

E LASTO -

DYNAMICS *

INSTRUCTOR:- Prof. Dr. Aftab Khan.

* We use tensor notations in this course.

⇒ Introduction: If \underline{A} is a vector, then

$$\underline{A} = A_1 \underline{i} + A_2 \underline{j} + A_3 \underline{k} = (A_1, A_2, A_3)$$

In tensor form it is written as

$$\underline{A} = A_i ; i=1,2,3$$

دو ضربی اینجی double understood Σ
 $\sum_{i=1}^3$

$$\underline{A} \cdot \underline{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$$

$$= \sum_{i=1}^3 A_i B_i ; i=1,2,3 \left. \vphantom{\sum_{i=1}^3} \right\} \text{In the tensor form}$$

$$= A_i B_i ; i=1,2,3$$

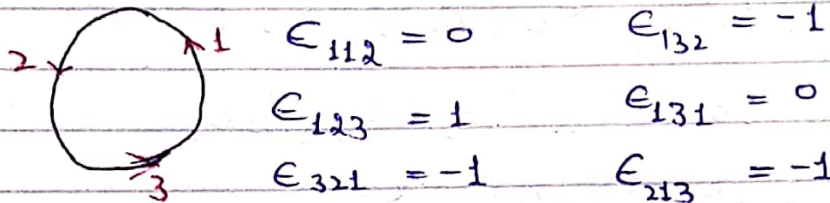
$$\underline{A} \times \underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \underline{i} \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} - \underline{j} \begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix} + \underline{k} \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}$$

$$= \underline{i} (A_2 B_3 - B_2 A_3) - \underline{j} (A_1 B_3 - B_1 A_3) + \underline{k} (A_1 B_2 - B_1 A_2)$$

q In the form of tensor

$$\underline{A} \times \underline{B} = \epsilon_{ijk} A_j B_k, \text{ where}$$

$\epsilon_{ijk} = 0$ if any two indices are equal
 $= 1$ if i, j, k are cyclic.
 $= -1$ if i, j, k are anti-cyclic.



$$\begin{aligned}
 \epsilon_{112} &= 0 & \epsilon_{132} &= -1 \\
 \epsilon_{123} &= 1 & \epsilon_{131} &= 0 \\
 \epsilon_{321} &= -1 & \epsilon_{213} &= -1
 \end{aligned}$$

Now, $\underline{A} \times \underline{B} = \epsilon_{ijk} A_j B_k$

For $i=1 \Rightarrow (\underline{A} \times \underline{B})_1 = \epsilon_{1jk} A_j B_k$

$$\Rightarrow (\underline{A} \times \underline{B})_1 = \epsilon_{11k} A_1 B_k + \epsilon_{12k} A_2 B_k + \epsilon_{13k} A_3 B_k$$

$$\begin{aligned}
 &= \epsilon_{121} A_2 B_1 + \epsilon_{122} A_2 B_2 + \epsilon_{123} A_2 B_3 \\
 &+ \epsilon_{131} A_3 B_1 + \epsilon_{132} A_3 B_2 + \epsilon_{133} A_3 B_3
 \end{aligned}$$

$$= \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2$$

$$= (A_2 B_3 - B_2 A_3) \because \epsilon_{123} = 1 \text{ \& } \epsilon_{132} = -1$$

For $i=2 \Rightarrow (\underline{A} \times \underline{B})_2 = \epsilon_{2jk} A_j B_k$

$$\Rightarrow (\underline{A} \times \underline{B})_2 = \epsilon_{21k} A_1 B_k + \epsilon_{22k} A_2 B_k + \epsilon_{23k} A_3 B_k$$

$$\begin{aligned}
 &= \epsilon_{211} A_1 B_1 + \epsilon_{212} A_1 B_2 + \epsilon_{213} A_1 B_3 \\
 &+ \epsilon_{231} A_3 B_1 + \epsilon_{232} A_3 B_2 + \epsilon_{233} A_3 B_3
 \end{aligned}$$

Put $i=1$ کرنا
 component نکالنے کے لیے
 $i=2, 3$ کرنے سے
 دوسرا اور غیر آگے
 ہوا
 double indices
 کے لیے ایسی ہی
 لائنیں لکھیں

$$\Rightarrow (\underline{A} \times \underline{B})_2 = \epsilon_{213} A_1 B_3 + \epsilon_{231} A_3 B_1$$

$$= (A_3 B_1 - A_1 B_3) \begin{cases} \because \epsilon_{231} = 1 & \text{cyclic} \\ \epsilon_{213} = -1 & \text{a-cyclic} \end{cases}$$

For $i=3 \Rightarrow (\underline{A} \times \underline{B})_3 = \epsilon_{3jk} A_j B_k$

$$\Rightarrow (\underline{A} \times \underline{B})_3 = \epsilon_{31k} A_1 B_k + \epsilon_{32k} A_2 B_k + \epsilon_{33k} A_3 B_k$$

$$= \epsilon_{311} A_1 B_1 + \epsilon_{312} A_1 B_2 + \epsilon_{313} A_1 B_3$$

$$+ \epsilon_{321} A_2 B_1 + \epsilon_{322} A_2 B_2 + \epsilon_{323} A_2 B_3$$

$$= \epsilon_{312} A_1 B_2 + \epsilon_{321} A_2 B_1$$

$$= (A_1 B_2 - B_1 A_2) \begin{cases} \because \epsilon_{312} = 1 \\ \epsilon_{321} = -1 \end{cases}$$

$$\Rightarrow \underline{A} \times \underline{B} = (A_2 B_3 - B_2 A_3) \underline{i} + (A_3 B_1 - A_1 B_3) \underline{j}$$

$$+ (A_1 B_2 - B_1 A_2) \underline{k}$$

* If $\underline{i}, \underline{j}, \underline{k}$ have bars $\underline{i}, \underline{j}, \underline{k}$ then they represent unit vectors otherwise they represent indices.

Partial Derivatives:

* If $u = u(x_1, x_2, x_3)$ is a scalar function then,

$$\frac{\partial u}{\partial x_1} = u_{,1} \quad , \quad \frac{\partial u}{\partial x_2} = u_{,2} \quad , \quad \frac{\partial u}{\partial x_3} = u_{,3}$$

* For a vector $\underline{A} = (A_1, A_2, A_3)$

$$\Rightarrow \underline{A} = A_i \quad ; \quad i=1, 2, 3$$

$$\frac{\partial \underline{A}}{\partial x} = A_{i,1} = (A_{1,1}, A_{2,1}, A_{3,1})$$

* $\phi(x, y, z)$

$$\Rightarrow \frac{\partial \phi}{\partial x} = \phi_{,1} ; \frac{\partial \phi}{\partial y} = \phi_{,2} ; \frac{\partial \phi}{\partial z} = \phi_{,3}$$

Similarly we can define double derivatives

$$\frac{\partial^2 \phi}{\partial x^2} = \phi_{,11} \quad , \quad \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) = \phi_{,21}$$

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} \right) = \frac{\partial}{\partial y} (\phi_{,1}) = \phi_{,12}$$

And similarly for 3rd, 4th and so on derivatives.

\Rightarrow Gradient :- (of a scalar function)

$$\nabla \phi = \frac{\partial \phi}{\partial x} \underline{i} + \frac{\partial \phi}{\partial y} \underline{j} + \frac{\partial \phi}{\partial z} \underline{k}$$

$$= \phi_{,1} \underline{i} + \phi_{,2} \underline{j} + \phi_{,3} \underline{k}$$

$$= (\phi_{,1}, \phi_{,2}, \phi_{,3}) \quad \text{in component form}$$

$$= \phi_{,i} ; i=1,2,3 \quad (\text{Tensor form})$$

\Rightarrow Divergence :- (of a vector function)

$$\nabla \cdot \underline{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$$

$$= f_{1,1} + f_{2,2} + f_{3,3}$$

$$= f_{i,i} ; i=1,2,3$$

⇒ **Curl** :- (of a vector function)

$$\nabla \times \underline{f} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix} = \epsilon_{ijk} \nabla_j f_k$$

$$= \underline{i} \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) - \underline{j} \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} \right) + \underline{k} \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right)$$

$$= \epsilon_{ijk} f_{k,j} \underline{i}$$

Results:- 1) $\underline{A} = A_i \underline{i}$

2) $\underline{A} \cdot \underline{B} = A_i B_i = A_j B_j = A_k B_k$

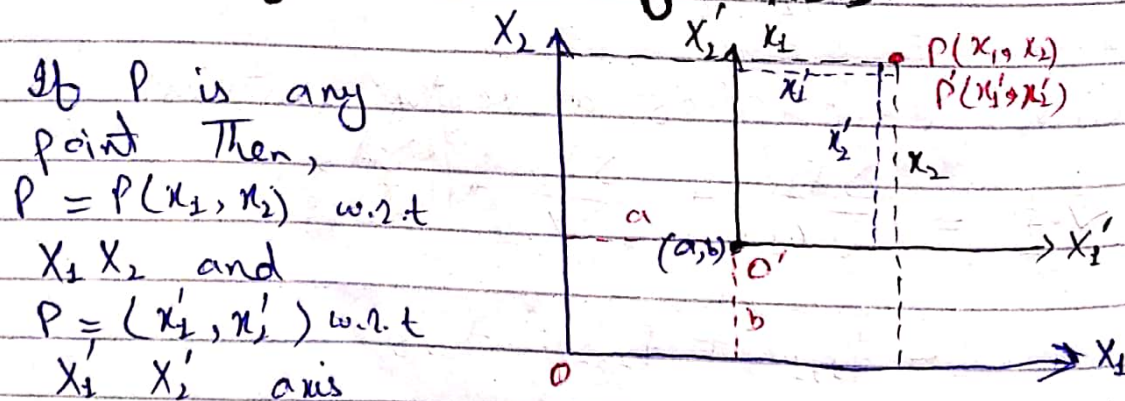
3) $\underline{A} \times \underline{B} = \epsilon_{ijk} A_j B_k \underline{i}$

4) $\text{Grad } \phi = \phi_{,i} \underline{i}$

5) $\text{Div } \underline{f} = f_{i,i}$

6) $\text{Curl } \underline{f} = \epsilon_{ijk} f_{k,j} \underline{i}$

⇒ **Transformation of Axis:-**



Transformations are

$$x_1 = a + x'_1 \longrightarrow \textcircled{1}$$

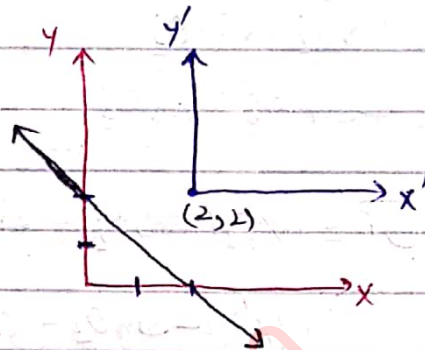
$$x_2 = b + x'_2 \longrightarrow \textcircled{2}$$

Example: * Let $a=2=b$

$$x + y = 2$$

$$\text{As } x = a + x'$$

$$\& y = b + y'$$



$$\Rightarrow x + y = 2$$

$$\Rightarrow (2 + x') + (2 + y') = 2 \quad \because a=2=b$$

$$\Rightarrow x' + y' + 4 = 2$$

$$\Rightarrow x' + y' = -2$$

* Again Let $a=2=b$

$$x^2 + y^2 = 4^2$$

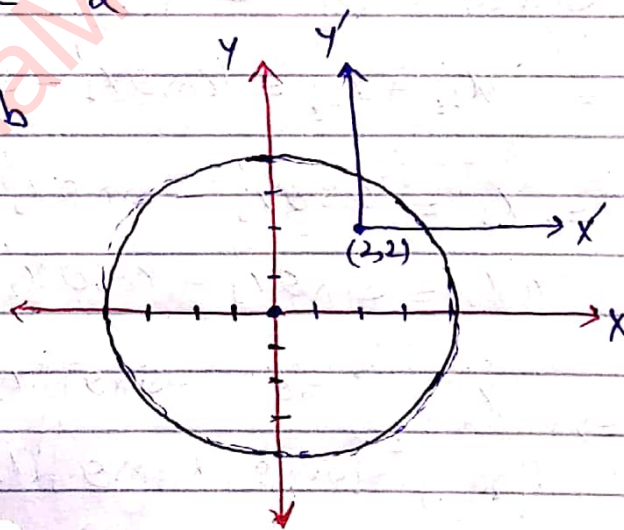
$$\Rightarrow (2 + x')^2 + (2 + y')^2 = 4^2$$

$$\Rightarrow (x' + 2)^2 + (y' + 2)^2 = 4^2$$

' is an equation of circle having centre

at $(-2, -2)$ w.r.t

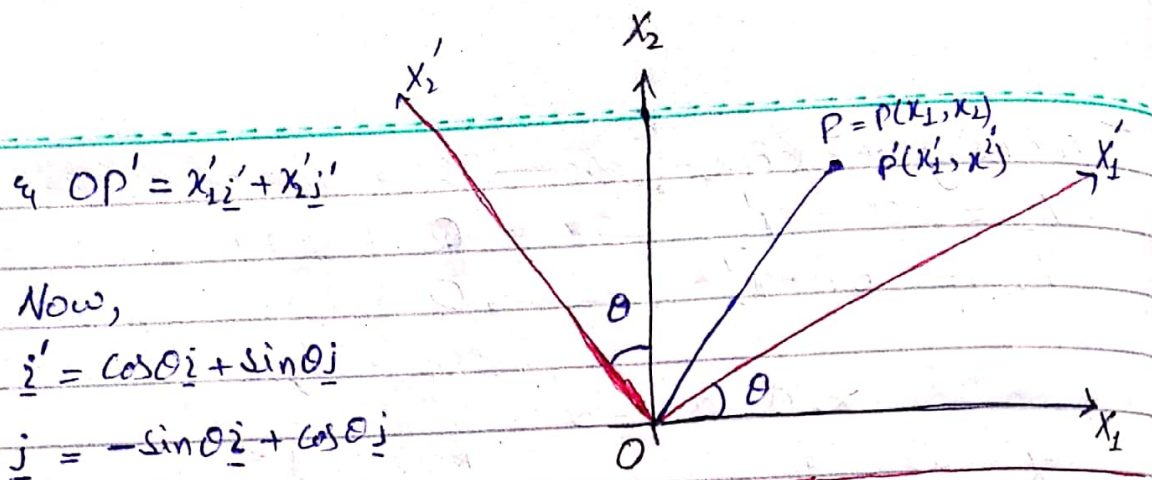
$x'y'$ (while original equation has centre at $(0, 0)$ w.r.t XY .)



\Rightarrow Rotation of Axis:

$$\overline{OP} = \overline{OP'} \longrightarrow \textcircled{1}$$

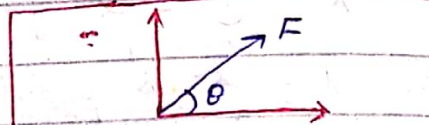
$$\overline{OP} = x_1 i + x_2 j \longrightarrow \textcircled{2}$$



$$\Rightarrow \overline{OP'} = (x_1')(\cos\theta \underline{i} + \sin\theta \underline{j}) + (x_2')(-\sin\theta \underline{i} + \cos\theta \underline{j})$$

$$\Rightarrow \overline{OP'} = (\cos\theta x_1' - \sin\theta x_2') \underline{i}$$

$$+ (\sin\theta x_1' + \cos\theta x_2') \underline{j} \rightarrow \textcircled{3}$$



$$F = |F| \cos\theta \underline{i} + |F| \sin\theta \underline{j}$$

$$\underline{i}' = |\underline{i}'| \cos\theta \underline{i} + |\underline{i}'| \sin\theta \underline{j}$$

$$\underline{j}' = |\underline{j}'| \cos\theta \underline{i} + |\underline{j}'| \sin\theta \underline{j}$$

Since $\overline{OP} = \overline{OP'}$

$$\Rightarrow x_1 \underline{i} + x_2 \underline{j} = (\cos\theta x_1' - \sin\theta x_2') \underline{i} + (\sin\theta x_1' + \cos\theta x_2') \underline{j}$$

$$\Rightarrow x_1 = \cos\theta x_1' - \sin\theta x_2'$$

$$\& x_2 = \sin\theta x_1' + \cos\theta x_2'$$

$\& \theta = 90^\circ$
 $\Rightarrow x_1 = -x_2' \&$
 $x_2 = x_1'$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

where $a_{11} = \cos\theta$
 $a_{12} = -\sin\theta, a_{21} = \sin\theta$
 $a_{22} = \cos\theta$

$$\Rightarrow x_1 = a_{11} x_1' + a_{12} x_2' \quad \text{and}$$

$$x_2 = a_{21} x_1' + a_{22} x_2'$$

$$\Rightarrow x_i = a_{ji} x_j' \quad \& x_2 = a_{2i} x_i' \quad ; i=1,2$$

$$\Rightarrow x_i = a_{ij} x'_j \longrightarrow \textcircled{*} \quad i, j = 1, 2$$

$\textcircled{*}$ is the equation of rotation of axis.

where $a_{ij} = \cos$ of angle between x_i and x'_j

$$\Rightarrow a_{11} = \cos \theta, \quad a_{12} = \cos(90 + \theta) = -\sin \theta$$

$$a_{21} = \cos(90 - \theta) = \sin \theta$$

$$\text{Similarly } a_{22} = \cos \theta$$

Results:- 1) Transformation $\Rightarrow x_i = x'_i + a_i$

2) Rotation $\Rightarrow x_i = a_{ij} x'_j$

\Rightarrow ELASTIC BODIES:- A body is said to be elastic if Hooke's Law holds in it, i.e. stress \propto strain, where stress is force per unit area and strain is measure of deformation

$$\Rightarrow \sigma_{ij} \propto \epsilon_{ij}$$

σ_{ij} is stress & ϵ_{ij} is strain tensor

$$\Rightarrow \sigma_{ij} = C_{ijkl} \epsilon_{kl}$$

σ_{ij} is a tensor of order 2.

Similarly ϵ_{ij} is a tensor of order 2.

and C_{ijkl} is a tensor/elastic constant of order 4.

It has $3^4 = 81$ components.

Also it is symmetric

w.r.t. indices

\Rightarrow By Voigt's Notation has 36 independent components

A_i has 3 component

A_{ij} has $9 = 3^2$ "

A_{ijk} has $27 = 3^3$ "

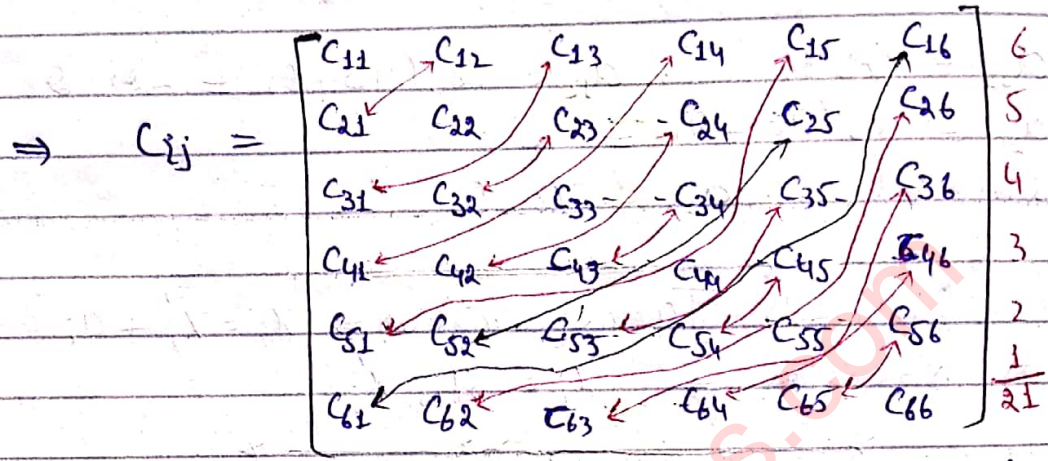
A_{ijkl} has $81 = 3^4$ "

$$\left[\begin{array}{c} \dots \\ 9 \times 9 \\ \dots \end{array} \right] \begin{array}{c} 9 \\ 8 \\ 7 \\ \dots \\ 1 \end{array} = 45 - 9 = 36$$

\equiv -2ans

$\Rightarrow 11 \rightarrow 1, 22 \rightarrow 2, 33 \rightarrow 3$
 $23 \rightarrow 4, 13 \rightarrow 5, 12 \rightarrow 6$

$\Rightarrow C_{1122} = C_{12}$, $C_{1223} = C_{64}$
 $C_{25} = C_{2213}$



Also C_{ij} is symmetric due to double commutative property of C_{ijkl} ,

i.e. $C_{ijkl} = C_{klij} \Rightarrow C_{ij} = C_{ji}$

$\Rightarrow C_{ij}$ has 21 independent components.

*** Note:-** C_{ijkl} has 21 components.

\Rightarrow Isotropic Material:- A material which has same properties in all directions is known as isotropic.

Now by using equation of Transformation

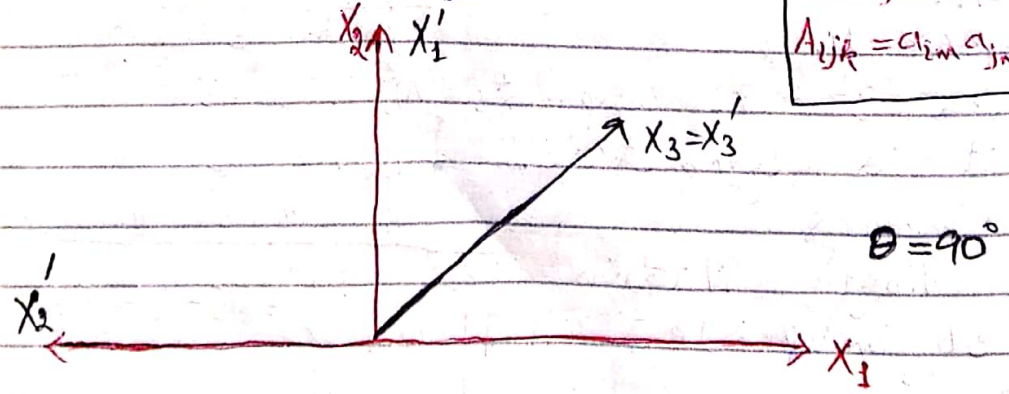
$C_{ij} = a_{im} a_{jn} C'_{mn}$

Equ of Transformation

$x_i = a_{ij} x'_j$

$A_{ij} = a_{im} a_{jn} A'_{mn}$

$A'_{ijk} = a_{im} a_{jn} a_{kp} A'_{mnp}$



$$C_{ij} = a_{im} a_{jn} C'_{mn}$$

$a_{ij} = \cos$ of angle b/w a_i and a_j

$$\Rightarrow C_{11} = a_{1m} a_{1n} C'_{mn}$$

$$= a_{11} a_{1n} C'_{1n} + a_{12} a_{1n} C'_{2n} + a_{13} a_{1n} C'_{3n}$$

$$= \cos 90^\circ a_{1n} C'_{1n} + \cos 180^\circ C'_{1n} C'_{2n} + \cos 90^\circ a_{1n} C'_{3n}$$

$\because \cos 90^\circ = 0$

$$= -1 a_{1n} C'_{2n}$$

$$= -1 \{ a_{11} C'_{21} + a_{12} C'_{22} + a_{13} C'_{23} \}$$

$$= -1 \{ \cos 90^\circ C'_{21} + \cos 180^\circ C'_{22} + \cos 90^\circ C'_{23} \}$$

$$= -1 \{ -1 C'_{22} \}$$

$$\Rightarrow C_{11} = C'_{22}$$

$$\Rightarrow \boxed{C_{11} = C_{22}}$$

\because for isotropic $C_{22} = C_{22}$

Now

$$a_{11} = \cos 90^\circ = 0, \quad a_{12} = \cos 180^\circ = -1$$

$$a_{13} = \cos 90^\circ = 0, \quad a_{21} = \cos 0 = 1$$

$$a_{22} = \cos 90^\circ = 0, \quad a_{23} = \cos 90^\circ = 0$$

$$a_{31} = \cos 90^\circ = 0, \quad a_{32} = \cos 90^\circ = 0$$

$$a_{33} = \cos 0 = 1$$

$$\Rightarrow C_{12} = a_{1m} a_{2n} C'_{mn}$$

$$= a_{11} a_{2n} C'_{1n} + a_{12} a_{2n} C'_{2n} + a_{13} a_{2n} C'_{3n}$$

$$= -1 \{ a_{21} C'_{21} + a_{22} C'_{22} + a_{23} C'_{23} \}$$

$$C_{12} = -C'_{21}$$

$$\Rightarrow \boxed{C_{12} = -C_{21}}$$

$$C_{13} = a_{1m} a_{3n} C'_{mn}$$

$$= a_{11} a_{3n} C'_{1n} + a_{12} a_{3n} C'_{2n} + a_{13} a_{3n} C'_{3n}$$

$$\Rightarrow C_{13} = -1 \{ a_{31} C_{21} + a_{32} C_{22} + a_{33} C_{23} \}$$

$$= -1 \{ C_{23} \}$$

$$\Rightarrow \boxed{C_{13} = C_{23}}$$

$$C_{14} = C_{1123}$$

$$\& C_{1123} = a_{1m} a_{1n} a_{2p} a_{3q} C_{mnpq}$$

$$= a_{12} a_{12} a_{2p} a_{3q} C_{22pq}$$

\therefore all other components are zero

$$= (-1)(-1) \{ a_{21} a_{3q} C_{221q} +$$

$$a_{22} a_{3q} C_{222q} + a_{23} a_{3q} C_{223q} \}$$

$$\Rightarrow C_{1123} = 0 \Rightarrow \boxed{C_{14} = 0}$$

$$C_{15} = C_{1113}$$

$$\& C_{1113} = a_{1m} a_{1n} a_{1p} a_{3q} C_{mnpq}$$

$$= a_{12} a_{12} a_{12} a_{3q} C_{111q}$$

\therefore all other combinations are zero.

$$= (-1)(-1)(-1) \{ a_{31} C_{1111} + a_{32} C_{1112} + a_{33} C_{1113} \}$$

$$= -1 \{ 1 C_{1113} \} = -C_{1113}$$

$$\Rightarrow C_{1113} = -C_{1113} \Rightarrow C_{15} = -C_{15}$$

$$\Rightarrow C_{15} = -C_{15} \Rightarrow 2C_{15} = 0$$

$$\Rightarrow \boxed{C_{15} = 0}$$

$$C_{16} = C_{1112} \quad \epsilon_1$$

$$C_{1112} = a_{1m} a_{1n} a_{1p} a_{2q} C'_{mnpq}$$

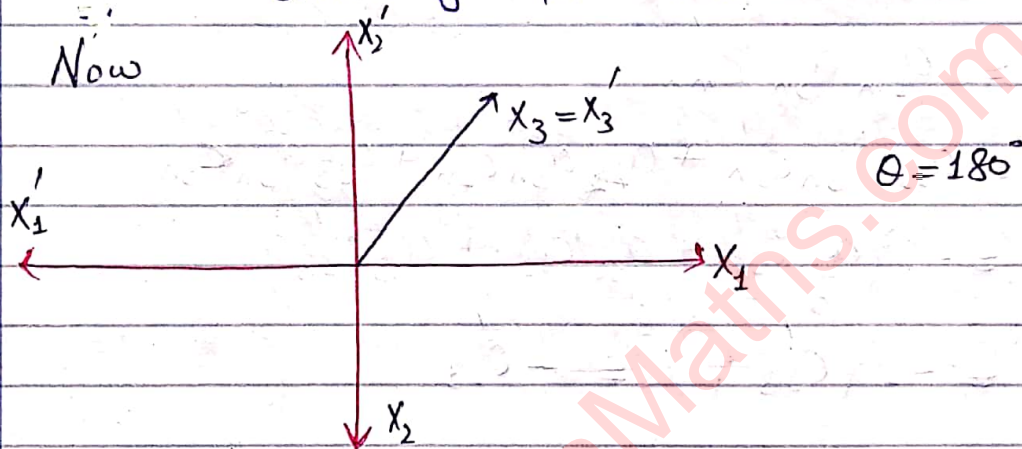
$$= a_{12} a_{12} a_{12} a_{22} C'_{1111q}$$

$$= -1 \{ a_{21} C'_{1111} + a_{22} C'_{1112} + a_{23} C'_{1113} \}$$

$$C_{1112} = -C'_{1111} \Rightarrow C_{16} = -C'_{11} \quad ? \text{ should be zero}$$

Similarly for others.

Now



$$a_{11} = \cos 180^\circ = -1$$

$$a_{12} = \cos 90^\circ = 0$$

$$a_{13} = \cos 90^\circ = 0$$

$$a_{21} = \cos 90^\circ = 0$$

$$a_{22} = \cos 180^\circ = -1$$

$$a_{23} = \cos 90^\circ = 0$$

$$a_{31} = \cos 90^\circ = 0$$

$$a_{32} = \cos 90^\circ = 0$$

$$a_{33} = \cos(0) = 1$$

$$C_{ij} = a_{im} a_{jn} C'_{mnp}$$

$$\Rightarrow C_{11} = a_{1m} a_{1n} C'_{mnp}$$

$$= a_{11} a_{11} C'_{11} + a_{12} a_{12} C'_{22} + a_{13} a_{13} C'_{33}$$

$$= -1 \{ a_{11} C'_{11} + a_{12} C'_{22} + a_{13} C'_{33} \}$$

$$= -1 \{ -C'_{11} \} = C'_{11}$$

$$\Rightarrow \boxed{C_{11} = C'_{11}} \quad \because \text{By isotropic } C'_{11} = C_{11}$$

$$\begin{aligned}
 C_{12} &= a_{1m} a_{2n} C_{mn} \\
 &= a_{11} a_{2n} C_{1n} + a_{12} a_{2n} C_{2n} + a_{13} a_{2n} C_{3n} \\
 &= -1 \{ a_{21} C_{11} + a_{22} C_{12} + a_{23} C_{13} \} \\
 &= -1 \{ -C_{12} \} = C_{12}
 \end{aligned}$$

$$\Rightarrow \boxed{C_{12} = C_{12}} \quad \because \text{By isotropy } C_{12} = C_{12}$$

$$\begin{aligned}
 C_{13} &= a_{1m} a_{3n} C_{mn} \\
 &= a_{11} a_{3n} C_{1n} + a_{12} a_{3n} C_{2n} + a_{13} a_{3n} C_{3n} \\
 &= -1 \{ a_{31} C_{11} + a_{32} C_{12} + a_{33} C_{13} \} \\
 &= -1 \{ C_{13} \} = -C_{13}
 \end{aligned}$$

$$\Rightarrow C_{13} = -C_{13} \Rightarrow C_{13} = ?$$

$$C_{14} = C_{1123}$$

$$\begin{aligned}
 \Rightarrow C_{1123} &= a_{1m} a_{1n} a_{2p} a_{3q} C_{mnpq} \\
 &= a_{11} a_{11} a_{22} a_{33} C_{1123} \\
 &= (-1)(-1)(-1)(1) C_{1123}
 \end{aligned}$$

$$C_{1123} = -C_{1123}$$

$$\Rightarrow C_{1123} = -C_{1123} \Rightarrow C_{1123} + C_{1123} = 0$$

$$\Rightarrow 2C_{1123} = 0$$

$$\Rightarrow C_{1123} = 0 \Rightarrow \boxed{C_{14} = 0}$$

$$\Rightarrow C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix}$$

$$\text{Lamé's Put: } \left. \begin{array}{l} C_{11} = \lambda + 2\mu \\ C_{12} = \lambda \end{array} \right\} \Rightarrow \frac{C_{11} - C_{12}}{2} = \mu$$

$$\Rightarrow C_{ij} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

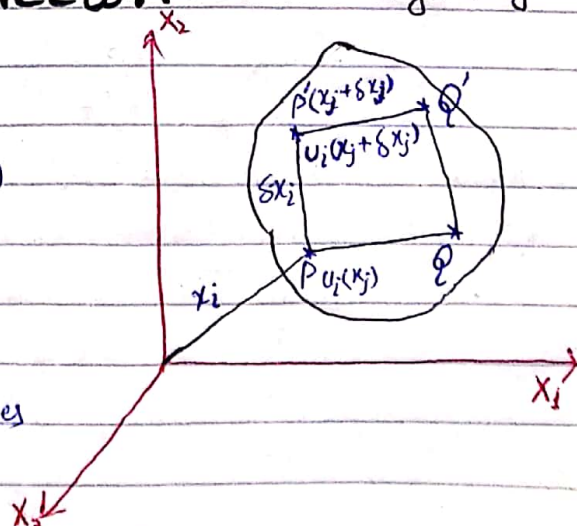
λ and μ are known as Lamé's Constant.

Displacement Vector:-

$$\begin{aligned} \delta \overline{PQ} &= \overline{P'Q'} - \overline{PQ} \\ &= u_i(x_j + \delta x_j) - u_i(x_j) \end{aligned}$$

$$\begin{aligned} &= u_i(x_j) + \frac{\partial u_i}{\partial x_j} \delta x_j + \dots \\ &\quad - u_i(x_j) \end{aligned}$$

\therefore By Taylor series



We consider the linear transformation.
Therefore neglecting higher terms

$$\begin{aligned}\delta \overline{PQ} &= u_i(x_j) + \frac{\partial u_i}{\partial x_j} \delta x_j - u_i(x_j) \\ &= \frac{\partial u_i}{\partial x_j} \delta x_j\end{aligned}$$

$$\Rightarrow \delta \overline{PQ} = \left\{ \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \right\} \delta x_j$$

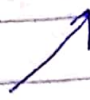
$$\Rightarrow \delta \overline{PQ} = \epsilon_{ij} \delta x_j + \xi_{ij} \delta x_j$$

where $\epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$ This is symmetric part and

$\xi_{ij} = \frac{1}{2} (u_{i,j} - u_{j,i})$ is anti-symmetric part due to rotation.

For Isotropic material

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix}$$

And Hook's Law is as follows 

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{33} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix}$$

Equation of Motion is as follows

$$\sigma_{ij,j} = \rho \ddot{u}_i$$

where ρ is mass density and σ_{ij} is the stress.

For $i=1$:- $\sigma_{1j,j} = \rho \ddot{u}_1$ (Solving for 2D)

$$\Rightarrow \sigma_{11,1} + \sigma_{12,2} = \rho \ddot{u}_1 \longrightarrow \textcircled{1}$$

And $\sigma_{11} = C_{11}\epsilon_{11} + C_{12}\epsilon_{22}$ from $\textcircled{2}$

$$\text{But } \epsilon_{ij} = \frac{1}{2}(u_{ij} + u_{ji})$$

$$\Rightarrow \epsilon_{11} = \frac{1}{2}(u_{1,1} + u_{1,1}) = u_{1,1}$$

$$\epsilon_{22} = u_{2,2}$$

$$\Rightarrow \sigma_{11} = C_{11}u_{1,1} + C_{12}u_{2,2}$$

$$\Rightarrow \sigma_{11,1} = C_{11}u_{1,11} + C_{12}u_{2,21}$$

$$\text{Also } \sigma_{12} = \frac{1}{2}(C_{11}-C_{12}) \cdot 2\epsilon_{12} \quad \text{from } \textcircled{2}$$

$$= (C_{11}-C_{12})\epsilon_{12}$$

$$= \frac{1}{2}(C_{11}-C_{12})(u_{1,2} + u_{2,1})$$

$$\Rightarrow \sigma_{12,2} = \frac{1}{2}(C_{11} - C_{12})(u_{1,22} + u_{2,12})$$

Thus equ (1) becomes

$$C_{11}u_{1,11} + C_{12}u_{2,12} + \frac{1}{2}(C_{11} - C_{12})(u_{1,22} + u_{2,12}) = \rho \ddot{u}_1 \quad \text{--- (A)}$$

For $i=2$,-

$$\sigma_{2j,j} = \rho \ddot{u}_2$$

$$\Rightarrow \sigma_{21,1} + \sigma_{22,2} = \rho \ddot{u}_2 \quad \text{--- (2)}$$

Now

$$\sigma_{21} = (C_{11} - C_{12})\epsilon_{12} \quad \because \sigma_{21} = \sigma_{12}$$

$$\Rightarrow \sigma_{21} = \frac{1}{2}(C_{11} - C_{12})(u_{1,2} + u_{2,1}) \quad \because \epsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1})$$

$$\Rightarrow \sigma_{21,1} = \frac{1}{2}(C_{11} - C_{12})(u_{1,21} + u_{2,11})$$

$$\text{And } \sigma_{22} = C_{12}\epsilon_{11} + C_{11}\epsilon_{22} \quad \because \text{for 2D}$$

$$= \frac{1}{2}C_{12}(u_{1,1} + u_{1,1}) + \frac{1}{2}C_{11}(u_{2,2} + u_{2,2})$$

$$= C_{12}u_{1,1} + C_{11}u_{2,2}$$

$$\Rightarrow \sigma_{22,2} = C_{12}u_{1,12} + C_{11}u_{2,22}$$

Thus equ (2) becomes

$$\frac{1}{2}(C_{11} - C_{12})(u_{1,21} + u_{2,11}) + C_{12}u_{1,12} + C_{11}u_{2,22} = \rho \ddot{u}_2 \quad \text{--- (B)}$$

$$\text{Let } u_i = Ae^{ik(x_j n_j - ct)} \quad \text{--- (3)}$$

Wave equation

$$\text{Wave equation} \Rightarrow u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt}$$

$$\Rightarrow u_{xx} = (ikn_x)^2 Ae^{-i(kx - \omega t)} \quad \text{--- } P_i$$

$$u_{yy} = (ikn_y)^2 Ae^{-i(kx - \omega t)} \quad \text{--- } P_i$$

$$u_{tt} = (-ikc)^2 Ae^{-i(kx - \omega t)} \quad \text{--- } P_i$$

Since k_1 for x & k_2 for y

Here A is the amplitude of the wave, k is the wave number, n_j is the propagation vector. P_i is the polarization vector and c is the speed of wave.

