Section 3.1

Prob. 2

If $x \perp y$ in an inner product space X, show that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

Extend the formula to m mutually orthogonal vectors.

Solution

If $x \perp y$, then $\langle x, y \rangle = 0$, and so

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + \langle x, y \rangle + \overline{\langle x, y \rangle} + \|y\|^2 \\ &= \|x\|^2 + \|y\|^2, \end{aligned}$$

as required.

From the above calculation we also have

$$\begin{aligned} \|x-y\|^2 &= \|x+(-y)\|^2 \\ &= \|x\|^2 + \|-y\|^2 \\ &= \|x\|^2 + \|(-1)y\|^2 \\ &= \|x\|^2 + |-1|^2\|y\|^2 \\ &= \|x\|^2 + \|y\|^2. \end{aligned}$$

Thus we have shown that, for any elements $x, y \in X$, if $x \perp y$, then

$$\|x \pm y\|^2 = \|x\|^2 + \|y\|^2.$$

Now suppose that, for some $m \in \mathbb{N}$, we have

$$\|x_1 \pm \cdots \pm x_m\|^2 = \|x_1\|^2 + \cdots + \|x_m\|^2,$$

where $x_1, \ldots, x_m \in X$ such that, for all $i, j \in \{1, \ldots, m\}$, we have $\langle x_i, x_j \rangle = 0$ if $i \neq j$.

Then using this equality and the above result for two vectors, we obtain

$$\begin{aligned}\|x_1 \pm \cdots \pm x_m \pm x_{m+1}\|^2 &= \|(x_1 \pm \cdots \pm x_m) \pm x_{m+1}\|^2 \\ &= \|x_1 \pm \cdots \pm x_m\|^2 + \|x_{m+1}\|^2 \\ &= \left(\|x_1\|^2 + \cdots + \|x_m\|^2\right) + \|x_{m+1}\|^2 \\ &= \|x_1\|^2 + \cdots + \|x_m\|^2 + \|x_{m+1}\|^2,\end{aligned}$$

where $x_1, \ldots, x_{m+1} \in X$ such that, for all $i, j \in \{1, \ldots, m+1\}$, we have $\langle x_i, x_j \rangle = 0$ if $i \neq j$.

For $m = 1$, the result is trivial.

Therefore by induction we can conclude that, for any natural number m, if $x_1, \ldots, x_m \in X$ such that, for all $i, j \in \{1, \ldots, m\}$, $\langle x_i, x_j \rangle = 0$ if $i \neq j$, then we must have

$$\|x_1 \pm \cdots \pm x_m\|^2 = \|x_1\|^2 + \cdots + \|x_m\|^2.$$

Prob. 3

If X in Prob. 2 is real, show that, conversely, the given relation implies that $x \perp y$. Show that this may not hold if X is complex. Give examples.

Solution

First, suppose that X is a *real* inner product space, and, for some $x, y \in X$, we have

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Upon expanding the norm on the left-hand side of this equation in terms of the inner product, this relation becomes

$$\|x\|^2 + \|y\|^2 + 2\langle x, y \rangle = \|x\|^2 + \|y\|^2.$$

which implies that
$$2\langle x, y\rangle = 0,$$
and hence
$$\langle x, y\rangle = 0,$$
showing that $x \perp y$.

However, if X is a *complex* inner product space, then we see that the relation
$$\|x+y\|^2 = \|x\|^2 + \|y\|^2,$$
after expansion of the norm on the left-hand side in terms of the inner product and cancellation of the terms on the two sides of the resulting equation, only yields
$$\Re\langle x, y\rangle = 0,$$
which only implies that $\langle x, y\rangle$ is a pure imaginary complex number.

As an example, let $X = \mathbb{C}$, the set of complex numbers, with the inner product defined as $\langle z, w\rangle = z\overline{w}$ for all $z, w \in \mathbb{C}$, and let $x = 1$ and $y = \iota$. Then
$$\|x+y\|^2 = \|1+\iota\|^2 = |1+\iota|^2 = 2,$$
and
$$\|x\|^2 + \|y\|^2 = \|1\|^2 + \|\iota\|^2 = |1|^2 + |\iota|^2 = 1 + 1 = 2,$$
showing that
$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$
But
$$\langle x, y\rangle = x\overline{y} = 1(-\iota) = -\iota \neq 0,$$
showing that $x \not\perp y$.

Prob. 4

If an inner product space X is real, show that the condition $\|x\| = \|x\|$ implies $\langle x+y, x-y\rangle = 0$. What does this mean geometrically if $X = \mathbb{R}^2$? What does the condition imply if X is complex?

Solution

First, suppose that X is a real inner product space, and x and y are some elements of X for which $\|x\| = \|y\|$. Then we see that

$$\begin{aligned}\langle x+y, x-y\rangle &= \langle x,x\rangle - \langle x,y\rangle + \langle y,x\rangle - \langle y,y\rangle \\ &= \|x\|^2 - \langle x,y\rangle + \langle x,y\rangle - \|y\|^2 \\ &= \|x\|^2 - \|y\|^2 \\ &= 0.\end{aligned}$$

Goemetrically, this means that, if we have a parallelogram in the plane whose two adjacent sides x and y are equal in length (i.e. $\|x\| = \|y\|$), then the diagonals $x+y$ and $x-y$ of this parallelogram are perpendicular (i.e. $\langle x+y, x-y\rangle = 0$).

Now suppose that X is a complex inner product space, and $x, y \in X$ such that $\|x\| = \|y\|$. Then we see that

$$\begin{aligned}\langle x+y, x-y\rangle &= \langle x,x\rangle - \langle x,y\rangle + \langle y,x\rangle - \langle y,y\rangle \\ &= \|x\|^2 - \langle x,y\rangle + \overline{\langle x,y\rangle} - \|y\|^2 \\ &= \|x\|^2 - \|y\|^2 - 2\iota\Im\langle x,y\rangle \\ &= -2\iota\Im\langle x,y\rangle.\end{aligned}$$

Thus, if X is a complex inner product space, then, for any elements $x, y \in X$, if $\|x\| = \|y\|$, then $\langle x+y, x-y\rangle = -2\iota\Im\langle x,y\rangle$.

Prob. 5

Verify by direct calculation that for any elements in an inner product space,

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\left\|z - \frac{1}{2}(x+y)\right\|^2.$$

Show that this identity can also be obtained from the parallelogram equality.

Solution

Let X be an inner product space, and let $x, y, z \in X$. Then

$$
\begin{aligned}
&\|z-x\|^2 + \|z-y\|^2 \\
&\qquad - \left[\frac{1}{2}\|x-y\|^2 + 2\left\|z - \frac{1}{2}(x+y)\right\|^2\right] \\
&= \langle z-x, z-x\rangle + \langle z-y, z-y\rangle \\
&\qquad - \left[\frac{1}{2}\langle x-y, x-y\rangle + 2\left\langle z - \frac{1}{2}(x+y), z - \frac{1}{2}(x+y)\right\rangle\right] \\
&= \|z\|^2 - 2\Re\langle z, x\rangle + \|x\|^2 + \|z\|^2 - 2\Re\langle z, y\rangle + \|y\|^2 \\
&\qquad - \frac{1}{2}\left(\|x\|^2 - 2\Re\langle x, y\rangle + \|y\|^2\right) \\
&\qquad - 2\left(\|z\|^2 - 2\Re\left\langle z, \frac{1}{2}(x+y)\right\rangle + \left\|\frac{1}{2}(x+y)\right\|^2\right) \\
&= 2\|z\|^2 - 2\Re\langle z, x\rangle + \|x\|^2 - 2\Re\langle z, y\rangle + \|y\|^2 \\
&\qquad - \frac{1}{2}\left(\|x\|^2 - 2\Re\langle x, y\rangle + \|y\|^2\right) \\
&\qquad - 2\left(\|z\|^2 - 2\Re\left(\frac{1}{2}\langle z, x+y\rangle\right) + \frac{1}{4}\|x+y\|^2\right) \\
&= 2\|z\|^2 - 2\Re\langle z, x\rangle + \|x\|^2 - 2\Re\langle z, y\rangle + \|y\|^2 \\
&\qquad - \frac{1}{2}\left(\|x\|^2 - 2\Re\langle x, y\rangle + \|y\|^2\right) \\
&\qquad - 2\|z\|^2 + 2\Re\langle z, x+y\rangle - \frac{1}{2}\|x+y\|^2 \\
&= 2\|z\|^2 - 2\Re\langle z, x\rangle + \|x\|^2 - 2\Re\langle z, y\rangle + \|y\|^2 \\
&\qquad - \frac{1}{2}\left(\|x\|^2 - 2\Re\langle x, y\rangle + \|y\|^2\right) \\
&\qquad - 2\|z\|^2 + 2\Re\left(\langle z, x\rangle + \langle z, y\rangle\right) - \frac{1}{2}\|x+y\|^2 \\
&= -2\Re\langle z, x\rangle + \|x\|^2 - 2\Re\langle z, y\rangle + \|y\|^2 \\
&\qquad - \frac{1}{2}\left(\|x\|^2 - 2\Re\langle x, y\rangle + \|y\|^2\right) + 2\Re\langle z, x\rangle \\
&\qquad + 2\Re\langle z, y\rangle - \frac{1}{2}\left(\|x\|^2 + 2\Re\langle x, y\rangle + \|y\|^2\right) \\
&= -2\Re\langle z, x\rangle + \|x\|^2 - 2\Re\langle z, y\rangle + \|y\|^2 \\
&\qquad - \frac{1}{2}\left(2\|x\|^2 - 2\Re\langle x, y\rangle + 5 2\Re\langle x, y\rangle + 2\|y\|^2\right) \\
&\qquad + 2\Re\langle z, x\rangle + 2\Re\langle z, y\rangle \\
&= \|x\|^2 + \|y\|^2 - \|x\|^2 - \|y\|^2 \\
&= 0.
\end{aligned}
$$

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Therefore,

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\left\|z - \frac{1}{2}(x+y)\right\|^2,$$

as required.

Now we know that if $u, v \in X$, then the parallelogram identity gives

$$2\|u\|^2 + 2\|v\|^2 = \|u+v\|^2 + \|u-v\|^2.$$

In this equality, put $u\colon = z-x$ and $v\colon = z-y$ and obtain

$$\begin{aligned} 2\|z-x\|^2 + 2\|z-y\|^2 &= \|(z-x)+(z-y)\|^2 + \|(z-x)-(z-y)\|^2 \\ &= \|2z-(x+y)\|^2 + \|-x+y\|^2 \\ &= 4\left\|z-\frac{1}{2}(x+y)\right\|^2 + \|x-y\|^2, \end{aligned}$$

and upon dividing both sides by 2, we get our desired equality.

Prob. 15

If X is a finite dimensional vector space and (e_j) is a basis for X, show that an inner product on X is completely determined by its values $\gamma_{jk} = \langle e_j, e_k \rangle$. Can we choose such scalars γ_{jk} in a completely arbitrary fashion?

Solution

Let X be a finite-dimensional vector space over the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Let K denote the field of scalars for X. Then K is either $\mathbb{R}$ or $\mathbb{C}$.

Let $n\colon = \dim X$, and let $\{e_1, \ldots, e_n\}$ be a basis for X. Then each element $u \in X$ can be uniquely represented as a linear combination of $e_1, \ldots, e_n$; that is, there exists a unique ordered n-tuple $(\alpha_1, \ldots, \alpha_n)$ of scalars such that

$$u = \alpha_1 e_1 + \cdots + \alpha_n e_n.$$

Let $x, y \in X$. Then there exist unique ordered n-tuples $(\mu_1, \ldots, \mu_n)$ and $(\nu_1, \ldots, \nu_n)$ of scalars such that

$$x = \mu_1 e_1 + \cdots + \mu_n e_n \quad \text{and} \quad y = \nu_1 e_1 + \cdots + \nu_n e_n.$$

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So if $\langle\cdot,\cdot\rangle$ is an inner product on X, then

$$\begin{aligned}\langle x,y\rangle &= \langle \mu_1 e_1+\cdots+\mu_n e_n\ ,\ \nu_1 e_1+\cdots+\nu_n e_n\rangle\\ &= \mu_1\langle e_1,\ \nu_1 e_1+\cdots+\nu_n e_n\rangle+\cdots+\mu_n\langle e_n,\ \nu_1 e_1+\cdots+\nu_n e_n\rangle\\ &= \mu_1\left(\overline{\nu_1}\langle e_1,e_1\rangle+\cdots+\overline{\nu_n}\langle e_1,e_n\rangle\right)\\ &\quad+\cdots+\mu_n\left(\overline{\nu_1}\langle e_n,e_1\rangle+\cdots+\overline{\nu_n}\langle e_n,e_n\rangle\right)\\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}\mu_j\overline{\nu_k}\langle e_j,e_k\rangle\\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}\mu_j\overline{\nu_k}\gamma_{jk},\end{aligned}$$

Thus, for any elements $x,y\in X$, the inner product $\langle x,y\rangle$ is completely determined once we know the values

$$\gamma_{jk}:\ =\langle e_j,e_k\rangle$$

for $j,k\in\{1,\dots,n\}$.

By IP3, we can conclude that, for any $j,k\in\{1,\dots,n\}$, we have

$$\langle e_j,e_k\rangle=\begin{cases}\langle e_k,e_j\rangle & \text{if } X \text{ is real,}\\ \overline{\langle e_k,e_j\rangle} & \text{if } X \text{ is complex.}\end{cases}$$

That is, for any $j,k\in\{1,\dots,n\}$,

$$\gamma_{jk}=\begin{cases}\gamma_{kj} & \text{if } X \text{ is real,}\\ \overline{\gamma_{kj}} & \text{if } X \text{ is complex.}\end{cases}\tag{0.1}$$

If X is not the trivial vector space consisting only of the zero vector, then $\dim X>0$, and the basis vectors $e_1,\dots,e_n$ are all non-zero; therefore by IP4 we have

$$\langle e_k,e_k\rangle>0$$

for all $k\in\{1,\dots,n\}$; that is, for all $k\in\{1,\dots,n\}$, we have

$$\gamma_{kk}>0\tag{0.2}$$

Thus (0.1) and (0.2) are the conditions that the γ_{jk} must satisfy.

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Section 3.2

Prob. 1

What is Schwarz inequality in $\mathbb{R}^2$ or $\mathbb{R}^3$? Give another proof of it in these cases.

Solution

The Shwarz inequality is as follows: For any elements x and y in an inner product space X,

$$|\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle},$$

that is,

$$|\langle x, y\rangle| \leq \|x\|\|y\|,$$

and the equality holds if and only if the set $\{x, y\}$ is linearly dependent, that is, if and only if $y = \alpha x$ for some scalar α. This is Lemma 3.2-1 (a) in Kreyszig.

In the case of the Euclidean plane or space, we can argue as follows: Let $\vec{x}$ and $\vec{y}$ be any two vectors, and let $\theta \in [0, \pi]$ be the angle between these vectors. Then we know that

$$\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}| \cos\theta,$$

and since $-1 \leq \cos\theta \leq 1$, therefore $0 \leq |\cos\theta| \leq 1$, and so

$$|\vec{x} \cdot \vec{y}| = |\vec{x}||\vec{y}||\cos\theta| \leq |\vec{x}||\vec{y}|,$$

and the equality holds if and only if $\cos\theta = \pm 1$, and this holds if and only if $\theta = 0$ or $\theta = \pi$. Thus equality holds if and only if $\vec{y} = \alpha\vec{x}$ for some scalar α.

Prob. 2

Give examples of subspaces of ℓ^2.

Solution

Recall that ℓ^2, by definition, is the inner product space consisting of all the sequences $(\alpha_n)_{n\in\mathbb{N}}$ of (real or) complex numbers for which the series $\sum |\alpha_n|^2$

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converges in $\mathbb{R}$, with the inner product defined by

$$\langle x, y\rangle :\ = \begin{cases} \sum_{n=1}^{\infty} \xi_n \eta_n & \text{if } X \text{ is real,} \\ \sum_{n=1}^{\infty} \xi_n \overline{\eta_n} & \text{if } X \text{ is complex} \end{cases}$$

for any elements $x:\ = (\xi_n)_{n\in\mathbb{N}}$ and $y:\ = (\eta_n)_{n\in\mathbb{N}}$ in ℓ^2, that is, for any sequences $x:\ = (\xi_n)_{n\in\mathbb{N}}$ and $y:\ = (\eta_n)_{n\in\mathbb{N}}$ of (real or) complex numbers for which the series $\sum |\xi_n|^2$ and $\sum |\eta_n|^2$ converge.

The (absolute) convergence of the series $\sum \xi_n \overline{\eta_n}$ or $\sum \xi_n \eta_n$ then follows from the Cauchy-Schwarz inequality (i.e. (11) Subsec. 1.2-3 in Kreyszig).

So the norm on ℓ^2 is given by

$$\|x\|_{\ell^2} :\ = \sqrt{\sum_{n=1}^{\infty} |\xi_n|^2}$$

for any element $x:\ = (\xi_n)_{n\in\mathbb{N}}$ in ℓ^2, and the metric on ℓ^2 is defined by

$$d_{\ell^2}(x, y) :\ = \sqrt{\sum_{n=1}^{\infty} |\xi_n - \eta_n|^2}$$

for any elements $x:\ = (\xi_n)_{n\in\mathbb{N}}$ and $y:\ = (\eta_n)_{n\in\mathbb{N}}$ in ℓ^2. And, with respect to this metric, the space ℓ^2 is a complete metric space (i.e. a Hilbert space). [Refer to Subsec. 1.5-4 in Kreyszig.]

Let N be a given natural number, and let

$$Y_N :\ = \left\{ (\xi_n)_{n\in\mathbb{N}} : \ \xi_n \in \mathbb{C}\ \forall n \in \mathbb{N},\ \xi_n = 0\ \forall n \in \mathbb{N} \text{ such that } n > N \right\}.$$

Check that this Y_N is a (vector) subspace of ℓ^2.

