).

f is also bounded. Since

$$|f(x)| = \left|\sum_1^\infty \xi_i \alpha_i\right| \underset{T.I}{\leq} \sum_1^\infty |\xi_i \alpha_i|$$

$$\underset{H.I}{\leq} \underbrace{\left(\sum_1^\infty |\xi_i|^2\right)^{1/2}}_{\|x\|} \underbrace{\left(\sum_1^\infty |\alpha_i|^2\right)^{1/2}}_{\|a\|}$$

$$\Rightarrow |f(x)| \leq \|x\| \|a\|$$

i.e $|f(x)| \leq C\|x\|$ where $c = \|a\|$

Define: Algebraic Dual Space

Let X be a vector space. Then the set of all linear functionals on X is denoted by X^* and is called The algebraic dual space of X.

$$f : X \longrightarrow \mathbb{R}$$

$$X^* = \{ f \mid f \text{ is linear functional on } X \}$$

It is easy to see that X^* forms a vector space under the operations

(i) $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

(ii) $(\alpha f)(x) = \alpha f(x)$

Second algebraic dual

We may go one step further and define linear functionals on X^*. The set of all linear functionals on X^* is denoted by X^{**} and is called 2nd algebraic dual of X.

$(X, +, \cdot)$	X^* (1st algebraic dual)	X^{**} (2nd Algebraic dual)
v. space.	V. space	V. space
x: general element	f (general element)	g (general element)
	$f(x)$: value at a point	$g(f)$: value at a point

Observations:- We can obtained a $g \in X^{**}$ by choosing a fixed $x \in X$ by setting

$$g(f) = g_x(f) = f(x) \quad \left(\begin{array}{l} x \in X \text{ is fixed,} \\ f \in X^* \text{ is variable} \end{array} \right.$$

Where the subscript that we got 'g' by using certain $x \in X$, we claim that g_x is linear. To see this, Consider

$$\begin{aligned} g_x(\alpha f_1 + \beta f_2) &= (\alpha f_1 + \beta f_2)(x) \\ &= \alpha f_1(x) + \beta f_2(x) \\ &= \alpha g_x(f_1) + \beta g_x(f_2) \end{aligned}$$

$\Rightarrow g$ is linear.

From above observation to each $x \in X$ there corresponding a $g_x \in X^{**}$. This define a mapping

$$C : X \longrightarrow X^{**}, \quad x \longrightarrow g_x$$

C is called canonical mapping or canonical embedding of x into X^{**}.

C is linear. Since

$$C(\alpha x+\beta y)(f) = g_{\alpha x+\beta y}(f)$$

$$= f(\alpha x+\beta y)$$

$$= \alpha f(x) + \beta f(y) \quad \because f \text{ is Linear}$$

$$= \alpha\, g_x(f) + \beta\, g_y(f)$$

$$= \alpha\, C(x)(f) + \beta\, C(y)(f)$$

$$= [\alpha\, C(x) + \beta\, C(y)](f), \ \forall f$$

$$\Rightarrow C(\alpha x+\beta y) = \alpha C(x) + \beta C(y)$$

$\Rightarrow C$ is linear.

Isomorphism

An isomorphism is bijective mapping of X into $\tilde{X}$ which preserve structure. Accordingly,

① If $(X,+,\cdot)$ & $(\tilde{X},\oplus,\odot)$ are two vector spaces, then an isomorphism

$T: X \longrightarrow \tilde{X}$ is a bijective mapping which preserves algebraic operations.

$$T(\alpha\cdot x+\beta\cdot y) = \alpha\odot Tx \oplus \beta\odot Ty$$

X & $\bar{X}$ are called isomorphic vector spaces.

2):- If (X, d) & $(\bar{X}, \bar{d})$ are two metric spaces then an isomorphism $T : (X, d) \longrightarrow (\bar{X}, \bar{d})$ is a bijective mapping which preserves distance.

i.e $\forall\ x, y \in X$

$$\bar{d}(Tx, Ty) = d(x, y)$$

We say that $\bar{X}$ is isomorphic with X.

3):- If $(X, \|\cdot\|)$ & $(\bar{X}, \|\cdot\|_1)$ are normed spaces. Then an isomorphism is a bijective mapping $T : (X, \|\cdot\|) \to (\bar{X}, \|\cdot\|_1)$ which preserve the algebraic operations of vector spaces and the norm.

i.e

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

& $$\|Tx - Ty\|_1 = \|x - y\|$$

Linear Operators & Functionals on finite dimensional spaces 6-5-14

Recall that: a matrix $A=(a_{ij})_{m\times n}$ can be used as an operator from $R^n \longrightarrow R^m$. i.e

$$(a_{ij}) : R^n \longrightarrow R^m$$

defined by

$$Ax = y \quad \text{where } x=\begin{pmatrix}\xi_1\\ \vdots\\ \xi_n\end{pmatrix} \;\&\; y=\begin{pmatrix}\eta_1\\ \vdots\\ \eta_n\end{pmatrix}$$

& $$\eta_i = \sum_{j=1}^{n} a_{ij}\,\xi_j$$

Conversely:- statement It can be shown that every linear operator defined on a finite dimensional vector space can be represented by a matrix. As shown by below,

Let X and Y be two finite dimensional vector spaces over the same field and $T : X \longrightarrow Y$ be a linear Operator.

Let $\dim X = n$ & $\dim Y = r$. Let $E=\{e_1, e_2, \ldots, e_n\}$ and $B=\{b_1, \ldots, b_r\}$ be basis for X and Y (resp.).

Then every $x \in X$ has a unique representation

$$x = \sum_{i=1}^{n} \xi_i e_i \longrightarrow (1)$$

$$Tx = T\left(\sum_{i=1}^{n} \xi_i e_i\right) = \sum_{i=1}^{n} \xi_i Te_i \quad (\because T \text{ is linear}) \longrightarrow (2)$$

Since representation (1) is unique, T is uniquely determined if the image (2) $y_i = Te_i$ if the n basis vectors $e_1, e_2, \dots, e_n$ are prescribed. Since y_i & $Te_i \in Y$, therefore they have unique representation of the form

$$\left.\begin{aligned} y &= \sum_{j=1}^{r} \eta_j b_j \\ \& \quad Te_i &= \sum_{j=1}^{r} \tau_{ji} b_j \end{aligned}\right\} \longrightarrow (3)$$

Now Consider

$$y = \sum_{j=1}^{r} \eta_j b_j \overset{(2)}{=} \sum_{i=1}^{n} \xi_i Te_i$$

Using (3)

$$y = \sum_{i=1}^{n} \xi_i \left(\sum_{j=1}^{r} \tau_{ji} b_j \right)$$

$$= \sum_{j=1}^{r} \left(\sum_{i=1}^{n} \tau_{ji} \xi_i \right) b_j$$

$$\Rightarrow \eta_j = \sum_{i=1}^{n} \tau_{ji} \xi_i \longrightarrow (4)$$

The image $y = Tx = \sum_{j=1}^{r} \eta_j b_j$ of $x = \sum_{j=1}^{n} \xi_j e_j$ can be obtained from (4).

Further (4) suggests a matrix.

$$T_{EB} = (T_{ji})_{r \times n}$$

So, operator can be represented by means of the matrix T_{EB}. By introducing the column ~~matrices~~ vectors,

$$x = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \quad y = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_r \end{pmatrix}$$

We can write (4) as

$$y = T_{EB}\, x$$

Dual Basis for X.

Consider the set of functionals on X, where X is a normed space with dim X = n and let $\{e_1, \dots, e_n\}$ be a basis for X. These functionals constitute the algebraic dual X^* of X then for every $f \in X^*$ & for every $x = \sum_{k=1}^{n} \xi_k e_k$

We have

$$f(x) = \sum_{k=1}^{n} \xi_k f(e_k)$$

$$= \sum_{k=1}^{n} \xi_k \alpha_k \xrightarrow{\text{where}} (1) \quad \alpha_k = f(e_k)$$

So, f is uniquely determined by its values α_k at the n basis vectors

$e_1, \ldots, e_n$ of X.

Conversely, every n-tuple of scalars defines a linear functional on X. In particular lets take n-tuples

$(1,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,\ldots,1)$

Then from

$$f_1(x) = \sum_{k=1}^{n} \xi_k \alpha_k = \xi_1$$

where $\alpha_k = \begin{cases} 1 & \text{if } k=1 \\ 0 & \text{if } k \neq 1 \end{cases}$

We get

$$f_1(x) = \xi_1$$

also $f_2(x) = \xi_2$

In this way, we get n-linear functionals $f_1, \ldots, f_n$ with

$$f_j(e_k) = \delta_{jk} = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j \neq k \end{cases}$$

We prove that the set of linear functionals satisfying $f_j(e_k) = \delta_{jk}$ form a dual basis for the basis $\{e_1, e_2, \ldots, e_n\}$ of X.

δ_{jk} is called the Kronecker delta.

Theorem: (Dimensions of X^*) 08-05-14

Let X be n-dimensional vector space and $E = \{e_1, e_2, \dots, e_n\}$ be a basis for X. Then $F = \{f_1, f_2, \dots, f_n\}$ satisfying
$f_j(e_k) = \delta_{jk}$ form a basis for the algebraic dual X^* of X and
$$\dim X^* = \dim X$$

Proof:- To show that F is linearly independent.

Let $\sum_{k=1}^{n} \beta_k f_k(x) = 0$, where β_k are scalars and $x \in X$ is arbitrary.

Let $x = e_j$ for some j. Then
$$\sum_{k=1}^{n} \beta_k f_k(e_j) = 0$$
$$\sum_{k=1}^{n} \beta_k \delta_{jk} = 0 \Rightarrow \beta_j = 0 \qquad j=k \quad \cancel{j \neq k}$$

So, that F is linearly independent.

To show that F span X^*.

