Introduce Sets:

Def (FIELD)

A field F is a non-empty set together with two binary operations addition and multiplication satisfying

1. Associative law for addition and multiplication
2. Commutative laws for addition and multiplication
3. Identity for addition and multiplication
4. Inverse for addition and multiplication
5. The distributive law.

e.g. Q is a field.

Def: Let $a, b, c \in R$ with $a < b$ we define

$(a,b) = \{x \in R : a < x < b\}$

$[a,b] = \{x \in R : a \leq x \leq b\}$

$[a,b) = \{x \in R : a \leq x < b\}$

$(a,b] = \{x \in R : a < x \leq b\}$

$(-\infty, a) = \{x \in R : x < a\}$

$(-\infty, a] = \{x \in R : x \leq a\}$

$[a, \infty) = \{x \in R : x \geq a\}$

$(a, \infty) = \{x \in R : x > a\}$

$(-\infty, \infty) = R.$

Methods of Proof:

(i) Induction (ii) Contradiction (iii) Contrapositive
$p \to q \Leftrightarrow \neg q \to \neg p$

Well-ordering property of N:

Every non-empty subset of N has a least element.

That is "If $S \subseteq N$: $S \neq \Phi$, then there exist $m \in S$ such that $m \leq k \ \forall \ k \in S$.

Principle of Mathematical Induction:

Let S be a subset of N that possesses the two properties

(i) The number $1 \in S$

(ii) For every $k \in N$, if $k \in S$, then $k+1 \in S$

Then we have $S = N$

Proof: Suppose to the contrary that $S \neq N$

Then $N \setminus S$ is non-empty. By well-ordering principle it has a least element (say) m.

Since $1 \in S$ by (i)

So $m > 1$ $\quad (\because m \in N \setminus S)$

imply $m-1$ is also a natural number.

Since $m-1 < m$ and m is least in $N \setminus S$

Thus $m-1 \in S$

Now by (ii) $(m-1)+1 \in S$

$\Rightarrow m \in S$, a contradiction

against the fact that $m \notin S$

Since m was obtained from the assumption that $N \setminus S$ is non-empty, a contradiction.

Hence $N \setminus S = \Phi$

That is $N = S$

Observation: The principle of Mathematical induction can be formulated as follows.

For each $n \in N$, let $P(n)$ be a statement about n. Suppose that

1. $P(1)$ is true (Inductive hypothesis)
2. For every $K \in N$, if $P(K)$ is true, then $P(K+1)$ is true. Then $P(n)$ is true $\forall n \in N$.

Example: Bernoulli's Inequality:

If $x > -1$, then

$$(1+x)^n \geq 1+nx \quad \forall n \in N$$

Proof: For $n = 1$

$$1+x \geq 1+x \quad \text{true.}$$

Suppose for $K \in N$

$$(1+x)^K \geq 1+Kx \quad ; \quad 1+x > 0$$

Consider

$$(1+x)^{K+1} = (1+x)^K(1+x)$$

$$\geq (1+Kx)(1+x) = 1+(K+1)x+Kx^2$$

$$\geq 1+(K+1)x \quad \because Kx^2 \geq 0$$

Hence by induction result is true.

Assignment: Prove the followings.

(i) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1} \quad \forall n\in N$

(ii) $1^3+2^3+\cdots+n^3=\left[\frac{n(n+1)}{2}\right]^2 \quad \forall n\in N$

(iii) $3+11+\cdots+(8n-5)=4n^2-n \quad \forall n\in N$

(iv) $1^2+3^2+\cdots+(2n-1)^2=\frac{4n^3-n}{3} \quad \forall n\in N$

(v) $1^2-2^2+3^2+\cdots+(-1)^{n+1}n^2=(-1)^{n+1}\frac{n(n+1)}{2} \quad \forall n\in N$

(vi) Show that n^3+5n is divisible by 6 $\forall n\in N$

(vii) Prove that $5^{2n}-1$ is divisible by 8 $\forall n\in N$

(ix) Prove that $n<2^n \quad \forall n\in N$

(x) Prove that $2^n<n! \quad n\in N\ ;\ n\geq 4$.

(xi) $1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}>\sqrt{n} \quad \forall n\in N$.

(xii) Conjecture a formula for the sum

$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}$ and

then prove your conjecture by using mathematical induction.

2 Contradiction:

THEOREM: There does not exist a rational number r such that $r^2 = 2$.

OR Show that $\sqrt{2}$ is an irrational number.

Proof: Suppose on contrary, that p and $q \neq 0$ are positive integers such that $\left(\frac{p}{q}\right)^2 = 2$ and $(p,q) = 1$

Since $p^2 = 2q^2$ —— ①

We see that p^2 is even, this implies that p is also even.

Let $p = 2t$ for some integer t.

put in ①, then $q^2 = 2t^2$ imply q^2 is even and is q.

Thus $(p,q) \geq 2$, a contradiction against the fact that $(p,q) = 1$

Thus there does not exist a rational number r such that $r^2 = 2$.

Assignment: Show that

① There does not exist a rational number r such that $r^2 = 3$

② There does not exist a rational number r such that $r^2 = 5$

③ There does not exist a rational number r such that $r^2 = 7$

Example: Show that $\sqrt{8}$ is not a rational number.

Solution: Let, if possible

$\sqrt{8} = \frac{p}{q}$ where $q \neq 0$ and p, q are positive integers prime to each other.

But $2 < \sqrt{8} < 3$

imply $2 < \frac{p}{q} < 3$

$\Rightarrow 2q < p < 3q$

or $0 < p - 2q < q$

Thus $p - 2q$ is a positive integer less than q,

So that $\sqrt{8}(p - 2q)$ is not an integer

i-e $\frac{p}{q}(p - 2q)$ is not an integer.

But $\sqrt{8}(p - 2q) = \frac{p}{q}(p - 2q)$

$= \frac{p^2}{q} - 2p$

$= \frac{p^2}{q^2} \cdot q - 2p$

$= 8q - 2p$

which is clearly an integer

imply $\sqrt{8}(p - 2q)$ is an integer.

Thus we arrive at a contradiction.

Hence $\sqrt{8}$ is not a rational number.

Assignment: Show that there is no rational number whose square is (i) 3 (ii) 6 (iii) 12 (iv) 18

THEOREM: $\sqrt{n}$ is irrational number.

PROOF: case I when n contain no square factor. let $\sqrt{n} = \frac{p}{q}$; $q \neq 0$, p, q are positive integers and $(p, q) = 1$

"SELF"

Case II Let n be a composite number.

Suppose $\sqrt{n} = \frac{p}{q}$, $q \neq 0$, p, q are positive integers which are co-prime to each other.

Then there exist consecutive integers r and $r+1$ such that

$$r < \sqrt{n} < r+1$$

$$\Rightarrow \quad r < \frac{p}{q} < r+1$$

$$\Rightarrow \quad rq < p < (r+1) q$$

$$\Rightarrow \quad 0 < p - rq < q$$

Shows that $p - rq$ is a positive integer less than q.

Then $\sqrt{n}(p - rq) = \frac{p}{q}(p - rq)$

is not an integer.

But $\sqrt{n}(p - rq) = \frac{p^2}{q} - rp$

$$= \frac{p^2}{q^2} \cdot q - rp$$

$$= nq - rp, \text{ an integer.}$$

arrives a contradiction.

Hence $\sqrt{n}$ is an irrational number.

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Assignment:

① If r is rational and x is irrational then $r+x$ and rx are irrational.

OR The sum and product of non-zero rational and irrational is irrational.

② Show that between any two rational numbers there is another rational number.

$r < s \Rightarrow r < \frac{s+r}{2}$ & $\frac{s+r}{2} < s \Rightarrow r < \frac{r+s}{2} < s.$

③ Show that between two irrational numbers there exist an irrational number.

④ Show that there exist infinite irrational between two irrational numbers. Same as above keeping r, s irrational

(Show $\frac{r+s}{2} < s$ contain another and ...)

⑤ Show that between a rational and irrational numbers, there exist an irrational number. (Same as 2)

⑥ Show that between any two rational there exist infinite number of rational numbers.

For two irrational, contain ration and conversely,

(i) $r < \frac{r+s}{2} < s \Rightarrow r < q < s \Rightarrow r < \sqrt{q} < s.$

(ii) $\sqrt{r} < \sqrt{s}$ ~~But~~ $\sqrt{r} < r$ —(i) & $\sqrt{s} < s$ —(ii)

$\Rightarrow \sqrt{r} < r - s < r \Rightarrow \sqrt{r} < r < r+s$

$\Rightarrow \sqrt{r} < r$

Contrapositive:

$p \to q \Leftrightarrow \neg q \to \neg p$

Give Detail

Example: If n^2 is odd then n is odd ineq

Its contrapositive statement is

if "n is not odd then n^2 is not odd"

That is "If n is even then n^2 is even"

Assignment:

① If n is an integer and n^3+5 is odd then n is even.

② Square of an even number is even

③ If n is an integer and $3n+2$ is even, then n is even.

④ $7n+4$ is even if n is even

⑤ $5n+6$ is odd if n is odd.

Terminology:

⓪ Let P be the non-empty subset of $\mathbb{R}$, called the set of positive real numbers. Then

① $\{a : a \in P\}$ = Set of positive real numbers

② $\{a : a \in P\} \cup \{0\}$ = Set of non-negative real numbers

③ $\{-a : a \in P\}$ = Set of negative real numbers

④ $\{-a : a \in P\} \cup \{0\}$ = Set of non-positive real numbers.

ORDERED SETS:

Definition: Let S be a set. An ordered on S is a relation, denoted by $<$, with the following two properties

(i) If $x, y \in S$ then one and only one of the statements $x < y$, $x = y$, $y < x$ is true (Law of Trichotomy)

(ii) If $x, y, z \in S$, if $x < y$ and $y < z$ then $x < z$ (Transitive law)

~~REMARK~~:

Definition: An ordered set is a set S in which an order is defined.

e.g Q is an order set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

Ordered Field:

A field S is an ordered field if it satisfy the following properties

1. Law of Trichotomy
2. Transitivity
3. Compatibility of order relation with addition composition

$$\forall \; a, b, c \in S$$
$$a > b \Rightarrow a + c > b + c$$

(4) $\forall\ a, b, c \in S$

$a > b$ and $c > 0$ imply $ac > bc$

e.g Q and R are ordered fields but N and Z ($\circ I$) are not ordered field.

REMARK: Ordinarily the order relation does not exist between the members of a general field, but as we are to deal with the field of real numbers, we can speak of one number being 'greater than (or less than)' the other. The order properties of R refer to the notions of positivity and inequalities between real numbers.

PROPOSITION: The axioms for addition (in the definition of Field) imply the following statements

(i) If $x + y = x + z$ then $y = z$

(ii) If $x + y = x$ then $y = 0$

(iii) If $x + y = 0$ then $y = -x$

(iv) If $-(-x) = x$. In particular $-(-1) = 1$

PROOF: Consider

$y = 0 + y$ (additive identy)

$= (-x + x) + y$

$= -x + (x + y)$ (Associative law)

$= -x + (x + z)$ Given

$= (-x+x)+z$ (Associative law)

$= 0+z$

$= z$ (additive identity)

Hence $y = z$

(ii) We know

$x+y = x+z \Rightarrow y = z$

put $z=0$, Then

$x+y = x+0 \Rightarrow y=0$

i-e $x+y = x \Rightarrow y=0$

(iii) We know $x+y = x+z$ imply $y=z$

put $z=-x$

Then $x+y = x+(-x)$ imply $y=-x$

i-e $x+y = 0 \Rightarrow y=-x$

(iv) Since

$-x+x=0$ Apply (iii)

Then $-x+x=0 \Rightarrow x=-(-x)$

put $x=1$, in particular

$-(-1) = 1$ proved.

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Proposition:

The axioms for multiplication (in the definition) imply the following statements.

(i) If $x \neq 0$ and $xy = xz$ then $y = z$

(ii) If $x \neq 0$ and $xy = x$ then $y = 1$

(iii) If $x \neq 0$ and $xy = 1$ then $y = \frac{1}{x}$

(iv) If $x \neq 0$ then $\frac{1}{\left(\frac{1}{x}\right)} = x$

(v) If $xy = 0$ then either $x = 0$ or $y = 0$

PROOF: To prove above proposition, First we will show if $x \neq 0$, then $\frac{1}{x} \neq 0$

Suppose $\frac{1}{x} = 0$

Then $1 = x \cdot \frac{1}{x}$

$= x \cdot 0$

$= 0$

$\Rightarrow 1 = 0$, not possible

Hence if $x \neq 0$, then $\frac{1}{x} \neq 0$

That is if $x \neq 0$, then $\frac{1}{x}$ exist

(i) Consider

$y = 1 \cdot y$ (multiplicative identity)

$= \left(x \cdot \frac{1}{x}\right) y$; $x \neq 0$

$= \left(\frac{1}{x} x\right) y$

$= \frac{1}{x}(xy)$ Associative law

$= \frac{1}{x}(xz)$ (Given)

$= \cancel{x} \cdot \frac{1}{\cancel{x}}$

$= \left(\frac{1}{x} \cdot x\right) z$ Associative law

$= 1 \cdot z$

$= z$ (Multiplicative identity)

Hence $y = z$

(ii) Take $z = 1$ in (i) then

$xy = x$ imply $y = 1$

Alternative:

Consider

$y = 1 \cdot y$ (Multiplicative identity)

$= \left(\frac{1}{x} \cdot x\right) y$

$= \frac{1}{x}(xy)$ (Associative law)

$= \frac{1}{x} x$ Given

$= 1$ (Inverse element)

Hence $y = 1$

(iii) Replace z by $\frac{1}{x}$ in (i)

$xy = x \cdot \frac{1}{x}$ imply $y = \frac{1}{x}$

or $xy = 1$ imply $y = \frac{1}{x}$

(iv) we know $\frac{1}{x} \cdot x = 1$

Then $\frac{1}{x} x = 1$ imply $x = \frac{1}{\frac{1}{x}}$

(Replace x by $\frac{1}{x}$ in iii).

(v) Let $xy = 0$ and $x \neq 0$, then $\frac{1}{x} \neq 0$

So that $\frac{1}{x} x = 1$

Then $y = 1 \cdot y$ (multiplicative identity)

$= \left(\frac{1}{x} \cdot x\right) y$

$= \frac{1}{x}(xy)$ (Associative law)

$= \frac{1}{x} \cdot 0$ (Given)

$= 0$

Hence $y = 0$

Again suppose ~~$xy=0$~~

$xy = 0$ and $y \neq 0$

then $\frac{1}{y} \neq 0$

So that $y \cdot \frac{1}{y} = 1$

Consider $x = x \cdot 1$ multiplicative identity

$= x\left(y \cdot \frac{1}{y}\right)$

$= (xy)\frac{1}{y}$ Associative law

$= 0 \cdot \frac{1}{y}$ Given

$= 0$

Hence $x = 0$

Proposition: The field axioms imply the following statements. For any $x, y, z \in F$

(i) $0x = 0$

(ii) If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$

(iii) $(-x)y = -(xy) = x(-y)$

(iv) $(-x)(-y) = xy$ In particular

$(-1)(-1) = 1 \cdot 1$

Assignment: Hint See Rudin

INEQUALITIES

THEOREM: Let a, b, c be any elements of $\mathbb{R}$

(i) If $a > b$ and $b > c$, then $a > c$

(ii) If $a > b$ then $a + c > b + c$

(iii) If $a > b$ and $c > 0$, then $ca > cb$

If $a > b$ and $c < 0$, then $ca < cb$

PROOF: Since $a > b$ and $b > c$

(i) So $a - b > 0$ and $b - c > 0$

Then $a - b + b - c > 0$

imply $a - c > 0$

$\Rightarrow a > c$.

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(ii) Consider

$$(a+c)-(b+c) = a-b$$

$$> 0 \quad \because a > b$$

then $(a+c)-(b+c) > 0$

i.e $a+c > b+c$

(iii) let $a > b$

$\Rightarrow a-b > 0$

$\Rightarrow c(a-b) > 0 \quad \because c > 0$

$\Rightarrow ca - cb > 0$

$\Rightarrow ca > cb$

if $c < 0$

then $c(a-b) < 0$

$\Rightarrow ca - cb < 0$

$\Rightarrow ca < cb.$

THEOREM: Prove the following.

(i) If $x > 0$ then $-x < 0$, and vice versa

(ii) If $x > 0$ then $\frac{1}{x} > 0$ and

if $x < 0$ then $\frac{1}{x} < 0$

(iii) If $x \neq 0$ then $x^2 > 0$

In particular $1 > 0$

(iv) If $0 < x < y$ then $0 < \frac{1}{y} < \frac{1}{x}$

(v) If x and y are positive then
$x < y$ iff $x^2 < y^2$ $\quad \forall x, y \in R.$

(vi) If $n \in N$, then $n > 0$

PROOF:

(i) If $x > 0$ then $0 = -x + x > -x + 0$
Thus $-x < 0$
If $x < 0$ then $0 = -x + x < -x + 0$
Thus $-x > 0$

(ii) Suppose $x > 0$ and $\frac{1}{x} \leq 0$
But $\frac{1}{x} = 0$ is not true because

$$0 = 0 \cdot x$$
$$= \frac{1}{x} \cdot x = 1$$

Hence $\frac{1}{x} < 0$

$\Rightarrow x \cdot \frac{1}{x} < 0 \cdot x \quad \because x > 0$

$\Rightarrow 1 < 0$ (not possible) a contradiction.

Hence $\frac{1}{x} > 0$

(iii) By law of trichotomy, if $x \neq 0$ then
either $x > 0$ or $x < 0$
If $x > 0$, then $x^2 = x \cdot x > 0$
If $x > 0$ then $-x > 0$ by (i)
so $x^2 = (-x)(-x) > 0$ Hence if $x \neq 0$ then $x^2 > 0$

Since $1 = 1^2$ and $1 \neq 0$

Hence $1 > 0$

(iv) Suppose $0 < x < y$ —— ①

Since $x > 0 \Rightarrow \frac{1}{x} > 0$

and $y > 0 \Rightarrow \frac{1}{y} > 0$

then $\frac{1}{x} \cdot \frac{1}{y} > 0$

Multiplying ① by $\frac{1}{x} \cdot \frac{1}{y}$, we get

$$0 < \frac{1}{x}\frac{1}{y} x < \frac{1}{x} \cdot \frac{1}{y} y$$

Then $0 < \frac{1}{y} < \frac{1}{x}$ Required.

(v) Suppose $x < y$

$\Rightarrow x^2 < xy$ —① $\because x > 0$

Again $x < y$

$\Rightarrow xy < y^2$ —— ② $\because y > 0$

① & ② imply $x^2 < xy < y^2$

Thus $x^2 < y^2$

Conversely suppose $x^2 < y^2$

then $y^2 - x^2 > 0$

i.e $(y-x)(y+x) > 0$ —— ①

Since $x > 0$ & $y > 0$ so

$x + y > 0$ imply $\frac{1}{x+y} > 0$

Then ① becomes

$$\frac{1}{x+y}(y+x)(y-x) > 0$$

imply $y-x>0$

i-e $y>x$

or $x<y$.

Conversely (Aliter):

Suppose $x^2<y^2$

But $x \geq y$

Then $x^2 \geq xy$ — ① $\because x>0$

and $x \geq y$

$\Rightarrow xy \geq y^2$ — ② $\because y>0$

① & ② imply $x^2 \geq xy \geq y^2$

i-e $x^2 \geq y^2$ contradiction

against $x^2<y^2$.

Hence $x<y$.

(vi) We use Mathematical induction,

The assertion for $n=1$ is true by (iii)

Suppose statement is true for $n=k$. ; $k \in N$

That is $k>0$

Then $k+1>0$ $\because 1>0$ & $k>0$

Hence by Mathematical Induction

$\forall n \in N$, then $n>0$

THEOREM: If $ab>0$, then either

(i) $a>0$ and $b>0$ or

(ii) $a<0$ and $b<0$

Proof: Suppose $ab>0$ then $a\neq 0$ and $b\neq 0$

otherwise $ab=0 \not> 0$

Since $a\neq 0$, by law of trichotomy

either $a>0$ or $a<0$

If $a>0$ then $\frac{1}{a}>0$ and therefore

$b=(\frac{1}{a})(ab)>0 \quad \because \frac{1}{a}>0$ & $ab>0$

Thus if $a>0$ then $b>0$ whenever $ab>0$

Similarly, if $a<0$ then $\frac{1}{a}<0$

then $b=\frac{1}{a}(ab)<0$

Thus if $a<0$ then $b<0$ whenever $ab>0$

Corollary: If $ab<0$ then either

(i) $a<0$ and $b>0$, or

(ii) $a>0$ and $b<0$

Proof: If $ab<0$ then $a\neq 0$ and $b\neq 0$

otherwise $ab=0 \not< 0$

If $a<0$ then $\frac{1}{a}<0$

Thus $b=\frac{1}{a}(ab)>0 \quad \because \frac{1}{a}<0$ & $ab<0$

Similarly (ii) can be treated.

Exercise: If $a < b+\varepsilon$ for every $\varepsilon > 0$ then $a \leq b$

Solution: Assume that $a > b$ then
$a - b > 0$ But for any $\varepsilon > 0$

$a < b+\varepsilon$ (Given) —— ①

Take $\varepsilon = a-b > 0$

then ① becomes

$a < b + a - b$ imply $a < a$

a contradiction. Hence $a > b$ must be false. therfore $a \leq b$.

Exercise: If $a \in R$ is such that $0 \leq a < \varepsilon$ for every $\varepsilon > 0$ then $a = 0$

Sol On contrary, suppose $a > 0$

Take $\varepsilon = \frac{1}{2} a$

then $0 < \varepsilon = \frac{1}{2}a < a$

i.e $\varepsilon < a$, contradiction

as $a < \varepsilon$.

Hence $a = 0$ $\because a \geq 0$ and $a > 0$ is false.

Assignment: Solve inequalities
BOOK T. Finny.

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BOUNDED AND UNBOUNDED SETS: SUPRIMUM, INFIMUM

DEFINITIONS:

① Suppose S is an ordered set and $E \subset S$ (non-em...
If there exist a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is bounded above, and call β an upper bound of E.

② Suppose S is an ordered set and E be a non-empty subset of S. If there exist a $\beta \in S$ such that $x \geq \beta$ for every $x \in E$. We say that E is bounded below, and call β as lower bound of E.

③ A set is said to be bounded if it is both bounded above and bounded below.

④ A set is said to be unbounded if it is not bounded.

⑤ Suppose S is an ordered set, $E \subset S$ and E is bounded above. Suppose there exist an $\alpha \in S$ with the following properties

(i) α is an upper bound of E

(ii) If $\gamma < \alpha$ then γ is not an upper bound of E.

Then α is called least upper bound of E (That there is atmost one such α is clear from (ii)) or the Suprimum

of E, and we write
$$\alpha = \sup E = l.u.b\, E$$

Alternative: Let E be a non-empty subset of an order set S. A number a in S is said to be least upper bound or sup for E if

(i) a is an upper bound for E.
i-e $x \leq a \quad \forall x \in E$.

(ii) If b is an upper bound for E, then $a \leq b$.

(6) Suppose S is an order set, $E \subset S$, and E is bounded below. Suppose there exist an $\alpha \in S$ with the following properties:

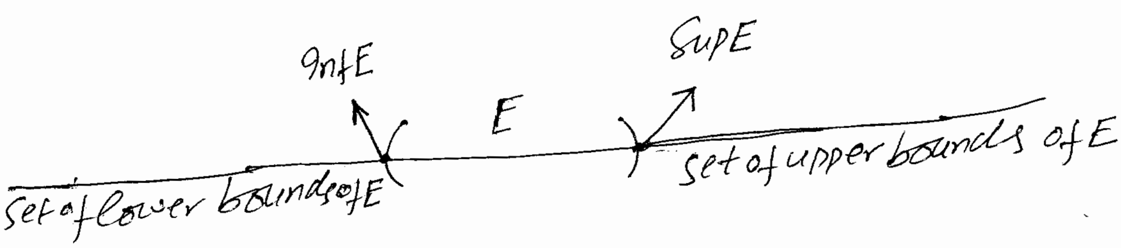
(i) α is a lower bound of E

(ii) If $\beta > \alpha$ then β is not a lower bound of E. Then $\alpha = \inf E = glb\, E$

Alternative: Let E be a non-empty subset of an ordered set S. A number a in S is said to be greatest lower bound for E if

(i) a is a lower bound for E

(ii) If b is a lower bound for E then $b \leq a$



REMARK:

① The set N of natural numbers is bounded below but not bounded above.

② The sets $\mathbb{Z}$, Q and R are not bounded.

③ Every finite set of numbers is bounded.

④ The set of positive ~~integers~~ numbers $\{x : x > 0, x \in R\}$ is not bounded above, but is bounded below. The infimum is zero is not the member of this set.

⑤ The infinite set $\{x : 0 < x < 1, x \in R\}$ is bounded with $\inf = 0$, $\sup = 1$ and both do not belong to set.

⑥ The infinite set $\{x : 0 \le x \le 1, x \in R\}$ is bounded with $\inf = 0$, $\sup = 1$ and both belong to set.

⑦ The infinite set $\{x : 0 < x \le 1\}$ is bounded with $\inf = 0$, $\sup = 1$ and inf does not belong to set.

⑧ The infinite set $\{x : 0 \le x < 1\}$ is bounded but $\sup = 1$ does not belong to set

⑨ ⑤ to ⑧ asserts $[a,b]$, (a,b), $(a,b]$ and $[a,b)$ all are bounded with

$$\sup[a,b] = \sup(a,b) = \sup(a,b] = \sup[a,b) = b$$

and $\inf[a,b] = \inf(a,b) = \inf(a,b] = \inf[a,b) = a$

⑩ If E is finite, then $\operatorname{lub} E = \max E$ & $\operatorname{glb} E = \min E$

Assignment:

(1) Find the infimum and Suprimum of the following sets. Which of these belong to the set. Which of these are bounded.

(i) $\{1, 3, 5, 7, 9\}$ (ii) $\{-1, -\frac{1}{2}, \dots\}$

(iii) $\{\frac{1}{n}; n \in N\}$ (iv) $\{\frac{(-1)^n}{n}; n \in N\}$

(v) $\{-2, -\frac{3}{2}, -\frac{4}{3}, \dots, -\frac{n+1}{n}, \dots\}$

(vi) $\{1 + \frac{(-1)^n}{n} : n \in N\}$

(vii) $]a, b[$ (viii) $[a, b)$

Ans (i) 1, 9 ; both (ii) $-1, 0$; infimum

(iii) 0, 1 ; sup (iv) $-1, \frac{1}{2}$; both

(v) $-2, -1$ Inf (vi) $0, \frac{3}{2}$; both

(vii) a, b ; none (viii) a, b ; inf

All bounded.

(2) Find the smallest and the greatest members (if they exist) for the sets.

(i) 1, 9 (ii) -1, does not exist

(iii) does not exist, 1 (iv) $-1, \frac{1}{2}$

(v) -2, does not exist (vi) $0, 3/2$

(vii) do not exist (viii) a, does not exist.

H M KHALID MAHMOOD

(ii) If S is infinite and ... upper bound ... $\in S$ then it is referred to as (greatest member) or maximum of S and define for min

Lemma: If it exists, the supremum (infimum) of a non-empty subset of an ordered field is unique.

Proof: Suppose that A is a subset of an ordered field. Suppose (if possible) s_1 and s_2 be the sups of A, i.e. $\sup A = s_1$ & $\sup A = s_2$

Since s_2 is an upper bound of A and $s_1 = \sup A$ therefore $s_1 \leq s_2$ —(1), similarly since s_1 is an upper bound of A and $s_2 = \sup A$, therefore $s_2 \leq s_1$ —(2)

Thus (1) & (2) asserts $s_1 = s_2$

For infimum go on parallel lines.

Exercise: Let n be a positive integer, then no positive integer m satisfy the inequality $n < m < n+1$

Sol: Suppose that there exist $m \in P$ (the set of positive integers) such that $n < m < n+1$

Then for $n < m$ imply $m - n \in P$

Also $m < n+1$ imply $m - n < 1$; $m - n \in P$

which is a contradiction against the fact that $n' \in P$ imply $n' \geq 1$

Hence there exist no integer $m \in P$ such that $n < m < n+1$.

(DO LATER) *****

THEOREM: The set of positive integers is not bounded above.

Proof: On contrary, suppose P (the set of positive integers) is bounded above. then suppose a be the least upper bound of P, imply $a-1$ is not an upper bound therefore their exist ~~for~~ $n \in N$

$$a-1 < n$$

$$\Rightarrow a < n+1 \text{ which contradicts}$$

~~imply a is not~~

the assumpsion that a is an upper bound for P. Hence P is not bounded above.

Definition (Complete ordered field)

An ordered field F is said to be complete if every non-empty subset of F which is bounded above, has a suprimum in F.

Least upper bound property (Suprimum property) of R.

Every non-empty set of real numbers that has an upper bound also has a suprimum in R.

Greatest lower bound (Infimam) property:

Every non-empty set of real numbers that has a lower bound also has infimum in R.

Remark: ① The sup and inf property of real numbers is also known as continuum property.

② The set of real numbers is the only set which is a complete ordered field.
(N, Z are not field, so are not complete ordered field).

THEOREM: The set of rational numbers is not a complete ordered field.

Proof: We will show that there exist a non-empty set S of Q which is bounded above but has no rational sup.

Let $S = \{x : x \in Q \,,\; x > 0 \;\&\; x^2 < 2\}$

Since $1 > 0$ & $1^2 < 2$ imply $1 \in S$
Hence S is non-empty. Further S is bounded above.

Let (if possible)
$$\sup S = \alpha \quad ; \quad \alpha \in Q.$$

Clearly α is positive.
Then by laws of trichotomy, there are three possibilities

(i) $\alpha^2 = 2$ (ii) $\alpha^2 < 2$ (iii) $\alpha^2 > 2$

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(i) We know that there does not exist any rational number whose square is 2. Thus case (i) is not possible.

(ii) Let y be the positive rational number s.t $y = \dfrac{4+3\alpha}{3+2\alpha}$

$\cancel{y^2 < 2}$

Then
$$\alpha - y = \alpha - \frac{4+3\alpha}{3+2\alpha}$$
$$= \frac{3\alpha + 2\alpha^2 - 4 - 3\alpha}{3+2\alpha}$$
$$= \frac{2(\alpha^2-2)}{3+2\alpha}$$
$$< 0 \quad \because \alpha^2 < 2 \text{ imply } \alpha^2 - 2 < 0.$$

i.e $\alpha - y < 0$

or $y > \alpha$. Also

$$2 - y^2 = 2 - \left(\frac{4+3\alpha}{3+2\alpha}\right)^2$$
$$= \frac{2-\alpha^2}{(3+2\alpha)^2} > 0 \quad \because 2-\alpha^2 > 0$$

$\Rightarrow y^2 < 2$ imply $y \in S$.