Let

$$Y_\infty :\ = \left\{ x :\ = (\xi_n)_{n\in\mathbb{N}} \ : \ \xi_n \in \mathbb{C},\ \exists N_x \in \mathbb{N} \text{ such that } \xi_n = 0\ \forall n > N_x \right\}.$$

That is, let Y_∞ be the set of all those sequences of complex numbers which have at most finitely many non-zero terms. We check that Y_∞ is a (vector) subspace of ℓ^∞.

First, note that if $x:\ = (\xi_n)_{n\in\mathbb{N}} \in Y_\infty$, then there exists a natural number N_x (where the subscript x means that this natural numbers depends on the

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particular x in Y_∞ and may be different for different x) such that $\xi_n = 0$ for all natural numbers $n > N_x$. In this case

$$\sum_{n=1}^{\infty} |\xi_n|^2 = \sum_{n=1}^{N_x} |\xi_n|^2 < +\infty,$$

showing that the series $\sum |\xi_n|^2$ converges in $\mathbb{R}$. Thus $x \in \ell^2$ and so $Y_\infty \subset \ell^2$.

If $x \colon = (0, 0, 0, \ldots)$, then $x \in Y_\infty$ because $\xi_n = 0$ for all natural numbers $n > 1$, for example.

Now let $x \colon = (\xi_n)_{n\in\mathbb{N}}$ and $y \colon = (\eta_n)_{n\in\mathbb{N}}$ be elements of Y_∞, and let α and β be some scalars (i.e. complex numbers).

Then x and y are sequences of complex numbers for which there are natural numbers N_x and N_y such that $\xi_n = 0$ for all $n > N_x$ and $\eta_n = 0$ for all $n > N_y$. Therefore, $\xi_n = 0 = \eta_n$ for all natural numbers $n > \max\{N_x, N_y\}$, and so $\alpha\xi_n + \beta\eta_n = 0$ for all natural numbers $n > \max\{N_x, N_y\}$, and as N_x and N_y are natural numbers, so is $\max\{N_x, N_y\}$. Now

$$\alpha x + \beta y = (\alpha\xi_n + \beta\eta_n)_{n\in\mathbb{N}}$$

is a sequence of complex numbers. And, as $\alpha\xi_n + \beta\eta_n = 0$ for all natural numbers $n > \max\{N_x, N_y\}$, so we can conclude that $\alpha x + \beta y \in Y_\infty$ whenever $x, y \in Y_\infty$ and α and β are any scalars. Hence Y_∞ is a (vector) subspace of ℓ^∞.

Similarly, show that

$$Y_{\text{odd}} \colon = \left\{ (\xi_n)_{n\in\mathbb{N}} \in \ell^2 \ : \ \xi_{2n-1} = 0 \ \forall n \in \mathbb{N} \right\}$$

and

$$Y_{\text{even}} \colon = \left\{ (\xi_n)_{n\in\mathbb{N}} \in \ell^2 \ : \ \xi_{2n} = 0 \ \forall n \in \mathbb{N} \right\}$$

are also subspaces of ℓ^2.

Prob. 3

Let X be the inner product space consisting of the zero polynomial and all real polynomials in t, of degree not exceeding 2, considered for real $t \in [a, b]$, with inner product defined by

$$\langle x, y \rangle \colon = \int_a^b x(t)y(t)\mathrm{d}t \ \forall x, y \in X.$$

Show that X is complete. Let Y consist of all $x \in X$ such that $x(a) = 0$. Is Y a subspace of X? Do all $x \in X$ of degree 2 form a subspace of X?

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Solution

If $x \in X$, then x is a real-valued function with domain $[a, b]$ defined by a formula of the form

$$x(t) = \alpha + \beta t + \gamma t^2 \ \forall t \in [a, b],$$

for some real numbers α, β, and γ.

Now let e_0, e_1, and e_2 be the real-valued functions defined on the closed interval $[a, b]$ by the formulas

$$e_0(t) :\ = 1, \qquad e_1(t) :\ = t, \qquad e_2(t) :\ = t^2$$

for all $t \in [a, b]$.

These e_0, e_1, and e_2 are all in X, and any $x \in X$ can be written as a linear combination of these as follows:

$$x = \alpha e_0 + \beta e_1 + \gamma e_2.$$

Therefore,

$$X = \text{span}\{e_0, e_1, e_2\}.$$

That is, X is a finite-dimensional inner product space and hence a finite-dimensional normed space. Therefore, by Theorem 2.4-2 in Kreyszig, X is complete.

The zero polynomial is in Y, and if $y_1, y_2 \in Y$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, then we note that

$$\begin{aligned}(\lambda_1 y_1 + \lambda_2 y_2)(a) &= \lambda_1 y_1(a) + \lambda_2 y_2(a) \\ &= \lambda_1 \times 0 + \lambda_2 \times 0 \\ &= 0,\end{aligned}$$

showing that $\lambda_1 y_1 + \lambda_2 y_2 \in Y$ also, and hence Y is a (vector) subspace of X.

The polynomials $x(t) = t^2 + 1$ and $y(t) = -t^2 + 1$ are both of degree 2, but their sum $x + y = 2$ is of degree 0. So the set of all the polynomials of degree 2 is *not* a subspace of X.

Prob. 4

Show that $y \perp x_n$ and $x_n \to x$ together imply $x \perp y$.

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Solution

Let X be an inner product space; let $x, y \in X$; and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $\langle y, x_n\rangle = 0$ for all $n \in \mathbb{N}$ and $x_n \to x$ in X as $n \to \infty$. We need to show that $\langle x, y\rangle = 0$.

Now let's take a real number $\varepsilon > 0$. As $(x_n)_{n\in\mathbb{N}}$ converges to x, so we can find a natural number N such that

$$\|x_n - x\| < \frac{\varepsilon}{1 + \|y\|}$$

for every natural number $n > N$.

So,

$$\begin{aligned}
&|\langle x, y\rangle| \\
&= |\langle x, y\rangle - \langle x_{N+1}, y\rangle| \qquad [\text{ as } \langle y, x_n\rangle = 0 \text{ for all } n, \text{ so } \langle x_n, y\rangle = 0 \text{ also }] \\
&= |\langle x - x_{N+1}, y\rangle| \\
&\le \|x - x_{N+1}\|\,\|y\| \qquad [\text{ by Schwarz inequality }] \\
&\le \frac{\varepsilon}{1 + \|y\|}\|y\| \\
&< \varepsilon.
\end{aligned}$$

Thus we have shown that

$$|\langle x, y\rangle| < \varepsilon$$

for every real number $\varepsilon > 0$. Therefore

$$|\langle x, y\rangle| = 0,$$

which implies that

$$\langle x, y\rangle = 0,$$

as required.

Prob. 5

Show that for a sequence (x_n) in an inner product space the conditions $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x, x\rangle$ imply convergence $x_n \to x$.

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Solution

Let X be an inner product space, let $x \in X$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that the sequence $(\|x_n\|)_{n\in\mathbb{N}}$ of norms converges in $\mathbb{R}$ to the real number $\|x\|$ and the sequence $(\langle x_n, x\rangle)_{n\in\mathbb{N}}$ of inner products converges in K to the inner product $\langle x, x\rangle$, where K denotes the field of scalars for X, and K is either $\mathbb{R}$ or $\mathbb{C}$ both with their usual norms; that is,

$$\lim_{n\to\infty} \|x_n\| = \|x\|, \quad \text{and} \quad \lim_{n\to\infty} \langle x_n, x\rangle = \langle x, x\rangle. \tag{0.3}$$

We need to show from this that the sequence $(x_n)_{n\in\mathbb{N}}$ converges in X to the point x; this convergence is with respect to the metric induced by the inner product on X.

We know that if a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of complex numbers converges to a complex number α, then the sequence $(\Re\alpha_n)_{n\in\mathbb{N}}$ of the real parts of the terms of $(\alpha_n)_{n\in\mathbb{N}}$ converges to the real part $\Re\alpha$ of α; that is,

$$\lim_{n\to\infty} \Re\alpha_n = \Re \lim_{n\to\infty} \alpha_n. \tag{0.4}$$

We will use this fact shortly.

We note that, for each $n \in \mathbb{N}$,

$$\begin{aligned}\|x_n - x\|^2 & \langle x_n - x\ ,\ x_n - x\rangle \\ &= \langle x_n, x_n\rangle - \langle x_n, x\rangle - \langle x, x_n\rangle + \langle x, x\rangle \\ &= \|x_n\|^2 - 2\Re\langle x_n, x\rangle + \|x\|^2.\end{aligned}$$

So,

$$\begin{aligned}\lim_{n\to\infty} \|x_n - x\|^2 &= \lim_{n\to\infty} \left(\|x_n\|^2 - 2\Re\langle x_n, x\rangle + \|x\|^2\right) \\ &= \lim_{n\to\infty} \|x_n\|^2 - \lim_{n\to\infty} 2\Re\langle x_n, x\rangle + \lim_{n\to\infty} \|x\|^2 \\ &\qquad [\text{ the limit of a sum is the sum of the limits }] \\ &= \lim_{n\to\infty} \|x_n\|^2 - 2\Re\left(\lim_{n\to\infty} \langle x_n, x\rangle\right) + \lim_{n\to\infty} \|x\|^2 \\ &\qquad [\text{ using (0.4) above }] \\ &= \|x\|^2 - 2\Re\langle x, x\rangle + \|x\|^2 \qquad [\text{ using (0.3) above }] \\ &= \|x\|^2 - 2\Re\|x\|^2 + \|x\|^2 \\ &= \|x\|^2 - 2\|x\|^2 + \|x\|^2 \\ &\qquad [\text{ note } \|x\|^2 = \langle x, x\rangle \text{ is real by IP4, so } \Re\|x\|^2 = \|x\|^2 \text{]} \\ &= 0.\end{aligned}$$

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Thus,

$$\lim_{n\to\infty} \|x_n - x\|^2 = 0.$$

This limit has been calculated in $\mathbb{C}$.

Let us take a real number $\varepsilon > 0$. Then we can find a natural number N such that

$$\left|\|x_n - x\|^2 - 0\right| = \left|\|x_n - x\|^2\right| = \|x_n - x\|^2 < \varepsilon^2$$

for any natural number $n > N$, which implies that

$$\|x_n - x\| < \varepsilon \tag{0.5}$$

for any natural number $n > N$.

Thus we have shown that, for every real number $\varepsilon > 0$, we can find a natural number N such that (0.5) holds. Hence the sequence $(x_n)_{n\in\mathbb{N}}$ converges in X to the point x.

Prob. 7

Show that in an inner product space, $x \perp y$ if and only if we have $\|x+\alpha y\| = \|x - \alpha y\|$ for all scalars α.

Solution

Let X be an inner product space, and let $x, y \in X$.

Suppose that $x \perp y$. Then $\langle x, y\rangle = 0$. So for every scalar α, we have

$$\begin{aligned}
\|x + \alpha y\|^2 &= \langle x + \alpha y, x + \alpha y\rangle \\
&= \langle x, x\rangle + \overline{\alpha}\langle x, y\rangle + \alpha\overline{\langle x, y\rangle} + \alpha\overline{\alpha}\langle y, y\rangle \\
&= \langle x, x\rangle + \overline{\alpha}\cdot 0 + \alpha\cdot\overline{0} + \alpha\overline{\alpha}\langle y, y\rangle \\
&= \|x\|^2 + |\alpha|^2\|y\|^2.
\end{aligned}$$

And,

$$\begin{aligned}
\|x - \alpha y\|^2 &= \langle x - \alpha y, x - \alpha y\rangle \\
&= \langle x, x\rangle - \overline{\alpha}\langle x, y\rangle - \alpha\overline{\langle x, y\rangle} + \alpha\overline{\alpha}\langle y, y\rangle \\
&= \langle x, x\rangle - \overline{\alpha}\cdot 0 - \alpha\cdot\overline{0} + \alpha\overline{\alpha}\langle y, y\rangle \\
&= \|x\|^2 + |\alpha|^2\|y\|^2.
\end{aligned}$$

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Therefore

$$\|x+\alpha y\|^2 = \|x\|^2 + |\alpha|^2\|y\|^2 = \|x-\alpha y\|^2$$

for every scalar α. Hence

$$\|x+\alpha y\| = \|x-\alpha y\|$$

for every scalar α.

Conversely, suppose that

$$\|x+\alpha y\| = \|x-\alpha y\|$$

for every scalar α. Then

$$\|x+\alpha y\|^2 = \|x-\alpha y\|^2$$

for every scalar α. Upon expanding the two sides of the last equation we obtain

$$\langle x,x\rangle + \overline{\alpha}\langle x,y\rangle + \alpha\overline{\langle x,y\rangle} + \alpha\overline{\alpha}\langle y,y\rangle = \langle x,x\rangle - \overline{\alpha}\langle x,y\rangle - \alpha\overline{\langle x,y\rangle} + \alpha\overline{\alpha}\langle y,y\rangle$$

for every scalar α. The last relation upon cancellation of the common terms on both sides yields

$$\overline{\alpha}\langle x,y\rangle + \alpha\overline{\langle x,y\rangle} = -\overline{\alpha}\langle x,y\rangle - \alpha\overline{\langle x,y\rangle}$$

for every scalar α, which implies that

$$2\overline{\alpha}\langle x,y\rangle + 2\alpha\overline{\langle x,y\rangle} = 0$$

for every scalar α. Therefore

$$\overline{\alpha}\langle x,y\rangle + \alpha\overline{\langle x,y\rangle} = 0 \tag{0.6}$$

for every scalar α.

Now if X is a real inner product space, then (0.6) takes the fomr

$$2\alpha\langle x,y\rangle = 0$$

for every scalar α, which upon putting $\alpha: \;= \frac{1}{2}$ gives

$$\langle x,y\rangle = 0,$$

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which is the same as $x \perp y$.

So suppose that X is a complex inner product space. By putting $\alpha \colon = \frac{1}{2}$ in (0.6) we obtain

$$\Re\langle x, y\rangle = \frac{1}{2}\left(\langle x, y\rangle + \overline{\langle x, y\rangle}\right) = 0, \tag{0.7}$$

and by putting $\alpha \colon = -\frac{1}{2\iota}$ in (0.6) we obtain

$$\Im\langle x, y\rangle = \frac{1}{2\iota}\left(\langle x, y\rangle - \overline{\langle x, y\rangle}\right) = 0, \tag{0.8}$$

From (0.7) and (0.8) we obtain $\langle x, y\rangle = 0$, which means the same as $x \perp y$.

Prob. 8

Show that in an inner product space, $x \perp y$ if and only if $\|x + \alpha y\| \geq \|x\|$ for all scalars α.

Solution

Let X be an inner product space, and let $x, y \in X$.

Suppose that $x \perp y$. This means that $\langle x, y\rangle = 0$. So for every scalar α, we have

$$\begin{aligned}
\|x + \alpha y\|^2 &= \langle x + \alpha y, x + \alpha y\rangle \\
&= \langle x, x\rangle + \overline{\alpha}\langle x, y\rangle + \alpha\overline{\langle x, y\rangle} + \alpha\overline{\alpha}\langle y, y\rangle \\
&= \langle x, x\rangle + \overline{\alpha}\cdot 0 + \alpha \cdot \overline{0} + \alpha\overline{\alpha}\langle y, y\rangle \\
&= \|x\|^2 + |\alpha|^2\|y\|^2 \\
&\geq \|x\|^2,
\end{aligned}$$

And,

$$\begin{aligned}
\|x - \alpha y\|^2 &= \langle x - \alpha y, x - \alpha y\rangle \\
&= \langle x, x\rangle - \overline{\alpha}\langle x, y\rangle - \alpha\overline{\langle x, y\rangle} + \alpha\overline{\alpha}\langle y, y\rangle \\
&= \langle x, x\rangle - \overline{\alpha}\cdot 0 - \alpha \cdot \overline{0} + \alpha\overline{\alpha}\langle y, y\rangle \\
&= \|x\|^2 + |\alpha|^2\|y\|^2 \\
&\geq \|x\|^2.
\end{aligned}$$

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Thus we have shown that

$$\|x \pm \alpha y\|^2 \geq \|x\|^2$$

for every scalar α, and both the norms involved are non-negative; so we can conclude that

$$\|x \pm \alpha y\| \geq \|x\| \tag{0.9}$$

for every scalar α.

Conversely, suppose that (0.9) holds. Upon squaring both sides, we obtain

$$\|x \pm \alpha y\|^2 \geq \|x\|^2$$

for every scalar α, and upon expanding the left-hand side of the last inequality we obtain

$$\|x\|^2 \pm \left(\overline{\alpha}\langle x, y\rangle + \alpha\overline{\langle x, y\rangle} \right) + |\alpha|^2\|y\|^2 \geq \|x\|^2$$

for every scalar α, which implies that

$$\pm \left(\overline{\alpha}\langle x, y\rangle + \alpha\overline{\langle x, y\rangle} \right) + |\alpha|^2\|y\|^2 \geq 0 \tag{0.10}$$

for every scalar α.

If $y = \mathbf{0}_X$, the zero vector in X, then we note that

$$\langle x, y\rangle = \langle x, \mathbf{0}_X\rangle = \langle x, 0x\rangle = \overline{0}\langle x, x\rangle = 0 \cdot \|x\|^2 = 0,$$

which is the same as $x \perp y$.