Let $f \in X^*$ and $x \in X$. then
$$x = \sum_{i=1}^{n} \xi_i e_i$$
Operating f on both sides
$$f(x) = \sum_{i=1}^{n} \xi_i f(e_i) \quad \because f \text{ is linear.}$$
$$= \sum_{1}^{n} \xi_i \alpha_i \quad , \text{where } \alpha_i = f(e_i)$$

Also, $f_j(x) = f_j\left(\sum_{1}^{n} \xi_i e_i\right)$
$$= \sum_{1}^{n} \xi_i \underbrace{f_j(e_i)}_{\delta_{ij}} = \xi_j$$

$$f(x) = \sum_{i=1}^{n} f_i(x)\,\alpha_i$$

Hence F spans X^*. So F is a basis for X^*. Since the basis for X^* lies in n elements. Therefore

$$\dim X = \dim X^*$$

(zero vector) Lemma:— Let X be a finite dimensional vector space. If $x_0 \in X$ has the property that $f(x_0) = 0 \quad \forall f \in X^*$, then $x_0 = 0$

Proof:— Let $\dim X = n$ & $\{e_1, e_2, \ldots, e_n\}$ be a basis for X. Then $x_0 \in X$ can be written as

$$x_0 = \sum_{i=1}^{n} \alpha_{i_0} e_i$$

$$\Rightarrow f(x_0) = \sum_{i=1}^{n} \alpha_{i_0} f(e_i) \quad (\because f \text{ is linear})$$

$$\Rightarrow f(x_0) = \sum_{i=1}^{n} \alpha_{i_0} \beta_i \quad \text{where } \beta_i = f(e_i)$$

Since $f(x_0) = 0 \quad \forall f \in X^*$, therefore

$$\sum_{i=1}^{n} \alpha_{i_0} \beta_i = 0 \quad \forall \beta_i$$

$$\Rightarrow \alpha_{i_0} = 0 \quad \text{for } i = 1, 2, \ldots, n$$

Hence

$$x_0 = \sum_{i=1}^{n} \alpha_{i_0} e_1 = 0$$

Book: A finite dimensional v.space is algebraically reflexive.

$$\dim X^{**} = \dim X^* = \dim X$$

Def:- $B(X,Y)$

Let X & Y be normed spaces over the same field. Then $B(X,Y)$ is the space of all bounded linear operators from X into Y.

or $B(X,Y) = \{T : X \to Y \mid T \text{ is linear \& bounded}\}$

$B(X,Y)$ forms a vector space under the operations.

$$(T_1 + T_2)(x) = T_1 x + T_2 x$$

$$(\alpha T)(x) = \alpha Tx$$

Also $B(X,Y)$ forms a normed space under the norm

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\| = 1}} \|Tx\|$$

Theorem: (Completeness)

If Y is a Banach space then $B(X,Y)$ is a Banach space.

Proof:- Let $(T_n)_1^\infty$ be a cauchy sequence in $B(X,Y)$. Then $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ such that

$$\|T_n - T_m\| < \varepsilon \quad \forall\ m, n \geq N$$

For any $x \in X$ & $m, n \geq N$ we have

$$\|T_n x - T_m x\| = \|(T_n - T_m)(x)\|$$

$$\leq \|T_n - T_m\| \|x\|$$

$$< \varepsilon \|x\| = \varepsilon_1$$

$\Rightarrow$ for each fixed x, $(T_n x)_1^\infty$ is a

Cauchy sequence in Y. Since Y is complete $\exists$ $y \in Y$ such that
$T_n x \longrightarrow y$.

Clearly the limit $y \in Y$ depends upon the choice of $x \in X$. This define an operator $T : X \longrightarrow Y$, where $y = Tx$

To show that $B(X, Y)$ is a Banach space, we need to prove that T is bounded, linear and
$$T_n \xrightarrow{\|\cdot\|} T.$$

★ To prove T is linear:

We have,
$$\lim_{n\to\infty} T_n x = Tx$$
$$\Rightarrow \quad T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y)$$
$$= \lim_{n\to\infty} (\alpha T_n x + \beta T_n y) \quad (\because T_n \text{ is linea})$$
$$= \alpha \lim_{n\to\infty} T_n x + \beta \lim_{n\to\infty} T_n y$$
$$= \alpha Tx + \beta Ty$$
$\Rightarrow$ T is linear.

★ To show that T is bounded.

We have for $m, n \geq N$
$$\|T_n x - T_m x\| \leq \varepsilon \|x\| \quad \text{from } (1)$$

$\|Tx\| \leq \|T\| \|x\|$

Fixing n and letting $m \to \infty$. We get,

$$\|T_n x - Tx\| \leq \varepsilon \|x\|$$

$$\Rightarrow \|(T_n - T)x\| \leq \varepsilon \|x\| \longrightarrow (2)$$

$\Rightarrow$ $T_n - T$ is bounded. So

$T = T_n - (T_n - T)$ is bounded. Since T_n and $T_n - T$ is bounded.

★ To show that

$$T_n \xrightarrow{\|\cdot\|_{B(X,Y)}} T$$

$$(2) \Rightarrow \frac{\|(T_n - T)x\|}{\|x\|} < \varepsilon \quad \forall\ x \neq 0$$

$$\sup_{\substack{x \in X \\ x \neq 0}} \frac{\|(T_n - T)x\|}{\|x\|} < \varepsilon$$

$$\Rightarrow \|T_n - T\| < \varepsilon$$

$$\Rightarrow T_n \xrightarrow{\|\cdot\|_{B(X,Y)}} T$$

Hence $B(X,Y)$ is a Banach space.

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Dual Space

Let X be a normed space. Then the set of all bounded linear functionals on X form a normed space with

$$\|f\| = \sup_{\substack{x \in D(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x \in D(f) \\ \|x\|=1}} |f(x)|$$

which is called dual space of X and denoted by X'.

i.e

$$X' = B(X,Y) = \{T : X \to F \mid T \text{ is bounded linear functional}\}$$

Remember that the algebraic dual X^* is the vector space of all linear functionals on X.

Remark:- From perivous theorem

$$B(X,Y) = \{T : X \to Y \mid T \text{ is bounded linear operator}\}$$

is a Banach space if Y is a Banach space.

$= \{f : X \to \mathbb{R} \text{ or } \mathbb{C} \mid f \text{ is bdd, linear fun}\}$

Since $X' = B(X, \mathbb{R} \text{ or } \mathbb{C})$ & $\mathbb{R}$ and $\mathbb{C}$ complete therefore X' is complete (Banach space). Where X is Banach space or not. So, Theorem:-

The dual space X' of the normed space X is a Banach space (whether or not X is).

In order to find duals of various spaces, we need the concept of isomorphism.

Isomorphic Normed Spaces:

An isomorphism of a norm space $(X, \|\cdot\|)$ onto to a norm space $(\tilde{X}, \|\cdot\|_{\sim})$ is a bijective linear operator. $T : X \longrightarrow \tilde{X}$ which preserve the norm. i.e $\forall\, x \in X$

$$\|Tx\|_{\sim} = \|x\|$$

X is called isomorphic with $\tilde{X}$ and X, $\tilde{X}$ is called isomorphic normed spaces.

Given a space X to find the dual X' of X, we establish an isomorphism $T : X \longrightarrow X'$

Example:- Show that dual of R^n is R^n itself, i.e

$$(R^n)' = R^n.$$

Sol:—

We know that

$\mathbb{R}^n = \{x = (\xi_1, \dots, \xi_n) \mid \xi_i \in \mathbb{R}\}$ and

$\|x\| = \sqrt{\sum_1^n |\xi_i|^2}$

Now

$(\mathbb{R}^n)'$ = Set of all bounded linear functionals of $\mathbb{R}^n$.

= Set of all linear functional on $\mathbb{R}^n = (\mathbb{R}^n)^*$

(because in a finite dimensional norm space every linear operator is bounded)

So we require to prove that

$(\mathbb{R}^n)' = (\mathbb{R}^n)^* = \mathbb{R}^n$

Let $x \in \mathbb{R}^n$, $f \in (\mathbb{R}^n)'$, then $f \in (\mathbb{R}^n)^*$ and $x = \sum_1^n \xi_i e_i$, where $\{e_1, \dots, e_n\}$ is basis for $\mathbb{R}^n$. Then, since f is linear

Therefore $f(x) = \sum_1^n \xi_i f(e_i) = \sum_1^n \xi_i (\gamma_i) \to (1)$

where $\gamma_i = f(e_i)$

Consequently corresponding to every $f \in (\mathbb{R}^n)'$, we are getting $(\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$ and vice versa.

This establish a mapping $T: (\mathbb{R}^n)' \longrightarrow \mathbb{R}^n$ defined by

$f \longmapsto c = (\gamma_1, \dots, \gamma_n)$ obivously this mapping is linear and bijective. We need to show that T is norm preserving. From ①, we have

$$|f(x)| \le \sum_{1}^{n} |\xi_i||\gamma_i|$$

$$\underset{\substack{H.I \\ \text{For } p=2}}{\le} \underbrace{\left(\sum_{1}^{n} |\xi_i|^2\right)^{1/2}}_{\|x\|} \underbrace{\left(\sum_{1}^{n} |\gamma_i|^2\right)^{1/2}}_{\|c\|}$$

$$\Rightarrow \frac{|f(x)|}{\|x\|} \le \|c\| \quad , \quad \forall x \ne 0$$

$$\Rightarrow \sup_{\substack{x \in D(f) \\ x \ne 0}} \frac{|f(x)|}{\|x\|} \le \|c\|$$

$$\Rightarrow \|f\|_{(R^n)'} \le \|c\|_{R^n} \longrightarrow ②$$

On the other hand if $x = c = (\gamma_1, \ldots, \gamma_n)$ then

$$\|f\| \ge \frac{|f(c)|}{\|c\|} = \frac{\left|\sum_{1}^{n} \gamma_i f(e_i)\right|}{\|c\|}$$

$$= \frac{\left|\sum_{1}^{n} \gamma_i^2\right|}{\|c\|} = \frac{\left(\sqrt{\sum_{1}^{n} \gamma_i^2}\right)^2}{\|c\|} = \frac{\|c\|^2}{\|c\|}$$

i.e $\|f\|_{(R^n)'} \ge \|c\|_{R^n} \longrightarrow ③$

$$② \ \& \ ③ \Rightarrow \|f\|_{(R^n)'} = \|c\|_{R^n}$$

Hence dual of R^n is R^n

i.e $(R^n)' = R^n$

Note:- In similar pattern $(C^n)' = C^n$

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Example:-

Show that the dual space of l^1 is l^∞.