By using (3) eqn (A) becomes

$$C_{11}(ikn_1)^2 P_1 A e^{ik(x_j n_j - ct)} + C_{12}(ikn_1)(ikn_2) P_2 A e^{ik(x_j n_j - ct)} \\ + \frac{1}{2}(C_{11} - C_{12}) \left\{ (ikn_1)^2 P_1 A e^{ik(x_j n_j - ct)} + (ikn_1)(ikn_2) P_2 A e^{ik(x_j n_j - ct)} \right\} \\ = \rho (-ikc)^2 P_1 A e^{ik(x_j n_j - ct)}$$

$$\Rightarrow A e^{ik(x_j n_j - ct)} \left[C_{11}(ikn_1)^2 P_1 + C_{12}(ikn_1)(ikn_2) P_2 + \frac{1}{2}(C_{11} - C_{12}) \left\{ (ikn_1)^2 P_1 + (ikn_1)(ikn_2) P_2 \right\} \right] = \rho (-ikc)^2 P_1 A e^{ik(x_j n_j - ct)}$$

$$\Rightarrow k^2 \left[C_{11}(in_1)^2 P_1 + C_{12}(in_1)(in_2) P_2 + \frac{1}{2}(C_{11} - C_{12}) \left\{ (in_1)^2 P_1 + (in_1)(in_2) P_2 \right\} \right] = -\rho k^2 c^2 P_1$$

$$\Rightarrow -C_{11}n_1^2 P_1 - C_{12}n_1 n_2 P_2 - \frac{1}{2}(C_{11} - C_{12})n_1^2 P_1 - \frac{1}{2}(C_{11} - C_{12})n_1 n_2 P_2 \\ = -\rho c^2 P_1$$

$$\Rightarrow \left\{ C_{11}n_1^2 + \frac{1}{2}(C_{11} - C_{12})n_1^2 - \rho c^2 \right\} P_1 + \left\{ C_{12}n_1 n_2 + \frac{1}{2}(C_{11} - C_{12})n_1 n_2 \right\} P_2 = 0 \quad \text{--- (4)}$$

Now substitute values of eqn (3) in (4)

$$\Rightarrow \frac{1}{2}(C_{11} - C_{12}) \left\{ n_1 n_2 P_1 + n_1^2 P_2 \right\} + C_{12}n_1 n_2 P_1 + C_{11}n_1^2 P_2 = \rho c^2 P_2$$

$$\Rightarrow \left\{ \frac{1}{2}(C_{11} + C_{12}) n_1 n_2 \right\} p_1 + \left\{ \frac{1}{2}(C_{11} - C_{12}) n_1^2 + C_{11} n_2^2 - \rho c^2 \right\} p_2 = 0 \quad \text{--- (8)}$$

$$\Rightarrow \begin{vmatrix} C_{11} n_1^2 + \frac{1}{2}(C_{11} - C_{12}) n_2^2 - \rho c^2 & C_{12} n_1 n_2 + \frac{1}{2}(C_{11} - C_{12}) n_1 n_2 \\ \frac{1}{2}(C_{11} + C_{12}) n_1 n_2 & \frac{1}{2}(C_{11} - C_{12}) n_1^2 + C_{11} n_2^2 - \rho c^2 \end{vmatrix} = 0$$

for simplicity choose $n = (1, 0)$

$$\Rightarrow \begin{vmatrix} C_{11} - \rho c^2 & 0 \\ 0 & \frac{1}{2}(C_{11} - C_{12}) - \rho c^2 \end{vmatrix} = 0$$

$$\begin{aligned} a_1 x_1 + a_2 x_2 &= 0 \quad \text{--- (9)} \\ b_1 x_1 + b_2 x_2 &= 0 \end{aligned}$$

from (9) $x_1 = -\frac{a_2}{a_1} x_2$

put in (10)

$$b_1 \left(-\frac{a_2}{a_1} \right) x_2 + b_2 x_2 = 0$$

$$\Rightarrow -\frac{b_1 a_2}{a_1} x_2 + b_2 x_2 = 0$$

$$a_1 b_2 - a_2 b_1 = 0$$

$$\Rightarrow \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$$

$$\Rightarrow (C_{11} - \rho c^2) \left[\frac{1}{2}(C_{11} - C_{12}) - \rho c^2 \right] = 0$$

$$\Rightarrow \rho c^2 = C_{11} \quad \left. \begin{aligned} \rho c^2 &= \frac{1}{2}(C_{11} - C_{12}) \\ \Rightarrow c^2 &= \frac{C_{11}}{\rho} \\ \Rightarrow c &= \sqrt{\frac{C_{11}}{\rho}} \end{aligned} \right\} \Rightarrow c = \sqrt{\frac{C_{11} - C_{12}}{2\rho}}$$

Lame's constant $\Rightarrow C_{11} = \lambda + 2\mu$
and $C_{12} = \lambda$

$$\Rightarrow c = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \sqrt{\frac{\mu}{\rho}}$$

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \quad (\text{By Lame's})$$

$$\Rightarrow \sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i})$$

$$\because \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Equation of motion is

$$\sigma_{ij,j} = \rho \ddot{u}_i$$

$$\Rightarrow \lambda u_{k,kj} \delta_{ij} + \mu (u_{ij,j} + u_{j,i,j}) = \rho \ddot{u}_i$$

$$\Rightarrow \lambda u_{k,kj} + \mu (u_{ij,j} + u_{j,i,j}) = \rho \ddot{u}_i$$

Let $u_i = A e^{ik(x_j n_j - ct)} P_i$

$$\Rightarrow \lambda P_k (in_k)(in_i) A e^{ik(x_j n_j - ct)}$$

$$+ \mu \left\{ P_i (in_j)^2 A e^{ik(x_j n_j - ct)} + P_j (in_i)(in_j) A e^{ik(x_j n_j - ct)} \right\}$$

$$= (-ick)^2 \rho A e^{ik(x_j n_j - ct)} P_i$$

$$\Rightarrow \lambda P_k (in_k)(in_i) + \mu \left\{ P_i (in_j)^2 + P_j (in_j)(in_i) \right\} = -c^2 \rho P_i$$

$$\Rightarrow \lambda P_k n_k n_i + \mu \left\{ P_i n_j^2 + P_j n_i n_j \right\} = \rho c^2 P_i \quad \boxed{n_j^2 = 1}$$

$$\Rightarrow \lambda (n_k P_k) n_i + \mu \left\{ P_i + (n_j P_j) n_i \right\} = \rho c^2 P_i$$

For Longitudinal wave $n_i \parallel P_i$

For Transverse wave $n_i \perp P_i$

* For \perp wave $n_i \perp P_i \Rightarrow n_i P_i = 0$

$$\Rightarrow \mu P_i = \rho c^2 P_i \Rightarrow \rho c^2 = \mu$$

$$\Rightarrow c = \sqrt{\frac{\mu}{\rho}}$$

* For $n_i \parallel P_i \Rightarrow n_i = P_i$

$$\Rightarrow \lambda n_i + \mu (n_i + n_i) = \rho c^2 n_i$$

$$\Rightarrow (\lambda + 2\mu) n_i = \rho c^2 n_i$$

$$\Rightarrow c = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

$$\boxed{n_i^2 = 1}$$

* Equation of Motion :-

$$\sigma_{ij,j} = \rho \ddot{u}_i$$

where ρ is mass density and σ_{ij} is the stress.

For $i=1$ $\sigma_{1j,j} = \rho \ddot{u}_1$ (solving for 3D)

$$\Rightarrow \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = \rho \ddot{u}_1 \quad \text{--- (1)}$$

Now

$$\sigma_{11} = C_{11} \epsilon_{11} + C_{12} \epsilon_{22} + C_{12} \epsilon_{33}$$

$$= C_{11} \cdot \frac{1}{2} (u_{1,1} + u_{1,1}) + C_{12} \cdot \frac{1}{2} (u_{2,2} + u_{2,2})$$

$$+ C_{12} \cdot \frac{1}{2} (u_{3,3} + u_{3,3}) \quad \because \epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$$

$$= C_{11} u_{1,1} + C_{12} u_{2,2} + C_{12} u_{3,3}$$

$$\Rightarrow \sigma_{11,1} = C_{11} u_{1,11} + C_{12} u_{2,21} + C_{12} u_{3,31}$$

$$\sigma_{12,2} = \frac{1}{2} (C_{11} - C_{12}) (u_{1,22} + u_{2,12})$$

And $\sigma_{13} = (C_{11} - C_{12}) \epsilon_{33} = \frac{1}{2} (C_{11} - C_{12}) (u_{1,33} + u_{3,13})$

$$\Rightarrow \sigma_{13,3} = \frac{1}{2} (C_{11} - C_{12}) (u_{1,33} + u_{3,13})$$

So equation (1) becomes

$$C_{11} u_{1,11} + C_{12} u_{2,21} + C_{12} u_{3,31} + \frac{1}{2} (C_{11} - C_{12}) (u_{1,22} + u_{2,12})$$

$$+ \frac{1}{2} (C_{11} - C_{12}) (u_{1,33} + u_{3,13}) \quad \longrightarrow \quad \text{--- (A)}$$

For $i=2$, $\sigma_{2ij} = f \ddot{u}_2$

$$\Rightarrow \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = f \ddot{u}_2 \longrightarrow \textcircled{2}$$

$$\sigma_{21,1} = \frac{1}{2} (C_{11} - C_{12}) (u_{2,11} + u_{1,21})$$

Now $\sigma_{22} = C_{12} \epsilon_{11} + C_{11} \epsilon_{22} + C_{12} \epsilon_{33}$

$$= \frac{1}{2} C_{12} (u_{1,11} + u_{1,11}) + \frac{1}{2} C_{11} (u_{2,22} + u_{2,22}) + \frac{1}{2} C_{12} (u_{3,33} + u_{3,33})$$

$$= C_{12} u_{1,11} + C_{11} u_{2,22} + C_{12} u_{3,33}$$

$$\Rightarrow \sigma_{22,2} = C_{12} u_{1,12} + C_{11} u_{2,22} + C_{12} u_{3,32}$$

And $\sigma_{23} = (C_{11} - C_{12}) \epsilon_{23} = \frac{1}{2} (C_{11} - C_{12}) (u_{2,33} + u_{3,23})$

$$\Rightarrow \sigma_{23,3} = \frac{1}{2} (C_{11} - C_{12}) (u_{2,33} + u_{3,23})$$

So equ $\textcircled{2}$ becomes

$$\frac{1}{2} (C_{11} - C_{12}) (u_{1,21} + u_{2,11}) + C_{12} u_{1,12} + C_{11} u_{2,22} + C_{12} u_{3,32}$$

$$+ \frac{1}{2} (C_{11} - C_{12}) (u_{2,33} + u_{3,23}) = f \ddot{u}_2 \longrightarrow \textcircled{B}$$

For $i=3$, $\sigma_{3ij} = f \ddot{u}_3$

$$\Rightarrow \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = f \ddot{u}_3 \longrightarrow \textcircled{3}$$

Now $\sigma_{31} = (C_{11} - C_{12}) \epsilon_{13} = \frac{1}{2} (C_{11} - C_{12}) (u_{1,33} + u_{3,11})$

$$\Rightarrow \sigma_{31,1} = \frac{1}{2} (C_{11} - C_{12}) (u_{1,31} + u_{3,11})$$

$$\sigma_{32} = \frac{1}{2} (C_{11} - C_{12}) (u_{2,33} + u_{3,22})$$

$$\Rightarrow \sigma_{32,2} = \frac{1}{2} (C_{11} - C_{12}) (u_{2,32} + u_{3,22})$$

And

$$\sigma_{33} = C_{12} \epsilon_{11} + C_{12} \epsilon_{22} + C_{11} \epsilon_{33}$$

$$= C_{12} u_{1,1} + C_{12} u_{2,2} + C_{11} u_{3,3}$$

$$\Rightarrow \sigma_{33,3} = C_{12} u_{1,13} + C_{12} u_{2,23} + C_{11} u_{3,33}$$

So eqn (3) becomes

$$\frac{1}{2} (C_{11} - C_{12}) (u_{1,31} + u_{3,11}) + \frac{1}{2} (C_{11} - C_{12}) (u_{2,32} + u_{3,22})$$

$$+ C_{12} u_{1,13} + C_{12} u_{2,23} + C_{11} u_{3,33} = \rho \ddot{u}_3 \quad \text{--- (3)}$$

Now Wave Equation

$$u_i = A e^{ik(x_j n_j - ct)} p_i \quad \text{--- (4)}$$

By using (4) eqn (3) becomes

$$C_{11} (ik n_2)^2 p_1 A e^{ik(x_j n_j - ct)} + C_{12} (ik n_1)(ik n_2) p_2 A e^{ik(x_j n_j - ct)}$$

$$+ C_{12} (ik n_3)(ik n_1) p_3 A e^{ik(x_j n_j - ct)} + \frac{1}{2} (C_{11} - C_{12}) \left\{ (ik n_2)^2 p_1 A e^{ik(x_j n_j - ct)} \right.$$

$$\left. + (ik n_1)(ik n_2) p_2 A e^{ik(x_j n_j - ct)} \right\} + \frac{1}{2} (C_{11} - C_{12}) \left\{ (ik n_3)^2 p_1 A e^{ik(x_j n_j - ct)} \right.$$

$$\left. + (ik n_1)(ik n_3) p_3 A e^{ik(x_j n_j - ct)} \right\}$$

$$= (-ikc)^2 \rho A e^{ik(x_j n_j - ct)} p_1$$

$$\Rightarrow C_{11} n_2^2 p_1 + C_{12} n_1 n_2 p_2 + C_{12} n_1 n_3 p_3 + \frac{1}{2} (C_{11} - C_{12})$$

$$\left\{ n_2^2 p_1 + n_1 n_2 p_2 \right\} + \frac{1}{2} (C_{11} - C_{12}) \left\{ n_3^2 p_1 + n_1 n_3 p_3 \right\}$$

$$= \rho c^2 p_1$$

$$\Rightarrow \left\{ C_{11} n_1^2 + \frac{1}{2}(C_{11} - C_{12}) n_2^2 + \frac{1}{2}(C_{11} - C_{12}) n_3^2 - \beta c^2 \right\} P_1$$

$$+ \left\{ C_{12} n_1 n_2 + \frac{1}{2}(C_{11} - C_{12}) n_1 n_2 \right\} P_2 + \left\{ C_{12} n_1 n_3 + \frac{1}{2}(C_{11} - C_{12}) n_1 n_3 \right\} P_3$$

$$= 0 \quad \longrightarrow \textcircled{5}$$

Now by using equ (A) equ (B) becomes

$$\frac{1}{2}(C_{11} - C_{12}) \left\{ (ikn_2)(ikn_1) P_1 A e^{ik(x_j n_j - ct)} + (ikn_1)^2 P_2 A e^{ik(x_j n_j - ct)} \right\}$$

$$+ C_{12} (ikn_1)(ikn_2) P_1 A e^{ik(x_j n_j - ct)} + C_{11} (ikn_2)^2 P_2 A e^{ik(x_j n_j - ct)}$$

$$+ C_{12} (ikn_3)(ikn_2) P_3 A e^{ik(x_j n_j - ct)} + \frac{1}{2}(C_{11} - C_{12}) \left\{ (ikn_3)^2 P_2 A e^{ik(x_j n_j - ct)} + (ikn_2)(ikn_3) P_3 A e^{ik(x_j n_j - ct)} \right\} = (-ikc)^2 \beta P_2 A e^{ik(x_j n_j - ct)}$$

$$\Rightarrow \frac{1}{2}(C_{11} - C_{12}) \{ n_1 n_2 P_1 + n_1^2 P_2 \} + C_{12} n_1 n_2 P_1 + C_{11} n_2^2 P_2 + C_{12} n_2 n_3 P_3$$

$$+ \frac{1}{2}(C_{11} - C_{12}) \{ n_3^2 P_2 + n_2 n_3 P_3 \} = \beta c^2 P_2$$

$$\Rightarrow \left\{ \frac{1}{2}(C_{11} - C_{12}) n_1 n_2 + C_{12} n_1 n_2 \right\} P_1 + \left\{ \frac{1}{2}(C_{11} - C_{12}) n_1^2 + C_{11} n_2^2 \right.$$

$$\left. + \frac{1}{2}(C_{11} - C_{12}) n_3^2 - \beta c^2 \right\} P_2 + \left\{ C_{12} n_2 n_3 + \frac{1}{2}(C_{11} - C_{12}) n_2 n_3 \right\} P_3 = 0 \quad \longrightarrow \textcircled{6}$$

Now by using equ (A) equ (C) becomes

$$\frac{1}{2}(C_{11} - C_{12}) \left\{ (ikn_3)(ikn_1) P_1 A e^{(\dots)} + (ikn_1)^2 P_3 A e^{(\dots)} \right\} +$$

$$\frac{1}{2}(C_{11} - C_{12}) \left\{ (ikn_2)(ikn_3) P_2 A e^{(\dots)} + (ikn_2)^2 P_3 A e^{(\dots)} \right\} +$$

$$C_{12} (ikn_1)(ikn_3) P_1 A e^{(\dots)} + C_{12} (ikn_2)(ikn_3) P_2 A e^{(\dots)}$$

$$+ C_{11} (ikn_3)^2 P_3 A e^{(\dots)} = \beta (-ikc)^2 P_3 A e^{(\dots)}$$

$$\Rightarrow \frac{1}{2}(G_{11} - G_{12}) \{ n_1 n_3 P_1 + n_3^2 P_3 \} + \frac{1}{2}(G_{11} - G_{12}) \{ n_2 n_3 P_2 + n_2^2 P_3 \} + G_{12} n_1 n_3 P_1 + G_{12} n_2 n_3 P_2 + G_{11} n_3^2 P_3 = \delta c^2 P_3$$

$$\Rightarrow \left\{ \frac{1}{2}(G_{11} - G_{12}) n_1 n_3 + G_{12} n_1 n_3 \right\} P_1 + \left\{ \frac{1}{2}(G_{11} - G_{12}) n_2 n_3 + G_{12} n_2 n_3 \right\} P_2 + \left\{ \frac{1}{2}(G_{11} - G_{12}) n_3^2 + \frac{1}{2}(G_{11} - G_{12}) n_3^2 + G_{11} n_3^2 - \delta c^2 \right\} P_3 = 0 \quad \text{--- (7)}$$

Solving eqn (5), (6) & (7) we get the value of c . i.e. (For simple take $n = (1, 0, 0)$)
For simplicity put $n = n(1, 0, 0)$

$$\text{So (5)} \Rightarrow (G_{11} n_1^2 - \delta c^2) P_1 = 0 \Rightarrow (G_{11} - \delta c^2) P_1 = 0$$

$$\text{(6)} \Rightarrow \left(\frac{1}{2}(G_{11} - G_{12}) - \delta c^2 \right) P_2 = 0$$

$$\text{(7)} \Rightarrow \left(\frac{1}{2}(G_{11} - G_{12}) - \delta c^2 \right) P_3 = 0$$

$$\Rightarrow \begin{vmatrix} G_{11} - \delta c^2 & 0 & 0 \\ 0 & \frac{1}{2}(G_{11} - G_{12}) - \delta c^2 & 0 \\ 0 & 0 & \frac{1}{2}(G_{11} - G_{12}) - \delta c^2 \end{vmatrix} = 0$$

$$\Rightarrow (G_{11} - \delta c^2) \left[\frac{1}{2}(G_{11} - G_{12}) - \delta c^2 \right] \left[\frac{1}{2}(G_{11} - G_{12}) - \delta c^2 \right] = 0$$

$$\Rightarrow G_{11} - \delta c^2 = 0 \quad \left\{ \begin{array}{l} \frac{1}{2}(G_{11} - G_{12}) - \delta c^2 = 0 \\ \Rightarrow \delta c^2 = \frac{1}{2}(G_{11} - G_{12}) \end{array} \right.$$

$$\Rightarrow \delta c^2 = G_{11}$$

$$\Rightarrow \boxed{c = \sqrt{\frac{G_{11}}{\delta}}}$$

$$\Rightarrow c^2 = \frac{G_{11} - G_{12}}{2\delta} \Rightarrow \boxed{c = \sqrt{\frac{G_{11} - G_{12}}{2\delta}}}$$

$$\Rightarrow \boxed{c = \sqrt{\frac{G_{11} - G_{12}}{2\delta}}}$$

⇒ Speed of Wave in a Rotating Isotropic Medium:-

$$\dot{u}_i \rightarrow \dot{u}_i + \underline{\Omega} \times \underline{u}$$

from linear → Angular

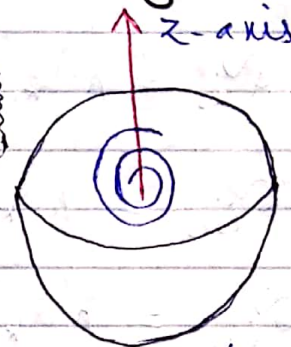
Similarly

$$\frac{d^2 u}{dt^2} = \ddot{u}_i$$

$$\downarrow \text{rotating medium}$$

$$\frac{d}{dt}(\dot{u}_i) \rightarrow \frac{d}{dt} \{ \dot{u}_i + \underline{\Omega} \times \underline{u} \}$$

$$\Rightarrow \{ \ddot{u}_i + \underline{\Omega} \times \dot{u}_i \} + \{ \underline{\Omega} \times (\dot{u}_i + \underline{\Omega} \times \underline{u}) \}$$



$$\underline{\Omega} = \Omega(0, 0, 1)$$

Ω is the angular velocity & \dot{u}_i is rate of change of displacement.

And $\frac{df}{dt} \Rightarrow \frac{d}{dt} f + \Omega \times f$

And $\ddot{u}_i \rightarrow \ddot{u}_i + 2 \underline{\Omega} \times \dot{u}_i + \underline{\Omega} \times (\underline{\Omega} \times \underline{u})$

$$= \ddot{u}_i + 2 \epsilon_{ijk} \Omega_j \dot{u}_k + (\underline{\Omega} \cdot \underline{u}) \underline{\Omega} - (\underline{\Omega} \cdot \underline{\Omega}) \underline{u}$$

$$= \ddot{u}_i + 2 \epsilon_{ijk} \Omega_j \dot{u}_k + \Omega_j u_j \Omega_i - \Omega^2 u_i$$

Thus the equation of motion in a rotating medium becomes

$$\sigma_{ij} = \{ \ddot{u}_i + 2 \epsilon_{ijk} \Omega_j \dot{u}_k + \Omega_j u_j \Omega_i - \Omega^2 u_i \} \rightarrow (*)$$

For Isotropic Medium

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2 \mu \epsilon_{ij}$$

where $\epsilon_{ij} = \frac{1}{2}(u_{ij} + u_{ji})$

$\Rightarrow \sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu (u_{ij} + u_{ji})$

$\Rightarrow \sigma_{i,jj} = \lambda u_{k,kj} \delta_{ij} + \mu (u_{ijj} + u_{jij})$

$\Rightarrow \sigma_{iij} = \lambda u_{k,ki} + \mu (u_{iij} + u_{jij})$

So equation (*) becomes as

$$\lambda u_{k,ki} + \mu (u_{iij} + u_{jij}) = \rho (\ddot{u}_i + 2\epsilon_{ijk} \Omega_j \dot{u}_k + \Omega_j u_j \Omega_i - \Omega^2 u_i) \rightarrow \text{D}$$

Let $u_i = A e^{i k(x_j n_j - ct)}$ Part A=1

So equ D becomes

$$\lambda (i n_k)(i n_i) A e^{i k(x_j n_j - ct)} + \mu \left\{ (i n_j)(i n_j) P_i A e^{i k(x_j n_j - ct)} + (i n_i)(i n_j) P_j A e^{i k(x_j n_j - ct)} \right\} = \rho \left\{ (-c^2) P_i A e^{i k(x_j n_j - ct)} + 2\epsilon_{ijk} \Omega_j (-i c) P_k A e^{i k(x_j n_j - ct)} - \Omega_j P_j \Omega_i A e^{i k(x_j n_j - ct)} - \Omega^2 P_i A e^{i k(x_j n_j - ct)} \right\}$$

$$\Rightarrow -\lambda (n_i n_k) P_k - \mu \left\{ n_j n_j P_i + n_i n_j P_j \right\} = \rho \left\{ -c^2 P_i + 2\epsilon_{ijk} \Omega_j (-i c) P_k + \Omega_j P_j \Omega_i - \Omega^2 P_i \right\}$$

$$\Rightarrow \left\{ \lambda n_i n_k P_k + \mu P_i + \mu n_j P_j n_i \right\} = \rho \left\{ c^2 P_i + 2\epsilon_{ijk} \Omega_j P_k + \Omega_j P_j \Omega_i + \Omega^2 P_i \right\}$$

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) p_i - 2\rho i c \epsilon_{ijk} \Omega_j p_k + (\lambda + \mu) n_k p_k n_i + \rho \Omega_j p_j \Omega_i = 0 \rightarrow \textcircled{2}$$

For Longitudinal Waves $n_i \parallel p_i \Rightarrow n_i p_i = 1$
 $\Rightarrow n_i = p_i$

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) p_i - 2\rho i c \epsilon_{ijk} \Omega_j p_k + (\lambda + \mu) p_i + \rho \Omega_j p_j \Omega_i = 0$$

For $i=1$

$$(\mu - \rho c^2 - \rho \Omega^2) p_1 - 2i\rho c \epsilon_{1jk} \Omega_j p_k + (\lambda + \mu) p_1 + \rho \Omega_j p_j \Omega_1 = 0$$

$$\text{Let } \Omega = \Omega(0, 0, 1)$$

i.e medium is rotating about z-axis

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) p_1 - 2i\rho c \epsilon_{1jk} \Omega_j p_k + (\lambda + \mu) p_1 + 0 = 0$$

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) p_1 - 2i\rho c \epsilon_{132} \Omega p_2 + (\lambda + \mu) p_1 = 0$$

$$\Rightarrow (\lambda + 2\mu - \rho c^2 - \rho \Omega^2) p_1 + 2i\rho c \Omega p_2 = 0 \quad \because \epsilon_{132} = -1 \rightarrow \textcircled{A}$$

For $i=2$

$$(\mu - \rho c^2 - \rho \Omega^2) p_2 - 2\rho i c \epsilon_{2jk} \Omega_j p_k + (\lambda + \mu) p_2 + \rho \Omega_j p_j \Omega_2 = 0$$

$$\Rightarrow \text{Let } \Omega = \Omega(0, 0, 1)$$

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) p_2 - 2\rho i c \epsilon_{2jk} \Omega_j p_k +$$

$$\cancel{(\lambda + \mu) p_2} + 0 = 0$$

$$\Rightarrow (\lambda + 2\mu - \rho c^2 - \rho \Omega^2) p_2 - 2\rho i c \epsilon_{231} \Omega p_3 = 0$$

$$\because \text{for } j=2, \Omega_j = 0$$

$$\Rightarrow (\lambda + 2\mu - \rho c^2 - \rho \Omega^2) P_2 - 2i\rho c \Omega P_1 = 0 \quad \longrightarrow \textcircled{B}$$

From equ ① & ②

$$\begin{vmatrix} \lambda + 2\mu - \rho c^2 - \rho \Omega^2 & 2i\rho c \Omega \\ -2i\rho c \Omega & \lambda + 2\mu - \rho c^2 - \rho \Omega^2 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda + 2\mu - \rho c^2 - \rho \Omega^2)^2 - 4\rho^2 c^2 \Omega^2 = 0$$

$$\Rightarrow (\lambda + 2\mu - \rho c^2 - \rho \Omega^2)^2 = 4\rho^2 c^2 \Omega^2$$

$$\Rightarrow \lambda + 2\mu - \rho c^2 - \rho \Omega^2 = \pm 2\rho c \Omega$$

$$\lambda + 2\mu - \rho c^2 - \rho \Omega^2 = 2\rho c \Omega \quad \lambda + 2\mu - \rho c^2 - \rho \Omega^2 = -2\rho c \Omega$$

$$\rho c^2 + \rho \Omega^2 + 2\rho c \Omega = \lambda + 2\mu \quad \rho c^2 + \rho \Omega^2 - 2\rho c \Omega = \lambda + 2\mu$$

$$\Rightarrow c^2 + \Omega^2 + 2c\Omega = \frac{\lambda + 2\mu}{\rho} \quad c^2 + \Omega^2 - 2c\Omega = \frac{\lambda + 2\mu}{\rho}$$

$$\Rightarrow (c + \Omega)^2 = \frac{\lambda + 2\mu}{\rho} \quad \Rightarrow (c - \Omega)^2 = \frac{\lambda + 2\mu}{\rho}$$

$$\Rightarrow c + \Omega = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad \Rightarrow c - \Omega = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

$\therefore c$ is +ve always

$$\Rightarrow c = \sqrt{\frac{\lambda + 2\mu}{\rho}} - \Omega$$

$$\Rightarrow c = \sqrt{\frac{\lambda + 2\mu}{\rho}} + \Omega$$

Now for Transverse Waves $n_i \perp P_i$

$$\Rightarrow n_i P_i = 0$$

$$\begin{aligned} (\mu - \rho c^2 - \rho \Omega^2) P_i - 2\rho c \Omega \epsilon_{ijk} \Omega_j P_k \\ + \rho \Omega_j P_j \Omega_i = 0 \end{aligned}$$

for $i=1$

$$(\mu - \rho c^2 - \rho \Omega^2) P_1 - 2\rho c \epsilon_{ijk} \Omega_j P_k + \rho \Omega_j P_j \Omega_1 = 0$$

Let $\underline{\Omega} = \Omega(0, 0, 1)$

i.e. medium is rotating about z-axis

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) P_1 - 2\rho c \epsilon_{132} \Omega P_2 + 0 = 0$$

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) P_1 + 2\rho c \epsilon_{132} \Omega P_2 = 0 \rightarrow \textcircled{3}$$

for $i=2$

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) P_2 - 2\rho c \epsilon_{231} \Omega P_1 + 0 = 0$$

Let $\underline{\Omega} = \Omega(0, 0, 1)$

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) P_2 - 2\rho c \epsilon_{231} \Omega P_1 + 0 = 0$$

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2) P_2 - 2\rho c \Omega P_1 = 0 \rightarrow \textcircled{4}$$

From equation $\textcircled{3}$ & $\textcircled{4}$

$$\begin{vmatrix} (\mu - \rho c^2 - \rho \Omega^2) & 2\rho c \Omega \\ -2\rho c \Omega & \mu - \rho c^2 - \rho \Omega^2 \end{vmatrix} = 0 \quad \left\{ \begin{array}{l} \text{for} \\ \text{trivial} \\ \text{solution} \end{array} \right.$$

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2)^2 - 4\rho^2 c^2 \Omega^2 = 0$$

$$\Rightarrow (\mu - \rho c^2 - \rho \Omega^2)^2 = 4\rho^2 c^2 \Omega^2$$

$$\Rightarrow \mu - \rho c^2 - \rho \Omega^2 = \pm 2\rho c \Omega$$

$$\Rightarrow \mu - \rho c^2 - \rho \Omega^2 = 2\rho c \Omega \quad \mu - \rho c^2 - \rho \Omega^2 = -2\rho c \Omega$$

$$\Rightarrow \rho c^2 + \rho \Omega^2 + 2\rho c \Omega = \mu \quad \Rightarrow \rho c^2 + \rho \Omega^2 - 2\rho c \Omega = \mu$$

$$\Rightarrow c^2 + \Omega^2 + 2c\Omega = \frac{u}{g} \Rightarrow c^2 + \Omega^2 - 2c\Omega = \frac{u}{g}$$

$$\Rightarrow (c + \Omega)^2 = \frac{u}{g} \quad \Rightarrow (c - \Omega)^2 = \frac{u}{g}$$

$$\Rightarrow c + \Omega = \sqrt{\frac{u}{g}} \quad \Rightarrow c - \Omega = \sqrt{\frac{u}{g}}$$

$$\Rightarrow c = \sqrt{\frac{u}{g}} - \Omega \quad \Rightarrow c = \sqrt{\frac{u}{g}} + \Omega$$

Now For 3D: For Longitudinal waves $n_i \perp p_i \Rightarrow n_i p_i = 0$

$$\Rightarrow (u - gc^2 - g\Omega^2)p_i - 2gic\epsilon_{ijk}\Omega_j p_k + (\lambda + u)p_i + g\Omega_j p_j \Omega_i = 0$$

for $i=3$

$$(u - gc^2 - g\Omega^2)p_3 - 0 + (\lambda + u)p_3 + g\Omega_3 p_3 \Omega_3 = 0$$

$$\because \underline{\Omega} = \Omega(0, 0, 1) \Rightarrow \epsilon_{ijk} = 0 \text{ for } j=1, 2 \text{ \& } i=j=3$$

$$\Rightarrow (u - gc^2 - g\Omega^2 + \lambda + u + g\Omega^2)p_3 = 0$$

$$\Rightarrow (\lambda + 2u - gc^2)p_3 = 0 \Rightarrow \lambda + 2u - gc^2 = 0$$

$$\Rightarrow gc^2 = \lambda + 2u \Rightarrow c = \sqrt{\frac{\lambda + 2u}{g}}$$

For Transverse Waves $n_i \parallel p_i \Rightarrow n_i p_i = 1 \Rightarrow n_i = p_i$
 eqn ① $\Rightarrow (u - gc^2 - g\Omega^2)p_i - 2gic\epsilon_{ijk}\Omega_j p_k + g\Omega_j p_j \Omega_i = 0$

for $i=3$

$$\Rightarrow (u - gc^2 - g\Omega^2)p_3 - 2gic\epsilon_{3jk}\Omega_j p_k + g\Omega_j p_j \Omega_3 = 0$$

$$\text{Let } \underline{\Omega} = \Omega(0, 0, 1)$$

$$\Rightarrow (u - gc^2 - g\Omega^2)p_3 - 0 + g\Omega_3 p_3 \Omega_3 = 0$$

$$\Rightarrow (u - gc^2 - g\Omega^2 + g\Omega^2)p_3 = 0$$

$$\Rightarrow (u - gc^2)p_3 = 0 \Rightarrow u - gc^2 = 0$$

$$\Rightarrow gc^2 = u \Rightarrow c = \sqrt{\frac{u}{g}}$$

Remaining all same as in 2D upto eqn ②
 & for $i=1, 2$. In 3D only $i=3$ is extra.

⇒ **Transversely Isotropic Medium:** A body / medium is said to be transversely isotropic medium if there exist an axis s.t. all the planes \perp to that axis are plane of isotropic. In transversely isotropic there are 5 independent elastic constants.

$$a_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix}$$

Equation of Motion is

$$\sigma_{ij,j} = \rho \ddot{u}_i$$

For $i=1$

$$\sigma_{1j,j} = \rho \ddot{u}_1$$

$$\Rightarrow \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = \rho \ddot{u}_1 \longrightarrow \text{①}$$

Now

$$\sigma_{11} = C_{11} \epsilon_{11} + C_{12} \epsilon_{22} + C_{13} \epsilon_{33}$$

$$\Rightarrow \sigma_{11} = C_{11} \cdot \frac{1}{2} (u_{1,1} + u_{1,1}) + \frac{1}{2} C_{22} (u_{2,2} + u_{2,2}) \\ + C_{33} \cdot \frac{1}{2} (u_{3,3} + u_{3,3})$$

$$\Rightarrow \sigma_{11} = C_{11} u_{1,1} + C_{22} u_{2,2} + C_{33} u_{3,3}$$

$$\Rightarrow \sigma_{11,1} = C_{11} u_{1,11} + C_{22} u_{2,21} + C_{33} u_{3,31}$$

$$\text{And } \sigma_{12} = (C_{11} - C_{22}) \epsilon_{12} = \frac{1}{2} (C_{11} - C_{22}) (u_{1,2} + u_{2,1})$$

$$\Rightarrow \sigma_{12,2} = \frac{1}{2} (C_{11} - C_{22}) (u_{1,22} + u_{2,12})$$

$$\text{And } \sigma_{13} = 2C_{55} \epsilon_{13} = 2C_{55} \cdot \frac{1}{2} (u_{1,3} + u_{3,1})$$

$$\Rightarrow \sigma_{13} = C_{55} (u_{1,3} + u_{3,1})$$

$$\Rightarrow \sigma_{13,3} = C_{55} (u_{1,33} + u_{3,13})$$

So eqn ① becomes

$$C_{11} u_{1,11} + C_{22} u_{2,21} + C_{33} u_{3,31} + \frac{1}{2} (C_{11} - C_{22}) (u_{1,22} + u_{2,12}) \\ + C_{55} (u_{1,33} + u_{3,13}) = \delta \ddot{u}_1 \longrightarrow \textcircled{A}$$

For $i=2$

$$\sigma_{2ji} = \delta \ddot{u}_2$$

$$\Rightarrow \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = \delta \ddot{u}_2 \longrightarrow \textcircled{B}$$

$$\text{Now } \sigma_{21} = (C_{11} - C_{22}) \epsilon_{12} \quad \because \sigma_{ij} = \sigma_{ji}$$

$$= \frac{1}{2} (C_{11} - C_{22}) (u_{1,2} + u_{2,1})$$

$$\Rightarrow \sigma_{21,1} = \frac{1}{2} (C_{11} - C_{22}) (u_{1,21} + u_{2,11})$$

$$\text{And } \sigma_{22} = C_{12} \epsilon_{11} + C_{11} \epsilon_{22} + C_{13} \epsilon_{33}$$

$$= C_{12} U_{1,1} + C_{11} U_{2,2} + C_{13} U_{3,3}$$

$$\Rightarrow \sigma_{22,2} = C_{12} U_{1,12} + C_{11} U_{2,22} + C_{13} U_{3,32}$$

$$\text{And } \sigma_{23} = 2C_{44} \epsilon_{23} = \frac{1}{2} \cdot 2C_{44} (U_{2,3} + U_{3,2})$$

$$= C_{44} (U_{2,3} + U_{3,2})$$

$$\Rightarrow \sigma_{23,3} = C_{44} (U_{2,33} + U_{3,23})$$

So equ ② becomes

$$\frac{1}{2} (C_{11} - C_{12}) (U_{1,21} + U_{2,11}) + C_{12} U_{1,12} + C_{11} U_{2,22} + C_{13} U_{3,32}$$

$$+ C_{44} (U_{2,33} + U_{3,23}) = \rho \ddot{u}_2 \longrightarrow \textcircled{B}$$

For $z=3$

$$\sigma_{3jj} = \rho \ddot{u}_3$$

$$\Rightarrow \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = \rho \ddot{u}_3 \longrightarrow \textcircled{C}$$

$$\text{Now } \sigma_{31} = 2C_{55} \epsilon_{13} = 2 \cdot \frac{1}{2} C_{55} (U_{1,3} + U_{3,1})$$

$$= C_{55} (U_{1,3} + U_{3,1})$$

$$\sigma_{32} = 2C_{44} \epsilon_{23} = 2 \cdot \frac{1}{2} C_{44} (U_{2,3} + U_{3,2})$$

$$= C_{44} (U_{2,3} + U_{3,2})$$

$$\text{And } \sigma_{33} = C_{13} \epsilon_{11} + C_{13} \epsilon_{22} + C_{11} \epsilon_{33}$$

$$= C_{13} U_{1,1} + C_{13} U_{2,2} + C_{11} U_{3,3}$$

$$\Rightarrow \sigma_{31,1} = C_{55} (U_{1,31} + U_{3,11})$$

$$\sigma_{32,2} = C_{44} (U_{2,32} + U_{3,22})$$

$$\sigma_{33,3} = G_{13} U_{1,13} + G_{23} U_{2,23} + G_{11} U_{3,33}$$

So equation (3) becomes

$$G_{55}(U_{1,31} + U_{3,11}) + G_{44}(U_{2,32} + U_{3,22}) + G_{13} U_{1,13} \\ + G_{23} U_{2,23} + G_{11} U_{3,33} = \rho \ddot{u}_3 \longrightarrow \textcircled{C}$$

$$\text{Let } u_i = A e^{ik(x_j n_j - ct)} P_i$$

Using u_i equ (A) becomes

$$G_{11} (ikn_1)(ikn_1) P_1 A e^{ik(x_j n_j - ct)} + G_{12} (ikn_2)(ikn_1) P_2 A e^{ik(x_j n_j - ct)} \\ + G_{13} (ikn_3)(ikn_1) P_3 A e^{ik(x_j n_j - ct)} + \frac{1}{2}(G_{11} - G_{12}) \{ (ikn_2)^2 P_1 A e^{ik(x_j n_j - ct)} \\ + (ikn_1)(ikn_2) P_2 A e^{ik(x_j n_j - ct)} \} + G_{55} \{ (ikn_3)^2 P_1 A e^{ik(x_j n_j - ct)} \\ + (ikn_1)(ikn_3) P_3 A e^{ik(x_j n_j - ct)} \} = \rho (-2kc)^2 P_1 A e^{2ik(x_j n_j - ct)}$$

$$\Rightarrow G_{11} n_1^2 P_1 + G_{12} n_2 n_2 P_2 + G_{13} n_1 n_3 P_3 + \frac{1}{2} (G_{11} - G_{12}) \\ \{ n_2^2 P_1 + n_1 n_2 P_2 \} + G_{55} \{ n_3^2 P_1 + n_1 n_3 P_3 \} \\ = \rho c^2 P_1$$

$$\Rightarrow \left\{ G_{11} n_1^2 + \frac{1}{2} (G_{11} - G_{12}) n_2^2 + G_{55} n_3^2 - \rho c^2 \right\} P_1 \\ + \left\{ G_{12} n_1 n_2 + \frac{1}{2} (G_{11} - G_{12}) n_1 n_2 \right\} P_2 + \left\{ G_{13} n_1 n_3 + G_{55} n_1 n_3 \right\} P_3 = 0$$

Now

using u_i equ (B) becomes

$$\frac{1}{2} (G_{11} - G_{12}) \{ (ikn_1)(ikn_2) P_1 + (ikn_1)^2 P_2 \} + G_{12} (ikn_1)(ikn_2) P_2$$

$$+G_1 (ikn_2)^2 P_2 + G_3 (ikn_2)(ikn_3) P_3 + G_{44} \{ (ikn_3)^2 P_2 + (ikn_2)(ikn_3) P_3 \} = (-2kc)^2 \rho P_2$$

$$\Rightarrow \frac{1}{2} (G_{11} - G_{12}) \{ n_1 n_2 P_1 + n_1^2 P_2 \} + G_{12} n_1 n_2 P_1 + G_{11} n_2^2 P_2 + G_{33} n_2 n_3 P_3 + G_{44} \{ n_3^2 P_2 + n_2 n_3 P_3 \} = \rho c^2 P_2$$

$$\Rightarrow \left\{ \frac{1}{2} (G_{11} - G_{12}) n_1 n_2 + G_{12} n_1 n_2 \right\} P_1 + \left\{ \frac{1}{2} (G_{11} - G_{12}) n_1^2 + G_{11} n_2^2 + G_{44} n_3^2 - \rho c^2 \right\} P_2 + \left\{ G_{33} n_2 n_3 + G_{44} n_2 n_3 \right\} P_3 = 0$$