So let's assume that y is not the zero vector in X. Then, by IP4,

$$\|y\|^2 = \langle y, y\rangle > 0.$$

By putting $\alpha :=\frac{\langle x,y\rangle}{\|y\|^2}$ in (0.10), we obtain

$$\pm \left(\overline{\frac{\langle x, y\rangle}{\|y\|^2}}\langle x, y\rangle + \frac{\langle x, y\rangle}{\|y\|^2}\overline{\langle x, y\rangle} \right) + \left| \frac{\langle x, y\rangle}{\|y\|^2} \right|^2 \|y\|^2 \geq 0,$$

which simplifies to

$$\pm 2\frac{|\langle x, y\rangle|^2}{\|y\|^2} + \frac{|\langle x, y\rangle|^2}{\|y\|^2} \geq 0,$$

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and as $\|y\|^2 > 0$, so we obtain

$$\pm 2\left|\langle x,y\rangle\right|^2 + \left|\langle x,y\rangle\right|^2 \geq 0;$$

that is,

$$3\left|\langle x,y\rangle\right|^2 \geq 0 \text{ and } -\left|\langle x,y\rangle\right|^2 \geq 0,$$

which implies that

$$\left|\langle x,y\rangle\right|^2 \leq 0 \leq \left|\langle x,y\rangle\right|^2;$$

so

$$\left|\langle x,y\rangle\right|^2 = 0,$$

which implies that

$$\left|\langle x,y\rangle\right| = 0,$$

and therefore

$$\langle x,y\rangle = 0,$$

which is the same as $x \perp y$.

Prob. 9

Let V be the vector space of all continuous complex-valued functions on $J = [a,b]$. Let $X_1 = (V, \|\cdot\|_\infty)$, where $\|x\|_\infty = \max_{t\in J}|x(t)|$, and let $X_2 = (V, \|\cdot\|_2)$, where

$$\|x\|_2 = \langle x,x\rangle^{1/2}, \qquad \langle x,y\rangle = \int_a^b x(t)\overline{y(t)}\mathrm{d}t.$$

Show that the identity mapping $x \mapsto x$ of X_1 onto X_2 is continuous. (It is not a homeomorphism. X_2 is not complete.)

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Solution

Let d denote the metric induced by the inner product on X_2. Then for any $x, y \in X_2$, we see that

$$\begin{aligned}
[d(x,y)]^2 &= \langle x-y, x-y \rangle \\
&= \int_a^b (x(t)-y(t))\overline{(x(t)-y(t))}\mathrm{d}t \\
&= \int_a^b |x(t)-y(t)|^2 \mathrm{d}t \\
&\leq \int_a^b \max_{s\in[a,b]} |x(s)-y(s)|^2 \mathrm{d}t \\
&= \max_{s\in[a,b]} |x(s)-y(s)|^2 \int_a^b \mathrm{d}t \\
&= \|x-y\|_\infty^2 (b-a).
\end{aligned}$$

So

$$d(x,y) \leq \|x-y\|_\infty \sqrt{b-a}$$

for all $x, y \in X_2$.

Let's choose a real number $\varepsilon > 0$, and let δ be any real number such that

$$0 < \delta \leq \frac{\varepsilon}{\sqrt{b-a}}.$$

Then for any elements $x, y \in X_1$ for which

$$\|x-y\|_\infty < \delta,$$

we have

$$d(x,y) \leq \|x-y\|_\infty \sqrt{b-a} < \delta\sqrt{b-a} \leq \frac{\varepsilon}{\sqrt{b-a}}\sqrt{b-a} = \varepsilon.$$

showing that the identity mapping $x \mapsto x$ of X_1 onto X_2 is *uniformly* continuous and hence continuous.

Prob. 10

Let X be a complex inner product space, and let $T\colon X \to X$ be a linear operator. If $\langle\, T(x)\,,\, x\rangle = 0$ for all $x \in X$, then show that T is the zero operator.

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Show that this result does not hold in a *real* inner product space. *Hint.* Consider a rotation of the Euclidean plane.

What if we take the same rotation of the complex plane?

Solution

Here X is a complex inner product space. This means that the field of scalars for X is the set $\mathbb{C}$ of complex numbers.

Let $u, v \in X$. Then, as $\langle\, T(x)\;,\; x\rangle = 0$ for all $x \in X$, so for any scalar (i.e. complex number) α, we have

$$\begin{aligned}
0 &= \langle\, T(u+\alpha v)\;,\; u+\alpha v\rangle \\
&\qquad [\text{ using the property of } T \text{ with } x :\; = u+\alpha v\ ] \\
&= \langle\, T(u)+\alpha T(v)\;,\; u+\alpha v\rangle \qquad [\text{ using the linearity of } T\ ] \\
&= \langle T(u), u\rangle + \overline{\alpha}\,\langle T(u), v\rangle + \alpha\,\langle T(v), u\rangle + \alpha\overline{\alpha}\,\langle T(v), v\rangle \\
&= 0 + \overline{\alpha}\,\langle T(u), v\rangle + \alpha\,\langle T(v), u\rangle + |\alpha|^2\cdot 0 \\
&= \overline{\alpha}\,\langle T(u), v\rangle + \alpha\,\langle T(v), u\rangle\,.
\end{aligned}$$

Thus we have shown that

$$\overline{\alpha}\,\langle T(u), v\rangle + \alpha\,\langle T(v), u\rangle = 0 \tag{0.11}$$

for every complex number α.

Now putting $\alpha :\; = 1$ in (0.11), we obtain

$$\langle T(u), v\rangle + \langle T(v), u\rangle = 0, \tag{0.12}$$

and by putting $\alpha :\; = -\iota$ in (0.11), we obtain

$$\overline{-\iota}\,\langle T(u), v\rangle - \iota\,\langle T(v), u\rangle = 0,$$

which simplifies to

$$\iota\,\langle T(u), v\rangle - \iota\,\langle T(v), u\rangle = 0,$$

which upon division by ι of both sides yields

$$\langle T(u), v\rangle - \langle T(v), u\rangle = 0. \tag{0.13}$$

Now adding (0.12) and (0.13) and then dividing the resulting equation by 2, we obtain

$$\langle\, T(u)\;,\; v\rangle = 0$$

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for any elements $u, v \in X$.

In the last equation, we put $v \colon = T(u)$ and obtain

$$\langle\, T(u)\ ,\ T(u)\,\rangle = 0,$$

which by IP4 implies that

$$T(u) = \mathbf{0}_X$$

for all $u \in X$, where $\mathbf{0}_X$ denotes the zero vector in X.

Now the domain of T is X, and $T(u) = \mathbf{0}_X$ for all $u \in X$. So T is the zero operator.

Now we turn to the real inner product space $\mathbb{R}^2$, which is the Euclidean plane with the dot product of vectors as the inner product.

Note that if P is the point in the plane given by

$$P \colon = (\xi, \eta) = (r\cos\theta, r\sin\theta),$$

then by rotating the segment $\overline{OP}$ counter-clockwise about the origin O, we obtain the point

$$Q = \left(\ r\cos\left(\theta + \frac{\pi}{2}\right)\ ,\ r\sin\left(\theta + \frac{\pi}{2}\right)\ \right) = (\ -r\sin\theta\ ,\ r\cos\theta\ ) = (-\eta, \xi).$$

So let's define a mapping $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(x) \colon = (-\xi_2, \xi_1)$$

for any point $x \colon = (\xi_1, \xi_2)$ in $\mathbb{R}^2$.

Then we note that

$$\begin{aligned}\langle\, T(x), x\rangle &= \langle\ (-\xi_2, \xi_1)\ \ ,\ (\xi_1, \xi_2)\ \rangle \\ &= -\xi_2\xi_1 + \xi_1\xi_2 \\ &= 0\end{aligned}$$

for any point $x \colon = (\xi_1, \xi_2)$ in $\mathbb{R}^2$.

But if we put $e_1 \colon = (1, 0)$ and $e_2 \colon = (0, 1)$, then $e_1, e_2 \in \mathbb{R}^2$, but

$$T(e_1) = (0, 1) = e_2 \neq (0, 0), \quad \text{and} \quad T(e_2) = (-1, 0) = -e_1 \neq (0, 0).$$

Thus T is not the zero operator on $\mathbb{R}^2$.

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We finally show that T is linear. The domain of T is the vector space $\mathbb{R}^2$. Let $u := (\mu_1, \mu_2)$ and $v := (\nu_1, \nu_2)$ be some points of $\mathbb{R}^2$, and let α and β be some real numbers. Then

$$\alpha u + \beta v = \alpha\,(\mu_1, \mu_2) + \beta\,(\nu_1, \nu_2) = (\ \alpha\mu_1 + \beta\nu_1\ ,\ \alpha\mu_2 + \beta\nu_2\ ),$$

and so

$$\begin{aligned} T(\alpha u + \beta v) &= T\,(\ (\ \alpha\mu_1 + \beta\nu_1\ ,\ \alpha\mu_2 + \beta\nu_2\ )\ ) \\ &= (\ -(\alpha\mu_2 + \beta\nu_2)\ ,\ \alpha\mu_1 + \beta\nu_1\ ) \\ &= (\ -\alpha\mu_2 - \beta\nu_2\ ,\ \alpha\mu_1 + \beta\nu_1\ ) \\ &= \alpha\,(-\mu_2, \mu_1) + \beta\,(-\nu_2, \nu_1) \\ &= \alpha T(u) + \beta T(v). \end{aligned}$$

Thus T is linear.

As $\mathbb{R}^2$ is a finite-dimensional normed space, so (by Theorem 2.7-8 in Kreyszig) every linear operator with $\mathbb{R}^2$ as its domain is bounded. Thus our T is also a bounded linear operator.

We now directly show that T is bounded as follows: Let $p := (\xi, \eta)$ be any point of $\mathbb{R}^2$. Then we note that

$$\begin{aligned} \|T(p)\| &= \|T\,(\ (\xi, \eta)\ )\| \\ &= \|(-\eta, \xi)\| \\ &= \sqrt{(-\eta)^2 + \xi^2} \\ &= \sqrt{\eta^2 + \xi^2} \\ &= \sqrt{\xi^2 + \eta^2} \\ &= \|(\xi, \eta)\| \\ &= \|p\|. \end{aligned}$$

and so

$$\frac{\|T(p)\|}{\|p\|} = 1$$

for all $p \in \mathbb{R}^2$ such that $p \neq (0, 0)$. Therefore, T is bounded and

$$\|T\| = \sup\left\{\ \frac{\|T(p)\|}{\|p\|}\ :\ p \in \mathbb{R}^2 \text{ such that } p \neq (0,0)\ \right\} = \sup\{\ 1\ \} = 1.$$

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Next, we take $X\colon = \mathbb{C}$, the set of complex numbers, regarded as a one-dimensional vector space over the field $\mathbb{C}$ (i.e. over itself), with the inner product defined by

$$\langle x, y\rangle \colon = x\overline{y}$$

for all $x, y \in X$.

Note that if $z = r(\cos\theta + \iota\sin\theta)$ is a non-zero complex number, then the complex number w obtained by increasing the argument of z by $\pi/2$ is given by

$$w = r(\cos(\theta + \pi/2) + \iota\sin(\theta + \pi/2)) = r(\cos\theta + \iota\sin\theta)\left(\cos\frac{\pi}{2} + \iota\sin\frac{\pi}{2}\right) = z\iota.$$

So let's define the mapping $T\colon \mathbb{C} \to \mathbb{C}$ by

$$T(z)\colon = \iota z$$

for all $z \in \mathbb{C}$. Then we note that, for any $z \in \mathbb{C}$,

$$\begin{aligned} \langle T(z), z\rangle &= \langle \iota z, z\rangle \\ &= (\iota z)\,\overline{z} \\ &= \iota|z|^2, \end{aligned}$$

so that

$$\langle T(z), z\rangle = 0$$

if and only if

$$|z|^2 = 0,$$

which holds if and only if $z = 0$. So this T does not satisfy the condition given in this problem.

We now show that this T is linear. The domain of T is the vector space $\mathbb{C}$. Suppose $w, z \in \mathbb{C}$, where $\mathbb{C}$ is regarded as a vector space, and suppose that $\alpha, \beta \in \mathbb{C}$, where we now regard $\mathbb{C}$ as the field of scalars for the vector space $\mathbb{C}$. (Recall that every field is a vector space of dimension 1 over itself.) Then

$$\begin{aligned} T(\alpha w + \beta z) &= \iota(\alpha w + \beta z) \\ &= \alpha(\iota w) + \beta(\iota z) \\ &= \alpha T(w) + \beta T(z), \end{aligned}$$

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which shows that T is linear.

As the domain of T is the finite-dimensional normed space $\mathbb{C}$, so (by Theorem 2.7-8 in Kreyszig) T is also bounded. This fact we show directly as follows:

For any complex number $z \neq 0$, we see that

$$\begin{aligned}\frac{\|T(z)\|}{\|z\|} &= \frac{\|\iota z\|}{\|z\|} \\ &= \frac{|\iota z|}{|z|} \qquad [\text{ using the definition of the norm in } \mathbb{C} \text{ }] \\ &= \frac{|\iota||z|}{|z|} \\ &= 1. \qquad [\text{ because } |\iota| = |0 + 1\iota| = \sqrt{0^2 + 1^2} = 1 \text{ }]\end{aligned}$$

Therefore, T is bounded and

$$\|T\| = \sup\left\{ \frac{\|T(z)\|}{\|z\|} \;:\; z \in \mathbb{C} \text{ and } z \neq 0 \right\} = \sup\{ 1 \} = 1.$$

And, this T is not the zero operator; for example, $T(1) = \iota \neq 0$.

Section 3.3

Prob. 1

Let H be a Hilbert space, $M \subset H$ a convex subset, and (x_n) a sequence in M such that $\|x_n\| \to d$, where $D = \inf_{x \in M} \|x\|$. Show that (x_n) converges in H.

Solution

Here we have a Hilbert space H, a convex subset M of H, and a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of M for which the sequence $(\|x_n\|)_{n \in \mathbb{N}}$ of norms converges in $\mathbb{R}$ to the real number D, where

$$D \colon = \inf\{ \|x\| \;:\; x \in M \}.$$

Note that as the norm of every element is non-negative, so the set $\{ \|x\| : x \in M \}$ is bounded below, with the real number 0 being a lower bound of this set, which implies that $D \geq 0$ because D is the *infimum* or the *greatest lower bound* of this set.

For each $n \in \mathbb{N}$, as $x_n \in M$ and as $D\colon = \inf\{\ \|x\| : x \in M\ \}$, so we must have

$$\|x_n\| \geq D.$$

As M is convex, so for every elements $x, y \in M$ and for every scalar $\alpha \in [0,1]$, the linear convex combination $(1-\alpha)x + \alpha y \in M$ also. In particular, for every elements $x, y \in M$, the linear convex combination

$$\frac{1}{2}(x+y) = \frac{1}{2}x + \frac{1}{2}y = \left(1 - \frac{1}{2}\right)x + \frac{1}{2}y \in M$$

also.

And, as the norm of H is induced by the inner product on H, so this norm satisfies the parallelogram identity. That is, for all $x, y \in H$,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Since $M \subset H$, therefore this identity holds for all $x, y \in M$ also.

As

$$\lim_{n\to\infty} \|x_n\| = D,$$

so

$$\lim_{n\to\infty} \|x_n\|^2 = D^2,$$

which implies that, for every real number $\varepsilon > 0$, we can find a natural number N such that

$$\left|\|x_n\|^2 - D^2\right| < \frac{\varepsilon^2}{4}$$

for any natural number $n > N$.

Let d denote the metric induced by the inner product on H. Then, for

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any natural numbers m and n such that $m > N$ and $n > N$, we have

$$\begin{aligned}
[d(x_m, x_n)]^2 &= \|x_m - x_n\|^2 \\
&= 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2 \\
&= 2\|x_m\|^2 + 2\|x_n\|^2 - \left\|2\left[\frac{1}{2}(x_m + x_n)\right]\right\|^2 \\
&= 2\|x_m\|^2 + 2\|x_n\|^2 - 4\left\|\frac{1}{2}(x_m + x_n)\right\|^2 \\
&\leq 2\|x_m\|^2 + 2\|x_n\|^2 - 4D^2 \\
&= 2\left(\|x_m\|^2 - D^2\right) + 2\left(\|x_n\|^2 - D^2\right) \\
&\leq 2\left|\|x_m\|^2 - D^2\right| + 2\left|\|x_n\|^2 - D^2\right| \\
&< 2\frac{\varepsilon^2}{4} + 2\frac{\varepsilon^2}{4} \\
&= \varepsilon^2,
\end{aligned}$$

which implies that $d(x_m, x_n) < \varepsilon$.

Thus we have shown that, for every real number $\varepsilon > 0$, we can find a natural number N such that $d(x_m, x_n) < \varepsilon$ for any natural numbers m and n such that $m > N$ and $n > N$. Therefore the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $M \subset H$ and hence a Cauchy sequence in the Hilbert space H.

Now as $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in H with respect to the metric induced by the inner product on H and as H is complete with respect to this metric, so this sequence converges in H.

Prob. 2

Show that the subset $M = \{\ y = (\eta_j)\ :\ \sum \eta_j = 1\ \}$ of complex space $\mathbb{C}^n$ is complete and convex. Find the vector of minimum norm in M.

Solution

Here $\mathbb{C}^n$ is the inner product space of all the ordered n-tuples of complex numbers, with the inner product defined by

$$\langle x, y\rangle :\, = \sum_{j=1}^{n} \xi_j \overline{\eta_j}$$

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for all $x \colon = (\xi_1, \ldots, \xi_n)$ and $y \colon = (\eta_1, \ldots, \eta_n)$ in $\mathbb{C}^n$.

And, the set M is given by

$$M \colon = \left\{ (\eta_1, \ldots, \eta_n) \in \mathbb{C}^n \ : \ \sum_{j=1}^{n} \eta_j = 1 \right\}.$$

We first show that this set M is convex. For this, let $x \colon = (\xi_1, \ldots, \xi_n)$ and $y \colon = (\eta_1, \ldots, \eta_n)$ be some elements of M, and let α be a (real) scalar such that $0 \leq \alpha \leq 1$.