Proof:- Recall that

$$l^1 = \{x = (\xi_i)_1^\infty \mid \sum_1^\infty |\xi_i|^1 < \infty\}$$

with $\|x\| = \sum_1^\infty |\xi_i|$, $x \in l^1$

$$l^\infty = \{x = (\xi_i)_1^\infty \mid \sup_i |\xi_i| < \infty\}$$

with $\|x\| = \sup_i |\xi_i|$, $x \in l^\infty$

Let $e_1, e_2, \dots$ be Schauder Basis for l^1 where, $e_1 = (1,0,0,\dots)$, $e_2 = (0,1,0,\dots)$, $\dots$ $e_n = (0,0,\dots,1,0,\dots)$, $\dots$ (1 in nth position)

Then $x \in l^1$ can be represented by

$$x = \sum_1^\infty \xi_k e_k$$

Let $f \in (l^1)'$, then f is bounded linear functional on l^1. So

$$f(x) = \sum_1^\infty \xi_k f(e_k) = \sum_1^\infty \xi_k \gamma_k, \quad \gamma_k = f(e_k)$$

Where the numbers $\gamma_k = f(e_k)$ are uniquely determined by f. Also $\|e_k\| = 1$ and

$$|\gamma_k| = |f(e_k)| \leq \|f\| \|e_k\| = \|f\|$$

$$\Rightarrow \sup_k |\gamma_k| \leq \|f\| < \infty \longrightarrow (A)$$

Hence $(\gamma_K)_1^\infty \in l^\infty$

Also for every $b=(\beta_K)\in l^\infty$, we can obtain a corresponding bounded linear functional g on l^1. Infact we may define g on l^1 by

$$g(x) = \sum_1^\infty \xi_K \beta_K \text{, where } (\beta_K)\in l^\infty,$$
$$x=(\xi_K)\in l^1$$

★ <u>To show g is linear:-</u>

$$g(\alpha x+\beta y) = \sum_1^\infty (\alpha\xi_K + \beta\eta_K)\beta_K$$
$$= \alpha \underbrace{\sum_1^\infty \xi_K\beta_K}_{g(x)} + \beta \underbrace{\sum_1^\infty \eta_K\beta_K}_{g(y)}$$
$$= \alpha g(x) + \beta g(y)$$

★ <u>To show g is bounded:</u>

$$|g(x)| = \left|\sum_1^\infty \xi_K\beta_K\right| \le \sum_1^\infty |\xi_K\beta_K|$$
$$\le \underbrace{\sup_K |\beta_K|}_{K} \underbrace{\sum |\xi_K|}_{\|x\|}$$

i.e
$$|g(x)| \le K\|x\|$$

$\Rightarrow g$ is bounded, then $g\in(l^1)'$

Hence $\exists$ a bijection $T:(l^1)'\to l^\infty$

We finally show that

$$\|f\|_{(l^1)'} = \|\gamma\|_{l^\infty}, \quad \gamma=(\gamma_K)_1^\infty \in l^\infty$$

We have

$$|f(x)| = \left|\sum \xi_K\gamma_K\right| \le \sup_K |\gamma_K| \underbrace{\sum_1^\infty |\xi_K|}_{\|x\|}$$

$$\Rightarrow \frac{|f(x)|}{\|x\|} \leq \sup_k |\gamma_k|$$

$$\Rightarrow \sup_{\substack{x \in \ell^1 \\ x \neq 0}} \frac{|f(x)|}{\|x\|} \leq \sup_k |\gamma_k|$$

i.e $\|f\|_{(\ell^1)'} \leq \|\gamma\|_{\ell^\infty} \longrightarrow (1)$

Also we have from (A)

$$\sup_k |\gamma_k| \leq \|f\|$$

i.e $\|\gamma\|_{\ell^\infty} \leq \|f\|_{(\ell^1)'} \longrightarrow (2)$

(1) & (2) $\Rightarrow$

$$\|f\|_{(\ell^1)'} = \|\gamma\|_{\ell^\infty}$$

Hence $(\ell^1)' = \ell^\infty$

Examples:- ~~x~~ not included

The dual space of ℓ^p is ℓ^q here $1 < p < +\infty$ and q is the conjugate of p, i.e $\frac{1}{p} + \frac{1}{q} = 1$.

Sol:- Consider any $f \in (\ell^p)'$, where $(\ell^p)'$ is dual of ℓ^p. Since f is linear and bounded, therefore let $x \in \ell^p \Rightarrow x = \sum_{k=1}^{\infty} \xi_k e_k$

$$f(x) = \sum_1^\infty \xi_k f(e_k) = \sum_1^\infty \xi_k \gamma_k \longrightarrow (1)$$

where $\gamma_k = f(e_k)$

A Schauder basis for l^p is (e_k) .., $e_k = (\delta_{kj})$

Let q be conjugate of p and

Consider $x_n = (\xi_k^{(n)})$, where

$$\xi_k^{(n)} = \begin{cases} \frac{|\gamma_k|^q}{\gamma_k} & \text{if } k \le n \text{ \& } \gamma_k \ne 0 \\ 0 & \text{if } k > n \text{ or } \gamma_k = 0 \end{cases}$$

Since $(\xi_k^{(n)})$ has finitely many non-zero terms, Therefore $\sum_1^\infty |\xi_k^{(n)}|^p < \infty$ so that

$(\xi_k^{(n)}) \in l^p$

Now,

$$f(x_n) = f\left(\sum_{k=1}^\infty \xi_k^{(n)} e_k\right)$$

$$= \sum_{k=1}^\infty \xi_k^{(n)} \underbrace{f(e_k)}_{\gamma_k}$$

$$= \sum_{k=1}^n \frac{|\gamma_k|^q}{\cancel{\gamma_k}} \cdot \cancel{\gamma_k} = \sum_{k=1}^n |\gamma_k|^q$$

i.e $f(x_n) = \sum_{k=1}^n |\gamma_k|^q \longrightarrow (*)$

Since f is bounded linear functional

therefore $|f(x_n)| \le \|f\| \|x_n\|$

$$= \|f\| \left(\sum_{k=1}^\infty |\xi_k^{(n)}|^p\right)^{1/p} \quad \left(\because x_n \in l^p \quad (\sum |\xi_i|^p)^{1/p}\right)$$

$$= \|f\| \left(\sum_{k=1}^n \left(\frac{|\gamma_k|^q}{\gamma_k}\right)^p\right)^{1/p}$$

$$= \|f\| \left(\sum_{k=1}^n |\gamma_k|^{(q-1)p}\right)^{1/p}$$

$$= \|f\| \left(\sum |\gamma_k|^q\right)^{1/p} \quad \because (q-1)p = q$$

i.e

$$|f(x_n)| \le \|f\| \left(\sum_{k=1}^{n} |\gamma_k|^q\right)^{1/p}$$

From ⊛.

$$\left(\sum_{k=1}^{n} |\gamma_k|^q\right) = |f(x_n)| \le \|f\| \left(\sum_{k=1}^{n} |\gamma_k|^q\right)^{1/p}$$

$$\Rightarrow \left(\sum_{k=1}^{n} |\gamma_k|^q\right)^{1-1/p} \le \|f\|$$

$$\Rightarrow \left(\sum_{k=1}^{n} |\gamma_k|^q\right)^{1/q} \le \|f\| < \infty \longrightarrow (**)$$

Since n is arbitrary, letting $n \to \infty$, we obtain $\sum_{k=1}^{\infty} |\gamma_k|^q < \infty$

$$\Rightarrow (\gamma_k)_1^{\infty} \in l^q$$

Conversely:- For any $b = (\beta_k) \in l^q$, we can get a corresponding bounded linear functional g on l^p. In fact we can define g on l^p by letting,

$$g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k \quad \text{where } \beta_k \in l^q \text{ \&}$$

$x = (\xi_k) \in l^p$.

We have define a mapping

$$T : (l^p)' \longrightarrow l^q$$

by

$$g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k$$

★ <u>To show g is linear:</u>

$$g(\alpha x_1 + \beta x_2) = \sum_{k=1}^{\infty} \left(\alpha \xi_k^{(1)} + \beta \xi_k^{(2)}\right)\beta_k$$

$$= \alpha \underbrace{\sum_{k=1}^{\infty} \xi_k^{(1)} \beta_k}_{g(x_1)} + \beta \underbrace{\sum_{1}^{\infty} \xi_k^{(2)} \beta_k}_{g(x_2)}$$

$$= \alpha\, g(x_1) + \beta\, g(x_2)$$

<u>To show g is bounded:</u>

$$|g(x)| = \left|\sum_{k=1}^{\infty} \xi_k \beta_k\right|$$

$$\leq \sum_{k=1}^{\infty} |\xi_k \beta_k|$$

$$\Rightarrow |g(x)| \leq \sup_k |\beta_k| \sum_{1}^{\infty} |\xi_k|$$

i.e $|g(x)| \leq K \|x\|$

$\Rightarrow g$ is bounded.

Hence $g \in (l^p)'$

We finally prove that

$$\|f\|_{(l^p)'} = \|\gamma_k\|_{l^q}, \quad \gamma_k \in l^q$$

From ① we have,

$$|f(x)| = \left|\sum_{1}^{\infty} \xi_k \gamma_k\right| \leq \sum_{1}^{\infty} |\xi_k| |\gamma_k|$$

$$\underset{H.I}{\leq} \left(\sum_{1}^{\infty} |\xi_k|^p\right)^{1/p} \left(\sum_{1}^{\infty} |\gamma_k|^q\right)^{1/q}$$

$$\Rightarrow \quad |f(x)| \le \|x\| \left(\sum_{1}^{\infty} |\gamma_K|^q\right)^{1/q}$$

$$\sup_{\substack{x \in D(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|} \le \left(\sum_{1}^{\infty} |\gamma_K|^q\right)^{1/q}$$

$$\|f\| \le \left(\sum_{1}^{\infty} |\gamma_K|^q\right)^{1/q} \longrightarrow (i)$$

From (**)

$$\left(\sum_{1}^{\infty} |\gamma_K|^q\right)^{1/q} \le \|f\| \longrightarrow (ii)$$

(i) & (ii) $\Rightarrow$

$$\|f\| = \left(\sum_{1}^{\infty} |\gamma_K|^q\right)^{1/q}$$

i.e $\|f\|_{(l^p)'} = \|\gamma_K\|_{l^q}$

Hence $(l^p)' = l^q$. Proved.

Chap # 3: Inner Product Spaces - 16-5-2014

The concept of dot product and orthogonality can be generalized to arbitrary vector spaces which leads to inner product spaces. A complete inner product space is Hilbert space.