→ ⑤

Using u_i eqn ④ becomes

$$G_{55} \{ (ikn_3)(ikn_1) P_1 + (ikn_1)^2 P_3 \} + G_{44} \{ (ikn_3)(ikn_2) P_2 + (ikn_2)^2 P_3 \} + G_{33} (ikn_1)(ikn_3) P_1 + G_{33} (ikn_2)(ikn_3) P_2 + (ikn_3)^2 P_3 = \rho (-2kc)^2 P_3$$

$$\Rightarrow \left\{ G_{55} n_1 n_3 + G_{33} n_1 n_3 \right\} P_1 + \left\{ G_{44} n_2 n_3 + G_{33} n_2 n_3 \right\} P_2 +$$

$$\left\{ G_{55} n_1^2 + G_{44} n_2^2 + G_{33} n_3^2 - \rho c^2 \right\} P_3 = 0 \quad \rightarrow ⑥$$

For simplicity put $n = n(1, 0, 0)$

$$\stackrel{\rho \neq 0}{\text{④}} \Rightarrow (G_{11} - \rho c^2) P_1 = 0 \quad \rightarrow ⑦$$

$$\text{⑤} \Rightarrow \left\{ \frac{1}{2} (G_{11} - G_{12}) - \rho c^2 \right\} P_2 = 0 \quad \rightarrow ⑧$$

$$\text{⑥} \Rightarrow (G_{55} - \rho c^2) P_3 = 0 \quad \rightarrow ⑨$$

from ⑦, ⑧ & ⑨
we have

$$\begin{vmatrix} G_{11} - \rho c^2 & 0 & 0 \\ 0 & \frac{1}{2}(G_{11} - G_{22}) - \rho c^2 & 0 \\ 0 & 0 & G_{55} - \rho c^2 \end{vmatrix} = 0$$

$$\Rightarrow (G_{11} - \rho c^2) \left\{ \frac{1}{2}(G_{11} - G_{22}) - \rho c^2 \right\} (G_{55} - \rho c^2) = 0$$

$$\Rightarrow \begin{cases} G_{11} - \rho c^2 = 0 \\ \frac{1}{2}(G_{11} - G_{22}) - \rho c^2 = 0 \\ G_{55} - \rho c^2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \rho c^2 = G_{11} \\ \rho c^2 = \frac{1}{2}(G_{11} - G_{22}) \\ \rho c^2 = G_{55} \end{cases}$$

$$\Rightarrow \begin{cases} c^2 = \frac{G_{11}}{\rho} \\ c^2 = \frac{G_{11} - G_{22}}{2\rho} \\ c^2 = \frac{G_{55}}{\rho} \end{cases}$$

$$\Rightarrow \boxed{c = \sqrt{\frac{G_{11}}{\rho}}} \Rightarrow \boxed{c = \sqrt{\frac{G_{11} - G_{22}}{2\rho}}} \Rightarrow \boxed{c = \sqrt{\frac{G_{55}}{\rho}}}$$

* For Isotropic Medium: *

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{G_{11} - G_{22}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{G_{11} - G_{22}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{G_{11} - G_{22}}{2} \end{bmatrix}$$

Two Independent C_{ij}

(*) 2D کے لئے solution کرنے کے لئے وزن کے دو اثری (یعنی 2D) سے (Rame سے) (تین زیادہ ہونگی اس کے بعد والی ساری فریم ہونگی) (ہر ڈیٹیم سے)

* For Transversely Isotropic Medium:-

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix} \quad \text{"5 independent"}$$

* For Orthotropic Medium:-

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \quad \text{"There are 9 independent"}$$

* For Monoclinic Medium:-

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0 \\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{56} & C_{66} \end{bmatrix} \quad \text{"There are 13 independent"}$$

* ————— *

⇒ Speed of Waves in a Non-Homogeneous Isotropic Medium =

Equation of Motion is

$$\sigma_{ij,j} = \rho \ddot{u}_i \quad \text{---} \quad (*)$$

Here $\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$

$$\Rightarrow \sigma_{ij} = \lambda \cdot \frac{1}{2} (u_{k,k} + u_{k,k}) \delta_{ij} + 2 \cdot \frac{1}{2} \mu (u_{i,j} + u_{j,i})$$

$$\therefore \epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$$

$$\Rightarrow \sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i})$$

$$\Rightarrow \sigma_{ij,j} = \left\{ \lambda u_{k,k} \delta_{ij} \right\}_{,j} + \left\{ \mu (u_{i,j} + u_{j,i}) \right\}_{,j}$$

$$= \left\{ \lambda u_{k,k} \right\}_{,i} + \left\{ \mu (u_{i,j} + u_{j,i}) \right\}_{,j}$$

$$\Rightarrow \sigma_{ij,j} = \lambda_{,i} u_{k,k} + \lambda u_{k,k,i} + \mu_{,j} (u_{i,j} + u_{j,i}) + \mu (u_{i,jj} + u_{j,ij})$$

So $(*)$ becomes

$$\lambda_{,i} u_{k,k} + \lambda u_{k,k,i} + \mu_{,j} (u_{i,j} + u_{j,i}) + \mu (u_{i,jj} + u_{j,ij}) = \rho \ddot{u}_i \quad \text{---} \quad (1)$$

Since for Non Homogeneous medium

$$\lambda = \lambda_0 e^{-m x_2}, \quad \mu = \mu_0 e^{-m x_2} \quad \left\{ \begin{array}{l} \text{Non-Homo} \\ \text{w.r.t } x_2 \end{array} \right.$$

$$\rho = \rho_0 e^{-m x_2}$$

So equation ① becomes

$$\begin{aligned} & (\lambda_0 e^{-mx_2})_{,i} u_{k,k} + (\lambda_0 e^{-mx_2}) u_{k,ki} + (\lambda_0 e^{-mx_2})_{,j} (u_{ij} \\ & + u_{j,i}) + (\lambda_0 e^{-mx_2}) \{u_{,ij} + u_{j,i}\} = \rho_0 e^{-mx_2} \ddot{u}_i \end{aligned} \quad \text{②}$$

$$\text{Let } u_i = A e^{i(x_j n_j - ct)} P_i \quad \therefore k=1$$

So equ ② becomes

$$(\lambda_0 e^{-mx_2})_{,i} i n_k A e^{i(x_j n_j - ct)} P_k + \lambda_0 e^{-mx_2} \{ (i n_k) (i n_i) \}$$

$$A e^{i(x_j n_j - ct)} P_k \} + (\lambda_0 e^{-mx_2})_{,j} \{ (i n_j) A e^{i(x_j n_j - ct)} P_i +$$

$$(i n_i) A e^{i(x_j n_j - ct)} P_j \} + (\lambda_0 e^{-mx_2}) \{ (i n_j)^2 A e^{i(x_j n_j - ct)} P_i +$$

$$+ (i n_i)^2 A e^{i(x_j n_j - ct)} P_j \} = \rho_0 e^{-mx_2} (-i c)^2 A e^{i(x_j n_j - ct)} P_i$$

$$\Rightarrow (\lambda_0 e^{-mx_2})_{,i} i n_k P_k - \lambda_0 e^{-mx_2} n_k n_i P_k +$$

$$(\lambda_0 e^{-mx_2})_{,j} \{ i n_j P_i + i n_i P_j \} - \lambda_0 e^{-mx_2} \{ n_j^2 P_i +$$

$$n_i n_j P_j \} = -\rho_0 e^{-mx_2} c^2 P_i \quad \text{③}$$

{2D}

for $i=1$

$$\{ \lambda_0 e^{-mx_2} \}_{,1} i n_k P_k - \lambda_0 e^{-mx_2} n_k n_1 P_k$$

$$+ (\lambda_0 e^{-mx_2})_{,j} \{ i n_j P_1 + i n_1 P_j \} - \lambda_0 e^{-mx_2} \{ n_j^2 P_1 +$$

$$n_1 n_j P_j \} = -\rho_0 e^{-mx_2} c^2 P_1$$

$$\Rightarrow -\lambda_0 e^{-mx_2} \{ n_1^2 P_1 + n_2 n_1 P_1 \} + (-m) \lambda_0 e^{-mx_2} \{ i n_2 P_1 +$$

$$i n_1 p_2 \} - u_0 e^{-m x_2} \{ n_1^2 p_1 + n_1^2 p_1 + n_2^2 p_1 + n_1 n_2 p_2 \}$$

$$= -c^2 \rho_0 e^{-m x_2} p_1$$

Put $\underline{n} = n(1, 0)$

$$\Rightarrow -\lambda_0 e^{-m x_2} p_1 - i m u_0 e^{-m x_2} p_2 - 2 u_0 e^{-m x_2} p_1$$

$$= -c^2 \rho_0 e^{-m x_2} p_1$$

$$\Rightarrow \{ (\lambda_0 + 2 u_0 - \rho_0 c^2) p_1 + (i m u_0) p_2 \} e^{-m x_2} = 0$$

$$\Rightarrow (\lambda_0 + 2 u_0 - \rho_0 c^2) p_1 + (i m u_0) p_2 = 0$$

choose $\underline{p} = p(1, 0)$

$$\Rightarrow \lambda_0 + 2 u_0 - \rho_0 c^2 = 0$$

$$\Rightarrow \rho_0 c^2 = \lambda_0 + 2 u_0$$

$$\Rightarrow c = \sqrt{\frac{\lambda_0 + 2 u_0}{\rho_0}}$$

for $i=2$

$$\Rightarrow (\lambda_0 e^{-m x_2})_{,2} i n_2 p_2 - \lambda_0 e^{-m x_2} \{ n_2 n_2 p_2 \}$$

$$+ (u_0 e^{-m x_2})_{,j} \{ i n_j p_2 + i n_2 p_j \} - u_0 e^{-m x_2} \{ n_j^2 p_2 + n_2 n_j p_j \} = -\rho_0 c^2 e^{-m x_2} p_2$$

$$\Rightarrow (-m) \lambda_0 e^{-m x_2} \{ i n_1 p_2 + i n_2 p_2 \} - \lambda_0 e^{-m x_2} \{ n_1 n_2 p_2 + n_2^2 p_2 \}$$

$$+ (-m) u_0 e^{-m x_2} \{ i n_2 p_2 + i n_2 p_2 \} - u_0 e^{-m x_2} \{ n_1^2 p_2 +$$

$$n_2^2 P_2 + n_2 n_1 P_1 + n_2^2 P_2 \} = -\rho_0 c^2 e^{-mx_2} P_2$$

$$\text{Put } n = n(1, 0)$$

$$\Rightarrow -2im\lambda_0 e^{-mx_2} P_1 - \mu_0 e^{-mx_2} P_2 = -\rho_0 c^2 e^{-mx_2} P_2$$

$$\Rightarrow \{+2im\lambda_0 e^{-mx_2} P_1\} + \{(\mu_0 - \rho_0 c^2) P_2\} e^{-mx_2} = 0$$

$$\Rightarrow 2im\lambda_0 P_1 + (\mu_0 - \rho_0 c^2) P_2 = 0$$

$$\text{Choose } P = P(0, 1)$$

$$\Rightarrow \mu_0 - \rho_0 c^2 = 0 \Rightarrow \rho_0 c^2 = \mu_0$$

$$\Rightarrow \boxed{c = \sqrt{\frac{\mu_0}{\rho_0}}}$$

⇒ **Orthotropic**:- Having three mutually perpendicular planes of at each point.

⇒ **Monoclinic**:- Having three unequal axis with two perpendicular and one oblique intersection.

MUHAMMAD TAHIR

FA15-RMT-007

M.S. MATH

Non-Homogeneous Isotropic Medium :- (w.r.t x_2 in 2D)

Equation of Motion is

$$\sigma_{ij,j} = f \ddot{u}_i \quad \text{--- (*)}$$

where for isotropic medium

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

$$= \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i})$$

$$\Rightarrow \sigma_{ij,j} = \left\{ \lambda u_{k,k} \delta_{ij} \right\}_{,j} + \left\{ \mu (u_{i,j} + u_{j,i}) \right\}_{,j}$$

$$= \left\{ \lambda u_{k,k} \right\}_{,i} + \left\{ \mu (u_{i,j} + u_{j,i}) \right\}_{,j}$$

$$\Rightarrow \sigma_{ij,j} = \lambda_{,i} u_{k,k} + \lambda u_{k,ki} + \mu_{,j} (u_{i,j} + u_{j,i}) + \mu (u_{i,jj} + u_{j,ij})$$

So (*) becomes

$$\lambda_{,i} u_{k,k} + \lambda u_{k,ki} + \mu_{,j} (u_{i,j} + u_{j,i}) +$$

$$\mu (u_{i,jj} + u_{j,ij}) = f \ddot{u}_i \quad \text{--- (1)}$$

for $i=1$

$$\Rightarrow \lambda_{,1} u_{k,k} + \lambda u_{k,k1} + \mu_{,j} (u_{1,j} + u_{j,1}) +$$

$$+ \mu (u_{1,jj} + u_{j,1j}) = f \ddot{u}_1$$

$$\therefore \lambda = \lambda_0 e^{-mx_2}, \quad \mu = \mu_0 e^{-mx_2}$$

$$f = f_0 e^{-mx_2}$$

$$\Rightarrow \lambda_{1,1}(u_{1,1} + u_{2,2}) + \lambda \{u_{1,1,1} + u_{2,2,1}\} + \mu_{1,1} \{u_{1,1} + u_{1,1}\} \\ + \mu_{1,2} \{u_{1,2} + u_{2,1}\} + \mu \{u_{1,1,1} + u_{1,1,1} + u_{1,2,2} + u_{2,1,2}\} \\ = \delta \ddot{u}_1$$

$$\text{Let } u_i = A e^{i(k_j n_j - ct)} P_i$$

$$\Rightarrow 0 + \lambda_0 e^{-m k_2} \{ (i n_1)^2 P_1 + (i n_1)(i n_2) P_2 \} + 0 + \\ (-m) \mu_0 e^{-m k_2} \{ (i n_2) P_1 + (i n_1) P_2 \} + \mu_0 e^{-m k_2} \{ 2(i n_1)^2 P_1 \\ + (i n_2)^2 P_1 + (i n_1)(i n_2) P_2 \} = (-i c)^2 \delta_0 e^{-m k_2}$$

$$\Rightarrow + \lambda_0 n_1^2 P_1 + n_1 n_2 \lambda_0 P_2 + 2 i m n_2 \mu_0 P_1 + 2 i m n_1 \mu_0 P_2 \\ + 2 \mu_0 n_1^2 P_1 + \mu_0 n_2^2 P_1 + n_1 n_2 \mu_0 P_2 = c^2 \delta_0$$

$$\Rightarrow \{ \lambda_0 n_1^2 + 2 i m n_2 \mu_0 + 2 \mu_0 n_1^2 + \mu_0 n_2^2 - \delta_0 c^2 \} P_1$$

$$\{ \lambda_0 n_1 n_2 + 2 i m n_1 \mu_0 + n_1 n_2 \mu_0 \} P_2 = 0$$

$$\Rightarrow \{ (\lambda_0 + 2 \mu_0) n_1^2 + \mu_0 n_2^2 + 2 i m \mu_0 n_1 - \delta_0 c^2 \} P_1$$

$$+ \{ (\lambda_0 + \mu_0) n_1 n_2 + 2 i m n_1 \mu_0 \} P_2 = 0 \quad \text{--- } \textcircled{A}$$

For $i=2$

$$\lambda_{2,2} u_{k,k} + \lambda u_{k,k,2} + \mu_{j,j} (u_{2,j} + u_{j,2}) +$$

$$\mu (u_{2,j,j} + u_{j,2,j}) = \delta \ddot{u}_2$$

$$\Rightarrow \lambda_{2,2} (u_{1,1} + u_{2,2}) + \lambda (u_{1,1,2} + u_{2,2,2}) + \mu_{1,1} (u_{2,1} + u_{1,2})$$

$$+ \mu_{1,2} (u_{2,2} + u_{2,2}) + \mu (u_{2,1,1} + u_{1,2,1} + u_{2,2,2} + u_{2,2,2}) \\ = \delta \ddot{u}_2$$

Since $\lambda = \lambda_0 e^{-mk_2}$, $\mu = \mu_0 e^{-mk_2}$

$f = f_0 e^{-mk_2}$

And $U_i = A e^{ik(x_j n_j - ct)}$

$$\Rightarrow (-m)\lambda_0((in_1)P_1 + (in_2)P_2) + \lambda_0((in_1)(in_2)P_1 + (in_2)^2 P_2) + (-m)\mu_0(2(in_2)P_2) + \mu_0((in_1)^2 P_2 + (in_1)(in_2)P_1 + 2(in_2)^2 P_2) = (-ic)^2 f_0 P_2$$

$$\Rightarrow (+imn_1 \lambda_0 + n_1 n_2 \lambda_0 + n_1 n_2 \mu_0) P_1 + (+imn_2 \lambda_0 + n_2^2 \lambda_0 + 2in_2 \mu_0 + n_1^2 \mu_0 + 2n_2^2 \mu_0 - f_0 c^2) P_2 = 0$$

$$\Rightarrow ((\lambda_0 + \mu_0)n_1 n_2 + imn_1 \lambda_0) P_1 + \{(\lambda_0 + 2\mu_0)n_2^2 + \mu_0 n_1^2 + im(2\mu_0 + \lambda_0)n_2 - f_0 c^2\} P_2 = 0$$

For Non trivial solution of eqn

$$\Rightarrow \begin{vmatrix} (\lambda_0 + \mu_0)n_1 n_2 + imn_1 \lambda_0 & (\lambda_0 + \mu_0)n_1 n_2 + imn_1 \mu_0 \\ (\lambda_0 + \mu_0)n_1 n_2 + im\lambda_0 n_1 & (\lambda_0 + 2\mu_0)n_2^2 + \mu_0 n_1^2 + im(2\mu_0 + \lambda_0)n_2 - f_0 c^2 \end{vmatrix} = 0$$

$$\Rightarrow \left\{ (\lambda_0 + 2\mu_0)n_2^2 + \mu_0 n_1^2 + im\lambda_0 n_2 - f_0 c^2 \right\} \left\{ (\lambda_0 + \mu_0)n_1 n_2 + im\lambda_0 n_1 \right\} - \left\{ (\lambda_0 + \mu_0)n_1 n_2 + imn_1 \mu_0 \right\} \left\{ (\lambda_0 + \mu_0)n_1 n_2 + im\lambda_0 n_1 \right\} = 0$$

$$\Rightarrow \left\{ (\lambda_0 + 2\omega_0)^2 n_1^2 n_2^2 + (\lambda_0 + 2\omega_0)\omega_0 n_1^4 + im(\lambda_0 + 2\omega_0)n_1^2 n_2 \right. \\ - (\lambda_0 + 2\omega_0)\beta_0 n_1^2 c^2 + (\lambda_0 + 2\omega_0)\omega_0 n_2^4 + \omega_0^2 n_1^2 n_2^2 + \\ im(\lambda_0 + 2\omega_0)\omega_0 n_2^3 \cdot \omega_0 \beta_0 c^2 n_2^2 + im(\lambda_0 + 2\omega_0)\omega_0 n_1 n_2^2 \\ + im\omega_0^2 n_1 n_1^2 - m^2(\lambda_0 + 2\omega_0)\omega_0 n_1 n_2 - im\omega_0 \beta_0 c^2 n_1 \\ - (\lambda_0 + 2\omega_0)\beta_0 c^2 n_2^2 - \omega_0 \beta_0 c^2 n_1^2 - im(\lambda_0 + \omega_0)\beta_0 c^2 n_2 \\ \left. + (\beta_0 c^2)^2 \right\} - \left\{ (\lambda_0 + \omega_0)^2 n_1^2 n_2^2 + im(\lambda_0 + \omega_0)\lambda_0 n_1^2 n_2 \right. \\ \left. + im(\lambda_0 + \omega_0)\omega_0 n_1^2 n_2 - m^2 \omega_0 \lambda_0 n_1^2 \right\} = 0$$

$$\Rightarrow (\beta_0 c^2)^2 + (\dots)\beta_0 c^2 + (\dots) = 0$$

$$\Rightarrow \beta_0 c^2 = \frac{\lambda_0 n_1^2}{2\beta_0} + \frac{3\omega_0 n_1^2}{2\beta_0} + \frac{m\lambda_0 n_2}{2\beta_0} + \frac{m\omega_0 n_2}{2\beta_0} \\ + \frac{\lambda_0 n_2^2}{2\beta_0} + \frac{3\omega_0 n_2^2}{2\beta_0} \pm A$$

$$\Rightarrow \beta_0 c^2 = \left\{ \frac{\lambda_0 + 2\omega_0}{2\beta_0} + \frac{m}{2\beta_0} (\lambda_0 + \omega_0) n_2 \right\} \pm A$$

where $A = \frac{1}{2\beta_0^2} \left(-\lambda_0 \beta_0 n_1^2 - 3\omega_0 n_1^2 - m\lambda_0 n_2 - m\omega_0 \beta_0 n_2 - \lambda_0 \beta_0 n_2^2 \right. \\ \left. - 3\omega_0 \beta_0 n_2^2 \right)^2 - 4\beta_0^2 \left(m\lambda_0^2 n_1 + m\lambda_0 \omega_0 n_1 + m\lambda_0 n_1^2 + \right. \\ \left. \lambda_0 \omega_0 n_1^4 + 2\omega_0^2 n_1^4 - \lambda_0^2 n_1 n_2 - 2\lambda_0 \omega_0 n_1 n_2 - \omega_0^2 n_1 n_2 - \right. \\ \left. \lambda_0 n_1^2 n_2 + m\lambda_0^2 n_1^2 n_2 - \omega_0 n_1^2 n_2 + 4m\lambda_0 \omega_0 n_1^2 n_2 + 3m\omega_0 n_1^2 n_2 \right. \\ \left. - m^2 \lambda_0 \omega_0 n_2^2 - 2m^2 \omega_0^2 n_2^2 + \lambda_0^2 n_1^2 n_2^2 + 4\lambda_0 \omega_0 n_1^2 n_2^2 \right. \\ \left. + 5\omega_0^2 n_1^2 n_2^2 + \lambda_0 n_2^2 + 2\omega_0^2 n_2^2 \right)^{1/2}$

Perticular Solutions

$$(i) \quad \eta = (1, 0) \quad P = (1, 0)$$

$$(A) \Rightarrow \lambda_0 + 2\mu_0 + im\mu_0\eta_1 - g_0 c^2 = 0$$

$$\text{Put } m=0$$

$$\Rightarrow \lambda_0 + 2\mu_0 = g_0 c^2$$

$$\Rightarrow c = \sqrt{\frac{\lambda_0 + 2\mu_0}{g_0}}$$

$$(B) \Rightarrow \mu_0 - g_0 c^2 = 0$$

$$\Rightarrow \mu_0 = g_0 c^2 \Rightarrow c = \sqrt{\frac{\mu_0}{g_0}}$$

$$\Rightarrow (ii) \quad \eta = (0, 1) \quad P = (0, 1)$$

$$(A) \Rightarrow \mu_0 - g_0 c^2 = 0$$

$$\Rightarrow \mu_0 = g_0 c^2 \Rightarrow c = \sqrt{\frac{\mu_0}{g_0}}$$

$$(B) \Rightarrow (\lambda_0 + 2\mu_0) + im(2\mu_0 + \lambda_0) - g_0 c^2 = 0$$

$$\text{Put } m=0$$

$$\Rightarrow \lambda_0 + 2\mu_0 - g_0 c^2 = 0$$

$$\Rightarrow g_0 c^2 = \lambda_0 + 2\mu_0$$

$$\Rightarrow c = \sqrt{\frac{\lambda_0 + 2\mu_0}{g_0}}$$

$$(3) \quad \underline{n} = (1, 0) \quad \underline{p} = (0, 1)$$

$$(A) \Rightarrow \text{im } \mu_0 = 0 \Rightarrow \text{imaginary}$$

$$(B) \Rightarrow \mu_0 - \rho_0 c^2 = 0$$

$$\Rightarrow \mu_0 = \rho_0 c^2 \Rightarrow c = \sqrt{\mu_0 / \rho_0}$$

$$(4) \quad \underline{n} = (0, 1) \quad \underline{p} = (1, 0)$$

$$(A) \Rightarrow \mu_0 - \rho_0 c^2 = 0$$

$$\Rightarrow c = \sqrt{\mu_0 / \rho_0}$$

$$(B) \Rightarrow 0$$

Question: - For a material it is given that $\lambda + 2\mu = 42.80 \times 10^9 \text{ N/m}^2$
 $\mu = 10.70 \times 10^9 \text{ N/m}^2$, $\rho = 5.35 \times 10^3 \text{ kg/m}^3$
 $\Omega = 2 \times 10^4$, 2×10^6 ; $\Omega = (0, 0, 1)$

$$(i) \quad \underline{n} = (1, 0, 0) \quad , \quad \underline{p} = (1, 0, 0)$$

$$(ii) \quad \underline{n} = (0, 1, 0) \quad , \quad \underline{p} = (1, 0, 0)$$

find speed of waves.

Solution

* For a Non Homogeneous material along x_2 - axis

$$\lambda = \lambda_0 e^{-mx_2}, \quad \mu = \mu_0 e^{-mx_2}$$

$$\rho = \rho_0 e^{-mx_2}$$

* Non-Homogeneous along x_i axis

$$\lambda = \lambda_0 e^{-m_i x_i}, \quad \mu = \mu_0 e^{-m_i x_i}$$

$$\rho = \rho_0 e^{-m_i x_i}$$

* For 3D

$$\lambda = \lambda_0 e^{-m_1 x_1 - m_2 x_2 - m_3 x_3}$$

$$\mu = \mu_0 e^{-m_1 x_1 - m_2 x_2 - m_3 x_3}$$

$$\rho = \rho_0 e^{-m_1 x_1 - m_2 x_2 - m_3 x_3}$$

Assignment: Find speed of waves in a Non-Homogeneous, w.r.t. x_1, x_2, x_3 axis, isotropic medium.

Solution

Equation of motion is

$$\sigma_{ij,jj} = \rho \ddot{u}_i \quad \text{--- (1)}$$

$$\text{where } \sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

$$= \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i})$$

$$\Rightarrow \sigma_{ij,jj} = \left\{ \lambda u_{k,k} \delta_{ij} \right\}_{,jj} + \left\{ \mu (u_{i,j} + u_{j,i}) \right\}_{,jj}$$

$$\Rightarrow \sigma_{ijj} = (\lambda u_{r,r})_{,i} + \{ \mu (u_{ijj} + u_{jji}) \}_{,i}$$

$$= \lambda_{,i} u_{r,r} + \lambda u_{r,r,i} + \mu_{,j} (u_{ijj} + u_{jji})$$

$$+ \mu (u_{ijj} + u_{jji})$$

Thus equation (1) becomes

$$\lambda_{,i} u_{r,r} + \lambda u_{r,r,i} + \mu_{,j} (u_{ijj} + u_{jji}) + \mu (u_{ijj} + u_{jji}) = \rho \ddot{u}_i \quad \longrightarrow (2)$$

for $i=1$

$$\Rightarrow \lambda_{,1} u_{r,r} + \lambda u_{r,r,1} + \mu_{,j} (u_{1jj} + u_{j,1j}) + \mu (u_{1jj} + u_{j,1j}) = \rho \ddot{u}_1$$

$$\Rightarrow \lambda_{,1} (u_{1,1} + u_{2,2} + u_{3,3}) + \lambda (u_{1,11} + u_{2,21} + u_{3,31})$$

$$+ \mu_{,1} (u_{1,11} + u_{1,11}) + \mu_{,2} (u_{1,12} + u_{2,11}) + \mu_{,3} (u_{1,13} + u_{3,11})$$

$$+ \mu (u_{1,11} + u_{1,11} + u_{1,22} + u_{2,12} + u_{1,33} + u_{3,13}) = \rho \ddot{u}_1 \quad \longrightarrow (3)$$

Since for 3D Non-Homogeneous medium

$$\lambda = \lambda_0 e^{-m_1 x_1 - m_2 x_2 - m_3 x_3}, \quad \mu = \mu_0 e^{-m_1 x_1 - m_2 x_2 - m_3 x_3}$$

$$\rho = \rho_0 e^{-m_1 x_1 - m_2 x_2 - m_3 x_3}$$

And also $u_i = A e^{i(k_j x_j - ct)}$

So equ (3) becomes

$$(-m_1) \lambda_0 (2i n_1 P_1 + 2i n_2 P_2 + 2i n_3 P_3) + \lambda_0 (2i n_1^2 P_1 - n_1 n_2 P_2 - n_1 n_3 P_3) + (-m_1) \mu_0 (2i n_1 P_1) - (m_2) \mu_0 (2i n_2 P_2 + 2i n_3 P_3)$$

$$+ (-m_3) \omega_0 (i n_3 P_1 + i n_1 P_3) = \omega_0 \{ 2 n_1^2 P_1 + n_2^2 P_1 + n_1 n_2 P_2 + n_3^2 P_1 + n_1 n_3 P_3 \} = \rho_0 c^2 P_1$$

$$\Rightarrow \{ i m_1 n_1 \lambda_0 + 2 i m_1 n_1 \omega_0 + 2 i m_2 n_2 \omega_0 + i m_3 n_3 \omega_0 + 2 n_1^2 \omega_0 + n_3^2 \omega_0 + n_2^2 \omega_0 - \rho_0 c^2 \} P_1 + \{ i m_1 n_2 \lambda_0 + i m_2 n_1 \omega_0 + n_1 n_2 \omega_0 + n_1 n_2 \lambda_0 \} P_2 + \{ i m_1 n_3 \lambda_0 + i m_1 m_3 \omega_0 + n_1 n_3 \omega_0 + n_1 n_3 \lambda_0 \} P_3 = 0$$

(A)

for $i=2$

$$\lambda_{,2} U_{k,k} + \lambda U_{k,k,2} + \mu_{,j} \{ U_{2,j} + U_{j,2} \} + \mu \{ U_{2,jj} + U_{j,2j} \} = \rho \ddot{U}_2$$

$$\Rightarrow \lambda_{,2} (U_{1,1} + U_{2,2} + U_{3,3}) + \lambda (U_{1,12} + U_{2,22} + U_{3,32}) + \mu_{,1} (U_{2,1} + U_{2,2}) + \mu_{,2} (U_{2,12} + U_{2,22}) + \mu_{,3} (U_{2,3} + U_{3,2}) + \mu (U_{2,11} + U_{3,21} + U_{2,22} + U_{2,22} + U_{2,33} + U_{3,23}) = \rho \ddot{U}_2$$

$$\Rightarrow m_2 \lambda_0 \{ i n_1 P_1 + i n_2 P_2 + i n_3 P_3 \} + \lambda_0 \{ n_1 n_2 P_1 + n_2^2 P_2 + n_2 n_3 P_3 \} + m_1 \omega_0 (i n_1 P_2 + i n_2 P_1) + 2 i m_2 \omega_0 P_2 + i m_3 \omega_0 n_3 P_2 + i m_3 n_2 \omega_0 P_3 + \omega_0 \{ n_1^2 P_2 + n_1 n_2 P_1 + 2 n_2^2 P_2 + n_3^2 P_2 + n_2 n_3 P_3 \} = \rho_0 (-i c)^2 P_2$$

$$\Rightarrow \{ i m_2 n_2 \lambda_0 + n_2^2 \lambda_0 + 2 i m_1 n_1 \omega_0 + 2 i m_2 n_2 \omega_0 + 2 i m_3 n_3 \omega_0 + n_1^2 \omega_0 + 2 n_2^2 \omega_0 + n_3^2 \omega_0 - \rho_0 c^2 \} P_2 + \{ 2 i n_1 m_2 \lambda_0 +$$

$$n_1 n_2 \lambda_0 + i m_1 n_2 u_0 + n_1^2 u_0 \} P_1 + \{ i m_2 n_3 \lambda_0 + n_2 n_3 u_0 + i m_2 m_3 u_0 + n_2 n_3 u_0 \} P_3 = 0 \longrightarrow \textcircled{B}$$

For $i=3$

$$\Rightarrow \lambda_{33} u_{k,k} + \lambda u_{k,k3} + u_{ij} (u_{3,ij} + u_{j,3i}) + u (u_{3,ij} + u_{j,3i}) = \rho \ddot{u}_3$$

$$\Rightarrow \lambda_{33} (u_{1,1} + u_{2,2} + u_{3,3}) + \lambda (u_{1,13} + u_{2,23} + u_{3,33}) + u_{1,1} (u_{3,1} + u_{1,3}) + u_{2,2} (u_{3,2} + u_{2,3}) + u_{3,3} (u_{3,3} + u_{3,3}) + u (u_{3,11} + u_{1,31} + u_{3,22} + u_{2,32} + u_{3,33} + u_{3,33}) = \rho \ddot{u}_3$$

Since $\lambda = \lambda_0 e^{-m_q x_q}$, $u = u_0 e^{-m_q x_q}$ where $q=1,2,3$
 $\rho = \rho_0 e^{-m_q x_q}$

And $u_i = A e^{i(x_j n_j - ct)} P_i$

So equation (4) becomes

$$-m_3 \lambda_0 \{ (in_1) P_1 + (in_2) P_2 + i(n_3) P_3 \} + \lambda_0 \{ 2in_1 (in_3) P_1 + (in_2)(in_3) P_2 + (in_3)^2 P_3 \} + (-m_1) u_0 (in_1 P_3 + in_3 P_1) + (-m_2) u_0 (in_2 P_3 + in_3 P_2) + (-m_3) u_0 (2in_3 P_3) + u_0 \{ (in_1)^2 P_3 + (in_1)(in_3) P_1 + (in_2)^2 P_3 + (in_2)(in_3) P_2 + 2(in_3)^2 P_3 \} = -\rho_0 c^2 P_3$$

$$\Rightarrow \{ i n_1 m_3 \lambda_0 + n_1 n_3 \lambda_0 + i m_1 n_3 u_0 + n_1 n_3 u_0 \} P_1 + \{ i n_2 m_3 \lambda_0 + n_1 n_3 \lambda_0 + i m_2 n_3 u_0 + n_2 n_3 u_0 \} P_2 + \{ i m_3 n_3 \lambda_0 + n_3^2 \lambda_0 + i m_1 n_1 u_0 + i m_2 n_2 u_0 + 2 i m_3 n_3 u_0 + n_1^2 u_0 + n_2^2 u_0 + 2 n_3^2 u_0 - \rho_0 c^2 \} P_3 = 0 \longrightarrow \textcircled{C}$$

Now let $n = (1, 0, 0)$ & $P = P(1, 0, 0)$ for equ (A)

So equ (A) becomes

$$\{ i m_1 \lambda_0 + i m_1 u_0 + 2 u_0 + \lambda_0 - \rho_0 c^2 \} = 0$$

Let $m_1 = 0$

$$\Rightarrow 2\mu_0 + \lambda_0 - \rho_0 c^2 = 0 \Rightarrow \rho_0 c^2 = \lambda_0 + 2\mu_0$$

$$\Rightarrow c = \sqrt{\frac{2\mu_0 + \lambda_0}{\rho_0}}$$

Now put $n = n(0, 1, 0)$ & $P = P(0, 1, 0)$.

Equ (B) \Rightarrow

$$(im_2 \lambda_0 + \lambda_0 + im_2 \mu_0 + (-\rho_0 c^2)) = 0$$

put $m_2 = 0$

$$\Rightarrow \lambda_0 + 2\mu_0 - \rho_0 c^2 = 0 \Rightarrow \rho_0 c^2 = \lambda_0 + 2\mu_0$$

$$\Rightarrow c = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho_0}}$$

Now put $n = n(0, 0, 1)$ & $P = P(0, 0, 1)$.

So equ (C) implies

$$im_3 \lambda_0 + \lambda_0 + im_3 \mu_0 + 2\mu_0 - \rho_0 c^2 = 0$$

$$\Rightarrow im_3(\lambda_0 + \mu_0) + \lambda_0 + 2\mu_0 - \rho_0 c^2 = 0$$

put $m_3 = 0$

$$\Rightarrow \lambda_0 + 2\mu_0 - \rho_0 c^2 = 0$$

$$\Rightarrow \rho_0 c^2 = \lambda_0 + 2\mu_0$$

$$\Rightarrow c^2 = \frac{\lambda_0 + 2\mu_0}{\rho_0}$$

$$\Rightarrow c = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho_0}}$$

⇒ Prove that: $\text{Div}(\text{curl } A) = 0$

~~Proof~~ $\text{Div}(\text{curl } A) = \text{Div}(\epsilon_{ijk} A_{k,j})$

$$\Rightarrow \text{Div}(\text{curl } A) = (\epsilon_{ijk} A_{k,j})_{,i} \quad \left\{ \begin{array}{l} \because \text{curl } A = \epsilon_{ijk} A_{k,j} \\ \text{Div } A = A_{i,i} \end{array} \right.$$

$$= \epsilon_{ijk} A_{k,j,i}$$

$$\Rightarrow \text{Div}(\text{curl } A) = \epsilon_{1jk} A_{k,j,1} + \epsilon_{2jk} A_{k,j,2} + \epsilon_{3jk} A_{k,j,3}$$

Now

$$\epsilon_{1jk} A_{k,j,1} = \epsilon_{11k} A_{k,1,1} + \epsilon_{12k} A_{k,2,1} + \epsilon_{13k} A_{k,3,1} \quad \rightarrow (*)$$

$$= \cancel{\epsilon_{121} A_{1,2,1}} + \epsilon_{122} A_{2,2,1} + \epsilon_{123} A_{3,2,1} +$$

$$\epsilon_{131} A_{1,3,1} + \epsilon_{132} A_{2,3,1} + \epsilon_{133} A_{3,3,1}$$

$$= A_{3,2,1} - A_{2,3,1} \quad \because \epsilon_{123} = 1 \text{ \& } \epsilon_{132} = -1$$

Now

$$\epsilon_{2jk} A_{k,j,2} = \epsilon_{21k} A_{k,1,2} + \epsilon_{22k} A_{k,2,2} + \epsilon_{23k} A_{k,3,2}$$

$$= \epsilon_{211} A_{1,1,2} + \epsilon_{212} A_{2,1,2} + \epsilon_{213} A_{3,1,2} +$$

$$\epsilon_{231} A_{1,3,2} + \epsilon_{232} A_{2,3,2} + \epsilon_{233} A_{3,3,2}$$

$$= -A_{3,1,2} + A_{1,3,2}$$

And

$$\epsilon_{3jk} A_{k,j,3} = \epsilon_{31k} A_{k,1,3} + \epsilon_{32k} A_{k,2,3} + \epsilon_{33k} A_{k,3,3}$$

$$= \epsilon_{311} A_{1,1,3} + \epsilon_{312} A_{2,1,3} + \epsilon_{313} A_{3,1,3} +$$

$$\epsilon_{321} A_{1,2,3} + \epsilon_{322} A_{2,2,3} + \epsilon_{323} A_{3,2,3}$$

$$= A_{2,1,3} - A_{1,2,3}$$

$$\Rightarrow \text{Div}(\text{Curl } A) = A_{3,21} - A_{2,31} - A_{3,12} + A_{1,32} + A_{2,13} - A_{1,23}$$

$$= A_{3,21} - A_{2,31} - A_{3,12} + A_{1,32} + A_{2,13} - A_{1,23}$$

$$\because A_{3,12} = A_{3,21}; A_{2,13} = A_{2,31} \quad \& \quad A_{1,23} = A_{1,32}$$

$$\Rightarrow \text{Div}(\text{Curl } A) = 0$$

Remark:- $\epsilon_{ijk} \psi_{k,ji} = 0$

Wave Equation:-

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad \text{--- (1)}$$

where 'c' is the speed of wave.
The solution of wave equation can be written as

$$u = A e^{ik(x_j n_j - ct)} \quad \text{--- (2)}$$

Now, we verify that it satisfies the wave equation.

$$\begin{aligned} \text{Now } \frac{\partial^2 u}{\partial x_1^2} &= (ik n_1)^2 A e^{ik(x_j n_j - ct)} \\ &= -k^2 n_1^2 A e^{ik(x_j n_j - ct)} \\ &= -k^2 A e^{ik(x_j n_j - ct)} \end{aligned}$$

⇒ Solution of Wave Equation By Decomposition Method:-

Wave equation in elastic medium may be written as

$$\sigma_{ij,j} = \rho \ddot{u}_i \quad \text{--- (1)}$$

where for isotropic medium

$$\sigma_{ij} = \lambda \epsilon_{r,r} \delta_{ij} + 2\mu \epsilon_{ij}$$

$$\Rightarrow \sigma_{ij,j} = \lambda u_{r,r} \delta_{ij} + \mu (u_{i,j} + u_{j,i})$$

Let $u_i = \text{Grad } \phi + \text{Curl } \psi$

i.e. $u_i = \phi_{,i} + \epsilon_{ijk} \psi_{k,i}$

Since $\sigma_{ij,j} = \lambda u_{r,r} \delta_{ij} + \mu (u_{i,j} + u_{j,i})$

Thus wave equation becomes

$$\lambda u_{r,r} \delta_{ij} + \mu (u_{i,j} + u_{j,i}) = \rho \ddot{u}_i \quad \text{--- (2)}$$

Now $u_r = \phi_{,r} + \epsilon_{rpq} \psi_{q,p}$

$$\Rightarrow u_{r,r} = \phi_{,r,r} + \epsilon_{rpq} \psi_{q,p,r}$$

Similarly $u_{i,j} = \phi_{,ij} + \epsilon_{ipq} \psi_{q,p,j}$

$$\& u_{j,i} = \phi_{,ji} + \epsilon_{jpr} \psi_{r,p,i}$$

Thus equ (2) becomes

$$\lambda \{ \phi_{,r,r} + \epsilon_{rpq} \psi_{q,p,r} \} + \mu (\phi_{,ij} + \epsilon_{ipq} \psi_{q,p,j}) + \mu (\phi_{,ji} + \epsilon_{jpr} \psi_{r,p,i}) = \rho (\ddot{\phi}_{,i} + \epsilon_{ipq} \ddot{\psi}_{q,p})$$

$$\Rightarrow \lambda \{ \phi_{,kk} + (\epsilon_{ipq} \psi_{q,pi})_{,i} \} + \mu (\phi_{,jij} + \epsilon_{ipq} \psi_{q,pij}) + \mu \{ \phi_{,jij} + (\epsilon_{ipq} \psi_{q,pi})_{,i} \} = \rho \{ \ddot{\phi}_{,i} + \epsilon_{ipq} \ddot{\psi}_{q,p} \}$$

$$\Rightarrow \frac{\partial}{\partial x_i} \{ \lambda \phi_{,kk} + \mu \phi_{,jij} + \mu \phi_{,jij} - \rho \ddot{\phi} \} +$$

$$\{ \mu \epsilon_{ipq} \psi_{q,pij} - \rho \epsilon_{ipq} \ddot{\psi}_{q,p} \} = 0$$

$$\Rightarrow \frac{\partial}{\partial x_i} \{ \lambda \phi_{,kk} + 2\mu \phi_{,jij} - \rho \ddot{\phi} \} +$$

$$\{ \mu \epsilon_{ipq} \psi_{q,pij} - \rho \epsilon_{ipq} \ddot{\psi}_{q,p} \} = 0$$

vector

$$\Rightarrow \lambda \phi_{,kk} + 2\mu \phi_{,jij} = \rho \ddot{\phi} \rightarrow \textcircled{1}$$

$$\rho \epsilon_{ipq} \{ \mu \psi_{q,pij} - \rho \ddot{\psi}_{q,p} \} = 0 \rightarrow \textcircled{2}$$

∴ scalar + vector = 0
 ⇒ scalar = 0 &
 vector = 0

$$\text{Eqn } \textcircled{1} \Rightarrow (\lambda + 2\mu) \nabla^2 \phi = \rho \ddot{\phi}$$

$$\Rightarrow \nabla^2 \phi = \frac{\rho}{\lambda + 2\mu} \ddot{\phi}$$

$$\Rightarrow c = \sqrt{\frac{\lambda + 2\mu}{\rho}} \rightarrow \text{longitudinal waves}$$

$$\text{Eqn } \textcircled{2} \Rightarrow \nabla^2 \psi_q - \frac{\rho}{\mu} \ddot{\psi}_q = 0$$

$$\Rightarrow \nabla^2 \psi_q = \frac{\rho}{\mu} \ddot{\psi}_q$$

$$\Rightarrow c = \sqrt{\frac{\mu}{\rho}} \rightarrow \text{transverse waves}$$



Since:- Wave Equation

$$\nabla^2 u = \frac{1}{c^2} \ddot{u}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

where c is the speed of waves.