Then

$$\sum_{j=1}^{n} \xi_j = 1 = \sum_{j=1}^{n} \eta_j,$$

and as $0 \leq \alpha \leq 1$, so $0 \leq 1 - \alpha \leq 1$ also. Now the element

$$(1-\alpha)x + \alpha y = (\ (1-\alpha)\xi_1 + \alpha\eta_1, \ldots, (1-\alpha)\xi_n + \alpha\eta_n\ ),$$

and

$$\sum_{j=1}^{n} [(1-\alpha)\xi_j + \alpha\eta_j] = (1-\alpha)\sum_{j=1}^{n} \xi_j + \alpha \sum_{j=1}^{n} \eta_j = (1-\alpha)\cdot 1 + \alpha \cdot 1 = 1,$$

which shows that $(1-\alpha)x + \alpha y \in M$ for any elements $x, y \in M$ and for any (real) scalar $\alpha \in [0, 1]$. Hence M is convex.

The norm induced by the inner product on $\mathbb{C}^n$ is defined by

$$\|x\| \colon = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{j=1}^{n} \xi_j \overline{\xi_j}} = \sqrt{\sum_{j=1}^{n} |\xi_j|^2}$$

for all $x \colon = (\xi_1, \ldots, \xi_n)$ in $\mathbb{C}^n$, and the metric induced by this norm is defined by

$$d(x, y) \colon = \|x - y\| = \sqrt{\sum_{j=1}^{n} |\xi_j - \eta_j|^2}$$

for all $x \colon = (\xi_1, \ldots, \xi_n)$ and $y \colon = (\eta_1, \ldots, \eta_n)$ in $\mathbb{C}^n$. We now show that set M is complete with respect to this metric.

For this, let $(x_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in M, where

$$x_m \colon = (\xi_{m1}, \ldots, \xi_{mn})$$

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for any natural number m.

For each $m \in \mathbb{N}$, as $x_m \in M$, so we must have

$$\sum_{j=1}^{n} \xi_{mj} = 1. \tag{0.14}$$

Let us take a real number $\varepsilon > 0$. Then there is a natural number N such that

$$d(x_k, x_m) < \varepsilon$$

for any natural numbers $k > N$ and $m > N$. That is,

$$\sqrt{\sum_{j=1}^{n} |\xi_{kj} - \xi_{mj}|^2} < \varepsilon$$

for any natural numbers $k > N$ and $m > N$.

So, for each $i = 1, \ldots, n$, we have

$$\begin{aligned} |\xi_{ki} - \xi_{mi}| &= \sqrt{|\xi_{ki} - \xi_{mi}|^2} \\ &\leq \sqrt{\sum_{j=1}^{n} |\xi_{kj} - \xi_{mj}|^2} \\ &= d(x_k, x_m) \\ &< \varepsilon \end{aligned}$$

for any natural numbers $k > N$ and $m > N$, from which it follows that the sequence $(\xi_{mi})_{m \in \mathbb{N}}$ is a Cauchy sequence in the usual metric space $\mathbb{C}$, and since the usual mettic space $\mathbb{C}$ is complete, the sequence $(\xi_{mi})_{m \in \mathbb{N}}$ converges in $\mathbb{C}$; let us put

$$\xi_i := \lim_{m \to \infty} \xi_{mi} \tag{0.15}$$

for each $i = 1, \ldots, n$.

And, let $x := (\xi_1, \ldots, \xi_n)$. Then $x \in \mathbb{C}^n$.

We show that this x is in M and that our original Cauchy sequence $(x_m)_{m \in \mathbb{N}}$ converges in $\mathbb{C}^n$ to this same x.

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From (0.15) and (0.14), we obtain

$$\begin{aligned}\sum_{i=1}^{n} \xi_i &= \sum_{i=1}^{n} \lim_{m\to\infty} \xi_{mi} \qquad [\text{ by } (0.15)\ ] \\ &= \lim_{m\to\infty} \sum_{i=1}^{n} \xi_{mi} \\ &= \lim_{m\to\infty} 1 \qquad [\text{ by } (0.14)\ ] \\ &= 1,\end{aligned}$$

which shows that $x = (\xi_1, \ldots, \xi_n) \in M$.

Now from (0.15) we can conclude that, for each $i = 1, \ldots, n$, we can find a natural number N_i such that

$$|\xi_{mi} - \xi_i| < \frac{\varepsilon}{\sqrt{n}}$$

for any natural number $m > N_i$.

So for any natural number $m > \max\{N_1, \ldots, N_n\}$, we see that

$$\begin{aligned}d(x_m, x) &= \sqrt{\sum_{i=1}^{n} |\xi_{mi} - \xi_i|^2} \\ &< \sqrt{\sum_{i=1}^{n} \frac{\varepsilon^2}{n}} \\ &= \sqrt{\varepsilon^2} \\ &= \varepsilon,\end{aligned}$$

Thus, for every real number $\varepsilon > 0$, we can find a natural number $N_0 := \max\{N_1, \ldots, N_n\}$ such that

$$d(x_m, x) < \varepsilon$$

for any natural number $m > N_0$. Hence $(x_m)_{m\in\mathbb{N}}$ converges to x in $\mathbb{C}^n$.

Thus we have shown that every Cauchy sequence in M converges to a point which also lies in M. Hence M is complete.

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In order to find the vector of minimum norm in M, you have to minimize

$$f(x) = \|x\|^2 = \sum_{j=1}^{n} |\xi_j|^2$$

subject to the constraint

$$\sum_{j=1}^{n} \xi_j = 1,$$

where each ξ_j is a complex variable, and so we can write it as $\xi_j = \Re\xi_j + \iota\Im\xi_j$, for each $j = 1, \ldots, n$. Our problems now take the following form.

Minimize

$$f(x) = \sum_{j=1}^{n} (\Re\xi_j)^2 + \sum_{j=1}^{n} (\Im\xi_j)^2$$

subject to the constraints

$$\sum_{j=1}^{n} \Re\xi_j = 1, \quad \text{and} \quad \sum_{j=1}^{n} \Im\xi_j = 0.$$

Thus now becomes a minimization problem of $2n$ real variables. Do it yourselves.

Prob. 3

(a) Show that the vector space X of all real-valued continuous functions on $[-1, 1]$ is the direct sum of the set of all even continuous functions and the set of all odd continuous functions on $[-1, 1]$.

(b) Give examples of representations of $\mathbb{R}^3$ as a direct sum (i) of a subspace and its orthogonal complement, (ii) of any complementary pair of subspaces.

Solution

(a) For each $f \in X$, let g and h be the functions defined on $[-1, 1]$ by

$$g(x) := \frac{f(x) + f(-x)}{2} \quad \text{and} \quad h(x) := \frac{f(x) - f(-x)}{2}$$

for all $x \in [-1, 1]$.

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Let the function $i\colon [-1,1] \to \mathbb{R}$ be defined by

$$i(x) :\ = x$$

for all $x \in [-1,1]$.

Then

$$g = \frac{1}{2}\left(f + f \circ (-i)\right) \quad \text{and} \quad h = \frac{1}{2}\left(f - f \circ (-i)\right). \tag{0.16}$$

As the constant (or scalar) multiple of a continuous function is continuous and as i is continuous, so is $-i$.

As the composite of two continuous functions is also continuous and as f and $-i$ are continuous, so is the composite function $f \circ (-i)$.
Refer to (0.16) above.

Finally, as any linear combination of two continuous functions is continuous and as f and $f \circ (-i)$ are continuous, so are g and h. Thus $g, h \in X$.

Now for any $x \in [-1,1]$, we see that

$$\begin{aligned} g(-x) &= \frac{1}{2}\left[\, f(-x) + f(-(-x))\,\right] \\ &= \frac{1}{2}\left(f(-x) + f(x)\right) \\ &= g(x), \end{aligned}$$

showing that g is even, and,

$$\begin{aligned} h(-x) &= \frac{1}{2}\left[\, f(-x) - f(-(-x))\,\right] \\ &= \frac{1}{2}\left(f(-x) - f(x)\right) \\ &= -\frac{1}{2}\left(f(x) - f(-x)\right) \\ &= -h(x), \end{aligned}$$

showing that h is odd.

Moreover, for all $x \in [-1,1]$,

$$g(x) + h(x) = \frac{1}{2}\left(f(x) + f(-x)\right) + \frac{1}{2}\left(f(x) - f(-x)\right) = f(x),$$

and so $f = g + h$.

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Now we show that this representation of f as a sum of an even (continuous) function and an odd (continuous) function is unique. For this, suppose that $f = g_1 + h_1$, where g_1 is an even (continuous) function defined on $[-1, 1]$ and h_1 is an odd (continuous) function defined on $[-1, 1]$. Then for all $x \in [-1, 1]$, we have

$$\begin{aligned} & f(x) = g_1(x) + h_1(x), \\ \text{and} \quad & f(-x) = g_1(-x) + h_1(-x), \\ \text{that is,} \quad & f(-x) = g_1(x) - h_1(x), \qquad [\text{ since } g_1 \text{ is even and } h_1 \text{ is odd }] \end{aligned}$$

and hence

$$f(x) + f(-x) = 2g_1(x) \quad \text{and} \quad f(x) - f(-x) = 2h_1(x),$$

which imply

$$g_1(x) = \frac{1}{2}\left(f(x) + f(-x)\right) = g(x) \quad \text{and} \quad h_1(x) = \frac{1}{2}\left(f(x) + f(-x)\right) = h(x),$$

which imply that $g_1 = g$ and $h_1 = h$, which is our desired uniqueness proof.

Let E denote the set of all the real-valued even continuous functions defined on $[-1, 1]$, and let O denote the set of all the real-valued odd continuous functions defined on $[-1, 1]$.

Thus we have shown that for every $f \in X$, there is a unique element $g \in E$ and a unique element $h \in O$ such that

$$f = g + h.$$

Hence

$$X = E \oplus O,$$

as required.

Exactly the same proof will work for complex-vlued functions too.

(b) Let the subsets U and V of $\mathbb{R}^3$ be defined as follows:

$$U := \left\{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \; : \; \xi_3 = 0 \right\},$$

and

$$V := \left\{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \; : \; \xi_1 = 0 = \xi_2 \right\}.$$

Then both U and V are (vector) subspaces of $\mathbb{R}^3$.

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First, note that $U \cap V = \{ (0,0,0) \}$.

Moreover, for each $x := (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, there exist unique elements $y := (\xi_1, \xi_2, 0) \in U$ and $z := (0,0,\xi_3) \in V$ such that $x = y + z$. Hence $\mathbb{R}^3 = U \oplus V$.

Moreover, for all $u := (\alpha, \beta, 0) \in U$ and $v := (0,0,\gamma) \in V$, we have

$$\langle u, v \rangle = \vec{u} \bullet \vec{v} = \alpha \cdot 0 + \beta \cdot 0 + 0 \cdot \gamma = 0,$$

which shows that $V \subset U^\perp$.

moreover, if $x = (\lambda_1, \lambda_2, \lambda_3) \in U^\perp$, then, for every $y := (\xi_1, \xi_2, 0) \in U$, we have $\langle x, y \rangle = 0$, that is,

$$\xi_1 \lambda_1 + \xi_2 \lambda_2 = 0 \tag{0.17}$$

for all $\xi_1, \xi_2 \in \mathbb{R}$.

By putting $\xi_1 = 1$ and $\xi_2 = 0$ in (0.17) we obtain $\lambda_1 = 0$, and by putting $\xi_1 = 0$ and $\xi_2 = 1$ we obtain $\lambda_2 = 0$. Therefore $x = (0,0,\lambda_3) \in V$. So $U^\perp \subset V$.

Therefore $V = U^\perp$, and so

$$\mathbb{R}^3 = U \oplus V = U \oplus U^\perp.$$

Now let W be the subset of $\mathbb{R}^3$ given by

$$W := \left\{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \; : \; \xi_1 = \xi_2 = \xi_3 \right\}.$$

Then again $U \cap W = \{ (0,0,0) \}$.

Moreover, for each element $x := (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, there are unique elements $u := (\xi_1 - \xi_3, \xi_2 - \xi_3, 0) \in U$ and $w := (\xi_3, \xi_3, \xi_3) \in W$ such that $x = u + w$. So $\mathbb{R}^3 = U \oplus W$.

However, we note that $(1,1,0) \in U$ and $(1,1,1) \in W$, but

$$\langle (1,1,0), (1,1,1) \rangle = 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 = 2,$$

showing that neither of U and W is orthogonal to the other.

Geometrically, U is our well-known xy-plane, V is the z-axis, and W is the straight line through the origin and the point $(1,1,1)$.

Prob. 5

Let $X = \mathbb{R}^2$. Find $M^\perp$ if M is (a) $\{ x \}$, where $x = (\xi_1, xi_2) \neq 0$, (b) a linearly independent set $\{ x_1, x_2 \} \subset X$.

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Solution

(a) By definition,

$$\begin{aligned} M^{\perp} &= \{\ y \in \mathbb{R}^2\ :\ \langle y, v\rangle = 0\ \text{ for all } v \in M\ \} \\ &= \{\ y \in \mathbb{R}^2\ :\ \langle y, x\rangle = 0\ \} \\ &= \{\ (\eta_1, \eta_2) \in \mathbb{R}^2\ :\ \langle (\eta_1, \eta_2), (\xi_1, \xi_2)\rangle = 0\ \} \\ &= \{\ (\eta_1, \eta_2) \in \mathbb{R}^2\ :\ \eta_1\xi_1 + \eta_2\xi_2 = 0\ \}, \end{aligned}$$

which is a straight line through the origin in the so-called xy-plane since $x = (\xi_1, \xi_2) \neq (0, 0)$. If $\xi_2 \neq 0$, then this line has slope equal to $\frac{-\xi_1}{\xi_2}$, and if $\xi_2 = 0$, then this is the vertical line through the origin (i.e. the so-called y-axis).

(b) Let us put $x_1 := (\alpha_{11}, \alpha_{12})$, and $x_2 := (\alpha_{21}, \alpha_{22})$. Then as the set $M = \{\ x_1, x_2\ \}$ is a linearly independent subset of $\mathbb{R}^2$, so neither of x_1 and x_2 is a scalar multiple of the other (and hence neither is the zero vector in $\mathbb{R}^2$).

So,

$$\begin{aligned} M^{\perp} &= \{\ y \in \mathbb{R}^2\ :\ \langle y, v\rangle = 0\ \text{ for all } v \in M\ \} \\ &= \{\ y \in \mathbb{R}^2\ :\ \langle y, x_1\rangle = 0 = \langle y, x_2\rangle\ \} \\ &= \{\ (\eta_1, \eta_2) \in \mathbb{R}^2\ :\ \langle (\eta_1, \eta_2), (\alpha_{11}, \alpha_{12})\rangle = 0 = \langle (\eta_1, \eta_2), (\alpha_{21}, \alpha_{22})\rangle\ \} \\ &= \{\ (\eta_1, \eta_2) \in \mathbb{R}^2\ :\ \eta_1\alpha_{11} + \eta_2\alpha_{12} = 0 = \eta_1\alpha_{21} + \eta_2\alpha_{22}\ \} \\ &= \{\ (\eta_1, \eta_2) \in \mathbb{R}^2\ :\ \alpha_{11}\eta_1 + \alpha_{12}\eta_2 = 0 = \alpha_{21}\eta_1 + \alpha_{22}\eta_2\ \} \end{aligned}$$

Thus, writing all the elements of $\mathbb{R}^2$ as *column vectors*, we can conclude that $M^{\perp}$ is the set of all the vectors $\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ in the plane that are the solutions to the following homogenous system of simultaneous linear equations:

$$\begin{aligned} \alpha_{11}\eta_1 + \alpha_{12}\eta_2 &= 0, \\ \alpha_{21}\eta_1 + \alpha_{22}\eta_2 &= 0; \end{aligned}$$

and, passing to matrices, $M^{\perp}$ is the set of all the vectors $\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ in the plane such that

$$\begin{bmatrix} \alpha_{11}\eta_1 + \alpha_{12}\eta_2 \\ \alpha_{21}\eta_1 + \alpha_{22}\eta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

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that is,

$$\begin{bmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}.$$

As x_1 and x_2 are linearly independent, so the rows of the matrix

$$\begin{bmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{bmatrix}$$

are linearly independent (and so neither is a scalar multiple of the other), which implies that this matrix is invertible, and multiplying both sides of the last matrix equation on the left by this inverse matrix yields the solution

$$y = \begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}.$$

Hence $M^{\perp} = \{\ (0,0)\ \}$.

Prob. 6

Show that $Y = \{\ x \mid x = (\xi_j) \in \ell^2,\ \xi_{2n} = 0,\ n \in \mathbb{N}\ \}$ is a closed subspace of ℓ^2 and find $Y^{\perp}$. What is $Y^{\perp}$ if $Y = \operatorname{span}\{e_1, \ldots, e_n\} \subset \ell^2$, where $e_j = (\delta_{jk})$?

Solution

Recall that ℓ^2, by definition, is the inner product space consisting of all the sequences $(\alpha_n)_{n\in\mathbb{N}}$ of (real or) complex numbers for which the series $\sum |\alpha_n|^2$ converges in $\mathbb{R}$, with the inner product defined by

$$\langle x, y\rangle :\ = \begin{cases}\sum_{n=1}^{\infty} \xi_n \eta_n & \text{if } X \text{ is real,}\\ \sum_{n=1}^{\infty} \xi_n \overline{\eta_n} & \text{if } X \text{ is complex}\end{cases}$$

for any elements $x :\ = (\xi_n)_{n\in\mathbb{N}}$ and $y :\ = (\eta_n)_{n\in\mathbb{N}}$ in ℓ^2, that is, for any sequences $x :\ = (\xi_n)_{n\in\mathbb{N}}$ and $y :\ = (\eta_n)_{n\in\mathbb{N}}$ of (real or) complex numbers for which the series $\sum |\xi_n|^2$ and $\sum |\eta_n|^2$ converge.