Definition:—

An inner Product space (or Pre-Hilbert space) is a vector space X together with an inner product defined on it. A Hilbert space is a complete inner Product Space (complete in the metric induced by inner Product)

Inner Product

An inner product on a vector Space $(X, +, \cdot)$ is a mapping from $X \times X$ into a scalar field K of X; that is, with every pair of vectors x & y there is associated a scalar which is written as;

$\langle x, y \rangle$ and is called the inner product of x and y such that $\forall\, x, y, z \in X$ & scalar α, we have

IP 1):— $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

IP2):– $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$

IP3):– $\langle x, y\rangle = \overline{\langle y, x\rangle}$

IP4):– $\langle x, x\rangle \geq 0$

$\langle x, x\rangle = 0 \Leftrightarrow x = 0$

$\Rightarrow$ The only vector which is $\perp$ to itself is 0.

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An inner product defines a norm on vector space X by

$$\|x\| = \sqrt{\langle x, x\rangle}$$

and metric d on X is given by

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}$$

Note:- Inner Product space is also a norm space But converse is not true in general.

Hence,

"Inner product spaces are normed spaces & Hilbert spaces are Banach spaces".

Definition: Orthogonality

An element x of an inner product space X is said to be orthogonal to an element $y \in X$ if

$$\langle x, y\rangle = 0$$

We also say that x & y are orthogonal, and we write $x \perp y$

Similarly, $A \perp B$ if $a \perp b$, $\forall\, a \in A$ and $b \in B$.

Result:- Every inner product space satisfies the parallelogram equality.

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

Proof:-

$$L.H.S = \|x+y\|^2 + \|x-y\|^2$$
$$= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$$
$$= \langle x, x+y\rangle + \langle y, x+y\rangle + \langle x, x-y\rangle - \langle y, x-y\rangle$$
$$= \langle x, x\rangle + \cancel{\langle x, y\rangle} + \cancel{\langle y, x\rangle} + \langle y, y\rangle + \langle x, x\rangle - \cancel{\langle x, y\rangle} - \cancel{\langle y, x\rangle} + \langle y, y\rangle$$
$$= 2\{\langle x, x\rangle + \langle y, y\rangle\}$$
$$= 2\{\|x\|^2 + \|y\|^2\}$$
$$= R.H.S$$

Example:- Consider $X = C[a,b]$ with $\|x\| = \max\limits_{t\in[a,b]} |x(t)|$. We know that

$(X, \|\cdot\|)$ is a normed space. We claim that $C[a,b]$ is not an inner product space.

Sol:- Choose $x(t) = 1$ & $y(t) = \frac{t-a}{b-a}$
$$= \frac{1}{b-a}(t-a) \in C[a,b]$$

Note that
$$y' = \frac{1}{b-a} > 0$$ then y is strictly increasing.

$$x+y = 1 + \frac{t-a}{b-a}$$
$$x-y = 1 - \frac{t-a}{b-a}$$

$$\|x\| = \max_{t\in[a,b]} 1 = 1$$

$$\|y\| = \max_{t\in[a,b]} |y(t)|$$

$$= \max_{t\in[a,b]} \left|\frac{t-a}{b-a}\right| = 1 \quad \left(\text{when } t=b \quad \frac{b-a}{b-a} = 1\right)$$

$$\|x+y\| = \max_{t\in[a,b]} \left|1 + \frac{t-a}{b-a}\right| = 2 \quad (\text{when } t=b)$$

$$\|x-y\| = \max_{t\in[a,b]} \left|1 - \frac{t-a}{b-a}\right| = 1 \quad (\text{when } t=a)$$

$$L.H.S = \|x+y\|^2 + \|x-y\|^2$$
$$= (2)^2 + (1)^2 = 5$$

But $R.H.S = 2(\|x\|^2 + \|y\|^2)$
$$= 2((1)^2 + (1)^2) = 4$$

As $L.H.S \neq R.H.S$

So, this $(C[a,b])$ is not an inner product space. In other words an inner product can not be defined on $C[a,b]$. Also, $C[a,b]$ is a Banach space but not a Hilbert space.

Example:- Consider $X = \ell^p$, $1 < p < \infty$ with $\|x\| = \left(\sum_{1}^{\infty} |\xi_i|^p\right)^{1/p}$ where $x = (\xi_i)_1^{\infty} \in \ell^p$, we know that $(\ell^p, \|\cdot\|)$ is a Banach (norm) space.

Sol:- Choose,

$$x = (1,1,0,0,\dots) \ \& \ y = (1,-1,0,0,\dots) \in \ell^p$$

$$x+y = (2,0,0,\dots) \ \& \ (x-y) = (0,2,0,0,\dots) \in \ell^p$$

P-Identity.

$$\|x\| = (|1|^p + |1|^p)^{1/p} = 2^{1/p}$$

$$\|y\| = (|1|^p + |-1|^p)^{1/p} = 2^{1/p}$$

$$\|x+y\| = (|2|^p)^{1/p} = 2$$

$$\|x-y\| = (|2|^p)^{1/p} = 2$$

$$L.H.S = \|x+y\|^2 + \|x-y\|^2 = (2)^2 + (2)^2 = 8$$

$$R.H.S = 2(\|x\|^2 + \|y\|^2) = 2\left[(2^{1/p})^2 + (2^{1/p})^2\right]$$

$$= 2(2^{2/p} + 2^{2/p})$$

$$= 4(2^{2/p})$$

So, $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

iff $p = 2$.

$\Rightarrow \ell^2$ is an inner product space but ℓ^p with $p \neq 2$ is not an I.P.S.

Example:- Show ℓ^2 with $\langle x,y\rangle = \sum_{1}^{\infty} \xi_i \bar{\eta}_i$, where $x = (\xi_i)_1^\infty$, $y = (\eta_i)_1^\infty \in \ell^2$ is an I.P.S.

Sol:- (i).

$$\langle x_1 + x_2, y\rangle = \sum_1^\infty (\xi_i^{(1)} + \xi_i^{(2)})\bar{\eta}_i$$

where $x_1 = (\xi_i^{(1)})_i^\infty$, $x_2 = (\xi_i^{(2)})_i^\infty$

$$= \sum_1^\infty \xi_i^{(1)}\bar{\eta}_i + \sum_1^\infty \xi_i^{(2)}\bar{\eta}_i$$

$$= \langle x_1, y\rangle + \langle x_2, y\rangle$$

(ii):- $\langle \alpha x, y\rangle = \sum_1^\infty \alpha\, \xi_i \bar{\eta}_i$

$$= \alpha \sum_1^\infty \xi_i \bar{\eta}_i$$

$$= \alpha \langle x, y\rangle$$

(iii):- $\langle x, y\rangle = \sum_1^\infty \xi_i \bar{\eta}_i \qquad \bar{\bar{\eta}} = \eta$

$$= \overline{\left(\sum_1^\infty \bar{\xi}_i \eta_i\right)}$$

$$= \overline{\langle y, x\rangle}$$

(iv):- $\langle x, x\rangle = \sum_1^\infty \xi_i \bar{\xi}_i = \sum_1^\infty |\xi_i|^2 \geq 0$

& $\langle x, x\rangle = \sum_1^\infty \xi_i \bar{\xi}_i = 0 \Leftrightarrow \xi_i = 0 \ \forall i$

$$\Leftrightarrow x = 0$$

(Self) Example:- $\mathbb{R}^n$ or $\mathbb{C}^n$ with inner product defined by $\langle x, y\rangle = \sum_1^n \xi_i \bar{\eta}_i$

where $x = \{\xi_1, \xi_2, \ldots, \xi_n\}$,

$y = \{\eta_1, \eta_2, \ldots, \eta_n\} \in \mathbb{R}^n$ or $\mathbb{C}^n$

is an inner product space.

Self

Example:-

$$C^2[a,b] = \{x \mid \int_a^b x^2(t)\,dt < \infty\}$$

with inner product defined by

$$\langle x, y\rangle = \int_a^b x(t)\,\overline{y(t)}\,dt$$

is an inner product space

Lemma:- 22-5-14

An inner product and the corresponding norm satisfy the schwartz inequality and the triangle inequality as follows;

a):- We have

$$|\langle x, y\rangle| \le \|x\|\,\|y\| \quad (S.I) \longrightarrow ①$$

Where the equality sign holds iff $\{x, y\}$ is linearly dependent.

b):- The norm satisfies

$$\|x+y\| \le \|x\| + \|y\| \longrightarrow ②$$

(Triangle inequality)

Where the equality holds if $y=0$ or $x = cy$ (c is real & ≥ 0)

Proof:- a):- If $y=0$ then ① holds trivially. Since

$$\langle x, 0\rangle = \langle x, 0.y\rangle = 0\langle x, y\rangle = 0$$

Let $y \ne 0$, then for every scalar α.

we have;

$$0 \le \|x-\alpha y\|^2 = \langle x-\alpha y, x-\alpha y\rangle$$

$$= \langle x, x\rangle - \bar{\alpha}\langle x, y\rangle - \alpha\langle y, x\rangle + \alpha\bar{\alpha}\langle y, y\rangle$$

$$= \langle x, x\rangle - \bar{\alpha}\langle x, y\rangle - \alpha\left[\langle y, x\rangle - \bar{\alpha}\langle y, y\rangle\right]$$

Since α is arbitrary, we choose α such that

$$\langle y, x\rangle - \bar{\alpha}\langle y, y\rangle = 0$$

i.e $\bar{\alpha} = \dfrac{\langle y, x\rangle}{\langle y, y\rangle}$ $\quad (\because y \neq 0)$

$$\therefore 0 \le \langle x, x\rangle - \bar{\alpha}\langle x, y\rangle$$

$$= \langle x, x\rangle - \frac{\langle y, x\rangle}{\langle y, y\rangle} \cdot \langle x, y\rangle$$

$$= \|x\|^2 - \frac{\overline{\langle x, y\rangle} \cdot \langle x, y\rangle}{\|y\|^2}$$

$$= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}$$

i.e $\dfrac{|\langle x, y\rangle|^2}{\|y\|^2} \le \|x\|^2$

$$|\langle x, y\rangle|^2 \le \|x\|^2 \|y\|^2$$

$$|\langle x, y\rangle| \le \|x\| \, \|y\|$$

In above discussion equality holds

$\boxed{y=0}$

iff $0 = \|x-\alpha y\|^2 \Rightarrow \|x-\alpha y\| = 0$

$\Rightarrow x - \alpha y = 0$

$\Rightarrow x = \alpha y$ which means that $\{x, y\}$ is L.D.