*

If wave equation is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \text{The speed of waves} = \frac{1}{\sqrt{10}}$$

⇒ Surface Waves / Rayleigh Waves / Plane Waves :-

Equation of motion is

$$\sigma_{ij,j} = \rho \ddot{u}_i \quad \text{---} \textcircled{*}$$

$$\Rightarrow \lambda u_{k,kj} + \mu (u_{i,jj} + u_{j,ij}) = \rho \ddot{u}_i$$

They are always in 2D, So

$$\text{For } i=1 \Rightarrow \lambda u_{k,k1} + \mu (u_{1,jj} + u_{j,1j}) = \rho \ddot{u}_1$$

$$\Rightarrow \lambda (u_{1,11} + u_{2,21}) + \mu (u_{1,11} + u_{1,11} + u_{1,22} + u_{2,12}) = \rho \ddot{u}_1$$

$$\Rightarrow (\lambda + 2\mu) u_{1,11} + \mu (u_{1,22}) + (\lambda + \mu) u_{2,12} = \rho \ddot{u}_1$$

$$\text{For } i=2 \Rightarrow \lambda u_{k,k2} + \mu (u_{2,jj} + u_{j,2j}) = \rho \ddot{u}_2 \quad \text{---} \textcircled{1}$$

$$\Rightarrow \lambda (u_{1,12} + u_{2,22}) + \mu (u_{2,11} + u_{1,21} + u_{2,22} + u_{2,22}) = \rho \ddot{u}_2$$

$$\left. \begin{aligned} \mathcal{L}(y'') &= s^2 \bar{y}(s) - sy'(0) - y(0) \\ \mathcal{L}(e^{at}) &= \frac{1}{s-a} \end{aligned} \right\}$$

$$\Rightarrow (\lambda + \mu) u_{1,12} + \mu u_{2,11} + (\lambda + 2\mu) u_{2,22} = g \ddot{u}_2 \rightarrow \textcircled{1}$$

$$ik(x_1 - ct)$$

$$\text{Let } u_i = \phi_i(kx_2) e^{ik(x_1 - ct)}$$

$$\Rightarrow u_{1,11} = -k^2 \phi_1(kx_2) e^{ik(x_1 - ct)}$$

$$u_{1,22} = +k^2 \phi_1''(kx_2) e^{ik(x_1 - ct)}$$

$$u_{2,12} = ik^2 \phi_1'(kx_2) e^{ik(x_1 - ct)}$$

So equ $\textcircled{1}$ implies

$$-(\lambda + 2\mu) k^2 \phi_1(kx_2) + \mu k^2 \phi_1''(kx_2) + (\lambda + \mu) ik^2 \phi_1' = -g c^2 \phi_1 k^2$$

$$\Rightarrow \mu \phi_1'' + i(\lambda + \mu) \phi_1' - (\lambda + 2\mu - g c^2) \phi_1 = 0 \rightarrow \textcircled{3}$$

Similarly for 2nd equation

$$(\lambda + \mu) ik^2 \phi_1' + \mu (ik)^2 \phi_2 + (\lambda + 2\mu) k^2 \phi_2'' = -g c^2 \phi_2$$

$$\Rightarrow (\lambda + 2\mu) \phi_2'' + i(\lambda + \mu) \phi_1' - (\mu - g c^2) \phi_2 = 0 \rightarrow \textcircled{4}$$

Now by Laplace Transform of equ $\textcircled{3}$ & $\textcircled{4}$

$$\text{From } \textcircled{3} \mu \{ s^2 \bar{\phi}_1(s) - s \phi_1(0) - \phi_1'(0) \} + i(\lambda + \mu) \{ s \bar{\phi}_2(s) - \phi_2(0) \} - \{ \lambda + 2\mu - g c^2 \} \bar{\phi}_1(s) = 0$$

$$\Rightarrow \{ \mu s^2 - (\lambda + 2\mu - g c^2) \} \bar{\phi}_1(s) + i(\lambda + \mu) s \bar{\phi}_2(s) = 0$$

$$= \mu \{ \phi_1(0) - \phi_1'(0) \} + i(\lambda + \mu) \phi_2(0)$$

$$\Rightarrow \{ \mu s^2 - (\lambda + 2\mu - g c^2) \} \bar{\phi}_1(s) - i(\lambda + \mu) s \bar{\phi}_2(s) = A_1$$

Now from eqn. (4)

$$(\lambda + 2\mu) \{ s^2 \bar{\phi}_2(s) - s \phi_2(0) - \phi_2'(0) \} + i(\lambda + \mu) \{ s \bar{\phi}_1(s) - \phi_1(0) \} - (\mu - g c^2) \bar{\phi}_2(s) = 0$$

$$\Rightarrow 2s(\lambda + \mu) \bar{\phi}_1(s) + \{ (\lambda + 2\mu) s^2 - (\mu - g c^2) \} \bar{\phi}_2(s) = A_2$$

⇒ Speed of Waves Under Gravity:-

$$G_i = \rho g U_{3,i}$$

Equation of motion in the presence of gravity can be written as

$$\sigma_{ij,j} + G_i = \rho \ddot{u}_i$$

where $G_i = \rho g U_{3,i}$

For isotropic medium

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

$$\text{or } \sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu (u_{ij} + u_{ji})$$

$$\Rightarrow \sigma_{ij,j} = \lambda u_{k,ki} + \mu (u_{ijj} + u_{j,ij})$$

Then equation of motion becomes

$$\lambda u_{k,ki} + \mu (u_{ijj} + u_{j,ij}) + \rho g U_{3,i} = \rho \ddot{u}_i$$

$$\text{Let } u_i = A e^{i(x_j n_j - ct)} P_i \quad \text{--- (1)}$$

Then equation (1) becomes

$$-\lambda n_i n_k P_k + \mu (-n_j n_j P_i - n_j P_j n_i) + \rho g n_i P_3 = -\rho c^2 P_i$$

$$\Rightarrow \{(\lambda + \mu) n_j P_j - 2\rho g P_3\} n_i + (\mu - \rho c^2) P_i = 0$$

For Longitudinal waves :- $n_{ii} P_i \Rightarrow n_i = P_i$

$$\Rightarrow \{(\lambda + \mu) - 2\rho g P_3\} + (\mu - \rho c^2) P_i = 0$$

$$\Rightarrow \lambda + 2\mu - \rho c^2 - 2\rho g P_3 = 0$$

* If $P_3 = 0$ (\Rightarrow in x-y plane)

$$\lambda + 2\mu - \rho c^2 = 0 \quad \Rightarrow \quad c = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

$$* \text{ If } P_3 \neq 0$$

$$\Rightarrow c = \sqrt{\frac{\mu + 2\mu - i\beta g P_3}{g}}$$

\Rightarrow There exist decaying waves

$$\text{Let } c = \alpha \pm i\beta$$

$$\Rightarrow U_i = A e^{i\{k_j n_j - (\alpha \pm i\beta)t\}} P_i$$

$$\Rightarrow U_i = A e^{-\beta t} e^{i(k_j n_j - \alpha t)} P_i$$

For Transverse waves $n_i \perp P_i \Rightarrow n_i P_i = 0$

$$\Rightarrow \mu P_i - i\beta g P_3 n_i - g c^2 P_i = 0$$

$$\Rightarrow (\mu - g c^2) P_i - i\beta g P_3 n_i = 0$$

Since $n_i \perp P_i$

$\Rightarrow n_i$ and P_i are linearly independent

$$\Rightarrow \mu - g c^2 = 0 \quad \text{And} \quad i\beta g P_3 = 0$$

$$\Rightarrow c = \sqrt{\frac{\mu}{g}} \quad \Bigg\} \quad \Rightarrow P_3 = 0$$

\Rightarrow In the case of Gravity Transverse waves only propagate in XY-plane.

\Rightarrow Speed of Waves Under magnetic Field

$$F_i = \mu_0 H_0^2 (U_{i,jj} - \mu_0 \epsilon_0 \ddot{U}_i)$$

Equation of motion

$$\sigma U_{i,jj} + F_i = \rho \ddot{U}_i$$

$$\Rightarrow \lambda U_{k,k_i} + \mu (U_{i,jj} + U_{j,ij}) + \mu_0 H_0^2 (U_{j,ij} - \epsilon_0 \ddot{u}_i) = \delta \ddot{u}_i$$

$$\Rightarrow \lambda U_{k,k_i} + \mu U_{i,jj} + (\mu + \mu_0 H_0^2) U_{j,ij} = (\delta + \mu_0^2 \epsilon_0 H_0^2) \ddot{u}_i$$

$$\text{Let } u_i = A e^{ik(x_j n_j - ct)} p_i$$

$$\begin{aligned} \Rightarrow \lambda n_i n_k p_k + \mu p_i + (\mu + \mu_0 H_0^2) n_j p_j n_i \\ = (\delta + \mu_0^2 \epsilon_0 H_0^2) c^2 p_i \end{aligned}$$

For Longitudinal Waves:- $n_i \parallel p_i \Rightarrow n_i = p_i$

$$\Rightarrow \{ \lambda + \mu + (\mu + \mu_0 H_0^2) - (\delta + \mu_0^2 \epsilon_0 H_0^2) c^2 \} p_i = 0$$

$$\begin{aligned} p_i \neq 0 \\ \Rightarrow C_L = \sqrt{\frac{\lambda + 2\mu + \mu_0 H_0^2}{\delta + \epsilon_0 \mu_0^2 H_0^2}} \end{aligned}$$

For Transverse Waves:- $n_i p_i = 0$

$$\Rightarrow \{ \mu - (\delta + \mu_0^2 \epsilon_0 H_0^2) c^2 \} p_i = 0$$

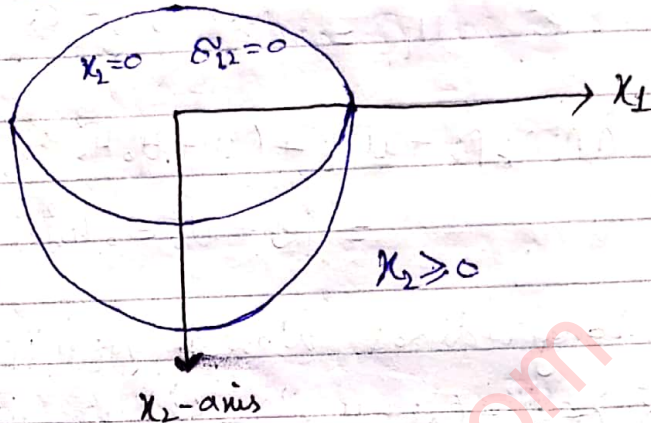
$$\Rightarrow C_T = \sqrt{\frac{\mu}{\delta + \mu_0^2 \epsilon_0 H_0^2}}$$

When $H_0 \longrightarrow \infty$

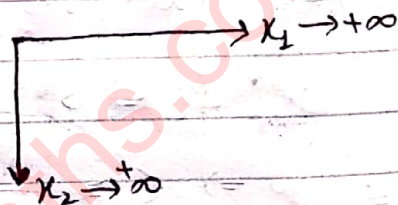
$$\Rightarrow C_T \longrightarrow 0$$



⇒ Rayleigh / Surface Waves in an Isotropic Medium (Without Rotation)



$$u_3 = 0 \quad u_{,3} = 0$$



For an isotropic medium, suppose that material lies in $x_2 \geq 0$ and free surface is $x_2 = 0$

$$\Rightarrow \sigma_{12} = 0 \quad \text{at } x_2 = 0$$

We know that the equation of motion for isotropic ~~medium~~ material is

$$\sigma_{ij,j} = \rho \ddot{u}_i, \quad \text{where } i,j = 1,2 \quad \text{--- (1)}$$

We also know that for an isotropic material

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \quad \text{--- (2)}$$

$$\text{Since } \epsilon_{ij} = \frac{1}{2} (u_{,ij} + u_{,ji}) \quad \text{--- (i)}$$

Put $i,j = k$ in (i) we get

$$\epsilon_{kk} = \frac{1}{2} (u_{,k,k} + u_{,k,k})$$

$$\Rightarrow \epsilon_{kk} = u_{,k,k} \quad \text{--- (ii)}$$

using (i) and (ii) in equ ② we get

$$\begin{aligned}\sigma_{ij} &= \lambda u_{k,k} \delta_{ij} + 2\mu \frac{1}{2} (u_{i,j} + u_{j,i}) \\ &= \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i})\end{aligned}$$

$$\Rightarrow \sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i}) \quad \longrightarrow \textcircled{3}$$

Comparing equation ① and ③ we get

$$\lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i}) = \rho \ddot{u}_i \quad \longrightarrow \textcircled{4}$$

Since $\int_j \delta_{ij} = \delta_i$ so equ ④ becomes

$$\lambda u_{k,ki} + \mu (u_{i,ij} + u_{j,ij}) = \rho \ddot{u}_i \quad \longrightarrow \textcircled{5}$$

Now we use the normal mode solution
Let $u_i = \phi_i(kx_2) e^{ik(x_1 - ct)} \quad \longrightarrow \textcircled{6}$

From equation ⑤

$$\lambda u_{k,ki} + \mu u_{i,ij} + \mu u_{j,ij} = \rho \ddot{u}_i$$

$$\lambda u_{k,ki} + \mu u_{j,ij} + \mu u_{i,ij} = \rho \ddot{u}_i$$

Put $j=k$ in 2nd term

$$\lambda u_{k,ki} + \mu u_{k,ik} + \mu u_{i,ij} = \rho \ddot{u}_i$$

$$\Rightarrow \lambda u_{k,ki} + \mu u_{k,ki} + \mu u_{i,ij} = \rho \ddot{u}_i$$

$$\Rightarrow (\lambda + \mu) u_{k,ki} + \mu u_{i,ij} = \rho \ddot{u}_i \quad \longrightarrow \textcircled{7}$$

Put $i=1$ in equ ⑦ we get

$$(\lambda + \mu) u_{k,k1} + \mu u_{1,ij} = \rho \ddot{u}_1 \quad \longrightarrow \textcircled{8}$$

Put $j,k = 1,2$ in equ ⑧ we get

$$(\lambda + \mu)(u_{1,11} + u_{2,21}) + \mu(u_{1,11} + u_{1,22}) = \rho \ddot{u}_1 \quad \longrightarrow \textcircled{9}$$

Now from equ (6)

$$u_i = \phi_i(kx_2) e^{ik(x_1 - ct)}$$

when $i=1$ then equation (6) becomes

$$u_1 = \phi_1(kx_2) e^{ik(x_1 - ct)} \longrightarrow (10)$$

$$u_{1,1} = \phi_1(kx_2) e^{ik(x_1 - ct)} (ik)$$

$$u_{1,1,1} = \phi_1(kx_2) e^{ik(x_1 - ct)} (ik)^2$$

$$= -k^2 \phi_1(kx_2) e^{ik(x_1 - ct)} \longrightarrow (iii)$$

when $i=2$, then equ (6) becomes

$$u_2 = \phi_2(kx_2) e^{ik(x_1 - ct)}$$

$$u_{2,2} = \phi_2'(kx_2) (k) e^{ik(x_1 - ct)}$$

$$u_{2,2,1} = \phi_2'(kx_2) k e^{ik(x_1 - ct)} (ik)$$

$$\Rightarrow u_{2,2,1} = ik^2 \phi_2'(kx_2) e^{ik(x_1 - ct)} \longrightarrow (iv)$$

from equation (10)

$$u_1 = \phi_1(kx_2) e^{ik(x_1 - ct)}$$

$$u_{1,1,2} = \phi_1'(kx_2) k e^{ik(x_1 - ct)}$$

$$u_{1,1,2,2} = \phi_1''(kx_2) k^2 e^{ik(x_1 - ct)}$$

$$\Rightarrow u_{1,1,2,2} = k^2 \phi_1''(kx_2) e^{ik(x_1 - ct)} \longrightarrow (v)$$

from equation (10)

$$u_1 = \phi_1(kx_2) e^{ik(x_1 - ct)}$$

Differentiate w.r.t 't'

$$\dot{u}_1 = \phi_1(kx_2) e^{ik(x_1 - ct)} (-ick)$$

$$\ddot{u}_1 = \phi_1(kx_2) e^{ik(x_1 - ct)} (-ick)^2$$

$$\Rightarrow \ddot{u}_1 = -c^2 k^2 \phi_1(kx_2) e^{ik(x_1 - ct)} \longrightarrow \text{(vi)}$$

using (iii), (iv), (v) and (vi) in equation
 (a) we get

$$(\lambda + \mu) \left[-k^2 \phi_1 e^{ik(x_1 - ct)} + ik^2 \phi_2' e^{ik(x_1 - ct)} \right] + \mu \left[-k^2 \phi_1 e^{ik(x_1 - ct)} + k^2 \phi_1'' e^{ik(x_1 - ct)} \right] = -c^2 k^2 \phi_1 e^{ik(x_1 - ct)}$$

$$\Rightarrow e^{ik(x_1 - ct)} (\lambda + \mu) \left[-k^2 \phi_1 + ik^2 \phi_2' \right] + e^{ik(x_1 - ct)} \mu \left[-k^2 \phi_1 + k^2 \phi_1'' \right] = -c^2 k^2 \phi_1 e^{ik(x_1 - ct)}$$

$$\left[-k^2 \phi_1 + k^2 \phi_1'' \right] = e^{ik(x_1 - ct)} \left[-c^2 k^2 \phi_1 \right]$$

$$\Rightarrow e^{ik(x_1 - ct)} \left[(\lambda + \mu) (-k^2 \phi_1 + ik^2 \phi_2') + \mu (-k^2 \phi_1 + k^2 \phi_1'') \right] = e^{ik(x_1 - ct)} \left[-c^2 k^2 \phi_1 \right]$$

$$\Rightarrow (\lambda + \mu) (-k^2 \phi_1 + ik^2 \phi_2') + \mu (-k^2 \phi_1 + k^2 \phi_1'') = -c^2 k^2 \phi_1$$

$$\Rightarrow k^2 \left[(\lambda + \mu) (-\phi_1 + i\phi_2') + \mu (-\phi_1 + \phi_1'') \right] = -c^2 k^2 \phi_1$$

$$\Rightarrow (\lambda + \mu) (-\phi_1 + i\phi_2') + \mu (-\phi_1 + \phi_1'') = -c^2 \phi_1$$

$$\Rightarrow \lambda (-\phi_1 + i\phi_2') + \mu (-\phi_1 + i\phi_2') + \mu (-\phi_1 + \phi_1'') = -c^2 \phi_1$$

$$\Rightarrow -\lambda \phi_1 + i\lambda \phi_2' - \mu \phi_1 + i\mu \phi_2' - \mu \phi_1 + \mu \phi_1'' = -c^2 \phi_1$$

$$\Rightarrow \mu \phi_1'' + i(\lambda + \mu) \phi_2' - 2\mu \phi_1 - \lambda \phi_1 = -c^2 \rho \phi_1$$

$$\Rightarrow \mu \phi_1'' + i(\lambda + \mu) \phi_2' - (2\mu + \lambda) \phi_1 = -c^2 \rho \phi_1$$

$$\Rightarrow \phi_1'' + i(\lambda + \mu) \phi_2' - (2\mu + \lambda) \phi_1 + c^2 \rho \phi_1 = 0$$

$$\mu \phi_2'' + i(\lambda + \mu) \phi_2' - (\lambda + 2\mu - \rho c^2) \phi_2 = 0 \quad \longrightarrow \textcircled{A}$$

From equ (7)

$$(\lambda + \mu) u_{k,k} + \mu u_{i,jj} = \rho \ddot{u}_i$$

Put $i=2$ in equ (7) we get

$$(\lambda + \mu) u_{k,k2} + \mu u_{2,jj} = \rho \ddot{u}_2 \quad \longrightarrow \textcircled{11}$$

Put $j, k = 1, 2$ in equ (11) we get

$$(\lambda + \mu) [u_{1,12} + u_{2,22}] + \mu [u_{2,11} + u_{2,22}] = \rho \ddot{u}_2$$

Now from equ (8) $\longrightarrow \textcircled{12}$

$$u_i = \phi_i(kx_2) e^{ik(x_1 - ct)}$$

$$\Rightarrow u_1 = \phi_1(kx_2) e^{ik(x_1 - ct)}$$

$$u_{1,1} = \phi_1(kx_2) e^{ik(x_1 - ct)} (ik)$$

$$u_{1,12} = \phi_1'(kx_2) k e^{ik(x_1 - ct)} (ik)$$

$$\Rightarrow u_{1,12} = ik^2 \phi_1'(kx_2) e^{ik(x_1 - ct)} \quad \longrightarrow \textcircled{vii}$$

Put $i=2$ in equ (8)

$$u_2 = \phi_2(kx_2) e^{ik(x_1 - ct)}$$

$$u_{2,2} = \phi_2'(kx_2) k e^{ik(x_1 - ct)}$$

$$u_{2,22} = \phi_2''(kx_2) k^2 e^{ik(x_1 - ct)}$$

$$u_{2,22} = k^2 \phi_2''(kx_2) e^{ik(x_1 - ct)} \quad \longrightarrow \textcircled{viii}$$

Now $U_2 = \phi_2(kx_2) e^{ik(x_1 - ct)}$

$$U_{2,1} = \phi_2(kx_2) e^{ik(x_1 - ct)} (ik)$$

$$U_{2,11} = \phi_2(kx_2) e^{ik(x_1 - ct)} (ik)^2$$

$$\Rightarrow U_{2,11} = -k^2 \phi_2(kx_2) e^{ik(x_1 - ct)} \quad \text{--- (ix)}$$

using relation (vi), (vii)

$$\ddot{u}_2 = -c^2 k^2 \phi_2(kx_2) e^{ik(x_1 - ct)} \quad \text{--- (x)}$$

using relation (vii), (viii), (ix) and (x) in eqn (12)

$$(\lambda + \mu) [u_{1,12} + u_{2,22}] + \mu [u_{2,11} + u_{1,22}] = \rho \ddot{u}_2$$

$$\Rightarrow (\lambda + \mu) \left[ik^2 \phi_1' e^{ik(x_1 - ct)} + k^2 \phi_2'' e^{ik(x_1 - ct)} \right]$$

$$+ \mu \left[-k^2 \phi_2 e^{ik(x_1 - ct)} + k^2 \phi_2'' e^{ik(x_1 - ct)} \right]$$

$$= \rho \left[-c^2 k^2 \phi_2 e^{ik(x_1 - ct)} \right]$$

$$\Rightarrow e^{ik(x_1 - ct)} \left[(\lambda + \mu) (ik^2 \phi_1' + k^2 \phi_2'') + \mu (-k^2 \phi_2 + k^2 \phi_2'') \right]$$

$$= e^{ik(x_1 - ct)} \left[-\rho c^2 k^2 \phi_2 \right]$$

$$\Rightarrow (\lambda + \mu) (ik^2 \phi_1' + k^2 \phi_2'') + \mu (-k^2 \phi_2 + k^2 \phi_2'') = -\rho c^2 k^2 \phi_2$$

$$\Rightarrow \lambda (ik^2 \phi_1' + k^2 \phi_2'') + \mu (ik^2 \phi_1' + k^2 \phi_2'') + \mu (-k^2 \phi_2 + k^2 \phi_2'')$$

$$= -\rho c^2 k^2 \phi_2$$

$$\Rightarrow i\lambda k^2 \phi_1' + \lambda k^2 \phi_2'' + i\mu k^2 \phi_1' + \mu k^2 \phi_2'' - \mu k^2 \phi_2$$

$$+ \mu k^2 \phi_2'' = -\rho c^2 k^2 \phi_2$$

Dividing through out by k^2

\Rightarrow

$$\Rightarrow i\lambda\phi_1' + \lambda\phi_2'' + iu\phi_1' + u\phi_2'' - u\phi_2 + u\phi_2'' = -g c^2 \phi_2$$

$$\Rightarrow i(\lambda+u)\phi_1' + \lambda\phi_2'' + 2u\phi_2'' - u\phi_2 = -g c^2 \phi_2$$

$$\Rightarrow i(\lambda+u)\phi_1' + (\lambda+2u)\phi_2'' - u\phi_2 + g c^2 \phi_2 = 0$$

$$\Rightarrow i(\lambda+u)\phi_1' + (\lambda+2u)\phi_2'' - (u - g c^2)\phi_2 = 0$$

$$\Rightarrow (\lambda+2u)\phi_2'' + i(\lambda+u)\phi_1' - (u - g c^2)\phi_2 = 0$$

—————→ (B)

From equation (A)

$$u\phi_1'' + i(\lambda+u)\phi_1' - (\lambda+2u - g c^2)\phi_1 = 0 \quad \text{—————→ (A)}$$

Taking Laplace transform on both sides of (A)

$$\text{Since } \mathcal{L}(\phi_1'') = s^2 \bar{\phi}_1(s) - s\phi_1(0) - \phi_1'(0) \quad \text{—————→ (xi)}$$

$$\mathcal{L}(\phi_1') = s\bar{\phi}_1(s) - \phi_1(0) \quad \text{—————→ (xii)}$$

$$\mathcal{L}(\phi_1) = \bar{\phi}_1(s) \quad \text{—————→ (xiii)}$$

using relation (xi), (xii) and (xiii) in (A) we get

$$u[s^2 \bar{\phi}_1(s) - s\phi_1(0) - \phi_1'(0)] + i(\lambda+u)[s\bar{\phi}_1(s) - \phi_1(0)] - (\lambda+2u - g c^2)\bar{\phi}_1(s) = 0$$

$$\Rightarrow us^2 \bar{\phi}_1(s) - us\phi_1(0) - u\phi_1'(0) + is(\lambda+u)\bar{\phi}_1(s) - i(\lambda+u)\phi_1(0) - \lambda\bar{\phi}_1(s) - 2u\bar{\phi}_1(s) + g c^2 \bar{\phi}_1(s) = 0$$

$$\Rightarrow us^2 \bar{\phi}_1(s) - \lambda\bar{\phi}_1(s) - 2u\bar{\phi}_1(s) + g c^2 \bar{\phi}_1(s) + is(\lambda+u)\bar{\phi}_1(s) - us\phi_1(0) - u\phi_1'(0) - i(\lambda+u)\phi_1(0) = 0$$

$$\Rightarrow [s^2 - \lambda - 2\mu + \beta c^2] \bar{\phi}_1(s) + i s (\lambda + \mu) \bar{\phi}_1(s) - \mu s \phi_1(0) - \mu \phi_1'(0) - i(\lambda + \mu) \phi_2(0) = 0 \longrightarrow \textcircled{C}$$

Now from eqn \textcircled{B}

$$(\lambda + 2\mu) \phi_2'' + i(\lambda + \mu) \phi_2' - (\mu - \beta c^2) \phi_2 = 0 \longrightarrow \textcircled{D}$$

Taking Laplace transform on both sides of \textcircled{B} . Since

$$\mathcal{L}\{\phi_2''\} = s^2 \bar{\phi}_2(s) - s \phi_2(0) - \phi_2'(0) \longrightarrow \textcircled{XIV}$$

$$\mathcal{L}\{\phi_2'\} = s \bar{\phi}_2(s) - \phi_2(0) \longrightarrow \textcircled{XV}$$

$$\mathcal{L}\{\phi_2\} = \bar{\phi}_2(s) \longrightarrow \textcircled{XVI}$$

using these values $\textcircled{XIV} \rightarrow \textcircled{XVI}$ in \textcircled{D} we get

$$(\lambda + 2\mu) [s^2 \bar{\phi}_2(s) - s \phi_2(0) - \phi_2'(0)] + i(\lambda + \mu) [s \bar{\phi}_2(s) - \phi_2(0)] + (\mu - \beta c^2) \bar{\phi}_2(s) = 0$$

$$\Rightarrow \lambda [s^2 \bar{\phi}_2(s) - s \phi_2(0) - \phi_2'(0)] + 2\mu [s^2 \bar{\phi}_2(s) - s \phi_2(0) - \phi_2'(0)] + i\lambda [s \bar{\phi}_2(s) - \phi_2(0)] + i\mu [s \bar{\phi}_2(s) - \phi_2(0)] + \mu \bar{\phi}_2(s) - \beta c^2 \bar{\phi}_2(s) = 0$$

$$\Rightarrow \lambda s^2 \bar{\phi}_2(s) - \lambda s \phi_2(0) - \lambda \phi_2'(0) + 2\mu s^2 \bar{\phi}_2(s) - 2\mu s \phi_2(0) - 2\mu \phi_2'(0) + i\lambda s \bar{\phi}_2(s) - i\lambda \phi_2(0) + i\mu s \bar{\phi}_2(s) - i\mu \phi_2(0) + \mu \bar{\phi}_2(s) - \beta c^2 \bar{\phi}_2(s) = 0$$

$$\Rightarrow \lambda s^2 \bar{\phi}_2(s) + 2\mu s^2 \bar{\phi}_2(s) + \mu \bar{\phi}_2(s) - \beta c^2 \bar{\phi}_2(s) + i\lambda s \bar{\phi}_2(s) + i\mu s \bar{\phi}_2(s) - \lambda s \phi_2(0) - 2\mu s \phi_2(0) - \lambda \phi_2'(0) - 2\mu \phi_2'(0) - i\lambda \phi_2(0) - i\mu \phi_2(0) = 0$$

$$\Rightarrow [\lambda s^2 + 2\mu s^2 + \mu - \delta c^2] \bar{\phi}_2(s) + [i\lambda s + i\mu s] \bar{\phi}_1(s) - [\lambda s + 2\mu s] \phi_2(0) - (\lambda + 2\mu) \phi_2'(0) - [i\lambda + i\mu] \phi_1(0) =$$

$$\Rightarrow [(\lambda + 2\mu)s^2 + \mu - \delta c^2] \bar{\phi}_2(s) + i(\lambda + \mu)s \bar{\phi}_1(s) - (\lambda + 2\mu)s \phi_2(0) - (\lambda + 2\mu) \phi_2'(0) - i(\lambda + \mu) \phi_1(0) \longrightarrow \textcircled{D}$$

From \textcircled{C} put $\mu s^2 - \lambda - 2\mu + \delta c^2 = a_1$

$i s(\lambda + \mu) = a_2$ and

$-\mu s \phi_1(0) - \mu \phi_1'(0) - i(\lambda + \mu) \phi_2(0) = -C_1$

we get

$$a_1 \bar{\phi}_1(s) + a_2 \bar{\phi}_2(s) - C_1 = 0 \longrightarrow \textcircled{E}$$

From \textcircled{D} put $(\lambda + 2\mu)s^2 + \mu - \delta c^2 = b_2$

$i(\lambda + \mu)s = b_1$ and

$-(\lambda + 2\mu)s \phi_2(0) - (\lambda + 2\mu) \phi_2'(0) - i(\lambda + \mu) \phi_1(0) = -C_2$

we get

$$b_2 \bar{\phi}_2(s) + b_1 \bar{\phi}_1(s) - C_2 = 0 \longrightarrow \textcircled{F}$$

Now from equation \textcircled{E} and \textcircled{F} we have

$$a_1 \bar{\phi}_1(s) + a_2 \bar{\phi}_2(s) = C_1$$

$$b_1 \bar{\phi}_1(s) + b_2 \bar{\phi}_2(s) = C_2$$

$$\Rightarrow \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} \bar{\phi}_1(s) \\ \bar{\phi}_2(s) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

Now we have to find $\bar{\phi}_1(s)$ and $\bar{\phi}_2(s)$
 \rightarrow by crammer Rule we have

$$\bar{\Phi}_1(s) = \frac{\begin{vmatrix} c_1 & a_2 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{c_1 b_2 - b_1 c_1}{a_1 b_2 - a_2 b_1}$$

And

$$\bar{\Phi}_2(s) = \frac{\begin{vmatrix} a_1 & c_1 \\ b_1 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{a_1 c_2 - b_1 c_1}{a_1 b_2 - a_2 b_1}$$

Here $a_1 b_2 - a_2 b_1$ is quadratic in s^2 .
Let s_1^2 and s_2^2 be the roots of $a_1 b_2 - a_2 b_1$

$$\Rightarrow a_1 b_2 - b_1 a_2 = (s^2 - s_1^2)(s^2 - s_2^2)$$

Thus we have

$$\bar{\Phi}_1(s) = \frac{1}{(s^2 - s_1^2)(s^2 - s_2^2)}$$

By using Partial fraction method we have

$$\bar{\Phi}_1(s) = \frac{A_1}{s - s_1} + \frac{A_2}{s + s_1} + \frac{A_3}{s - s_2} + \frac{A_4}{s + s_2} \quad (13)$$

Now by taking Laplace inverse transform on both sides.

$$\therefore \mathcal{L}^{-1}[\bar{\Phi}_1(s)] = \phi_1(x, x_2) \rightarrow (XVII)$$

$$\& \mathcal{L}^{-1}\left[\frac{1}{s \mp a}\right] = e^{\pm ax} \rightarrow (XVIII)$$

using (XVII) & (XVIII) in eqn (13) we get

$$\phi_1(kx_2) = A_1 e^{s_1 x_2} + A_2 e^{-s_1 x_2} + A_3 e^{s_2 x_2} + A_4 e^{-s_2 x_2} \quad \text{--- (14)}$$

For bounded solution $A_1 = A_3 = 0$

as $x_2 \rightarrow \infty$, $\phi_1(kx_2) \rightarrow 0$

So equ (14) becomes

$$\phi_1(kx_2) = A_2 e^{-s_1 x_2} + A_4 e^{-s_2 x_2} \quad \text{--- (15)}$$

Let $A_2 = A$ & $A_4 = B$

$$\Rightarrow \phi_1(kx_2) = A e^{-s_1 x_2} + B e^{-s_2 x_2} \quad \text{--- (16)}$$

differentiate equ (16) twice w.r.t x_2 we get

$$\phi_1' = -s_1 A e^{-s_1 x_2} - s_2 B e^{-s_2 x_2}$$

$$\phi_1'' = s_1^2 A e^{-s_1 x_2} + s_2^2 B e^{-s_2 x_2} \quad \text{--- (17)}$$

From equ (A)

$$u \phi_1'' + i(\lambda + u) \phi_1' - (\lambda + 2u - \delta c^2) \phi_1 = 0 \quad \text{--- (A)}$$

using values from (16) and (17) in (A) we get

$$u [s_1^2 A e^{-s_1 x_2} + s_2^2 B e^{-s_2 x_2}] + i(\lambda + u) [-s_1 A e^{-s_1 x_2} - s_2 B e^{-s_2 x_2}] - (\lambda + 2u - \delta c^2) [A e^{-s_1 x_2} + B e^{-s_2 x_2}] = 0$$

$$\Rightarrow i(\lambda + u) \phi_1' = (\lambda + 2u - \delta c^2) [A e^{-s_1 x_2} + B e^{-s_2 x_2}] - u [s_1^2 A e^{-s_1 x_2} + s_2^2 B e^{-s_2 x_2}]$$

$$\Rightarrow i(\lambda + u) \phi_1' = (\lambda + 2u - \delta c^2) A e^{-s_1 x_2} + (\lambda + 2u - \delta c^2) B e^{-s_2 x_2} - u s_1^2 A e^{-s_1 x_2} - u s_2^2 B e^{-s_2 x_2}$$

$$\Rightarrow i(\lambda + u)\phi_2' = (\lambda + 2u - gc^2 - us_1^2)Ae^{-s_1x_2} + (\lambda + 2u - gc^2 - us_2^2)Be^{-s_2x_2}$$

Multiplying both sides by i

$$-i(\lambda + u)\phi_2' = i(\lambda + 2u - gc^2 - us_1^2)Ae^{-s_1x_2} + i(\lambda + 2u - gc^2 - us_2^2)Be^{-s_2x_2}$$

Multiplying both sides by -1 we get

$$(\lambda + u)\phi_2' = -i(\lambda + 2u - gc^2 - us_1^2)Ae^{-s_1x_2} - i(\lambda + 2u - gc^2 - us_2^2)Be^{-s_2x_2}$$

$$\Rightarrow \phi_2' = \frac{-i}{\lambda + u} [\lambda + 2u - gc^2 - us_1^2] Ae^{-s_1x_2} - \frac{i}{\lambda + u} [\lambda + 2u - gc^2 - us_2^2] Be^{-s_2x_2}$$

Integrating both sides w.r.t x_2 we get

$$\phi_2 = \frac{-iA}{\lambda + u} [\lambda + 2u - gc^2 - us_1^2] \frac{e^{-s_1x_2}}{-s_1} - \frac{iB}{\lambda + u} [\lambda + 2u - gc^2 - us_2^2] \frac{e^{-s_2x_2}}{-s_2}$$

$$\Rightarrow \phi_2(x_2) = \frac{iA}{(\lambda + u)s_1} [\lambda + 2u - gc^2 - us_1^2] e^{-s_1x_2} + \frac{iB}{(\lambda + u)s_2} [\lambda + 2u - gc^2 - us_2^2] e^{-s_2x_2}$$

⇒ Thermal Effects On Surface:

Field equations in the presence of thermal effect (Temperature difference) can be written as

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl} - \beta_{ij} (T - T_0)$$

with heat equation

$$\frac{\partial}{\partial x_i} \left\{ K_{ij} \frac{\partial}{\partial x_j} (T - T_0) \right\} = \rho C_v \frac{\partial}{\partial t} (T - T_0)$$

β_{ij} is the coefficients of linear thermal expansion. This is conductivity of the material.