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Since $y \in S$, $y > \alpha$. Thus α cannot be an upper bound of S and hence there is a contradiction.

(iii) If $\alpha^2 > 2$ then consider a positive real number y s.t

$$y = \frac{4+3\alpha}{3+2\alpha}$$

then
$$\alpha - y = \alpha - \frac{4+3\alpha}{3+2\alpha}$$
$$= \frac{2(\alpha^2-2)}{3+2\alpha}$$
$$> 0 \quad \because \alpha > 0 \text{ \& } \alpha^2 - 2 > 0$$

$\Rightarrow \quad y < \alpha.$

Also
$$2 - y^2 = \frac{2-\alpha^2}{(3+2\alpha)^2}$$
$$< 0 \quad \because 2 - \alpha^2 < 0$$

$\Rightarrow \quad y^2 > 2.$

$\Rightarrow$ y is an upper bound of S.

Thus there exist an upper bound of S s.t $y < \alpha = l.u.b.$

Which is a contradiction.
Thus in all three cases there does not exist $\alpha \in Q$ s.t $\sup S = \alpha \notin Q$.
that is S is bounded above and has no sup in Q. Hence Q is not complete ordered field.

THEOREM: (Existance of irrational number)

There exist a positive real number α such that $\alpha^2 = 2$.

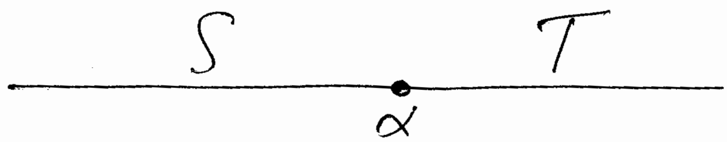
OR Define $S = \{x : x \in Q, x > 0 \text{ and } x^2 < 2\}$
Find $\sup S$.

PROOF: Suppose $\sup S = \alpha$.
Then (i) $\alpha^2 < 2$ (ii) $\alpha^2 > 2$ (iii) $\alpha^2 = 2$.

(i) & (ii) are not possible as explain. Hence $\sup S = \alpha = \sqrt{2}$. On the other hand (For existance of $\bar{Q}$)

(iii) $\alpha^2 = 2$ i.e l.u.b α exist whose square is 2 and $\alpha \notin Q$.
It follows that α is an irrational.
Thus there exist an irrational α s.t $\alpha^2 = 2$.

Remark: There is a close relation between greatest lower bounds and least upper bounds, and that every ordered set with the least upper bound property also has the greatest-lower-bound property



eg If S is bounded above then T, the set of upper bound of S is bounded below and both are the subset of an ordered field F, so have the l.u.b and g.l.b property. ~~for a com~~ Thus in this case

$$\sup S = \inf T = \alpha.$$

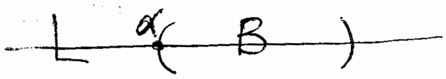
THEOREM: Suppose S is an ordered set with the least upper bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the all set of lower bounds of B, then

$$\alpha = \sup L \text{ exist in } S \quad \text{and}$$

$$\alpha = \inf B.$$

In particular $\inf B$ exists in S.

PROOF:



Since B is bounded below and L is the set of all lower bounds of B, so L is non-empty subset of S.

Since L consists of exactly those $y \in S$ which satisfy $y \leq x$ for every $x \in B$.

We see that every $x \in B$ is an upper bound of L. Thus L is bounded above.

Since L is a non-empty subset of S and is bounded above and S has least upper bound property, so supremum of L exists in S. Let it be α. Then $\sup L = \alpha$.

Now, if $\gamma < \alpha$, then γ is not an upper bound of L because $\sup L = \alpha$. Then $\gamma \notin B$

Thus $\alpha \leq x \ \forall \ x \in B$.

It follows that α is a lower bound of B.

Now, if $\beta > \alpha$ then β is an upper bound of L. Since $\sup L = \alpha$, therefore $\beta \notin L$.

But L is the set of lower bounds of B.

Hence β is not a lower bound of B

that is, we have shown

(i) α is a lower bound of B

(ii) If $\beta > \alpha$, then β is not a lower bound of B

thus by def

$$\alpha = \inf B.$$

or $$\inf B = \sup L.$$

Remark: Above theorem shows that least upper bound property and greatest lower bound property are equivalent.

THEOREM: Let A be a non-empty subset of real numbers which is bounded above. Define

$$-A = \{-x : x \in A\}, \text{ then } \sup(A) = -\inf(-A)$$

PROOF: Since A is a non-empty subset of real numbers ~~which~~ and is bounded above, so $\sup A$ exists in R because R has ~~greatest~~ least ~~lower~~ upper bound property. Let it be α i.e $\sup A = \alpha$.

$$\Rightarrow \alpha \geq x \quad \forall x \in A$$

$$\Rightarrow -\alpha \leq -x \quad \forall -x \in -A$$

$\Rightarrow -\alpha$ is a lower bound of $-A$.

Thus $-A$ is bounded below.

Since $-A$ is a non-empty, bounded below subset of R, so $\inf(-A)$ exists in R because R has glb property. Let it be β. i.e $\beta = \inf(-A)$

Since $-\alpha$ is a lower bound of $-A$ and β is the greatest lower bound of $-A$

Therefore $-\alpha \leq \beta$

$$\Rightarrow \alpha \geq -\beta \quad \text{——— (1)}$$

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Since $\beta = \inf(-A)$

$\Rightarrow \beta \leq -x \quad \forall -x \in -A$

$\Rightarrow -\beta \geq x \quad \forall x \in A$

$\Rightarrow -\beta$ is an upper bound of A.

and $\alpha = \sup A =$ least upper bound of A.

$\Rightarrow -\beta \geq \alpha$.

or $\alpha \leq -\beta$ —— (2)

(1) & (2) gives

$\alpha = -\beta$

Thus $\sup A = -\inf(-A)$.

Which complete the proof.

H.M KHALID MAHMOOD.

THEOREM:

Let A be a non-empty subset of real numbers which is bounded below, Define $-A = \{-x : x \in A\}$. Then $\inf A = -\sup(-A)$.

Proof: Since A is a non-empty subset of real numbers and is bounded above, so $\inf A$ exist in $\mathbb{R}$ because $\mathbb{R}$ has greatest lower bound property. Let it be α. i.e $\inf A = \alpha$.

$$\Rightarrow \alpha \leq x \quad \forall\, x \in A$$

$$\Rightarrow -\alpha \geq -x \quad \forall\, -x \in -A$$

It follows that $-\alpha$ is an upper bound of $-A$. Thus $-A$ is bounded above.

~~Then~~ Since $-A$ is a non-empty, bounded below subset of $\mathbb{R}$, so $\sup(-A)$ exists in $\mathbb{R}$ because $\mathbb{R}$ has least upper bound property. Let it be β. i.e $\sup(-A) = \beta$

Since $-\alpha$ is an upper bound of $-A$ and $\beta = \sup(-A)$, so $-\alpha \geq \beta$

$$\Rightarrow \alpha \leq -\beta \quad \text{--- (1)}$$

Again

Since $\beta = \sup(-A)$

$$\Rightarrow \beta \geq -x \quad \forall\, -x \in -A.$$

$$\Rightarrow -\beta \leq x \quad \forall\, x \in A.$$

$\Rightarrow -\beta$ is a lower bound of A

and $\alpha = \operatorname{Inf} A$

Thus $-\beta \leq \alpha$ —— (2)

(1) & (2) imply

$$\alpha = -\beta$$

i.e ~~$\operatorname{Sup} A = -\operatorname{Inf}(-A)$~~.

$$\operatorname{Inf} A = -\operatorname{Sup}(-A)$$

Corollary:

If a non-empty subset of a complete ordered field is bounded below, then it has inf in F.

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Corollary: If a non-empty subset of a complete ordered field F is bounded below, then it has infimum in F.

Proof: Let E be a non-empty subset of a complete ordered field F such that it is bounded below.

Then $-E$ is bounded above.

We know $\text{Inf } E = -\text{Sup}(-E)$ —— ①

Since $-E$ is bounded above and F possesses least upper bound property, so $\text{Sup}(-E)$ exist in F and by using ①. $\text{Inf } E$ exist in F $\because$ F is a field (i-e for $\alpha \in F$ imply $-\alpha \in F$)

Which complete the proof.

THEOREM: Suppose that S is a non-empty set of real numbers which is bounded above and $\alpha > 0$ then
$$\sup_{x \in S} \alpha x = \alpha \sup_{x \in S} x$$

OR $\text{Sup } \alpha S = \alpha \text{ Sup } S$.

PROOF: Suppose $\sup_{x \in S} x = \beta$

i-e $\text{Sup } S = \beta$.

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Then $x \leq \beta \quad \forall x \in S$ ——— ①

Define $T = \{\alpha x : x \in S\} (= \alpha S)$

By ① $\alpha x \leq \alpha \beta \quad \because \alpha > 0$

Then $\alpha\beta$ is an upper bound of T, so T is bounded above. Further T is non-empty because $\alpha > 0$ and S is non-empty.

Since T is non-empty, bounded above subset of $\mathbb{R}$, so by continuum property, it has least upper bound (Suprimum). Let it be γ.

i-e $\quad \sup T = \gamma$

or $\quad \sup\limits_{x \in S} \alpha x = \gamma$

We will show $\gamma = \alpha\beta$.

Since $\alpha\beta$ is an upper bound for T and γ is the least upper bound for T, so

$$\gamma \leq \alpha\beta \quad ——— ①$$

Since $\alpha > 0$, so α^{-1} exist and we conclude from T

$$S = \alpha^{-1} T$$

$$= \{\alpha^{-1} \cdot \alpha x : x \in S\}$$

$$= \{\alpha^{-1} y : y \in T\}$$

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Since $\sup T = \gamma$

$\Rightarrow \quad \gamma \geq y \quad \forall y \in T$

$\Rightarrow \quad \alpha^{-1}\gamma \geq \alpha^{-1}y$

i.e $\quad \alpha^{-1}y \leq \alpha^{-1}\gamma \quad \forall y \in T$

imply $\alpha^{-1}\gamma$ is an upper bound for T (S)

But $\sup \overset{S}{T} = \beta$

Therefore $\quad \alpha^{-1}\gamma \geq \beta$

or $\quad \gamma \geq \alpha\beta$ —— ②

Then ① & ② yields

$\gamma = \alpha\beta$, which complete the proof.

THEOREM: Suppose S is a non-empty set of real numbers which is bounded below and $\alpha > 0$. Then $\inf\limits_{x \in S} \alpha x = \alpha \inf\limits_{x \in S} x$

i.e $\quad \inf \alpha S = \alpha \inf S$

PROOF:

Left as an assignment.

THEOREM: Let S be a non-empty bounded set in $\mathbb{R}$. let $\alpha < 0$. then

(i) $Sup(\alpha S) = \alpha \, Inf \, S$

(ii) $Inf(\alpha S) = \alpha \, Sup \, S$

PROOF: Suppose $Inf \, S = \beta$

Then $\beta \leq x \quad \forall \, x \in S$. —— ①

Define $T = \{\alpha x : x \in S\}$; $\alpha < 0$

Since $\alpha < 0$, ① imply

$$\alpha\beta \geq \alpha x$$

i-e $\alpha x \leq \alpha\beta$; $x \in S$

Thus T is bounded above by $\alpha\beta$.

Further T is non-empty because $\alpha < 0$ and S is non-empty.

Since T is non-empty, bounded above subset of $\mathbb{R}$, so by continuum property, it has least upper bound (Suprimum).

let it be γ. i-e $\underset{x \in S}{Sup}(\alpha x) = \gamma$.

Will will show $\gamma = \alpha\beta$

Since $\alpha\beta$ is an upper bound for T and γ is the least upper bound for T

so $\gamma \leq \alpha\beta$ —— ①

Since $\alpha < 0$ imply $\alpha \neq 0$ imply α^{-1} exist and $\alpha^{-1} < 0$

Then from T, we have

$$S = \alpha^{-1} T$$
$$= \{\alpha^{-1} \alpha x : x \in S\}$$
$$= \{\alpha^{-1} y : y \in T\}$$

Since $\sup T = \gamma$

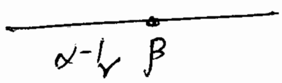
So $\gamma \geq y \quad \forall\, y \in T$

$\Rightarrow \alpha^{-1}\gamma \leq \alpha^{-1} y \quad \because \alpha^{-1} < 0$

or $\alpha^{-1} y \geq \alpha^{-1}\gamma$

i-e $\alpha^{-1}\gamma$ is a lower bound for S.

But $\inf S = \beta$, greatest lower bound for S.



So $\alpha^{-1}\gamma \leq \beta$

$\Rightarrow \gamma \geq \alpha\beta \quad \because \alpha^{-1} < 0$ —— ②

① & ② imply $\gamma = \alpha\beta$

i-e $\sup T = \alpha \inf S$

or $\sup(\alpha S) = \alpha \inf S$.

(ii) Left as an assignment.

EXERCISE: Let S be a non-empty bounded subset of real numbers and a be any number in R. Define $a+S=\{a+s : s\in S\}$, Then

(i) $\sup(a+S) = a+\sup S$

(ii) $\inf(a+S) = a+\inf S$

Sol: Suppose $\alpha = \sup S$, then $x \le \alpha$ $\forall x\in S$.

So that $a+x \le a+\alpha$ for $a\in R$

Then $a+\alpha$ is an upper bound of $a+S$.

Thus $a+S$ is non-empty bounded above subset of R, by Completeness property it has sup in R. Let it be ν.

i-e $\sup(a+S) = \nu$. (least upper bound)

Then $\nu \le a+\alpha$ —① $\because$ $a+\alpha$ is an upper bound.

Since $\sup(a+S) = \nu$

$\Rightarrow$ $a+s \le \nu$ $\forall s\in S$.

$\Rightarrow$ $s \le \nu - a$ $\forall s\in S$.

$\Rightarrow$ $\nu - a$ is an upper bound for S

and α is the least upper bound for S

Thus $\alpha \le \nu - a$

or $a+\alpha \le \nu$ —②

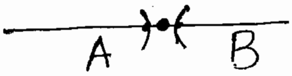
① & ② imply $\nu = a+\alpha \Rightarrow \sup(a+S) = a+\sup S$

(ii) Left as an assignment.

Exercise:

① Suppose A and B are non-empty subsets of R that satisfy the property $a \leq b$ $\forall a \in A$ & $\forall b \in B$ then $\sup A \leq \inf B$.

Solution:



We are given, For any b

$$a \leq b \quad \forall a \in A$$

imply b is an upper bound of A,

So that $\sup A \leq b$ —— ①

Since ① hold for all $b \in B$

So $\sup A$ is a lower bound for B.

Then $\sup A \leq \inf B$ = Greatest lower bound for B

i-e $\sup A \leq \inf B$.

② Suppose A and B ~~are~~ non-empty subsets of R that satisfy the property $a \geq b$ $\forall a \in A$ & $\forall b \in B$

Then $\inf A \geq \sup B$

Sol Left as an assignment.

H.M KHALID MAHMOOD

Exercise: Let A and B be bounded non-empty subsets of $\mathbb{R}$, and let

$$A+B = \{a+b : a \in A, b \in B\}$$

Then

(i) $\sup(A+B) = \sup A + \sup B$

(ii) $\inf(A+B) = \inf A + \inf B$

Solution (i) Suppose $\sup A = \alpha$ & $\sup B = \beta$

Then $\alpha \geq a \quad \forall a \in A$

& $\beta \geq b \quad \forall b \in B$

Then $a+b \leq \alpha+\beta$

But $a+b \in A+B$

Thus $\alpha+\beta$ is an upper bound of $A+B$, That is $A+B$ is bounded above.

Thus $\sup(A+B) \leq \alpha+\beta$ —— ①*

Let $\sup(A+B) = t$ i.e $t \leq \alpha+\beta$ —— ①

Then $t \geq a'+b' \quad \forall a' \in A$ & $\forall b' \in B$

imply $t \geq a'$ and $t \geq b'$

Thus t is an upper bound for both A & B. But α, β are their (A & B) respective lub's

So $t \geq \alpha$ and $t \geq b$

imply $t \geq$

imply $a'+b' \le t \quad \forall\, a' \in A$ & $b' \in B$

$\Rightarrow a' \le t-b' \quad \forall\, a' \in A$

$\Rightarrow t-b'$ is an upper bound for A and $\operatorname{Sup} A = \alpha$. (l.u.b)

Thus $\alpha \le t-b'$

$\Rightarrow b' \le t-\alpha \quad \forall\, b' \in B$

$\Rightarrow t-\alpha$ is an upper bound for B and $\operatorname{Sup} B = \beta$. (l.u.b)

Thus $\beta \le t-\alpha$

or $\alpha+\beta \le t$ ——— (2)

From (1) & (2)

$t = \alpha+\beta$

i-e $\operatorname{Sup}(A+B) = \operatorname{Sup} A + \operatorname{Sup} B$.

(ii) Left as an assignment.

(iii) Check $\operatorname{Sup}(AB) = \operatorname{Sup} A \operatorname{Sup} B$. &
$\operatorname{Inf}(AB) = \operatorname{Inf} A \operatorname{Inf} B$

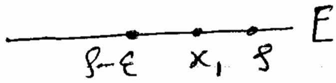
Left as an assignment

(iv) If A and B are two non-empty Subsets of R and for $\alpha \in R$ if $a+b \le \alpha \quad \forall\, a \in A$ & $b \in B$ Then $\operatorname{Sup} A + \operatorname{Sup} B \le \alpha$ (Hint proceede as in $\operatorname{Sup} A + \operatorname{Sup} B = \operatorname{Sup}(A+B)$)

Given $a \le \alpha - b \Rightarrow \operatorname{Sup} A \le \alpha - b$

THEOREM: Suppose that E is a subset of R which is bounded above and that $s = \sup E$. Then for every real number $\varepsilon > 0$ (epsilon), however small, there is an element $x_1 \in E$ such that $s - \varepsilon < x_1$

PROOF:



Let $\varepsilon > 0$ be any arbitrary real number.

Suppose there is no $x \in E$ such that $s - \varepsilon < x$.

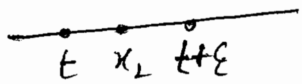
Then $x \leq s - \varepsilon \;\; \forall \, x \in E$.

It follows that $s - \varepsilon$ is an upper bound of E. Thus $\sup E \leq s - \varepsilon$.

i.e $s \leq s - \varepsilon$ which is absurd

Hence there exist atleast one element $x_1 \in E$ such that $s - \varepsilon < x_1$

(G.M)
Assignment: Suppose that E is a subset of R which is bounded below and that $t = \inf E$. Then for every real number $\varepsilon > 0$, however small, there is an element $x_2 \in E$ such that $x_2 < t + \varepsilon$



THEOREM (B.M) Every set of natural number which is non-empty has a minimum.

PROOF: Assume that the distance between ~~two~~ distinct natural numbers is atleast one.

Let S be a non-empty set of natural numbers.

Given that all natural numbers are positive, 0 is a lower bound for S. Thus S is bounded below. It follows that S has a largest lower bound (say) b.

Since b is the largest lower bound, $b+1$ is not a lower bound. therefor for some ~~$n\in N$~~ $n\in S$ $n< b+1$. If n is the minimum of S, there is nothing to prove. If not, then, for some $m\in S$, $m< n$, we obtain

$$b \leq m < n < b+1$$ follows the

contradiction $0 < n-m < 1$

(Hint $m < n < b+1$
$0 < n-m < b+1-b$ $\because m \geq b$)
i-e $0 < n-m < 1$
$4 < 5$
$4-2 < 5-1$ $\because 2 > 1$

thus m must be the min of S.

which complete the proof.

THEOREM (B.M) The set N of natural numbers is unbounded above.

PROOF: Suppose that theorem is false.
Then N is bounded above. Then it has smallest upper bound (say) b.
Since b is the smallest upper bound, so $b-1$ is not an ~~lower~~ upper bound for N.
then there exist some $n\in N$ s.t

$$n > b-1$$
$$\Rightarrow n+1 > b$$

But $n+1 \in N$ $\because n\in N$
and b is an upper bound of N
(Hint if b is an upper bound of n the $b \geq n$ $\forall n\in N$) but $b < n+1 \in N$)

This is a contradiction. Hence N is unbounded above.

① Exp (B.M) Prove that the set $S = \{n^{-1} : n \in N\}$ is bounded below with largest lower bound 0.

Aliter (G.M) Self

THEOREM (B.M) Prove that any non-empty set of integers which is bounded above has a maximum and that any non-empty set of integers which is bounded below has a minimum.

~~PROOF:~~

THEOREM (G.M) Let T be a non-empty subset of Z

(i) If T is bounded above, Sup T exist and belongs to T. That is T has maximum element

(ii) If T is bounded below, Inf T exists and belong to T. That is T has minum element.

Since T is an arbitrary subset of Z, so we have alternative statement defined above.

PROOF: Suppose T be a non-empty subset of Z and is bounded above. By least upper bound property, Sup T exists.

Let $\text{Sup}\, T = t$

Since N is unbounded above, so there exist some $n \in N$ such that $t < n$

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~~Then~~ Since $\sup T = t$, so

$$x \leq t < n \quad \forall\, x \in T,\ n \in N$$

$$\Rightarrow n - x > 0 \quad \forall\, x \in T$$

Thus $\{n - x : n \in N, x \in T\}$ is a non-empty subset of N. So it has a minimum.

Let it be $n - m$ where $m \in T$

Therefore $n - m \leq n - x \quad \forall\, x \in T$

$$\Rightarrow -m \leq -x$$

$$\text{or} \quad m \geq x$$

$$\text{or} \quad x \leq m \quad \forall\, x \in T$$

Thus ~~Since~~ $x \leq m\ \forall\, x \in T$ and $m \in T$

which clearly shows that m is the maximum element of T.

Thus in this case $\sup T = m = t$

Hence $\sup T$ exist and belongs to t.

(ii) Assignment.

Exercise: (B.M) Prove that the set $\left\{\frac{n-1}{n} : n \in N\right\} = S$ is bounded above with smallest upper bound 1.

Does S has a maximum?

(Hint) Let $y = \frac{n-1}{n} = 1 - \frac{1}{n} < 1 \Rightarrow y < 1 \quad \forall\, y \in S$

Thus S is bounded above.

Assignment.

H.M KHALID MAHMOOD

Archimedian Principle

Statement (G.M): For any real number x, there is an integer n such that $x < n$.

PROOF: Suppose that there does not exist any integer n such that $x < n$

then $n \leq x \quad \forall\, n \in \mathbb{Z}$. —— (1)

Then $\mathbb{Z}$ is bounded above by x.

Then $\sup \mathbb{Z}$ exists. Let $\sup \mathbb{Z} = b$

Since $n \leq x \quad \forall n \in \mathbb{Z}$

imply $n+1 \leq x \quad \because n+1 \in \mathbb{Z}$

$\Rightarrow n+1 \leq b \quad \because \sup \mathbb{Z} = b$

$\Rightarrow n \leq b-1 \quad \forall n \in \mathbb{Z}$.

$\Rightarrow \sup \mathbb{Z} = b-1$, a contradiction against the fact that $\sup \mathbb{Z} = b$

Hence there exist an integer n such that $x < n$ for any real number x.

Archimedian Property (RU):

Statement: If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0$ then there is a positive integer n such that $nx > y$

Proof: Case when $y > 0$

Consider the set $A = \{nx : x > 0,\ n \in \mathbb{Z}^+\}$

Suppose that there does not exist any positive integer n such that $nx > y$

then $nx \le y \quad \forall \, nx \in A$

This shows that y is an upper bound of A, so A is bounded above.

Since A is a non-empty, bounded above, subset of R and R is complete ordered field, so A has sup in R.

let it be α. i.e $\alpha = \sup A$.

Since $x > 0$

imply $-x < 0$

$\Rightarrow \alpha - x < \alpha$

Therefore $\alpha - x$ is not an upper bound of A, so there exist some positive integer m such that

$\alpha - x < mx$

$\Rightarrow \alpha < mx + x$

$\Rightarrow \alpha < (m+1)x$ —— ①

But $(m+1)x \in A$.

Thus ① yields a contradiction to the fact $\alpha = \sup A$.

Hence $nx > y$

Case 2 If $y \le 0$

then for $x > 0$ and $n \in Z^+$

$nx > 0$

or $y \le 0 < nx$

$\Rightarrow nx > y$

Condensation property (RU)

Statement: If $x, y \in R$ and $x < y$, then there exist a $p \in Q$ such that $x < p < y$

(May be stated by saying Q is dense in R)

OR Show that between any two real numbers there is a rational number.

PROOF: Let $x, y \in R$ with $x < y$

then $y - x > 0$

Since $1, y-x \in R$ and $y-x > 0$

So by Archimedean property there exist a positive integer n such that

$$n(y-x) > 1$$

$$\Rightarrow \quad ny > nx+1$$

$$\text{or} \quad nx+1 < ny \quad \text{—— ①}$$

Since $-nx, nx, 1 \in R$, so by Archimedean property, there exist positive integers m_1 and m_2 such that

$$m_1 \cdot 1 > nx \quad \& \quad m_2 \cdot 1 > -nx \quad \text{with } 1 > 0$$

$$\Rightarrow \quad m_1 > nx \quad \& \quad nx > -m_2$$

$$\text{or} \quad -m_2 < nx < m_1 \quad \text{—— ②}$$

For $-m_2, m_1$ there must exist integer m such that

$$-m_2 < m < m_1$$

then there exist an integer m with $-m_2 < m < m_1$

Such that $m-1 < nx < m$ — ①* (Since $-m_2$ & m_1 are arbitrary. If we choose however large then it is possible to find m ; $-m_2 < m < m_1$ with $m-1 < nx < m$ because $nx \in R$.)

$\Rightarrow$ $m < nx+1 < m+1$

$\Rightarrow$ $m < nx+1$ — ②

By ① & ②* and ②

$\cancel{m < nx+1 <}$

$nx < m < nx+1 < ny$

or $nx < m < ny$

$\Rightarrow$ $x < \frac{m}{n} < y$; $m, n \in Z$

Take $\frac{m}{n} = p$ then $p \in Q$.

Thus for $x, y \in R$, $x < y$ there exist a $p \in Q$ such that $x < p < y$.

(G.M)
Corollary: If $x, y \in R$ and $x < y$ then there is an irrational number μ such that $x < \mu < y$.

PROOF: Let $x, y \in R$ and α be a positive irrational number. then $\frac{x}{\alpha}, \frac{y}{\alpha} \in R$

Since $x < y$ & $\alpha > 0$

So $\frac{x}{\alpha} < y/\alpha$

By condensation property, we can find a rational number p such that $\frac{x}{\alpha} < p < \frac{y}{\alpha}$.

imply $x < \alpha p < y$

let $\alpha p = u$, then u is irrational because, product of rational and irrational is irrational.

Hence for $x, y \in R$, there exist an irrational u s.t $x < u < y$.

THEOREM (G.M) For every real number x there is a set E of rational numbers such that $x = \sup E$.

OR Every real number is the supremum of some subset of R.

PROOF: Let x be an arbitrary real number.

Define $E = \{p \mid p \in Q \wedge p < x\}$ ~~$(\frac{P}{E})_x$~~

Obviously E is a non-empty subset of R. and is bounded above. Thus E has sup in R because R possesses lub property.

Let it be α. That is $\sup E = \alpha$.

Then ~~$\alpha \geq x$~~ $x \leq \alpha$, $\forall x \in E$

~~If $\alpha = x$, then there is nothing to prove~~

~~If $x < \alpha$ then we can find a rational number q such that $x < q < \alpha$. by condensation property.~~

~~Thus $x < r$ imply $r \not> x$ $\forall x \in E$~~

Then $\alpha \leq x$ by def of E.
(Since $\sup E = x$ or $\sup E < x$ by def of E).

If $\alpha = x$, then there is nothing to prove.

If $\alpha < x$, then by condensation property we can find a rational number r such that $\alpha < r < x$ —— (1)

Since $r < x$ imply $r \in E$.

and $\alpha < r$ with $\alpha = \sup E$,

Thus (1) yields a contradiction

Hence the case $\alpha < x$ is not possible. Thus $x = \sup E$.

Exercise: (1) Every real number is the infimum of some subset of R.

(2) For any $\varepsilon > 0$, there exist a positive integer n s.t $\frac{1}{n} < \varepsilon$ (use Archemedian property S.A)

Assignment

H.M KHALID MAHMOOD.

THEOREM (RU) For every real number $x>0$ and every integer $n>0$ there is one and only one real number y such that $y^n = x$ <u>OR</u>

Show that nth root of every positive real number exists and is unique.

PROOF: Consider the set

$$A = \{t : t \in R \text{ and } t^n < x\}$$

let $t = \frac{x}{x+1}$

then obviously $t < x$ and $0 < t < 1$

imply $t^n < t$ and $t < x$

thus $t^n < x$

imply $t \in A$. thus A is a non-empty subset of R.

Now consider $t' = 1 + x$

obviously $t' > x$ & $t' > 1$

then $(t')^n > t'$ & $t' > x$

imply $(t')^n > x$

NOTE: $t^n > t$ if $t > 1$

imply $t' \notin A$

It is obvious that t' is an upper bound of A. Thus A is bounded above.

Since A is non-empty subset of R and is bounded above, so due to least upper bound property of R, A has suprimum in R. Let it be y. i.e $\sup A = y$

Then there are three possibilities

(i) $y^n < x$ (ii) $y^n = x$ (iii) $y^n > x$

Assume $y^n < x$ and choose a number h so that $0 < h < 1$ and $h < \dfrac{x - y^n}{n(y+1)^{n-1}}$ ——①*

Moreover $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$ yields the inequality

① —— $b^n - a^n < (b-a)nb^{n-1}$ when $0 < a < b$

put $a = y$, $b = y+h$ in ①, we get

$$(y+h)^n - y^n < nh(y+h)^{n-1}$$
$$< nh(y+1)^{n-1} \quad \because h < 1$$
$$< x - y^n \quad \text{using } ①^*$$

Thus $(y+h)^n - y^n < x - y^n$

imply $(y+h)^n < x$

$\Rightarrow$ $y+h \in A$. Since $y+h > y$, this contradicts the fact that y is an upper bound of A.

Assume that $y^n > x$.

put ② — $K = \dfrac{y^n - x}{n y^{n-1}}$, then $0 < K < y$

put $a = y - K$ & $b = y$ in ①, we get

$$y^n - (y-K)^n < nKy^{n-1} = y^n - x \text{ by ②}$$

Thus $y^n - (y-K)^n < y^n - x$

or $-(y-K)^n < -x$

imply $(y-K)^n > x$

$\Rightarrow y - K \notin A$

Which is a contradiction to the fact y is the Sup A.