not include

(b):- $\|x+y\|^2 = \langle x+y, x+y\rangle$

$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$

$\underset{S.I}{\leq} \|x\|^2 + \|x\|\,\|y\| + \|y\|\,\|x\| + \|y\|^2$

$= \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$ | Always $x \leq |x|$

$= (\|x\| + \|y\|)^2$

i.e $\|x+y\| \leq \|x\| + \|y\|$

Equality holds:- iff

$\langle x, y\rangle + \langle y, x\rangle = 2\|x\|\,\|y\|$

$\Rightarrow \langle x, y\rangle + \overline{\langle x, y\rangle} = 2\|x\|\,\|y\|$

$\Rightarrow \cancel{2}\,\mathrm{Re}\langle x, y\rangle = \cancel{2}\|x\|\,\|y\|$ | $z + \bar{z} = 2\,\mathrm{Re}(z)$

$\Rightarrow \mathrm{Re}\langle x, y\rangle = \|x\|\,\|y\| \geq |\langle x, y\rangle|$

But $\mathrm{Re}\langle x, y\rangle \leq |\langle x, y\rangle|$

So, $\mathrm{Re}\langle x, y\rangle = |\langle x, y\rangle| \longrightarrow *$

$\|x\|\,\|y\| = |\langle x, y\rangle| \iff$ either $y = 0$ or $x = cy$ (by (a))

From (*) $Re\langle x,y\rangle = |\langle x,y\rangle|$, therefore

$Im\langle x,y\rangle = 0$ So that

$|\langle x,y\rangle| = Re\langle x,y\rangle \geq 0$

So,

$\langle cy, y\rangle \geq 0$

$\Rightarrow c\langle y,y\rangle \geq 0$

$\Rightarrow c\|y\|^2 \geq 0$

$\Rightarrow c \geq 0$

Lemma:- If in an inner product space X, $x_n \longrightarrow x$ & $y_n \longrightarrow y$ then

$\langle x_n, y_n\rangle \longrightarrow \langle x,y\rangle$

Proof:-

$|\langle x_n,y_n\rangle - \langle x,y\rangle| = |\langle x_n,y_n\rangle - \langle x_n,y\rangle + \langle x_n,y\rangle - \langle x,y\rangle|$

$= |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle|$

$\underset{T.I}{\leq} |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|$

$\underset{S.I}{\leq} \|x_n\|\,\|y_n - y\| + \|x_n - x\|\,\|y\| \longrightarrow 0$ as $n \longrightarrow \infty$

i.e $|\langle x_n,y_n\rangle - \langle x,y\rangle| \to 0$ as $n \to \infty$ $\quad (\because x_n \longrightarrow x$ & $y_n \longrightarrow y)$

$\Rightarrow \langle x_n, y_n\rangle \longrightarrow \langle x,y\rangle$

(Orthogonality) 26-5-14

earlier written = Page 141

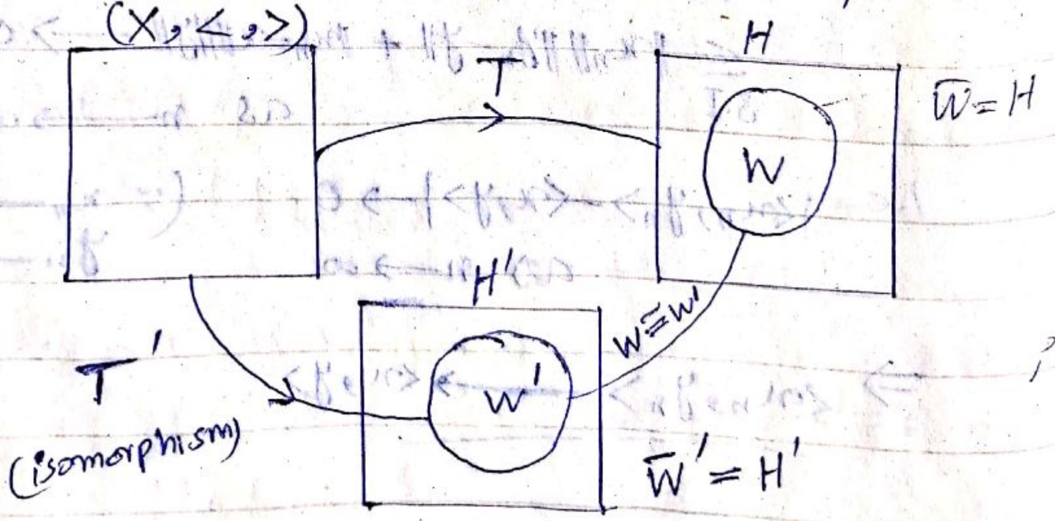
Def: Isomorphism

An isomorphism T of inner product space $(X, \langle\cdot,\cdot\rangle)$ onto an inner product space $(\tilde{X}, \langle\cdot,\cdot\rangle)$ over the same field K is a bijective linear operator $T: X \longrightarrow \tilde{X}$ which preserves the corresponding inner products,

$$\langle x, y\rangle = \langle Tx, Ty\rangle \quad \forall\ x, y \in X$$

Theorem: (Completion Theorem)

For any inner product space X $\exists$ a hilbert space H & an isomorphism T from X onto a dense subspace $W \subset H$. Then space H is unique except for isomorphism



Def:-

A subspace Y of an inner product space X is defined to be a vector subspace of X with the inner product on X restricted to $Y \times Y$.

Similarly, hilbert space H is defined to be subspace of H regarded a inner product space. Note that Y need not be a hilbert space.

★ The following result follows immediatly by perivious results.

Theorem:-

Let Y be a subspace of a hilbert space H. Then,

(a):- Y is complete iff Y is closed in H.

b):- If Y is finite dimensional then Y is complete.

C):- If H is seperable, so is Y.

i.e In general every subset of a separable inner product space is separable.

Existance & Uniqueness Problem.

Consider existance and uniqueness problem for hilbert space and formulate key theorem, we need two related concepts which are of general interest.

Book

The ~~Segment~~ joining two given elements x and y of a vector space X is defined to be the set of all $z \in X$ of the form

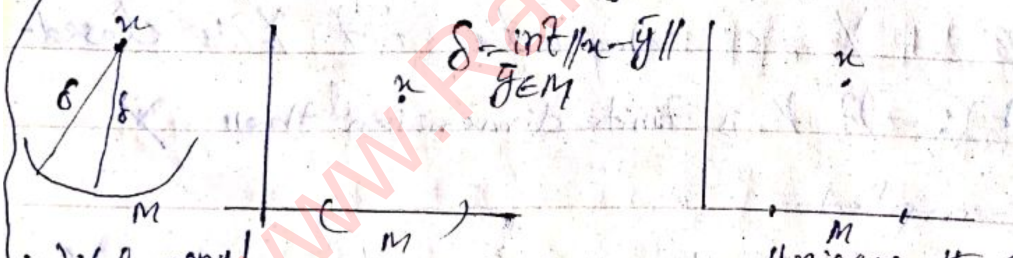
$$z = \alpha x + (1-\alpha) y \qquad (\alpha \in \mathbb{R}, 0 \le \alpha \le 1)$$

A subset M of X is said to be Convex if for every $x, y \in M$ the segment joining x & y is contained in M.

For instance, every subspace Y of X is convex, and the intersection of convex sets is a convex set.

i.e $\forall\ x, y \in M$

$$\alpha x + (1-\alpha) y \in M \ , \text{ where } \alpha \in [0,1]$$



infitely many y's are available. | no y available | Unique y exist

Let $\{C_i\}$ be a collection of convex sets. Let $x, y \in \bigcap_{i=1}^{\infty} C_i$

$\Rightarrow x, y \in C_i \ \forall i$

$\Rightarrow \alpha x + (1-\alpha) y \in C_i, \ \forall i$ $\quad$ ($\because C_i$ is convex $\forall i$)
$\alpha \in [0,1]$

$\Rightarrow \alpha x + (1-\alpha) y \in \bigcap_{i=1}^{\infty} C_i$

Theorem:- Minimizing Vector. 29-5-14

Let X be an inner product space & $M \neq \phi$ a convex subset which is complete (in the metric induced by inner product). Then for every $x \in X$ $\exists$ a $\boxed{\text{unique}}$ $y \in M$ s.t

$$\delta = \inf_{\tilde{y} \in M} \|x - \tilde{y}\| = \|x - y\|$$

Proof:- a:- Existance of y

We have

$$\delta = \inf_{\tilde{y} \in M} \|x - \tilde{y}\|$$

By definition of infimum, $\exists$ a seq. (y_n) in M such that $\delta_n \to \delta$.

where $\delta_n = \|x - y_n\| \longrightarrow (1)$

We show that y_n is a cauchy sequence in M. Let $v_n = x - y_n$ then $\|v_n\| = \|x - y_n\| = \delta_n$

and $\|v_n + v_m\| = \|(x - y_n) + (x - y_m)\|$

$$= \|2x - (y_n + y_m)\|$$

$$= 2\|x - \tfrac{1}{2}(y_n + (1 - \tfrac{1}{2}) y_m)\|$$

$\in M$

$$\geq 2\delta$$

(By definition of inf) ($\because$ M is convex)

i.e

$$\|v_n + v_m\| \geq 2\delta \longrightarrow (2)$$

$$\|y_n - y_m\|^2 = \|x - v_n - (x - v_m)\|^2$$
$$= \|v_m - v_n\|^2$$
$$= -\|v_m + v_n\|^2 + 2\left(\|v_m\|^2 + \|v_n\|^2\right)$$
(By parallelogram law)
$$\leq -4\delta^2 + 2\left(\delta_m^2 + \delta_n^2\right) \Rightarrow$$
$$-4\delta^2 + 2(\delta^2 + \delta^2) = 0$$
as $m, n \to \infty$

$\Rightarrow (y_n)$ is a c. sequence in M. Since M is complete. $\exists$ $y \in M$ such that $y_n \to y$ since $y \in M$, $\|x - y\| \geq \delta \to (*)$