The (absolute) convergence of the series $\sum \xi_n \overline{\eta_n}$ or $\sum \xi_n \eta_n$ then follows from the Cauchy-Schwarz inequality (i.e. (11) Subec. 1.2-3 in Kreyszig).

So the norm on ℓ^2 is given by

$$\|x\|_{\ell^2} :\ = \sqrt{\sum_{n=1}^{\infty} |\xi_n|^2}$$

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for any element $x\colon = (\xi_n)_{n\in\mathbb{N}}$ in ℓ^2, and the metric on ℓ^2 is defined by

$$d_{\ell^2}(x,y)\colon = \sqrt{\sum_{n=1}^{\infty} |\xi_n - \eta_n|^2}$$

for any elements $x\colon = (\xi_n)_{n\in\mathbb{N}}$ and $y\colon = (\eta_n)_{n\in\mathbb{N}}$ in ℓ^2. And, with respect to this metric, the space ℓ^2 is a complete metric space (i.e. a Hilbert space). [Refer to Subsec. 1.5-4 in Kreyszig.]

Here

$$Y = \left\{ (\xi_n)_{n\in\mathbb{N}} \in \ell^2 \ : \ \xi_{2n} = 0 \ \forall \ n \in \mathbb{N} \right\}.$$

First, note that the zero sequence

$$\mathbf{0}_{\ell^2}\colon = (0, 0, 0, \ldots)$$

is in Y. Suppose that $x\colon = (\xi_n)_{n\in\mathbb{N}}$ and $y\colon = (\eta_n)_{n\in\mathbb{N}}$ are some elements of Y and α and β are some scalars (i.e. some real or complex numbers). Then $x, y \in \ell^2$, and

$$\xi_{2n} = \eta_{2n} = 0$$

for all $n \in \mathbb{N}$.

Then the sequence $\alpha x + \beta y$ also belongs to ℓ^2 [Refer to Subsecs. 1.2-3 and 2.2-3 in Kreyszig.],

$$\alpha x + \beta y = (\alpha\xi_n + \beta\eta_n)_{n\in\mathbb{N}},$$

and, for all $n \in \mathbb{N}$, we have

$$\alpha\xi_{2n} + \beta\eta_{2n} = \alpha \cdot 0 + \beta \cdot 0 = 0,$$

which shows that $\alpha x + \beta y \in Y$ also. Thus Y is a (vector) subspace of ℓ^2.

In order to show that Y is closed in ℓ^2, let $(x_m)_{m\in\mathbb{N}}$ be a sequence in Y and suppose that this sequence converges in ℓ^2 to a point x. Let's put

$$x_m\colon = (\xi_{m,n})_{n\in\mathbb{N}} \quad \text{for all } m \in \mathbb{N},$$

and

$$x\colon = (\xi_n)_{n\in\mathbb{N}}.$$

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Suppose a real number $\varepsilon > 0$ is given. Then there exists a natural number N such that

$$d(x_m, x) = \sqrt{\sum_{n=1}^{\infty} |\xi_{m,n} - \xi_n|^2} = \sqrt{\lim_{n\to\infty} \sum_{j=1}^{n} |\xi_{m,j} - \xi_j|^2} < \varepsilon$$

for any natural number $m > N$.

Let k be an arbitrary natural number. Then we see that

$$\begin{aligned}
&|\xi_{m,k} - \xi_k| \\
&= \sqrt{|\xi_{m,k} - \xi_k|^2} \\
&\leq \sqrt{\sum_{j=1}^{n} |\xi_{m,j} - \xi_j|^2} \quad \text{for any natural number } n \geq k \\
&\leq \sqrt{\lim_{n\to\infty} \sum_{j=1}^{n} |\xi_{m,j} - \xi_j|^2} \\
&\text{[because the sequence of partial sums is monotonically increasing]} \\
&= \sqrt{\sum_{n=1}^{\infty} |\xi_{m,n} - \xi_n|^2} \\
&= d(x_m, x) \\
&< \varepsilon
\end{aligned}$$

for any natural number $m > N$, and thus it follows that the sequence $(\xi_{m,k})_{m\in\mathbb{N}}$ of (real or complex) numbers converges to ξ_k in the usual metric space $\mathbb{R}$ or $\mathbb{C}$.

Thus for all $k \in \mathbb{N}$, we have

$$\lim_{m\to\infty} \xi_{m,k} = \xi_k.$$

But, for each $m \in \mathbb{N}$, as $x_m = (\xi_{m,n})_{n\in\mathbb{N}} \in Y$, so

$$\xi_{m,2n} = 0 \quad \text{for all} \quad n \in \mathbb{N},$$

and therefore

$$\xi_{2n} = \lim_{m\to\infty} \xi_{m,2n} = 0 \quad \text{for all} \quad n \in \mathbb{N},$$

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which shows that $x = (\xi_n)_{n\in\mathbb{N}} \in Y$.

Thus we have shown that the limit of every convergent sequence of points of Y also belongs to Y. Hence (by Theorem 1.4-6(a) in Kreyszig) Y is closed in ℓ^2.

Thus we have shown that Y is a (vector) subspace of ℓ^2 and that Y is also closed in ℓ^2.

Now, by definition,

$$
\begin{aligned}
&Y^{\perp}\\
&= \left\{\ x \in \ell^2\ :\ \langle x, y\rangle = 0\ \forall y \in Y\ \right\}\\
&= \left\{\ (\xi_n)_{n\in\mathbb{N}} \in \ell^2\ :\ \left\langle (\xi_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}} \right\rangle = 0\ \forall\, (\eta_n)_{n\in\mathbb{N}} \in Y\ \right\}\\
&= \left\{\ (\xi_n)_{n\in\mathbb{N}} \in \ell^2\ :\ \sum_{n=1}^{\infty} \xi_n \overline{\eta_n} = 0\ \forall\, (\eta_n)_{n\in\mathbb{N}} \in Y\ \right\}\\
&= \left\{\ (\xi_n)_{n\in\mathbb{N}} \in \ell^2\ :\ \sum_{n=1}^{\infty} \xi_n \overline{\eta_n} = 0\ \forall\, (\eta_n)_{n\in\mathbb{N}} \in \ell^2 \text{ such that } \eta_{2n} = 0\ \forall n \in \mathbb{N}\ \right\}\\
&=\\
&\left\{\ (\xi_n)_{n\in\mathbb{N}} \in \ell^2\ :\ \sum_{n=1}^{\infty} \xi_{2n-1} \overline{\eta_{2n-1}} = 0\ \forall\ (\eta_n)_{n\in\mathbb{N}} \in \ell^2 \text{ such that } \eta_{2n} = 0\ \forall\ n \in \mathbb{N}\ \right\}.
\end{aligned}
$$

Thus $Y^{\perp}$ is given by

$$
\begin{aligned}
&Y^{\perp} =\\
&\left\{\ (\xi_n)_{n\in\mathbb{N}} \in \ell^2\ :\ \sum_{n=1}^{\infty} \xi_{2n-1} \overline{\eta_{2n-1}} = 0\ \forall\ (\eta_n)_{n\in\mathbb{N}} \in \ell^2 \text{ such that } \eta_{2n} = 0\ \forall\ n \in \mathbb{N}\ \right\}.
\end{aligned}
\tag{0.18}
$$

Let S be the subset of ℓ^2 given by

$$
S:\ = \left\{\ (\xi_n)_{n\in\mathbb{N}} \in \ell^2\ :\ \xi_{2n-1} = 0\ \forall n \in \mathbb{N}\ \right\}. \tag{0.19}
$$

We first show that $S \subset Y^{\perp}$. Let $x:\ = (\xi_n)_{n\in\mathbb{N}}$ be any point of S. Then $x \in \ell^2$ and $\xi_{2n-1} = 0$ for all $n \in \mathbb{N}$.

Let $y:\ = (\eta_n)_{n\in\mathbb{N}}$ be any element of Y. Then $y \in \ell^2$ and $\eta_{2n} = 0$ for all $n \in \mathbb{N}$.

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Then, by the definition of ℓ^2, we have

$$\begin{aligned}\langle x, y\rangle &= \left\langle (\xi_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}}\right\rangle \\ &= \sum_{n=1}^{\infty} \xi_n \overline{\eta_n} \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} \xi_j \overline{\eta_j} \\ &= \lim_{n\to\infty} \underbrace{[0\cdot\eta_1 + \xi_2\cdot 0 + 0\cdot\eta_3 + \xi_4\cdot 0 + 0\cdot\eta_5 + \xi_6\cdot 0 + \cdots]}_{n \text{ terms}} \\ &= 0\end{aligned}$$

Thus, we have shown that $\langle x, y\rangle = 0$ for all $y \in Y$, which implies that $x \in Y^{\perp}$ and hence $S \subset Y^{\perp}$.

We now show that $Y^{\perp} \subset S$ also. For this we show that $S^c \subset \left(Y^{\perp}\right)^c$. Let $z\colon = (\zeta_n)_{n\in\mathbb{N}}$ be any element of ℓ^2 such that $z \notin S$. Then there exists a natural number n such that $\zeta_{2n-1} \neq 0$. Refer to the definition of set S in (0.19) above. Let N be the smallest natural number such that $\zeta_{2N-1} \neq 0$. Let $y\colon = (\eta_n)_{n\in\mathbb{N}}$, where

$$\eta_n\colon = \begin{cases} 1 & \text{if } n = 2N-1, \\ 0 & \text{otherwise.} \end{cases}$$

This y belongs to Y, and moreover,

$$\langle z, y\rangle = \zeta_{2N-1} \neq 0, .$$

which implies that $z \notin Y^{\perp}$. So $z \in \left(Y^{\perp}\right)^c$. Thus $S^c \subset \left(Y^{\perp}\right)^c$, which implies that $Y^{\perp} \subset S$.

Hence

$$Y^{\perp} = S = \left\{ (\xi_n)_{n\in\mathbb{N}} \in \ell^2 \; : \; \xi_{2n-1} = 0 \;\forall n \in \mathbb{N} \right\}.$$

Now let n be a given natural number, and let $e_1, \ldots, e_n$ be defined as

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follows:

$$
\begin{aligned}
e_1 &:= (1, 0, 0, \ldots)\\
e_2 &:= (0, 1, 0, 0, \ldots)\\
e_3 &:= (0, 0, 1, 0, 0, \ldots)\\
&\vdots\\
e_n &:= \left(\underbrace{0, \ldots, 0,}_{n-1 \text{ terms}} \; 1, 0, 0, \ldots \right).
\end{aligned}
$$

That is, for each $j = 1, \ldots, n$, we define $e_j := (\delta_{jk})_{k\in\mathbb{N}}$. Here the function $\delta\colon \mathbb{N}\times\mathbb{N} \to \{\, 0, 1 \,\}$ is defined by

$$
\delta_{jk} := \begin{cases} 1 & \text{if j = k,} \\ 0 & \text{if } j \neq k \end{cases}
$$

for all $(j, k) \in \mathbb{N}\times\mathbb{N}$, and it is called the *Kronecker's delta* function.

You can easily verify that all of the e_j belong to ℓ^2.

Now let Y be ths span (i.e. the set of all the finite linear combinations) of the set $\{\, e_1, \ldots, e_n \,\}$. Then

$$
\begin{aligned}
Y &= \{\, \eta_1 e_1 + \cdots + \eta_n e_n \;:\; \eta_1, \ldots, \eta_n \text{ are complex numbers} \,\}\\
&= \{\, (\eta_1, \ldots, \eta_n, 0, 0, \ldots) \;:\; \eta_1, \ldots, \eta_n \text{ are complex numbers} \,\}\\
&= \left\{\, (\eta_k)_{k\in\mathbb{N}} \;:\; (\eta_1 \ldots, \eta_n) \in \mathbb{C}^n, \eta_k = 0 \text{ for } k > n \,\right\}\\
&= \left\{\, (\eta_k)_{k\in\mathbb{N}} \in \ell^2 \;:\; \eta_k = 0 \text{ for } k > n \,\right\}.
\end{aligned}
$$

Now

$$
\begin{aligned}
&Y^\perp\\
&= \{\, x \in \ell^2 \;:\; \langle x, y\rangle = 0 \;\forall\; y \in Y \,\}\\
&= \left\{\, (\xi_k)_{k\in\mathbb{N}} \in \ell^2 \;:\; \left\langle (\xi_k)_{k\in\mathbb{N}}, (\eta_k)_{k\in\mathbb{N}} \right\rangle = 0 \;\forall\; (\eta_k)_{k\in\mathbb{N}} \in Y \,\right\}\\
&= \left\{\, (\xi_k)_{k\in\mathbb{N}} \in \ell^2 \;:\; \left\langle (\xi_k)_{k\in\mathbb{N}}, (\eta_k)_{k\in\mathbb{N}} \right\rangle = 0 \;\forall\; (\eta_k)_{k\in\mathbb{N}} \in \ell^2 \text{ such that } \eta_k = 0 \text{ for all } k > n \,\right\}\\
&= \left\{\, (\xi_k)_{k\in\mathbb{N}} \in \ell^2 \;:\; \sum_{k=1}^{\infty} \xi_k \overline{\eta_k} = 0 \;\forall\; (\eta_k)_{k\in\mathbb{N}} \in \ell^2 \text{ such that } \eta_k = 0 \text{ for all } k > n \,\right\}\\
&= \left\{\, (\xi_k)_{k\in\mathbb{N}} \in \ell^2 \;:\; \sum_{k=1}^{n} \xi_k \overline{\eta_k} = 0 \;\forall\; (\eta_k)_{k\in\mathbb{N}} \in \ell^2 \text{ such that } \eta_k = 0 \text{ for all } k > n \,\right\}.
\end{aligned}
$$

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Let S be the subset of ℓ^2 given by

$$S\colon = \left\{ \, (\xi_k)_{k\in\mathbb{N}} \in \ell^2 \, : \, \xi_k = 0 \text{ for } k = 1, \ldots, n \, \right\}.$$

We show that $Y^{\perp} = S$.

Let $x\colon = (\xi_k)_{k\in\mathbb{N}}$ be any element of S. Then $(\xi_k)_{k\in\mathbb{N}} \in \ell^2$ such that $\xi_k = 0$ for each $k = 1, \ldots, n$.

And, let $y\colon = (\eta_k)_{k\in\mathbb{N}} \in Y$. Then $(\eta_k)_{k\in\mathbb{N}} \in \ell^2$ such that $\eta_k = 0$ $k > n$. So we have

$$\begin{aligned}
\langle x, y\rangle &= \left\langle (\xi_k)_{k\in\mathbb{N}}, (\eta_k)_{k\in\mathbb{N}} \right\rangle \\
&= \sum_{k=1}^{\infty} \xi_k \overline{\eta_k} \\
&= \lim_{k\to\infty} \sum_{j=1}^{k} \xi_j \eta_j \\
&= \lim_{k\to\infty} \underbrace{[0\cdot\eta_1 + \cdots + 0\cdot\eta_n + \xi_{n+1}\cdot 0 + \xi_{n+2}\cdot 0 + \cdots]}_{k \text{ terms}} \\
&= 0.
\end{aligned}$$

Thus we have shown that $\langle x, y\rangle = 0$ for all $y \in Y$. So $x \in Y^{\perp}$. Therefore $S \subset Y^{\perp}$.

Now we show that $Y^{\perp} \subset S$. For this we show that $S^c \subset \left(Y^{\perp}\right)^c$.

Let $x\colon = (\xi_k)_{k\in\mathbb{N}}$ be any element of ℓ^2 such that $x \notin S$. Then, there exists a natural number $k \in \{1, \ldots, n\}$ such that $\xi_k \neq 0$; let K be the smallest such natural number. Let $y\colon = (\eta_k)_{k\in\mathbb{N}}$ be a sequence of complex numbers such that

$$\eta_k\colon = \begin{cases} 1 & \text{if } k = K, \\ 0 & \text{otherwise.} \end{cases}$$

This $y \in Y$ because $\eta_k = 0$ for all $k \in \mathbb{N}$ such that $k > n$. And, we note that

$$\langle x, y\rangle = \xi_K \neq 0,$$

which shows that $x \notin Y^{\perp}$ and so $x \in \left(Y^{\perp}\right)^c$, showing that $S^c \subset \left(Y^{\perp}\right)^c$. Therefor $Y^{\perp} \subset S$.

Hence

$$Y^{\perp} = S = \left\{ \, (\xi_k)_{k\in\mathbb{N}} \in \ell^2 \, : \, \xi_k = 0 \text{ for } k = 1, \ldots, n \, \right\}.$$

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Section 3.4

Prob. 8

Show that an element x of an inner product space X cannot have "too many" Fourier coefficients $\langle x, e_k \rangle$ which are "big"; here, (e_k) is a given orthonormal sequence; more precisely, show that the number n_m of $\langle x, e_k \rangle$ such that $|\langle x, e_k \rangle| > 1/m$ must satisfy $n_m \leq m^2 \|x\|^2$.