Thus there is only possibility $y^n = x$.

For uniqueness. Suppose y_1 and y_2 be two real numbers such that

$y_1^n = x$ & $y_2^n = x$

Then $y_1^n = y_2^n$

imply $y_1 = y_2$

Corollary: If a and b are positive real numbers and n is a positive integer, then $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$

$\alpha = a^{1/n}$ & $\beta = b^{\frac{1}{n}}$

then $\alpha^n = a$ & $\beta^n = b.$

imply
$$ab = \alpha^n \beta^n$$
$$= (\alpha\beta)^n$$
or
$$\alpha\beta = (ab)^{\frac{1}{n}}$$
imply
$$a^{\frac{1}{n}} b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$$
or
$$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$$

THE EXTENDED REAL NUMBER SYSTEM: (RU)

Def: The extended real number system consists of the real field R and two symbols, $+\infty$ and $-\infty$. We preserve the orignal order in R and define $-\infty < x < \infty$ for every $x \in R$.

REMARK:

① In Extended real number system $+\infty$ is an upper bound of every non-empty subset and $\sup E = \infty$, where E be a non-empty subset of extended real number system and is not bounded above in R. Similarly, let F be a non-empty subset of extended real number system which is not bounded below in R. Then $-\infty$ is its lower bound. Also $\inf F = -\infty$.

② The extended real number system does not form a field but satisfies the following axioms.

(i) If x is real then

$$x+\infty=\infty, \quad x-\infty=-\infty, \quad \frac{x}{+\infty}=\frac{x}{-\infty}=0$$

(ii) If $x>0$ then $x\cdot(+\infty)=+\infty$

$$x\cdot(-\infty)=-\infty.$$

(iii) If $x<0$ then $x\cdot(+\infty)=-\infty$

$$x\cdot(-\infty)=+\infty.$$

ABSOLUTE VALUE OF A REAL NUMBER:

Def: The absolute value (the numerical value) or the modulus of a real number x is denoted by $|x|$ and is defined as

$$|x|=\begin{cases} x & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -x & \text{if } x<0 \end{cases}$$

REMARK: (i) $|x|\geq 0$

(ii) $|-x|=|x|$

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THEOREM: For $x, y \in R$

(i) $|xy| = |x|\,|y|$

(ii) $|x|^2 = x^2$ or $|x|^2 = \sqrt{x^2}$

(iii) If $a \geq 0$, then
$|x| \leq a$ iff $-a \leq x \leq a$

(iv) $-|x| \leq x \leq |x|$

(v) $|x+y| \leq |x| + |y|$

(vi) $\big||x| - |y|\big| \leq |x-y|$

(vii) $|x-y| \leq |x| + |y|$

(viii) $|x-z| \leq |x-y| + |y-z| \quad \forall z \in R.$

(ix) $|x-y| \geq |x| - |y|$

PROOF:
(i) If x or y equal to 0, then
both sides are zero.
If $x > 0$ & $y > 0$, so $|x| = x$ & $|y| = x$

Then $xy > 0$
imply $|xy| = xy$
$= |x|\,|y|$

Remaining are same.

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(ii) Since $x^2 > 0$, so

$$x^2 = |x^2| = |x\,x| = |x|\,|x| = |x|^2$$

imply $|x|^2 = x^2$ or $|x| = \sqrt{x^2}$.

(iii) If $|x| \le a$ then

$x \le a$ and $-x \le a$ by def of $|x|$

$x \le a$ and $x \ge -a$

or $-a \le x \le a$.

Conversely if $-a \le x \le a$

then $x \le a$ & $-x \le a$

imply $|x| \le a$.

(iv) Take $a = |x|$ in (iii)

we get

$$-|x| \le x \le |x|.$$

(v)–(viii) Self.

(ix) $\big||x| - |y|\big| \le |x - y|$

$$\Rightarrow \quad -|x-y| \le |x| - |y| \le |x - y|$$

$$\Rightarrow \quad |x| - |y| \le |x - y|$$

or $\quad |x - y| \geq |x| - |y|$

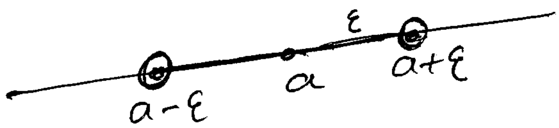
Remark:
Geometrically $|a|$ of an element a in R is regarded as the distance from a to the origin 0. Moreover, the distance between elements a and b in R is denoted by $|a-b|$.

Def (BR) Let $a \in R$ and $\varepsilon > 0$, then the ε-neighborhood of a is the set

$$V_\varepsilon(a) = \{x \in R : |x - a| < \varepsilon\}$$

$$= \{x \in R : a - \varepsilon < x < a + \varepsilon\}$$

$$= N_\varepsilon(a)$$



Or simply the interval

$(a-\varepsilon, a+\varepsilon)$ is called

ε-neighborhood of a and is denoted by $N_\varepsilon(a)$.

neighborhood (z_{P})

The set $(x_0-\varepsilon, x_0) \cup (x_0, x_0+\varepsilon)$ is called a deleted ε-neighborhood of the point x_0 (That is x_0 has been deleted from the neighborhood $N_\varepsilon(x_0)$). It is denoted by $N_\varepsilon^*(x_0)$

Definition (RU):

A complex number is an ordered pair (a,b) of real numbers. Ordered means that (a,b) and (b,a) are regarded as distinct if $a \neq b$.

The set of complex numbers is denoted by $\mathbb{C}$.

Let $x = (a,b)$ &
$y = (c,d)$ be two complex numbers, we define addition and multiplication as

$$x+y = (a+c, b+d)$$
$$xy = (ac-bd, ad+bc)$$

Definition (RU) If a, b are real and $z = a + ib$, then the complex number $\bar{z} = a - ib$ is called the conjugate of z. The numbers a and b are the real part and the imaginary part of z, respectively.

THEOREM (RU)

let z and ω be complex numbers, then

(a) $|z| > 0$ unless $z = 0$, $|0| = 0$

(b) $|\bar{z}| = |z|$

(c) $|z\omega| = |z|\,|\omega|$

(d) $|\mathrm{Re}\, z| \leq |z|$

(e) $|z + \omega| \leq |z| + |\omega|$

(f) $\overline{z + \omega} = \bar{z} + \bar{\omega}$

(g) $\overline{z\omega} = \bar{z}\,\bar{\omega}$

(h) $\bar{z} + z = 2\,\mathrm{Re}(z)$, $z - \bar{z} = 2i\,\mathrm{Im}(z)$

(i) $z\bar{z}$ is real positive. (except when $z = 0$)

Schwarz inequality:

THEOREM (RU): If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are complex numbers, then

$$\left|\sum_{j=1}^{n} a_j \bar{b}_j\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

PROOF: Put $A = \sum_{j=1}^{n} |a_j|^2$, $B = \sum_{j=1}^{n} |b_j|^2$

and $$C = \sum_{j=1}^{n} a_j \bar{b}_j$$

If $B = 0$ then $b_1 = \cdots = b_n = 0$ and the conclusion is trivial.

Assume therefore that $B > 0$, then

$$\sum_{j=1}^{n} |B a_j - C b_j|^2 = \sum_{j=1}^{n} (B a_j - C b_j)(\overline{B a_j - C b_j})$$

$$= \sum_{j=1}^{n} (B a_j - C b_j)(B \bar{a}_j - \bar{C} \bar{b}_j)$$

$$= B^2 \sum_{j=1}^{n} |a_j|^2 - B\bar{C} \sum_{j=1}^{n} a_j \bar{b}_j$$

$$- BC \sum_{j=1}^{n} \bar{a}_j b_j + |C|^2 \sum_{j=1}^{n} |b_j|^2$$

$$= B^2 A - B|C|^2 = B(BA - |C|^2)$$

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Since each term in the first sum is non-negative, we see that

$$B(AB-|C|^2) \geq 0$$

Since $B>0$, it follows that

$$AB-|C|^2 \geq 0$$

or $\quad |C|^2 \leq AB$

putting values of A, B and C to get desired inequality.

EUCLIDEAN Space (R^K):

Def: let R^K be the set of all ordered K-tuples, for each positive integer K, of the form

$$\underline{x} = (x_1, x_2, \ldots, x_K)$$

Where $x_1, x_2, \ldots, x_K$ are real numbers, called coordinate of $\underline{x}$

Define addition and scalar multiplication in R^K as

For $\underline{x}, \underline{y} \in R^K$; $\quad \underline{x} = (x_1, \ldots, x_K)$
$\underline{y} = (y_1, \ldots, y_K)$

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$\underline{x}+\underline{y} = (x_1+y_1, \dots, x_K+y_K)$

If α is a real number, then

$\alpha \underline{x} = (\alpha x_1, \alpha x_2, \dots, \alpha x_K)$

The set R^K with addition and scalar multiplication defined above form a vector space over R.

Inner Product:

For $\underline{x}, \underline{y} \in R^K$;

$\underline{x} = (x_1, x_2, \dots, x_K)$

$\underline{y} = (y_1, y_2, \dots, y_K)$

inner product is denoted by $\underline{x} \cdot \underline{y}$

and $\underline{x} \cdot \underline{y} = \sum_{i=1}^{K} x_i \cdot y_i$

NORM: For $\underline{x} \in R^K$; $\underline{x} = (x_1, x_2, \dots, x_K)$ the norm of $\underline{x}$ is denoted by $\|\underline{x}\|$

and $\|\underline{x}\| = (\underline{x} \cdot \underline{x})^{1/2}$

$= \left(\sum_{i=1}^{K} x_i^2\right)^{1/2}$

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Remark: If it is understood that $\underline{x}$ is in R^K (the Euclidean space) then for sake of convenience $\|\underline{x}\|$ is sometime simply denoted by $|\underline{x}|$

THEOREM (RU):

let $\underline{x}, \underline{y}, \underline{z} \in R^K$ and $\alpha \in R$, then

(a) $\|x\| \geq 0$

(b) $\|\underline{x}\| = 0$ iff $\underline{x} = 0$

(c) $\|\alpha \underline{x}\| = |\alpha| \, \|\underline{x}\|$

(d) $\|\underline{x} \cdot \underline{y}\| \leq \|\underline{x}\| \, \|\underline{y}\|$

(e) $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$

(f) $\|\underline{x} - \underline{z}\| \leq \|\underline{x} - \underline{y}\| + \|\underline{y} - \underline{z}\|$

PROOF: (a) Since $x_i^2 \geq 0 \quad \forall \; i = 1, 2, \ldots K$

imply $\sum_{i=1}^{K} x_i^2 \geq 0$

$$\Rightarrow \left(\sum_{i=1}^{K} x_i^2 \right)^{1/2} \geq 0$$

i.e $\|\underline{x}\| \geq 0$

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(b) $\|\underline{x}\| = 0$ iff $\left(\sum_{i=1}^{K} x_i^2\right)^{1/2} = 0$

iff $\sum_{i=1}^{K} x_i^2 = 0$

iff $x_i^2 = 0$

iff $x_i = 0 \quad \forall i = 1, 2, \dots, K$

iff $\underline{x} = 0$

Thus $\|\underline{x}\| = 0 \Leftrightarrow \underline{x} = 0$

(c) $\|\alpha \underline{x}\| = \left(\sum_{i=1}^{K} (\alpha x_i)^2\right)^{1/2}$

$= \left(\alpha^2 \sum_{i=1}^{K} x_i^2\right)^{1/2}$

$= |\alpha| \left(\sum_{i=1}^{K} x_i^2\right)^{1/2}$

$= |\alpha| \, \|\underline{x}\|$

(d) By Schwarz Inequality

$\left|\sum_{i=1}^{K} x_i \bar{y}_i\right|^2 \le \sum_{i=1}^{K} |x_i|^2 \sum_{i=1}^{K} |y_i|^2$

Since y_i are real $\forall$

So $\overline{y_i} = y_i \quad \forall\, i = 1, 2, \ldots K$

imply
$$\left|\sum_{i=1}^{K} x_i \cdot y_i\right|^2 \leq \sum_{i=1}^{K} |x_i|^2 \sum_{i=1}^{K} |y_i|^2$$

$$\Rightarrow \quad \|\underline{x} \cdot \underline{y}\|^2 \leq \|\underline{x}\|^2 \|\underline{y}\|^2 \quad \text{by def}$$

$$\Rightarrow \quad \|\underline{x} \cdot \underline{y}\| \leq \|\underline{x}\| \, \|\underline{y}\|$$

(e) $\|\underline{x} + \underline{y}\|^2 = (\underline{x} + \underline{y}) \cdot (\underline{x} + \underline{y})$

$$= \underline{x} \cdot \underline{x} + \underline{x} \cdot \underline{y} + \underline{y} \cdot \underline{x} + \underline{y} \cdot \underline{y}$$

$$= \|\underline{x}\|^2 + \underline{x} \cdot \underline{y} + \underline{x} \cdot \underline{y} + \|\underline{y}\|^2$$

$$= \|\underline{x}\| + 2\, \underline{x} \cdot \underline{y} + \|\underline{y}\|^2$$

$$\leq \|\underline{x}\|^2 + 2\, \|\underline{x}\| \, \|\underline{y}\| + \|\underline{y}\|^2 \quad \text{using (d)}$$

Or $\|\underline{x} + \underline{y}\|^2 \leq (\|\underline{x}\| + \|\underline{y}\|)^2$

$$\Rightarrow \quad \|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$$

(f) we have

$$\|\underline{x}+\underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$$

Replace $\underline{x}$ by $\underline{x}-\underline{y}$

and $\underline{y}$ by $\underline{y}-\underline{z}$

to get

$$\|\underline{x}-\underline{z}\| \leq \|\underline{x}-\underline{y}\| + \|\underline{y}-\underline{z}\|$$

Dedekind's property:

If L and U are two subsets of R such that

(i) $L \neq \Phi$, $U \neq \Phi$ (each class has atleast one member)

(ii) $L \cup U = R$ (Every real number has a class)

(iii) Every member of L is less than every member of U, i-e

$x \in L$ and $y \in U$ imply

$$x < y$$

or $x < y \quad \forall\, x \in L$ and $\forall\, y \in U$

Then either L has the greatest member or U has the smallest member.

Remark: (S.A)

The two forms of completeness are equivalant. That is

THEOREM:
Order Completeness property iff Dedekind's property

PROOF: Suppose that the set R has the order completeness property. i-e every non-empty subset of R which is bounded above (below) has the supremum (Infimum).

Let L, U be two subsets of R such that

(i) $L \neq \varphi$, $U \neq \varphi$

(ii) $L \cup U = R$

(iii) $x < y \quad \forall x \in L \;\&\; \forall y \in U$

Then we have to show that either L has the greatest member or U has the smallest member.

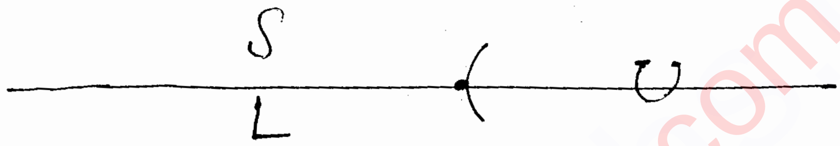
By (iii) $x < y \quad \forall x \in L$.

Thus y is an upper bound of L. Thus the non-empty set L is bounded above. If L has the greatest member, it establishes the result. If L has no greatest member, then by the order completeness property, the set of its upper bounds, which coincides with U, has the smallest member. Thus either L has the greatest member or U has the smallest member.

Converselly, Suppose that R Satisfy the Dedekind's property. We Shall Show that R also Satisfies the order Completeness property. That is every non-empty Subset of R, which is bounded above has Suprimum, let it be S. That is S be a non-empty set of real numbers and is bounded above.

let $L = \{x : x \text{ is not an upper bound of } S\}$

$U = \{x : x \text{ is an upper bound of } S\}$



Since S is bounded above, so there exist at least one upper bound α.

Then $\alpha \in U$ by def of U

imply $U \neq \Phi$

Since $S \neq \phi$ is bounded above and α is an upperer bound of S.

So $x \le \alpha \quad \forall x \in S.$

That is there must exist $x_1 \in S$ Such that x_1 is not an upper bound of S. Hence by def $x_1 \in L$.

imply $L \neq \Phi$

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Let $r \in R$, then either r is an upper bound of S or it not an upper bound.

Therefor either $r \in U$ or $r \in L$

imply $r \in L \cup U$

Since r was arbitrary.

So $R \subseteq L \cup U$

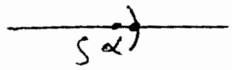
But $L \cup U \subseteq R$

Hence $L \cup U = R$.

Now let $x \in L$ and $y \in U$

Since y is an upper bound of S, ~~therefore~~ so

$y \geq s \quad \forall s \in S.$



~~imply $y \geq s$,~~

But for all $x \in L$, x is not an upper bound of S, so there exist some $\alpha \in S$ such that $\alpha > x$.

and $y \geq \alpha \quad \because \alpha \in S.$

Thus $y \geq \alpha > x$

$\Rightarrow y > x \quad \forall x \in L \ \& \ \forall y \in U$

Aliter: By def of L & U It is obvious $x < y \quad \forall x \in L$ & $\forall y \in U$

Then Dedekind's property, either L has the greatest number or U has the smallest member.

Let, if possible, L have the greatest member say ξ. Then $\xi \in L$ imply ξ is not an upper bound of S. then there exist an $a \in S$ such that $\xi < a$

Moreover $\xi < \frac{\xi + a}{2}$ $\because \xi < a$

$< a$ $\Rightarrow \xi + a < 2a$

$\Rightarrow \frac{\xi + a}{2} < a.$

imply $\frac{\xi+a}{2} > \xi$ which is greatest member of L.

Thus $\frac{\xi+a}{2} \in U$ —— (1)

imply $\frac{\xi+a}{2}$ is an upper bound of S.

~~Also~~ Also $\frac{\xi+a}{2} < a$ for $a \in S$

$\Rightarrow \frac{\xi+a}{2}$ is not an upper bound of S. Therefore $\frac{\xi+a}{2} \in L$. —(2)

Hence (1) & (2) provide a contradiction

It follows that L has no greatest member. Thus by Dedekind's property U, the set of upper bounds of S has smallest member. i.e S has supremum.

Smallest member. i.e the set S has Suprimum. Since S was arbitrary so every non-empty subset of R which is bounded above has suprimum. therefore R is complete.

Dedekind's Theorem: (Zip)

R is complete.

PROOF: Choose arbitrary non-empty subset S

s.t $S = \{x \in Q : x^2 < 2\}$

then S is bounded above.

then show $\sup S = \sqrt{2} \in R$.

thus R is complete.

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13/9/2003.

Some Important Definitions:

Def (ZIP) Let A and B be two non-empty sets. A function from A to B (or a mapping from A into B) is a rule which associates with each element $x \in A$, a unique element $y \in Y$. That is

If $x = y$ then $f(x) = f(y)$ or

If $f(x) \neq f(y)$ then $x \neq y$.

The element b associated with a given element $a \in A$ is denoted by $f(a)$ and we write

$b = f(a)$ or

$y = f(x)$; y is called the image of x under f and x is called a preimage of y. and the function is often denoted by $f: A \to B$

Notice that the set A is called domain of the function and is denoted by D_f.

The subset of B containing elements which are the images of the elements of A, called the range of the function and is denoted by R_f. If $f: X \to Y$

Then $D_f = X$

and $R_f = f(X) = \{ y \in Y : y = f(x) \text{ for some } x \in X \}$

$\subseteq Y$

Moreover the set Y is called Co-domain of f.

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Inclusion Function (SHAR):

If $B \subset A$, then the function
$i : B \to A : i(b) = b \ \forall b \in B$
is called the inclusion function.

Restriction and Extension of a Function: (SH)

If $f : X \to Y$ and $A \subset X$, then the mapping
$g : A \to Y$ defined by
$g(x) = f(x) \ \forall x \in A$
is called the restriction of f to A and is denoted by $f|A$ or f_A. Then f is called an extension of g.

Onto (Surjective) function:

Let $f : X \to Y$ be a function such that
$f(X) = Y$. That is
R_f = co-domain, or
Every element of Y has atleast one pre-image in X. Then f is called onto (or surjective) function (i.e f is onto Y)

One-One (or Injective) function:

If every $y \in f(X)$ has a unique $x \in X$ such that $y = f(x)$.
That is every image in Y has a unique pre-image in X, then f is called 1-1 function.

One-one Correspondance (Bijective) function:

If f is both injective and surjective, then it is called bijective and then f is called bijection.

Inverse Function (BR):

If $f: X \to Y$ is 1-1, we define a function $f^{-1}: f(X) \to X$ by the following

For each $y \in f(X)$ define $f^{-1}(y)$ to be the unique preimage of y under f.

Then $f^{-1}(y) = x$ iff $y = f(x)$.

It is clear that f^{-1} is onto X and that f^{-1} is one-to-one. Moreover if $f: X \to Y$ is bijective, then $f^{-1}: Y \to X$ is also bijective. That is a one-to-one correspondance between Y and X.

Example: let ① $f_1 : [-1,2] \to [0,4]$: $f_1(x) = \frac{4}{3}(x+1)$

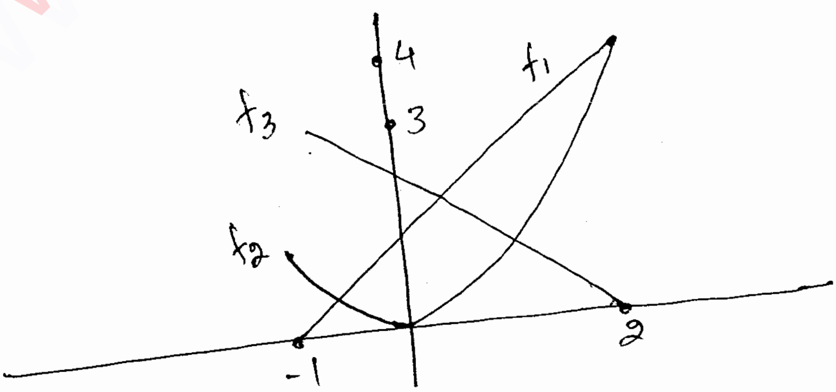
② $f_2 : [-1,2] \to [0,4]$: $f_2(x) = x^2$

③ $f_3 : [-1,2] \to [0,4]$: $f_3(x) = -x+2$

Then ① f_1 is bijective

② f_2 is onto but not 1-1.

③ f_3 is 1-1 but not onto



Horizontal line test for 1-1.

vertical line test for function.

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Even & odd functions.

Composition of functions: (BR)

If $f: A \to B$ and $g: B \to C$ and if $R_f \subseteq D_g = B$, then the composition function gof is the function from A into C defined by $(gof)(x) = g(f(x)) \ \forall x \in A$.

Example:

$f(x) = 2x$ & $g(x) = 3x^2 - 1$

then $D_g = R$, $R_f = E^* \subseteq R = D_g$

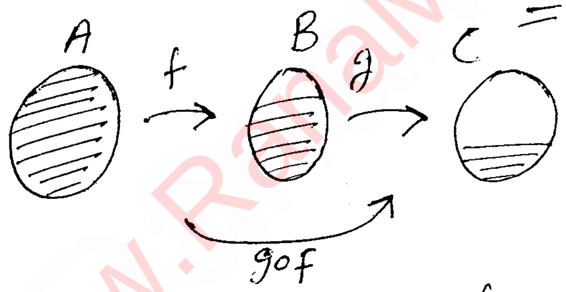
$E^* = \{y \in R : y = 2x ; x \in R\}$

then $D_{gof} = R$.

and $(gof)_{(x)} = g(f(x)) = g(2x)$

$= 3(2x)^2 - 1$

$= 12x^2 - 1$



Exp: let $f: R \to R$ by $f(x) = x^2$ then f is not bijective and inverse does not exist. However, if we restrict f to the set $A_1 = \{x \in R : x \geq 0\}$ Then the restriction $f|A_1$ is a bijection of A_1 onto A_1. Therefore this restriction has an inverse function, which is the positive square root function.

Remark: We can express the connection between f and its inverse f^{-1} by noting that

$$D_f = R_{f^{-1}} \quad \text{and} \quad R_f = D_{f^{-1}} \quad \text{and that}$$

$$b = f(a) \text{ iff } a = f^{-1}(b)$$

Example:

$$f(x) = \frac{2x}{x-1}$$

is a bijection of $A = \{x \in R : x \neq 1\}$ onto the set $B = \{y \in R : y \neq 2\}$

The function inverse to f is given by

$$f^{-1}(y) = \frac{y}{y-2} \quad \text{for } y \in B.$$

Def (BR): ① A set S is said to be denumerable (or countably infinite) if there exist a bijection of N onto S.

② A set S is said to be countable if it is either finite or denumerable.

③ A set S is said to be uncountable if it is not countable.

Exp ① The set $E = \{2n : n \in N\}$ is countable, becs it is denumerable.

② $S = \{1, 2, 3\}$, it is countable

③ R is not countable.

Metric (SH):

Let X be a non-empty set. A mapping d of $X \times X$ into R (the set of real numbers) is said to be a metric (or distance function) iff d satisfies the following axioms

M_1 : $d(x,y) \geq 0 \quad \forall\ x, y \in X$

M_2 : $d(x,y) = 0$ iff $x = y$

M_3 : $d(x,y) = d(y,x) \quad \forall\ x, y \in X$

M_4 : $d(x,y) \leq d(x,z) + d(y,z)$

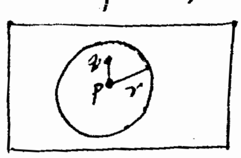
Example: $d(x,y) = |x-y| \quad \forall\ x, y \in R$ is a metric on R.

Remark: The set X together with metric on X, i.e. (X, d) is called a metric space.

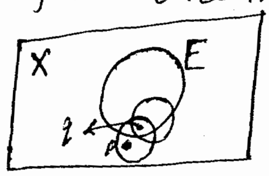
Definition:

Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X

(a) A neighborhood of a point p is a set $N_r(p)$ consisting of all points q such that $d(p,q) < r$. The number r is called the radius of $N_r(p)$



(b) A point p is ~~the~~ a limit point of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$

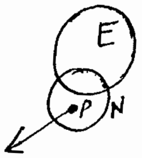


(c) If $p \in E$ and p is not a limit point of E, then p is called an isolated point.

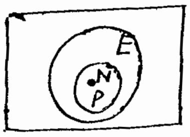
(d) E is closed if every limit point of E is a point of E.

(e) A point p is an interior point of E if there is a neighborhood N of p such that $N \subset E$

$N \not\subset E$



not interior



(f) "E is open if every point of E is an interior point of E." That is for each $p \in E$, there is a $N_r(p)$ such that $N_r(p) \subset E$.

(g) The complement of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.

(h) E is perfect if E is close and if every point of E is a limit point of E.

(i) E is bounded if there is a real number M and a point $q \in X$ such that $d(p,q) < M \ \forall p \in E$

(j) E is dense in X if every point of X is a limit point of E, or a point of E (or both).

Open Cover: By an open cover of a set E in a metric space X, we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$

$E \subset \bigcup_{\alpha=1}^{3} G_\alpha$

Compact Set:

A subset E of a metric space X is said to be compact if every open cover of E contains a finite subcover.

That is If $\{G_\alpha\}_{\alpha\in\Omega}$ is an open cover of E then there are finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $E \subset \bigcup_{i=1}^{n} G_{\alpha_i}$

OR If $E = \bigcup_\alpha G_\alpha$, then

$$E = \bigcup_{i=1}^{n} G_{\alpha_i}$$

Exp: $[a,b]$ is compact in R.

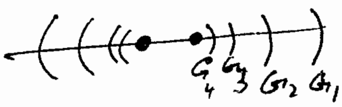
Since $\{]a-\frac{1}{n}, b+\frac{1}{n}[; n\in N\}$ is an open cover and $\{]a-\frac{1}{r}, b+\frac{1}{r}[\}_{r=1,2,3,4.}$ ~~such that~~ which obviously its finite subcover contain $[a,b]$ in their union.

let $G_1 = (a-1, b+1)$

$G_2 = (a-\frac{1}{2}, b+\frac{1}{2})$

$G_3 = (a-\frac{1}{3}, b+\frac{1}{3})$

$G_4 = (a-\frac{1}{4}, b+\frac{1}{4})$



Thus $[a,b] \subset \bigcup_{i=1}^{4} G_i$.

② $G_n =]a+\frac{1}{n}, b-\frac{1}{n}[$ is an open cover of $]a,b[$ but has no finite subcover.
Thus $]a,b[$ is not compact.

③ R with $d=|x-y|\ \forall x,y\in R$ is not compact because $\bigcup_{n=1}^{\infty}\{]-n,n[; n\in N\} = R$
Thus $G_n = \{]-n,n[; n\in N\}$ is an open cover but has no subcover which is finite

④ then family of open intervals $\{(\frac{1}{n},1) ; n=2,3,\cdots\}$ is an open cover of $(0,1)$ because

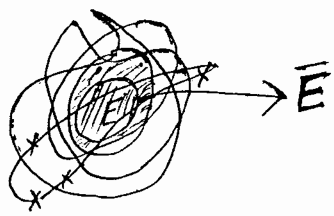
$$\bigcup_{n=2}^{\infty}]\tfrac{1}{n},1[=]0,1[$$

But has no finite subcover.

Closure of a Set:
If X is a metric space, if $E\subset X$ and if E' denote the set of all limit points of E in X, then the closure of E is the set

$$\bar{E} = E\cup E' \text{ imply } E\subseteq \bar{E}$$

Thus $\bar{E}$ is the smallest closed superset of E.



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Separated Sets:

Two sets A and B in a metric space X are said to separated if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty, i.e if no point of A lies in the closure of B and no point of B lies in the closure of A

Connected Set:

A set $E \subset X$ is said to be connected if E is not a union of two non-empty separated sets.

Disconnected:

A subset A of a metric space (X, d) is said to be connected if it cannot be expressed as the union of two non-empty separated sets. If A is not connected, then it is said to be disconnected.

Remark: Two disjoint non-empty sets are not necessarily separated. e.g

$$A = \{x : -\infty < x < 0\}$$
$$B = \{x : 0 \le x < \infty\}$$

Then A and B are disjoint but not separated.

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SEQUENCES (CH-2)

Sequence: A sequence is a function whose domain is the set of natural numbers.

Real Sequence:

A sequence of real numbers (or a sequence in R) is a function defined on the set $N = \{1, 2, 3, \cdots,\}$ of natural numbers whose range is contained in the set R of real numbers.