Now
$$\|x - y\| \underset{T.I}{\leq} \|x - y_n\| + \|y_n - y\|$$
$$= \delta_n + \|y_n - y\| \to \delta + 0$$
as $y_n \to y$ as $n \to \infty$

i.e
$$\|x - y\| \leq \delta \to (**)$$

$(*)$ & $(**)$ $\Rightarrow$
$$\delta = \|x - y\|$$

b):- Uniqueness

Let $y_1, y_2 \in M$ s.t
$$\delta = \|x - y_1\| = \|x - y_2\|$$
$$\|y_1 - y_2\|^2 = \|y_1 - x + x - y_2\|^2$$
$$= \|(y_1 - x) - (y_2 - x)\|^2$$

$$= -\|(y_1 - x) + (y_2 - x)\|^2 + 2(\|y_1 - x\|^2 + \|y_2 - x\|^2)$$

(By parallelogram law)

$$= -\|-2x + (y_1 + y_2)\|^2 + 2(\|y_1 - x\|^2 + \|y_2 - x\|^2)$$

$$= -4\|x - \underbrace{(\tfrac{1}{2} y_1 + (1 - \tfrac{1}{2}) y_2)}_{\in M}\| + 2(\underbrace{\|y_1 - x\|^2}_{\delta^2} + \underbrace{\|y_2 - x\|^2}_{\delta^2})$$

(∵ M is convex)

$$\le -4\delta^2 + 2(\delta^2 + \delta^2) = 0$$

$\Rightarrow \|y_1 - y_2\|^2 = 0$ (square never be negative) $\|y_1 - y_2\|^2 \nless 0$

$\Rightarrow \|y_1 - y_2\| = 0$

$\Rightarrow y_1 = y_2$

Theorem:— Let $(X, \langle , \rangle)$ be an inner product space and let Y be a complete subspace of X & $x \in X$ a fixed point then $z = x - y$ is orthogonal to Y.

Proof:— Suppose $z \perp Y$ is not true. Then $\exists\, y_1 \in Y$ such that $\langle z, y_1 \rangle = \beta \neq 0$

Now for any scalar,

Consider $\|z - \alpha y_1\|^2 = \langle z - \alpha y_1, z - \alpha y_1 \rangle$

$= \langle z, z - \alpha y_1 \rangle - \alpha \langle y_1, z - \alpha y_1 \rangle$

$= \langle z,z\rangle - \bar{\alpha}\langle z, y_1\rangle - \alpha\langle y_1, z\rangle + \alpha\bar{\alpha}\langle y_1, y_1\rangle$

$= \langle z,z\rangle - \alpha\langle y_1, z\rangle + \bar{\alpha}[\alpha\langle y_1, y_1\rangle - \langle z, y_1\rangle]$

Since α is arbitrary, we may choose α s.t

$$\alpha\langle y_1, y_1\rangle - \langle z, y_1\rangle = 0$$

i.e $\alpha = \dfrac{\langle z, y_1\rangle}{\langle y_1, y_1\rangle}$

$\Rightarrow \|z - \alpha y_1\|^2 = \langle z,z\rangle - \dfrac{\langle z, y_1\rangle\langle y_1, z\rangle}{\langle y_1, y_1\rangle}$

$= \|z\|^2 - \dfrac{\langle z, y_1\rangle\overline{\langle z, y_1\rangle}}{\|y_1\|^2}$

$= \|z\|^2 - \dfrac{|\langle z, y_1\rangle|^2}{\|y_1\|^2}$

$= \|z\|^2 - \dfrac{\beta^2}{\|y_1\|^2} < \|z\|^2$

But $\|z\| = \|x - y\| = \delta$ so that

$\|z - \alpha y_1\|^2 < \delta^2 \longrightarrow (1)$

We claim that (1) can not happens

$\|z - \alpha y_1\| = \|x - y - \alpha y_1\| = \|x - \underbrace{(y + \alpha y_1)}_{y_2}\|$

$\because Y$ is subspace of M

$= \|x - y_2\| \geq \delta$

$\Rightarrow \|z - \alpha y_1\|^2 \geq \delta^2$ which contradicts (1)

Hence $z \perp Y$.

Direct Sum

30-5-14

A vector space X is said to be direct sum of two subspaces Y & Z of X: written $X = Y \oplus Z$ if each $x \in X$ has a unique representation

$$x = y + z \quad , \text{where } y \in Y,\ z \in Z$$

Z is called algebraic complement of Y & vice versa and Y, Z is called a complementry pair of subspaces in X.

Def: Orthogonal complement

Let X be an inner product space and Y a subspace of X then orthogonal complement of Y is defined as

$$Y^{\perp} = \{z \in X \mid z \perp Y\}$$

$$\langle z, y \rangle = 0 \ , \forall\, y \in Y$$

(Not Proof)

Theorem:- Let Y be any closed subspace of a Hilbert space H. Then

$$H = Y \oplus Z$$

Recall:- A sequence (e_n) in an inner product space X is said to be orthonormal if

$$\langle e_n, e_m \rangle = \begin{cases} 0 \ , & m \neq n \\ 1 \ , & m = n \end{cases}$$

Riesz's Theorem:- (Functionals on HS)

Every bounded linear functional f on a hilbert space H can be represented ~~as~~ in terms of inner product by

$$f(x) = \langle x, z \rangle \longrightarrow (1)$$

where z which depends upon f, is uniquely determined by f. Moreover

$$\|f\| = \|z\| \longrightarrow (2)$$

(z is a vector)

Proof:- We have to prove that

a):- f has representation (1).

b):- z in (1) is unique

c):- (2) holds.

a):⇒ If $f=0$, then (1) & (2) holds, if we choose $z=0$ (Trivial case).

Let $f \neq 0$. To motivate the idea of proof, we investigate that what properties z must have if a representation (1) exist. First of all $z \neq 0$ since otherwise $f=0$.

Second if for some x,

$$f(x) = \langle x, z \rangle = 0$$

then $x \in N(f)$ & $z \perp N(f)$ i.e $z \in N(f)^{\perp}$

We know that $N(f)$ is a vector

see page = 101

(self) space and is closed. So, we have

$$H = N(f) \oplus N(f)^{\perp} \quad (\because \text{ If } Y \text{ is closed subspace of a H.S } (H) \text{ then } H = Y \oplus Y^{\perp})$$

Note that

$N(f) \neq H$ since $f \neq 0$ also

$N(f)^{\perp} \neq \{0\}$ since otherwise $N(f) \equiv H$

Choose $0 \neq z_0 \in N(f)^{\perp}$ and we set

$$v = f(x)\, z_0 - f(z_0)\, x \quad , \; x \in H$$

$$\Rightarrow f(v) = f(x) f(z_0) - f(z_0) f(x) = 0$$

$$\Rightarrow v \in N(f)$$

Note that $\langle z_0, z_0 \rangle = \|z_0\|^2 \neq 0$

Now since $z_0 \perp N(f)$ & $v \in N(f)$

therefore

$$0 = \langle v, z_0 \rangle$$

$$= \langle f(x)\, z_0 - f(z_0)\, x ,\, z_0 \rangle$$

$$= f(x) \langle z_0, z_0 \rangle - f(z_0) \langle x, z_0 \rangle$$

$$\Rightarrow f(x) = \frac{f(z_0)}{\langle z_0, z_0 \rangle} \langle x, z_0 \rangle \quad (\because \langle z_0, z_0 \rangle \neq 0)$$

$$= \left\langle x, \frac{\overline{f(z_0)}}{\langle z_0, z_0 \rangle} z_0 \right\rangle$$

$$= \langle x, z \rangle \quad \text{where } z = \frac{\overline{f(z_0)}}{(z_0, z_0)} z_0$$

(B) Since $x \in H$ was arbitrary, (1) is proved.

b):– To show that z is unique-

Suppose that $\forall x \in H$, we have

$$f(x) = \langle x, z_1 \rangle = \langle x, z_2 \rangle$$

$$\Rightarrow \langle x, z_1 \rangle - \langle x, z_2 \rangle = 0, \quad \forall x \in H$$

$$\Rightarrow \langle x, z_1 - z_2 \rangle = 0 \quad \forall x \in H$$

choose $x = z_1 - z_2$, So that

$$\langle z_1 - z_2, z_1 - z_2 \rangle = 0$$

$$\Rightarrow \|z_1 - z_2\|^2 = 0$$

$$\Rightarrow z_1 = z_2 = z \quad \text{(the uniqueness)}$$

C):– To show that $\|f\| = \|z\| \longrightarrow (2)$

(Trivially) If $f = 0$ then $z = 0$ & (2) holds.

Let $f \neq 0$ then $z \neq 0$. Then from (1) with $x = z$ we have

$$f(z) = \langle z, z \rangle \quad \text{i.e}$$

$$\|z\|^2 = f(z) \leq \|f\| \, \|z\|$$

$$\|z\| \leq \|f\| \longrightarrow (3) \quad (\because \|z\| \neq 0)$$

Now $|f(x)| = |\langle x, z \rangle| \underset{S.I}{\leq} \|x\| \|z\|$

$$\Rightarrow \frac{|f(x)|}{\|x\|} \leq \|z\|$$

$$\sup_{\substack{x \in D(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|} \leq \|z\|$$

$$\|z\| \le \|y\| \longrightarrow (4)$$

(3) & (4) $\Rightarrow$

$$\|z\| = \|y\|$$

Lemma:- (Equality)

If $\langle v_1, w\rangle = \langle v_2, w\rangle \ \forall\ w$ in an inner product space X. Then $v_1 = v_2$. In particular

$$\langle v_1, w\rangle = 0 \ \forall\ w \in X \Rightarrow v_1 = 0$$

Proof:- We have

$$\langle v_1, w\rangle = \langle v_2, w\rangle \quad \forall\ w \in X$$

$$\Rightarrow \langle v_1 - v_2, w\rangle = 0$$

For $w = v_1 - v_2$, we have

$$\langle v_1 - v_2, v_1 - v_2\rangle = 0 \Rightarrow \|v_1 - v_2\|^2 = 0$$

$$\Rightarrow v_1 = v_2$$

If $\langle v_1, w\rangle = 0, \ \forall\ w \in X$ then for $w = v_1$, we have

$$\langle v_1, v_1\rangle = 0$$

$$\Rightarrow \|v_1\|^2 = 0$$

$$\Rightarrow \|v_1\| = 0$$

$$\Rightarrow \boxed{v_1 = 0}$$

Def:- (Sesquilinear Form)

Let X & Y be vector spaces over the same field K ($= \mathbb{R}$ or $\mathbb{C}$). Then a sesquilinear form (or functional) h

on $X \times Y$ is a mapping

$$h : X \times Y \longrightarrow K$$

such that for all $x, x_1, x_2 \in X$ & $y, y_1, y_2 \in Y$ and all scalars α, β, we have

a):- $h(x_1 + x_2, y) = h(x_1, y) + h(x_2, y)$

b):- $h(x, y_1 + y_2) = h(x, y_1) + h(x, y_2)$

c):- $h(\alpha x, y) = \alpha h(x, y)$

d):- $h(x, \beta y) = \bar{\beta} h(x, y)$

(B) Hence h is linear in the first argument and conjugate linear in the second one. If X and Y are real ($K = \mathbb{R}$), then (d) is simply $h(x, \beta y) = \beta h(x, y)$ and h is called bilinear since it is linear in both arguments.