Solution

As $(e_k)_{k \in \mathbb{N}}$ is an orthonormal sequence in the inner product space X, so by Theorem 3.4-6 (Bessel Inequality) in Kreyszig, for any $x \in X$, the series $\sum |\langle x, e_k \rangle|^2$ converges in $\mathbb{R}$, and

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

Let $x \in X$, let m be a given natural number, and let $A_m(x)$ be ths subset of $\mathbb{N}$ given by

$$A_m(x) : = \left\{ k \in \mathbb{N} : \ |\langle x, e_k \rangle| > \frac{1}{m} \right\}.$$

Let n_m denote the cardinality of the set $A_m(x)$ (which is the same as the number of elements in the set $A_m(x)$ if $A_m(x)$ is finite), where

$$n_m \in \{ 0 \} \cup \mathbb{N} \cup \{ \aleph_0 \},$$

where $\aleph_0$ (pronounced "aleph null") denotes the cardinality of the set $\mathbb{N}$ of natural numbers, because the set $A_m(x)$ can be empty, non-empty but finite, or countably infinite. Furthermore, as $A_m(x) \subset \mathbb{N}$ and as $\mathbb{N}$ is countable, so $A_m(x)$ cannot be uncountable.

If $x = \mathbf{0}_X$, the zero vector in X, then

$$\langle x, e_k \rangle = 0$$

for all $k \in \mathbb{N}$, and so the set $A_m(x)$ is empty, and therefore

$$n_m = 0 = m^2 \cdot 0 = m^2 \|x\|^2.$$

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So let's suppose that x is not the zero vector in X, and suppose also that $n_m \geq m^2 \|x\|^2$. Then as

$$|\langle x, e_k \rangle|^2 \geq 0$$

for all $k \in \mathbb{N}$ and as

$$|\langle x, e_k \rangle|^2 > \frac{1}{m^2}$$

for all $k \in A_m(x)$, so we note that

$$\begin{aligned}
\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 &= \sum_{k \in \mathbb{N}} |\langle x, e_k \rangle|^2 \\
&= \sum_{k \in A_m(x)} |\langle x, e_k \rangle|^2 + \sum_{k \in \mathbb{N} - A_m(x)} |\langle x, e_k \rangle|^2 \\
&\geq \sum_{k \in A_m(x)} |\langle x, e_k \rangle|^2 \\
&> \frac{n_m}{m^2} \\
&\geq \frac{m^2 \|x\|^2}{m^2} \\
&= \|x\|^2,
\end{aligned}$$

which contradicts the Bessel's inequality. Hence we must have

$$n_m < m^2 \|x\|^2,$$

as required.

In the above calculation, we have used the equality

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 = \sum_{k \in \mathbb{N}} |\langle x, e_k \rangle|^2 .$$

This is because of Theorem 3.55 in the book *Principles of Mathematical Analysis* by Walter Rudin, 3rd edition, which says that if a series of complex numbers converges absolutely, then, by altering the order of the terms of that series in any way whatsoever, we obtain a series that also converges absolutely and has the same sum as the sum of the original series.

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Section 3.5

Prob. 1

Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in an inner product space X, let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of scalars, and let $x \in X$. If the series $\sum \alpha_n e_n$ converges in X and has sum x, show that then the series $\sum |\alpha_n|^2$ converges in $\mathbb{R}$ and has sum $\|x\|^2$.

Solution

Suppose the series $\sum \alpha_n e_n$ converges in X and

$$\sum_{n=1}^{\infty} \alpha_n e_n = x.$$

Let $(s_n)_{n\in\mathbb{N}}$ be the sequence of the partial sums of the series $\sum \alpha_n e_n$; that is, let

$$s_n := \sum_{j=1}^{n} \alpha_j e_j$$

for all $n \in \mathbb{N}$. Then by our supposition the sequence $(s_n)_{n\in\mathbb{N}}$ converges in X to the point x.

Thus, for every real number $\varepsilon > 0$, we can find a natural number N such that

$$\|s_n - x\| < \varepsilon$$

for every natural number $n > N$.

But we know that

$$|\|u\| - \|v\|| \leq \|u - v\|$$

for all $u, v \in X$.

So, we can conclude that

$$|\|s_n\| - \|x\|| < \varepsilon$$

for every natural number $n > N$.

Thus it follows that the sequence $(\|s_n\|)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ to the real number $\|x\|$; so the sequence $\left(\|s_n\|^2\right)_{n\in\mathbb{N}}$ of squares of the norms converges

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to the real number $\|x\|^2$. But, for each $n \in \mathbb{N}$,

$$\begin{aligned} \|s_n\|^2 &= \langle s_n, s_n \rangle \\ &= \langle \alpha_1 e_1 + \cdots + \alpha_n e_n, \alpha_1 e_1 + \cdots + \alpha_n e_n \rangle \\ &= \alpha_1 \overline{\alpha_1} \langle e_1, e_1 \rangle + \cdots + \alpha_n \overline{\alpha_n} \langle e_n, e_n \rangle \\ &= |\alpha_1|^2 + \cdots + |\alpha_n|^2 \end{aligned}$$

Thus we can conclude that the sequence $\left(\|s_n\|^2\right)_{n\in\mathbb{N}}$ is in fact the sequence $\left(\sum_{j=1}^{n} |\alpha_j|^2\right)_{n\in\mathbb{N}}$ of the partial sums of the series $\sum |\alpha_n|^2$, and as the former sequence converges in $\mathbb{R}$ to the real number $\|x\|^2$, we can conclude that so does the latter sequence, which implies that the series $\sum |\alpha_n|^2$ of non-negative real numbers converges in $\mathbb{R}$ and has the sum $\|x\|^2$.

Thus we have shown that, if the series $\sum \alpha_n e_n$ converges in X, and if

$$\sum_{n=1}^{\infty} \alpha_n e_n = x,$$

then the series $\sum |\alpha_n|^2$ converges in $\mathbb{R}$ and

$$\sum_{n=1}^{\infty} |\alpha_n|^2 = \|x\|^2,$$

as required.

Prob. 3

Illustrate with an example that a convergent series $\sum \langle x, e_k \rangle e_k$ need not have the sum x.

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Solution

Let $X = \ell^2$, and let the sequence $(e_k)_{k\in\mathbb{N}}$ in ℓ^2 be defined as follows:

$$\begin{aligned}
e_1 &:= (0,1,0,0,0,0,\ldots),\\
e_2 &:= (0,0,0,1,0,0,\ldots),\\
e_3 &:= (0,0,0,0,0,1,0,0,\ldots),\\
&\vdots,\\
e_j &:= (0,\ldots,0,\underbrace{1}_{2j\text{-th term}},0,0,\ldots),\\
&\vdots.
\end{aligned}$$

Let $x := (\xi_k)_{k\in\mathbb{N}}$, where

$$\xi_k := \frac{1}{2^k}$$

for all $k \in \mathbb{N}$. The sum

$$\sum_{k=1}^{\infty} |\xi_k|^2 = \sum_{k=1}^{\infty} \left|\frac{1}{2^k}\right|^2 = \sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1/4}{1-1/4} = \frac{1}{3} < +\infty,$$

so

$$x = \left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots\right) \in \ell^2.$$

Then

$$\begin{aligned}
\langle x, e_1\rangle &= \frac{1}{2^2},\\
\langle x, e_2\rangle &= \frac{1}{2^4},\\
\langle x, e_3\rangle &= \frac{1}{2^6},\\
&\vdots\\
\langle x, e_j\rangle &= \frac{1}{2^{2j}},\\
&\vdots\ .
\end{aligned}$$

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For all $k, r \in \mathbb{N}$, we note that

$$\langle e_k, e_r \rangle = \begin{cases} 1 & \text{if } k = r, \\ 0 & \text{if } k \neq r; \end{cases}$$

that is, the sequence $(e_k)_{k\in\mathbb{N}}$ is an orthonormal sequence in ℓ^2. By Theorem 3.4-6 (Bessel's inequality) in Kreyszig, the series $\sum |\langle x, e_k \rangle|^2$ converges in $\mathbb{R}$ with the sum

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2,$$

and since ℓ^2 is a Hilbert space, therefore by Theorem 3.5-2(a) in Kreyszig, we can conclude that the series $\sum \langle x, e_k \rangle e_k$ also converges in ℓ^2, and we find the sum of this series as follows:

$$\begin{aligned}
\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k &= \lim_{k\to\infty} \sum_{j=1}^{k} \langle x, e_j \rangle e_j \\
&= \lim_{k\to\infty} \sum_{j=1}^{k} \frac{1}{2^{2j}} (0, \ldots, 0, \underbrace{1}_{2j\text{-th term}}, 0, 0, \ldots) \\
&= \lim_{k\to\infty} \sum_{j=1}^{k} \left(0, \ldots, 0, \underbrace{\frac{1}{2^{2j}}}_{2j\text{-th term}}, 0, 0, \ldots \right) \\
&= \lim_{k\to\infty} \left(0, \frac{1}{2^2}, 0, \frac{1}{2^4}, \ldots, 0, \underbrace{\frac{1}{2^{2k}}}_{2k\text{-th term}}, 0, \ldots \right) \\
&= \left(0, \frac{1}{2^2}, 0, \frac{1}{2^4}, 0, \frac{1}{2^6}, 0, \frac{1}{2^8}, 0, \ldots \right) \\
&\neq \left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots \right) \\
&= x.
\end{aligned}$$

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Let us now show that

$$\lim_{k\to\infty}\left(0, \frac{1}{2^2}, 0, \frac{1}{2^4}, \dots, 0, \underbrace{\frac{1}{2^{2k}}}_{2k\text{-th term}}, 0, \dots\right)$$
$$= \left(0, \frac{1}{2^2}, 0, \frac{1}{2^4}, 0, \frac{1}{2^6}, 0, \frac{1}{2^8}, 0, \dots\right).$$

Let us take a real number $\varepsilon > 0$. Let's put

$$y_k \colon = \left(0, \frac{1}{2^2}, 0, \frac{1}{2^4}, \dots, 0, \underbrace{\frac{1}{2^{2k}}}_{2k\text{-th term}}, 0, \dots\right)$$

for all $k \in \mathbb{N}$, and let

$$y \colon = \left(0, \frac{1}{2^2}, 0, \frac{1}{2^4}, 0, \frac{1}{2^6}, 0, \frac{1}{2^8}, 0, \dots\right).$$

Then, for each $k \in \mathbb{N}$, we have

$$y - y_k = \left(\underbrace{0, \dots, 0}_{2k+1\text{ terms}}, \frac{1}{2^{2k+2}}, 0, \frac{1}{2^{2k+4}}, 0, \dots\right),$$

and so

$$\begin{aligned}\|y - y_k\|^2 &= \frac{1}{2^{4k+4}} \sum_{i=0}^{\infty} \frac{1}{2^{4i}} \\ &= \frac{1}{16^{k+1}} \sum_{i=0}^{\infty} \frac{1}{16^i} \\ &= \frac{1}{16^{k+1}} \frac{1}{1 - \frac{1}{16}} \\ &= \frac{1}{16^k} \frac{1}{15},\end{aligned}$$

which implies

$$\|y - y_k\|_{\ell^2} = \frac{1}{4^k} \frac{1}{\sqrt{15}}.$$

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As the set $\mathbb{N}$ of natural numbers is not bounded from above in $\mathbb{R}$, so we can find a natural number K such that $K > 1/\varepsilon$. And, for any natural number k, we can prove by induction that $4^k > k$.

So, for any natural number $k > K$, we have

$$4^k\sqrt{15} > 4^k > k > K > 1/\varepsilon,$$

which implies that

$$\|y - y_k\|_{\ell^2} = \frac{1}{4^k\sqrt{15}} < \varepsilon$$

for any natural number $k > K$.

Since ε was an arbitrary positive real number, we can conclude that

$$\lim_{k\to\infty}\left(0, \frac{1}{2^2}, 0, \frac{1}{2^4}, \ldots, 0, \underbrace{\frac{1}{2^{2k}}}_{2k\text{-th term}}, 0, \ldots\right)$$
$$= \left(0, \frac{1}{2^2}, 0, \frac{1}{2^4}, 0, \frac{1}{2^6}, 0, \frac{1}{2^8}, 0, \ldots\right).$$

Prob. 4

If (x_i) is a sequence in an inner product space X such that the series $\|x_1\| + \|x_2\| + \cdots$ converges, show that (s_n) is a Cauchy sequence, where $s_n = x_1 + \cdots + x_n$.

Solution

This result holds in any normed space X. Let $(\sigma_n)_{n\in\mathbb{N}}$ be the sequence of the partial sums of the convergent series

$$\|x_1\| + \|x_2\| + \cdots .$$

Then

$$\sigma_n = \|x_1\| + \|x_2\| + \cdots + \|x_n\|$$

for all $n \in \mathbb{N}$.

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First of all, we note that, for any natural numbers m and n such that $m < n$, we have

$$\begin{aligned}|\sigma_m - \sigma_n| &= \left|\sum_{j=1}^{m} \|x_j\| - \sum_{j=1}^{n} \|x_j\|\right| \\ &= \left|-\sum_{j=m+1}^{n} \|x_j\|\right| \\ &= \left|\sum_{j=m+1}^{n} \|x_j\|\right| \\ &= \sum_{j=m+1}^{n} \|x_j\|.\end{aligned}$$

As $(\sigma_n)_{n\in\mathbb{N}}$ is a convergent sequence, so it is also Cauchy. Therefore, for every real number $\varepsilon > 0$, we can find a natural number N such that

$$|\sigma_m - \sigma_n| < \varepsilon$$

for any natural numbers m and $n > N$.

So, for any natural numbers m and n such that $n > m > N$, we have

$$\sum_{j=m+1}^{n} \|x_j\| < \varepsilon,$$

which implies that

$$\begin{aligned}\|s_m - s_n\| = \|s_n - s_m\| &= \left\|\sum_{j=1}^{n} x_j - \sum_{j=1}^{m} x_j\right\| \\ &= \left\|\sum_{j=m+1}^{n} x_j\right\| \\ &\leq \sum_{j=m+1}^{n} \|x_j\| \\ &< \varepsilon.\end{aligned}$$

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And, for any natural numbers m and n such that $m > n > N$, we interchange the roles of m and n in the last calculation and obtain again

$$\|s_m - s_n\| < \varepsilon.$$

And, for $m = n$, we see that

$$\|s_m - s_n\| = 0 < \varepsilon.$$

Thus we have shown that, for every real number $\varepsilon > 0$, there exists a natural number N such that

$$\|s_m - s_n\| < \varepsilon$$

for any natural numbers m and n such that $m > N$ and $n > N$. Hence $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.

Note that in the above proof, we have used only the properties of X as a *normed* space.

Prob. 5

Show that in a Hilbert space H, convergence of $\sum \|x_j\|$ implies convergence of $\sum x_j$.

Solution

This result holds in any Banach space. Suppose the series $\sum \|x_j\|$ converges (in $\mathbb{R}$). Then, as in the preceding problem (i.e. Prob. 4, Sec. 3.5, in Kreyszig), the sequence $(s_n)_{n\in\mathbb{N}}$ of the partial sums of the series $\sum x_j$ is a Cauchy sequence, and since X is complete (with respect to the metric induced by the norm on X), the sequence $(s_n)_{n\in\mathbb{N}}$ is convergent, and this implies that the series $\sum x_j$ converges in X.

Prob. 6

Let (e_j) be an orthonormal sequence in a Hilbert space H. Show that if

$$x = \sum_{j=1}^{\infty} \alpha_j e_j \quad \text{and} \quad y = \sum_{j=1}^{\infty} \beta_j e_j, \qquad \text{then} \quad \langle x, y\rangle = \sum_{j=1}^{\infty} \alpha_j \overline{\beta_j},$$

the series being absolutely convergent.

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Solution

For each $n \in \mathbb{N}$, let

$$x_n := \sum_{j=1}^{n} \alpha_j e_j, \quad \text{and} \quad y_n := \sum_{j=1}^{n} \beta_j e_j.$$

Thus, $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are the sequences of the partial sums of the series $\sum \alpha_j e_j$ and $\sum \beta_j e_j$, respectively, and therefore, by our hypothesis, the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge in H to the points x and y, respectively.

Therefore, (by Lemma 3.2-2 in Kreyszig) the sequence $(\langle x_n, y_n\rangle)_{n\in\mathbb{N}}$ of the inner products converges (in $\mathbb{R}$ or $\mathbb{C}$) to the (real or complex) number $\langle x, y\rangle$.

But, as the sequence $(e_j)_{j\in\mathbb{N}}$ is an orthonormal sequence in H, so, for each $n \in \mathbb{N}$, we have

$$\langle x_n, y_n\rangle = \langle \alpha_1 e_1 + \cdots + \alpha_n e_n, \beta_1 e_1 + \cdots + \beta_n e_n\rangle = \alpha_1\overline{\beta_1} + \cdots + \alpha_n\overline{\beta_n}.$$

So

$$\lim_{n\to\infty} \left(\alpha_1\overline{\beta_1} + \cdots + \alpha_n\overline{\beta_n}\right) = \lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle;$$

that is, the series $\sum \alpha_j\overline{\beta_j}$ converges, and

$$\sum_{j=1}^{\infty} \alpha_j\overline{\beta_j} = \langle x, y\rangle.$$

By the Cauchy-Schwarz inequality (i.e. (11), Subsec. 1.2-3, in Kreyszig), we see that

$$\begin{aligned}\sum_{j=1}^{\infty} \left|\alpha_j\overline{\beta_j}\right| &\leq \sqrt{\sum_{j=1}^{\infty} |\alpha_j|^2}\sqrt{\sum_{j=1}^{\infty} \left|\overline{\beta_j}\right|^2} \\ &= \sqrt{\sum_{j=1}^{\infty} |\alpha_j|^2}\sqrt{\sum_{j=1}^{\infty} |\beta_j|^2} \\ &= \|x\|\|x\|. \qquad \text{[Refer to Prob. 1, Sec. 3.5, in Kreyszig]}\end{aligned}$$

Thus the series $\sum \alpha_j \overline{\beta_j}$ converges absolutely, and the sum

$$\sum_{j=1}^{\infty} \alpha_j \overline{\beta_j} = \langle x, y \rangle.$$

Note that, in the above proof, we have not used the completeness of the Hilbert space H. Thus the above result holds in *any* inner product space.