Examples:

① $\{a_n\} = \{(-1)^n ; n \in N\}$ ② $\{a_n\} = \{\frac{1}{n} ; n \in N\}$

③ $\{a_n\}$ where $a_n = (1+\frac{1}{n})^n$, $n \in N$

④ $\{a_n\}$ where $a_n = 1 + (-1)^n$, $n \in N$

⑤ $\{a_n\}$ where $a_n = 1$; $n \in N$

⑥ $\{a_n\}$ where $a_n = \left\{\frac{(-1)^{n-1}}{n!}\right\}$, $n \in N$.

A sequence may be defined by given an explicit formula ~~for~~ the nth term as in Example 1 to 6.

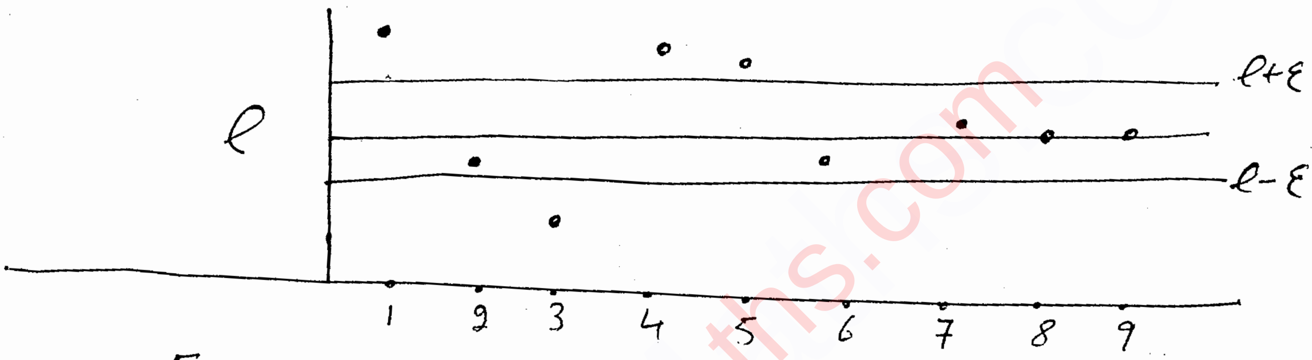
A sequence may also be defined inductively as $a_1 = 0$, $a_2 = 1$, $a_{n+2} = \frac{a_n + a_{n+1}}{2}$; $n = 1, 2, \cdots$

Give the simple meanings of convergence

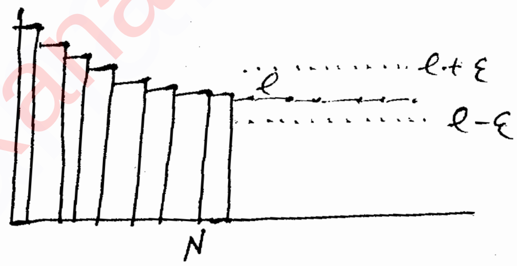
Definition of Convergence (BM)

A sequence $\{a_n\}$ is said to converge to the limit l iff the following criterion is satisfied

"Given any $\varepsilon > 0$, we can find an N such that, for any $n > N$, $|a_n - l| < \varepsilon$."



For $n > 4$, $|a_n - l| < \varepsilon$.



Remark: If we have choosen smaller value of $\varepsilon > 0$ then, we have to choose bigger value of N. Thus the value of N depend on ε and sometime, we write $N(\varepsilon)$.

Alter: $n > N \Rightarrow |a_n - l| < \varepsilon$ iff $l - \varepsilon < a_n < l + \varepsilon$

Thus if for every $n > N$ imply a_n belong to $]l-\varepsilon, l+\varepsilon[$ then $\{a_n\}$ converges to l and l is called limit of $\{a_n\}$

Then we write $a_n \to \ell$ as $n \to \infty$ or

$$\lim_{n\to\infty} a_n = \ell.$$

Def: If a sequence has a limit, we say that sequence is convergent; if it has no limit we say that the sequence is divergent.

Examples:

① $\lim_{n\to\infty} \frac{1}{n} = 0$

By Archimedian property, given any $\varepsilon > 0$ there is a natural number N such that $\frac{1}{N} < \varepsilon$. Now if $n > N$

then $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$

$\left|\frac{1}{n} - 0\right| < \varepsilon.$

② $a_n = 2 - \frac{1}{2^n}$ then $\lim_{n\to\infty} a_n = 2$

By Archemedian property, given $\varepsilon > 0$ there exist a natural number $n > N$ such that $\frac{1}{N} < \varepsilon$. Now if $n > N$

then

$$|a_n - 2| = \frac{1}{2^n} = \frac{1}{(1+1)^n} \le \frac{1}{1+n(1)} \quad \text{by Bernoulli's inequality}$$

$$< \frac{1}{n}$$

$$< \frac{1}{N}$$

$$< \varepsilon.$$

Thus $|a_n - 2| < \varepsilon$ for $n > N$ & $\varepsilon > 0$ imply $\lim_{n\to\infty} a_n = 2$.

③ Let $a_n = \dfrac{2n^2+1}{n^2+3n}$, then $\lim_{n\to\infty} a_n = 2$.

For $\varepsilon > 0$ (Given)

$$\left|\frac{2n^2+1}{n^2+3n} - 2\right| = \left|\frac{1-6n}{n^2+3n}\right|$$

$$= \frac{6n-1}{n^2+3n} < \frac{6n}{n^2} = \frac{6}{n} \quad \text{——— ①}$$

Choose $N \in \mathbb{N}$ so large that $\dfrac{1}{N} < \dfrac{\varepsilon}{6}$

Then for $n > N$, we have

$$|a_n - 2| < \frac{6}{n} < \frac{6}{N} < \varepsilon.$$

or $\lim_{n\to\infty} a_n = 2$.

Assignment ① $\lim_{n\to\infty} \dfrac{1}{n^2+1} = 0$ ② $\lim_{n\to\infty} \dfrac{3n+2}{n+1} = 3$.

③ $\lim_{n\to\infty} \dfrac{1}{n+2} = 0$ ④ $\lim_{n\to\infty} 2 - \dfrac{1}{n} = 2$

⑤ $\lim_{n\to\infty} \dfrac{n}{n+2} = 1$ ⑥ $\lim_{n\to\infty} \dfrac{2n}{n+2} = 2$

⑦ $\lim_{n\to\infty} \dfrac{(-1)^n}{n} = 0$

④ For $\varepsilon = 0.01$, Find the value of N s.t $n > N$ & $a_n = \dfrac{n}{n+2}$

We suspect $\lim_{n\to\infty} a_n = 1$

then For $n > N$ imply $\left|\dfrac{n}{n+2} - 1\right| < 0.01$

or $\dfrac{2}{n+2} < 0.01 \Rightarrow n > 198$

Thus $N \geq 199$.

Thus $n \geq 199 \Rightarrow |a_n - 1| < 0.01$.

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Generally, How to find N.

For $\varepsilon > 0$, we need to find $N \in \mathbb{N}$ such that

$$n > N \Rightarrow |a_n - 1| < \varepsilon.$$

$$|a_n - 1| = \left|\frac{n}{n+2} - 1\right| = \frac{2}{n+2}.$$

$$|a_n - 1| < \varepsilon \Rightarrow \frac{2}{n+2} < \varepsilon.$$

$$\Rightarrow n > \frac{2 - 2\varepsilon}{\varepsilon}$$

Therefore if we pick $N > \frac{2-2\varepsilon}{\varepsilon}$ imply $\frac{1}{N} < \frac{1}{\frac{2-2\varepsilon}{\varepsilon}}$.

Then for $n > N$, we have

$$|a_n - 1| = \frac{2}{n+2} < \frac{2}{N+2} < \frac{2}{\frac{2-2\varepsilon}{\varepsilon} + 2} = \varepsilon.$$

So $|a_n - 1| < \varepsilon$. Hence $\lim_{n\to\infty} a_n = 1$.

(43) THEOREM: (zip) A sequence in $\mathbb{R}$ can have atmost one limit. That is If $\lim_{n\to\infty} a_n$ exists, then it is unique.

PROOF: Suppose $\lim_{n\to\infty} a_n = A$ &

$$\lim_{n\to\infty} a_n = A'.$$

Given any $\varepsilon > 0$ there is a natural number N_1 such that $|a_n - A| < \varepsilon/2$ whenever $n > N_1$ and there is a natural number N_2 such that

$$|a_n - A'| < \varepsilon \text{ whenever } n > N_2$$

If $N > \max(N_1, N_2)$, then for $n \geq N$

$$|a_n - A| < \varepsilon/2 \quad \&$$

$$|a_n - A'| < \varepsilon/2$$

By triangular inequality

$$|A-A'| = |A-a_n+a_n-A'|$$
$$\leq |a_n-A| + |a_n-A'|$$
$$< \varepsilon/_2 + \varepsilon/_2 = \varepsilon.$$

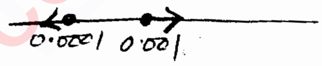
imply $|A-A'| < \varepsilon$

Since $\varepsilon > 0$ is an arbitrary positive number, we conclude that $A-A'=0 \Rightarrow A=A'$

Explaination: ε- arbitrary mean, say $\varepsilon = 0.0001$ (It may be any real number)

then $|A-A'| < 0.0001$

But by def $|A-A'| \geq 0$

It may be $|A-A'| > 0.001$ 

thus to way out is ~~for~~ $|A-A'| = 0$

Assignment (BR)

let $X = \{x_n\}$ be a sequence of real numbers and let $x \in R$, then the following statements are equivalent.

(a) X Converges to x

(b) For every $\varepsilon > 0$, there exist a natural number K such that for all $n \geq K$, $|x_n - x| < \varepsilon$.

(c) For every $\varepsilon > 0$, there exist a natural number K such that for all $n \geq K$, $x - \varepsilon < x_n < x + \varepsilon$.

(d) For every ε-neighborhood $V_\varepsilon(x)$ of x, there exist a natural number K such that for all $n \geq K$, the term $x_n \in V_\varepsilon(x)$.

PROOF:

Consider

(d) $|u-x| < \varepsilon \Leftrightarrow x-\varepsilon < u < x+\varepsilon \Leftrightarrow u \in V_\varepsilon(x)$, Take $u = \{x_n\}$

Definition (BR):

If $X = (x_1, x_2, \cdots, x_n, \cdots)$ is a sequence of real numbers and if m is a given natural number, then the m-tail of X is the sequence

$$X_m = \{x_{m+n} : n \in N\}$$
$$= (x_{m+1}, x_{m+2}, \cdots)$$

For example, the 3-tail of the sequence $(2, 4, 6, 8, 10, \cdots, 2n, \cdots)$ is the sequence $X_3 = (8, 10, \cdots)$

REMARK (KOS): Some obvious observations that should be made include the followings.

① A sequence $\{a_n\}$ is identical to a sequence $\{b_n\}$ if they both attain the same values in exactly the same order.

② If $\{a_n\}$ and $\{b_n\}$ differ from each other in only a finite number of terms, then both sequence converges to the same value or they both diverges.

③ If $\{a_n\}$ converges to A, then $\{a_n\}_{n=K}^{\infty}$ for any $K \in N$ also converges to A. That is $\lim_{n\to\infty} a_n = A$ iff $\lim_{n\to\infty} a_{n+K} = A$; $K = 0, 1, \cdots$

Equivalently, if $\{b_n\}$ is a sequence with $b_n = a_{n+K}$, then $\{b_n\}$ must also converges to A. Therefore, no matter where we start a converging sequence, the new sequence will converge to the same value as the original sequence did.

④ If the sequence $\{a_n\}$ converges and $a_n = A$ for infinitely many values of n, where A is a constant, then $\{a_n\}$ converges to A.

⑤ If $a_n = A$, for infinitely many values of n, wher A is constant, then $\{a_n\}$ does not necessarily have to converge to A. For example $(-1)^n = a_n$

⑥ The $\lim_{n\to\infty} a_n = A$ implies for any $\varepsilon > 0$, there exist $n^* \in N$ such that for $n \geq n^*$, $|a_n - A| < K\varepsilon$ where K is any constant.

⑦ The $\lim_{n\to\infty} a_n = A$ implies that there exist $n^* \in N$ such that for all $n \geq n^*$, $|a_n - A| < K$ for any constant $K > 0$

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Definition (KOS):

A sequence $\{a_n\}$ is said to be bounded iff there exist a positive real number M such that $|a_n| \le M \ \forall n \in N$.

Thus, the sequence $\{a_n\}$ is bounded iff the set $\{a_n : n \in N\}$ of its values is a bounded subset of R.

If a sequence is not bounded, then it is said to be unbounded.

THEOREM:(BR) A convergent sequence of real numbers is bounded.

PROOF:(ZIP) If $\varepsilon = 1 > 0$, then there is a natural number N such that $|a_n - A| < 1$ whenever $n > N$.

It follows that

$$|a_n| - |A| \le |a_n - A| < 1$$

imply $|a_n| < 1 + |A|$ for all $n > N$.

Chose $M = \max(|a_1|, |a_2|, \dots, |a_N|, 1+|A|)$

Clearly $|a_n| \le M$ for every natural number.

and so $\{a_n\}$ is bounded.

Remark:(ZIP) Converse of above statement may not be true. That is a bounded sequence may not be convergent.

let $a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$

Then obviously $\{a_n\}$ is bounded because $|a_n| \le 1 \ \forall n \in N$.

Suppose $\lim\limits_{n\to\infty} a_n = A$

such that $|a_n - A| < \varepsilon$ whenever $n > N$.

Taking $\varepsilon = \frac{1}{2}$, then

$$|0 - A| < \frac{1}{2} \quad \text{and} \quad |1 - A| < \frac{1}{2}$$

$|0 - A| < \frac{1}{2}$ implies $|A| < \frac{1}{2}$ —— ①

Also $1 - |A| \leq |1 - A| < \frac{1}{2}$ imply

$1 - |A| < \frac{1}{2}$ imply

$|A| > \frac{1}{2}$ —— ②

① & ② yields a contradiction. This contradiction shows that there can be no A such that $\lim_{n\to\infty} a_n = A$. Hence $\{a_n\}$ is divergent.

THEOREM: (KOS)

Consider a sequence $\{a_n\}$ that converges to a non-zero constant A. Then there exist N such that $a_n \neq 0$ for all $n \geq N$. In fact $|a_n| \geq \frac{|A|}{2}$ for all $n \geq N$.

PROOF: Since $\lim_{n\to\infty} a_n = A$, ~~for~~ so there exist a natural number N, such that

$|a_n - A| < \frac{|A|}{2}$ whenever $n \geq N$. —— ①

~~we know $\frac{|A|}{2} > |a_n - A| \geq |a_n| - |A|$~~

~~or $|a_n| <$~~

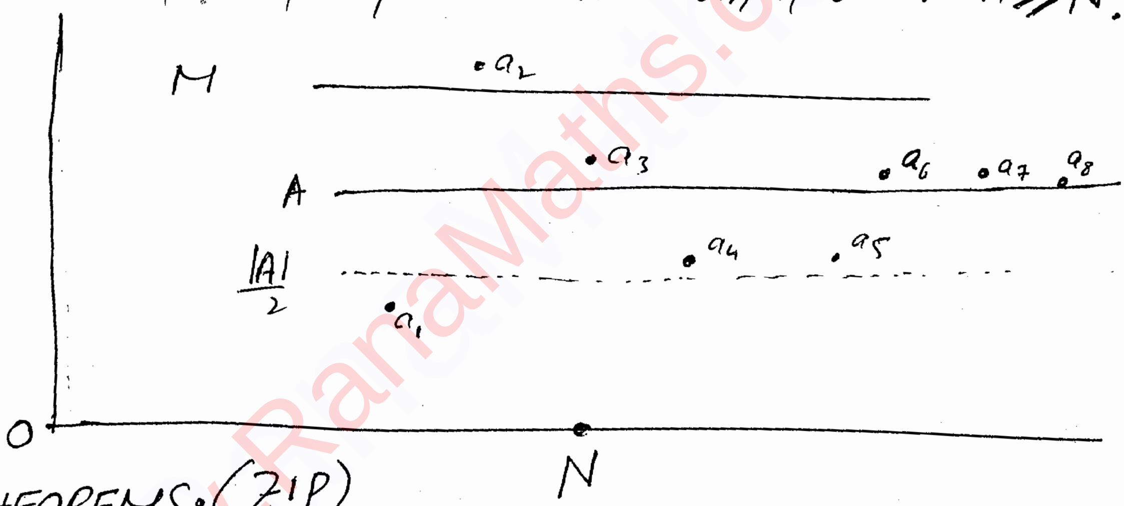
Thus for $n \geq N$

$$|a_n| = |a_n - A + A|$$
$$= |A - (A - a_n)|$$
$$\geq |A| - |A - a_n| = |A| - |a_n - A|$$

or $|a_n| > |A| - \frac{|A|}{2} = \frac{|A|}{2} > 0$

Thus $|a_n| > \frac{|A|}{2} > 0$

or $|a_n| \neq 0 \Leftrightarrow a_n \neq 0 \quad \forall\, n \geq N.$



Limit THEOREMS: (21P)

If $\lim_{n\to\infty} a_n = A$ & $\lim_{n\to\infty} b_n = B$ then

(a) $\lim_{n\to\infty} (a_n + b_n) = A + B$

(b) $\lim_{n\to\infty} K a_n = KA$ for any real number K.

(c) $\lim_{n\to\infty} a_n b_n = AB$

(d) $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{B}$; $b_n \neq 0$ & $B \neq 0$

(e) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$; $b_n \neq 0$, $B \neq 0$

PROOF: (a) Given $\varepsilon > 0$, there is a natural number N_1 such that $|a_n - A| < \varepsilon/2$ whenever $n > N_1$ and there is a natural number N_2 such that $|b_n - B| < \varepsilon/2$ whenever $n > N_2$. Now, if $n > \max(N_1, N_2)$, then we have

$$\begin{aligned} |(a_n + b_n) - (A+B)| &= |a_n - A + b_n - B| \\ &\leq |a_n - A| + |b_n - B| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence $\lim_{n\to\infty} (a_n + b_n) = A+B$

(b) If $K=0$, the result is obviously true since $|Ka_n - KA| = 0 < \varepsilon$ for every natural number. Suppose $K \neq 0$ and let $\varepsilon > 0$ be given. Then there is a natural number N such that $|a_n - A| < \frac{\varepsilon}{|K|}$ whenever $n > N$.

Now, if $n > N$, we have

$$\begin{aligned} |Ka_n - KA| &= |K|\,|a_n - A| \\ &< |K| \cdot \frac{\varepsilon}{|K|} = \varepsilon. \end{aligned}$$

Thus $\lim_{n\to\infty} Ka_n = KA.$

(c) Since $\{a_n\}$ is convergent, so it is bounded. (Say) $|a_n| \leq M$ for every natural number n. where M is a positive real number.

Now let $\varepsilon > 0$ be given. There is a natural number N_1 such that $|a_n - A| < \frac{\varepsilon}{2(1+|B|)}$ whenever $n > N_1$

(Notice that the ~~bounds~~ for $|a_n - A|$ was choosen to be $\frac{\varepsilon}{2(1+|B|)}$ instead of $\frac{\varepsilon}{2|B|}$ because B may be zero).

and there is a natural number N_2 such that

$$|b_n - B| < \frac{\varepsilon}{2M} \quad \text{whenever } n > N_2$$

Now if $n > \text{Max}(N_1, N_2)$ then we have

$$\begin{aligned}|a_n b_n - AB| &= |a_n b_n - a_n B + a_n B - AB| \\ &\leq |a_n b_n - a_n B| + |a_n B - AB| \\ &= |a_n|\,|b_n - B| + |B|\,|a_n - A| \\ &< M \frac{\varepsilon}{2M} + |B| \cdot \frac{\varepsilon}{2(1+|B|)} < \varepsilon\end{aligned}$$

Thus $\lim\limits_{n\to\infty} a_n b_n = AB$.

(d) We are assuming here that B and each b_n is non-zero. Let $\varepsilon > 0$ be given. There is a natural number N_1 such that $|b_n - B| < \frac{|B|}{2}$ whenever $n > N_1$. Hence if $n > N_1$, then

$$\frac{|B|}{2} > |b_n - B| = |B - b_n| \geq |B| - |b_n| \text{ and so}$$

$$|b_n| > \frac{|B|}{2}.$$ Also there is a natural number N_2 such that

$$|b_n - B| < \frac{\varepsilon \cdot |B|^2}{2} \quad \text{whenever } n > N_2$$

Therefore, if $n > Max(N_1, N_2)$, we have

$$\left|\frac{1}{b_n} - \frac{1}{B}\right| = \frac{|B - b_n|}{|b_n|\,|B|}$$

$$= \frac{|b_n - B|}{|b_n|\,|B|}$$

$$< \frac{\left(\varepsilon \cdot \frac{|B|^2}{2}\right)}{\frac{|B|}{2} \cdot |B|} = \varepsilon.$$

Thus $\lim\limits_{n\to\infty} \frac{1}{b_n} = \frac{1}{B}$.

(e) Given $\lim\limits_{n\to\infty} \frac{1}{b_n} = \frac{1}{B}$ & $\lim\limits_{n\to\infty} a_n = A$

Consider

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} \frac{1}{b_n}$$ $\because A$ & B both are definite.

$$= A \cdot \frac{1}{B}$$

$$= \frac{A}{B}.$$

e.g. $$\lim_{n\to\infty} \frac{4n^3 + 3n^2 + 2n + 1}{7n^3 - 2n^2 + 3} = \lim_{n\to\infty} \frac{4 + \frac{3}{n} + \frac{2}{n^2} + \frac{1}{n^3}}{7 - \frac{2}{n} + \frac{3}{n^3}}$$

Apply limit theorem simultaneously, to get

$$\lim_{n\to\infty} a_n = 4/7.$$

Assignment:

① Let $\{x_n\}$ be a sequence of real numbers and let $x \in R$. If $\{a_n\}$ is a sequence of positive real numbers with $\lim_{n\to\infty} a_n = 0$ and if for some constant $C > 0$ and some $m \in N$, we have

$$|x_n - x| \le C a_n \quad \forall\ n \geq m$$

then it follows that $\lim_{n\to\infty} x_n = x$. (BAR)

② If $\{x_n\}$ is a convergent sequence of real numbers and if $x_n \geq 0$ $\forall\ n \in N$ then $x = \lim_{n\to\infty} x_n \geq 0$ (BAR)

③ If $\{x_n\}$ and $\{y_n\}$ are convergent sequences of real numbers and if $x_n \le y_n$ for all $n \in N$ then $\lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n$ (BAR)

④ If $\{x_n\}$ is a convergent sequence and if $a \le x_n \le b$ $\forall\ n \in N$ then $a \le \lim_{n\to\infty} x_n \le b$

⑤ If $\lim_{n\to\infty} a_n = 0$ and $0 \le b_n \le a_n$ $\forall n \in N$ then $\lim_{n\to\infty} b_n = 0$ (SW) * For no hint *

⑥ If $\lim_{n\to\infty} a_n = A$, then $\lim_{n\to\infty} a_n^P = P^A$ (KOS) $\lim_{n\to\infty} a_n b_n = AB$.

H.M. KHALID MAHMOOD

Remark: If $\lim\limits_{n\to\infty} a_n = A$ & $\lim\limits_{n\to\infty} B_n = B$ and ~~atleast~~ either A or B is not definite. Then $\lim\limits_{n\to\infty} a_n b_n \neq \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

that is factorization is not allowed.

Example: $\lim\limits_{n\to\infty} \dfrac{n^2}{9n-1} \sin\left(\dfrac{1}{n}\right)$

let $a_n = \dfrac{n^2}{9n-1}$, $b_n = \sin\left(\dfrac{1}{n}\right)$

$\lim\limits_{n\to\infty} a_n = \infty$, $\lim\limits_{n\to\infty} b_n = 0$

Now $\lim\limits_{n\to\infty} \dfrac{n^2}{9n-1} \cdot \sin\left(\dfrac{1}{n}\right)$

$= \lim\limits_{n\to\infty} \dfrac{n^2}{9n-1} \cdot \sin\left(\dfrac{1}{n}\right) = \infty \cdot 0$
$= ?$

Consider

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{9}{n} - \frac{1}{n^2}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{-\frac{1}{n^2}}\left(9 - \frac{2}{n}\right)}$$

$$= \frac{1}{9} \quad \underline{\underline{\text{Ans}}}$$

Example (S.W):

① If $h>0$, then $\lim\limits_{K\to\infty} \frac{1}{1+Kh} = 0$ or $\frac{1}{1+Kh} \to 0$.

② If $|x|<1$, then $x^K \to 0$

③ If $P>0$, then $\sqrt[K]{P} \to 1$

④ If $\sqrt[K]{K} \to 1$

Solution: Since $h>0$, so

① $0 < Kh < 1+Kh$

imply $0 < \frac{1}{1+Kh} < \frac{1}{Kh} = \left(\frac{1}{K}\right)\left(\frac{1}{h}\right)$

But $\lim\limits_{K\to\infty} \frac{1}{K} = 0$

Thus $\lim\limits_{K\to\infty} \frac{1}{1+Kh} = 0$

② If $|x|<1$, then

$|x| = \frac{1}{1+h}$; $h>0$

Then $|x^K| = |x|^K$

$= \frac{1}{(1+h)^K}$

$\leq \frac{1}{1+Kh}$ By Bernoulli's inequality

$< \frac{1}{Kh}$

Thus $x^K \to 0$

If $|x| = 0$, then $x^K = 0 \;\forall K \in N$

So $\lim\limits_{K\to\infty} x^K = \lim\limits_{K\to\infty} 0 = 0$.

That is $x^K \to 0$ whenever $|x| < 1$.

(3) For $P = 1$, Result is trivial. So suppose $P > 1$

Then $\sqrt[K]{P} > 1$ and, we can write

$$\sqrt[K]{P} = 1 + h_K \quad \text{where } h_K > 0$$

or
$$P = (1 + h_K)^K$$
$$\geq 1 + K h_K \quad \text{By Bernoulli's inequality.}$$

So that $0 < h_K \leq \frac{P-1}{K}$.

That is $h_K < (P-1) \cdot \frac{1}{K}$

But $\frac{1}{K} \to 0$ as $K \to \infty$

Thus $h_K \to 0$ as $K \to \infty$

i.e. $\sqrt[K]{P} - 1 \to 0$ as $K \to \infty$

or $\sqrt[K]{P} \to 1$ as $K \to \infty$.

(4) ~~Notice that $K > 1$ (because $K \to \infty$)~~

If $0 < P < 1$, then $\frac{1}{P} > 1$ imply

$$\left(\frac{1}{P}\right)^{1/K} = \frac{1}{P^{1/K}} \to 1 \qquad \because \frac{1}{P} > 1 \text{ \& apply quotient rule.}$$

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Now by quotient rule

$$\frac{1}{p^{\frac{1}{k}}} \to \frac{1}{1} \text{ implies } p^{\frac{1}{k}} \to 1.$$

OR

$$\lim \frac{1}{p^{\frac{1}{k}}} \to 1 \quad \& \quad \lim 1 \to 1$$

Thus $$\lim \frac{1}{\frac{1}{p^{\frac{1}{k}}}} \to \frac{1}{1}$$

or $$\lim p^{\frac{1}{k}} \to 1$$

(4) Note that for $k>1$, $\sqrt[k]{k} > 1$ so that

$$\sqrt[k]{k} = 1 + h_k \quad ; \quad h_k > 0$$

or $$k = (1+h_k)^k$$

$$= \sum_{j=0}^{k} \binom{k}{j} h_k^j$$

$$> \binom{k}{2} h_k^2 = \frac{k(k-1)}{2} h_k^2$$

imply $$h_k^2 < \frac{2}{k-1}$$

But $\frac{2}{k-1} \to 0$ as $k \to \infty$

Thus $h_k^2 \to 0$ as $k \to \infty$

i.e $h_k \to 0$ as $k \to \infty$

or $\sqrt[k]{k} - 1 \to 0$ as $k \to \infty$

or $\sqrt[k]{k} \to 1$ as $k \to \infty$

(21P) THEOREM (Sandwich or Sequeeze Theorem):

If $a_n \le b_n \le c_n$ for every $n \in N$ and $\lim_{n\to\infty} a_n = A = \lim_{n\to\infty} c_n$ then $\lim_{n\to\infty} b_n = A$

Where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of real number.

PROOF: Suppose $a_n \le b_n \le c_n$ for every $n \in N$.

Then $a_n - A \le b_n - A \le c_n - A$.

It follows that

$$b_n - A \le |c_n - A| \quad \text{—— (1)}$$

and $-(b_n - A) \le -(a_n - A) \le |a_n - A|$ —— (2)

Now given any $\varepsilon > 0$, since $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} c_n = A$ there exist natural number N_1 such that

$|a_n - A| < \varepsilon$ whenever $n > N_1$ and

there exist a natural number N_2 such that

$|c_n - A| < \varepsilon$ whenever $n > N_2$

If $n > \max(N_1, N_2)$, then by (1)

$$b_n - A \le |c_n - A| < \varepsilon$$

So $b_n - A < \varepsilon$ —— $(1)^*$

Also by (2). $-(b_n - A) \le |a_n - A| < \varepsilon$.

or $-\varepsilon \le b_n - A$ —— $(2)^*$

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By ①* and ②*

$$-\varepsilon < b_n - A < \varepsilon$$

i.e. $|b_n - A| < \varepsilon$ whenever $n > \max(N_1, N_2)$

Hence $\lim_{n\to\infty} b_n = A$.

Applications: ① $\lim_{n\to\infty} \frac{\sin n}{n} = 0$

we know $-1 \le \sin n \le 1 \quad \forall n \in N$.

then $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$

But $\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$

Hence by Sandwich theorem

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0$$

② Sequence $\{n\}$ is divergent.

Suppose $\{n\}$ is convergent.

Then it is bounded.

imply $|n| \le M$ for som fixed positive M.

$\Rightarrow -M \le n \le M$

~~But $\lim -M \ne \lim M$~~

~~Hence $\lim n$ does not exist.~~

which is a contradiction against archimedian property becs $n \in N$ it may be sufficiently large.

Thus, it is divergent.