C_v is specific heat.

T_0 is the reference (initial) temperature.

For isotropic.

$$K_{ij} = \kappa \delta_{ij}, \quad \beta_{ij} = \beta \delta_{ij}$$

For transversely isotropic medium

$$K_{ij} = K_1 \delta_{ij} + (K_2 - K_1) \delta_{i3} \delta_{j3}$$

$$\beta_{ij} = \beta_1 \delta_{ij} + (\beta_2 - \beta_1) \delta_{i3} \delta_{j3}$$

Equation of motion for isotropic medium becomes

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} - \beta \delta_{ij} (T - T_0)$$

$$\Rightarrow \sigma_{ij,j} = \lambda u_{k,k,i} + \mu (u_{i,jj} + u_{j,ij}) - \beta T_{,i}$$

Thus equation of motion

$$\sigma_{ij,j} = \rho \ddot{u}_i$$

becomes

$$\lambda u_{k,k,i} + \mu (u_{i,jj} + u_{j,ij}) - \beta T_{,i} = \rho \ddot{u}_i \quad \longrightarrow \textcircled{1}$$

And heat equation becomes

$$k T_{,ij} = \rho C_v \frac{\partial T}{\partial t} \longrightarrow \textcircled{1}$$

$$u_i = \phi_i(kx_2) e^{ik(x_1 - ct)} \quad ; i=1,2$$

$$T = \phi_3(kx_2) e^{ik(x_1 - ct)}$$

from equation ①

for $i=1$

$$\lambda u_{k,k1} + \mu (u_{i,jj} + u_{j,ij}) - \beta T_{,1} = \rho \ddot{u}_1$$

$$\Rightarrow \lambda (u_{1,11} + u_{2,21}) + \mu (u_{1,11} + u_{1,22} + u_{1,11} + u_{2,12}) - \beta T_{,1} = \rho \ddot{u}_1$$

$$\Rightarrow \lambda \{ -k^2 \phi_1(kx_2) + ik \phi_2'(kx_2) \} + \mu \{ -2k^2 \phi_1(kx_2) + \phi_1'' + ik \phi_2' \} - \beta ik \phi_3 = \rho (-k^2 c^2) \phi_1$$

$$\Rightarrow \mu \phi_1'' + (\lambda + \mu) ik \phi_2' - (\lambda k^2 + 2k^2 \mu - \rho k^2 c^2) \phi_1 - ik \beta \phi_3 = 0 \longrightarrow \textcircled{A}$$

for $i=2$

$$\lambda u_{k,k2} + \mu (u_{2,ij} + u_{j,ij}) - \beta T_{,2} = \rho \ddot{u}_2$$

$$\rightarrow \lambda (u_{1,12} + u_{2,22}) + \mu (u_{2,11} + u_{2,22} + u_{1,21} + u_{2,22}) - \beta T_{,2} = \rho \ddot{u}_2$$

$$\Rightarrow \lambda \{ ik \phi_2'(kx_2) + \phi_2''(kx_2) \} + \mu \{ -k^2 \phi_2 + 2\phi_2'' + ik \phi_1' \} - \beta \phi_3' = \rho (-k^2 c^2) \phi_2$$

$$\Rightarrow (\lambda + 2\mu) \phi_2'' + (\lambda + \mu) ik \phi_1' - k^2 (\mu - \rho c^2) \phi_2 - \beta \phi_3' = 0 \longrightarrow \textcircled{B}$$

Now from equation ②

$$T_{11} + T_{22} = -\frac{\delta C_V}{K} \frac{\delta T}{\delta t}$$

$$\Rightarrow -K^2 \phi_3 + \phi_3'' = \frac{\delta C_V}{K} i k c \phi_3$$

$$\Rightarrow \phi_3'' - \left(K^2 + \frac{2k c \delta C_V}{K}\right) \phi_3 = 0 \quad \text{--- } \textcircled{C}$$

Taking Laplace Transform of eqn (A)

$$u \{s^2 \bar{\phi}_3(s) - s \phi_3(0) - \phi_3'(0)\} + 2k(\lambda + u) \{s \bar{\phi}_2(s) - \phi_2(0)\}$$

$$- \{ \lambda k^2 + 2k^2 u - \delta k^2 c^2 \} \bar{\phi}_1 - 2k \beta \bar{\phi}_3 = 0$$

$$\Rightarrow (u s^2 - \lambda k^2 - 2k^2 u + \delta k^2 c^2) \bar{\phi}_1(s) + 2s k (\lambda + u) \bar{\phi}_2(s)$$

$$- 2k \beta \bar{\phi}_3(s) = u s \phi_3(0) + u \phi_3'(0) + 2k(\lambda + u) \phi_2(0)$$

Similarly taking Laplace Transform of (B)

$$(\lambda + 2u) \{s^2 \bar{\phi}_2(s) - s \phi_2(0) - \phi_2'(0)\} + 2k(\lambda + u) \{s \bar{\phi}_1(s) - \phi_1(0)\}$$

$$- k^2 (u - \delta c^2) \bar{\phi}_2 - \beta \{s \bar{\phi}_3(s) - \phi_3(0)\} = 0$$

$$\Rightarrow \{ \lambda s^2 + 2u s^2 - u k^2 + \delta c^2 k^2 \} \bar{\phi}_2(s) + 2k s (\lambda + u) \bar{\phi}_1(s)$$

$$- \beta s \bar{\phi}_3(s) = -(\lambda + 2u) s \phi_2(0) + (\lambda + 2u) \phi_2'(0)$$

$$\Rightarrow 2k(\lambda + u) \phi_1(0) + \beta \phi_3(0) = 0 \quad \text{--- } \textcircled{**}$$

Now taking Laplace Transform of equation (C)

$$\Rightarrow s^2 \bar{\phi}_3(s) - s \phi_3(0) - \phi_3'(0) + \xi \bar{\phi}_3(s) = 0$$

$$\Rightarrow (s^2 + \xi) \bar{\phi}_3(s) = s \phi_3(0) + \phi_3'(0) \quad \text{--- } \textcircled{***}$$

from $\textcircled{*}$ put $(\mu s^2 - \lambda k^2 - 2k^2 \mu + \beta k^2 c^2) = a_1$,
 $i s k (\lambda + \mu) = a_2$; $-i k \beta = a_3$ &
 $\mu s \phi_3(0) + \mu \phi_1'(0) + i k (\lambda + \mu) \phi_2(0) = d_1$

from $\textcircled{**}$ put; $(\lambda s^2 + 2\mu s^2 - \mu k^2 + \beta c^2 k^2) = b_1$,
 $i s k (\lambda + \mu) = b_2$, $-\beta s = b_3$ and
 $-(\lambda + 2\mu)s + (\lambda + 2\mu) - i k (\lambda + \mu) + \beta = d_2$

from $\textcircled{***}$ $s^2 + \xi = c_3$ and $s \phi_3(0) + \phi_3'(0) = d_3$

$\textcircled{*} \Rightarrow a_1 \bar{\phi}_1(s) + a_2 \bar{\phi}_2(s) + a_3 \bar{\phi}_3(s) = d_1 \quad \textcircled{3}$

$\textcircled{**} \Rightarrow b_1 \bar{\phi}_1(s) + b_2 \bar{\phi}_2(s) + b_3 \bar{\phi}_3(s) = d_2 \quad \textcircled{4}$

$\textcircled{***} \Rightarrow c_3 \bar{\phi}_3(s) = d_3 \quad \textcircled{5}$

$$\Rightarrow \bar{\phi}_1(s) = \frac{\begin{vmatrix} d_1 & a_2 & a_3 \\ d_2 & b_2 & b_3 \\ d_3 & 0 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix}} = \frac{\begin{vmatrix} d_1 & a_2 & a_3 \\ d_2 & b_2 & b_3 \\ d_3 & 0 & c_3 \end{vmatrix}}{c_3 (a_1 b_2 - b_1 a_2)}$$

$$\bar{\phi}_2(s) = \frac{\begin{vmatrix} a_1 & d_1 & a_3 \\ b_1 & d_2 & b_3 \\ 0 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix}} = \frac{\begin{vmatrix} a_1 & d_1 & a_3 \\ b_1 & d_2 & b_3 \\ 0 & d_3 & c_3 \end{vmatrix}}{c_3 (a_1 b_2 - b_1 a_2)}$$

$$\bar{\phi}_3(s) = \frac{\begin{vmatrix} a_1 & a_2 & d_1 \\ b_1 & b_2 & d_2 \\ 0 & 0 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix}} = \frac{\begin{vmatrix} a_1 & a_2 & d_1 \\ b_1 & b_2 & d_2 \\ 0 & 0 & d_3 \end{vmatrix}}{c_3 (a_1 b_2 - b_1 a_2)}$$

Here $c_3(a_1b_2 - b_1a_2)$ is quadratic in s^2

Let s_1^2 and s_2^2 be the roots of
 $c_3(a_1b_2 - b_1a_2)$

$$\Rightarrow c_3(a_1b_2 - b_1a_2) = c^3(s^2 - s_1^2)(s^2 - s_2^2)$$

Thus we have

$$\bar{\phi}_1(s) = \frac{1}{c^3(s^2 - s_1^2)(s^2 - s_2^2)}$$

$$\Rightarrow \bar{\phi}_1(s) = \frac{A_1}{c_3} + \frac{A_2}{s - s_1} + \frac{A_3}{s - s_2} + \frac{A_4}{s + s_1} + \frac{A_5}{s + s_2}$$

Now by taking inverse Laplace Transform
on both sides

$$\mathcal{L}^{-1}\{\bar{\phi}_1(s)\} = \mathcal{L}^{-1}\left\{ \frac{A_1}{c_3} + \frac{A_2}{s - s_1} + \frac{A_3}{s - s_2} + \frac{A_4}{s + s_1} + \frac{A_5}{s + s_2} \right\}$$

$$\Rightarrow \phi_1(x_1, x_2) = A$$

⇒ Reflection of p-waves as a p-wave:-

P-waves → Longitudinal waves
 SV-waves → Transverse waves

$$C_L = \sqrt{\frac{\lambda + \mu}{\rho}} \quad , \quad G = \sqrt{\frac{\mu}{\rho}}$$

with boundary conditions

- 1) Clamped Boundary Conditions:- In these boundary conditions displacement u_i vanish at boundary i.e. $u_i = 0$ at $x_2 = 0$
- 2) Free Boundary Conditions:- In these boundary conditions stress temperature vanish at the boundary i.e. $\sigma_{ij} = 0$ at $x_2 = 0$

* Now Consider Reflection of p-waves as a p-wave:- Let θ_0 be the angle of incident, A_0 is the amplitude of incident wave and k_0 is the wave number. Similarly θ_1 , A_1 and k_1 are angle of the reflected wave, amplitude of the reflected wave and wave number of the reflected wave respectively.

Let

$$U_i^{(0)} = A_0 e^{i k_0 (n_j^{(0)} x_j - c t)} \quad P_i$$

$$U_i^{(1)} = A_1 e^{i k_1 (n_j^{(1)} x_j - c t)} \quad P_i$$

be the displacement vector for incident and reflected waves.

Here
$$n_i^{(0)} = (n_1^{(0)}, n_2^{(0)}, 0)$$

$$= (\cos(\frac{\pi}{2} - \theta_0), \sin(\frac{\pi}{2} - \theta_0), 0)$$

$$\Rightarrow n_i^{(0)} = (\sin \theta_0, \cos \theta_0, 0)$$

$$\Rightarrow P_i^{(0)} = (\sin \theta_0, \cos \theta_0, 0) \quad \begin{array}{l} \because \text{we are considering} \\ \text{P-waves} \\ \rightarrow n_i \parallel P_i \end{array}$$

Also
$$n_i^{(1)} = (\sin \theta_1, -\cos \theta_1, 0)$$

$$\Rightarrow P_i^{(1)} = (\sin \theta_1, -\cos \theta_1, 0) \quad \because n_i \parallel P_i$$

Now for Clamped Boundary Conditions-

i.e. $U_i = 0$ at $x_2 = 0$

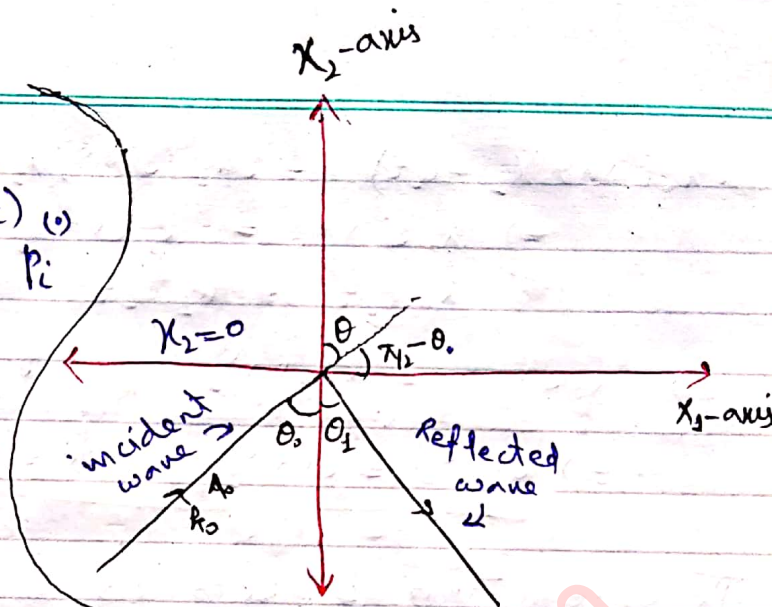
$$\Rightarrow U_i^{(0)} + U_i^{(1)} = 0 \quad \text{at } x_2 = 0$$

for $i=1$
$$U_1^{(0)} + U_1^{(1)} = 0 \quad \text{at } x_2 = 0$$

$$\Rightarrow A_0 P_1 e^{i k_0 (n_1^{(0)} x_1 - c t)} + A_1 P_1 e^{i k_1 (n_1^{(1)} x_1 - c t)} = 0$$

$$\Rightarrow A_0 \sin \theta_0 e^{i k_0 (\sin \theta_0 x_1 - c t)} + A_1 \sin \theta_1 e^{i k_1 (\sin \theta_1 x_1 - c t)} = 0$$

by stress Law



$$ik_0(\sin\theta_0 x_1 - ct) = ik_1(\sin\theta_1 x_1 - ct)$$

$$\Rightarrow k_0 \sin\theta_0 = k_1 \sin\theta_1$$

$$\& k_0 ct = k_1 ct$$

$$\Rightarrow k_0 \sin\theta_0 = k_1 \sin\theta_1 \quad \& k_0 = k_1$$

$$\Rightarrow \boxed{\sin\theta_0 = \sin\theta_1 \quad \& k_0 = k_1}$$

Then equation (1) becomes

$$A_0 \sin\theta_0 = A_1 \sin\theta_1$$

$$\Rightarrow A_0 = A_1 \quad \text{or} \quad A_1 = A_0$$

Now for free Boundary Conditions

$$\text{i.e. } \sigma_{i2} = 0 \text{ at } x_2 = 0$$

$$\Rightarrow \sigma_{i2}^{(0)} + \sigma_{i2}^{(1)} = 0 \text{ at } x_2 = 0$$

for isotropic medium

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

$$\Rightarrow \sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}$$

$$\Rightarrow \sigma_{12} = 2\mu \epsilon_{12} + \mu(u_{1,2} + u_{2,1})$$

$$\Rightarrow \sigma_{12}^{(0)} = \mu(u_{1,2}^{(0)} + u_{2,1}^{(0)})$$

$$\sigma_{12}^{(1)} = \mu(u_{1,2}^{(1)} + u_{2,1}^{(1)})$$

for $i=1$

$$\sigma_{12}^{(0)} + \sigma_{12}^{(1)} = 0$$

$$\Rightarrow \{u_{1,2}^{(0)} + u_{2,1}^{(0)}\} + \{u_{1,2}^{(1)} + u_{2,1}^{(1)}\} = 0$$

$$\Rightarrow A_0 \{n_2 P_1^{(0)} + n_1 P_2^{(0)}\} e^{ik_0(n_1 x_1 - ct)} + A_1 \{n_2 P_1^{(1)} +$$

$$n_1 P_2^{(1)}\} e^{ik_1(n_1 x_1 - ct)} = 0$$

By stress Law

$$k_0 n_1^{(0)} = k_1 n_1^{(1)} \quad \text{and} \quad k_0 = k_1$$

$$\Rightarrow n_1^{(0)} = n_1^{(1)} \quad \text{and} \quad k_0 = k_1$$

$$\Rightarrow \sin \theta_0 = \sin \theta_1, \quad k_0 = k_1$$

$$\Rightarrow \boxed{\theta_0 = \theta_1 \quad \& \quad k_0 = k_1} \quad \Rightarrow A_0 - A_1 = 0$$

for $i=2$

$$\sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}$$

$$\begin{aligned} \Rightarrow \sigma_{22} &= \lambda \epsilon_{kk} + 2\mu \epsilon_{22} \\ &= \lambda \epsilon_{11} + (\lambda + 2\mu) \epsilon_{22} \end{aligned}$$

$$\Rightarrow \sigma_{22} = \lambda u_{1,1} + (\lambda + 2\mu) u_{2,2}$$

Now

$$\sigma_{22}^{(0)} + \sigma_{22}^{(1)} = 0$$

$$\Rightarrow \left\{ \lambda u_{1,1}^{(0)} + (\lambda + 2\mu) u_{2,2}^{(0)} \right\} + \left\{ \lambda u_{1,1}^{(1)} + (\lambda + 2\mu) u_{2,2}^{(1)} \right\} = 0$$

$$\Rightarrow A_0 \left\{ \lambda n_1^{(0)} p_1^{(0)} + (\lambda + 2\mu) n_2^{(0)} p_2^{(0)} \right\} + A_1 \left\{ \lambda n_1^{(1)} p_1^{(1)} + (\lambda + 2\mu) n_2^{(1)} p_2^{(1)} \right\} = 0$$

$$\Rightarrow A_0 \left\{ \lambda \sin^2 \theta_0 + (\lambda + 2\mu) \cos^2 \theta_0 \right\} + A_1 \left\{ \lambda \sin^2 \theta_1 + (\lambda + 2\mu) \cos^2 \theta_1 \right\} = 0$$

$\because \theta_0 = \theta_1$

$$\Rightarrow \left. \begin{aligned} A_0 + A_1 &= 0 \\ A_0 - A_1 &= 0 \end{aligned} \right\} \Rightarrow A_1 = 0$$

Then we have seen in both cases the problem can not be solved by assuming only a reflected p-wave.



⇒ Reflection of S.V. waves as a P-Wave:-

Incident wave is SV-wave and reflected wave is P-wave
The displacement vector of the wave are given

$$u_i^{(0)} = A_0 e^{ik_0(n_j^{(0)}x_j - ct)} p_i^{(0)}$$

Incident wave:-

$$n_i^{(0)} \perp p_0 \quad n^{(0)} = (\sin\theta, \cos\theta, 0)$$

$$p_0 = (-\cos\theta, \sin\theta, 0)$$

The incident wave is S.V wave

$$u_i^{(0)} = A_0 e^{ik_0(n_j^{(0)}x_j - ct)} p_i^{(0)}$$

$$\Rightarrow u_1^{(0)} = A_0 e^{ik_0(\sin\theta_0 x_1 + \cos\theta_0 x_2 - ct)} (-\cos\theta_0)$$

$$u_2^{(0)} = A_0 e^{ik_0(x_1 \sin\theta_0 + x_2 \cos\theta_0 - ct)} (\sin\theta_0)$$

Reflected wave:-

$$n^{\perp} \parallel p^{\perp}$$

$$n^{(\perp)} = (\sin\theta, -\cos\theta, 0) = p^{(\perp)}$$

$$u_i^{(\perp)} = A_1 e^{ik_1(n_j^{(\perp)}x_j - ct)} p_i^{(\perp)}$$

$$u_1^{(\perp)} = A_1 e^{ik_1(x_1 \sin\theta_1 - x_2 \cos\theta_1 - ct)} (\sin\theta_1)$$

$$u_2^{(\perp)} = A_1 e^{ik_1(x_1 \sin\theta_1 - x_2 \cos\theta_1 - ct)} (-\cos\theta_1)$$

For Free Boundary Condition:-

Boundary $x_2 = 0$ is free mean
 $\sigma_{i2} = 0$ where σ_{ij} is stress
tensor

$$\begin{aligned}
 \sigma_{12} &= \hat{n} \epsilon_{ijk} \delta_{ij} + 2\mu \epsilon_{12} \\
 &= (0) + 2\mu (u_{1,2} + u_{2,1}) \cdot \frac{1}{2} \quad \because \delta_{12} = 0 \\
 &= \mu (u_{1,2} + u_{2,1}) \\
 &= \mu (u_{1,2}^{(0)} - u_{1,2}^{(1)} + u_{2,1}^{(0)} - u_{2,1}^{(1)}) \\
 &= \mu \left\{ -A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \left[ik_0(x_1 \sin \theta_0 - ct) \right] \right. \\
 &\quad \left. + A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \left[ik_1(x_1 \sin \theta_1 - ct) \right] \right. \\
 &\quad \left. + \sin \theta_1 \cos \theta_1 + A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \right. \\
 &\quad \left. - ik_0 \sin \theta_0 \sin \theta_0 - A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \cos \theta_1 \sin \theta_1 \right\} \\
 &= \mu \left\{ -A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \left[ik_0(\cos^2 \theta_0 - \sin^2 \theta_0) \right] \right\}
 \end{aligned}$$

$$\sigma_{12} = 0 \Rightarrow$$

$$- \mu A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} (\cos^2 \theta_0 - \sin^2 \theta_0) = 0$$

$$\Rightarrow A_0 \cos 2\theta_0 = 0$$

$$\Rightarrow A_0 = 0 \Rightarrow \text{Reflection does not exist.}$$

→ Reflection of SV-waves as SV-waves:-

The incident wave is SV and the reflected wave is SV. The displacement vector of the wave are given

$$u_i^{(0)} = A_0 e^{ik_0(x_j n_j - ct)} p_i^{(0)}$$

$$u_i^{(1)} = A_1 e^{ik_1(x_j n_j - ct)} p_i^{(1)}$$

$$\hat{n}^0 \perp \hat{p}^0$$

$$\hat{n}^0 = (\sin \theta_0, \cos \theta_0, 0), \hat{p}^0 = (-\cos \theta_0, \sin \theta_0, 0)$$

$$\hat{n}^1 = (\sin \theta_1, -\cos \theta_1, 0) \Rightarrow \hat{p}^1 = (\cos \theta_1, \sin \theta_1, 0)$$

For free Boundary Conditions:-

$$\sigma_{12} = 0, \quad \sigma_{22} = 0$$

$$\sigma_{12} = \lambda \epsilon_{kk} S_{12} + 2\mu \epsilon_{12}$$

$$= \lambda \epsilon_{kk} (0) + 2\mu (U_{1,2} + U_{2,1})$$

$$= \mu (U_{1,2} + U_{1,2} + U_{2,1} + U_{2,1})$$

$$= \mu \left\{ -A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \right. \\ \left. - A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \right\} \frac{ik_0 \cos \theta_0 \cos \theta_0}{ik_1 (\cos \theta_1 \cos \theta_1) + A_0 e^{ik_0(x_1 \sin \theta_0 - ct)}}$$

$$= \mu \left\{ -A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \right. \\ \left. - A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \right\} \frac{ik_0 (\cos^2 \theta_0 - \sin^2 \theta_0)}{ik_1 (\cos^2 \theta_1 - \sin^2 \theta_1)}$$

$$= \mu \left\{ -A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \right. \\ \left. - A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \right\} \frac{ik_0 \cos 2\theta_0}{ik_1 \cos 2\theta_1}$$

$$= \mu \left\{ -A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \right. \\ \left. - A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \right\} \frac{ik_0 \cos 2\theta_0}{ik_1 \cos 2\theta_1}$$

$$\text{Now } \sigma_{12} = 0 \Rightarrow$$

$$-A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \frac{ik_0 \cos 2\theta_0}{ik_1 \cos 2\theta_1} - A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \frac{ik_1 \cos 2\theta_1}{ik_1 \cos 2\theta_1} = 0$$

To make exponents identical

$$k_0 \sin \theta_0 = k_1 \sin \theta_1$$

This is possible only if $\theta_0 = \theta_1$ & $k_0 = k_1$

If it holds then

$$-A_0 \cos 2\theta_0 - A_1 \cos 2\theta_0 = 0$$

$$\Rightarrow \boxed{A_0 = -A_1}$$

Now

$$\sigma_{22} = \lambda(\epsilon_{11} + \epsilon_{22}) + 2\mu(u_{2,2})$$

$$= \lambda[u_{1,1} + u_{2,2}] + 2\mu(u_{2,2})$$

$$\Rightarrow \sigma_{22} = \lambda[u_{1,1}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(0)} + u_{2,2}^{(1)}] + 2\mu(u_{2,2}^{(0)} + u_{2,2}^{(1)})$$

$$= \lambda \left[A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} + A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} + A_0 e^{ik_0(x_2 \sin \theta_0 \cos \theta_0 + A_1 e^{ik_1(x_2 \sin \theta_1 \cos \theta_1 - ct)} \right]$$

$$+ A_1 e^{ik_1(x_2 \sin \theta_1 \cos \theta_1 - ct)} + 2\mu \left[A_0 e^{ik_0(x_2 \sin \theta_0 \cos \theta_0 - A_1 e^{ik_1(x_2 \sin \theta_1 \cos \theta_1 - ct)} \right]$$

$$+ A_1 e^{ik_1(x_2 \sin \theta_1 \cos \theta_1 - ct)} + 2\mu \left[A_0 e^{ik_0(x_2 \sin \theta_0 \cos \theta_0 - A_1 e^{ik_1(x_2 \sin \theta_1 \cos \theta_1 - ct)} \right]$$

$$+ A_1 e^{ik_1(x_2 \sin \theta_1 \cos \theta_1 - ct)} + 2\mu \left[A_0 e^{ik_0(x_2 \sin \theta_0 \cos \theta_0 - A_1 e^{ik_1(x_2 \sin \theta_1 \cos \theta_1 - ct)} \right]$$

$$= \lambda(0) + 2\mu \left[A_0 e^{ik_0(x_2 \sin \theta_0 \cos \theta_0 - A_1 e^{ik_1(x_2 \sin \theta_1 \cos \theta_1 - ct)} \right]$$

$$+ A_1 e^{ik_1(x_2 \sin \theta_1 \cos \theta_1 - ct)} \sin \theta_1 \cos \theta_1]$$

Now $\sigma_{22} = 0$ implies

$$2\mu A_0 \cos \theta_0 \sin \theta_0 - 2\mu A_1 \cos \theta_0 \sin \theta_0 = 0$$

$$\Rightarrow \boxed{A_0 = A_1}$$

Thus we have seen in both cases the problem can not be solved by assuming only a reflected SV-wave.

⇒ Reflection of P-waves as SV-wave + P-wave:-

Here we assume a reflected P-wave, there is a reflected SV-wave too.

The displacement vector $u_i^{(2)}$ is given

$$u_i^{(2)} = A_2 e^{ik_2(x_1 n_1^{(2)} - ct)} p_i^{(2)}$$

$$n^2 = (\sin \theta_2, -\cos \theta_2, 0)$$

$$p^2 = (\cos \theta_2, \sin \theta_2, 0) \quad \because n^2 \perp p^2$$

Boundary is free:-

Boundary $x_2 = 0$ is free mean $\sigma_{i2} = 0$, where σ_{ij} is stress tensor.

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}, \text{ for } j=2$$

$$\begin{aligned} \sigma_{12} &= \lambda(0) + 2\mu \epsilon_{12} \\ &= 2\mu \left(\frac{u_{1,2} + u_{2,1}}{2} \right) \end{aligned}$$

$$\Rightarrow \sigma_{12}^{(2)} = \mu (u_{1,2}^{(2)} + u_{2,1}^{(2)})$$

$$\Rightarrow \sigma_{12}^{(2)} = -\mu i k_2 A_2 \cos 2\theta_2 e^{ik_2(x_1 \sin \theta_2 - ct)}$$

$$\sigma_{12} = \mu (u_{1,2}^{(0)} + u_{1,2}^{(1)} + u_{1,2}^{(2)} + u_{2,1}^{(0)} + u_{2,1}^{(1)} + u_{2,1}^{(2)})$$

$$\sigma_{12} = 0 \text{ at } x_2 = 0 \quad \forall x, t$$

$$\begin{aligned} \Rightarrow 0 &= \mu \left\{ A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} + A_1 e^{ik_1 \sin \theta_0 \cos \theta_0 - ct} + A_2 e^{ik_2(x_1 \sin \theta_2 - ct)} \right\} \\ &\quad i k_1 (\sin \theta_1 \cos \theta_1) - A_2 e^{ik_2(x_1 \sin \theta_2 - ct)} \quad i k_2 \cos \theta_2 \cos \theta_2 \end{aligned}$$

$$+ A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} - A_1 e^{ik_1(x_1 \sin \theta_1 - ct)}$$

$$+ A_2 e^{ik_2(x_1 \sin \theta_2 - ct)} - A_2 e^{ik_2(\sin \theta_2 \sin \theta_2)}$$

$$\Rightarrow 0 = \mu_0 A_0 i k_0 (\cos \theta_0 \sin \theta_0 + \cos \theta_0 \sin \theta_0) e^{ik_0(x_1 \sin \theta_0 - ct)} - \mu A_1 i k_1 \sin 2\theta_1 e^{ik_1(x_1 \sin \theta_1 - ct)} - \mu i k_2 A_2 (\cos^2 \theta_2 - \sin^2 \theta_2)$$

$$\Rightarrow 0 = \mu_0 A_0 i k_0 \sin 2\theta_0 e^{ik_0(x_1 \sin \theta_0 - ct)} - \mu A_1 i k_1 \sin 2\theta_1 e^{ik_1(x_1 \sin \theta_1 - ct)} - \mu A_2 i k_2 \cos 2\theta_2 e^{ik_2(x_1 \sin \theta_2 - ct)}$$

To make exponent identical

$$k_0 \sin \theta_0 = k_1 \sin \theta_1 = k_2 \sin \theta_2$$

$$\text{And } k_0 c = k_1 c = k_2 c$$

This holds if $k_0 = k_1$, $\theta_0 = \theta_1$

$$\Rightarrow k_0 c = k_2 c \Rightarrow k_2 = k_0 \frac{c}{c}$$

$$\Rightarrow k_2 = k_0 k \Rightarrow k = \frac{k_2}{k_0}$$

$$\Rightarrow A_0 k_0 \sin 2\theta_0 - A_1 k_0 \sin 2\theta_0 - A_2 k_2 \cos 2\theta_2 = 0 \quad \text{--- (a)}$$

for $i=2$

$$\sigma_{22} = \lambda \epsilon_{kk} (1) + 2\mu (\epsilon_{22})$$

$$= \lambda (\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2\mu (\epsilon_{22})$$

$$= \lambda (u_{2,1} + u_{2,2}) + 2\mu (u_{2,2})$$

$$\Rightarrow \sigma_{22} = \lambda (u_{1,1}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(0)} + u_{2,2}^{(1)} + u_{1,1}^{(2)} + u_{2,2}^{(2)}) + 2\mu (u_{2,2}^{(0)} + u_{2,2}^{(1)} + u_{2,2}^{(2)})$$

$$\sigma_{22} = 0 \quad \text{at } x_2 = 0 \quad \forall x, t$$

$$\Rightarrow 0 = \lambda \left\{ \begin{aligned} & A_0 i k_0 \sin \theta_0 e^{i k_0 (x_1 \sin \theta_0 - ct)} \sin \theta_0 + A_1 i k_1 \sin \theta_1 e^{i k_1 (x_1 \sin \theta_1 - ct)} \sin \theta_1 \\ & + A_0 i k_0 \cos \theta_0 e^{i k_0 (x_1 \sin \theta_0 - ct)} \cos \theta_0 + A_1 i k_1 \cos \theta_1 e^{i k_1 (x_1 \sin \theta_1 - ct)} \cos \theta_1 \\ & - A_2 i k_2 \cos \theta_2 e^{i k_2 (x_1 \sin \theta_2 - ct)} \cos \theta_2 \end{aligned} \right\}$$

$$\Rightarrow 0 = (\lambda + 2\mu \cos^2 \theta_0) A_0 i k_0 e^{i k_0 (x_1 \sin \theta_0 - ct)} + (\lambda + 2\mu \cos^2 \theta_1) A_1 i k_1 e^{i k_1 (x_1 \sin \theta_1 - ct)} - \mu \sin 2\theta_2 A_2 i k_2 e^{i k_2 (x_1 \sin \theta_2 - ct)}$$

Therefore equation gives

$$0 = (\lambda + 2\mu \cos^2 \theta_0) A_0 k_0 + (\lambda + 2\mu \cos^2 \theta_1) A_1 k_1 - \mu A_2 k_2 \sin 2\theta_2 \quad \text{--- (b)}$$

from (A)

$$A_0 k_0 \sin 2\theta_0 - A_1 k_0 \sin 2\theta_0 - A_2 k_2 \cos 2\theta_2 = 0$$

$$\Rightarrow \frac{A_1}{A_0} \sin 2\theta_0 + \frac{A_2 k_2}{A_0 k_0} \cos 2\theta_2 = \sin 2\theta_0$$

$$\Rightarrow \frac{A_1}{A_0} \sin 2\theta_0 + \frac{A_2 k}{A_0} \cos 2\theta_2 = \sin 2\theta_0 \quad \text{--- (c)}$$

$$\begin{aligned} \text{From } \textcircled{B} \quad A_1 k_0 (\lambda + 2\mu \cos^2 \theta_0) - \mu A_2 k_2 (\sin 2\theta_2) \\ = -A_0 k_0 (\lambda + 2\mu \cos^2 \theta_0) \end{aligned}$$

$$\Rightarrow \frac{A_1}{A_0} (\lambda + 2\mu \cos^2 \theta_0) - \mu \frac{A_2}{A_0} \frac{k_2}{k_0} \sin 2\theta_2 = -(\lambda + 2\mu \cos^2 \theta_0)$$

$$\Rightarrow \frac{A_1}{A_0} (\lambda + 2\mu \cos^2 \theta_0) - \mu \frac{A_2}{A_0} k \sin 2\theta_2 = -(\lambda + 2\mu \cos^2 \theta_0)$$

$$\therefore \frac{k_2}{k_0} = \frac{C}{C} = k$$

$$\Rightarrow k = \sqrt{\frac{\lambda + 2\mu}{\mu}} \Rightarrow k^2 = \frac{\lambda + 2\mu}{\mu}$$

$$\Rightarrow k^2 \mu - 2\mu = \lambda$$

Put in above equation

$$\frac{A_1}{A_0} (k^2 \mu - 2\mu + 2\mu \cos^2 \theta_0) - \mu \frac{A_2}{A_0} k \sin 2\theta_2 = -(k^2 \mu - 2\mu + 2\mu \cos^2 \theta_0)$$

$$\Rightarrow \frac{A_1}{A_0} (k^2 \mu - 2\mu(1 - \cos^2 \theta_0)) - \mu \frac{A_2}{A_0} k \sin 2\theta_2 = -(k^2 \mu - 2\mu \sin^2 \theta_0)$$

$$\Rightarrow \frac{A_1}{A_0} (k^2 \mu - 2\mu \sin^2 \theta_0) - \mu k \sin 2\theta_2 \frac{A_2}{A_0} = -(k^2 \mu - 2\mu \sin^2 \theta_0)$$

$$\therefore \sin^2 \theta_2 = \frac{1}{k^2} \sin^2 \theta_0$$

$$\Rightarrow \frac{A_1}{A_0} (k^2 \mu - 2\mu \sin^2 \theta_0) - \mu k \sin 2\theta_2 \frac{A_2}{A_0} = -k^2 \mu (1 - 2\sin^2 \theta_2)$$

$$\Rightarrow (k^2 - 2\sin^2 \theta_0) \frac{A_1}{A_0} - k \sin 2\theta_2 \frac{A_2}{A_0} = -k^2 \cos 2\theta_2$$

→ \textcircled{d}

Now multiplying equation \textcircled{d} by $\sin 2\theta_2$

∴ eqn (d) by $\cos 2\theta_2$ and add

$$\Rightarrow \sin 2\theta_0 \sin 2\theta_2 \frac{A_1}{A_0} + k \cos 2\theta_2 \sin 2\theta_0 \frac{A_2}{A_0} = \sin 2\theta_2 \sin 2\theta_0$$

$$(k^2 - 2 \sin^2 \theta_0) \cos 2\theta_2 \frac{A_1}{A_0} - k \cos 2\theta_2 \sin 2\theta_0 \frac{A_2}{A_0} = -k^2 \cos^2 2\theta_2$$

$$\Rightarrow \frac{A_1}{A_0} \left[\sin 2\theta_0 \sin 2\theta_2 + (k^2 - 2 \sin^2 \theta_0) \cos 2\theta_2 \right] = \frac{\sin 2\theta_2 \sin 2\theta_0}{-k^2 \cos^2 2\theta_2}$$

$$\Rightarrow \boxed{\frac{A_1}{A_0} = \frac{\sin 2\theta_0 \sin 2\theta_2 - k^2 \cos^2 2\theta_2}{\sin 2\theta_0 \sin 2\theta_2 + k^2 \cos^2 2\theta_2}}$$

Put it in eqn (c)

$$\Rightarrow \frac{A_2}{A_0} = \frac{2k \cos 2\theta_2 \sin 2\theta_0}{\sin 2\theta_0 \sin 2\theta_2 + k^2 \cos^2 2\theta_2}$$

⇒ Reflection of SV-waves as P-waves + SV-waves:

Consider the reflection of an incident SV-wave propagation in $x_1 x_2$ -plane.

We assume that on reflection there is a P-wave and SV-wave.