Prob. 7

Let (e_k) be an orthonormal sequence in a Hilbert space H. Show that for every $x \in H$, the vector

$$y = \sum_{k=1}^{\infty} \langle x, e_k \rangle \, e_k$$

exists in H and $x - y$ is orthogonal to every e_k.

Solution

As $(e_k)_{k \in \mathbb{N}}$ is an orthonormal sequence in the inner product space H, so (by Theorem 3.4-6 in Kreyszig) the series $\sum |\langle x, e_k \rangle|^2$ converges (in $\mathbb{R}$); therefore (by Theorem 3.5-2 (a) or (c) in Kreyszig) the series $\sum \langle x, e_k \rangle \, e_k$ converges in H. Let

$$y : = \sum_{k=1}^{\infty} \langle x, e_k \rangle \, e_k.$$

Then $y \in H$.

Now, for each $n \in \mathbb{N}$, let

$$y_n : = \sum_{j=1}^{n} \langle x, e_j \rangle \, e_j.$$

Thus $(y_n)_{n \in \mathbb{N}}$ is the sequence of the partial sums of the series $\sum \langle x, e_k \rangle \, e_k$ and therefore this sequence converges to the point y in H.

Now let k be an arbitrary but fixed natural number, and let us take a

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natural number $n > k$. Then

$$\begin{aligned}
\langle x - y_n, e_k \rangle &= \langle x, e_k \rangle - \langle y_n, e_k \rangle \\
&= \langle x, e_k \rangle - \left\langle \sum_{j=1}^{n} \langle x, e_j \rangle \, e_j, \ e_k \right\rangle \\
&= \langle x, e_k \rangle - \sum_{j=1}^{n} \langle x, e_j \rangle \, \langle e_j, \ e_k \rangle \\
&= \langle x, e_k \rangle - \langle x, e_k \rangle \, \langle e_k, e_k \rangle \\
&= \langle x, e_k \rangle - \langle x, e_k \rangle \cdot 1 \\
&= 0,
\end{aligned}$$

and so

$$\lim_{n\to\infty} \langle x - y_n, e_k \rangle = 0. \tag{0.20}$$

But as $y_n \to y$ in H as $n \to \infty$, so the sequence $(x - y_n)_{n\in\mathbb{N}}$ converges to $x - y$, and so the sequence $(\langle x - y_n, e_k \rangle)_{n\in\mathbb{N}}$ of inner products converges (in $\mathbb{R}$ or $\mathbb{C}$) to the inner product $\langle x - y, e_k \rangle$. That is,

$$\lim_{n\to\infty} \langle x - y_n, e_k \rangle = \langle x - y, e_k \rangle \,. \tag{0.21}$$

But, in any metric space, the limit of a convergent sequence is unique. So from (0.20) and (0.21) we can conclude that

$$\langle x - y, e_k \rangle = 0.$$

But as $k \in \mathbb{N}$ was arbitrary, so we can conclude that $x - y$ is orthogonal to every e_k.

Prob. 8

Let (e_k) be an orthonormal sequence in a Hilbert space H, and let $M = \operatorname{span}(e_k)$. Show that for any $x \in H$ we have $x \in \overline{M}$ if and only if x can be represented by

$$x = \sum_{k=1}^{\infty} \alpha_k e_k$$

with coefficients $\alpha_k = \langle x, e_k \rangle$.

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Solution

If

$$x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k,$$

then x is the limit (in H) of the sequence $(x_n)_{n\in\mathbb{N}}$, where

$$x_n := \sum_{k=1}^{n} \langle x, e_k \rangle e_k$$

for all $n \in \mathbb{N}$, and $(x_n)_{n\in\mathbb{N}}$ is a sequence in $M := \operatorname{span}(e_k)$, which (by Theorem 1.4-6 (a) in Kreyszig) implies that $x \in \overline{M} = \overline{\operatorname{span}(e_k)}$.

Conversely, suppose that $x \in \overline{M}$. Then, for every real number $\varepsilon > 0$, we can find a point $v \in M$ such that

$$\|x - v\| < \varepsilon. \tag{0.22}$$

As $v \in M = \operatorname{span}(e_k)$, so v can be written as a (finite) linear combination of the terms of the orthonormal sequence $(e_k)_{k\in\mathbb{N}}$; that is, there is a natural number N and an N-tuple $(\beta_1, \ldots, \beta_N)$ of scalars such that

$$v = \beta_1 e_1 + \cdots + \beta_N e_N.$$

Now this v belongs to the span of $(e_1, \ldots, e_N)$, so (by Prob. 6, Sec. 3.4, in Kreyszig) we can conclude that

$$\left\| x - \sum_{j=1}^{N} \langle x, e_j \rangle e_j \right\| \leq \|x - v\|,$$

which together with (0.22) implies that

$$\left\| x - \sum_{j=1}^{N} \langle x, e_j \rangle e_j \right\| < \varepsilon. \tag{0.23}$$

Let

$$y_N := \sum_{j=1}^{N} \langle x, e_j \rangle e_j.$$

Then $y_N \in \operatorname{span}(e_1, \ldots, e_N)$.

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And, for any natural number $n > N$, we note that

$$\{e_1, \ldots, e_N\} \subset \{e_1, \ldots, e_n\},$$

and so

$$\operatorname{span}(e_1, \ldots, e_N) \subset \operatorname{span}(e_1, \ldots, e_n),$$

which implies that $y_N \in \operatorname{span}(e_1, \ldots, e_n)$, and therefore (again by Prob. 6, Sec. 3.4, in Kreyszig) we can conclude that

$$\left\| x - \sum_{j=1}^{n} \langle x, e_j \rangle e_j \right\| \leq \| x - y_N \|.$$

Then (0.23) yields

$$\left\| x - \sum_{j=1}^{n} \langle x, e_j \rangle e_j \right\| < \varepsilon.$$

Thus, for every real number $\varepsilon > 0$, we can find a natural number N such that

$$\left\| x - \sum_{j=1}^{n} \langle x, e_j \rangle e_j \right\| < \varepsilon$$

for any natural number $n > N$. Therefore the sequence $\left(\sum_{j=1}^{n} \langle x, e_j \rangle e_j \right)_{n \in \mathbb{N}}$ of the partial sums of the series $\sum \langle x, e_k \rangle e_k$ converges to the point x, which means that the series $\sum \langle x, e_k \rangle e_k$ converges and has sum x.

That is,

$$\lim_{n \to \infty} \sum_{j=1}^{n} \langle x, e_j \rangle e_j = x,$$

and so we can write

$$x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k,$$

as required.

Note that for the above proof, we have not needed the completeness of H. So this result holds even if H is any inner product space that is not necessarily a Hilbert space.

Prob. 9

Let (e_n) and $(\tilde{e}_n)$ be orthonormal sequences in a Hilbert space H, and let $M_1 = \text{span}(e_n)$ and $M_2 = \text{span}(\tilde{e}_n)$. Using Prob. 8, show that $\overline{M_1} = \overline{M_2}$ if and only if

$$\text{(a)} \quad e_n = \sum_{m=1}^{\infty} \alpha_{nm}\tilde{e}_m, \qquad \text{(b)} \quad \tilde{e}_n = \sum_{m=1}^{\infty} \overline{\alpha_{mn}} e_m, \qquad \alpha_{nm} = \langle e_n, \tilde{e}_m \rangle .$$

Solution

Let us define

$$\alpha_{nm} \colon = \langle e_n, \tilde{e}_m \rangle \tag{0.24}$$

for all $(n, m) \in \mathbb{N} \times \mathbb{N}$.

If $\overline{M_1} = \overline{M_2}$, then $\overline{M_1} \subset \overline{M_2}$ and $\overline{M_2} \subset \overline{M_1}$.

For each $n \in \mathbb{N}$, as $e_n \in \text{span}\,(e_m)_{m\in\mathbb{N}} = M_1$ and as $M_1 \subset \overline{M_1} \subset \overline{M_2}$, so $e_n \in \overline{M_2} = \overline{\text{span}\,(\tilde{e}_m)_{m\in\mathbb{N}}}$, and therefore (by Prob. 8, Sec. 3.5, in Kreyszig) we have

$$e_n = \sum_{m=1}^{\infty} \langle e_n, \tilde{e}_m \rangle \, \tilde{e}_m = \sum_{m=1}^{\infty} \alpha_{nm}\tilde{e}_m,$$

using (0.24) above.

Similarly, for each $n \in \mathbb{N}$, as $\tilde{e}_n \in \text{span}\,(\tilde{e}_m)_{m\in\mathbb{N}} = M_2$ and as $M_2 \subset \overline{M_2} \subset \overline{M_1}$, so $\tilde{e}_n \in \overline{M_1} = \overline{\text{span}\,(e_m)_{m\in\mathbb{N}}}$, and therefore (again by Prob. 8, Sec. 3.5, in Kreyszig) we have

$$\begin{aligned} \tilde{e}_n &= \sum_{m=1}^{\infty} \langle \tilde{e}_n, e_m \rangle \, e_m \\ &= \sum_{m=1}^{\infty} \overline{\langle e_m, \tilde{e}_n \rangle} e_m \\ &= \sum_{m=1}^{\infty} \overline{\alpha_{mn}} e_m, \qquad [\text{ using } (0.24) \text{ }] \end{aligned}$$

as required.

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Prob. 10

Let X be an inner product space, let M be a non-empty orthonormal subset of X, and let $x \in X$. Let $M(x)$ be the subset of M defined as follows:

$$M(x) \colon = \{ \, v \in M \; : \; \langle x, v \rangle \neq 0 \, \}. \tag{0.25}$$

Then set $M(x)$ is at most countable (i.e. either finite or countable).

Solution

Recall that a set S is said to be *finite* if either S is empty or, for some natural number N, there exists a bijective function $f \colon \{ \, 1, \ldots, N \, \} \to S$; otherwise S is said to be *infinite*. And, recall also that the set S is said to be *countable* if there exists a bijective function $f \colon \mathbb{N} \to S$. If the set S is neither finite nor countable, then S is said to be *uncountable*. [Refer to Sections 6 and 7 in the book *Topology* by James R. Munkres, 2nd edition. Or, refer to Definitions 2.3 and 2.4 in the book *Principles of Mathematical Analysis* by Walter Rudin, 3rd edition.]

If $x = \mathbf{0}_X$, the zero vector in X, then we note that, for any element $y \in X$,

$$\langle x, y \rangle = \langle \mathbf{0}_X, y \rangle = \langle 0y, y \rangle = 0 \langle y, y \rangle = 0.$$

So $\langle x, v \rangle = 0$ for all $v \in M$ also, and in this case the set $M(x)$ in (0.25) is empty.

So let's suppose that $x \neq \mathbf{0}_X$.

If the vector space X is finite-dimensional, then (by definition) every linearly independent subset of X has at most finitely many elements (the number of elements in any linearly independent subset of X not exceeding the dimension of X), and as M is orthonormal, so M is linearly independent and thus finite, which implies that the subset $M(x)$ of M is also finite.

So let's assume that X is infinite-dimensional. If M is countable, then every subset of M is either finite or countable (by Corollary 7.3 in the book *Topology* by James R. Munkres, 2nd edition, or Theorem 2.8 in the book *Principles of Mathematical Analysis* by Walter Rudin, 3rd edition), and therefore $M(x)$ is also either finite or countable (i. e. at most countable).

So we assume that X is infinite-dimensional and also that set M is uncountable.

We know that, for any real or complex number α, the following holds: $\alpha = 0$ if and only if $|\alpha| = 0$; therefore we can conclude that $\alpha \neq 0$ if and only if $|\alpha| \neq 0$. But $|\alpha| \not< 0$. So we can also conclude that $\alpha \neq 0$ if and only if $|\alpha| > 0$.

For any elements u and v in X, as the inner product $\langle u, v \rangle$ of u and v is a real or complex number, so we can rewrite the set $M(x)$ as follows:

$$M(x) = \{ \ v \in M \ : \ |\langle x, v \rangle| > 0 \ \}. \tag{0.26}$$

Refer to (0.25) above.

For each $n \in \mathbb{N}$, let $M_n(x)$ be the subset of M defined as follows:

$$M_n(x) \colon = \left\{ \ v \in M \ : \ |\langle x, v \rangle| > \frac{1}{n} \ \right\}. \tag{0.27}$$

Let r be any given real number. If $r > 0$, then we can find a natural number n such that $nr > 1$, by Theorem 1.20 (a) in the book *Principles of Mathematical Analysis* by Walter Rudin, 3rd edition. Therefore $r > 1/n$. Conversely, if there exists a natural number n such that $r > 1/n$, then as $n > 0$, so $1/n > 0$ also and therefore $r > 0$.

Thus we have shown that, for any real number r, we have $r > 0$ if and only if there exists a natural number n such that $r > 1/n$. Using this result, we can now conclude that

$$M(x) = \bigcup_{n \in \mathbb{N}} M_n(x). \tag{0.28}$$

Refer to (0.26) and (0.27) above.

If each of the sets of the collection

$$\{ \ M_n(x) \ : \ n \in \mathbb{N} \ \}$$

is at most countable, then so is their union $M(x)$. [Refer to Theorem 7.5 in the book *Topology* by James R. Munkres, 2nd edition, or Theorem 2.12 in the book *Principles of Mathematical Analysis* by Walter Rudin, 3rd edition.]

So we assume that there exists a natural number m such that the set $M_m(x)$ is uncountable. We show that this assumption leads to a contradiction.

We construct a sequence $(e_k)_{k \in \mathbb{N}}$ of distinct points in set $M_m(x)$ as follows:

As $M_m(x)$ is uncountable by our assumption, so this set is non-empty; thus we can choose an element from this set and call that element e_1.

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Now suppose that $k \in \mathbb{N}$, and suppose that we have chosen some distinct elements $e_1, \dots, e_k$ from set $M_m(x)$.

As set $M_m(x)$ is uncountable, so it is infinite, and therefore the set

$$M_m(x) - \{e_1, \dots, e_k\}$$

is non-empty; so we can choose an element from this set and call that element e_{k+1}. Then

$$e_{k+1} \notin \{e_1, \dots, e_k\},$$

which implies that $e_{k+1} \neq e_j$ for any natural number $j \in \{ 1, \dots, k \}$.

By "induction", we have chosen a sequence $(e_k)_{k \in \mathbb{N}}$ of points of set $M_m(x)$ such that, if j and k are any two natural numbers such that $j < k$, then we have $e_j \neq e_k$.

Thus we have a sequence $(e_k)_{k \in \mathbb{N}}$ of distinct points from set $M_m(x)$. And, as $M_m(x) \subset M$ and as M is orthonormal by our hypothesis, so is $M_m(x)$.

Thus $(e_k)_{k \in \mathbb{N}}$ is an orthonormal sequence in X. So (by Theorem 3.4-6 in Kreyszig), the series $\sum |\langle x, e_k \rangle|^2$ converges in $\mathbb{R}$ and we have the inequality

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2. \tag{0.29}$$

By the definition of the set $M_m(x)$, we note that

$$|\langle x, v \rangle| > \frac{1}{m},$$

for each $v \in M_m(x)$. Refer to (0.27) above. And, for each $k \in \mathbb{N}$, as $e_k \in M_m(x)$, so we must also have

$$|\langle x, e_k \rangle| > \frac{1}{m}. \tag{0.30}$$

As the set $\mathbb{N}$ of natural numbers is uncounded from above, so we can choose a natural number greater than any given real number. Let us choose a natural number N such that

$$N > \left(m^2 + 1\right) \|x\|^2. \tag{0.31}$$

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Then using (0.30), we conclude that

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$$

$$\geq \sum_{k=1}^{N} |\langle x, e_k \rangle|^2$$

[note that the sequence of the partial sums of $\sum |\langle x, e_k \rangle|^2$ is a monotonically increasing sequence; so

$\lim_{n\to\infty} \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 = \sup \left\{ \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \ : \ n \in \mathbb{N} \right\}$]

$$> \sum_{k=1}^{N} \frac{1}{m^2} \qquad \text{[using (0.30) above]}$$

$$= \frac{N}{m^2}$$

$$> \frac{(m^2+1)\,\|x\|^2}{m^2} \qquad \text{[using (0.31) above]}$$

$$\geq \|x\|^2,$$

which contradicts (0.29).

Thus our assumption that some set $M_m(x)$ in (0.27) is uncountable has led to a contradiction.

Hence all the sets in (0.27) are at most countable, and so by (0.28) we can conclude that set $M(x)$ is also at most countable.

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Q.1 Let H_1 and H_2 be hilbert space and $T: H_1 \longrightarrow H_2$ a bounded linear operator. If $M_1 \subset H_1$ and $M_2 \subset H_2$ are such that $T(M_1) \subset M_2$, Prove that $M_1^{\perp} \supset T^*(M_2^{\perp})$.

Proof: To prove: $M_1^{\perp} \supset T^*(M_2^{\perp})$

Step 1: Let $z \in T^*(M_2^{\perp})$. Then there exist $y \in M_2^{\perp}$ such that $z = T^*y$.

Step 2: By Definition of Hilbert-adjoint operators for any $x \in M_1$ one has

$$\langle x, T^*y \rangle = \langle Tx, y \rangle = 0$$

Step 3: Since $Tx \in T(M_1) \subset M_2$ (given).

and $y \in M_2^{\perp}$

Step 4: Therefore $z = T^*y \in M_1^{\perp}$ so that

$$M_1^{\perp} \supset T^*(M_2^{\perp})$$

Note:

$M_1^{\perp} = T - M_1 = M_2$

$M_2^{\perp} = T - M_2 = M_1$

If $M_1 = N(T) = \{x \mid Tx = 0\}$, in given Q. (1)

Then (a) $T^*(H_2) \subset M_1^\perp$

m:

$T(M_1) = 0$ and $H_2 = \{0\}^\perp$.