THEOREM: Let the sequence $\{x_n\}$ converges to x. Then the sequence $\{|x_n|\}$ of absolute values converges to $|x|$. That's, if $x = \lim_{n\to\infty} x_n$, then $|x| = \lim_{n\to\infty} |x_n|$

PROOF: For given $\varepsilon > 0$, there is a natural number N such that

$$|x_n - x| < \varepsilon \quad \text{whenever} \quad n > N$$

From triangular inequality, we follow,

$$\big||x_n| - |x|\big| \leq |x_n - x| < \varepsilon.$$

That is

$$\big||x_n| - |x|\big| < \varepsilon \quad \text{whenever} \quad n > N.$$

Therefore $\{|x_n|\}$ is convergent and

$$\lim_{n\to\infty} |x_n| = |x|.$$

Assignment (BAR)

Let $\{x_n\}$ be a sequence of real numbers that converges to x and suppose $x_n \geq 0$. Then the sequence $\{\sqrt{x_n}\}$ of positive square root converges and

$$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$$

(KOS) MONOTONE SEQUENCES:-

A sequence $\{a_n\}$ is said to be

(a) Increasing iff for all $n, m \in N$ and $n \le m$, we have $a_n \le a_m$

(b) Eventually increasing iff there exist $n^* \in N$ such that for all $n, m \in N$ and $n^* \le n \le m$, we have $a_n \le a_m$

(c) Strictly increasing if and only if ~~there exist n^*~~ $\in N$ for all $n, m \in N$ and $n < m$, we have $a_n < a_m$

(d) Eventually strictly increasing iff there exist $n^* \in N$ such that for all $n, m \in N$ and $n^* \le n < m$, we have $a_n < a_m$.

In a similar fashion, we can define decreasing, eventually decreasing, strictly decreasing and eventually strictly decreasing.

Example

① $1, 1, 2, 2, 3, 3, \cdots$

② $4, 2, 1, 3, 3, 5, 5, \cdots$

③ $1, 3, 5, 7, \cdots$

④ $4, 3, 5, 7, \cdots$

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(KOS) **Remark:** Observe that the following are all equivalent statements for a sequence $\{a_n\}$

(a) $\{a_n\}$ is increasing

(b) $\{a_n\}$ $a_n \leq a_{n+1}$ for all $n \in N$.

(c) $a_{n+1} - a_n \geq 0$ for all $n \in N$.

If in addition $a_n > 0$, then the above are equivalent to

(d) $\dfrac{a_n+1}{a_n} \geq 1$ for all n.

** Explain Difference See P-84 KOS **

Example: See G.M P-58

(1) $\left(1+\frac{1}{n}\right)^n$ is increasing; $n \geq 1$

(2) $\left(1+\frac{1}{n}\right)^{n+1}$ is decreasing; $n \geq 1$

(3) $\dfrac{n-1}{n^2+2}$, $n \geq 3$ is decreasing

(4) ~~$\frac{1}{n}\cos n\pi$~~ Assignment

Monotone: $\{a_n\}$ is said to be monotone if it is either increasing or decreasing.

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(2iP) THEOREM:

If $\{a_n\}$ is monotone increasing and bounded above then $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = \sup_{n\in N}\{a_n\}$$

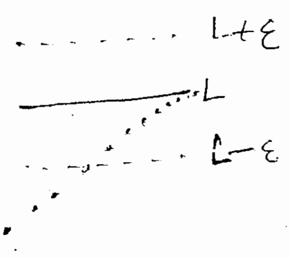
PROOF: Since $\{a_n : n\in N\}$ is bounded above, so by least-upper-bound property of real numbers the set $\{a_n : n\in N\}$ has a supremum. let it be L.

Given any $\varepsilon>0$ there is an natural number n^* such that $L-\varepsilon < a_{n^*} \leq L$, for otherwise $L-\varepsilon$ would be an upper bound of $\{a_n : n\in N\}$ less than the least upper bound L. Now if $n>n^*$, then

$$L-\varepsilon < a_{n^*} \leq a_n \quad \because a_n \text{ is increasing}$$

$$\leq L$$

$$< L+\varepsilon.$$



Thus $L-\varepsilon < a_n < L+\varepsilon$

iff $|a_n - L| < \varepsilon$.

Therefore $\lim_{n\to\infty} a_n = L = \sup\{a_n : n\in N\}$

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THEOREM: (BIN)

If $\{a_n\}$ is monotone decreasing and bounded below, then

$$\lim_{n\to\infty} a_n = l = \inf\{a_n : n \in N\}$$

Give Proof as above

Alternative See BAR

(BAR) Monotone Convergence theorem:

Statement:

A monotone sequence of real numbers is convergent iff it is bounded.

Remark: (ZIP)

Monotone Convergence theorem is quite helpfull to determine the convergence of a sequence despite it is difficult to determine exactly what the limit of a given sequence is.

(ZIP) Euler Number

Example: $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.$

let $e_n = \left(1+\frac{1}{n}\right)^n$, we will show that $\{e_n\}$ is bounded above and mononoton, then it is convergent.

We know

$$(1+p)^n > 1+np \; ; \; p \in R, \; p \geq -1, \; p \neq 0 \; \& \; n \geq 2.$$

put $p = -\frac{1}{n^2}$, $n \geq 2$, then

$$\left(1-\frac{1}{n^2}\right)^n > 1 - \frac{n}{n^2}$$

or $\left(1-\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)^n > \left(1-\frac{1}{n}\right)$

imply $\left(1+\frac{1}{n}\right)^n > \left(1-\frac{1}{n}\right)^{1-n} = \left(\frac{n-1}{n}\right)^{1-n}$

$$= \left(\frac{n}{n-1}\right)^{n-1}$$

$$= \left(1-\frac{1}{n-1}\right)^{n-1}$$

Therefore $\left(1+\frac{1}{n}\right)^n > \left(1-\frac{1}{n-1}\right)^{n-1}$

or $e_n > e_{n-1}$ and $n > n-1$.

Hence $\{e_n\}$ is increasing.

Moreover

$$e_n = \left(1+\frac{1}{n}\right)^n$$

$$= 1 + n\cdot\frac{1}{n} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{1}{n}\right)^k + n\left(\frac{1}{n}\right)^{n-1} + \left(\frac{1}{n}\right)^n$$

$$e_n \leq 1 + \frac{1}{1!} + \cdots + \frac{1}{K!} + \cdots + \frac{1}{n!} \quad ; n \geq 1.$$

$$\leq 1 + \frac{1}{2^0} + \cdots + \frac{1}{2^{K-1}} + \cdots + \frac{1}{2^{n-1}} \quad \because 2^{n-1} \leq n! \text{ implies } \frac{1}{2^{n-1}} \leq \frac{1}{n!} \; n \in N.$$

$$= 1 + \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}$$

$$< 1 + \frac{1}{1 - \frac{1}{2}} = 3$$

i.e $\{e_n\} < 3$. Also $e_n \geq 2 \; \forall n \geq 1$.

which shows that $\{e_n\}$ is bounded above.

Thus $\{e_n\}$ is convergent and its limit does not exceed 3. and always greater than 2 for every $n \in N$.

We call the limit of this sequence the real number e and write

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

Recursive (or Inductive) Steps to find limits (if exists):

Example (KOS) Consider a sequence defined by

$$a_1 = \sqrt{2} \quad \text{and} \quad a_{n+1} = \sqrt{2 + a_n} \text{ for } n \in N$$

Determine convergence or divergence.

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Sol By induction

$$a_2 = \sqrt{2+\sqrt{2}}$$
$$> \sqrt{2} = a_1$$

Suppose $a_{k+1} > a_k$

Consider

$$a_{k+2} = \sqrt{2 + a_{k+1}}$$
$$> \sqrt{2 + a_k} = a_{k+1}$$

Thus $a_{k+2} > a_{k+1} \quad \forall k \in N.$

Hence this sequence is strictly increasing for all $n \in N$. Next, we will show that $\{a_n\}$ is bounded above, by (say) 3.

Note that $a_1 = \sqrt{2} < 3$.

Suppose $a_k < 3$

Consider $a_{k+1} = \sqrt{2 + a_k}$
$$< \sqrt{2+3}$$
$$< 3.$$

Therefore $a_n < 3 \quad \forall n \in N.$

Therefore $\{a_n\}$ is bounded above increasing sequence.

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Then $\{a_n\}$ is convergent and its limit exist. let it be a.

Then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = a$.

So that

$$\lim_{n\to\infty} a_{n+1} = \sqrt{2 + \lim_{n\to\infty} a_n}$$

$$a = \sqrt{2+a}$$

imply $a^2 = 2 + a$

or $a^2 - a - 2 = 0$

or $a = -1, 2$

Since $\{a_n\}$ is certainly positive

Hence $\lim_{n\to\infty} a_n = 2$.

Assignment: Find limits (if exists)

① $y_1 = 1,\ y_{n+1} = \frac{1}{4}(2y_n + 3)$; $n \geq 1$

② $z_1 = 1,\ z_{n+1} = \sqrt{2 z_n}$; $n \in N$

③ Calculation of Square root.

$$S_{n+1} = \frac{1}{2}\left(S_n + \frac{a}{S_n}\right),\ a>0 \text{ and } n \in N.$$

Hint See (BAR) P-71

④ Show by inductively $\lim_{n\to\infty} r^n = 0$; $0<r<1$

(KOS) ⑤ For any $x \in R$, there exist n^* such that $n^* > x$ (A-Principal) use Induction

SUBSEQUENCES:

Def: The sequence $\{b_n\}_{n=i}^{\infty}$ is a subsequence of $\{a_n\}_{n=k}^{\infty}$ with $i, k \in N$, if and only if there exist a strictly increasing function f, where

$$f : Z_i \to Z_k \quad \text{and} \quad b_n = a_{f(n)} \quad \forall\, n \in Z_i.$$

OR A sequence $\{b_k\}$ is called a subsequence of the sequence $\{a_n\}$ if there are natural numbers $n_1 < n_2 < n_3 < \cdots$ such that

$$b_k = a_{n_k} \text{ for } k = 1, 2, 3, \cdots$$

Examples:

$n_1 \quad n_2 \quad n_3 \quad n_r$

$b_1 = a_{n_1} \quad b_2 = a_{n_2} \quad b_r = a_{n_r}$

(1) Let $a_n = n$, $b_n = 2n$ and $f(n) = 2n$

To see $\{b_n\}$ is a subsequence of $\{a_n\}$, write out some terms

$$a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ a_4 = 4,\ a_5 = 5,\ a_6 = 6, \ldots$$

$$b_1 = 2 \qquad b_2 = 4 \qquad b_3 = 6, \ldots$$

because $b_n = a_{f(n)}$ by def.

Therefore $b_n = a_{2n}$ $\forall\, n \in N$ and

Clearly 'f' is strictly increasing.

KOS (2) $a_n = \frac{1}{n}$, $b_n = \frac{1}{n^2}$ and $f(n) = n^2$

To visualize, we write,

$a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, $a_4 = \frac{1}{4}$, ...

$b_1 = 1$ $\quad$ $b_2 = \frac{1}{4}$, ...

Therefore $b_n = a_{n^2}$ $\forall n \in N$. Thus

$b_n = a_{f(n)}$ and 'f' is strictly increasing.

(3) Let $a_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \frac{1}{n} & \text{if } n \text{ is even} \end{cases}$, $b_n = \frac{1}{n}$ and $f(n) = 2n$.

Clearly $b_n = a_{2n}$ $\forall n \in N$ and 'f' is strictly increasing.

KOS REMARK: we obtain subsequence $\{b_n\}$ from the sequence $\{a_n\}$. Notice that only certain values are used since the terms of subsequence comes from a_n. These values of n are in ascending order ~~and~~ with their own subscripts indicating the position in the subsequence $\{b_n\} = \{a_{n_k}\}$. The term a_{n_1} is the first term of the subsequence, the term a_{n_2} is the 2nd term of the subsequence. Therefore if $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$, then $n_k \geq k$ for each $k \in N$.

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(SW)

Proposition: If $\{x_k\}$ converges to x, then every subsequence $\{x_{n_k}\}$ of $\{x_k\}$ also converges to x.

PROOF: Since $\lim_{k\to\infty} x_k = x$, For given $\varepsilon > 0$ there exist a natural number n such that

$k \geq n$ implies $|x_k - x| < \varepsilon$.

Now $k \geq n$ implies $n_k \geq k$. For $\varepsilon > 0$,

$n_k \geq k \geq n$ implies $|x_{n_k} - x| < \varepsilon$.

or $n_k \geq n$ implies $|x_{n_k} - x| < \varepsilon$.

Hence $x_{n_k} \to x$ as $k \to \infty$.

(SW) **Urysohn Property:**

Statement: The sequence $\{x_k\}$ converges to x iff, every subsequence of $\{x_k\}$ has a subsequence that converges to x.

PROOF: Suppose $\{x_k\}$ converges to x. Then the subsequence $\{x_{n_k}\}$ converges to x (See proof given above). Now, since $\{x_{n_k}\}$ converges to x, then by argument stated ~~proved~~ above ~~[illegible]~~, the subsequence of $\{x_{n_k}\}$ also converges to x.

To prove its converse, we will prove the contrapositive statement of converse statement, that is actually the inverse statement of first statement.

Suppose $\{x_k\}$ does not converges to x.

So, there must be atleast one $\varepsilon > 0$ such that $|x_k - x| \geq \varepsilon$ for infinitely many k.

That is for each positive integer j, there is n_j with $|x_{n_j} - x| \geq \varepsilon$. ——(1) More over n_j can be choosen so that $n_{j+1} \geq n_j$.

Thus $\{x_{n_j}\}$ is a subsequence of $\{x_k\}$ and no subsequence of $\{x_{n_j}\}$ can converge to x by argument as in (1). Which complete the proof.

Modified Form:

(KOS) THEOREM: A sequence $\{x_n\}$ converges to A if and only if each of its subsequences converges to A.

PROOF: If every subsequence of $\{x_n\}$ converges to A, then $\{x_n\}$ converges to A since it is subsequence of itself.

Conversely, suppose $\{x_n\}$ converges to A. Choose arbitrary subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Since it is convergent and converges to A (proved) so, we establish the result.

(SW) **Proposition:** Let $\{x_K\}$ be a real sequence. If both subsequences $\{x_{2K}\}$ and $\{x_{2K-1}\}$ converges to x, then $\{x_K\}$ converges to x.

PROOF: Let $\varepsilon > 0$ be given. Then there are n_1 and n_2 such that $K \geq n_1$ implies $|x_{2K} - x| < \varepsilon$ and $K \geq n_2$ implies $|x_{2K-1} - x| < \varepsilon$. If $n = \max\{2n_1, 2n_2 - 1\}$ then $K \geq n$ implies $|x_K - x| < \varepsilon$.

Hence $\{x_K\}$ converges to x.

(BR) **REMARK:**

A sequence $\{x_n\}$ is divergent if it satisfy any of the following properties

① $\{x_n\}$ has two convergent subsequences whose limits are not equal.

② $\{x_n\}$ is unbounded.

Example: Let $C_n = \left(1 - \frac{1}{n}\right) \sin \frac{n\pi}{2}$

Then obviously $\{C_n\}$ is divergent.

Consider $C_{2K} = \left(1 - \frac{1}{2K}\right) \sin\left(2K\frac{\pi}{2}\right) = 0$; $K = 1, 2, \cdots$

Thus $\{C_{2K}\}$ converges to 0.

Also $C_{2K+1} = \left(1 - \frac{1}{2K+1}\right) \sin\left((2K+1)\frac{\pi}{2}\right)$ $\quad K = \cancel{0}, 1, 3, 5, \cdots$

$= \left(1 - \frac{1}{2K+1}\right)(-1)$

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Thus $\lim_{K\to\infty} c_{2K+1} = -1$

Since $\lim_{K\to\infty} c_{2K} \neq \lim_{K\to\infty} c_{2K+1}$

Thus $\{c_n\}$ is divergent.

(BR) Monotone Subsequence theorem:

THEOREM: If $\{x_n\}$ is a sequence of real number, then there is a subsequence of $\{x_n\}$ that is monotone.

(BIN) OR Every Sequence has a monotone subsequence.

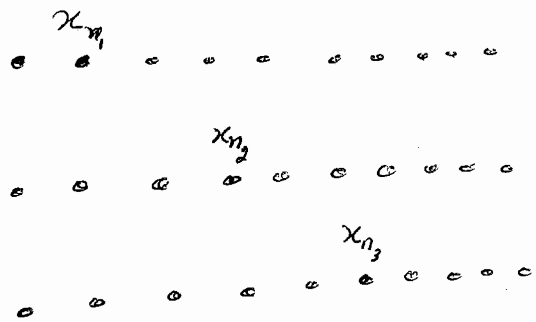
PROOF: Let $\{x_n\}$ be a sequence of real numbers. We must construct a subsequence of $\{x_{n_r}\}$ which is either increasing or decreasing. We have two cases

Case I: Every set $\{x_n : n > N\}$ has a maximum. In this case we can find a sequence n_r of natural numbers such that

$$x_{n_1} = \max_{n>1} x_n$$

$$x_{n_2} = \max_{n>n_1} x_n$$

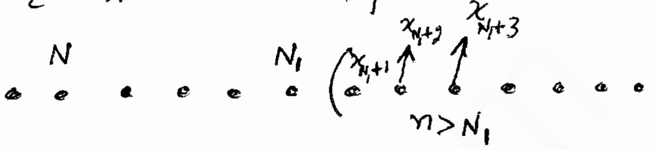
$$x_{n_3} = \max_{n>n_2} x_n$$



and so on. Obviously $n_1 < n_2 < n_3 \cdots$. Thus at each stage we are taking the maximum of the smaller set that at the previous stage. Hence $\{x_{n_r}\}$ is a decreasing subsequences of $\{x_n\}$.

case 2: Suppose that it is not true that all the sets $\{x_n : n > N\}$ have a maximum.

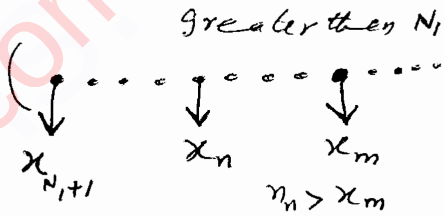
Then for some N_1, the set $\{x_n : n > N_1\}$ has no maximum.



It follows that given any x_m with $m > N_1$, we can find x_n such that $x_n > x_m$ (otherwise the biggest of $x_{N_1+1}, \ldots, x_m$ is the max of $\{x_n : n > N_1\}$.

{That is if there exist no x_n between x_{N_1+1} and x_m. (arbitrary)

Then biggest of $x_{N_1+1}, \ldots, x_m$ must be the maximum of $x_n : n > N_1\}$



Define $x_{n_1} = x_{N_1+1}$ and let x_{n_2} be the first term following (after x_{n_1}) x_{n_1} for which $x_{n_2} > x_{n_1}$

Now let x_{n_3} be the first term following x_{n_2} for which $x_{n_3} > x_{n_2}$ (obviously $x_{n_3} > x_{n_1}$)

$(2, 4, 1, 1, 6, \ldots)$

$x_{n_1} = x_{N_1+1}$ $\quad$ x_{n_2}

6 is the first term after 4 such that $6 > 4$.

and so on. Thus we obtain an increasing subsequence $\{x_{n_r}\}$ of $\{x_n\}$. Thus in either case, we have the proof.

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(BR) The Bolzano-Weierstrass Theorem:

(BIN) Statement:

A bounded sequence of real numbers has a convergent subsequence.

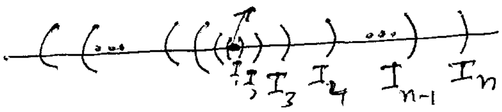
PROOF: Let $\{x_n\}$ be a bounded sequence. Then by Monotone subsequence theorem, $\{x_n\}$ has a monotone subsequence $\{x_{n_r}\}$. Since $\{x_n\}$ is bounded, so $\{x_{n_r}\}$ is a bounded monotone subsequence of $\{x_n\}$. Therefore by Monotone convergent theorem $\{x_{n_r}\}$ is convergent. Hence every bounded sequence has a convergent subsequence.

Alternative PROOF:

(ZIP) THEOREM (NESTED INTERVAL PROPERTY)

Statement: Suppose that $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, $I_3 = [a_3, b_3], \ldots$, where $I_1 \supseteq I_2 \supseteq I_3 \ldots$ and $\lim_{n\to\infty}(b_n - a_n) = 0$, Then there is exactly one real number common to all the intervals I_n.

PROOF:



Given $\{a_n\}$ is a sequence which is monotone increasing and bounded above (b_1 is an upper bound). Therefore $\lim_{n\to\infty} a_n$ exists. Let it be A. Moreover $\lim_{n\to\infty} a_n = \sup_n a_n$ because $\{a_n\}$ is increasing and bounded above. Then $A = \lim_{n\to\infty} a_n \le b_k$ for each natural k.

Since $A = \sup_n a_n$, so

$$a_k \leq A \leq b_k \text{ for every } k \in \mathbb{N}$$

That is A is contained in each interval I_k. Suppose $B \in I_n$ for every natural number n. Then $a_n \leq B \leq b_n \ \forall n$.

Imply $0 \leq B - a_n \leq b_n - a_n$ for each n.

But $\lim_{n\to\infty} 0 = \lim_{n\to\infty} (b_n - a_n) = 0$

By Sequeeze theorem

$$\lim_{n\to\infty} (B - a_n) = 0$$

or $$B = \lim_{n\to\infty} a_n = A.$$

So A is the only real number common to all the intervals I_n.

(ZIP) Bolzano-Weierstrass theorem

Statement: Every bounded sequence of real numbers has a convergent subsequence.

PROOF: If $\{a_n\}$ is bounded, then there is a positive real number M such that $|a_n| \leq M$ for every natural number n.

Hence $a_n \in [-M, M]$, $\forall n \in \mathbb{N}$. Consider the intervals $[-M, 0]$ and $[0, M]$. Atleast one of these two intervals must contain a_n for infinitely many natural numbers n. (Note that a_n are not necessarily distinct) call such an interval I_0. Next divide I_0 into two subintervals of equal length; ~~then~~ atleast one of these subintervals must contain a_n for infinitely many ~~values of~~ natural number n. call such an interval I_1. continue in this manner to obtain a sequence of intervals $I_0, I_1, \cdots$ with $I_0 \supset I_1 \supset I_2 \ldots$. The length of I_n is $\frac{M}{2^n}$.

and $\lim_{n\to\infty} \frac{M}{2^n} = 0$. Therefore by nested interval property, there is exactly one point 'A' common to all these intervals. Choose $a_{n_1} \in I_1$, $a_{n_2} \in I_2$ with $n_2 > n_1$, $a_{n_3} \in I_3$ with $n_3 > n_2$ ... (these selection can be made since each I_K contains a_n for infinitely natural number n)

Then $\{a_{n_K}\}$ is a subsequence of a_n and a_{n_K} and A are both contained in I_K.

thus $|a_{n_K} - A| < \frac{M}{2^K}$

and so $\lim_{K\to\infty} a_{n_K} = A$ which complete the proof

Assignment (BR)

Let $\{x_n\}$ be a bounded sequence of real numbers and let $x \in R$ have the property that every convergent subsequence of $\{x_n\}$ converges to x. Then the sequence $\{x_n\}$ converges to x.

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KOS CAUCHY SEQUENCE

Definition:

A sequence $\{a_n\}$ is called a Cauchy sequence if and only if for each $\varepsilon > 0$ there exist $n^* \in N$ such that $|a_m - a_n| < \varepsilon$ for all $m, n \geq n^*$.

Example: Suppose that the sequence $\{a_n\}$ satisfied $|a_{n+1} - a_n| \leq \frac{1}{2^n}$. Show that $\{a_n\}$ is cauchy.

Sol: let m and n be two distinct natural numbers. Without the loss of generality, suppose that $m > n$. Consequently, we have

$$|a_m - a_n| \leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+2} - a_{n+1}| + |a_{n+1} - a_n|$$

$$\leq \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^{n+1}} + \frac{1}{2^n}$$

$$= \frac{1}{2^n}\left(\frac{1}{2^{m-n-1}} + \frac{1}{2^{m-n-2}} + \cdots + \frac{1}{2} + 1\right)$$

$$< \frac{1}{2^n}(2) = \frac{1}{2^{n-1}}$$

$$|a_m - a_n| < \frac{1}{2^{n-1}} \quad \text{——} \ (1)$$

let $\varepsilon > 0$ be given. Since $\lim\limits_{n \to \infty} \frac{1}{2^{n-1}} = 0$,

there exist n^* such that $\frac{1}{2^{n-1}} < \varepsilon \ \forall n \geq n^*$

(1) imply $|a_m - a_n| < \varepsilon \Rightarrow \{a_n\}$ is cauchy.

(BR) ② $\{\frac{1}{n}\}$ is a Cauchy sequence.

Let $\varepsilon > 0$ be given. Choose a natural number n^* depending on ε such that $\frac{1}{n^*} < \varepsilon/2$. Then if $m, n \geq n^*$

then $\frac{1}{m} \leq \frac{1}{n^*} < \varepsilon/2$

Similarly $\frac{1}{n} < \varepsilon/2$.

then

$$\left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{m} + \frac{1}{n}$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \text{for all } m, n \geq n^*$$

Hence $\{\frac{1}{n}\}$ is a Cauchy sequence.

(BR) REMARK: To prove a sequence $\{x_n\}$ is Cauchy we have no need to find relation between m and n because the inequality $|x_m - x_n| < \varepsilon$ for all $m, n \geq n^*(\varepsilon)$. But to prove a sequence is not Cauchy sequence, we may specify a relation between n and m as long as arbitrary large values of n and m can be choosen so that

$$|x_n - x_m| \geq \varepsilon_0$$

Example: Show that $\{(-1)^n\}$ is not a Cauchy sequence.

let $a_n = (-1)^n$. let $m = n+1$. for all n.

We have
$$|a_m - a_n| = |a_{n+1} - a_n|$$
$$= 2 > 1 = \varepsilon \text{ (say)}$$

Thus $\{(-1)^n\}$ is not a Cauchy sequence.

(Imp) Lemma: Every convergent sequence of real numbers is a cauchy sequence.

Proof: Suppose $\lim_{n\to\infty} a_n = A$. Then given $\varepsilon > 0$ there is a natural number n^* such that

$$|a_n - A| < \varepsilon/2 \text{ whenever } n > n^*.$$

Therefore if $m, n > n^*$, we have

$$|a_m - a_n| = |a_m - A + A - a_n|$$
$$\leq |a_m - A| + |a_n - A|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

and so $\{a_n\}$ is a cauchy sequence.

Completeness of $\mathbb{R}$:

Every Cauchy sequence of real numbers converges. (It is obvious that it will converge to a point of $\mathbb{R}$).

(2ip) THEOREM: Every cauchy sequence is bounded.

PROOF: Suppose $\{a_n\}$ is a cauchy sequence. Then for $\epsilon = 1$, there is a natural number n^* such that $|a_m - a_n| < 1$ whenever $m, n > n^*$.

Choose $n_0 > n^*$ and observe that

$$|a_n| = |a_n - a_{n_0} + a_{n_0}|$$
$$\leq |a_n - a_{n_0}| + |a_{n_0}|$$
$$< 1 + |a_{n_0}| \quad \text{whenever } n > n^*$$

let $M = \max(|a_1|, |a_2|, \dots, 1+|a_{n_0}|)$

Clearly $|a_n| \leq M$. for every natural number n and hence $\{a_n\}$ is a bounded sequence.

(2ip) THEOREM: Every cauchy sequence of real numbers converges.

PROOF: Suppose $\{a_n\}$ is a cauchy sequence of real numbers. Then $\{a_n\}$ is bounded because every cauchy sequence is bounded. Then by Bolzano-Weierstrass theorem $\{a_n\}$ has a subsequence, (say) $\{a_{n_k}\}$ which

converges. Suppose that $\lim_{k\to\infty} a_{n_k} = A$.
We will show that $\lim_{n\to\infty} a_n = A$.
Let $\varepsilon > 0$ be given. Since $\{a_n\}$ is a Cauchy sequence, there exist natural number n^* such that $|a_m - a_n| < \varepsilon/2$ whenever $m, n > n^*$.
Choose a natural number n_k so large that

$$|a_{n_k} - A| < \varepsilon/2 \; ; \; n_k > n^*.$$

Then if $n > n^*$, we have

$$|a_n - A| = |a_n - a_{n_k} + a_{n_k} - A|$$
$$\leq |a_n - a_{n_k}| + |a_{n_k} - A|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus $\{a_n\}$ converges to A.
That is $\lim_{n\to\infty} a_n = A$.
Since $\{a_n\}$ was arbitrary, so every (BR) Cauchy sequence of real numbers converges.

Cauchy Convergence Criterion

Statement: A sequence of real numbers is convergent if and only if it is a Cauchy sequence. (Assignment)

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Contractive Sequence

Def: A sequence $\{a_n\}$ of real numbers is said to be contractive sequence if and only if there exists a constant K, with $K \in (0,1)$ such that $|a_{n+2} - a_{n+1}| \leq K|a_{n+1} - a_n| \; \forall n \in \mathbb{N}$.

Example: Define $a_1 = 1$, $a_{n+1} = 1 + \frac{1}{2} a_n \; \forall n \in \mathbb{N}$. Then $\{a_n\}$ is a contractive sequence.

Consider

$$|a_{n+2} - a_{n+1}| = \left|\left(1 + \frac{1}{2} a_{n+1}\right) - \left(1 + \frac{1}{2} a_n\right)\right|$$

$$= \frac{1}{2}|a_{n+1} - a_n|$$

Since $0 < K = \frac{1}{2} < 1$, the sequence $\{a_n\}$ is contractive.

Contraction Principle

Statement: Every contractive sequence is a Cauchy sequence, and hence convergent.

PROOF: Let $\{a_n\}$ be a contractive sequence. Then for $K \in (0,1)$, apply inequality of definition successively, we have

$$|a_{n+2} - a_{n+1}| \leq K|a_{n+1} - a_n| \leq K^2|a_n - a_{n-1}|$$

$\ldots$, we get

$$|a_{n+2} - a_{n+1}| \leq K^n |a_2 - a_1| \quad \text{——(1)}$$

Without any loss of generality, assume $m > n$ and we write

$$|a_m - a_n| \leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n|$$

$$\leq (K^{m-2} + K^{m-3} + \cdots + K^{n-1}) |a_2 - a_1| \text{ by (1)}$$

$$= K^{n-1}(K^{m-n-1} + K^{m-n-2} + \cdots + 1) |a_2 - a_1|$$

$$= K^{n-1}\left(\frac{1 - K^{m-n}}{1-K}\right) |a_2 - a_1|$$

$$< K^{n-1}\left(\frac{1}{1-K}\right) |a_2 - a_1| \quad \text{——(2)}$$

Since $\frac{|a_2 - a_1|}{1-K}$ is constant and

$$\lim_{n\to\infty} K^{n-1} = 0 \quad ; \quad 0 < K < 1$$

Then for large n,

$$\frac{K^{n-1}}{1-K} |a_2 - a_1| < \varepsilon > 0$$

and 2 gives

$|a_m - a_n| < \varepsilon$ for large n with $m > n$.

Hence $\{a_n\}$ is a Cauchy sequence and hence convergent. H.M. KHALID MAHMOOD

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(BR) Corollary (Assignment)

Suppose $\{a_n\}$ is a contractive sequence. If $\{a_n\}$ converges to A and $K \in (0,1)$, then

(a) $|a_n - A| \le \dfrac{K^{n-1}}{1-K} |a_2 - a_1| \quad \forall n \in N.$

(b) $|a_n - A| \le \dfrac{K}{1-K} |a_n - a_{n-1}|$, if $n > 1$.