Incident wave:-

$$u_i^{(0)} = A_0 e^{ik_0(x_j n_j^{(0)} - Ct)} P_i^{(0)}$$

$$n_i^{(0)} = (\sin \theta_0, \cos \theta_0, 0), \quad P_i^{(0)} = (-\cos \theta_0, \sin \theta_0, 0)$$

$$\therefore n_i^{(0)} \perp P_i^{(0)}$$

Reflected P-wave:-

$$u_i^{(1)} = A_1 e^{ik_1(x_j n_j^{(1)} - C_1 t)} P_i^{(1)}$$

$$\eta_i^{(1)} = (\sin\theta_1, -\cos\theta_1, 0), \quad p_i^{(1)} = (\sin\theta_1, -\cos\theta_1, 0)$$

Reflected s-v waves

$$u_i^{(2)} = A_2 e^{ik_2(x_j \eta_j^{(2)} - c_1 t)} p_i^{(2)}$$

$$\eta_i^{(2)} = (\sin\theta_2, -\cos\theta_2, 0), \quad p_i^{(2)} = (\cos\theta_2, \sin\theta_2, 0)$$

For free Boundary Conditions:-

$x_2 = 0$ is free mean $\sigma_{i2} = 0$

$$\Rightarrow \sigma_{12} = 0, \quad \sigma_{22} = 0, \quad \sigma_{32} = 0$$

$$\sigma_{12} = \lambda \epsilon_{kk} \delta_{12} + 2\mu \epsilon_{12}$$

$$= 2\mu \left(\frac{u_{1,2} + u_{2,1}}{2} \right)$$

$$= \mu (u_{1,2} + u_{2,1})$$

$$= \mu (u_{1,2}^{(0)} + u_{1,2}^{(1)} + u_{1,2}^{(2)} + u_{2,1}^{(0)} + u_{2,1}^{(1)} + u_{2,1}^{(2)})$$

$$\Rightarrow 0 = \mu \left\{ -A_0 e^{ik_0(x_1 \sin\theta_0 - c_1 t)} \left[ik_0(\cos^2\theta_0 - \sin^2\theta_0) - A_1 2ik_1 \right] \right.$$

$$e^{ik_1(x_1 \sin\theta_1 - c_1 t)} (\sin\theta_1 \cos\theta_1 + \cos\theta_1 \sin\theta_1) - A_2 2ik_2$$

$$\left. e^{ik_2(x_1 \sin\theta_2 - c_1 t)} \cos 2\theta_2 \right\} \longrightarrow \text{ⓐ}$$

$$\sigma_{22} = \lambda \epsilon_{kk} \delta_{22} + 2\mu \epsilon_{22}$$

$$= \lambda (\epsilon_{11} + \epsilon_{22}) + 2\mu (u_{2,2})$$

$$= \lambda (u_{1,1} + u_{2,2}) + 2\mu (u_{2,2})$$

$$\sigma_{22} = 0 \quad \text{at } x_2 = 0$$

$$\sigma_{22} = \sigma_{22}^{(0)} + \sigma_{22}^{(1)} + \sigma_{22}^{(2)} = 0$$

implies

$$0 = \lambda \left\{ -A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \right. \\ \left. + A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \right. \\ \left. + A_2 e^{ik_2(x_1 \sin \theta_2 - ct)} \right\} \\ + 2\mu \left\{ A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \right. \\ \left. + A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \right. \\ \left. + A_2 e^{ik_2(x_1 \sin \theta_2 - ct)} \right\} \\ \cos \theta_0 \sin \theta_0 + A_1 \sin \theta_1 \cos \theta_1 + A_2 \sin \theta_2 \cos \theta_2$$

$$\Rightarrow 0 = \lambda A_0 \sin 2\theta_0 e^{ik_0(x_1 \sin \theta_0 - ct)} + (\lambda + 2\mu) A_0 \cos^2 \theta_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \\ - \mu A_1 \sin 2\theta_1 e^{ik_1(x_1 \sin \theta_1 - ct)} - \mu A_2 \sin 2\theta_2 e^{ik_2(x_1 \sin \theta_2 - ct)}$$

To satisfy these equation, exponent must be identical

$$\Rightarrow k_0 \sin \theta_0 = k_1 \sin \theta_1 = k_2 \sin \theta_2$$

$$\& k_0 c = k_1 c = k_2 c$$

$$\Rightarrow \boxed{k_0 = k_2, \theta_0 = \theta_2}$$

$$\Rightarrow k_1 = k_0 \frac{c}{c} \Rightarrow k_1 = \frac{k_0}{K}$$

$$\Rightarrow \sin \theta_1 = K \sin \theta_0$$

$$K = \frac{c}{c} = \sqrt{\frac{\lambda + 2\mu}{\mu}}$$

$$\Rightarrow K^2 = \frac{\lambda + 2\mu}{\mu}$$

$$\Rightarrow \lambda = k^2 u - 2u$$

$$\text{equ } \textcircled{1} \Rightarrow -A_0 k_0 \cos 2\theta_0 - A_1 k_1 \sin 2\theta_1 - A_2 k_2 \cos 2\theta_2 = 0$$

$$\Rightarrow A_1 k_1 \sin 2\theta_1 + A_2 k_2 \cos 2\theta_2 = -A_0 k_0 \cos 2\theta_0$$

$$k_2 = k_0, \quad \theta_2 = \theta_0$$

$$\Rightarrow A_1 k_1 \sin 2\theta_1 + A_2 k_0 \cos 2\theta_0 = -A_0 k_0 \cos 2\theta_0$$

$$\Rightarrow \frac{A_2 k_0}{A_0 k_0} \cos 2\theta_0 + \frac{A_1 k_1}{A_0 k_0} \sin 2\theta_1 = -\cos 2\theta_0$$

$$\Rightarrow \frac{A_2}{A_0} \cos 2\theta_0 + \frac{A_1}{A_0} \cdot \frac{1}{k} \sin 2\theta_1 = -\cos 2\theta_0 \quad \text{--- } \textcircled{2}$$

$$\textcircled{2} \Rightarrow u A_0 k_0 \sin 2\theta_0 + (\lambda + 2u \cos^2 \theta_1) A_1 k_1 - u \sin 2\theta_2 A_2 k_2 = 0$$

$$\Rightarrow (\lambda + 2u \cos^2 \theta_1) A_1 k_1 - u \sin 2\theta_2 A_2 k_2 = -u A_0 k_0 \sin 2\theta_0$$

$$\therefore \lambda = k^2 u - 2u$$

$$\Rightarrow (k^2 u - 2u + 2u \cos^2 \theta_1) A_1 k_1 - u \sin 2\theta_2 A_2 k_2 = -u A_0 k_0 \sin 2\theta_0$$

$$\Rightarrow (k^2 u - 2u(1 - \cos^2 \theta_1)) A_1 k_1 - u \sin 2\theta_2 A_2 k_2 = -u A_0 k_0 \sin 2\theta_0$$

$$\Rightarrow (k^2 u - 2u \sin^2 \theta_1) A_1 k_1 - u \sin 2\theta_2 A_2 k_2 = -u A_0 k_0 \sin 2\theta_0$$

$$\Rightarrow (k^2 u - 2u \sin^2 \theta_1) \frac{A_1 k_1}{A_0 k_0} - u \sin 2\theta_2 \frac{A_2 k_0}{A_1 k_0} = -u \sin 2\theta_0$$

$$\Rightarrow (k^2 - 2 \sin^2 \theta_1) \frac{A_1}{A_0} - \frac{A_2}{A_1} k \sin 2\theta_0 = -k \sin 2\theta_0$$

$$\therefore \sin \theta_1 = k \sin \theta_0$$

$$\Rightarrow (k^2 - 2k^2 \sin^2 \theta_0) \frac{A_1}{A_0} - \frac{A_2}{A_0} k \sin 2\theta_0 = -k \sin 2\theta_0$$

$$\Rightarrow k \frac{A_1}{A_0} \cos 2\theta_0 - \frac{A_2}{A_0} \sin 2\theta_0 = -\sin 2\theta_0 \quad \text{--- (4)}$$

Multiply equ (3) by $\sin 2\theta_0$ & equ (4) by $k \cos 2\theta_0$.

$$\Rightarrow \frac{A_1}{A_0} \sin 2\theta_0 \sin 2\theta_0 + \frac{A_2}{A_0} k \cos 2\theta_0 \sin 2\theta_0 = -k \cos 2\theta_0 \sin 2\theta_0$$

$$\frac{A_1}{A_0} k^2 \cos 2\theta_0 \cos 2\theta_0 - \frac{A_2}{A_0} \sin 2\theta_0 \cos 2\theta_0 k = -k \sin 2\theta_0 \cos 2\theta_0$$

$$\Rightarrow \frac{A_1}{A_0} (\sin 2\theta_0 \sin 2\theta_0 + k^2 \cos^2 2\theta_0) = -2k \sin 2\theta_0 \cos 2\theta_0$$

$$\Rightarrow \boxed{\frac{A_1}{A_0} = \frac{-k \sin 4\theta_0}{\sin 2\theta_0 \sin 2\theta_0 + k^2 \cos^2 2\theta_0}}$$

Put in equ (3)

$$\frac{-k \sin 4\theta_0 \sin 2\theta_0}{\sin 2\theta_0 \sin 2\theta_0 + k^2 \cos^2 2\theta_0} + \frac{A_2}{A_0} k \cos 2\theta_0 = -k \cos 2\theta_0$$

$$\Rightarrow \frac{A_2}{A_0} k \cos 2\theta_0 = -k \cos 2\theta_0 + \frac{k \sin 4\theta_0 \sin 2\theta_0}{\sin 2\theta_0 \sin 2\theta_0 + k^2 \cos^2 2\theta_0}$$

$$\Rightarrow \frac{A_2}{A_0} = -1 + \frac{k \sin 4\theta_0 \sin 2\theta_0}{\sin 2\theta_0 \sin 2\theta_0 + k^2 \cos^2 2\theta_0} \cdot \frac{1}{k \cos 2\theta_0}$$

$$= -1 + \frac{2 \sin 2\theta_0 \sin 2\theta_0}{\sin 2\theta_0 \sin 2\theta_0 + k^2 \cos^2 2\theta_0}$$

$$\Rightarrow \frac{A_2}{A_0} = \frac{-\sin 2\theta_0 \sin 2\theta_1 - k^2 \cos^2 2\theta_0 + 2 \sin 2\theta_0 \sin 2\theta_1}{\sin 2\theta_0 \sin 2\theta_1 + k^2 \cos^2 2\theta_0}$$

$$\Rightarrow \frac{A_2}{A_0} = \frac{\sin 2\theta_0 \sin 2\theta_1 - k^2 \cos^2 2\theta_0}{\sin 2\theta_0 \sin 2\theta_1 + k^2 \cos^2 2\theta_0}$$

\Rightarrow Reflection of P-wave as SV-wave:-

The incident wave is P-wave and reflected wave is SV-wave. The displacement vector $u_i^{(0)}$ for incident wave and displacement vector $u_i^{(1)}$ for reflected wave.

$$n_i^{(0)} = p_i^{(0)} = (\sin \theta_0, \cos \theta_0, 0) \quad \because n^{(0)} \parallel p^{(0)}$$

$$n_i^{(1)} = (\sin \theta_1, -\cos \theta_1, 0), \quad p_i^{(1)} = (\cos \theta_1, \sin \theta_1, 0)$$

$$\because p_i^{(1)} \perp n_i^{(1)}$$

$$u_i^{(0)} = A_0 e^{ik_0(x_1 \sin \theta_0 + x_2 \cos \theta_0 - ct)} \quad p_i^{(0)}$$

$$\Rightarrow u_1^{(0)} = A_0 e^{ik_0(x_1 \sin \theta_0 + x_2 \cos \theta_0 - ct)} \quad \sin \theta_0$$

$$u_1^{(1)} = A_1 e^{ik_1(x_1 \sin \theta_1 - x_2 \cos \theta_1 - ct)} \quad \cos \theta_1$$

$$u_2^{(0)} = A_0 e^{ik_0(x_1 \sin \theta_0 + x_2 \cos \theta_0 - ct)} \quad \cos \theta_0$$

$$u_2^{(1)} = A_1 e^{ik_1(x_1 \sin \theta_1 - x_2 \cos \theta_1 - ct)} \quad \sin \theta_1$$

Free Boundary

$$\sigma_{12} = \sigma_{12}^{(0)} + \sigma_{12}^{(1)}$$

$$\begin{aligned} \sigma'_{12} = 0 &\Rightarrow \sigma'_{12} = 2\mu (\epsilon_{12}) \\ &= 2\mu (u_{1,2} + u_{2,1}) \\ &= \mu (u_{1,2} + u_{2,1}) = 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow 0 = \mu \left\{ A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \right. & \left. + A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \right. \\ & \left. + A_0 e^{ik_0 \cos \theta_0 \cos \theta_1} + A_1 e^{ik_1 \cos \theta_1 \cos \theta_0} \right. \\ & \left. + A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \right. & \left. + A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \right\} \\ & \left. + A_1 e^{ik_1 \sin \theta_1 \sin \theta_0} \right\} \end{aligned}$$

$$0 = \mu \left\{ A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \right. & \left. + A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \right. \\ & \left. + A_1 e^{ik_1 \sin 2\theta_0} + A_0 e^{ik_0 \sin 2\theta_1} \right\}$$

To make exponents identical

$$k_0 \sin \theta_0 = k_1 \sin \theta_1$$

$$\Rightarrow \boxed{\theta_0 = \theta_1 \Rightarrow k_0 = k_1}$$

$$\Rightarrow A_0 e^{ik_0 ct} + A_1 e^{ik_0 ct} \longrightarrow \textcircled{1}$$

$$\sigma'_{22} = \lambda \epsilon_{kk} + 2\mu \epsilon_{22}$$

$$= \lambda (\epsilon_{11} + \epsilon_{22}) + 2\mu \epsilon_{22}$$

$$= \lambda (u_{1,1} + u_{2,2}) + 2\mu (u_{2,2})$$

$$= \lambda [u_{1,1}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(0)} + u_{2,2}^{(1)}] + 2\mu [u_{2,2}^{(0)} + u_{2,2}^{(1)}]$$

$\sigma'_{22} = 0$ implies

$$\begin{aligned} 0 = \lambda & \left\{ A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \right. & \left. + A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \right. \\ & \left. + A_0 e^{ik_0 \cos \theta_0 \cos \theta_0} + A_1 e^{ik_1 \cos \theta_1 \cos \theta_1} \right. \\ & \left. + A_1 e^{ik_1 \sin \theta_1 \cos \theta_1} \right. & \left. + A_0 e^{ik_0 \sin \theta_0 \cos \theta_0} \right\} \end{aligned}$$

$$+ 2\mu \left\{ A_0 e^{ik_0(x, \sin\theta_0 - ct)} \quad ik_0 \cos\theta_0 \cos\theta_0 - A_1 e^{ik_1(x, \sin\theta_1 - ct)} \quad \sin\theta_1 \cos\theta_1 \right\}$$

$$\Rightarrow 0 = (\lambda + 2\mu \cos^2\theta_0) ik_0 A_0 e^{ik_0(x, \sin\theta_0 - ct)} - \mu A_1 e^{ik_1(x, \sin\theta_1 - ct)} \sin 2\theta_1 \quad \text{--- (2)}$$

Multiply eqn (1) by μ and add in (2)

$$\begin{array}{r} \mu A_0 e^{ik_0 ct} + \mu A_1 e^{ik_1 ct} = 0 \\ (\lambda + 2\mu \cos^2\theta_0) A_0 e^{ik_0 ct} - \mu A_1 e^{ik_1 ct} = 0 \\ \hline \Rightarrow (\mu + \lambda + 2\mu \cos^2\theta_0) A_0 e^{ik_0 ct} = 0 \end{array}$$

$\Rightarrow A_0 = 0$
 \Rightarrow Reflection does not exist.

\Rightarrow ~~Transversely~~ ^{*} ~~Isotropic~~ ^{*} Material

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11} - C_{12}}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix}$$

\Rightarrow Reflection of P-waves as P-waves.

$$\begin{aligned} n^0 \parallel p^0 \quad \text{and} \quad n^1 \parallel p^1 \\ n^0 = (\sin\theta_0, \cos\theta_0, 0) = p^0 \\ n^1 = (\sin\theta_1, -\cos\theta_1, 0) = p^1 \end{aligned}$$

Free Boundary: - Boundary $x_2 = 0$ is free
 mean $\sigma_{i2} = 0$, where σ_{ij} is stress tensor.

$$\sigma_{12} = \left(\frac{G_1 - G_2}{2}\right) \gamma \epsilon_{12}$$

$$= (G_1 - G_2) \epsilon_{12} = G_1 - G_2 \left(\frac{U_{3,2} + U_{2,1}}{2}\right)$$

$$\Rightarrow \sigma_{12} = \left(\frac{G_1 - G_2}{2}\right) \left[U_{1,2}^{(0)} + U_{1,2}^{(1)} + U_{2,1}^{(0)} + U_{2,1}^{(1)} \right]$$

$$= \left[\frac{G_1 - G_2}{2}\right] \left[A_0 i k_0 \cos \theta_0 e^{i k_0 (x_1 \sin \theta_0 - ct)} \sin \theta_0 - A_1 i k_1 \cos \theta_1 e^{i k_1 (x_1 \sin \theta_1 - ct)} \sin \theta_1 + A_0 i k_0 \cos \theta_0 e^{i k_0 (x_1 \sin \theta_0 - ct)} \cos \theta_0 - A_1 i k_1 \cos \theta_1 e^{i k_1 (x_1 \sin \theta_1 - ct)} \cos \theta_1 \right]$$

$$= \left[\frac{G_1 - G_2}{2}\right] \left[A_0 i k_0 2 \sin \theta_0 \cos \theta_0 e^{i k_0 (x_1 \sin \theta_0 - ct)} - A_1 i k_1 2 \sin \theta_1 \cos \theta_1 e^{i k_1 (x_1 \sin \theta_1 - ct)} \right]$$

To make exponents identical

$k_0 \sin \theta_0 = k_1 \sin \theta_1$. This is possible only if

$$k_0 = k_1, \theta_0 = \theta_1$$

$$\sigma_{12} = 0 \Rightarrow \boxed{A_0 - A_1 = 0}$$

Now solve σ_{22}

$$\sigma_{22} = C_{12} \epsilon_{11} + C_{22} \epsilon_{22} + C_{23} \epsilon_{33}$$

$$= G_2 \epsilon_{11} + G_{22} \epsilon_{22}$$

$$= G_2 (U_{1,1}^{(0)} + U_{1,1}^{(1)}) + G_{22} (U_{2,2}^{(0)} + U_{2,2}^{(1)})$$

$$\sigma_{22} = 0 \text{ implies}$$

$$0 = G_2 \left[A_0 i k_0 \sin \theta_0 e^{i k_0 (x_1 \sin \theta_0 - ct)} \sin \theta_0 + A_1 i k_1 \sin \theta_1 e^{i k_1 (x_1 \sin \theta_1 - ct)} \sin \theta_1 \right] + C_{22} \left[A_0 i k_0 \cos^2 \theta_0 e^{i k_0 (x_1 \sin \theta_0 - ct)} + A_1 i k_1 \cos^2 \theta_1 e^{i k_1 (x_1 \sin \theta_1 - ct)} \right]$$

$$\Rightarrow (G_2 \sin^2 \theta_0 + C_{22} \cos^2 \theta_0) i k_0 e^{i k_0 (x_1 \sin \theta_0 - ct)} A_0 + (C_{22} \cos^2 \theta_1 + G_2 \sin^2 \theta_1) i k_1 e^{i k_1 (x_1 \sin \theta_1 - ct)} A_1 = 0$$

$$\therefore \theta_0 = \theta_1, \quad k_0 = k_1$$

$$\Rightarrow \left. \begin{array}{l} A_0 + A_1 = 0 \\ A_0 - A_1 = 0 \end{array} \right\} \Rightarrow A_0 = 0$$

Hence no reflecting P-wave.

Reflection of SV-wave as SV-wave:-

For Transversely isotropic medium
Incident wave SV:-

$$u_i^{(0)} = A_0 e^{i k_0 (x_j n_j^{(0)} - ct)} p_i^{(0)}$$

$$n^0 \perp p^0; \quad n^0 = (\sin \theta_0, \cos \theta_0, 0), \quad p^0 = (-\cos \theta_0, \sin \theta_0, 0)$$

Reflecting wave SV:-

$$u_i^{(1)} = A_1 e^{i k_1 (x_j n_j^{(1)} - ct)} p_i^{(1)}$$

$$n^1 \perp p^1; \quad n^1 = (\sin \theta_1, -\cos \theta_1, 0), \quad p^1 = (\cos \theta_1, \sin \theta_1, 0)$$

Free Boundary:-

$$\sigma_{12} = \left(\frac{G_1 - G_2}{2} \right) \gamma \epsilon_{12} = G_1 - G_2 \left(\frac{u_{1,2} + u_{2,1}}{2} \right)$$

$$= \left(\frac{G_1 - G_2}{2} \right) (u_{1,2}^0 + u_{1,2}^1 + u_{2,1}^0 + u_{2,1}^1)$$

$$\Rightarrow \sigma_{12} = \left[\frac{C_{11} - C_{12}}{2} \right] \begin{bmatrix} -A_0 e^{ik_0(x, \sin \theta_0 - \omega t)} & ik_0 \cos 2\theta_0 \\ -A_1 e^{ik_1(x, \sin \theta_1 - \omega t)} & ik_1 \cos 2\theta_1 \end{bmatrix}$$

$$\Rightarrow \left[\frac{C_{11} - C_{12}}{2} \right] \begin{pmatrix} ik_0(x, \sin \theta_0 - \omega t) & ik_0 \cos 2\theta_0 \\ ik_1(x, \sin \theta_1 - \omega t) & ik_1 \cos 2\theta_1 \end{pmatrix} = 0$$

To make exponents identical.

$k_0 \sin \theta_0 = k_1 \sin \theta_1$ This is possible only if
 $\theta_0 = \theta_1$ and $k_0 = k_1$ if holds this

Then $-A_0 \cos 2\theta_0 - A_1 \cos 2\theta_0 = 0$

$$\Rightarrow \boxed{A_0 = -A_1}$$

Now

$$\sigma_{22} = C_{22} \epsilon_{11} + C_{22} \epsilon_{22} + C_{23} \epsilon_{33}$$

$$= C_{22} (u_{1,1}^{(0)} + u_{1,1}^{(1)}) + C_{22} (u_{2,2}^{(0)} + u_{2,2}^{(1)})$$

$$\Rightarrow \sigma_{22} = C_{12} \begin{bmatrix} -A_0 e^{ik_0(x, \sin \theta_0 - \omega t)} & ik_0 \sin \theta_0 \cos \theta_0 + A_1 ik_1 e^{ik_1(x, \sin \theta_1 - \omega t)} \\ ik_1(x, \sin \theta_1 - \omega t) \sin \theta_1 \cos \theta_1 \end{bmatrix} + C_{22} \begin{bmatrix} A_0 e^{ik_0(x, \sin \theta_0 - \omega t)} & ik_0 \sin \theta_0 \\ ik_1(x, \sin \theta_1 - \omega t) \cos \theta_1 - A_1 e^{ik_1(x, \sin \theta_1 - \omega t)} & ik_1 \cos \theta_1 \sin \theta_1 \end{bmatrix}$$

$\sigma_{22} = 0$; $\theta_0 = \theta_1$, $k_0 = k_1$ implies

$$-(C_{12} - C_{22}) A_0 e^{ik_0(x, \sin \theta_0 - \omega t)} ik_0 \sin \theta_0 \cos \theta_0 + (C_{12} - C_{22})$$

$$A_1 ik_1 e^{ik_1(x, \sin \theta_1 - \omega t)} \sin \theta_1 \cos \theta_1 = 0$$

$$\Rightarrow -(C_{12} - C_{22}) A_0 + (C_{12} - C_{22}) A_1 = 0$$

$$\Rightarrow -(C_{12} - C_{22}) A_0 = -(C_{12} - C_{22}) A_1$$

$$\Rightarrow A_0 = A_1 \longrightarrow \textcircled{1}$$

from $\textcircled{1}$ & $\textcircled{2}$

$$\left. \begin{array}{l} A_0 + A_1 = 0 \\ A_0 - A_1 = 0 \end{array} \right\} \Rightarrow A_0 = 0$$

\Rightarrow SV \rightarrow SV does not exist.

\Rightarrow Reflection of P-wave as SV-wave :-

$$u_i^{(0)} = A_0 e^{ik_0(x_1 \sin \theta_0 + x_2 \cos \theta_0 - ct)} \quad P_i^{(0)}$$

$$n_i^{(0)} = P_i^{(0)} = (\sin \theta_0, \cos \theta_0, 0)$$

$$u_i^{(1)} = A_1 e^{ik_1(x_1 \sin \theta_1 - x_2 \cos \theta_1 - ct)} \quad P_i^{(1)}$$

$$n_i^1 \perp P_i^1; \quad n_i^1 = (\sin \theta_1, -\cos \theta_1, 0), \quad P_i^1 = (\cos \theta_1, \sin \theta_1, 0)$$

Free Boundary :- $\sigma_{12} = 0$ at $x_2 = 0$

$$\Rightarrow \sigma_{12} = 0 \quad \& \quad \sigma_{22} = 0$$

$$\sigma_{12} = \left(\frac{G_1 - G_2}{2} \right) \epsilon_{12} = (G_1 - G_2) \left[\frac{u_{1,2} + u_{2,1}}{2} \right]$$

$$= \left(\frac{G_1 - G_2}{2} \right) \left[u_{1,2}^{(0)} + u_{1,2}^{(1)} + u_{2,1}^{(0)} + u_{2,1}^{(1)} \right]$$

$$= \left[\frac{G_1 - G_2}{2} \right] \left[A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \quad ik_0 \sin \theta_0 \cos \theta_0 - A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \right]$$

$$\left(x_2 \sin \theta_1 - ct \right) ik_1 \cos^2 \theta_1 + A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \quad ik_0 \cos \theta_0 \sin \theta_0$$

$$+ A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \quad ik_1 \sin^2 \theta_1$$

$$= \left[\frac{G_1 - G_2}{2} \right] \left[A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \quad ik_0 \sin 2\theta_0 - \right]$$

$$A_1 e^{ik_1(x, \sin \theta_1 - ct)} \quad ik_1 (\cos^2 \theta_1 - \sin^2 \theta_1)$$

$\sigma_{12} = 0$ implies

$$\Rightarrow \left[\frac{C_{11} - C_{12}}{2} \right] \left[A_0 e^{ik_0(x, \sin \theta_0 - ct)} \quad ik_0 \sin 2\theta_0 - A_1 e^{ik_1(x, \sin \theta_1 - ct)} \quad ik_1 \cos 2\theta_1 \right] = 0$$

To make exponent identical
 $k_0 \sin \theta_0 = k_1 \sin \theta_1$ & $k_0 c = k_1 c$

This is possible if

$k_0 = k_1$, $\theta_0 = \theta_1$ if this holds then

$$\Rightarrow \left[\frac{C_{11} - C_{12}}{2} \right] \left[A_0 e^{ik_0(x, \sin \theta_0 - ct)} \quad ik_0 \sin 2\theta_0 - A_1 e^{ik_1(x, \sin \theta_0 - ct)} \quad ik_0 \cos 2\theta_0 \right] = 0$$

$$\Rightarrow A_0 \sin 2\theta_0 - A_1 \cos 2\theta_0 = 0$$

$$\Rightarrow A_0 = A_1 \frac{\cos 2\theta_0}{\sin 2\theta_0} \quad \text{--- (1)}$$

Now

$$\sigma_{22} = C_{12} \epsilon_{11} + C_{22} \epsilon_{22} + C_{33} \epsilon_{33}$$

$$= C_{12} (u_{1,1}^{(1)} + u_{1,1}^{(2)}) + C_{22} (u_{2,2}^{(1)} + u_{2,2}^{(2)})$$

$$\Rightarrow \sigma_{22} = C_{12} \left[A_0 e^{ik_0(x, \sin \theta_0 - ct)} \quad ik_0 \sin^2 \theta_0 + A_1 ik_1 e^{ik_1(x, \sin \theta_0 - ct)} \right. \\ \left. \sin \theta_1 \cos \theta_1 \right] + C_{22} \left[A_0 e^{ik_0(x, \sin \theta_0 - ct)} \quad ik_0 \cos^2 \theta_0 - \right. \\ \left. A_1 ik_1 e^{ik_1(x, \sin \theta_0 - ct)} \quad \cos \theta_1 \sin \theta_1 \right]$$

$$= (C_{12} \sin^2 \theta_0 + C_{22} \cos^2 \theta_0) A_0 e^{ik_0(x, \sin \theta_0 - ct)} \quad ik_0$$

$$+ (G_2 \sin \theta_1 \cos \theta_1 - G_{22} \cos \theta_1 \sin \theta_1) A_1 e^{ik_1(x, \sin \theta_1 - Gt)}$$

To make exponent identical

$$k_0 \sin \theta_0 = k_1 \sin \theta_1 \quad \text{— this is possible only if}$$

$$k_0 = k_1, \quad \theta_0 = \theta_1 \quad \text{if this holds then}$$

$$k_0 G = k_1 G \Rightarrow G = G$$

$\partial_{22} = 0$ implies

$$(G_2 \sin^2 \theta_0 + G_{22} \cos^2 \theta_0) A_0 e^{ik_0(x, \sin \theta_0 - Gt)} +$$

$$(G_2 \sin \theta_0 \cos \theta_0 - G_{22} \cos \theta_0 \sin \theta_0) A_1 e^{ik_0(x, \sin \theta_0 - Gt)} = 0$$

$$\Rightarrow (G_2 \sin^2 \theta_0 + G_{22} \cos^2 \theta_0) A_0 + (G_2 \sin \theta_0 \cos \theta_0 - G_{22} \cos \theta_0 \sin \theta_0) A_1 = 0$$

$$\Rightarrow (G_2 \sin^2 \theta_0 + G_{22} \cos^2 \theta_0) A_1 \frac{\cos 2\theta_0}{\sin 2\theta_0} + (G_2 - G_{22}) \sin 2\theta_0 A_1 = 0$$

* ————— *

MUHAMMAD TAHIR

(FA15-RMT-007)

M.S. MATH

COMSATS UNIVERSITY

⇒ Reflection of SH-wave as SH-wave :-

A transverse wave (shear or secondary wave) is a wave in which P_i is perpendicular to n_i , where (P_i is polarization vector and n_i is propagation vector)

Consider the reflection of SH-wave in x_1x_2 -plane, by definition of SH-wave, the displacement component will be in x_3 -direction only.

In the figure,

θ_0 is angle of incident,

θ_1 is angle of reflection,

$n^{(0)}$ is unit propagation vector

along incident ray, $n^{(1)}$ is unit propagation vector along reflected ray.

Incident ray is as follows

$$u_3^{(0)} = A_0 e^{ik_0(n_j^{(0)}x_j - \bar{t})} P_3 \quad \text{--- (1)}$$

where $n_i^{(0)} = (n_1^{(0)}, n_2^{(0)}, n_3^{(0)})$, $\bar{x} = (x_1, x_2, 0)$, $P_i = (0, 0, 1)$

$$\Rightarrow n_i^{(0)} = (\sin\theta_0, -\cos\theta_0, 0)$$

So eqn (1) becomes

$$u_3^{(0)} = A_0 e^{ik_0(x_1 \sin\theta_0 - x_2 \cos\theta_0 - \bar{t})}$$

Similarly for reflected wave

$$u_3^{(1)} = A_1 e^{ik_1(x_1 \sin\theta_1 - x_2 \cos\theta_1 - \bar{t})}$$

For Clamped Boundary conditions

For clamped boundary conditions we have,

$$U_3^{(0)} + U_3^{(1)} = 0 \text{ at } x_2 = 0 \text{ for all } x_1, t$$

$$\Rightarrow A_0 e^{ik_0(x_1 \sin \theta_0 - Gt)} + A_1 e^{ik_1(x_1 \sin \theta_1 - Gt)} = 0 \quad \text{--- (2)}$$

A_0 , k_0 & θ_0 are given, we choose A_1 , k_1 and θ_1 in such a way that eqn (2) is satisfied

To make exponent identical

$k_0 \sin \theta_0 = k_1 \sin \theta_1$ This is possible only if $k_0 = k_1$ and $\theta_0 = \theta_1$. If this holds then

$$A_0 + A_1 = 0 \Rightarrow A_0 = -A_1$$

For Free Boundary:- Boundary $x_2 = 0$ is free means $\sigma_{i2} = 0$, where σ_{ij} is stress tensor.

By Hooke's Law for isotropic material

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}, \text{ for } i, j = 1, 2, 3$$

$$\sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}, \text{ where } \epsilon_{ij} = \frac{1}{2}(u_{ij} + u_{ji})$$

$$\sigma_{12} = 0 \quad \because u_{1,2} = u_{2,1} = 0$$

$$\sigma_{22} = \lambda(u_{1,1} + u_{2,2} + u_{3,3}) + \mu(u_{1,2} + u_{2,1})$$

$$= \lambda u_{3,3} \quad \because u_{1,2} = u_{2,1} = 0$$

$$\Rightarrow \sigma_{22} = \lambda \frac{\partial u_3}{\partial x_3} = 0 \quad \because x_3 \neq u_3$$

And

$$\sigma_{32} = \mu(u_{3,2} + u_{2,3})$$

$$= \mu(u_{3,2})$$

So the only non-zero component for σ_{12} is σ_{32}

This implies $\sigma_{32}^{(0)} = \mu(U_{3,2}^{(0)})$, $\sigma_{32}^{(1)} = \mu(U_{3,2}^{(1)})$

$$\Rightarrow \sigma_{32}^{(0)} = \mu A_0 i k_0 \cos \theta_0 e^{i k_0 (x_1 \sin \theta_0 + x_2 \cos \theta_0 - Gt)}$$

$$\sigma_{32}^{(1)} = -\mu A_1 i k_1 \cos \theta_1 e^{i k_1 (x_1 \sin \theta_1 - x_2 \cos \theta_1 - Gt)}$$

If boundary $x_2 = 0$ is free then

$$\sigma_{32}^{(0)} + \sigma_{32}^{(1)} = 0 \text{ at } x_2 = 0 \quad \forall x_1, t$$

$$\Rightarrow \mu A_0 i k_0 \cos \theta_0 e^{i k_0 (x_1 \sin \theta_0 - Gt)} - \mu A_1 i k_1 \cos \theta_1 e^{i k_1 (x_1 \sin \theta_1 - Gt)} = 0 \quad \text{--- } \textcircled{2}$$

To make exponent identical $k_0 \sin \theta_0 = k_1 \sin \theta_1$, This is possible only if $k_0 = k_1$, $\theta_0 = \theta_1$; if this holds then

$$A_0 - A_1 = 0 \Rightarrow A_0 = A_1$$

\Rightarrow In a reflected SH-wave if angle of reflection is equal to the angle of incident then its wave number and amplitude remains the same.

Clamped Boundary:-

For clamped boundary condition, we have

$$U_1^{(0)} + U_1^{(1)} + U_1^{(2)} = 0 \text{ at } x_2 = 0 \quad \forall x_1, t$$

and

$$U_1^{(0)} + U_1^{(1)} + U_1^{(2)} = 0 \text{ at } x_2 \quad \forall x_1, t$$

Putting values in equations () & ()
we have

$$A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \sin \theta_0 + A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \sin \theta_1 +$$

$$A_2 e^{ik_2(x_1 \sin \theta_2 - ct)} \cos \theta_2 = 0$$

$$\Rightarrow A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \cos \theta_0 - A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} +$$

$$A_2 e^{ik_2(x_1 \sin \theta_2 - ct)} \sin \theta_2 = 0$$

To satisfy these equations exponents
must be identical, so

$$k_0 \sin \theta_0 = k_1 \sin \theta_1 = k_2 \sin \theta_2$$

$$\text{and } k_0 c = k_1 c = k_2 c$$

This holds if

$$k_0 = k_1, \quad \theta_0 = \theta_1$$

$$\Rightarrow k_2 = k_0 \frac{c}{c} = k_0 k, \quad \text{where } k = \frac{c}{c} > 1$$

So we have

$$\frac{A_1}{A_0} \sin \theta_0 + \frac{A_2}{A_0} \cos \theta_2 = -\sin \theta_0 \quad \text{--- (a)}$$

$$-\frac{A_1}{A_0} \cos \theta_0 + \frac{A_2}{A_0} \sin \theta_2 = -\cos \theta_0 \quad \text{--- (b)}$$

by solving eqn (a) & (b), we have

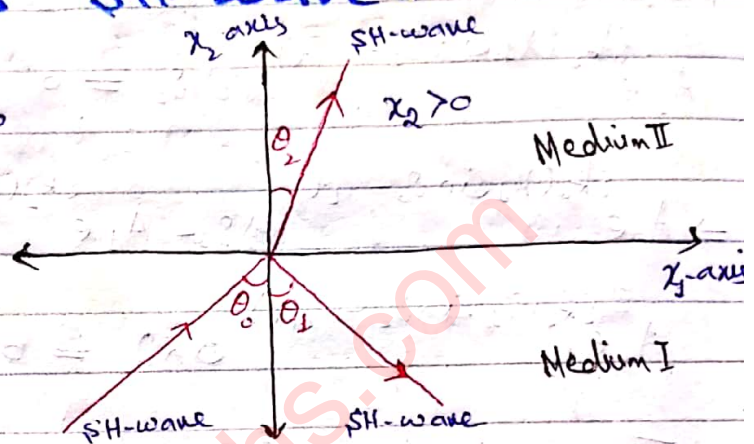
$$\frac{A_1}{A_0} = \frac{\cos \theta_0 \cos \theta_2 - \sin \theta_0 \sin \theta_2}{\cos \theta_0 \cos \theta_2 + \sin \theta_0 \sin \theta_2} = \frac{\cos(\theta_2 + \theta_0)}{\cos(\theta_2 - \theta_0)}$$

$$\frac{A_2}{A_0} = \frac{-\sin 2\theta_0}{\cos \theta_0 \cos \theta_2 + \sin \theta_0 \sin \theta_2} = \frac{-\sin 2\theta_0}{\cos(\theta_2 - \theta_0)}$$

⇒ Transmission:-

⇒ Reflection and Transmission of SH-wave as SH-wave:-

If a space $x_2 > 0$ is filled with another medium (may liquid or solid) then waves transmit across $x_2 = 0$.



Let an SH-wave incident at $x_2 = 0$ i.e. at the interface of two mediums. Both reflected and transmitted waves are considered SH-waves.