Def:- (norm of h)

If X & Y are normed spaces and if $\exists$ a positive real number C such that, $\forall\ x, y$

$$|h(x, y)| \leq C \|x\| \|y\|$$

then h is said be bounded and the number

$$\|h\| = \sup_{\substack{x \in X - \{0\} \\ y \in Y - \{0\}}} \frac{|h(x, y)|}{\|x\| \|y\|} = \sup_{\substack{\|x\| = 1 \\ \|y\| = 1}} |h(x, y)|$$

2-6-14

is called the norm of h. and we have

$$|h(x,y)| \le \|h\| \, \|x\| \, \|y\|$$

Theorem:- (Riesz Representation)

Let H_1 & H_2 be hilbert spaces and $h: H_1 \times H_2 \longrightarrow K$ a bounded sesquilinear form. Then h has representation

$$\boxed{h(x,y) = \langle Sx, y \rangle} \longrightarrow ①$$

where $S: H_1 \longrightarrow H_2$ is $\boxed{\text{bounded}}$ and $\boxed{\text{linear}}$ operator. S is $\boxed{\text{uniquely}}$ determind by h and has norm,

$$\boxed{\|S\| = \|h\|}$$

Proof:- Consider $\overline{h(x,y)}$ for $x \in H_1$ fixed. This is linear in y because of the bar.

Then by Riesz representation theorem for functional, for fixed $x \in H_1$,

$$f(y) = \overline{h(x,y)} = \langle y, z \rangle \quad \because f(x) = \langle x, z \rangle$$

where z is uniquely determind by f with $\|z\| = \|f\|$ then we have,

$$h(x,y) = \langle z, y \rangle \longrightarrow ②$$

where $z \in H_2$ is unique and depends upon our fixed $x \in H_1$. It follows that ② with variable x defines an operator $S : H_1 \to H_2$ by $Sx = z$.

Substituting $z = Sx$ in ② we have

$h(x, y) = \langle Sx, y \rangle$ which is ①.

To show S is linear.

From ① we have

$$\langle S(\alpha x_1 + \beta x_2), y \rangle = h(\alpha x_1 + \beta x_2, y)$$

$$= \alpha h(x_1, y) + \beta h(x_2, y)$$

$$= \alpha \langle Sx_1, y \rangle + \beta \langle Sx_2, y \rangle$$

$$= \langle \alpha Sx_1 + \beta Sx_2, y \rangle$$

We know that

$$\langle v_1, w \rangle = \langle v_2, w \rangle, \ \forall w$$
$$\Rightarrow v_1 = v_2$$

So, $S(\alpha x_1 + \beta x_2) = \alpha Sx_1 + \beta Sx_2$

$\Rightarrow S$ is linear.

To show that S is bounded.

If $S = 0$, it is bounded. (Trivially)

If $S \neq 0$, then

$$\|h\| = \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|h(x, y)|}{\|x\| \|y\|} = \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|\langle Sx, y \rangle|}{\|x\| \|y\|}$$

By fixing y.

$$\geq \sup_{x\neq 0} \frac{|\langle Sx, Sx\rangle|}{\|x\| \, \|Sx\|}$$

$$= \sup_{x\neq 0} \frac{\|Sx\|^2}{\|x\|\|Sx\|} = \sup_{x\neq 0} \frac{\|Sx\|}{\|x\|}$$

$$= \|S\|$$

i.e $\|S\| \leq \|h\| < \infty$

$\Rightarrow$ S is bounded.

To show that $\|S\| = \|h\|$

$$\|S\| \leq \|h\| \longrightarrow (a)$$

We have

$$\|h\| = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|h(x,y)|}{\|x\| \, \|y\|} = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|\langle Sx, y\rangle|}{\|x\| \, \|y\|}$$

$$\underset{S.I}{\leq} \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{\|Sx\| \, \|y\|}{\|x\| \, \|y\|}$$

$$= \sup_{x\neq 0} \frac{\|Sx\|}{\|x\|}$$

$$= \|S\|$$

i.e $\|h\| \leq \|S\| \longrightarrow (b)$

(a) & (b) $\Rightarrow$ $\|h\| = \|S\|$

To Show S is Unique.

Suppose $\exists$ bounded linear operator $T: H_1 \longrightarrow H_2$ s.t $\forall\, x \in H_1, y \in H_2$

We have $h(x,y) = \langle Sx, y\rangle$
$= \langle Tx, y\rangle$

$\Rightarrow Sx = Tx \quad \forall x \in H_1$ | $\because \langle v_1, w\rangle = \langle v_2, w\rangle \Rightarrow v_1 = v_2$

$\Rightarrow S = T$

$\Rightarrow S$ is unique.

Def:- Hilbert-Adjoint Operator T^*

Let $T : H_1 \longrightarrow H_2$ be a bounded linear operator, where H_1 & H_2 are hilbert spaces. Then the Hilbert-Adjoint Operator T^* of T is the operator

$$T^* : H_2 \longrightarrow H_1$$

such that $\forall\, x \in H_1$ & $y \in H_2$,

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

$$\|T\| = \|T^*\|$$

Theorem: (Existance)

The Hilbert adjoint operator T^* of $T : H_1 \longrightarrow H_2$ where H_1 & H_2 are Hilbert spaces, exists, is unique and is $\boxed{\text{bounded}}$ linear operator with norm $\boxed{\|T^*\| = \|T\|}$

Proof:- Consider $h : H_2 \times H_1 \longrightarrow K$ defined by $h(y, x) = \langle y, Tx\rangle \longrightarrow (1)$

Note that h is sesquilinear form because it is defined in terms of inner product, Ⓑ is sesquilinear and T is linear. In fact conjugate linearity of the form is seen from

$$h(y, \alpha x_1 + \beta x_2) = \langle y, T(\alpha x_1 + \beta x_2)\rangle$$

$$= \langle y, \alpha T x_1 + \beta T x_2 \rangle \quad \text{T is linear}$$

$$= \bar{\alpha}\langle y, T x_1\rangle + \bar{\beta}\langle y, T x_2\rangle$$

$$= \bar{\alpha} h(y, x_1) + \bar{\beta} h(y, x_2)$$

h is bounded. 5-6-14

Since

$$|h(y,x)| = |\langle y, Tx\rangle|$$

$$\underset{S.I}{\leq} \|y\| \, \|Tx\|$$

$$\underset{T \text{ is bdd}}{\leq} \|y\| \, \|T\| \, \|x\| \longrightarrow (*)$$

$\Rightarrow$ h is a bounded sequilinear form.

So, that Riesz's Representation theorem, for sesquilinear form, we obtain by replacing S by T^*

$$h(y,x) = \langle T^* y, x\rangle$$

But $\quad h(y,x) = \langle y, Tx\rangle$

$$\Rightarrow \langle T^* y, x\rangle = \langle y, Tx\rangle$$

$$\Rightarrow \overline{\langle T^* y, x\rangle} = \overline{\langle y, Tx\rangle}$$

$$\Rightarrow \langle x, T^* y\rangle = \langle Tx, y\rangle$$

i.e. $\langle Tx, y\rangle = \langle x, T^* y\rangle$

$\Rightarrow T^* : H_2 \to H_1$ is a Hilbert adjoint operator. Also by perivious theorem T^* is uniquely determine, bounded linear operator with $\|T^*\| = \|h\|$

From $(*)$ we have

$$\frac{|h(y, x)|}{\|x\| \, \|y\|} \le \|T\|$$

$$\Rightarrow \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|h(y, x)|}{\|x\| \, \|y\|} \le \|T\|$$

$$\Rightarrow \|h\| \le \|T\| \longrightarrow (a)$$

Also

$$\|h\| = \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|h(y, x)|}{\|y\| \, \|x\|}$$

$$= \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|\langle y, Tx\rangle|}{\|y\| \, \|x\|}$$

$$\geq \sup_{x \neq 0} \frac{|\langle Tx, Tx\rangle|}{\|Tx\| \, \|x\|}$$

$$= \sup_{x \neq 0} \frac{\|Tx\|^{\cancel{2}}}{\cancel{\|Tx\|}\ \|x\|} = \|T\|$$

i.e $\|h\| \geq \|T\| \longrightarrow$ (b)

(a) & (b) $\Rightarrow$ $\|h\| = \|T\|$

But $\|T^*\| = \|h\|$

S.t $\|T^*\| = \|T\|$

Theorem: (Properties of Hilbert Adjoint Operator)

Let H_1 & H_2 are hilbert spaces and $T: H_1 \longrightarrow H_2$ & $S: H_1 \longrightarrow H_2$ be bound linear operators and α be any scalar then,

a):- $\langle T^* y, x \rangle = \langle y, Tx \rangle$, $x \in H_1, y \in H_2$

Proof:- L.H.S $\langle T^* y, x \rangle = \overline{\langle x, T^* y \rangle}$

$= \overline{\langle Tx, y \rangle}$ $\because \langle Tx, y \rangle = \langle x, T^* y \rangle$

$= \langle y, Tx \rangle$

b):- $(S+T)^* = S^* + T^*$

Proof:- $\langle x, (S+T)^* y \rangle = \langle (S+T)x, y \rangle$

$= \langle Sx + Tx, y \rangle$

$= \langle Sx, y \rangle + \langle Tx, y \rangle$

$= \langle x, S^* y \rangle + \langle x, T^* y \rangle$

$= \langle x, (S^* + T^*) y \rangle$

$\Rightarrow (S+T)^* y = (S^* + T^*) y \quad \forall y$ | $\langle v_1, w\rangle = \langle v_2, w\rangle \quad \forall w$ $\Rightarrow v_1 = v_2$