PROPERLY DIVERGENT SEQUENCES

(KOS) Infinite Limits:

Def: A sequence $\{a_n\}$ diverges to $+\infty$ (tends to $+\infty$) if and only if for any $M>0$ there exist $n^* \in N$ such that $a_n > M \; \forall n > n^*$.

If this is the case, then the limit exists and we can write $\lim\limits_{n\to\infty} a_n = +\infty$.

Notice that M need not be an integer. ~~and positive because def will work again if M is negative.~~ Moreover

" A sequence $\{a_n\}$ diverging to $-\infty$ iff for any $K>0$ there exist n^* such that

$a_n < -K \quad \forall n \ge n^*.$

Remark: If $\{a_n\}$ tends to $+\infty$, then $\{-a_n\}$ tends to $-\infty$, and vice versa. To prove $\lim\limits_{n\to\infty} a_n =$ constant, we need to start with an arbitrary $\varepsilon > 0$, but to prove that a sequence diverges to infinity, we need to start with an arbitrary $M > 0$, which usually represents a large number. Also note that if $\{a_n\}$ diverges to ∞, then the limit exists, even though its value is $+\infty$ or $-\infty$.

Oscillate Sequence:

A divergent sequence that does not tends to infinity is called an oscillate sequence.

e.g $\quad a_n = (-1)^n$.

Observe that not all oscillating sequences are bounded. (Find examples Assignment). In addition, it turns out that any desired sequence can be written as the sum of two properly choosen oscillating sequences.

e.g $\quad a_n = (-1)^n, \quad b_n = (-1)^{n+1}$. Then $\{a_n\}$ and $\{b_n\}$ both divergent. But there sum is convergent. Similarly we can discuss other cases.

Note that if $\{a_n\}$ and $\{b_n\}$ are divergent so that their limits exists, then $\{a_n + b_n\}$ is also divergent.

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Properly Divergent:

A sequence $\{a_n\}$ is properly divergent if either $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$

(RJ) **Proposition:** let $\lim_{k\to\infty} a_k = \infty = \lim_{k\to\infty} b_k$

then $\lim_{k\to\infty} \{a_k + b_k\} = \infty$.

Proof: let M be a real number. There exist a positive integer n_1 such that

if $k \geq n_1$, then $\frac{M}{2} < a_k$ — ①

Also there exist a positive integer n_2 such that

if $k \geq n_2$, then $\frac{M}{2} < b_k$. — ②

If $n \geq \max(n_1, n_2)$, then

$$M = \frac{M}{2} + \frac{M}{2} < a_k + b_k \quad \text{by ① \& ②}$$

thus $a_k + b_k > M$ for $k \geq n$.

Therefore $\{a_k + b_k\}$ is divergent sequence and $\lim_{k\to\infty} \{a_k + b_k\} = \infty$ by def.

(SW) **Proposition:** let $x_k \to \infty$

(i) If $\{y_k\}$ is bounded below, then

$$\lim (x_k + y_k) = \infty.$$

(ii) If $t > 0$, then $t x_k \to \infty$

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(iv) If $x_K > 0$ and $x_K \to 0$, then $\frac{1}{x_K} \to \infty$.

PROOF:

(i) Suppose $y_K \geq b$ $\forall K$ (because y_K is bounded below).

let $M > 0$, there is n such that $K \geq n$ implies $x_K \geq M - b$ (since $x_K > M$)

then $K \geq n$ implies $x_K + b \geq M$

$\Rightarrow x_K + y_K \geq M$.

$\Rightarrow \lim (x_K + y_K) = \infty$.

(ii) If $M > 0$, there exist n such that

$x_K > \frac{M}{t}$ for $K \geq n$

imply $t x_K > M$ for $K \geq n$.

Thus by def $\lim t x_K = \infty$

(iii) let $\varepsilon > 0$ be given. Then there exist n such that ~~$|x_K| < \varepsilon$~~. let $M = \frac{1}{\varepsilon}$

~~imply~~ $x_K > \frac{1}{\varepsilon}$

then $\frac{1}{x_K} < \varepsilon \Rightarrow \left|\frac{1}{x_K} - 0\right| < \varepsilon$.

imply $\frac{1}{x_K} \to 0$

(iv) If $x_K \to 0$ and $x_K > 0$, then $\frac{1}{x_K} \to \infty$.

Proof let $M > 0$. Then there exist n such that

$x_K < \frac{1}{M}$ for $K \geq n$

$\Rightarrow \frac{1}{x_K} > M$ imply $\frac{1}{x_K} \to \infty$.

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(KOS) THEOREM Consider a sequence $\{a_n\}$ where $a_n > 0 \ \forall n$. Then $\{a_n\}$ diverges to $+\infty$ iff the sequence $\left\{\frac{1}{a_n}\right\}$ converges to zero.

(Assignment)

② If $\lim a_n = \infty = \lim b_n$, then $\lim a_n b_n = \infty$.

Example: let $a_n = r^n$; $r > 1$

Then $\lim a_n = \infty$.

Sol: Since $r > 1$

let $r = 1 + h$; $h > 0$

$$\Rightarrow r^n = (1+h)^n$$

$$\geq 1 + nh \quad \text{by bernoulli's}$$

$$\Rightarrow a_n \geq 1 + nh$$

Since $b_n = 1 + nh$ diverges to ∞

So by comparison theorem

$$\lim a_n = \infty.$$

Comparison Theorem

(KOS) Statement: If a sequence $\{a_n\}$ diverges to ∞ and $a_n \leq b_n$ for all $n > n_1$, then the sequence $\{b_n\}$ also diverges to ∞.

Proof: Since $\lim a_n = \infty$, so for $K > 0$, there exist n_1 such that $a_n > K$ for all $n \geq n_1$.

So $b_n \geq a_n > K$

imply $b_n > K$ for all $n \geq n_1$

Thus $\lim b_n = \infty$.

(BR) Assignment: If $\lim y_n = -\infty$ and $x_n \le y_n$ for all n, then $\lim x_n = -\infty$.

THEOREM: Consider a sequence $\{a_n\}$ of non-zero terms such that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \alpha$, α a constant.

(a) If $\alpha < 1$, then $\lim_{n\to\infty} a_n = 0$

(b) If $\alpha > 1$, then $\lim_{n\to\infty} a_n = \infty$

(c) If $\alpha = 1$, then may converge, diverge to infinity or oscillate.

PROOF: Suppose $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \alpha$ where $\alpha < 1$.

Clearly $\alpha \ge 0$ (since it is the limit of absolute value). Thus there exist n_1 such that for all $n \ge n_1$, we have $\left|\frac{a_{n+1}}{a_n}\right| \le \beta$ for some $\beta \in (\alpha, 1)$. So for $n \ge n_1$, we can write

$$|a_n| = \frac{|a_n|}{|a_{n-1}|}\cdot\left|\frac{a_{n-1}}{a_{n-2}}\right|\left|\frac{a_{n-2}}{a_{n-3}}\right|\cdots\left|\frac{a_{n_1+1}}{a_{n_1}}\right|\cdot|a_{n_1}|$$

$$\le \beta\cdot\beta\cdot\beta\cdots\beta|a_{n_1}| = \beta^{n-n_1}|a_{n_1}|$$

$$= \beta^n\left|\frac{a_{n_1}}{\beta^{n_1}}\right| = c\,\beta^n$$

If $n=7$

$\frac{a_7}{a_6}\cdot\frac{a_6}{a_5}\cdot\frac{a_5}{a_4}$

$\beta\cdot\beta\cdot\beta$

$= \beta^3$

$= \beta^{7-4}$

imply $|a_n| \le c\,\beta^n$ —— (2) where $c = \frac{|a_{n_1}|}{\beta^{n_1}}$, a constant.

Since $0 < \beta < 1$. Thus $\lim_{n\to\infty}\beta^n = 0$

Thus by sandwich theorem

$\lim_{n\to\infty}|a_n| = 0$ Because $|a_n| \ge 0$ always

We know if $|a_n| \to 0$, then $a_n \to 0$

Hence $\lim_{n\to\infty} a_n = 0$

(b) Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \alpha$ where $\alpha > 1$.

Define a new sequence

$b_n = \frac{1}{|a_n|}$. Then $\left|\frac{b_{n+1}}{b_n}\right| = \frac{1}{\alpha} < 1$

and so, by part (a) $\lim_{n\to\infty} b_n = 0$

Therefore $\left\{\frac{1}{b_n}\right\}$ diverges to $+\infty$.

i.e $\{|a_n|\}$ diverges to $+\infty$.

Hence $\{a_n\}$ diverges to $+\infty$.

(c) To justify part (c), we will give three examples. See exercise (8.2) T. Firmy

<u>Example</u> of a sequence $\{d_n\}$ defined by

$$d_n = \frac{n^p}{b^n} \quad \text{for } p > 0 \text{ and } b > 1$$

$$\lim_{n\to\infty} \left|\frac{d_{n+1}}{d_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^p}{b^{n+1}} \cdot \frac{b^n}{n^p}\right|$$

$$= \frac{1}{b} \lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right)^p\right|$$

$$= \frac{1}{b}(1)$$

$$< 1 \quad \because b > 1$$

Thus $\lim_{n\to\infty} d_n = 0$

(G.M) LIMIT INFERIOR AND LIMIT SUPERIOR

Def: Let $\{s_n\}$ be a sequence which is bounded below. Define

$$u_1 = \inf\{s_n : n \geq 1\}$$
$$u_2 = \inf\{s_n : n \geq 2\}$$
$$\vdots$$
$$u_K = \inf\{s_n : n \geq K\}$$
$$\vdots$$

Then lt inferior of $\{s_n\}$, denoted by $\lim\limits_{n\to\infty} \inf s_n$ or $\underline{\lim}_{n\to\infty} s_n$, is defined as

$$\lim_{n\to\infty} \inf s_n = \lim_{K\to\infty} u_K.$$

Moreover if $\{s_n\}$ is unbounded below, then

$$\lim_{n\to\infty} \inf s_n = -\infty$$

Def: Let $\{t_n\}$ be a sequence which is bounded above.

Define
$$v_1 = \sup\{t_n : n \geq 1\}$$
$$v_2 = \sup\{t_n : n \geq 2\}$$
$$\vdots$$
$$v_K = \sup\{t_n : n \geq K\}$$
$$\vdots$$

Then limit superior of $\{t_n\}$, denoted by $\lim\limits_{n\to\infty} \sup t_n$ or $\overline{\lim}_{n\to\infty} t_n$, is defined as

$$\lim_{n\to\infty} \sup t_n = \lim_{K\to\infty} v_K.$$

If $\{t_n\}$ is unbounded above, then

$$\lim_{n\to\infty} \sup t_n = \infty$$

Examples:

① let $s_n = \{(-1)^n : n \in N\}$

Obviously it is bounded.
$\{s_n\}$ has two subsequential limits -1 & 1. i.e $L = \{-1, 1\}$

Then $\lim_{n\to\infty} \sup (-1)^n = 1$

and $\lim_{n\to\infty} \inf (-1)^n = -1$

(B.M)
② let $a_n = 1 + (-1)^n$; $n \in N$, then

Define

$$U_K = \inf \{a_n : n \geq K\}$$

$$U_K = \inf \{1 + (-1)^K, 1 + (-1)^{K+1}, \ldots\}$$

$$U_K = 0$$

$$\lim_{n\to\infty} \inf a_n = \lim_{K\to\infty} U_K = 0$$

Also $V_K = \sup \{a_n : n \geq K\}$

$$= \sup \{1 + (-1)^K, 1 + (-1)^{K+1}, \ldots\}$$

$$= 2$$

$$\lim_{n\to\infty} \sup a_n = \lim_{K\to\infty} V_K = 2.$$

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(G.N) THEOREM: A bounded sequence has a unique limit inferior and a unique limit superior. Moreover $\lim_{n\to\infty} \inf a_n \le \lim_{n\to\infty} \sup a_n$

Where $\{a_n\}$ is a bounded sequence.

PROOF: Define $v_K = \sup\{a_n : n \geq K\}$ &

$u_K = \inf\{a_n : n \geq K\}$

Then $\{v_K\}$ is a decreasing bounded below and $\{u_K\}$ is an increasing bounded above sequences. (because $e_1 = \{\sup a_n : n \geq 1\}$ & $e_2 = \sup\{a_n : n \geq 2\}$ imply $e_1 \geq e_2$).

Thus $\lim_{K\to\infty} v_K = \inf\{v_K\}$, always unique.

i.e $\lim_{K\to\infty} v_K = \lim_{n\to\infty} \sup a_n$ is unique.

Similarly $\sup\{u_K\} = \lim_{K\to\infty} u_K = \lim_{n\to\infty} \inf a_n$ is unique.

Now $\inf\{v_K\} \geq u_K$ by def of v_K & u_K.

imply $\inf\{v_K\} \geq \sup\{u_K\}$

i.e $\lim_{n\to\infty} \sup a_n \geq \lim_{n\to\infty} \inf a_n$

or $\lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \sup a_n$.

(R.B) Alternate:

If $\{a_n\}$ is not bounded above, then $\lim_{n\to\infty} \sup a_n = \infty$

and $\lim_{n\to\infty} \inf a_n \leq \infty = \lim_{n\to\infty} \sup a_n$

imply $\lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \sup a_n$

Again, if $\{a_n\}$ is not bounded below, then

$$\lim_{n\to\infty} a_n = -\infty \leq \lim_{n\to\infty} \sup a_n$$

imply $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} \sup a_n$

(21P) DEFINITION (Cluster point):

A real number x is called a clusterpoint of the sequence $\{a_n\}$ provided some subsequence of $\{a_n\}$ converges to x.

Remark: Every bounded sequence has atleast one clusterpoint. If $\lim_{n\to\infty} a_n = A$ then the set of all cluster point of $\{a_n\}$ denoted by C is of course $C = \{A\}$ Morove C may be empty or an infinite subset of R. If $\{a_n\}$ is bounded the C must be non-empty and we can define

limit sup a_n = $\sup C$ and limit inf a_n = $\inf C$. Thus $\overline{\text{limit}}\, a_n$ is the largest cluster point of $\{a_n\}$ and $\underline{\text{limit}}\, a_n$ is the smallest cluster point of $\{a_n\}$.

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(Zip) THEOREM

If $L =$ limit sup a_n and $l =$ limit inf a_n then given any $\varepsilon > 0$ there is a natural number N such that $l - \varepsilon < a_n < L + \varepsilon$. whenever $n > N$.

PROOF: Since l and L are real number and $l - \varepsilon < a_n < L + \varepsilon$, so $\{a_n\}$ is bounded.

Let $\varepsilon > 0$ and suppose $a_n \geq L + \varepsilon$ for infinitely many natural numbers n. Then then there exist natural numbers $n_1 < n_2 < n_3 \cdots$ such that

$a_{n_k} \geq L + \varepsilon$ for $k = 1, 2, 3, \ldots$ (by supposition)

Clearly $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ and so $\{a_{n_k}\}$ is bounded. By the Bolzano-Weierstrass theorem, $\{a_{n_k}\}$ has a cluster point x, which is then a cluster point of $\{a_n\}$. But $a_{n_k} \geq L + \varepsilon$ for all k.

implies that $x \geq L + \varepsilon$

$\Rightarrow x - \varepsilon \geq L$ which is a contradiction because $L =$ limit sup a_n i.e ~~G~~ Greatest cluster point of a_n.

Hence $a_n \geq L + \varepsilon$ for only finitely many natural number n. Consequently, there must be a natural number N_1 such that

$a_n < L + \varepsilon$ for all $n > N_1$

or $a_n < L + \varepsilon$ whenever $n > N_1$. —— (1)

Similarly, Suppose $a_n \leq l - \varepsilon$ for infinitely many natural numbers n. Then there exist natural numbers $n_1 < n_2 < n_3 \cdots$ such that $a_{n_k} \leq l - \varepsilon$ for $k = 1, 2, \ldots$

clearly $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ and so $\{a_{n_k}\}$ is bounded. By the Bolzano-Weierstrass theorem, $\{a_{n_k}\}$ has a cluster point y, which is then a clusterpoint of $\{a_n\}$. But

$$a_{n_k} \leq l-\varepsilon \quad \text{for all } k.$$

implies that $y \leq l-\varepsilon$

$$\Rightarrow \quad y+\varepsilon \leq l.$$

Which is a contradiction because

$l = \text{limit Inf } a_n$

i-e Smallest cluster point.

Hence $a_n \not\leq l-\varepsilon$ for only finitely many natural number n. consequently, there must be a natural number N_2 such that

$$a_n > l-\varepsilon \quad \text{whenever } n > N_2 \quad \text{—— (2)}$$

Let $N = \max(N_1, N_2)$;

then (1) and (2) imply

$$l-\varepsilon < a_n < L+\varepsilon \quad \text{for all natural number } n > N.$$

Corollary: $\lim\limits_{n\to\infty} a_n = A$ iff $\text{limitSup } a_n = A = \lim\limits_{n\to\infty} \text{inf } a_n$

Proof: Suppose $\text{limit Sup } a_n = A = \text{lt Inf } a_n$ —— (1)

we know if $\text{limit Sup} = L$ &

$\text{limit Inf} = l$ and let $\varepsilon > 0$ be given, then there is a natural number N s.t $l-\varepsilon < a_n < L+\varepsilon$ whenever $n > N$

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imply $A-\varepsilon < a_n < A+\varepsilon$

$\Rightarrow -\varepsilon < a_n - A < \varepsilon$

iff $|a_n - A| < \varepsilon$ whenever $n > N$.

Therefore $\lim\limits_{n\to\infty} a_n = A$.

Conversely, suppose that

$\lim\limits_{n\to\infty} a_n = A$. That is for $\varepsilon > 0$

there exisit a natural number N such that

$|a_n - A| < \varepsilon$ whenever $n > N$.

iff $A-\varepsilon < a_n < A+\varepsilon$. But

$l-\varepsilon < a_n < L+\varepsilon$ for given conditions

Where $L = \lim \sup a_n$

$l = \lim \inf a_n$

Therefore $l = L = A$

That is $\lim \inf a_n = \lim \sup a_n = A$.

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(R.B) THEOREM:

let $\{a_n\}$ and $\{b_n\}$ be bounded sequence such that $a_n \le b_n$ for every positive integer n. Then (i) $\lim_{n\to\infty} \sup a_n \le \lim_{n\to\infty} \sup b_n$

(ii) $\lim_{n\to\infty} \inf a_n \le \lim_{n\to\infty} \inf b_n$

PROOF: let $\{a_n\}$ be a bounded real sequence and let C, denote the set of all 'l' such that

$l = \lim_{k\to\infty} a_{n_k}$ where $\{a_{n_k}\}$ is a

Convergent subsequence of $\{a_n\}$. (Notice that if $\lim_{n\to\infty} a_n = l$, then $C = \{l\}$. Thus if $\lim_{n\to\infty} a_n \ne l$ then there is a concept of $\inf C$ and $\sup C$).

let $\{b_{n_k}\}$ be a subsequence of $\{b_n\}$ (not necessarily convergent). Then $\{b_{n_k}\}$ is bounded. By Bolzano-W $\{b_{n_k}\}$ has a convergent subsequence. let it be $\{b_{n_{k_j}}\}$ or $\{b_{n_{k,j}}\}$

Since $a_n \le b_n$ for positive n.

imply $a_{n_{k_j}} \le b_{n_{k_j}}$, we have

$$l = \lim_{k\to\infty} a_{n_k} = \lim_{j\to\infty} a_{n_{k_j}}$$
$$\le \lim_{j\to\infty} b_{n_{k_j}}$$
$$\le \lim_{n\to\infty} \sup b_n$$

imply $\quad l \le \lim_{n\to\infty} \sup b_n$

Which shows that $\lim_{n\to\infty} \sup b_n$ is an upper bound of C.

Thus $\quad l.u.b\, C \le \lim_{n\to\infty} \sup b_n$

or $\quad \sup C \le \lim_{n\to\infty} \sup b_n$

or $\quad \lim_{n\to\infty} \sup a_n \le \lim_{n\to\infty} \sup b_n$

(ii) Left as an exercise.

(G.M) THEOREM:

Let $\{s_n\}$ and $\{t_n\}$ be bounded sequences and α, β are positive numbers, then

(i) $\lim_{n\to\infty} \sup(\alpha s_n + \beta t_n) \le \alpha \lim_{n\to\infty} \sup s_n + \beta \lim_{n\to\infty} \sup t_n$

(ii) $\lim_{n\to\infty} \inf(\alpha s_n + \beta t_n) \ge \alpha \lim_{n\to\infty} \inf s_n + \beta \lim_{n\to\infty} \inf t_n$

PROOF: Let $U_K = \sup\{s_n : n \ge K\}$ and

$V_K = \sup\{t_n : n \ge K\}$

Since $\alpha > 0$ and $\beta > 0$, so

$$\alpha s_n + \beta s_n \le \alpha U_K + \beta V_K \quad \forall\, n \ge K$$

imply $\quad \cancel{\lim_{n\to\infty}} \sup(\alpha s_n + \beta t_n) \le \alpha U_K + \beta V_K$

$$\Rightarrow \lim_{n\to\infty} \sup(\alpha s_n + \beta t_n) \le \alpha \lim_{n\to\infty} U_K + \lim_{n\to\infty} V_K$$

$\because$ R.H.S is independent of n.

imply $\lim_{n\to\infty} \sup(\alpha s_n + \beta t_n) \le \lim_{n\to\infty} \alpha \sup s_n + \lim_{n\to\infty} \beta \sup t_n$

or $\lim_{n\to\infty} \sup(\alpha s_n + \beta t_n) \le \alpha \lim_{n\to\infty} \sup s_n + \beta \lim_{n\to\infty} \sup t_n$.

(ii) left as an exercise.

case If $\alpha = \beta = 1$ then ?

(R.B)
(Cauchy first theorem on limits)
Example: Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} a_n = l$, then $\lim_{n\to\infty} \frac{a_1+a_2+\cdots a_n}{n} = l$

Proof ~~Solution~~: Since $\lim_{n\to\infty} a_n = l$, For given $\varepsilon > 0$ there exist a natural number N such that

$$l-\varepsilon < a_n < l+\varepsilon \quad \text{whenever } n \ge N. \quad (1)$$

Let $b_n = \frac{a_1+a_2+\cdots+a_n}{n}$ for $n \ge N$.

Then $b_n = \frac{a_1+a_2+\cdots+a_N}{n} + \frac{a_{N+1}+\cdots+a_n}{n}$

and since

$$\frac{(n-N)(l-\varepsilon)}{n} < \frac{a_{N+1}+\cdots+a_n}{n} < \frac{(n-N)(l+\varepsilon)}{n} \quad \text{(see def } (1))$$

$$\Rightarrow \frac{C}{n} + \frac{(n-N)(l-\varepsilon)}{n} < \frac{C}{n} + \frac{a_{N+1}+\cdots+a_n}{n} < \frac{(n-N)(l+\varepsilon)}{n} + \frac{C}{n}$$

where $C = a_1+a_2+\cdots+a_N$

therefore $\frac{C}{n} + \frac{(n-N)(l-\varepsilon)}{n} < b_n < \frac{(n-N)(l+\varepsilon)}{n} + \frac{C}{n}$.

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imply

$$\lim_{n\to\infty} \sup\left(\frac{C}{n}+\frac{(n-N)(l-\varepsilon)}{n}\right) \le \lim_{n\to\infty} \sup b_n$$

$$\le \lim_{n\to\infty} \sup\left(\frac{(n-N)(l+\varepsilon)}{n}+\frac{C}{n}\right)$$

or $$l-\varepsilon \le \lim_{n\to\infty} \sup b_n \le l+\varepsilon.$$

$$\Rightarrow \lim_{n\to\infty} \sup b_n = l$$

$\left(\begin{array}{l} \because\ l-\varepsilon \le a_n \le l+\varepsilon \\ \Rightarrow l \le a_n \le l \\ \Rightarrow a_n = l. \end{array}\right)$

~~or $\lim_{n\to\infty}$ Sup~~

Similarly $\lim_{n\to\infty} \inf b_n = l$

Since $l = \lim_{n\to\infty} \sup b_n = \lim_{n\to\infty} \inf b_n$

Therefore $\lim_{n\to\infty} b_n = l$

or $$\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = l$$

Example: Let $s_n = (-1)^n$, $t_n = (-1)^{n+1}$

Then $\lim_{n\to\infty} \sup s_n = 1$, $\lim_{n\to\infty} \sup t_n = 1$

and $\lim_{n\to\infty} \sup(s_n + t_n) = 0$

Thus $$\lim_{n\to\infty} \sup(s_n+t_n) \ne \lim_{n\to\infty} \sup s_n + \lim_{n\to\infty} \sup t_n.$$

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Example:

$$\lim_{n\to\infty} \frac{1+2^{\frac{1}{2}}+3^{\frac{1}{3}}+\cdots+n^{\frac{1}{n}}}{n} = 1$$

$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

So by Cauchy's first theorem

$$\lim_{n\to\infty} \frac{1}{n}(a_1+a_2+\cdots+a_n) = 1$$

or $\lim_{n\to\infty} \frac{1}{n}\left(1+2^{\frac{1}{2}}+3^{\frac{1}{3}}+\cdots+n^{\frac{1}{n}}\right) = 1$

Cauchy's Second theorem on limit

If all the terms of a sequence $\{a_n\}$ are positive and if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists then so does $\lim_{n\to\infty} (a_n)^{\frac{1}{n}}$ and the two limits are equal. i.e $\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

Example Find $\lim_{n\to\infty} \left[\frac{(3n)!}{(n!)^3}\right]^{\frac{1}{n}}$.

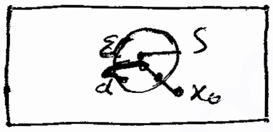
let $a_n = \frac{(3n)!}{(n!)^3}$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 27$.

Hence $\lim_{n\to\infty} \left[\frac{(3n)!}{(n!)^3}\right]^{\frac{1}{n}} = 27$

(ii) $\lim_{n\to\infty} \left(\frac{n^n}{(n+1)(n+2)\cdots(n+n)}\right)^{\frac{1}{n}} = ?$

(iii) $\lim_{n\to\infty} \frac{n^p}{(1+a)^n} = 0$

(ZIP) Def: A point x_0 is called a limit point (or an accumulation point) of the set S if given $\varepsilon > 0$ there is a point $x \in S$ with $x \neq x_0$ and $|x - x_0| < \varepsilon$.



Remark: It is not necessary that $x_0 \in S$. For example $S = (0,1)$. Then the points 0 and 1 both are the limit points of S and not in S.
The set $\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$ has exactly one limit point namely $x_0 = 1$. The points in this set cluster up around the point $x_0 = 1$ whereas the sequence $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$ has the single cluster point $x_0 = 1$ (since lt of this sequence is 1). Thus for this example it is difficult to distinguish between limit point and cluster point. On the other hand the sequence $1, 2, 1, 2, 1, 2, \dots$ has two cluster points 1 and 2 but the range of the sequence $\{1, 2\}$ has no limit point. Where the range of the sequence $1, \frac{1}{2}, \frac{2}{3}, \dots$ was precisely contained in the set $\{1, \frac{1}{2}, \frac{2}{3}, \dots\}$. Thus from above discussion, we conclude the followings.

(1) A finite set has no limit point. despite an infinite set may also fail to have a limit point.

e.g the set of natural numbers. because natural numbers do not cluster up around any real number.

Explaination: let x_0 be an arbitrary real number. let ε be the distance from x_0 to the nearest natural number (other than x_0 if x_0 is itself a natural number). Then there is no natural number n with the property that $n \neq x_0$ and $|n - x_0| < \varepsilon$. Consequently x_0 is not a limit point of N.

(2) Think over the following statement.

Every limit point is a cluster point but converse may not be hold. That is every cluster point may not be a limit point.

H.M. KHALID MAHMOOD.

THEOREM (Bolzano-Weierstrass for sets)

Every bounded infinite set has atleast one limit point.

PROOF: Let S be a bounded infinite set. Since S is infinite, we can select a sequence $\{a_n\}$ of distinct elements from S provided $a_n \in S$ for every $n \in N$ and $a_i \neq a_j$ for $i \neq j$. Then $\{a_n\}$ is bounded (because S is bounded) and so by the Bolzano-Weierstrass theorem $\{a_n\}$ has a convergent subsequence. Let it be $\{a_{n_k}\}$ and $\lim_{k\to\infty} a_{n_k} = x_0$

We will show that x_0 is a limit point of S. Since $\lim_{k\to\infty} a_{n_k} = x_0$. For $\varepsilon > 0$ (given) there is a natural number N such that

$$|a_{n_k} - x_0| < \varepsilon \quad \text{whenever } k > N.$$

Choose $k > N$ such that $a_{n_k} \neq x_0$ (This is possible, since $\{a_{n_k}\}$ is a sequence of distinct elements)

Clearly $|a_{n_k} - x_0| < \varepsilon$ and $a_{n_k} \in S$

Since $a_{n_k} \neq x_0$ and $\varepsilon > 0$ was arbitrary, it follows that x_0 is a limit point of S.

(2P) Remark:

Boundedness is not necessary in order for an infinit set S to have a limit point.
e.g the set $\{\frac{1}{2}, 2, \frac{1}{3}, 3, \ldots\}$ is unbounded and infinite and has the limit point 0.
The unbounded interval $(1, \infty)$ has infinitely many limit points; infact, each element in $[1, \infty)$ is a limit point of the set.

(2P)

THEOREM: A set F is closed iff F contains all its limit points.

PROOF: Assume F is closed. let $x_0 \in F^c$
It suffices to show that x_0 is not a limit point of F (That is whenever $x_0 \notin F$, it is not a limit point of F, then ultemately all limit points of F are in F).
Since F^c is open, there is an $\varepsilon > 0$ such that $N_\varepsilon(x_0) \subseteq F^c$. Hence there is no $x \in F$ with $|x - x_0| < \varepsilon$ and so x_0 is not a limit point of F.
Converseley, Suppose F is a set which contains all its limit points. let $x_0 \in F^c$, then x_0 is not a limit point of F, there must exist an $\varepsilon > 0$ for which no element $x \in F$ satisfies $|x - x_0| < \varepsilon$.
Hence $N_\varepsilon(x_0) \subseteq F^c$, and F^c is an open set.
It follows that F is closed.

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Def: A collection of open sets G_i is called an open covering of the set S provided

$$S \subseteq \bigcup_i G_i$$

Heine-Borel (Covering) theorem: $\mathcal{G}$

(2p) Statement: If $G = \{G_i\}$ is an open covering of the closed bounded set F then there is a finite subset of G which is also an open covering of $\mathcal{G}$.