Let incident wave makes an angle θ_0 with vertical then

$$n_i^{(0)} = (\sin \theta_0, \cos \theta_0, 0)$$

Similarly reflected and transmitted waves makes angle θ_1 and θ_2 respectively

$$\Rightarrow n_i^{(1)} = (\sin \theta_1, -\cos \theta_1, 0)$$

$$q \cdot n_i^{(2)} = (\sin \theta_2, \cos \theta_2, 0)$$

Since all waves are SH-waves,

$$\Rightarrow P_i^{(0)} = P_i^{(1)} = P_i^{(2)} = (0, 0, 1)$$

$$\text{Let } u_i^{(0)} = A_0 e^{i k_0 (x_1 n_j - q t)}$$

$$\Rightarrow u_3^{(0)} = A_0 e^{i k_0 (x_1 \sin \theta_0 + x_2 \cos \theta_0 - q t)} \quad \because P_3 = 1$$

Similarly

$$U_3^{(1)} = A_1 e^{ik_1(x_1 \sin \theta_1 - x_2 \cos \theta_1 - ct)}$$

$$\& U_3^{(2)} = A_2 e^{ik_2(x_1 \sin \theta_2 + x_2 \cos \theta_2 - ct)}$$

We shall use the following boundary conditions at the interface $x_2 = 0$

- (i) U_i is continuous at $x_2 = 0$
- (ii) σ_{iz} is continuous at $x_2 = 0$

1) U_i is continuous at $x_2 = 0$

$$\Rightarrow U_3^{(2)} = U_3^{(0)} + U_3^{(1)} \quad \text{at } x_2 = 0$$

$$\Rightarrow A_2 e^{ik_2(x_1 \sin \theta_2 - ct)} = A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} + A_1 e^{ik_1(x_1 \sin \theta_1 - ct)}$$

By Snell's law implies that ①

$$k_2 \sin \theta_2 = k_0 \sin \theta_0 = k_1 \sin \theta_1$$

$$\& k_2 G^p = k_0 G = k_1 G$$

$$\Rightarrow \boxed{k_0 = k_1} \quad \text{and} \quad \boxed{k_2 = k_0 \frac{G}{G^p}}$$

$$\Rightarrow \boxed{\theta_0 = \theta_1} \quad \text{and} \quad \sin \theta_2 = \frac{k_0}{k_2} \sin \theta_0$$

$$\Rightarrow \boxed{\sin \theta_2 = \frac{G^p}{G} \sin \theta_0}$$

Thus equation ① implies

$$A_2 = A_0 + A_1 \quad \text{---} \quad \text{②}$$

2) σ_{iz} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{iz}^{(2)} = \sigma_{iz}^{(0)} + \sigma_{iz}^{(1)}$$

$$\text{Since } \sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

$$\Rightarrow \sigma_{12} = 2\mu \epsilon_{12} = 2\mu \cdot \frac{1}{2}(u_{1,2} + u_{2,1})$$

$$= \mu(u_{1,2} + u_{2,1})$$

$$\Rightarrow \sigma_{12}^{(0)} = \sigma_{12}^{(1)} = \sigma_{12}^{(2)} = 0 \quad (\because u_1, u_2 = 0)$$

Now

$$\sigma_{22} = \lambda \epsilon_{kk} + 2\mu \epsilon_{22}$$

$$= \lambda \epsilon_{11} + (\lambda + 2\mu) \epsilon_{22} + \lambda \epsilon_{33}$$

$$= \lambda u_{1,1} + (\lambda + 2\mu) u_{2,2} + \lambda u_{3,3}$$

$$\Rightarrow \sigma_{22} = 0 \quad \because u_1, u_2 = 0 \text{ and } x_3 \neq u_3$$

$$\Rightarrow \sigma_{22}^{(0)} = \sigma_{22}^{(1)} = \sigma_{22}^{(2)} = 0$$

Now

$$\sigma_{23} = 0 + 2\mu \epsilon_{23}$$

$$= \mu(u_{2,3} + u_{3,2})$$

$$\Rightarrow \sigma_{23} = \mu u_{3,2}$$

$$\text{Thus } \sigma_{23}^{(2)} = \sigma_{23}^{(0)} + \sigma_{23}^{(1)} \quad \text{at } x_2 = 0$$

$$\Rightarrow \mu u_{3,2}^{(2)} = \mu \{ u_{3,2}^{(0)} + u_{3,2}^{(1)} \} \quad \text{at } x_2 = 0$$

$$\Rightarrow \mu \cos \theta_2 (ik_2 A_2) e^{ik_2 (x_1 \sin \theta_2 - ct)} = \mu \left\{ \cos \theta_0 (ik_0 A_0) e^{ik_0 (x_1 \sin \theta_0 - ct)} \right.$$

$$\left. + (-\cos \theta_1) (ik_1 A_1) e^{ik_1 (x_1 \sin \theta_1 - ct)} \right\}$$

$$\Rightarrow k_2 \mu \cos \theta_2 A_2 = k_0 \mu \cos \theta_0 A_0 = k_1 \mu \cos \theta_1 A_1$$

$$\text{Put } \theta_1 = 0 \text{ and } k_1 = k_0$$

$$\Rightarrow k_2 u^\beta \cos \theta_2 A_2 = k_0 u \cos \theta_0 (A_0 - A_1) \quad \text{---} \rightarrow \textcircled{*}_2$$

Solving $\textcircled{*}_1$ and $\textcircled{*}_2$

$$\textcircled{*}_2 \Rightarrow \frac{k_2 u^\beta \cos \theta_2 A_2}{k_0 u \cos \theta_0} = A_0 - A_1 \quad \text{---} \rightarrow \textcircled{a}$$

Adding $\textcircled{*}_1$ and \textcircled{a} implies

$$\left(1 + \frac{k_2 u^\beta \cos \theta_2}{k_0 u \cos \theta_0}\right) A_2 = 2A_0$$

$$\Rightarrow A_2 = \left(\frac{2k_0 u \cos \theta_0}{k_0 u \cos \theta_0 + k_2 u^\beta \cos \theta_2}\right) A_0 \quad \text{---} \rightarrow \textcircled{b}$$

Put the value of A_2 in \textcircled{a} implies

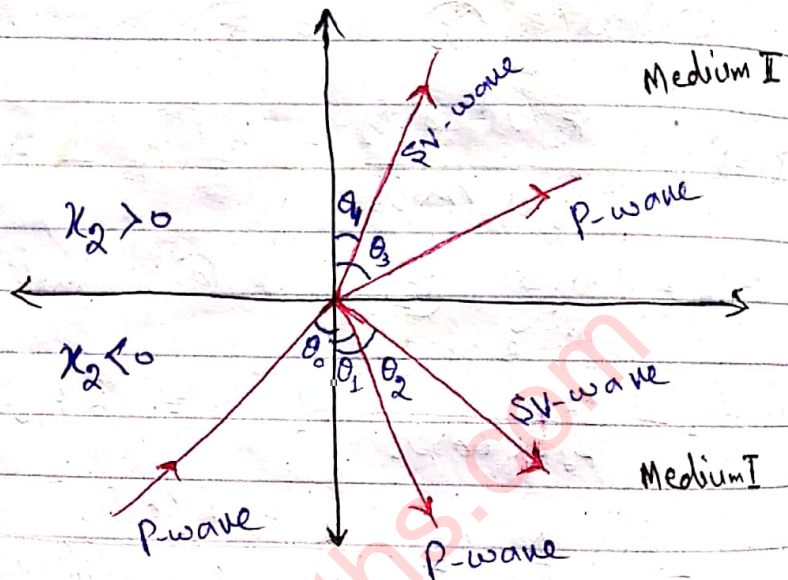
$$\frac{k_2 u^\beta \cos \theta_2}{k_0 u \cos \theta_0} \cdot \frac{2(k_0 u \cos \theta_0)}{k_0 u \cos \theta_0 + k_2 u^\beta \cos \theta_2} \cdot A_0 = A_0 - A_1$$

$$\Rightarrow A_1 = \left(1 - \frac{2k_2 u^\beta \cos \theta_2}{k_0 u \cos \theta_0 + k_2 u^\beta \cos \theta_2}\right) A_0$$

$$= \left(\frac{k_0 u \cos \theta_0 + k_2 u^\beta \cos \theta_2 - 2k_2 u^\beta \cos \theta_2}{k_0 u \cos \theta_0 + k_2 u^\beta \cos \theta_2}\right) A_0$$

$$\Rightarrow A_1 = \left(\frac{k_0 u \cos \theta_0 - k_2 u^\beta \cos \theta_2}{k_0 u \cos \theta_0 + k_2 u^\beta \cos \theta_2}\right) A_0 \quad \text{---} \rightarrow \textcircled{c}$$

⇒ Reflection and Transmission of P-wave as P-wave and SV-wave:



* Boundary Conditions: - U_i and σ_{i2} are continuous at $x_2 = 0$

$$\eta_i^{(0)} = (\sin\theta_0, \cos\theta_0, 0) = P_i^{(0)}$$

$$\eta_i^{(1)} = (\sin\theta_1, -\cos\theta_1, 0) = P_i^{(1)}$$

$$\eta_i^{(2)} = (\sin\theta_2, -\cos\theta_2, 0), P_i^{(2)} = (\cos\theta_2, \sin\theta_2, 0)$$

$$\eta_i^{(3)} = (\sin\theta_3, \cos\theta_3, 0) = P_i^{(3)}$$

$$\eta_i^{(4)} = (\sin\theta_4, \cos\theta_4, 0), P_i^{(4)} = (\cos\theta_4, -\sin\theta_4, 0)$$

This implies

$$U_i^{(0)} = A_0 e^{ik_0(x_1 \sin\theta_0 + x_2 \cos\theta_0 - ct)} (\sin\theta_0, \cos\theta_0, 0)$$

$$U_i^{(1)} = A_1 e^{ik_1(x_1 \sin\theta_1 - x_2 \cos\theta_1 - ct)} (\sin\theta_1, -\cos\theta_1, 0)$$

$$U_i^{(2)} = A_2 e^{ik_2(x_1 \sin\theta_2 - x_2 \cos\theta_2 - ct)} (\cos\theta_2, \sin\theta_2, 0)$$

$$U_i^{(3)} = A_3 e^{ik_3(x_1 \sin \theta_3 + x_2 \cos \theta_3 - Ct)} (\sin \theta_3, \cos \theta_3, 0)$$

$$U_i^{(4)} = A_4 e^{ik_4(x_1 \sin \theta_4 + x_2 \cos \theta_4 - Ct)} (\cos \theta_4, -\sin \theta_4, 0)$$

1) U_i is continuous at $x_2 = 0$

$$\Rightarrow U_i^{(0)} + U_i^{(1)} + U_i^{(2)} = U_i^{(3)} + U_i^{(4)}$$

for $i=1$

$$\Rightarrow U_1^{(0)} + U_1^{(1)} + U_1^{(2)} = U_1^{(3)} + U_1^{(4)}$$

$$\begin{aligned} \Rightarrow A_0 e^{ik_0(x_1 \sin \theta_0 - Ct)} \sin \theta_0 + A_1 e^{ik_1(x_1 \sin \theta_1 - Ct)} \sin \theta_1 \\ + A_2 e^{ik_2(x_1 \sin \theta_2 - Ct)} \cos \theta_2 = A_3 e^{ik_3(x_1 \sin \theta_3 - Ct)} \sin \theta_3 \\ + A_4 e^{ik_4(x_1 \sin \theta_4 - Ct)} \cos \theta_4 \rightarrow \textcircled{A} \end{aligned}$$

By Snell's law

$$k_0 \sin \theta_0 = k_1 \sin \theta_1 = k_2 \sin \theta_2 = k_3 \sin \theta_3 = k_4 \sin \theta_4$$

And $k_0 C = k_1 C = k_2 C = k_3 C = k_4 C$

$$\Rightarrow \boxed{k_1 = k_0} \Rightarrow \boxed{\theta_1 = \theta_0}$$

$$\Rightarrow \boxed{k_2 = \frac{C}{C'} k_0} \Rightarrow \boxed{k_3 = \frac{C}{C''} k_0}$$

$$\boxed{k_4 = \frac{C}{C'''} k_0}$$

$$\sin \theta_2 = \frac{k_0}{k_2} \sin \theta_0 \Rightarrow \boxed{\frac{C'}{C} \sin \theta_0 = \sin \theta_2}$$

$$\sin \theta_3 = \frac{k_0}{k_3} \sin \theta_0 \Rightarrow \boxed{\sin \theta_3 = \frac{C''}{C} \sin \theta_0}$$

$$\sin \theta_4 = \frac{P_0}{P_4} \sin \theta_0 \Rightarrow \boxed{\sin \theta_4 = \frac{P}{Q} \sin \theta_0}$$

So (A) implies

$$A_0 \sin \theta_0 + A_1 \sin \theta_1 + A_2 \cos \theta_2 - A_3 \cos \theta_3 - A_4 \cos \theta_4 = 0 \quad \text{--- (1)}$$

for $i=2$

$$\begin{aligned} & U_2^{(0)} + U_2^{(1)} + U_2^{(2)} = U_2^{(3)} + U_2^{(4)} \\ \Rightarrow A_0 e^{i k_0 (x_1 \sin \theta_0 - ct)} \cos \theta_0 - A_1 e^{i k_1 (x_1 \sin \theta_1 - ct)} \cos \theta_1 \\ & + A_2 e^{i k_2 (x_1 \sin \theta_2 - ct)} \sin \theta_2 = A_3 e^{i k_3 (x_1 \sin \theta_3 - ct)} \cos \theta_3 \\ & - A_4 e^{i k_4 (x_1 \sin \theta_4 - ct)} \sin \theta_4 \end{aligned}$$

By Snell's law

$$A_0 \cos \theta_0 - A_1 \cos \theta_1 + A_2 \sin \theta_2 - A_3 \cos \theta_3 + A_4 \sin \theta_4 = 0 \quad \text{--- (2)}$$

2) σ_{iz} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{iz}^{(0)} + \sigma_{iz}^{(1)} + \sigma_{iz}^{(2)} = \sigma_{iz}^{(3)} + \sigma_{iz}^{(4)} \quad \text{--- (*)}$$

$$\text{Since } \sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

$$\Rightarrow \sigma_{iz} = \lambda \epsilon_{kk} \delta_{iz} + 2\mu \epsilon_{iz}$$

for $i=1$

$$\sigma_{1z} = 2\mu \epsilon_{1z} = \mu (U_{1,2} + U_{2,1})$$

$$\begin{aligned} \Rightarrow \mu \{ & (U_{1,2}^{(0)} + U_{2,1}^{(0)}) + (U_{1,2}^{(1)} + U_{2,1}^{(1)}) + (U_{1,2}^{(2)} + U_{2,1}^{(2)}) \} \\ & = \mu \{ (U_{1,2}^{(3)} + U_{2,1}^{(3)}) + (U_{1,2}^{(4)} + U_{2,1}^{(4)}) \} \end{aligned}$$

$$\Rightarrow \mu \{ A_0 k_0 (\sin \theta_0 \cos \theta_0 + \sin \theta_0 \cos \theta_0) - A_1 k_1 (\sin \theta_1 \cos \theta_1 + \sin \theta_1 \cos \theta_1) \}$$

$$+ A_2 k_2 (-\cos^2 \theta_2 + \sin^2 \theta_2) = u^\beta \{ A_3 k_3 (\sin \theta_3 \cos \theta_3 + \sin \theta_3 \cos \theta_3) \\ + A_4 k_4 (\cos^2 \theta_4 - \sin^2 \theta_4) \}$$

$$\Rightarrow u [A_0 k_0 \sin 2\theta_0 - A_1 k_1 \sin 2\theta_1 - A_2 k_2 \cos 2\theta_2] = \\ u^\beta [A_3 k_3 (\sin 2\theta_3) + A_4 k_4 \cos 2\theta_4]$$

$$\Rightarrow A_0 k_0 \sin 2\theta_0 - A_1 k_1 \sin 2\theta_1 - A_2 k_2 \cos 2\theta_2 = \frac{u^\beta}{u} [A_3 k_3 \sin 2\theta_3 \\ + A_4 k_4 \cos 2\theta_4] \quad \text{--- (3)}$$

For i=2

$$\sigma_{22} = \lambda \epsilon_{kk} + 2\mu \epsilon_{22} \\ = \lambda (\epsilon_{11} + \epsilon_{22}) + 2\mu \epsilon_{22}$$

$$\sigma_{22} = \lambda (u_{1,1} + u_{2,2}) + 2\mu (u_{2,2})$$

So (*) implies

$$\lambda (u_{1,1}^{(0)} + u_{2,2}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(1)} + u_{1,1}^{(2)} + u_{2,2}^{(2)}) \\ + 2\mu (u_{2,2}^{(0)} + u_{2,2}^{(1)} + u_{2,2}^{(2)}) = \lambda^\beta (u_{1,1}^{(3)} + u_{2,2}^{(3)} + \\ u_{1,1}^{(4)} + u_{2,2}^{(4)}) + 2\mu^\beta (u_{2,2}^{(3)} + u_{2,2}^{(4)})$$

$$\Rightarrow \lambda \{ A_0 k_0 \sin^2 \theta_0 + A_0 k_0 \cos^2 \theta_0 + A_1 k_1 \sin^2 \theta_1 + A_1 k_1 \cos^2 \theta_1 + \\ A_2 k_2 \cos \theta_2 \sin \theta_2 - A_2 k_2 \cos \theta_2 \sin \theta_2 \} + 2\mu \{ A_0 k_0 \cos^2 \theta_0 \\ + A_1 k_1 \cos^2 \theta_1 + A_2 k_2 \sin \theta_2 \cos \theta_2 \} = \lambda^\beta \{ A_3 k_3 \sin^2 \theta_3 + \\ A_3 k_3 \cos^2 \theta_3 + A_4 k_4 \sin \theta_4 \cos \theta_4 - A_4 k_4 \sin \theta_4 \cos \theta_4 \} \\ + 2\mu^\beta \{ A_3 k_3 \cos^2 \theta_3 - A_4 k_4 \sin \theta_4 \cos \theta_4 \}$$

$$\Rightarrow \lambda(A_0 k_0 + A_1 k_1) + 2u(A_0 k_0 \cos^2 \theta_0 + A_1 k_1 \cos^2 \theta_1 + A_2 k_2 \cos^2 \theta_2 \sin \theta_2) = \lambda^\beta (A_3 k_3) + 2u^\beta (A_3 k_3 \cos^2 \theta_3 - A_4 k_4 \sin \theta_4 \cos \theta_4)$$

$$\Rightarrow (\lambda + 2u \cos^2 \theta_0) A_0 k_0 + (\lambda + 2u \cos^2 \theta_1) A_1 k_1 - u \sin 2\theta_2 A_2 k_2 - (\lambda^\beta + 2u^\beta \cos^2 \theta_3) A_3 k_3 + u^\beta \sin 2\theta_4 A_4 k_4 = 0 \quad (4)$$

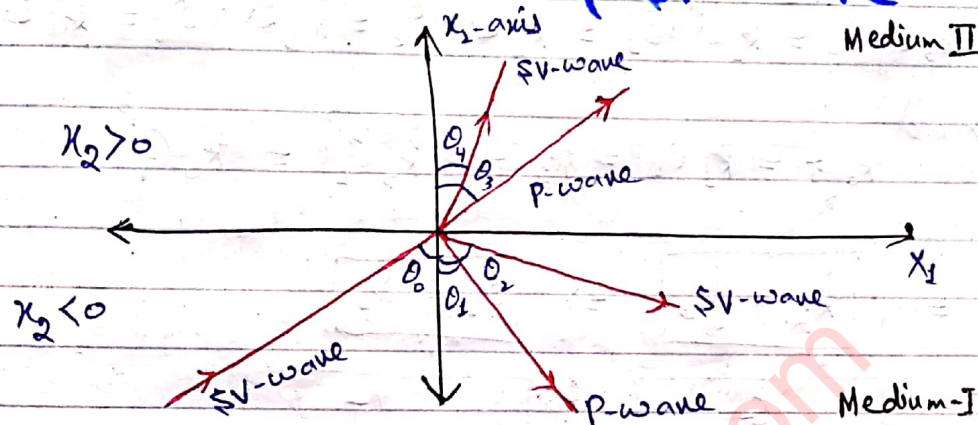
from equation (1) \rightarrow (4)

$$\begin{bmatrix} \sin \theta_1 & \cos \theta_2 & -\sin \theta_3 & -\cos \theta_4 \\ -\cos \theta_1 & \sin \theta_2 & -\cos \theta_3 & \sin \theta_4 \\ -k_1 \sin 2\theta_1 & -k_2 \cos 2\theta_2 & -\frac{u^\beta}{u} k_3 \sin 2\theta_3 & -\frac{u^\beta}{u} k_4 \cos 2\theta_4 \\ k(\lambda + 2u \cos^2 \theta_0) & -k_1 u \sin 2\theta_1 & -k_3 (\lambda^\beta + 2u^\beta \cos^2 \theta_3) & k_4 u^\beta \sin 2\theta_4 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

$$= \begin{bmatrix} -A_0 \sin \theta_0 \\ -A_0 \cos \theta_0 \\ -A_0 k_0 \sin 2\theta_0 \\ -(\lambda + 2u \cos^2 \theta_0) A_0 k_0 \end{bmatrix}$$

Solve above system of equations (matrix) and find the values of the unknowns (A_1, A_2, A_3, A_4)
This is our solution.

⇒ Reflection And Transmission of SV-wave as P-wave & SV-wave.



* Boundary Conditions:- u_i and σ_{ij} are continuous at $x_2 = 0$

$$u_i^{(0)} = A_0 e^{ik_0(x_1 \sin \theta_0 + x_2 \cos \theta_0 - ct)} \quad (-\cos \theta_0, \sin \theta_0, 0)$$

$$u_i^{(1)} = A_1 e^{ik_1(x_1 \sin \theta_1 - x_2 \cos \theta_1 - ct)} \quad (\sin \theta_1, -\cos \theta_1, 0)$$

$$u_i^{(2)} = A_2 e^{ik_2(x_1 \sin \theta_2 - x_2 \cos \theta_2 - ct)} \quad (\cos \theta_2, \sin \theta_2, 0)$$

$$u_i^{(3)} = A_3 e^{ik_3(x_1 \sin \theta_3 + x_2 \cos \theta_3 - ct)} \quad (\sin \theta_3, \cos \theta_3, 0)$$

$$u_i^{(4)} = A_4 e^{ik_4(x_1 \sin \theta_4 + x_2 \cos \theta_4 - ct)} \quad (-\cos \theta_4, \sin \theta_4, 0)$$

1) u_i is continuous at $x_2 = 0$

$$\Rightarrow u_i^{(0)} + u_i^{(1)} + u_i^{(2)} = u_i^{(3)} + u_i^{(4)}$$

for $i=1$

$$u_1^{(0)} + u_1^{(1)} + u_1^{(2)} = u_1^{(3)} + u_1^{(4)}$$

$$\Rightarrow -A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \cos \theta_0 + A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \sin \theta_1$$

$$+ A_2 e^{ik_2(x_1 \sin \theta_2 - ct)} \cos \theta_2 = A_3 e^{ik_3(x_1 \sin \theta_3 - ct)} \sin \theta_3$$

$$- A_4 e^{ik_4(x_1 \sin \theta_4 - ct)} \cos \theta_4 \quad \text{--- (A)}$$

By Snell's law

$$k_0 \sin \theta_0 = k_1 \sin \theta_1 = k_2 \sin \theta_2 = k_3 \sin \theta_3 = k_4 \sin \theta_4$$

And

$$k_0 c_T = k_1 c_L = k_2 c_T = k_3 c_L^{\beta} = k_4 c_T^{\beta}$$

$$\Rightarrow \boxed{k_2 = k_0} \Rightarrow \boxed{\theta_2 = \theta_0}$$

$$\Rightarrow \boxed{k_1 = \frac{c_T}{c_L} k_0} \quad \boxed{k_3 = \frac{c_T}{c_L^{\beta}} k_0}$$

$$\boxed{k_4 = \frac{c_T}{c_T^{\beta}} k_0}$$

$$\sin \theta_1 = \frac{k_0}{k_1} \sin \theta_0 \Rightarrow \boxed{\sin \theta_1 = \frac{c_L}{c_T} \sin \theta_0}$$

$$\sin \theta_3 = \frac{k_0}{k_3} \sin \theta_0 \Rightarrow \boxed{\sin \theta_3 = \frac{c_L^{\beta}}{c_T} \sin \theta_0}$$

$$\sin \theta_4 = \frac{k_0}{k_4} \sin \theta_0 \Rightarrow \boxed{\sin \theta_4 = \frac{c_T^{\beta}}{c_T} \sin \theta_0}$$

So equation (A) implies

$$-A_0 \cos \theta_0 + A_1 \sin \theta_1 + A_2 \cos \theta_2 = A_3 \sin \theta_3 - A_4 \cos \theta_4$$

$$\Rightarrow -A_0 \cos \theta_0 + A_1 \sin \theta_1 + A_2 \cos \theta_2 - A_3 \sin \theta_3 + A_4 \cos \theta_4 = 0$$

—————→ (1)

for $i=2$

$$U_2^{(0)} + U_2^{(1)} + U_2^{(2)} = U_2^{(3)} + U_2^{(4)}$$

$$\Rightarrow A_0 \sin \theta_0 - A_1 \cos \theta_1 + A_2 \sin \theta_2 = A_3 \cos \theta_3 + A_4 \sin \theta_4$$

$$\Rightarrow A_0 \sin \theta_0 - A_1 \cos \theta_1 + A_2 \sin \theta_2 - A_3 \cos \theta_3 - A_4 \sin \theta_4 = 0$$

\(\xrightarrow{\quad}\) ②

② σ_{ij} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{ij}^{(0)} + \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} = \sigma_{ij}^{(3)} + \sigma_{ij}^{(4)} \quad \xrightarrow{\quad} (*)$$

Since $\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$

$$\Rightarrow \sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

for $i=1$

$$\sigma_{12} = 2\mu \epsilon_{12} = \mu (U_{1,2} + U_{2,1})$$

$$\Rightarrow \mu \left\{ (U_{1,2}^{(0)} + U_{2,1}^{(0)}) + (U_{1,2}^{(1)} + U_{2,1}^{(1)}) + (U_{1,2}^{(2)} + U_{2,1}^{(2)}) \right\}$$

$$= \mu \left\{ (U_{1,2}^{(3)} + U_{2,1}^{(3)}) + (U_{1,2}^{(4)} + U_{2,1}^{(4)}) \right\}$$

$$\Rightarrow \mu \left\{ A_0 k_0 (-\cos^2 \theta_0 + \sin^2 \theta_0) + A_1 k_1 (-\sin \theta_1 \cos \theta_1 - \sin \theta_1 \cos \theta_1) + A_2 k_2 (\cos^2 \theta_2 + \sin^2 \theta_2) \right\} = \mu \left\{ A_3 k_3 (\sin \theta_3 \cos \theta_3 + \sin \theta_3 \cos \theta_3) \right.$$

$$\left. + A_4 k_4 (-\cos^2 \theta_4 + \sin^2 \theta_4) \right\}$$

$$\Rightarrow \mu \left\{ -A_0 k_0 \cos 2\theta_0 - A_1 k_1 \sin 2\theta_1 - A_2 k_2 \cos 2\theta_2 \right\}$$

$$= \mu \left\{ A_3 k_3 \sin 2\theta_3 - A_4 k_4 \cos 2\theta_4 \right\}$$

$$\Rightarrow -A_0 k_0 u \cos 2\theta_0 - A_1 k_1 u \sin 2\theta_1 - A_2 k_2 u \cos 2\theta_2$$

$$- A_3 k_3 u^\beta \sin 2\theta_3 + A_4 k_4 u^\beta \cos 2\theta_4 = 0 \quad \text{---} \quad (3)$$

for $i=2$

$$\sigma_{22} = \lambda \epsilon_{kk} + 2\mu \epsilon_{22}$$

$$= \lambda (\epsilon_{11} + \epsilon_{22}) + 2\mu \epsilon_{22}$$

$$= \lambda (U_{1,1} + U_{2,2}) + 2\mu (U_{2,2})$$

So (*) implies

$$\lambda (U_{1,1}^{(0)} + U_{2,2}^{(0)} + U_{1,1}^{(1)} + U_{2,2}^{(1)} + U_{1,1}^{(2)} + U_{2,2}^{(2)}) +$$

$$2\mu (U_{2,2}^{(0)} + U_{2,2}^{(1)} + U_{2,2}^{(2)}) = \lambda^\beta (U_{1,1}^{(3)} + U_{2,2}^{(3)} +$$

$$U_{1,1}^{(4)} + U_{2,2}^{(4)}) + 2\mu^\beta (U_{2,2}^{(3)} + U_{2,2}^{(4)})$$

$$\Rightarrow \lambda \{ A_0 k_0 \sin \theta_0 \cos \theta_0 + A_0 k_0 \sin \theta_0 \cos \theta_0 + A_1 k_1 \sin^2 \theta_1$$

$$+ A_1 k_1 \cos^2 \theta_1 + A_2 k_2 \sin \theta_2 \cos \theta_2 - A_2 k_2 \sin \theta_2 \cos \theta_2 \}$$

$$+ 2\mu \{ A_0 k_0 \sin \theta_0 \cos \theta_0 + A_1 k_1 \cos^2 \theta_1 - A_2 k_2 \sin \theta_2 \cos \theta_2 \}$$

$$= \lambda^\beta \{ A_3 k_3 \sin^2 \theta_3 + A_3 k_3 \cos^2 \theta_3 - A_4 k_4 \sin \theta_4 \cos \theta_4$$

$$+ A_4 k_4 \sin \theta_4 \cos \theta_4 \} + 2\mu^\beta \{ A_3 k_3 \cos^2 \theta_3 +$$

$$A_4 k_4 \sin \theta_4 \cos \theta_4 \}$$

$$\Rightarrow \lambda \{ A_1 k_1 \} + A_0 k_0 u \sin 2\theta_0 + A_1 k_1 2\mu \cos^2 \theta_1$$

$$- A_2 k_2 u \sin 2\theta_2 = \lambda^\beta A_3 k_3 + A_3 k_3 2\mu^\beta \cos^2 \theta_3 +$$

$$A_4 k_4 u^\beta \sin 2\theta_4$$

$$\Rightarrow A_0 k_0 u \sin 2\theta_0 + A_1 k_1 (\lambda + 2\mu \cos^2 \theta_1) - A_2 k_2 u \sin 2\theta_2 - A_3 k_3 (\lambda + 2\mu \cos^2 \theta_3) - A_4 k_4 u \sin 2\theta_4 \quad (4)$$

from equation (1) \rightarrow (4)

$$\begin{bmatrix} \sin \theta_1 & \cos \theta_1 & -\sin \theta_3 & \cos \theta_3 \\ -\cos \theta_1 & \sin \theta_1 & -\cos \theta_3 & -\sin \theta_3 \\ -k_1 u \sin 2\theta_1 & -k_1 u \cos 2\theta_1 & -k_3 u \sin 2\theta_3 & k_3 u \cos 2\theta_3 \\ k_1 (\lambda + 2\mu \cos^2 \theta_1) & -k_1 u \sin 2\theta_1 & -k_3 (\lambda + 2\mu \cos^2 \theta_3) & -k_3 u \sin 2\theta_3 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

$$= \begin{bmatrix} A_0 \cos \theta_0 \\ -A_0 \sin \theta_0 \\ A_0 k_0 u \cos 2\theta_0 \\ -A_0 k_0 u \sin 2\theta_0 \end{bmatrix}$$

By solving this matrix (system of equations) we will find the four unknowns (A_1, A_2, A_3, A_4)

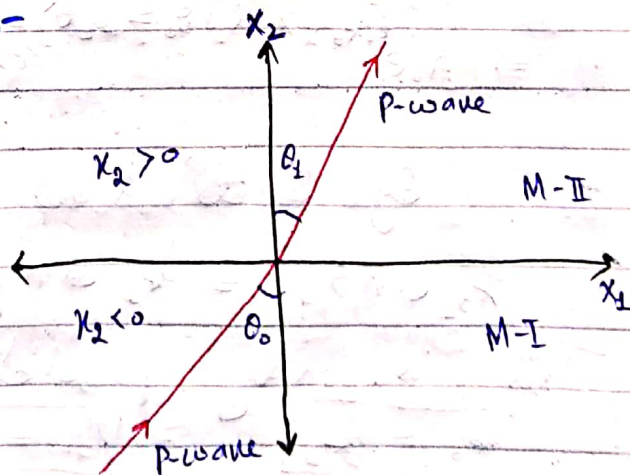
* Refraction (Transmission) of P-wave as P-wave :-

$$n_i^{(0)} = (\sin \theta_0, \cos \theta_0, 0)$$

$$P_i^{(0)} = (\sin \theta_0, \cos \theta_0, 0)$$

$$n_i^{(1)} = (\sin \theta_1, \cos \theta_1, 0)$$

$$P_i^{(1)} = (\sin \theta_1, \cos \theta_1, 0)$$



$$U_i^{(0)} = A_0 e^{i k_0 (x_1 \sin \theta_0 + x_2 \cos \theta_0 - c_0 t)} (\sin \theta_0, \cos \theta_0, 0)$$

$$U_i^{(1)} = A_1 e^{i k_1 (x_1 \sin \theta_1 + x_2 \cos \theta_1 - c_1 t)} (\sin \theta_1, \cos \theta_1, 0)$$

1) U_i is continuous at $x_2 = 0$

$$\Rightarrow U_i^{(0)} = U_i^{(1)}$$

for $i=1$

$$A_0 e^{i k_0 (x_1 \sin \theta_0 - c_0 t)} \sin \theta_0 = A_1 e^{i k_1 (x_1 \sin \theta_1 - c_1 t)} \sin \theta_1$$

By Snell's law

$$k_0 \sin \theta_0 = k_1 \sin \theta_1 \quad \text{or} \quad k_0 c_1 = k_1 c_0$$

$$\Rightarrow k_1 = \frac{c_1}{c_0} k_0$$

$$\Rightarrow \sin \theta_1 = \frac{k_0}{k_1} \sin \theta_0 \Rightarrow \sin \theta_1 = \frac{c_0}{c_1} \sin \theta_0$$

$$\Rightarrow A_0 = A_1 \quad \text{--- (1)}$$

for $i=2$

$$U_2^{(0)} = U_2^{(1)}$$

$$\Rightarrow A_0 e^{i k_0 (x_1 \sin \theta_0 - c_0 t)} \cos \theta_0 = A_1 e^{i k_1 (x_1 \sin \theta_1 - c_1 t)} \cos \theta_1$$

By Snell's law

$$k_0 \sin \theta_0 = k_1 \sin \theta_1$$

This is only possible if

$$\boxed{k_0 = k_1} \quad \text{or} \quad \boxed{c_0 = c_1} \quad \text{if this holds}$$

then

$$c_0 = c_1$$

$$\Rightarrow A_0 \cos \theta_0 = A_1 \cos \theta_0$$

$$\Rightarrow A_0 = A_1 \longrightarrow \textcircled{2}$$

② σ_{i2} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{i2}^{(0)} = \sigma_{i2}^{(1)} \longrightarrow \textcircled{*}$$

where $\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$

$$\Rightarrow \sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}$$

for $i=1$

$$\sigma_{12} = 2\mu \epsilon_{12} = \mu (u_{1,2} + u_{2,1})$$

$$\text{So } \mu \{ u_{1,2}^{(0)} + u_{2,1}^{(0)} \} = \mu^{\beta} \{ u_{1,2}^{(1)} + u_{2,1}^{(1)} \}$$

$$\Rightarrow \mu \{ A_0 k_0 (\sin \theta_0 \cos \theta_0 + \sin \theta_0 \cos \theta_0) \} = \mu^{\beta} \{ A_1 k_1 (\sin \theta_1 \cos \theta_1 + \sin \theta_1 \cos \theta_1) \}$$

$$\Rightarrow A_0 k_0 \mu \sin 2\theta_0 = A_1 k_1 \mu^{\beta} \sin 2\theta_1 \longrightarrow \textcircled{3}$$

for $i=2$

$$\sigma_{22} = \lambda \epsilon_{kk} + 2\mu \epsilon_{22}$$

$$= \lambda (\epsilon_{11} + \epsilon_{22}) + 2\mu \epsilon_{22}$$

$$= \lambda (u_{1,1} + u_{2,2}) + 2\mu (u_{2,2})$$

$$\Rightarrow \lambda (u_{1,1}^{(0)} + u_{2,2}^{(0)}) + 2\mu (u_{2,2}^{(0)}) = \lambda^{\beta} (u_{1,1}^{(1)} + u_{2,2}^{(1)}) + 2\mu^{\beta} u_{2,2}^{(1)}$$

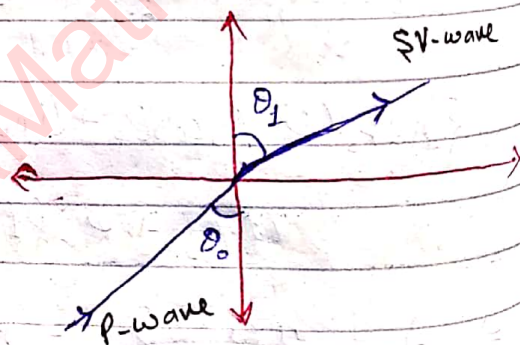
$$\Rightarrow \lambda \{ A_0 k_0 (\sin^2 \theta_0 + \cos^2 \theta_0) \} + 2\mu A_0 k_0 \cos^2 \theta_0 =$$

$$\lambda^{\beta} \{ A_1 k_1 (\sin^2 \theta_1 + \cos^2 \theta_1) \} + 2\mu^{\beta} \{ A_1 k_1 \cos^2 \theta_1 \}$$

$$\Rightarrow A_0 k_0 \lambda + 2 A_0 k_0 \mu \cos^2 \theta_0 = A_1 k_1 \lambda + 2 A_1 k_1 \mu \cos^2 \theta_1 \quad (4)$$

There are four equations and one unknown. So there does not exist any unique solution.
Hence Refraction of P-wave as a P-wave not exist.

⇒ Transmission of P-wave as SV-wave:-



$$u_i^{(0)} = A_0 e^{i k_0 (\kappa_1 \sin \theta_0 + \kappa_2 \cos \theta_0 - c t)} \quad (\sin \theta_0, \cos \theta_0, 0)$$

$$u_i^{(1)} = A_1 e^{i k_1 (\kappa_1 \sin \theta_1 + \kappa_2 \cos \theta_1 - c t)} \quad (-\cos \theta_1, \sin \theta_1, 0)$$

① u_i is continuous at $\kappa_2 = 0$

for $i=1$ ⇒ $u_i^{(0)} = u_i^{(1)}$

$$\Rightarrow A_0 e^{i k_0 (\kappa_1 \sin \theta_0 - c t)} \sin \theta_0 = -A_1 e^{i k_1 (\kappa_1 \sin \theta_1 - c t)} \cos \theta_1$$

To make exponent identical
 $k_0 \sin \theta_0 = k_1 \sin \theta_1$ And $k_0 c_1 = k_1 c_1^\beta$

This is only possible if
 $k_0 = k_1$ & $\theta_0 = \theta_1 \Rightarrow c_1 = c_1^\beta$

So eqn (A) implies

$$A_0 \sin \theta_0 = -A_1 \cos \theta_1$$

$$\Rightarrow A_0 \sin \theta_0 + A_1 \cos \theta_1 = 0 \longrightarrow \textcircled{1}$$

for $i=2$

$$u_2^{(0)} = u_2^{(1)}$$

$$\Rightarrow A_0 \cos \theta_0 = A_1 \sin \theta_1$$

$$\Rightarrow A_0 \cos \theta_0 - A_1 \sin \theta_1 = 0 \longrightarrow \textcircled{2}$$

(2) σ_{ij} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{ij}^{(0)} = \sigma_{ij}^{(1)}$$

$$\text{Since } \sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

for $i=1$

$$\sigma_{12}^{(0)} = \sigma_{12}^{(1)}$$

$$\Rightarrow \mu \{ u_{1,2}^{(0)} + u_{2,1}^{(0)} \} = \mu^\beta \{ u_{1,2}^{(1)} + u_{2,1}^{(1)} \}$$

$$\Rightarrow \mu \{ A_0 k_0 (\sin \theta_0 \cos \theta_0 + \sin \theta_0 \cos \theta_0) \} = \mu^\beta \{ A_1 k_1 (-\cos^2 \theta_1 + \sin^2 \theta_1) \}$$

$$\Rightarrow \mu A_0 k_0 \sin 2\theta_0 = -\mu^\beta A_1 k_1 \cos 2\theta_1$$

$$\Rightarrow A_0 k_0 \mu \sin 2\theta_0 + A_1 k_1 \mu^\beta \cos 2\theta_1 = 0 \longrightarrow \textcircled{3}$$

for $i=2$

$$\sigma_{22}^{(0)} = \sigma_{22}^{(1)}$$

$$\Rightarrow \lambda (U_{1,1}^{(0)} + U_{2,2}^{(0)}) + 2\mu (U_{2,1}^{(0)}) = \lambda (U_{1,1}^{(1)} + U_{2,2}^{(1)}) + 2\mu U_{2,1}^{(1)}$$

$$\Rightarrow \lambda \{A_0 k_0 (\sin^2 \theta_0 + \cos^2 \theta_0)\} + 2\mu A_0 k_0 \cos^2 \theta_0 =$$

$$\lambda \{A_1 k_1 (-\sin \theta_1 \cos \theta_1 + \sin \theta_1 \cos \theta_1)\} + 2\mu A_1 k_1 \sin \theta_1 \cos \theta_1$$

$$\Rightarrow \lambda A_0 k_0 + A_0 k_0 2\mu \cos^2 \theta_0 - A_1 k_1 \mu \sin 2\theta_1 = 0$$

$$\Rightarrow A_0 k_0 (\lambda + 2\mu \cos^2 \theta_0) - A_1 k_1 \mu \sin 2\theta_1 = 0 \quad \text{--- (4)}$$

There are four equations and one unknown, so there does not exist any unique solution. Hence Refraction of P-wave as SV-wave do not possible.