$T^*(M_2^\perp) \subset M_1^\perp$.

Here $H_2 = M_2^\perp$, $M_2 = \{0\}$

~~$T^*(H_2)$~~

Since $T(M_1) \subset M_2$.

$T(M_1) = 0$

so $M_2 = 0$

$M_2^\perp = \{0\}^\perp$.

i/e $T^*(\{0\}^\perp) \subset M_1^\perp$

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(b) show that $[T(H_1)]^{\perp} \subset N(T^*)$

proof:- let $x \in [T(H_1)]^{\perp}$ —— (1)

Then $\langle y, x \rangle = 0$ for $y = Tz \in T(H_1)$

By the definition of the adjoint operator T^*

$$0 = \langle y, x \rangle = \langle Tz, x \rangle$$

$$= \langle z, T^* x \rangle \text{ for any } z \in H_1$$

$$\Rightarrow (z, T^*(x)) = \langle z, T^* x \rangle = 0$$

Therefore we have $T^* x = 0 \Rightarrow$ implies $x \in N(T^*)$ —— (2)

Hence, $x \in [T(H_1)]^{\perp}$
$\Rightarrow x \in N(T^*)$] from (1) and (2) —— (3)

From (3), we have,

$$[T(H_1)]^{\perp} \subset (N(T^*)$$

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(c) Show that $M_1 = [T^*(H_2)]^\perp$

proof: From (a), we have.

$$T^*(H_2) \subset M_1^\perp$$

$$[T^*(H_2)]^\perp \supset (M_1^\perp)^\perp$$

Theorem:- If H is an inner product space and if $M_1 \subset M_2 \subset H$, then we have $M_2^\perp \subset M_1^\perp$

proof: let $x \in M_2^\perp$, then

$$\langle x, y \rangle = 0 \text{ for all } y \in M_2$$

Since $M_1 \subset M_2$, we have.

$$\langle x, z \rangle = 0 \text{ for all } z \in M_1.$$

$\Rightarrow x \in M_1^\perp$

implies $\boxed{M_2^\perp \subset M_1^\perp}$

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$\Rightarrow [T^*(H_2)]^\perp \supset (M_1^\perp)^\perp = M_1.$ —— (1)

Now let $x \in [T^*(H_2)]^\perp$. Then

$$0 = \langle x, T^*y \rangle = \langle Tx, y \rangle \text{ for any } y \in H_2$$

$$\langle Tx, y \rangle = 0 \text{ for all } y \in H_2.$$

$$\Rightarrow Tx = 0$$

$$\Rightarrow x \in N(T) = M_1 \quad \left\{ \text{Since } M_1 = N(T) = \{x \mid Tx = 0\} \right\}$$

Thus

$x \in [T^*(H_2)]^\perp$

From (1), $x \in M_1$.

$$\Rightarrow [T^*(H_2)]^\perp \subset M_1 \quad \text{—— (2)}$$

From (1) and (2), we have

$$[T^*(H_2)]^\perp = M_1.$$

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Q. If S and T are bounded self adjoint linear operators on a Hilbert space H and α and β are real, show that

$T = \alpha S + \beta T$ is self-adjoint.

proof:- S and T are self-adjoint are given

So, $S = S^*$ and $T = T^*$

Now, for all $x \in H$

$\langle (\alpha S + \beta T)x, x \rangle$

$= \langle x, (\alpha S + \beta T)^* x \rangle$

$= \langle x, (\alpha S^* + \beta T^*) x \rangle$

α and β are real, $\alpha^* = \alpha, \beta^* = \beta$

$= \langle (\alpha S + \beta T)^* x, x \rangle.$

$= \langle x, (\alpha S + \beta T) x \rangle,$

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$\Rightarrow \langle ((\alpha S + \beta T)x - (\alpha S + \beta T)^* x, x \rangle = 0$

$\Rightarrow \langle ((\alpha S + \beta T) - (\alpha S + \beta T)^*) x, x \rangle = 0$

$\therefore \boxed{\alpha S + \beta T = (\alpha S + \beta T)^*}$

Hence $\alpha S + \beta T$ is self-adjoint.

(2) Let (T_n) be a sequence of bdd self-adjoint l.o. $T_n : H \to H$ or $T_n \to T$, $\|T_n - T\| \to 0$.

$\langle T_n x, x \rangle$ is real for every $x \in H$, since T_n's are self adjoint. $\forall n \in \mathbb{N}$

$\langle T_n x, x \rangle = \langle x, T_n x \rangle$. (Self adjoint).

Now. $\langle Tx, x \rangle = \langle \lim_{n\to\infty} T_n x, x \rangle = \lim_{n\to\infty} \langle T_n x, x \rangle$.

$= \lim_{n\to\infty} \langle x, T_n x \rangle$.

$= \langle x, \lim_{n\to\infty} T_n x \rangle = \langle x, Tx \rangle$.

Thus T is self adjoint (since $\langle Tx, x \rangle$ is real $\forall x \in H$).

(*) [we can do the above step since, $\langle -, - \rangle$ is cont]

(3). Let $T : H \to H$ be a bdd self adjoint l.o.

Now consider

$\langle T^n x, x \rangle = \langle T(T^{n-1} x), x \rangle$

$= \langle T^{n-1} x, Tx \rangle$ since T is self-adjo.

$= \langle T(T^{n-2} x), Tx \rangle$

$= \langle T^{n-2} x, T^2 x \rangle$ since T is self ad

$\vdots$

$= \langle x, T^n x \rangle$.

Thus $\langle T^n x, x \rangle$ is real for all $x \in H$, $\forall n \in \mathbb{N}$.

Thus T^n is bdd self adjoint op, $\forall n \in \mathbb{N}$.

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P207, 4. Show that for any bounded linear operator T on H, the operators

$$T_1 = \frac{1}{2}(T + T^*) \qquad \text{and} \qquad T_2 = \frac{1}{2i}(T - T^*)$$

are self-adjoint. Show that

$$T = T_1 + iT_2, \qquad T^* = T_1 - iT_2.$$

Show uniqueness, that is $T_1 + iT_2 = S_1 + iS_2$ implies $S_1 = T_1$ and $S_2 = T_2$; here S_1 and S_2 are self-adjoint by assumption.

Proof. Let T be a bounded linear operator on H. Set $T_1 = \frac{1}{2}(T + T^*)$ and $T_2 = \frac{1}{2i}(T - T^*)$. Then,

$$T_1^* = (\frac{1}{2}(T + T^*))^* = \frac{1}{2}(T^* + (T^*)^*) = \frac{1}{2}(T^* + T) = T_1$$
$$T_1^* = (\frac{1}{2i}(T + T^*))^* = -\frac{1}{2i}(T^* - (T^*)^*) = \frac{1}{2i}(-T^* + T) = T_2$$

So, T_1 and T_2 are self-adjoint. Moreover, it is easy to check that

$$T = T_1 + iT_2, \qquad T^* = T_1 - iT_2.$$

Assume $T_1 + iT_2 = S_1 + iS_2$ with S_1, S_2 are self-adjoint. Then,

$$(T_1 + iT_2)^* = (S_1 + iS_2)^* \Leftrightarrow T_1 - iT_2 = S_1 - iS_2.$$

Therefore, $T_1 = S_1$ and $T_2 = S_2$. ☒

(5) $T : \mathbb{C}^2 \to \mathbb{C}^2$.

$T(\xi_1, \xi_2) = (\xi_1 + i\xi_2, \xi_1 - i\xi_2)$

$T(1, 0) = (1, 1)$

$T(0, 1) = (i, -i)$

Basis of $\mathbb{C}^2 = \{(1,0), (0,1)\}$ over $\mathbb{C}$.

$\therefore$ matrix is $\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} = A$ (say)

$A^* = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$

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$\therefore T^*(z_{\varphi_1}, z_{\varphi_2})$

$= \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} z_{\varphi_1} \\ z_{\varphi_2} \end{bmatrix}$

$= \left[z_{\varphi_1} + z_{\varphi_2} , -i z_{\varphi_1} + i z_{\varphi_2} \right]^t .$

$\therefore T^*(z_{\varphi_1}, z_{\varphi_2})$

$= \left(z_{\varphi_1} + z_{\varphi_2} , -i z_{\varphi_1} + i z_{\varphi_2} \right) .$

$T_1 (z_{\varphi_1}, z_{\varphi_2}) = \left(z_{\varphi_1} + \frac{1+i}{2} z_{\varphi_2} , \frac{1-i}{2} z_{\varphi_1} \right)$

$T_2 (z_{\varphi_1}, z_{\varphi_2}) = \left(\frac{1+i}{2} z_{\varphi_2} , \frac{1-i}{2} z_{\varphi_1} - z_{\varphi_2} \right) .$

Now, $T^*T(z_{\varphi_1}, z_{\varphi_2})$

$= T^*(z_{\varphi_1} + i z_{\varphi_2} , z_{\varphi_1} - i z_{\varphi_2})$

$= (z_{\varphi_1} + i z_{\varphi_2} + z_{\varphi_1} - i z_{\varphi_2} , -i z_{\varphi_1} + z_{\varphi_2} + i z_{\varphi_1} + z_{\varphi_2}$

$= (2 z_{\varphi_1} , 2 z_{\varphi_2}) = 2 (z_{\varphi_1}, z_{\varphi_2})$

$\therefore T^*T = 2I .$

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$T T^*(z_1, z_2)$

$= T(z_1 + z_2, -iz_1 + iz_2)$

$= (z_1 + z_2 + z_1 - z_2, z_1 + z_2 - z_1 + z_2)$

$= 2(z_1, z_2)$. $\therefore T T^* = 2I$.

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Q6

Solution.

Let $T : H \to H$ be a bdd self adjoint operator

> ⇒ A self adjoint linear transformation A is non-negative if $(Ax, x) \geq 0$ for any vector x and positive definite if $(Ax, x) > 0$ $\forall x \neq 0$

⇒ Let λ be an eigen value of T, and let V be a non-zero vector in H sot

$T v = \lambda v$ Then

$$\lambda \|v\|^2 = \lambda < v, v >$$
$$= < \lambda v, v >$$
$$= < T v, v > = < v, T v >$$

we have give $T \neq 0$

$$\Rightarrow < T v, v > 0$$

$$\Rightarrow T v \neq 0 \Rightarrow T v = u$$

Thus, $T^2(v) = T(Tv) = T(u) \neq 0$

$$\Rightarrow T^3(v) = T(T^2(v)) = T(v^2)$$

Thus, $T^n \neq 0$ $\quad \forall n = 2, 4, 8 \ldots$

& $T^n \neq 0$ $\quad \forall n \in N$

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Q.7

Q7 Let A be a Unitary matrix of order n.

$$\Rightarrow A^* A = A.A^* = I \quad \text{——— (1)}$$

Where, $*$ is define by Conjugate transpose.

& $|\lambda| = 1$

here λ be an eigenvalue of A

Let $e_1, e_2, \ldots, e_n$ be the columns of A.

i.e. $A = (e_1, e_2, \ldots, e_n)$.

by Inner product property on C^n

$$\langle e_i, e_i \rangle = \|e_i\|^2 = 1$$

by the eq (1)

$$A.A^* = I \Rightarrow \langle e_i . e_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Thus, $e_1, e_2, \ldots, e_n$ are orthonormal basis on C^n

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(8) Let $T: H \to H$ be an isometric linear opreator.

Then By definition of isometric linear opreator.

$$\|Tx\| = \|x\| \quad , \quad \forall x \in H$$

$$\Rightarrow \langle Tx, Tx \rangle = \langle x, x \rangle$$

$$\Rightarrow \langle (T^*T)(x), x \rangle = \langle x, x \rangle$$

$$\Rightarrow \langle (T^*T)(x), x \rangle = \langle Ix, x \rangle$$ (where I is an identity opreator on H.

$$\Rightarrow \langle (T^*T)(x), x \rangle - \langle I(x), x \rangle = 0$$

$$\Rightarrow \langle (T^*T - I)(x), x \rangle = 0 \quad \forall x \in H$$

$$\Rightarrow (T^*T - I) = 0 \quad \ldots \text{(By Lemma)}$$

$$\Rightarrow T^*T = I$$

#

(9) ~~Let M = H~~ Consider $T: H \to H$ be an isometric linear opreator which is not unitary.

Let $M = T(H)$ be the range of T.

Then M is a subspace of H. Let $y \in \overline{M}$

Then, $\exists$ a sequence $\{y_n\}$ in M such that $y_n \to y$

Set $Tx_n = y_n$

Since T is isometric, we have

$$\|x_m - x_n\| = \|T(x_m - x_n)\|$$
$$= \|Tx_m - Tx_n\|$$
$$= \|y_m - y_n\|$$
$$\to 0 \quad \text{as } m, n \to \infty$$

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This shows that $\{x_n\}$ is a cauchy sequence in H. But H is complete because H is a Hilbert space.

Therefore, $y = Tx \in M$.

Hence M is closed.

Further, T is not unitary then $M \neq H$

Hence M is a closed subspace of H and the result follows.

#

(10) Let X be an inner product space and $T: X \to X$ is an isometric linear opreator and $\dim X < \infty$.

$$T^*T = I \quad \text{(From (8) Part)}$$

$$(T^*T)(x) = I(x) \quad \forall x \in H$$

$$\Rightarrow \langle (T^*T)(x), y \rangle = \langle I(x), y \rangle \quad \forall x, y \in H$$

$$\Rightarrow \langle Tx, Ty \rangle = \langle x, y \rangle . \quad \forall x, y \in H.$$

Hence $\langle (T^*T)(x), y \rangle = \langle Tx, Ty \rangle = \langle x, y \rangle \quad \forall x, y \in H$

Holds for every $x, y \in H$.

Hence $T^*T = I$

Since the space is finite dimensional, we must also have $T^*T = I$

$$\Rightarrow T^*T = TT^* = I$$

$\Rightarrow$ T is unitary #

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P207, 11.(Unitary equivalence) Let S and T be linear operators on a Hilbert space H. The operator S is said to be unitarily equivalent to T if there is a unitary operator U on H such that

$$S = UTU^{-1} = UTU^*.$$

If T is self-adjoint, show that S is self-adjoint.

Proof. Suppose T is self-adjoint and S is unitarily equivalent to T. Then, it follows from the properties of Hilbert-adjoint operators that

$$S^* = (UTU^*)^* = (U^*)^*(UT)^* = UT^*U^* = UTU^* = S.$$

Therefore, S is also self-adjoint. □

P208, 13. If $T_n : H \to H (n = 1, 2, \cdots)$ are normal linear operators and $T_n \to T$, show that T is a normal linear operator.

Proof. It is clear that T is a bounded linear operator. It follows from the properties of Hilbert-adjoint operators that

$$\begin{aligned} \|T_n^*T_n - T^*T\| &\leq \|T_n^*T_n - T_n^*T\| + \|T_n^*T - T^*T\| \\ &\leq \|T_n^*\|\|T_n - T\| + \|T_n^* - T^*\|\|T\| \\ &= \|T_n\|\|T_n - T\| + \|T_n - T\|\|T\| \to 0, \quad \text{as} \quad n \to +\infty, \end{aligned}$$

since $T_n \to T$. Then, since T_n is normal, i.e. $T_nT_n^* = T_n^*T_n$, it holds that

$$\begin{aligned} \|TT^* - T^*T\| &\leq \|TT^* - T_nT_n^*\| + \|T_nT_n^* - T^*T\| \\ &= \|(T^*T - T_n^*T_n)^*\| + \|T_n^*T_n - T^*T\| \\ &= 2\|T_n^*T_n - T^*T\| \to 0, \quad \text{as} \quad n \to +\infty. \end{aligned}$$

Therefore, $TT^* = T^*T$, i.e. T is normal. □

P208, 14. If S and T are normal linear operators satisfying $ST^* = T^*S$ and $TS^* = S^*T$, show that their sum $S + T$ and product ST are normal.

Proof.

$$\begin{aligned} (S+T)(S+T)^* &= (S+T)(S^* + T^*) \\ &= SS^* + ST^* + TS^* + TT^* \\ &= S^*S + T^*S + S^*T + T^*T \\ &= (S^* + T^*)(S+T) \\ &= (S+T)^*(S+T), \end{aligned}$$

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and

$$(ST)(ST)^* = STT^*S^* = ST^*TS^* = T^*SS^*T = T^*S^*ST = (ST)^*ST.$$

Therefore, $S+T, ST$ are normal. □

P208, 15. Show that a bounded linear operator $T : H \to H$ on a complex Hilbert space H is normal if and only if $\|T^*x\| = \|Tx\|$ for all $x \in H$. Using this, show that for a normal linear operator,

$$\|T^2\| = \|T\|^2.$$

Proof. By the definition of adjoint operator,

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle$$

and

$$\|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle x, TT^*x\rangle.$$

Then T is normal, i.e. $TT^* = T^*T$ if and only if $\|Tx\| = \|T^*x\|$.
Since, for any $x \in H$,

$$\|T^2x\| \leq \|T\|\|Tx\| \leq \|T\|\|T\|\|x\|,$$

it yields that

$$\|T^2\| \leq \|T\|^2.$$

On the other hand, for any $x \in H$, it holds that

$$\begin{aligned}\|T^2x\|^2 &= \langle T^2x, T^2x\rangle = \langle Tx, T^*T^2x\rangle \\ &= \langle Tx, TT^*Tx\rangle = \langle T^*Tx, T^*Tx\rangle \\ &= \|T^*Tx\|^2.\end{aligned}$$

Then,

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*Tx\|\|x\| \leq \|T^2x\|\|x\| \leq \|T^2\|\|x\|^2,$$

that is,

$$\|T\|^2 \leq \|T^2\|.$$

Hence, $\|T^2\| = \|T\|^2$. □