PROOF: We assume that ~~infinite~~ no finite subset of $\mathcal{G}$ covers F. Since F is bounded, there is a positive real number M such that $F \subseteq [-M, M]$. Consider the two intervals $[-M, 0]$ and $[0, M]$ provided atleast one of these intervals must contain a portion of F which cannot be covered by a finite number of sets from $\mathcal{G}$. otherwise if G_1 from $\mathcal{G}$ is finite and covers that part of F in $[-M, 0]$ and G_2 from $\mathcal{G}$ is finite and covers that part of F in $[0, M]$. Then $G_1 \cup G_2$ is finite and covers all of F, a contradiction to our assumption. Let I_0 be one of the intervals $[-M, 0]$ and $[0, M]$ which has the property that the portion of F inside it cannot be covered by a finite number of sets from $\mathcal{G}$. Now subdivide I_0 into two closed intervals of equal length as before, at least one of these two intervals must contain a portion of F which cannot be covered by a finite number of sets from $\mathcal{G}$. Call such an interval I_1. If we continue this process indefinitely we obtain a sequence of closed intervals $I_0 \supset I_1 \supset I_2 \cdots$ with the property that length of I_K is $\frac{M}{2^K}$ and the portion of F in I_K cannot be covered by finitely many sets from $\mathcal{G}$, for each $K = 0, 1, 2, \cdots$

Moreover $\lim_{K\to\infty} \frac{M}{2^K} = 0$. Then by the Nested-Interval property there is a unique x_0 common to each of the closed intervals I_K.

We will show x_0 is a limit point of F.

Let $\varepsilon > 0$ be given. Choose a natural number n so large that $\frac{M}{2^n} < \varepsilon$. (That is length of I_n is less than ε). But $x_0 \in I_n$ for each n. It follows that $I_n \subset N_\varepsilon(x_0)$

$x_0-\varepsilon \quad (I_n) \quad x_0+\varepsilon$

$|(x_0+\varepsilon)-(x_0-\varepsilon)| = 2\varepsilon > \varepsilon > |I_n|$

But I_n contains infinitely many points of F (because if $F \cap I_n$ were finite set then surely it could be covered by finitely many sets from $\mathcal{G}$). Hence there must be an $x \in F$ with $x \neq x_0$ and $|x - x_0| < \varepsilon$. Therefore x_0 is a limit point of F. Since F is closed, $x_0 \in F$.

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Since $\mathcal{G}$ is an open covering of F, there is a set $G_{i_0} \in \mathcal{G}$ such that $x_0 \in G_{i_0}$. But G_{i_0} is an open set so there is a $\delta > 0$ such that $N_\delta(x_0) \subseteq G_{i_0}$. Like above choose the natural number m so large that $I_m \subset N_\delta(x_0)$ then $I_m \subseteq G_{i_0}$. That is I_m is covered by a finite number of sets (namely one) from $\mathcal{G}$ and so clearly that portion of F in I_m is covered by a finite number of sets from $\mathcal{G}$. This is a contradiction to the construction of the sequence of closed intervals $\{I_K\}$. Hence our first supposition is wrong which complete the proof.

Def: A set $K \subseteq R$ is called compact if every open covering $\mathcal{G}$ of K admits a finite subcovering.

H-B theorem (Alternative Statement):

"Every closed bounded subset of R is compact."

Notice that the converse to the Heine-Borel theorem is also true. That is " If K is compact subset of R, then K is both closed and bounded.

Alternative Heine-Borel. "Every subset of R is compact iff it is closed and bounded."

(Imp) Def: Let $\{a_K\}$ be a sequence of real numbers. A series, denoted by $\sum_{K=1}^{\infty} a_K$ is defined to be the sequence $\{S_n\}$ where $S_n = \sum_{K=1}^{n} a_K$. The numbers a_K are called the terms of the series, and the numbers S_n are called the partial sums of the series.

Explaination:

$$S_1 = a_1, \quad S_2 = a_1 + a_2, \ldots$$

$$S_n = \sum_{K=1}^{n} a_K, \ldots, S_\infty = \sum_{K=1}^{\infty} a_K$$

$$\text{or} \quad \sum_{K=1}^{\infty} a_K = \lim_{n\to\infty} S_n$$

(KP) Convergence (Possess a Sum)

Def: The series $\sum_{n=1}^{\infty} a_n$ is convergent if the sequence of partial sums is convergent; the series is divergent if the sequence of partial sums is divergent. If the series is convergent and the sequence of partial sums S_n converges to S, then S is called the sum of the series and, we write $\sum_{n=1}^{\infty} a_n = S$

Observe that

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

$$= \sum_{i=1}^{\infty} a_i$$

$$= \sum_{n=1}^{\infty} a_n$$

imply $\sum_{n=1}^{\infty} a_n = S$.

Properly Divergent:

If $\lim_{n\to\infty} S_n = \infty$ or $-\infty$, then $\sum_{n=1}^{\infty} a_n$ is said to be properly divergent.

(KS) THEOREM (nth-term test)

If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

OR If $\sum_{k=1}^{\infty} a_k$ converges, then the sequence $\{a_k\}$ converges to zero.

Proof: Suppose $\sum_{k=1}^{\infty} a_k$ converges, say to S, then

$\lim_{n\to\infty} S_k = S$. Then

$$a_k = S_k - S_{k-1} \text{ for } k \geq 2.$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1}$$

$$= S - S$$

$$= 0$$

Remark: Converse of above theorem may not be true. That is if $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ may converge or diverge.

Example: Harmonic Series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

$$S_{2^1} = 1 + \frac{1}{2}$$

$$S_{2^2} = S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2}$$

$$S_{2^3} = S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= 1 + \frac{3}{2}$$

$$S_{2^4} > 1 + \frac{4}{2}$$

$$\vdots$$

$$S_{2^n} > 1 + \frac{n}{2}.$$

It follows that the sequence $\{S_n\}$ is unbounded and hence diverges. Therefore $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, yet $\lim_{k \to \infty} \frac{1}{k} = 0$

(KOS)
THEOREM (Divergence Test) OR [Nth term test (KP)]

If a sequence $\{a_k\}$ does not converges to zero, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

PROOF: Use contrapositive

$$\sum_{n=1}^{\infty} n^{\frac{1}{n}} = 1 + \sqrt{2} + 3^{1/3} + \cdots$$

$$\lim_{n \to \infty} n^{1/n} = 1 \neq 0$$

thus $\sum_{n=1}^{\infty} n^{\frac{1}{n}}$ diverges.

OR If $\lim_{n \to \infty} a_k \neq 0$ or ∞.

Then $\sum_{k=1}^{\infty} a_k$ diverges.

① Introduce Telescoping Series.

② Convergence of Geometric Series

We know
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \quad \text{by Telescoping Series.}$$

Alternative method for the divergence of Harmonic Series by Bernoulli

$$1 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \cdots$$

$$1 - \frac{1}{1\cdot 2} = \frac{1}{2} = \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$

$$\frac{1}{2} - \frac{1}{2\cdot 3} = \frac{1}{3} = \frac{1}{3\cdot 4} + \cdots \quad \text{and so on.}$$

Adding above

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots = \frac{1}{1\cdot 2} + \frac{2}{2\cdot 3} + \frac{3}{3\cdot 4} + \cdots$$
$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

imply $1 + \alpha = \alpha$ if $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \alpha$ (say)

$\Rightarrow$ $1 = 0$, a paradoxical equation.

Thus α, must be finite.

Thus we cannot assign a definite number α to $\frac{1}{2} + \frac{1}{3} + \cdots$

Therefore $\sum_{k=1}^{\infty} \frac{1}{k}$ is not possible. That is diverges.

(R.B) THEOREM: let $\sum_{n=1}^{\infty} a_n$ be a series with non-negative terms. Then $\sum_{n=1}^{\infty} a_n$ converges iff the sequence of partial sums is bounded.

In this case $\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k$

$$= \sup\{S_k : k \in N\}$$

Proof: Since $a_n \geq 0$ for all n.

Therefore the sequence $\{S_k\}$ of partial sums is obviously increasing.

By Monotone Convergence theorem $\{S_k\}$ Converges iff it is bounded. Thus $\{S_k\}$ is bounded. Since $\{S_k\}$ is bounded increasing, therefore

$$\lim_{k\to\infty} S_k = \sup\{S_k : k \in N\}$$

Example: $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Sol: Since partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ are monotone, so we will find a bounded subsequence of $\{S_k\}$

let $k_1 = 2^1 - 1 = 1$, then $S_{k_1} = 1$

$k_2 = 2^2 - 1 = 3$, $\quad S_{k_2} = 1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) < 1 + \frac{2}{2^2}$

$k_3 = 2^3 - 1 = 7$ $\quad S_{k_3} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{7^2}$

$< S_{k_2} + \frac{4}{4^2} = 1 + \frac{1}{2} + \frac{1}{2^2}$

let $K_j = 2^j - 1$, then

$$0 < S_{K_j} \leq 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{j-1}} = 2.$$

imply $S_{K_j} < 2$.

Since $\{S_{K_j}\}$ is bounded

Hence $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

(KP) THEOREM If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent with sums A and B respectively and K is constant, then

① $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

② $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

③ $\sum_{n=1}^{\infty} (K a_n) = K \sum_{n=1}^{\infty} a_n = KA.$

PROOF: let $A_K = \sum_{n=1}^{K} a_n$, $B_K = \sum_{n=1}^{K} b_n$

Then $S_K = A_K + B_K = (a_1+b_1) + \cdots + (a_K + b_K) = \sum_{n=1}^{\infty}(a_n+b_n)$

$$\lim_{K\to\infty} S_K = \lim_{K\to\infty} A_K + \lim_{K\to\infty} B_K = A + B$$

$$\lim_{K\to\infty} (A_K + B_K) = A + B$$

$$\lim_{K\to\infty} \sum_{n=1}^{K} (a_n + b_n) = A + B \Rightarrow \sum_{n=1}^{\infty} (a_n + b_n) = A + B$$

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(G.105) Cauchy criterion

Statement: The series $\sum a_n$ is convergent if and only if given $\varepsilon > 0$ there exist a natural number N such that

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \varepsilon \text{ for every } n \geq N \text{ and every } p \in \mathbb{N}.$$

or $$\left|\sum_{k=n}^{m} a_k\right| < \varepsilon \text{ for } m > n > N.$$

PROOF: Assume that the series is convergent and let S_n denote the nth partial sum of the series. This sequence is convergent and so must satisfy the Cauchy criterion for sequences. Thus for any $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that

$$|S_{n+p} - S_n| < \varepsilon \text{ whenever } n > N \text{ \& } p \in \mathbb{N}$$

imply $$|S_{n+p} - S_n| = \left|\sum_{k=n+1}^{n+p} a_k\right| < \varepsilon.$$

Conversely, suppose $\left|\sum_{k=n+1}^{n+p} a_k\right| < \varepsilon$ for $n > N$ & $p \in \mathbb{N}$

Then $|S_{n+p} - S_n| < \varepsilon$ for $n > N$ & $p \mathbb{N}$

Thus $\{S_n\}$ is a Cauchy sequence, so it is convergent. Hence $\sum a_n$ is convergent.

(GCS/ZIP) Remarks: The above theorem asserts that if $\sum a_n$ is to have a sum, then 'blocks' of terms of arbitrary length must, after a certain stage, sum to a negligible quantity. That is for each N, the Nth tail of the series approaches 0. $\left(\left|\sum_{n=N+1}^{\infty} a_n\right| < \varepsilon\right)$. In other words the size of the first few terms of an infinite series has no effect on whether or not the series converges. OR the convergence character of a series is completely determined in "the tail of the series".

Consequence of Cauchy Criterion:

"If $\sum_{k=1}^{\infty} a_k$ converges, then for any natural number N, the series $\sum_{k=N}^{\infty} a_k$ converges. If $\sum_{k=N}^{\infty} a_k$ converges for some natural number N, then $\sum_{k=1}^{\infty} a_k$ converges."

(RU) THEOREM (2^n-Test or Cauchy condensation test):

Suppose $a_1 \geq a_2 \geq a_3 \cdots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

converges.

PROOF: First assume that $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

Now $a_1 \le a_1 = 2^0 a_{2^0}$

$a_2 + a_3 \le a_2 + a_2 = 2a_2 = 2^1 a_{2^1}$ ∵ decreasing sequence.

$a_4 + a_5 + a_6 + a_7 \le 4a_4 = 2^2 a_{2^2}$

⋮ and in general

$$a_{2^n} + a_{2^n+1} + \cdots + a_{2^{n+1}-1} \le 2^n a_{2^n}$$

Add above inequalities, we get

$$S_{2^{n+1}-1} = \sum_{k=1}^{2^{n+1}-1} a_k \le \sum_{k=0}^{n} 2^k a_{2^k} \le \sum_{k=0}^{\infty} 2^k a_{2^k}$$

Thus the subsequence $\{S_{2^{n+1}-1}\}$ of the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ is bounded by $\sum_{k=0}^{\infty} 2^k a_{2^k}$ ($= \ell$ ∵ it is convergent).

Since $\{S_n\}$ is monotone, it follows that the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ is bounded. Thus $\sum_{n=1}^{\infty} a_n$ converges as the sequence of partial sums is itself bounded.

Conversely, Assume that $\sum_{n=1}^{\infty} a_n$ converges.

then $a_3 + a_4 \geq 2a_4$ $\because \{a_n\}$ is decreasing

$a_5 + a_6 + a_7 + a_8 \geq 4a_8$

and in general

$$a_{2^n+1} + \cdots + a_{2^{n+1}} \geq 2^n a_{2^{n+1}}$$

So $$\sum_{k=3}^{\infty} a_k \geq \frac{1}{2} \sum_{k=1}^{n} 2^{k+1} a_{2^{k+1}}$$

Thus the sequence of partial sums of the series $\sum_{k=1}^{\infty} 2^k a_{2^k}$ is bounded by $2\sum_{k=3}^{\infty} a_k$. Hence $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

Remark:

Cauchy condensation test for divergence case can be prove by

① Give first contrapositive Statement

② Proof of above theorem.

Example Discuss the convergence of $\sum \frac{1}{n^p}$, $p>0$

Solution: Since $p>0$ ~~implying~~

Then $\frac{1}{n^r} > \frac{1}{n^s}$; $0<r<s$;

Thus $\{\frac{1}{n^p}\}$ is decreasing. We will apply condensation test.

Then the condensed series is

$$\sum 2^n a_{2^n} = \sum 2^n \frac{1}{(2^n)^p}$$

$$= \sum \frac{1}{2^{-n} 2^{np}}$$

$$= \sum \left(\frac{1}{2^{p-1}}\right)^n$$

which is a geometric series with $r = \frac{1}{2^{p-1}}$

Thus $|r|<1$ if $p-1 \nless 0$

i-e $p-1>0$

or $p>1$

Therefore $\sum 2^n a_{2^n}$ converges if $p>1$ and diverges if $p \leq 1$.

Thus $\sum \frac{1}{n^p}$; $p>0$ converges if $p>1$ and diverges if $p \leq 1$.

e.g $\sum \frac{1}{n^2}$ is convergent.

(RU) Example: If $P>1$, $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges; if $P \le 1$, the series diverges.

Where $\log n$ denotes the logrithm of n to the base e.

Sol obviously $\log n$ increasing, then $\frac{1}{n\log n}$ decreasing. Then by cauchy condensation, the condensed series

$$\sum_{K=1}^{\infty} 2^K \cdot \frac{1}{2^K (\log 2^K)^p} = \frac{1}{(\log 2)^p} \sum_{K=1}^{\infty} \frac{1}{K^p}$$

But $\sum_{K=1}^{\infty} \frac{1}{K^p}$ converges if $P>1$

and diverges if $P \le 1$

Therefore $\sum \frac{1}{n(\log n)^p}$ converges if $P>1$

and diverges $P \le 1$.

(RU) Assignment:

Test the convergence of the series

(1) $\sum_{n=3}^{\infty} \frac{1}{n \log n \log \log n}$ diverges

(2) $\sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^2}$ converges.

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③ If $a_n \geq 0$ for $n = 1, 2, \dots$, the the series $\sum_{n=1}^{\infty} a_n$ is either convergent or properly divergent. (See Kaplan)

④ (Give presentation) e is irrational.

Def: $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ —①

Suppose e is rational. Then $e = \frac{p}{q}$

Where p and q are positive integers; $q \neq 0$

From①

$$S_q = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{q!}$$

Then $0 < e - S_q = \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \dots$

$$= \frac{1}{(q+1)!}\left[1 + \frac{1}{q+2} + \frac{1}{(q+3)(q+2)} + \dots\right]$$

$$< \frac{1}{(q+1)!}\left(1 + \frac{1}{q+1} + \frac{1}{(q+1)^2} + \dots\right)$$

$$= \frac{1}{q!} \cdot \frac{1}{q}$$

by (G.P).

imply $0 < q!(e - S_q) < \frac{1}{q}$ —①*

Moreover $q!\,S_q = q!\left(1 + 1 + \frac{1}{2!} + \dots + \frac{1}{q!}\right)$

is an integer.

Therefore $q!(e - S_q)$ is an integer.

But $q \geq 1$, so ①* imply that there is an integer between 0 and 1, a contradiction.

Hence e is irrational.

(ZIP) Comparison test:

Statement: If $0 < a_K \le b_K$ (i.e $\sum a_K$ & $\sum b_K$ both are non-negative) for every $K \ge n$ where $n \in N$, then

① Convergence of $\sum_{K=1}^{\infty} b_K$ implies that $\sum_{K=1}^{\infty} a_K$ converges

② Divergence of $\sum_{K=1}^{\infty} a_K$ implies that $\sum_{K=1}^{\infty} b_K$ diverges.

Proof: Note that ① & ② are the contrapositive statements of each other. We will prove ②.

Since a_K and b_K are positive terms for every $K \ge n$. Therefore the sequences of partial sums $\{A_n\}$ and $\{B_n\}$ are strictly increasing ~~for every $n \in N$~~. Where

$$A_n = \sum_{K=1}^{n} a_K \ , \ B_n = \sum_{K=1}^{n} b_K$$

Since $a_K \le b_K$ for $K = 1, 2, \dots$, so

$$A_n \le B_n \text{ for every } n \in N.$$

Now if $\sum_{K=1}^{\infty} a_K$ diverges, then $\{A_n\}$ diverges and so is an unbounded sequence. Hence $\{B_n\}$ is an unbounded sequence and therefore diverges.

It follows that $\sum_{k=1}^{\infty} b_k$ diverges.

For the proof ①, we have proved its contrapositive statement.

e.g $\sum_{k=1}^{\infty} \frac{1}{k!}$

By induction

$$k! \geq 2^{k-1} \quad \text{for every } k.$$

imply $\frac{1}{k!} \leq \frac{1}{2^{k-1}}$

But $\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$ is a geometric series with $|r| < 1$.

Therefore $\sum \frac{1}{2^{k-1}}$ is convergent.

Hence by comparison test

$\sum_{k=1}^{\infty} \frac{1}{k!}$ is convergent.

Remark: The p-series and geometric series are helpful to test the convergence or divergence of given series by comparison test.

(9) $\sum_{K=1}^{\infty} \frac{1}{\sqrt{K}}$ diverges, comparison with $\sum_{K=1}^{\infty} \frac{1}{K}$.

(2P)

Corollary:
If $a_K > 0$, $b_K > 0$ for $K = 1, 2, 3, \cdots$ and $\left\{\frac{a_K}{b_K}\right\}$, $\left\{\frac{b_K}{a_K}\right\}$ are ~~both~~ bounded sequences then $\sum_{K=1}^{\infty} a_K$ and $\sum_{K=1}^{\infty} b_K$ either both converges or both diverges.

Proof: Suppose $a_K > 0$, $b_K > 0$, $\frac{a_K}{b_K} \le M_1$ and $\frac{b_K}{a_K} \le M_2$ for every natural number K, where $M_1 > 0$, $M_2 > 0$.
then $0 < \alpha \le b_K / a_K \le \beta$ hold for every $K \in N$ where $\alpha = \frac{1}{M_1}$ and $\beta = M_2$

Since $a_K > 0$ for every K, so

$$0 \le \alpha a_K \le b_K \le \beta a_K.$$

Now if $\sum_{K=1}^{\infty} b_K$ converges, then $\sum_{K=1}^{\infty} \alpha a_K$ converges and therefore $\sum_{K=1}^{\infty} a_K$ converges.
If $\sum_{K=1}^{\infty} a_K$ converges, then $\sum_{K=1}^{\infty} \beta a_K$ converges and therefore by comparison test $\sum_{K=1}^{\infty} b_K$ converges. Which complete the proof.

H.M KHALID MAHMOOD

(2p) (6Ms) Corollary 2 (Limit comparison test)

If $a_K > 0$, $b_K > 0$ for $K = 1, 2, 3, \cdots$ and $\lim_{K \to \infty} b_K / a_K$ exists and is a positive real number, then $\sum_{K=1}^{\infty} a_K$ and $\sum_{K=1}^{\infty} b_K$ either both converges or both diverges.

Proof: Let $\lim_{K \to \infty} \frac{b_K}{a_K} = l$, l is a positive real number. then for given $\varepsilon > 0$ such that $l > \varepsilon > 0$, we can find a natural number N such that

$$l - \varepsilon < \frac{a_n}{b_n} < l + \varepsilon \text{ whenever } n > N.$$

imply $(l-\varepsilon) b_n < a_n < (l+\varepsilon) b_n \quad \because b_n > 0 \ \forall n.$

If $\sum b_n$ converges, then $\sum (l+\varepsilon) b_n$ also converges and by comparison test $\sum a_n$ converges.

If $\sum a_n$ converges, then $\sum (l-\varepsilon) b_n$ converges by comparison and therefore $\sum b_n$ converges.

Case If $l = 0$, then we have

$$-\varepsilon b_n < a_n < \varepsilon b_n$$

Therefore if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} \varepsilon b_n$ converges and by comparison test $\sum a_n$ converges. But we have no idea about the convergence of $\sum -\varepsilon b_n$ when $\sum a_n$ converges because $\sum -\varepsilon b_n$ is negative term series.

*** Case If $l = \infty$. and $\sum b_n$ is divergent. Then $\{B_n\}$, sequence of partial sums is divergent. and so $\{A_n\}$ is divergent, therefore $\sum a_n$ is divergent.

(Just give information not write)

Hence we have the limit comparison test.

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GAS **Limit Comparison test:**

Statement: let $\sum a_n$ and $\sum b_n$ be two series of positive terms such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l, \text{ then}$$

(i) If $l \neq 0$, then both the series either converge or both of them diverge.

(ii) If $l = 0$ and $\sum b_n$ converges then $\sum a_n$ converges.

(iii) If $l = \infty$ and $\sum b_n$ diverges then $\sum a_n$ also diverges.

GAS <u>Remark</u> If the limit is non-zero and one of the series converges ~~then~~ (diverges) the other is also converges (diverges).

Examples: ① $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$

We know $\ln n > 1 \quad \forall n > 2.$

imply $\dfrac{1}{\ln n} < 1$

imply $\dfrac{1}{n^p \ln n} < \dfrac{1}{n^p} \quad \because n \in N$

But $\sum \frac{1}{n^p}$ converges if $p > 1$

Hence by comparison test $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ converges for $P>1$

(2) $\sum \frac{1}{\log n}$ is divergent.

let $a_n = \frac{1}{\log n}$, $b_n = \frac{1}{n}$

$$\frac{a_n}{b_n} = \frac{1}{\log n} \times n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\log n}}{\frac{1}{n}} = \infty.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent,

by the comparison test $\sum \frac{1}{\log n}$ is also divergent.

Alternative: $a_n = \frac{1}{\log n}$, $2^n a_{2^n} = \frac{2^n}{\log 2^n} = \frac{1}{\log 2} \frac{2^n}{n}$

But $\frac{2^n}{n} > \frac{1}{n}$ $\forall n \geq 1$

But $\sum \frac{1}{n}$ diverges, so $\sum \frac{2^n}{n}$ diverges.

Since $\sum 2^n a_{2^n}$ diverges,

Therefore $\sum \frac{1}{\log n}$ diverges.

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① Assignment: Use Cauchy condensation test to establish the divergence of the series

(a) $\sum \frac{1}{n \ln n}$ (b) $\sum \frac{1}{n(\ln n)(\ln \ln n)}$

(c) $\sum \frac{1}{n(\ln n)(\ln \ln n)(\ln \ln \ln n)}$

② Show that if $c > 1$, then the following series are convergent.

(a) $\sum \frac{1}{n(\ln n)^c}$ (b) $\sum \frac{1}{n(\ln n)(\ln \ln n)^c}$

For eample (b)

$$2^n a_{2^n} = \frac{\cancel{2^n}}{\cancel{2^n}\, n \ln 2\, (\ln n \ln 2)}$$

again

$$2^n b_{2^n} = \frac{\cancel{2^n}}{\cancel{2^n} \ln 2\, (\ln 2^n \ln 2)}$$

$$= \frac{1}{\ln 2 \cdot n(\ln 2 \ln 2)}$$

$$= \frac{1}{(\ln 2)^3 n}$$ But $\sum \frac{1}{n}$

is divergent, so going back $\sum \frac{1}{n \ln n(\ln \ln n)}$ is divergent.

Example: $\sum \frac{1}{n} \sin^2\left(\frac{x}{n}\right)$ is convergent $x \neq 0$.

let $a_n = \frac{1}{n} \sin^2\left(\frac{x}{n}\right)$, $b_n = \frac{1}{n^3}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin^2\left(\frac{x}{n}\right)}{\frac{1}{n^2}}$

$= x^2 \neq 0$.

Since $\sum \frac{1}{n^3}$ is convergent, so by comparison test $\sum \frac{1}{n} \sin^2\left(\frac{x}{n}\right)$ is convergent.

PRESENTATION:

① Ratio Test (D-Alembert's)

② Root Test (Cauchy root test)

T. Finny

$\sum \frac{1}{n}$ $\quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

$\sum \frac{1}{n^2}$ $\quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

But $\sum \frac{1}{n^2}$ is convergent.

ABSOLUTE AND CONDITIONALLY CONVERGENT SERIES:

G10.8

Def A series, $\sum a_n$ is called absolutely convergent provided $\sum |a_n|$ is a convergent series. A series $\sum a_n$ is called conditionally convergent if $\sum a_n$ is convergent, but $\sum |a_n|$ is divergent.

e.g $\sum \frac{(-1)^n}{n}$ is conditionally / non-absolutely convergent, while the series $\sum \frac{(-1)^n}{n^2}$ is an absolutely convergent series.

(2iP) **THEOREM:** If $\sum_{K=1}^{\infty} |a_K|$ converges, then $\sum_{K=1}^{\infty} a_K$ converges. That is, "Every absolutely convergent series is convergent."

PROOF: If $\sum_{K=1}^{\infty} |a_K|$ converges, then by Cauchy Criterion, For given $\varepsilon > 0$, there is a natural number N such that

$$\Big| |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}| \Big| < \varepsilon \quad \text{for every } n \geq N \text{ and for each } p \in N$$

Since $|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| \leq |a_{n+1}| + \cdots + |a_{n+p}| < \varepsilon$.

So again by Cauchy criterion $\sum_{K=1}^{\infty} a_K$ converges.

Example $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2+1}$

We know

$$\left|\frac{\sin nx}{n^2+1}\right| \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}$$

But $\sum \frac{1}{n^2}$ is convergent

Therefore by comparison test

$\sum \left|\frac{\sin nx}{n^2+1}\right|$ is convergent.

Hence $\sum \frac{\sin nx}{n^2+1}$ is convergent.

Since every absolute convergent series is convergent.

KOS

Def Suppose that $\{a_n\}$ is any sequence of real numbers. For all $n \in \mathbb{N}$, Define

$$a_n^+ = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases} \quad \text{and} \quad a_n^- = \begin{cases} a_n & \text{if } a_n \leq 0 \\ 0 & \text{if } a_n > 0 \end{cases}$$

Example: Suppose $\{a_n\}$ consists of the values $2, 5, -\frac{1}{2}, 0, -2, 3, 1, \frac{1}{2}, -4$. Then

$\{a_n^+\} = \{2, 5, 0, 0, 0, 3, 1, \frac{1}{2}, 0, \ldots\}$

$\{a_n^-\} = \{0, 0, -\frac{1}{2}, 0, -2, 0, 0, 0, -4, \ldots\}$

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Remark ① $\{a_n\} = \{a_n^+\} + \{a_n^-\}$ $\forall\, n \in N$

② $S_n^+ = \sum_{K=1}^{n} a_K^+$ and $S_n^- = \sum_{K=1}^{n} a_K^-$ then

$$S_n = \sum_{K=1}^{n} a_K = S_n^+ + S_n^-$$

and

$$S_n^* = \sum_{K=1}^{n} |a_K| = S_n^+ - S_n^-$$

③ Sequences $\{S_n^+\}$ and $\{S_n^-\}$ both converge if and only if $\{S_n^*\}$ converges. That is, if $\sum a_K^+$ and $\sum a_K^-$ both converges, then so does $\sum |a_K|$ i.e, $\sum a_K$ converges absolutely. Furthermore, if either $\sum a_K^+$ or $\sum a_n^-$ but not both diverges then both $\sum |a_K|$ and $\sum a_K$ diverges. Moreover if both $\sum a_K^+$ and $\sum a_K^-$ diverge, then $\sum |a_K|$ diverges, but $\sum a_K$ need not diverge in which case $\sum a_K$ would converge conditionally.

Example Find two infinite series $\sum a_K$ and $\sum b_K$ where $\sum a_K^+$, $\sum a_K^-$, $\sum b_K^+$, $\sum b_K^-$ all diverge, but $\sum a_K$ converges, while $\sum b_K$ diverges.

Sol: $\sum \frac{(-1)^{n+1}}{n}$, $\sum (-1)^n$.

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~~Regrouping~~ Grouping of Series:

To obtain a new series, we insert the pair of parentheses. The new series is called grouping.

THEOREM: If $\sum a_K$ converges to s and $\sum b_K$ is a regrouping of $\sum a_K$, then $\sum b_K$ also converges to s.

PROOF: Since $\sum a_K$ converges, so

$\lim\limits_{n\to\infty} S_n$ exist where $S_n = \sum\limits_{K=1}^{n} a_K$

and $\lim\limits_{n\to\infty} S_n = s$ because $\sum\limits_{K=1}^{n} a_K = s$ converges to s.