* Reflection & Transmission of P-wave as P-wave *

$$\eta_i^{(0)} = (\sin \theta_0, \cos \theta_0, 0)$$

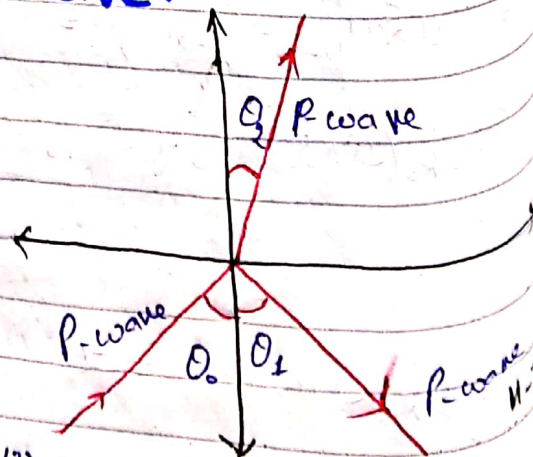
$$= P_i^{(0)}$$

$$\eta_i^{(1)} = (\sin \theta_1, -\cos \theta_1, 0)$$

$$= P_i^{(1)}$$

$$\eta_i^{(2)} = (\sin \theta_2, \cos \theta_2, 0)$$

$$= P_i^{(2)}$$



$$\Rightarrow U_i^{(0)} = A_0 e^{ik_0(x_1 \sin \theta_0 + x_2 \cos \theta_0 - ct)} \quad (\sin \theta_0, \cos \theta_0, 0)$$

$$U_i^{(1)} = A_1 e^{ik_1(x_1 \sin \theta_1 - x_2 \cos \theta_1 - ct)} \quad (\sin \theta_1, -\cos \theta_1, 0)$$

$$U_i^{(2)} = A_2 e^{ik_2(x_1 \sin \theta_2 + x_2 \cos \theta_2 - ct)} \quad (\sin \theta_2, \cos \theta_2, 0)$$

① U_i is continuous at $x_2 = 0$

$$\Rightarrow U_i^{(0)} + U_i^{(1)} = U_i^{(2)}$$

for $i=1$

$$U_1^{(0)} + U_1^{(1)} = U_1^{(2)}$$

$$\Rightarrow A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \sin \theta_0 + A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \sin \theta_1 = A_2 e^{ik_2(x_1 \sin \theta_2 - ct)} \sin \theta_2$$

By Snell's Law

$$k_0 \sin \theta_0 = k_1 \sin \theta_1 = k_2 \sin \theta_2$$

$$\text{And } k_0 c = k_1 c = k_2 c$$

$$\Rightarrow \boxed{k_1 = k_0} \Rightarrow \boxed{k_2 = \frac{c}{c'} k_0}$$

$$\Rightarrow \boxed{\theta_1 = \theta_0}$$

$$\sin \theta_2 = \frac{k_0}{k_2} \sin \theta_0 \Rightarrow \boxed{\sin \theta_2 = \frac{c'}{c} \sin \theta_0}$$

$$\Rightarrow A_0 \sin \theta_0 + A_1 \sin \theta_1 = A_2 \sin \theta_2$$

$$\Rightarrow A_0 \sin \theta_0 + A_1 \sin \theta_1 - A_2 \sin \theta_2 = 0 \quad \text{①}$$

for $i=2$

$$U_2^{(0)} + U_2^{(1)} = U_2^{(2)}$$

$$\Rightarrow A_0 \cos \theta_0 - A_1 \cos \theta_1 = A_2 \cos \theta_2$$

$$\Rightarrow A_0 \cos \theta_0 - A_1 \cos \theta_1 - A_2 \cos \theta_2 = 0 \quad \text{--- (2)}$$

② σ_{i2} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{i2}^{(0)} + \sigma_{i2}^{(1)} = \sigma_{i2}^{(2)} \quad \text{--- (3)}$$

since $\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$

$$\Rightarrow \sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}$$

for $i=1$

$$\sigma_{12} = 2\mu \epsilon_{12} = \mu (u_{1,2} + u_{2,1})$$

So (3) implies

$$\mu (u_{1,2}^{(0)} + u_{2,1}^{(0)} + u_{1,2}^{(1)} + u_{2,1}^{(1)}) = \mu^\beta (u_{1,2}^{(2)} + u_{2,1}^{(2)})$$

$$\Rightarrow \mu \{ A_0 k_0 (\sin \theta_0 \cos \theta_0 + \sin \theta_0 \cos \theta_0) + A_1 k_1 (-\sin \theta_1 \cos \theta_1 - \sin \theta_1 \cos \theta_1) \} = \mu^\beta \{ A_2 k_2 (\sin \theta_2 \cos \theta_2 + \sin \theta_2 \cos \theta_2) \}$$

$$\Rightarrow A_0 k_0 \mu \sin 2\theta_0 - A_1 k_1 \mu \sin 2\theta_1 - A_2 k_2 \mu^\beta \sin 2\theta_2 = 0$$

for $i=2$

$$\sigma_{22} = \lambda \epsilon_{kk} + 2\mu \epsilon_{22}$$

$$= \lambda (\epsilon_{11} + \epsilon_{22}) + 2\mu \epsilon_{22}$$

$$= \lambda (u_{1,1} + u_{2,2}) + 2\mu (u_{2,2})$$

So (3) becomes

$$\lambda (u_{1,1}^{(0)} + u_{2,2}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(1)}) + 2\mu (u_{2,2}^{(0)} + u_{2,2}^{(1)})$$

$$= \lambda^\beta (u_{1,1}^{(2)} + u_{2,2}^{(2)}) + 2\mu^\beta (u_{2,2}^{(2)})$$

$$\Rightarrow \lambda \left\{ A_0 k_0 (\sin^2 \theta_0 + \cos^2 \theta_0) + A_1 k_1 (\sin^2 \theta_1 + \cos^2 \theta_1) \right\} + 2\mu \left\{ A_0 k_0 \cos^2 \theta_0 + A_1 k_1 \cos^2 \theta_1 \right\} = \lambda^\beta \left\{ A_2 k_2 (\sin^2 \theta_2 + \cos^2 \theta_2) \right\} + 2\mu^\beta A_2 k_2 \cos^2 \theta_2$$

$$\Rightarrow \lambda \left\{ A_0 k_0 + A_1 k_1 \right\} + 2\mu \left\{ A_0 k_0 \cos^2 \theta_0 + A_1 k_1 \cos^2 \theta_1 \right\} + \lambda^\beta \left\{ A_2 k_2 \right\} + 2\mu^\beta A_2 k_2 \cos^2 \theta_2$$

$$\Rightarrow A_0 k_0 (\lambda + 2\mu \cos^2 \theta_0) + A_1 k_1 (\lambda + 2\mu \cos^2 \theta_1) - A_2 k_2 (\lambda^\beta + 2\mu^\beta \cos^2 \theta_2) = 0 \quad \text{--- (4)}$$

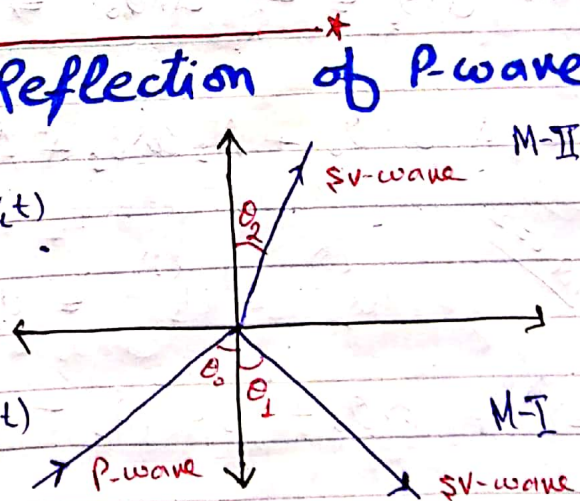
There are four equations and two unknowns, so there does not exist any unique solution.
Hence Refraction of and Reflection of P-wave as a P-wave does not exist.

\Rightarrow Transmission & Reflection of P-wave as SV-wave :-

$$(0) \quad i k_0 (x_1 \sin \theta_0 + x_2 \cos \theta_0 - ct) \\ u_i = A_0 e^{i k_0 (x_1 \sin \theta_0 + x_2 \cos \theta_0 - ct)}$$

$$(1) \quad i k_1 (x_1 \sin \theta_1 - x_2 \cos \theta_1 - ct) \\ u_i = A_1 e^{i k_1 (x_1 \sin \theta_1 - x_2 \cos \theta_1 - ct)}$$

$$(2) \quad i k_2 (x_1 \sin \theta_2 + x_2 \cos \theta_2 - ct) \\ u_i = A_2 e^{i k_2 (x_1 \sin \theta_2 + x_2 \cos \theta_2 - ct)}$$



1) U_i is continuous at $x_2 = 0$

$$\text{for } i=1 \Rightarrow U_i^{(0)} + U_i^{(1)} = U_i^{(2)}$$

$$\Rightarrow A_0 e^{ik_0(x_1 \sin \theta_0 - Ct)} \sin \theta_0 + A_1 e^{ik_1(x_1 \sin \theta_1 - Ct)} \cos \theta_1 = -A_2 e^{ik_2(x_1 \sin \theta_2 - Ct^p)} \cos \theta_2$$

By Snell's law

$$k_0 \sin \theta_0 = k_1 \sin \theta_1 = k_2 \sin \theta_2$$

$$\text{And } k_0 C = k_1 C = k_2 C^p$$

This is possible if

$$\boxed{k_1 = k_0} \quad \& \quad \boxed{\theta_1 = \theta_0} \Rightarrow C = C^p$$

$$\Rightarrow \boxed{k_2 = \frac{C}{C^p} k_0}$$

$$\text{And } \sin \theta_2 = \frac{k_0}{k_2} \sin \theta_0 \Rightarrow \boxed{\sin \theta_2 = \frac{C^p}{C} \sin \theta_0}$$

$$\Rightarrow A_0 \sin \theta_0 + A_1 \cos \theta_1 + A_2 \cos \theta_2 = 0 \quad \text{--- (1)}$$

$$\text{for } i=2 \Rightarrow U_2^{(0)} + U_2^{(1)} = U_2^{(2)}$$

$$\Rightarrow A_0 \cos \theta_0 + A_1 \sin \theta_1 - A_2 \sin \theta_2 = 0 \quad \text{--- (2)}$$

② σ_{i2} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{i2}^{(0)} + \sigma_{i2}^{(1)} = \sigma_{i2}^{(2)} \quad \text{--- (3)}$$

$$\text{since } \sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

$$\Rightarrow \sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}$$

for i=1

$$\sigma_{12} = 2\mu \epsilon_{12} = \mu(u_{1,2} + u_{2,1})$$

So \oplus implies

$$\mu \{ u_{1,2}^{(0)} + u_{2,1}^{(0)} + u_{1,2}^{(1)} + u_{2,1}^{(1)} \} = \mu^\beta \{ u_{1,2}^{(2)} + u_{2,1}^{(2)} \}$$

$$\Rightarrow \mu \{ A_0 k_0 (\sin \theta_0 \cos \theta_0 + \sin \theta_0 \cos \theta_0) + A_1 k_1 (-\cos^2 \theta_1 + \sin^2 \theta_1) \}$$

$$= \mu^\beta \{ A_2 k_2 (-\cos^2 \theta_2 + \sin^2 \theta_2) \}$$

$$\Rightarrow A_0 k_0 \mu \sin 2\theta_0 - A_1 k_1 \mu \cos 2\theta_1 + A_2 k_2 \mu \cos 2\theta_2 = 0 \quad \text{--- } \textcircled{3}$$

for i=2

$$\sigma_{22} = \lambda(u_{1,1} + u_{2,2}) + 2\mu(u_{2,2})$$

$$\Rightarrow \lambda \{ u_{1,1}^{(0)} + u_{2,2}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(1)} \} + 2\mu \{ u_{2,2}^{(0)} + u_{2,2}^{(1)} \}$$

$$= \lambda^\beta \{ u_{1,1}^{(2)} + u_{2,2}^{(2)} \} + 2\mu^\beta (u_{2,2}^{(2)})$$

$$\Rightarrow \lambda \{ A_0 k_0 (\sin^2 \theta_0 + \cos^2 \theta_0) + A_1 k_1 (\sin \theta_1 \cos \theta_1 - \sin \theta_1 \cos \theta_1) \}$$

$$+ 2\mu \{ A_0 k_0 \cos^2 \theta_0 - A_1 k_1 \sin \theta_1 \cos \theta_1 \} = \lambda^\beta \{ A_2 k_2 (-\sin^2 \theta_2$$

$$\cos \theta_2 + \sin \theta_2 \cos \theta_2) \} + 2\mu^\beta (A_2 k_2 \sin \theta_2 \cos \theta_2)$$

$$\Rightarrow \lambda A_0 k_0 + 2\mu A_0 k_0 \cos^2 \theta_0 - \mu A_1 k_1 \sin 2\theta_1 = \mu^\beta A_2 k_2 \sin 2\theta_2$$

$$\Rightarrow A_0 k_0 (\lambda + 2\mu \cos^2 \theta_0) - A_1 k_1 \mu \sin 2\theta_1 - A_2 k_2 \mu^\beta \sin 2\theta_2 = 0 \quad \text{--- } \textcircled{4}$$

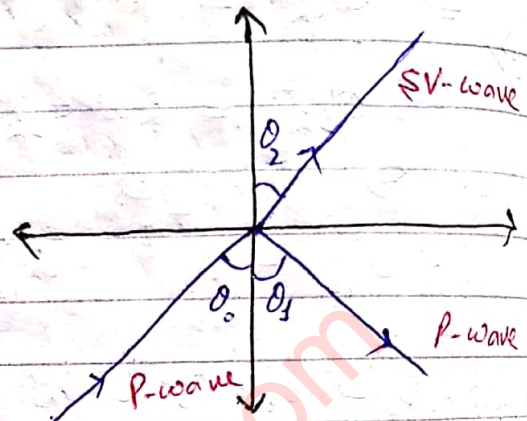
⇒ Reflection of P-wave as P-wave with refracted SV-wave also:

$$n_i^{(0)} = (\sin \theta_0, \cos \theta_0, 0)$$

$$= p_i^{(0)}$$

$$n_i^{(1)} = (\sin \theta_1, -\cos \theta_1, 0)$$

$$= p_i^{(1)}$$



$$n_i^{(2)} = (\sin \theta_2, \cos \theta_2, 0)$$

$$p_i^{(2)} = (-\cos \theta_2, \sin \theta_2, 0)$$

$$\Rightarrow u_i^{(0)} = A_0 e^{ik_0(x_1 \sin \theta_0 + x_2 \cos \theta_0 - ct)} (\sin \theta_0, \cos \theta_0, 0)$$

$$u_i^{(1)} = A_1 e^{ik_1(x_1 \sin \theta_1 - x_2 \cos \theta_1 - ct)} (\sin \theta_1, -\cos \theta_1, 0)$$

$$u_i^{(2)} = A_2 e^{ik_2(x_1 \sin \theta_2 + x_2 \cos \theta_2 - ct)} (-\cos \theta_2, \sin \theta_2, 0)$$

① u_i is continuous at $x_2 = 0$

$$\Rightarrow u_i^{(0)} + u_i^{(1)} = u_i^{(2)}$$

$$u_1^{(0)} + u_1^{(1)} = u_1^{(2)}$$

$$\Rightarrow A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \sin \theta_0 + A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \sin \theta_1$$

$$= -A_2 e^{ik_2(x_1 \sin \theta_2 - ct)} \cos \theta_2$$

By Snell's Law

$$k_0 \sin \theta_0 = k_1 \sin \theta_1 = k_2 \sin \theta_2$$

$$\text{And } k_0 C_L = k_1 C_L = k_2 C^P$$

$$\Rightarrow k_0 = k_1 \Rightarrow \theta_1 = \theta_0$$

$$\Rightarrow k_2 = \frac{C_L}{C^P} k_0$$

$$\Rightarrow \sin \theta_2 = \frac{k_0}{k_2} \sin \theta_0 \Rightarrow \sin \theta_2 = \frac{C^P}{C_L} \sin \theta_0$$

$$\Rightarrow A_0 \sin \theta_0 + A_1 \sin \theta_1 + A_2 \cos \theta_2 = 0 \quad \text{--- (1)}$$

for $i=2$

$$u_2^{(0)} + u_2^{(1)} = u_2^{(2)}$$

$$\Rightarrow A_0 \cos \theta_0 - A_1 \cos \theta_1 = A_2 \sin \theta_2$$

$$\Rightarrow A_0 \cos \theta_0 - A_1 \cos \theta_1 - A_2 \sin \theta_2 = 0 \quad \text{--- (2)}$$

(2) σ_{i2} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{i2}^{(0)} + \sigma_{i2}^{(1)} = \sigma_{i2}^{(2)} \quad \text{--- (*)}$$

$$\text{And } \sigma_{i2} = \lambda k_k \delta_{i2} + 2\mu \epsilon_{i2}$$

for $i=1$

$$\sigma_{12} = \mu (u_{1,2} + u_{2,1})$$

$$\Rightarrow \mu \{ u_{1,2}^{(0)} + u_{2,1}^{(0)} + u_{1,2}^{(1)} + u_{2,1}^{(1)} \} = \mu \{ u_{1,2}^{(2)} + u_{2,1}^{(2)} \}$$

$$\Rightarrow \mu \{ A_0 k_0 (\sin \theta_0 \cos \theta_0 + \sin \theta_0 \cos \theta_0) + A_1 k_1 (-\sin \theta_1 \cos \theta_1 - \sin \theta_1 \cos \theta_1) \}$$

$$= \mu \{ A_2 k_2 (-\cos^2 \theta_2 + \sin^2 \theta_2) \}$$

$$\Rightarrow \mu A_0 k_0 \sin 2\theta_0 - \mu A_1 k_1 \sin 2\theta_1 = -\mu A_2 k_2 \cos 2\theta_2$$

$$\Rightarrow A_0 k_0 u \sin 2\theta_0 - A_1 k_1 u \sin 2\theta_1 + A_2 k_2 u^\beta \cos 2\theta_2 = 0 \quad \text{--- (3)}$$

~~Exercise 2~~

$$\sigma'_{22} = \lambda (U_{1,11} + U_{2,22}) + 2\mu (U_{2,12})$$

$$\begin{aligned} \Rightarrow \lambda \{ U_{1,11}^{(0)} + U_{2,22}^{(0)} + U_{1,11}^{(1)} + U_{2,22}^{(1)} \} + 2\mu \{ U_{2,12}^{(0)} + U_{2,12}^{(1)} \} \\ = \lambda^\beta \{ U_{1,11}^{(2)} + U_{2,22}^{(2)} \} + 2\mu^\beta \{ U_{2,12}^{(2)} \} \end{aligned}$$

$$\begin{aligned} \Rightarrow \lambda \{ A_0 k_0 (\sin^2 \theta_0 + \cos^2 \theta_0) + A_1 k_1 (\sin^2 \theta_1 + \cos^2 \theta_1) \} \\ + 2\mu \{ A_0 k_0 \cos^2 \theta_0 + A_1 k_1 \cos^2 \theta_1 \} = \lambda^\beta \{ A_2 k_2 (-\sin \theta_2 \cos \theta_2 \\ + \sin \theta_2 \cos \theta_2) \} + 2\mu^\beta \{ A_2 k_2 \sin \theta_2 \cos \theta_2 \} \end{aligned}$$

$$\begin{aligned} \Rightarrow \lambda \{ A_0 k_0 + A_1 k_1 \} + 2\mu A_0 k_0 \cos^2 \theta_0 + 2\mu A_1 k_1 \cos^2 \theta_1 \\ = \mu^\beta A_2 k_2 \sin 2\theta_2 \end{aligned}$$

$$\begin{aligned} \Rightarrow A_0 k_0 (\lambda + 2\mu \cos^2 \theta_0) + A_1 k_1 (\lambda + 2\mu \cos^2 \theta_1) \\ - \mu^\beta A_2 k_2 \sin 2\theta_2 = 0 \quad \text{--- (4)} \end{aligned}$$

Solution not exist

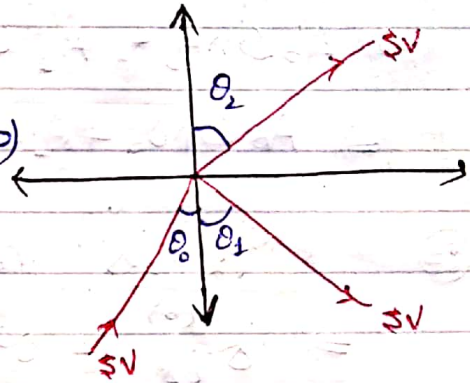


⇒ Reflection And Transmission of SV-wave as SV-wave:-

$$U_i^{(0)} = A_0 e^{ik_0(x_1 \sin \theta_0 + x_2 \cos \theta_0 - Gt)} \quad (-\cos \theta_0, \sin \theta_0, 0)$$

$$U_i^{(1)} = A_1 e^{ik_1(x_1 \sin \theta_1 - x_2 \cos \theta_1 - Gt)} \quad (\cos \theta_1, \sin \theta_1, 0)$$

$$U_i^{(2)} = A_2 e^{ik_2(x_1 \sin \theta_2 + x_2 \cos \theta_2 - Gt^\beta)} \quad (-\cos \theta_2, \sin \theta_2, 0)$$



① U_i is continuous at $x_2 = 0$

$$\Rightarrow U_i^{(0)} + U_i^{(1)} = U_i^{(2)}$$

for $x_2 = 0$

$$U_1^{(0)} + U_1^{(1)} = U_1^{(2)}$$

$$\Rightarrow -A_0 e^{ik_0(x_1 \sin \theta_0 - Gt)} \cos \theta_0 + A_1 e^{ik_1(x_1 \sin \theta_1 - Gt)} \cos \theta_1 = -A_2 e^{ik_2(x_1 \sin \theta_2 - Gt^\beta)} \cos \theta_2$$

By Snell's Law

$$k_0 \sin \theta_0 = k_1 \sin \theta_1 = k_2 \sin \theta_2$$

$$\text{And } k_0 c_T = k_1 c_T = k_2 c_T^\beta$$

$$\Rightarrow \boxed{k_0 = k_1} \quad \text{or} \quad \boxed{\theta_0 = \theta_1}$$

$$\Rightarrow \boxed{k_2 = \frac{c_T}{c_T^\beta} k_0}$$

$$\Rightarrow \sin \theta_2 = \frac{k_0}{k_2} \sin \theta_0 \Rightarrow \boxed{\sin \theta_2 = \frac{c_T^\beta}{c_T} \sin \theta_0}$$

$$\Rightarrow -A_0 \cos \theta_0 + A_1 \cos \theta_1 + A_2 \cos \theta_2 = 0$$

→ ②

for i=2

$$u_2^{(0)} + u_2^{(1)} = u_2^{(2)}$$

$$\Rightarrow A_0 \sin \theta_0 + A_1 \sin \theta_1 = A_2 \sin \theta_2$$

$$\Rightarrow A_0 \sin \theta_0 + A_1 \sin \theta_1 - A_2 \sin \theta_2 = 0 \quad \text{--- (2)}$$

(2) σ_{i2} is continuous at $x_2=0$

$$\Rightarrow \sigma_{i2}^{(0)} + \sigma_{i2}^{(1)} = \sigma_{i2}^{(2)}$$

where $\sigma_{i2} = \lambda \epsilon_{kk} \delta_{i2} + 2\mu \epsilon_{i2}$

for i=1

$$\sigma_{12} = \mu (u_{1,2} + u_{2,1})$$

$$\Rightarrow \mu \{ u_{1,2}^{(0)} + u_{2,1}^{(0)} + u_{1,2}^{(1)} + u_{2,1}^{(1)} \} = \mu \{ u_{1,2}^{(2)} + u_{2,1}^{(2)} \}$$

$$\Rightarrow \mu \{ A_0 k_0 (-\cos^2 \theta_0 + \sin^2 \theta_0) + A_1 k_1 (-\cos^2 \theta_1 + \sin^2 \theta_1) \}$$

$$= \mu \{ A_2 k_2 (-\cos^2 \theta_2 + \sin^2 \theta_2) \}$$

$$\Rightarrow A_0 k_0 \mu \cos 2\theta_0 - A_1 k_1 \mu \cos 2\theta_1 + A_2 k_2 \mu \cos 2\theta_2 = 0$$

for i=2

$$\sigma_{22} = \lambda (u_{1,1} + u_{2,2}) + 2\mu (u_{2,2})$$

$$\Rightarrow \lambda \{ u_{1,1}^{(0)} + u_{2,2}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(1)} \} + 2\mu \{ u_{2,2}^{(0)} + u_{2,2}^{(1)} \}$$

$$= \lambda \{ u_{1,1}^{(2)} + u_{2,2}^{(2)} \} + 2\mu \{ u_{2,2}^{(2)} \}$$

$$\Rightarrow \lambda \{ A_0 k_0 (-\cos \theta_0 \sin \theta_0 + \cos \theta_0 \sin \theta_0) + A_1 k_1 (\cos \theta_1 \sin \theta_1 - \cos \theta_1 \sin \theta_1) \} + 2\mu \{ A_0 k_0 \cos \theta_0 \sin \theta_0 - A_1 k_1 \cos \theta_1 \sin \theta_1 \}$$

$$= \lambda \{ A_2 k_2 (-\sin \theta_2 / \cos \theta_2 + \sin \theta_2 / \cos \theta_2) \} + 2\mu \{ A_2 k_2 \sin \theta_2 \cos \theta_2 \}$$

$$\Rightarrow \mu A_0 k_0 \sin 2\theta_0 - \mu A_1 k_1 \sin 2\theta_1 = \mu A_2 k_2 \sin 2\theta_2$$

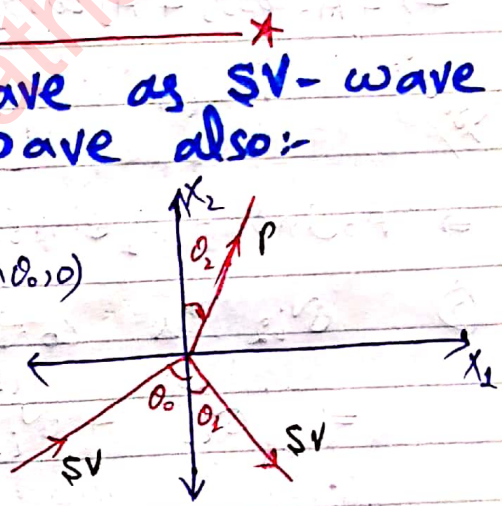
$$\Rightarrow A_0 k_0 \mu \sin 2\theta_0 - A_1 k_1 \mu \sin 2\theta_1 - A_2 k_2 \mu^\beta \sin 2\theta_2 = 0 \quad \text{--- (4)}$$

\Rightarrow Reflection of SV-wave as SV-wave with refracted P-wave also:-

(0) $U_i = A_0 e^{ik_0(x_1 \sin \theta_0 + x_2 \cos \theta_0 - ct)}$
 direction: $(-\cos \theta_0, \sin \theta_0, 0)$

(1) $U_i = A_1 e^{ik_1(x_1 \sin \theta_1 - x_2 \cos \theta_1 - ct)}$
 direction: $(\cos \theta_1, \sin \theta_1, 0)$

(2) $U_i = A_2 e^{ik_2(x_1 \sin \theta_2 + x_2 \cos \theta_2 - ct)}$
 direction: $(\sin \theta_2, \cos \theta_2, 0)$



(1) U_i is continuous at $x_2 = 0$

$$\Rightarrow U_i^{(0)} + U_i^{(1)} = U_i^{(2)}$$

For $i=1$

$$-A_0 e^{ik_0(x_1 \sin \theta_0 - ct)} \cos \theta_0 + A_1 e^{ik_1(x_1 \sin \theta_1 - ct)} \cos \theta_1 = A_2 e^{ik_2(x_1 \sin \theta_2 - ct)} \sin \theta_2$$

By Snell's Law

$$k_0 \sin \theta_0 = k_1 \sin \theta_1 = k_2 \sin \theta_2$$

$$\text{And } k_0 v = k_1 v = k_2 c_L^P$$

$$\Rightarrow \boxed{k_0 = k_1} \quad \text{And } \boxed{\theta_1 = \theta_0}$$

$$\Rightarrow \boxed{k_2 = \frac{v}{c_L^P} k_0}$$

$$\text{And } \sin \theta_2 = \frac{k_0}{k_2} \sin \theta_0 \Rightarrow \boxed{\sin \theta_2 = \frac{c_L^P}{v} \sin \theta_0}$$

$$\Rightarrow -A_0 \cos \theta_0 + A_1 \cos \theta_1 - A_2 \sin \theta_2 = 0 \quad \text{--- (1)}$$

For $i=2$

$$u_1^{(0)} + u_2^{(1)} = u_2^{(2)}$$

$$\Rightarrow A_0 \sin \theta_0 + A_1 \sin \theta_1 - A_2 \cos \theta_2 = 0 \quad \text{--- (2)}$$

(2) σ_{i2} is continuous at $x_2 = 0$

$$\Rightarrow \sigma_{i2}^{(0)} + \sigma_{i2}^{(1)} = \sigma_{i2}^{(2)}$$

where $\sigma_{i2} = \lambda k k S_{i2} + 2\mu E_{i2}$

For $i=2$

$$\sigma_{22} = \mu (u_{1,2} + u_{2,1})$$

$$\Rightarrow \mu \{ u_{1,2}^{(0)} + u_{2,1}^{(0)} + u_{1,2}^{(1)} + u_{2,1}^{(1)} \} = \mu^P \{ u_{1,2}^{(2)} + u_{2,1}^{(2)} \}$$

$$\Rightarrow \mu \{ A_0 k_0 (-\cos^2 \theta_0 + \sin^2 \theta_0) + A_1 k_1 (-\cos^2 \theta_1 + \sin^2 \theta_1) \} = \mu^P \{ A_2 k_2 (\sin \theta_2 \cos \theta_2 + \sin \theta_2 \cos \theta_2) \}$$

$$\Rightarrow \mu A_0 k_0 \cos 2\theta_0 - \mu A_1 k_1 \cos 2\theta_1 = \mu^P A_2 k_2 \sin 2\theta_2$$

$$\Rightarrow -A_0 k_0 u \cos 2\theta_0 - A_1 k_1 u \cos 2\theta_1 - A_2 k_2 u^\beta \sin 2\theta_2 = 0 \quad \text{②}$$

for $i=2$

$$\sigma_{22} = \lambda(u_{1,1} + u_{2,2}) + 2\mu(u_{2,2})$$

$$\Rightarrow \lambda \{ u_{1,1}^{(0)} + u_{2,2}^{(0)} + u_{1,1}^{(1)} + u_{2,2}^{(1)} \} + 2\mu \{ u_{2,2}^{(0)} + u_{2,2}^{(1)} \}$$

$$= \lambda^\beta \{ u_{2,2}^{(0)} + u_{2,2}^{(1)} \} + 2\mu^\beta \{ u_{2,2}^{(1)} \}$$

$$\Rightarrow \lambda \{ A_0 k_0 (-\sin\theta_0 \cos\theta_0 + \sin\theta_0 \cos\theta_0) + A_1 k_1 (\sin\theta_1 \cos\theta_1 - \sin\theta_1 \cos\theta_1) \} + 2\mu \{ A_0 k_0 \sin\theta_0 \cos\theta_0 - A_1 k_1 \sin\theta_1 \cos\theta_1 \}$$

$$= \lambda^\beta \{ A_2 k_2 \sin^2\theta_2 + A_2 k_2 \cos^2\theta_2 \} + 2\mu^\beta \{ A_2 k_2 \cos^2\theta_2 \}$$

$$\Rightarrow \mu A_0 k_0 \sin 2\theta_0 - \mu A_1 k_1 \sin 2\theta_1 = A_2 k_2 \lambda^\beta + 2\mu^\beta A_2 k_2 \cos^2\theta_2$$

$$\Rightarrow A_0 k_0 \mu \sin 2\theta_0 - A_1 k_1 \mu \sin 2\theta_1 - (\lambda^\beta + 2\mu^\beta \cos^2\theta_2) A_2 k_2 = 0 \quad \text{④}$$

Solution not exist

⇒ Equation of Motion for Homogeneous Transversely Isotropic Medium:-

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{22}-C_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix}$$

Here axis of symmetry is x_1 -axis.
We have 5-independent elastic constants
For Fibre Reinforced Material

$$\begin{aligned} \sigma_{ij} = & \lambda \epsilon_{kk} \delta_{ij} + 2\mu_T \epsilon_{ij} + \alpha (a_k a_m \epsilon_{km} \delta_{ij} \\ & + \epsilon_{kk} a_i a_j) + 2(\mu_L - \mu_T) (a_i a_k \epsilon_{kj} + \\ & a_j a_k \epsilon_{ki}) + \beta (a_k a_m \epsilon_{km} \delta_{ij}) \end{aligned}$$

Equation of motion is

$$\sigma_{ij,j} = \rho \ddot{u}_i$$

$$\Rightarrow \left. \begin{aligned} \sigma_{11,1} + \sigma_{12,2} &= \rho \ddot{u}_1 \\ \sigma_{21,1} + \sigma_{22,2} &= \rho \ddot{u}_2 \\ \sigma_{31,1} + \sigma_{32,2} &= \rho \ddot{u}_3 \end{aligned} \right\} \text{--- } \textcircled{1}$$

Now $\sigma_{11} = C_{11} \epsilon_{11} + C_{12} \epsilon_{22}$

$$\begin{aligned} \Rightarrow \sigma_{11} &= C_{11} \cdot \frac{1}{2} (u_{1,1} + u_{1,1}) + \frac{1}{2} C_{12} (u_{2,2} + u_{2,2}) \\ &= C_{11} u_{1,1} + \frac{1}{2} C_{12} (u_{2,2} + u_{2,2}) \end{aligned}$$

$$\Rightarrow \sigma_{11,1} = C_{11} U_{1,11} + C_{12} U_{2,21} \rightarrow \textcircled{a}$$

$$\begin{aligned} * \sigma_{12} &= 2 C_{66} C_{12} = 2 C_{66} \cdot \frac{1}{2} (U_{1,2} + U_{2,1}) \\ &= C_{66} (U_{1,2} + U_{2,1}) \end{aligned}$$

$$\Rightarrow \sigma_{12,2} = C_{66} (U_{1,22} + U_{2,12}) \rightarrow \textcircled{b}$$

$$\begin{aligned} \sigma_{21} &= 2 C_{66} \epsilon_{12} \\ &= C_{66} (U_{1,2} + U_{2,1}) \end{aligned}$$

$$\Rightarrow \sigma_{21,1} = C_{66} (U_{1,21} + U_{2,11}) \rightarrow \textcircled{c}$$

$$\begin{aligned} \sigma_{22} &= C_{12} \epsilon_{11} + C_{22} \epsilon_{22} \\ &= C_{12} U_{1,1} + C_{22} U_{2,2} \end{aligned}$$

$$\Rightarrow \sigma_{22,2} = C_{12} U_{1,12} + C_{22} U_{2,22} \rightarrow \textcircled{d}$$

$$\begin{aligned} \sigma_{31} &= 2 C_{66} \epsilon_{13} \\ &= C_{66} (U_{1,3} + U_{3,1}) \\ &= C_{66} U_{3,1} \quad \because U_{1,3} = 0 \end{aligned}$$

$$\Rightarrow \sigma_{31,1} = C_{66} U_{3,11} \rightarrow \textcircled{e}$$

$$\begin{aligned} \sigma_{32} &= \frac{1}{2} (C_{22} - C_{23}) \cdot 2 \epsilon_{23} \\ &= \frac{1}{2} (C_{22} - C_{23}) (U_{2,3} + U_{3,2}) \\ &= \frac{1}{2} (C_{22} - C_{23}) U_{3,2} \quad \because U_{2,3} = 0 \end{aligned}$$

$$\Rightarrow \sigma_{32,2} = \frac{1}{2} (C_{22} - C_{23}) U_{3,22} \rightarrow \textcircled{f}$$

using $\textcircled{a} \rightarrow \textcircled{b}$ eqn $\textcircled{1}$ becomes

$$\begin{aligned} C_{11} U_{1,11} + C_{12} U_{2,21} + C_{66} U_{1,22} + C_{66} U_{2,21} &= \rho \ddot{u}_1 \\ C_{22} U_{2,22} + C_{12} U_{1,12} + C_{66} U_{2,11} + C_{66} U_{1,12} &= \rho \ddot{u}_2 \\ C_{66} U_{3,11} + \frac{1}{2}(C_{22} - C_{23}) U_{3,22} &= \rho \ddot{u}_3 \end{aligned} \rightarrow \textcircled{2}$$

Now for fibre reinforced material
Let $a_i = (a, 0, 1)$

$$\begin{aligned} \sigma_{11} &= \lambda \epsilon_{kk} + 2\mu_T \epsilon_{11} \\ &= \lambda \epsilon_{kk} + 2\mu_T \epsilon_{11} \\ &= \lambda \epsilon_{11} + \lambda \epsilon_{22} + 2\mu_T \epsilon_{11} \\ &= \lambda U_{1,1} + \lambda U_{2,2} + 2\mu_T U_{1,1} \end{aligned}$$

$$\Rightarrow \sigma_{11,1} = (\lambda + 2\mu_T) U_{1,11} + \lambda U_{2,21} \rightarrow \textcircled{3}$$

$$\sigma_{12} = 2\mu_T \epsilon_{12} = \mu_T (U_{1,2} + U_{2,1})$$

$$\Rightarrow \sigma_{12,2} = \mu_T U_{1,22} + \mu_T U_{2,12} \rightarrow \textcircled{4}$$

$$\Rightarrow (\lambda + 2\mu_T) U_{1,11} + (\lambda + \mu_T) U_{2,21} + \mu_T U_{1,22} = \rho \ddot{u}_1$$

Comparing equation 1 of $\textcircled{2}$ & $\textcircled{3}$, we have

$$\boxed{\lambda + 2\mu_T = C_{11}}$$

$$\boxed{C_{66} = \mu_T}$$

$$C_{12} + C_{66} = \lambda + \mu_T$$

$$\Rightarrow \boxed{C_{12} = \lambda}$$

similarly we can find others of waves speed

$$\sigma_{21} = 2\mu_T \epsilon_{21} = \mu_T (u_{2,1} + u_{1,2})$$

$$\Rightarrow \sigma_{21,1} = \mu_T u_{2,11} + \mu_T u_{1,21} \longrightarrow \textcircled{1}$$

$$\begin{aligned} \sigma_{22} &= \lambda \epsilon_{kk} + 2\mu_T \epsilon_{22} \\ &= \lambda u_{1,1} + \lambda u_{2,2} + 2\mu_T u_{2,2} \end{aligned}$$

$$\Rightarrow \sigma_{22,2} = (\lambda + 2\mu_T) u_{2,22} + \lambda u_{1,12} \longrightarrow \textcircled{2}$$

$$\Rightarrow \mu_T u_{2,11} + (\lambda + \mu_T) u_{1,12} + (\lambda + 2\mu_T) u_{2,22} = \delta_{ij} \ddot{u}_j \longrightarrow \textcircled{4}$$

Comparing equation 2 of $\textcircled{2}$ & $\textcircled{4}$ we have

$$\boxed{C_{66} = \mu_T} \quad C_{12} + C_{66} = \lambda + \mu_T$$

$$\Rightarrow \boxed{C_{12} = \lambda} \quad \therefore C_{66} = \mu_T$$

$$\boxed{C_{22} = \lambda + 2\mu_T}$$

$$\begin{aligned} \sigma_{31} &= 2\mu_T \epsilon_{31} + 2(\mu_L - \mu_T) \epsilon_{31} \\ &= \mu_T (u_{3,1} + u_{1,3}) + (\mu_L - \mu_T) (u_{3,1} + u_{1,3}) \end{aligned}$$

$$\Rightarrow \sigma_{31,1} = \mu_T u_{3,11} + (\mu_L - \mu_T) u_{3,11}$$

$$\Rightarrow \sigma_{31,1} = \mu_L u_{3,11} \longrightarrow \textcircled{3}$$

$$\begin{aligned} \sigma_{32} &= 2\mu_T \epsilon_{32} + 2(\mu_L - \mu_T) \epsilon_{32} \\ &= \mu_T (u_{3,2} + u_{2,3}) + (\mu_L - \mu_T) (u_{3,2} + u_{2,3}) \end{aligned}$$

$$\Rightarrow \sigma_{32,2} = \mu_T u_{3,22} + (\mu_L - \mu_T) u_{3,22}$$

$$\Rightarrow \sigma_{32,2} = \mu_L u_{3,22} \longrightarrow \textcircled{5}$$

$$\Rightarrow \mu_L u_{3,11} + \mu_L u_{3,22} = \rho \ddot{u}_3 \longrightarrow \textcircled{5}$$

Comparing equation 3 of ② & ③
we have

$$\boxed{C_{66} = \mu_L} \quad \& \quad \frac{1}{2}(C_{22} - C_{23}) = \mu_L$$

$$\therefore C_{22} = \lambda + 2\mu_T$$

$$\Rightarrow \frac{1}{2}(\lambda + 2\mu_T - C_{23}) = \mu_L$$

$$\Rightarrow \frac{1}{2}\lambda + \mu_T - \mu_L = \frac{1}{2}C_{23}$$

$$\Rightarrow \frac{1}{2}\lambda = \frac{1}{2}C_{23}$$

$$\begin{aligned} \because C_{66} = \mu_T \quad \& \quad C_{66} = \mu_L \\ \Rightarrow \mu_T = \mu_L \end{aligned}$$

$$\Rightarrow \boxed{C_{23} = \lambda}$$