By the def. of functions.

$$\Rightarrow (S+T)^* = S^* + T^*$$

c):- $(\alpha T)^* = \bar{\alpha} T^*$

Proof:- $\langle (\alpha T)^* y, x\rangle = \langle y, \alpha T x\rangle$

$$= \bar{\alpha} \langle y, Tx\rangle$$

$$= \bar{\alpha} \langle T^* y, x\rangle$$

$$= \langle \bar{\alpha} T^* y, x\rangle$$

$$\Rightarrow (\alpha T)^* y = (\bar{\alpha} T^*) y, \quad \forall y$$

$$\Rightarrow (\alpha T)^* = \bar{\alpha} T^*$$

d):- $(T^*)^* = T$

Proof:- $\langle (T^*)^* y, x\rangle = \langle y, T^* x\rangle$

$$= \langle Ty, x\rangle, \quad \forall x, y$$

$$\Rightarrow (T^*)^* y = Ty, \forall y.$$

$$\Rightarrow (T^*)^* = T$$

e):- $\|T^* T\| = \|T T^*\| = \|T\|^2$

Proof:- $\|Tx\|^2 = \langle Tx, Tx\rangle$

$$= \langle T^* T x, x\rangle$$

$$\underset{S.I}{\leq} \|T^* Tx\| \, \|x\|$$

$$\underset{T \,\&\, T^* \text{ bdd.}}{\leq} \|T^*T\| \, \|x\|^2$$

$$\Rightarrow \frac{\|Tx\|^2}{\|x\|^2} \leq \|T^*T\|$$

$$\Rightarrow \left(\sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}\right)^2 \leq \|T^*T\|$$

$\|T_1 T_2\| \leq \|T_1\| \|T_2\|$ page# 103

$$\Rightarrow \|T\|^2 \leq \|T^*T\| \leq \|T^*\| \, \|T\| = \|T\|^2$$

$$\Rightarrow \|T^*T\| = \|T\|^2 \longrightarrow (*)$$

Interchanging T & T^* in $(*)$.

$$\Rightarrow \|TT^*\| = \|T^*\|^2 = \|T\|^2 \longrightarrow (**)$$

$(*)$ & $(**)$ $\Rightarrow$

$$\|T^*T\| = \|TT^*\| = \|T\|^2$$

7:- $T^*T = 0 \Leftrightarrow T = 0$

Proof:- Suppose $T^*T = 0 \Leftrightarrow \|T^*T\| = 0$

i.e $\|T\|^2 = 0 \Leftrightarrow T = 0$

8:- $(ST)^* = T^* S^*$ (Assuming $H_1 = H_2$)

Proof:- $\langle x, (ST)^* y\rangle = \langle (ST)x, y\rangle$

$= \langle Tx, S^* y\rangle$

$= \langle x, (T^* S^*)y\rangle \quad \forall x, y$

$$\Rightarrow (ST)^* y = (T^* S^*) y \quad \forall y.$$

$$\Rightarrow (ST)^* = T^* S^*$$

Self Adjoint, Unitary & Normal Operator.

A bounded linear operator $T: H \longrightarrow H$ on a hilbert space H is said to be self Adjoint or Hermitian if $T^* = T$,

Unitary if T is bijective & $T^* = T^{-1}$

normal if $T^* T = T T^*$

Book: We know

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

But if T is self-adjoint, then

$$\langle Tx, y \rangle = \langle x, Ty \rangle$$

Remarks:- If T is self-adjoint or unitary then it is normal. But a normal operator need not be unitary or self-adjoint.

Example:- Consider $I: H \longrightarrow H$ Define T on H by $T = 2iI$. Then T is normal, since

$$T^* T = (2iI)^* (2iI)$$
$$= 2\bar{i}\, I^* \cdot 2iI$$

$$= -2\bar{i}\,T^{*}.2iI$$
$$= -4i^{2}\,I$$
$$= 4\,I$$

Also $TT^{*} = 4I$

i.e $TT^{*} = T^{*}T$

But $T^{*} = 2\bar{i}\,I \neq T$

$\Rightarrow$ T is not self-adjoint.

&

$T^{-1} = \frac{1}{2i} I = \frac{-i}{2} I \neq T^{*}$

$\Rightarrow$ T is not unitary.

Matrix Representation of T^{*} 6-6-14

Suppose $T: \mathbb{C}^{n} \longrightarrow \mathbb{C}^{n}$ be a linear operator. Then T is bounded ($\because \mathbb{C}^{n}$ is finite dimensional). Let T be represented by matrix say A and T^{*} be represented by matrix say B. The inner product on $\mathbb{C}^{n}$ is

$$\langle x, y\rangle = \sum_{1}^{n} \xi_i \bar{\eta}_i \;,\; x = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}, y = \begin{pmatrix} \bar{\eta}_1 \\ \vdots \\ \bar{\eta}_2 \end{pmatrix}$$
$$= x^{T}\bar{y}$$

Now $T^{*}: \mathbb{C}^{n} \longrightarrow \mathbb{C}^{n}$ is bounded linear operator. Then $\langle Tx, y\rangle = \langle x, T^{*}y\rangle$

$$\Rightarrow \langle Ax, y\rangle = \langle x, By\rangle$$

By def.

$$\Rightarrow (Ax)^T \bar{y} = (x)^T (\overline{By})$$

$$\Rightarrow x^T A^T \bar{y} = x^T \bar{B} \bar{y}$$

$$\Rightarrow A^T = \bar{B}$$

Taking conjugate on both sides.

$$\Rightarrow B = \overline{A^T}$$

$$\Rightarrow B = \bar{A}^T$$

Theorem:- Let $T : H \longrightarrow H$ be a bounded linear operator on a hilbert space H. Then

a):- If T is self-adjoint, then $\langle Tx, x\rangle$ is real $\forall\, x \in H$.

b):- If H is complex and $\langle Tx, x\rangle$ is real $\forall\, x \in H$, then operator T is self-adjoint.

Proof: a):- Given that

$T^* = T$ then

if a is real $\Rightarrow \bar{a} = a$

$$\overline{\langle Tx, x\rangle} = \overline{\langle x, T^*x\rangle}$$

$$= \overline{\langle x, Tx\rangle} \quad \because T^* = T$$

$$= \langle Tx, x\rangle$$

$\Rightarrow \langle Tx, x\rangle$ is real.

b):- Suppose H is Complex and $\langle Tx, x\rangle$ is real. i.e $\overline{\langle Tx, x\rangle} = \langle Tx, x\rangle$

$$\Rightarrow \overline{\langle x, T^*x\rangle} = \langle Tx, x\rangle$$

$\Rightarrow \langle T^* x, x\rangle = \langle Tx, x\rangle$

$\Rightarrow T^* x = Tx, \ \forall x$ | $\langle V_1, W\rangle = \langle V_2, W\rangle \Rightarrow V_1 = V_2 \ \forall W.$

$\Rightarrow T^* = T$

$\Rightarrow$ T is self-adjoint.

Theorem:- The product of two bounded self-adjoint linear operators S and T on a hilbert space H is a self-adjoint iff the operators commute. i.e

$$ST = TS$$

Proof:- $\Rightarrow$ Given that

$S^* = S$ & $T^* = T$

Suppose ST is self-adjoint

i.e $(ST)^* = ST$

$\Rightarrow T^* S^* = ST$

$\Rightarrow TS = ST \quad \because S^* = S$ & $T^* = T$

$\Leftarrow$ Suppose that $ST = TS$

Then $(ST)^* = T^* S^* = TS = ST$

i.e $(ST)^* = ST$

$\Rightarrow$ ST is self-adjoint.

Theorem:- (Seq. of Self-adjoint operators)

Let (T_n) be a seq. of bounded self-adjoint linear operators $T_n : H \to H$ on a Hilbert space H. Suppose that (T_n) converges, say,

$$T_n \xrightarrow{\|\cdot\|_{B(H,H)}} T \quad \text{i.e } \|T_n - T\| \to 0 \text{ as } n \to \infty$$

Then the limit operator T is a bounded self-adjoint linear operator on H.

Proof:- Consider

$$\|T - T^*\| \underset{T.I}{\leq} \|T - T_n\| + \|T_n - \cancel{T_n^*}\| + \|T_n^* - T^*\|$$

$$= \|T - T_n\| + \|T_n - T\| \to 0 \text{ as } n \to \infty$$

$(\because \|T_n^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\| \quad \because \|T^*\| = \|T\|)$

$$\Rightarrow \|T - T^*\| = 0$$

$$\Rightarrow T^* = T$$

Now $\|T\| \underset{TI}{\leq} \|T - T_n\| + \|T_n\| < \infty$

$\Rightarrow T$ is bounded.

Functional Analysis
BS VI (Regular, SSI & SSII)
Final Term Examination

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Sir Yaseen

Max Marks: 60 Time Allowed: 2 Hours. Date:16/06/2014

Q. No. 1(12 Marks)

Answer the following short questions:

(i) If T is a bounded linear operator, then show that the null space $\mathcal{N}(T)$ is closed.

(ii) Suppose the matrix $A = (a_{ij})_{m\times n}$ defines a linear operator $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $Ax = y$. Show that T is bounded.

(iii) Show that for a sequence (x_n) in an inner product space the conditions $\|x_n\| \to \|x\|$ and $< x_n, x > \to < x, x >$ imply $x_n \to x$.

(iv) Show by giving an example that an infinite dimensional subspace need not be closed.

Q. No. 2(12 Marks)

Show that the dual space of l^p is l^q, where $1 < p < \infty$ and q is conjugate of p.

Q. No. 3(12 Marks)

If Y is a Banach space, then prove that $B(X, Y)$ is a Banach space.

Q. No. 4(12 Marks)

Let X be an inner product space and $M \neq \phi$ a convex subset which is complete (in the metric induced by the inner product). Then prove that for every given $x \in X$, there exists a unique $y \in M$ such that

$$\delta = \inf_{\tilde{y}\in M} \|x - \tilde{y}\| = \|x - y\|$$

Q. No. 5(12 Marks)

Let H_1 and H_2 be Hilbert spaces and $h : H_1 \times H_2 \to K$ a bounded sesquilinear form. Then prove that h has representation $h(x, y) =< Sx, y >$, where $S : H_1 \to H_2$ is a bounded linear operator S uniquely determined by h and has the norm $\|S\| = \|h\|$.