Define $b_1 = a_1 + a_2 + \cdots + a_{K_1}$

$b_2 = a_{K_1+1} + \cdots + a_{K_2}$

$\vdots$

Define its partial sums S_n^*

$S_1^* = b_1 = a_1 + a_2 + \cdots + a_{K_1} = S_{K_1}$

$S_2^* = b_1 + b_2 = a_1 + a_2 + \cdots + a_{K_1} + a_{K_1+1} + \cdots + a_{K_2} = S_{K_2}$

$\vdots$

Therefore $\{S_n^*\} = \{S_{K_n}\}$ is a subsequence of $\{S_n\}$. Therefore $\lim\limits_{n\to\infty} S_{K_n} = \lim\limits_{n\to\infty} S_n = s$.

Hence $\{S_{K_n}\} = \{S_n^*\}$ is convergent.

Therefore $\sum b_K$ is convergent and converges to s.

Rearrangement of Series:

Def Suppose that $f: N \to N$ is some bijection. Then $\sum a_{f(k)}$ is called a rearrangement of the series $\sum a_k$.

The idea is very simple. We start with the $\sum a_k$ and change the order of its terms. The resulting series is a rearrangement of $\sum a_k$. (Give example)

(KAP) Rearrangement theorem

Statement: If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and $\sum_{m=1}^{\infty} b_m$ is a rearrangement of $\sum_{n=1}^{\infty} a_n$, then $\sum_{m=1}^{\infty} b_m$ is absolutely convergent and has the same sum as $\sum_{n=1}^{\infty} a_n$.

PROOF: clearly $\sum_{m=1}^{\infty} |b_m| \leq \sum_{n=1}^{\infty} |a_n|$ — ①

Since every b_m equals an a_n for appropriate n and no two m's correspond to the same n. Hence the series $\sum_{m=1}^{\infty} |b_m|$ converges so that $\sum b_m$ is absolutely convergent (since $\sum a_n$ is absolutely converget satisfying ①) let S_n and S be the nth partial sum of sum for $\sum a_n$ and let S'_m and S' be the corresponding quantities for $\sum b_m$.

For given positive ε, let N be choosen so large that

$$|S_n - S| < \varepsilon/2 \text{ and } \sum_{k=n+1}^{p} |a_k| < \varepsilon/2$$

for $n > N$ and $p \in \mathbb{N}$

observe that such an N can be found, since $\sum a_n$ converges to S and since the cauchy criterion hold for $\sum |a_n|$. For m sufficiently large S_m' will be sum of the terms including $a_1, a_2, \dots, a_n$ and perhaps more

$$S_m' = S_n + a_{k_1} + a_{k_2} + \dots + a_{k_s} \quad \text{(Due to rearrangement)} \quad \text{—②}$$

Where $k_1, k_2, \dots, k_s$ are all larger than n.

let $k_0 = n + p_0$ be the largest of these identities. Then ② imply

$$|S_m' - S_n| = |a_{k_1} + \dots + a_{k_s}|$$
$$\leq |a_{k_1}| + \dots + |a_{k_s}|$$
$$\leq |a_{n+1}| + |a_{n+2}| + \dots + |a_{n+p_0}| < \varepsilon/2.$$

i.e $|S_m' - S_n| < \varepsilon/2$

Now $|S_m' - S| = |S_m' - S_n + S_n - S|$
$$\leq |S_m' - S_n| + |S_n - S|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

It follows that S_m' converges to S so that $S' = S$.

TESTS FOR ABSOLUTE CONVERGENCE

BR ① Limit comparison test:

Suppose $\{x_n\}$ and $\{y_n\}$ are non-zero real sequences and suppose

$$\lim_{n\to\infty} \left|\frac{x_n}{y_n}\right| = r \quad \text{exist in } R.$$

(a) If $r \neq 0$, then $\sum x_n$ is absolutely convergent if and only if $\sum y_n$ is absolutely convergent.

(b) If $r=0$ and if $\sum y_n$ is absolutely convergent, then $\sum x_n$ is absolutely convergent.

KP

② Cauchy root test:

let a series $\sum_{n=1}^{\infty} a_n$ be given and let $\lim_{n\to\infty} (|a_n|)^{\frac{1}{n}} = r$, Then

(i) If $r<1$, the series is absolutely convergent

(ii) If $r>1$, the series diverges

(iii) If $r=1$, the test fails.

③ Ratio test (d' Alembrt's ratio test)

If $a_n \neq 0$ for $n = 1, 2, \ldots$ and

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r \quad \text{then}$$

(i) If $r < 1$, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $r = 1$, test fails

(iii) If $r > 1$, $\sum_{n=1}^{\infty} a_n$ is divergent.

KOS

④ Raabe's test

Suppose that $\{a_n\}$ is a sequence of real numbers and $n^* \in \mathbb{N}$ If $a_n > 0$ for all $n > n^*$ and

(a) $\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) > 1$, then $\sum a_k$ Converges

(b) $\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) < 1$, then $\sum a_k$ diverges.

e.g. $a_n = \dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$ $\quad \dfrac{(2n-1)!}{(2n)!}$

⑤ Bertrand's test

Suppose $\{a_n\}$ is a sequence of real numbers and $n^* \in \mathbb{N}$ If $a_n > 0$ for all $n \geq n^*$ then

(a) If $\lim_{n\to\infty} n \ln n \left(1 - \frac{1}{n} - \frac{a_{n+1}}{a_n}\right) > 1$, then $\sum a_n$ Converges

(b) If $\lim_{n\to\infty} n \ln n \left(1 - \frac{1}{n} - \frac{a_{n+1}}{a_n}\right) < 1$, then $\sum a_n$ diverges.

(6) Gauss's Test

let $\sum a_n$ be a series of positive terms and suppose that there is a bounded sequence $\{b_n\}$ and a constant K such that

$$\frac{a_n}{a_{n+1}} = 1 + \frac{K}{n} + \frac{b_n}{n^2}$$ Then

(i) $\sum a_n$ converges if $K > 1$

(ii) $\sum a_n$ diverges if $K \leq 1$.

Kos e.g. See Gra. p-350

(7) First log test

Suppose a series $\sum a_K$ of positive term of real numbers and

$$L_n = \frac{\ln\left(\frac{1}{a_n}\right)}{\ln n}$$

(i) If $\lim_{n\to\infty} L_n > 1$, Then $\sum a_n$ converges

Kos (ii) If $\lim_{n\to\infty} L_n \leq$, then $\sum a_K$ diverges.

(8) Second log test

Suppose a series $\sum a_K$ of positive real numbers is to be tested for convergence and $M_n = \ln\left(\frac{1}{n a_n}\right) / \ln(\ln n)$

(i) If $\lim_{n\to\infty} M_n > 1$, then $\sum a_K$ converges

(ii) If $\lim_{n\to\infty} M_n \leq 1$ (eventually) the $\sum a_K$ diverges

Assignment: Give two examples of each from (1) to (8) test.

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(KOS) Alternating Series:

Def: If $\{a_n\}$ is a sequence of non-negative real numbers, then series

$$\sum (-1)^{k+1} a_k \quad \text{or} \quad \sum (-1)^k a_k$$

is called an alternating series

Alternating Series test (Leibniz's test):

(ZIP) **Statement:** If $\{a_k\}$ is a monotone decreasing sequence with $\lim_{k\to\infty} a_k = 0$, then the alternating series $\sum_{k=1}^{\infty} (-1)^k a_k$ converges.

PROOF: let $S_n = \sum_{k=1}^{n} (-1)^k a_k$ and examine first the even numbered partial sums.

$$S_2 = a_2 - a_1 \le 0 \quad \left(\because \underset{\text{m-decreasing}}{a_1 \ge a_2} \right)$$

$S_4 - S_2 = a_4 - a_3 \le 0$ imply $S_4 \le S_2$

$S_6 - S_4 = a_6 - a_5 \le 0$ and so $S_6 \le S_4$

In general

$$S_{2n} - S_{2n-2} = a_{2n} - a_{2n-1} \le 0 \text{ imply } S_{2n} \le S_{2n-2}$$

Consequently $\{S_{2n}\}$ is monotone decreasing sequence. Moreove

$$S_{2n} = -a_1 + a_2 - a_3 + a_4 - a_5 + \cdots + a_{2n-2} - a_{2n-1} + a_{2n}$$

$$= -a_1 + (a_2 - a_3) + (a_4 - a_5) + \cdots + (a_{2n-2} - a_{2n-1}) + a_{2n}$$

$$\ge -a_1 \qquad \because a_i - a_{i+1} \ge 0$$

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Hence $\{S_{2n}\}$ is bounded below and therefore must converge. (monotone decreasing and bounded below)

let $\lim_{n\to\infty} S_{2n} = S$.

Now $S_{2n-1} = S_{2n} - a_{2n}$ e.g. $S_2 - S_1 = -a_1 + a_2 + a_1 = a_2$.

and so $\lim_{n\to\infty} S_{2n-1} = \lim_{n\to\infty} S_{2n} - \lim_{n\to\infty} a_{2n}$

$= S - 0$ ($\because a_n$ is decreasing so $\lim_{n\to\infty} a_{2n} = 0$) Also Given $\lim_{n\to\infty} a_n = 0$

$= S$

Since $\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} S_{2n-1} = S$.

Therefore $\lim_{n\to\infty} S_n = S$.

and therefore $\sum (-1)^k a_k$ is convergent.

Example: $\sum \frac{(-1)^{k+1}}{k}$ converges.

Since $\frac{1}{k}$ is decreasing.

Also $\lim_{k\to\infty} \frac{1}{k} = 0$

By alternating series test. $\sum \frac{(-1)^{k+1}}{k}$ converges.

KBS Assignment Test the convergence of

$\sum (-1)^{k+1} \frac{\ln k}{k}$ converges.

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(BR) **Abel's lemma**

Let $\{x_n\}$ and $\{y_n\}$ be sequences in R and let the partial sums of $\sum y_n$ be denoted by $\{s_n\}$ with $s_0 = 0$. If $m > n$, then

$$\sum_{k=n+1}^{m} x_k y_k = x_m s_m - x_{n+1} s_n + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k$$

Proof: Since $y_k = s_k - s_{k-1}$ for $k = 1, 2, 3, \cdots$

so
$$\sum_{k=n+1}^{m} x_k y_k = \sum_{k=n+1}^{m} x_k (s_k - s_{k-1})$$

$$= \sum_{k=n+1}^{m} x_k s_k - \sum_{k=n+1}^{m} x_k s_{k-1}$$

$$= (x_{n+1} s_{n+1} + x_{n+2} s_{n+2} + \cdots + x_m s_m)$$

$$- (x_{n+1} s_n + x_{n+2} s_{n+1} + \cdots + x_m s_{m-1})$$

$$= (x_m s_m - x_{n+1} s_n) + (x_{n+1} s_{n+1} - x_{n+2} s_{n+1})$$

$$+ (x_{n+2} s_{n+2} - x_{n+3} s_{n+2}) + \cdots + (x_{m-1} s_{m-1} - x_m s_{m-1})$$

$$= (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k .$$

Required.

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(BR) Dirichlet's Test:

Statement: If $\{x_n\}$ is a decreasing sequence with $\lim_{n\to\infty} x_n = 0$ and if the partial sums $\{S_n\}$ of $\sum y_n$ are bounded, then the series $\sum x_n y_n$ is convergent.

Proof: Since $\{S_n\}$ is bounded, so

let $|S_n| \le B \quad \forall n \in \mathbb{N}$

Since $\{x_n\}$ is decreasing, so

$$x_k \geq x_{k+1} \quad \forall k.$$

$$\text{or} \quad x_k - x_{k+1} \geq 0 \quad \text{——— (1)}$$

If $m > n$, then by Abel's lemma

$$\left| \sum_{k=n+1}^{m} x_k y_k \right| \le (x_m + x_{n+1})B + \sum_{k=n+1}^{m-1} (x_k - x_{k+1})B$$

$|A+B| \le |A| + |B|$

$$= (x_m + x_{n+1} + x_{n+1} - x_m)B$$

$$= 2x_{n+1}B.$$

Since $\lim_{n\to\infty} x_n = 0$, so by Cauchy criterion $\sum x_k y_k$ converges.

$n+1 \to k$ imply $1 \to k$.

NOTE See Cauchy criterion.

Abel's Test

Statement

If $\{x_n\}$ is convergent monotone sequence and the series $\sum y_n$ is convergent, then the series $\sum x_n y_n$ is also convergent.

PROOF: If $\{x_n\}$ is decreasing with limit x,

Define $U_n = x_n - x$, $n \in N$, Then $\{U_n\}$ decreasing to zero. i.e $\lim\limits_{n\to\infty} U_n = 0$.

Moreover by the convergence of $\sum y_n$ the sequence of partial sums of $\sum y_n$ is bounded. By Dirichlet's test $\sum U_n y_n$ is convergent. Also $\sum x y_n$ is convergent by the convergence of $\sum y_n$.

Therefore

$$U_n y_n = x_n y_n - x y_n$$

Or $$x_n y_n = U_n y_n + x y_n$$

imply $\sum x_n y_n = \sum U_n y_n + \sum x y_n$ is convergent, by the sum of two convergent series.

Proceed self for decreasing case.

H.M KHALID MAHMOOD

LIMITS (LIMITS OF FUNCTIONS)

(GIS|KS)

Def (Limits at infinity)

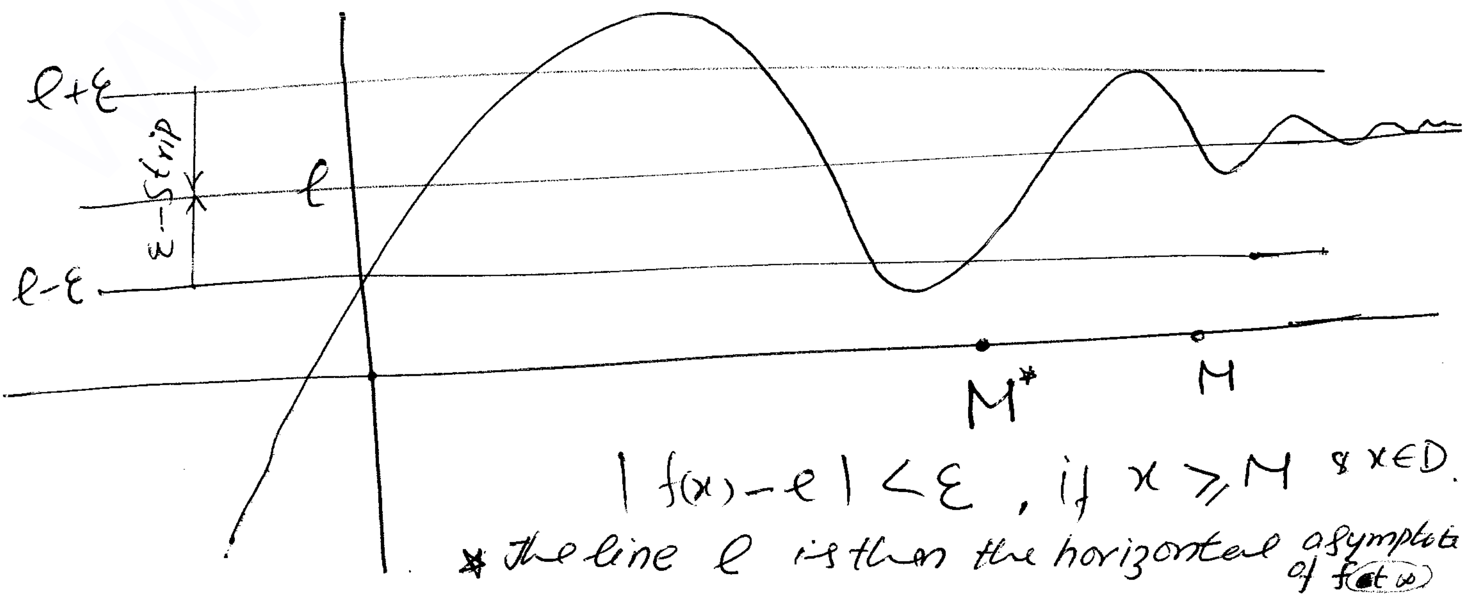
Let 'f' be a function with domain $D \subseteq R$, which contains arbitrary large values. We say that f has a limit as x tends to ∞ provided there exist $l \in R$ such that for every positive ε, there is an $M \in R$ such that $x \in D$ and $x > M$ implies

$$|f(x) - l| < \varepsilon.$$

If such an l exists, we say l is the limit of the function f as x tends to ∞, or simply, l is the limit of f at plus infinity, or l is the limiting value at ∞ and we write

$$\lim_{x \to \infty} f(x) = l$$

where x is a dummy variable.



$|f(x) - l| < \varepsilon$, if $x \geq M$ & $x \in D$.

* The line l is then the horizontal asymptote of f at ∞

Remark: The difference between ~~the~~ limit of sequence and the def defined above is

(i) Domain for sequences is set of natural numbers. Therefore the condition $n \geq N$; $N \in \mathbb{N}$ is obviously true.

(ii) Domain for sequence is a subset of real number. Therefore $(M, \infty) \cap D$ may be empty.

i-e $x > M$ might not be selected.

(iii) Thus there aris diff between both definitions when the domin of function (not in seq case) is bounded.

*(iv) In case of bounded domain any real number will serve for limit at ∞.

GIS
Example $f(x) = \dfrac{1}{x+1}$, $x \in Q \cap [0, \infty)$

Sol $|f(x) - 0| = \left|\dfrac{1}{x+1} - 0\right| = \dfrac{1}{x+1}$; $x \in Q \cap [0, \infty)$

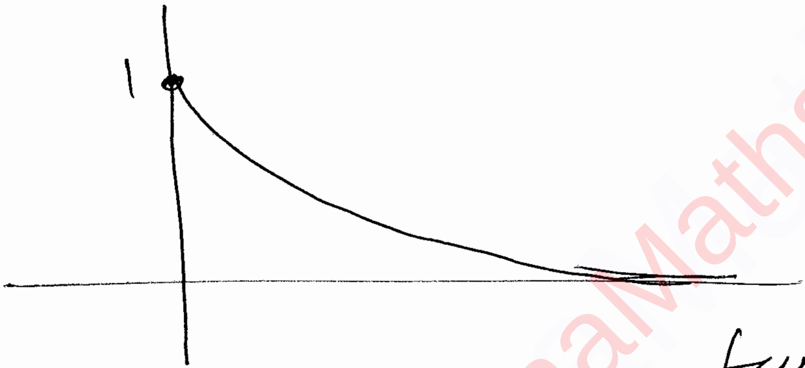
let $M = \dfrac{1}{\varepsilon}$ if $x > M$ then $x + 1 > x$
$> \dfrac{1}{\varepsilon}$

i-e $\frac{1}{x+1} < \varepsilon$ if $x > M$

Therefore

$$|f(x) - 0| = \frac{1}{x+1}$$
$$< \frac{1}{x}$$
$$< \varepsilon. \quad \text{if } x > M.$$

Therefore lt is 0.



Note that above function has infinitely many holes.

* Limit is same if we consider the sequence $\left\{\frac{1}{n+1}\right\}$. $Q \cap [0, \infty)$ is unbounded.

Example Find $\lim\limits_{x\to\infty} f(x) = \cancel{\frac{1}{x^2}}\ x^2$ on $[0, 10^{10000}]$.

★ Assignment.

Negation of the limit definition:

A function 'f', having domain D, will fail to have a limit as x tends to infinity provided for every $l \in R$, there exist $\varepsilon > 0$ such that for every positive $M \in R$ there exist $x \in D$ such that $x > M$ and $|f(x) - l| \geq \varepsilon$.

** If a function fails to have a limit as $x \to \infty$, then the domain D, cannot be bounded above.

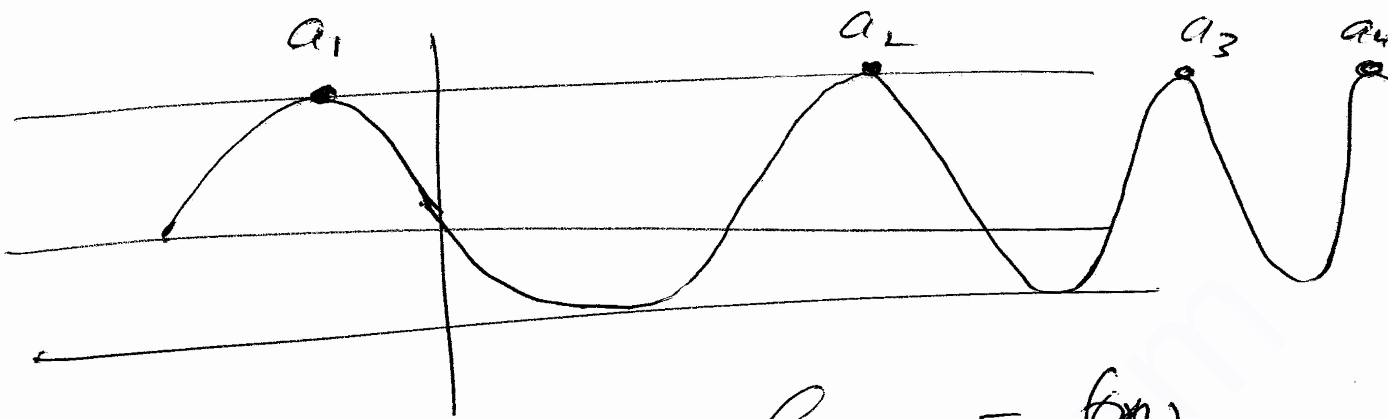
KOS

REMARK:

To evaluate limit for sequences (at ∞), we use technique for evaluating more general limits of functions but not conversely. That is If $\lim\limits_{x \to \infty} f(x) = l$ then this limits hold for any $x \in R$ which tends to infinity, in particular, $x = n$, $n \in N$. Therefore $\lim\limits_{x \to \infty} f(x) = l$ implies $\lim\limits_{n \to \infty} f(n) = l$.

But $\lim\limits_{n \to \infty} f(n) = l \not\Rightarrow \lim\limits_{x \to \infty} f(x) = l$

e.g $f(n) = \sin n$.



The sequence $a_n = \sin n = f(n)$ is convergent but $f(x) = \sin x$ is divergent.

because $f(x) = \sin x$, then

$$\sin\left(\frac{\pi}{2} + 2n\pi\right) = 1$$

$$\sin\left(\frac{3\pi}{2} + 2\pi x\right) = -1$$

Take $\varepsilon = \frac{1}{2}$. and fix real number M. Clearly many values $x \in R^+$ exist that satisfy

$$|f(x) - L| \geq \frac{1}{2} \quad (\text{whatever } L)$$

But if we take $f(n) = \sin n\pi$ then $x_n = n$; $n \in N$ is divergent and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin(n\pi) = 0$ is convergent.

Therefore to prove that $\lim_{x\to\infty} f(x)$ does not exist, we often show that $\lim_{n\to\infty} f(n)$ does not exist and often this can be achieved by finding two subsequences that converges to two different values.

G.S

THEOREM

Let f be a function with domain D that is not bounded above. If f has a limit at ∞, then this limit is unique.

PROOF: Since 'f' has limit at ∞. let it ℓ. Let ℓ' be any other limit. If $\ell \neq \ell'$, then we may assume that $\ell > \ell' \Rightarrow \ell - \ell' > 0$. let $\varepsilon = \frac{\ell - \ell'}{2}$ — ①*

Since ℓ, ℓ' are the limits, so we can find M_1 and M_2 depending upon ℓ & ℓ' such that

$|f(x) - \ell| < \varepsilon$ whenever $x > M_1$; $x \in D$

& $|f(x) - \ell'| < \varepsilon$ whenever $x > M_2$; $x \in D$

Let $M = \max(M_1, M_2)$.

Now, D is not bounded above, so there is an $x_0 \in D$ with $x_0 > M$ such that

$f(x_0) < \ell' + \varepsilon \underset{\text{by ①*}}{=} \ell - \varepsilon < f(x_0)$, which is absurd. Hence $\ell = \ell'$

THEOREM:

Suppose $D \subseteq R$ is an unbounded domain of the function 'f' (That is D contains arbitrary large values). Then $\lim_{x \to \infty} f(x) = \ell$ iff for every sequence $\{x_n\}$ in D that diverges to ∞ (i.e $\lim_{n\to\infty} x_n = \infty$), the sequence $\{f(x_n)\}$ converges to ℓ.

PROOF: Suppose that $\lim_{x\to\infty} f(x) = \ell$. Let $\{x_n\}$ be any sequence in D diverging to plus infinity and consider the sequence $\{f(x_n)\}$. We will prove that $\{f(x_n)\}$ converges to ℓ. Let an arbitrary $\varepsilon > 0$ be given. Since $\lim_{x\to\infty} f(x) = \ell$, there exist $M > 0$ such that $|f(x) - \ell| < \varepsilon$ if $x \geq M$, $x \in D$.
Pick $n^* \geq M$. Then $x_n \geq n^* \geq M$ ($\because x_n \to \infty$ diverging to ∞)
Thus $|f(x_n) - \ell| < \varepsilon$ whenever $x_n \geq M$.
Hence $\{f(x_n)\}$ converges to ℓ.

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Conversely, suppose that for every sequence $\{x_n\}$ in D that diverges to infinity, the sequence $\{f(x_n)\}$ converges to l. On contrary suppose that $\lim_{x\to\infty} f(x) \neq l$. then there exists an $\varepsilon > 0$ such that for every M there exist an $x \geq M$ with $x \in D$,

$$|f(x) - l| \geq \varepsilon.$$

In particular for each integer n there exist $x_n \in D$ such that

$$|f(x_n) - l| \geq \varepsilon \text{ for } x_n \geq n.$$

Which is a contradiction against our hypothesis. therefore $\lim_{x\to\infty} f(x) = l$.

Assignment:

Study P-118, Theorem 3.1.12 without proof and Example 3.1.13.

H.M KHALID MAHMOOD

Limit of a Function at a Real number

Definition:

Let $A \subseteq R$ and let c be a cluster point of A (i.e there is a sequence in A which converges to c). For a function $f: A \to R$, a real number l is said to be a limit of f at c if, given any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then

$$|f(x) - l| < \varepsilon.$$

OR

Let $f(x)$ be defined on an open interval about x_0, except possibily at x_0 itself. We say that $f(x)$ approaches the limit l as x approaches x_0 and write $\lim_{x \to x_0} f(x) = l$ if

For every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon$

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REMARK:

① Since the value of δ usually depends on ε, we sometime write $\delta(\varepsilon)$ instead of δ to emphasize dependence.

② The inequality $0<|x-c|$ is equivalent to saying $x \neq c$

③ The ε-δ definition of limit does not tell how to find the limit of a function, but it enables us to verify that a suspected limit is correct.

④ If l is a limit of 'f' at c, we also say that f converges to l at c

We often write $l = \lim\limits_{x \to c} f(x)$ or

$$l = \lim_{x \to c} f.$$

or $f(x) \to l$ as $x \to c$

⑤ If the limit of 'f' at c does not exist, we say that 'f' diverges at c.

R
Assignment
)
If $f: A \to \mathbb{R}$ and if c is a cluster point of A, then 'f' can have only one limit at c.

... following statements are equivalent.

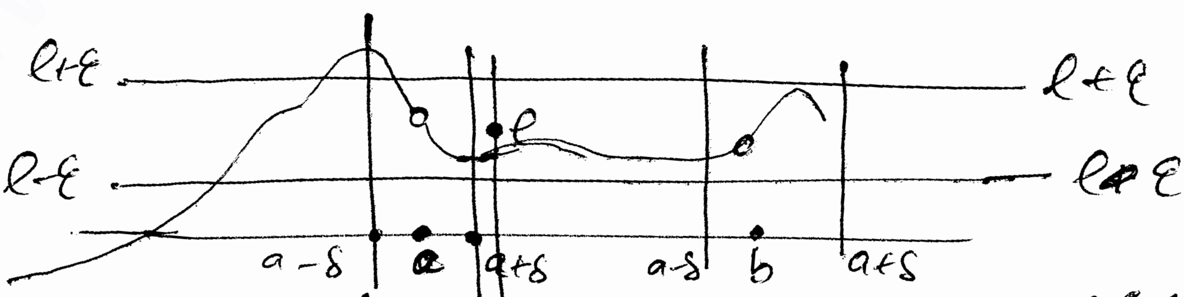
(i) $\lim_{x \to c} f(x) = \ell$

(ii) Given any ε-neighborhood $V_\varepsilon(\ell)$ of ℓ, there exist a δ-neighborhood $V_\delta(c)$ of c such that if $x \neq c$ is any point in $V_\delta(c) \cap A$, then $f(x) \in V_\varepsilon(\ell)$.

T.F How to find a δ for a given f, ℓ, x_0 and $\varepsilon > 0$ algebraically.

<u>Step I</u> Solve the inequality $|f(x) - \ell| < \varepsilon$ to find an open interval (a, b) about x_0 on which the inequality holds for all $x \neq x_0$

<u>Step 2</u> Find a value of $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 ~~include~~ inside the interval (a, b). The inequality $|f(x) - \ell| < \varepsilon$ will hold for all $x \neq x_0$ in this δ-interval.



Rectangle does not cover all values at $(a-\delta, a+\delta)$

All values corresponding to δ-strip lies in ε-strip.

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T.F **Examples**

① Prove that $\lim_{x\to 2} f(x) = 4$ if

$$f(x)=\begin{cases} x^2, & x\neq 2 \\ 1, & x=2 \end{cases}$$

Sol Our task is to show that given $\varepsilon>0$ there exist a $\delta>0$ such that for all x

$$0<|x-2|<\delta \Rightarrow |f(x)-4|<\varepsilon$$

Step I Solve the inequality $|f(x)-4|<\varepsilon$ to find an open interval about $x_0=2$ on which the inequality hold for all $x\neq x_0$.

For $x\neq x_0=2$, we have $f(x)=x^2$, then

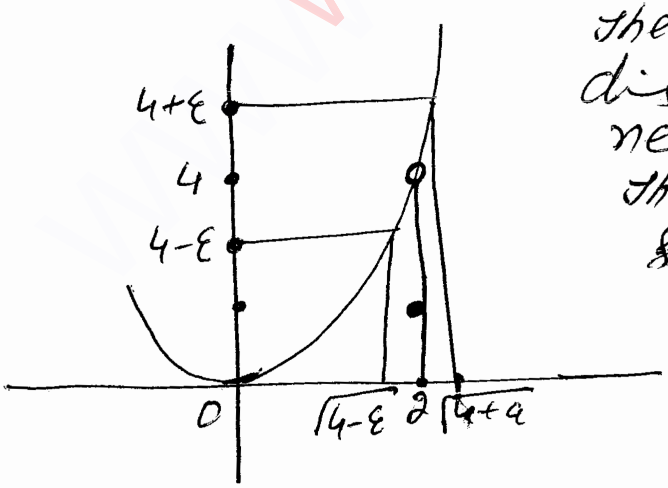
$$|x^2-4|<\varepsilon \text{ iff } 4-\varepsilon<x^2<4+\varepsilon$$

$$\Rightarrow \sqrt{4-\varepsilon}<x<\sqrt{4+\varepsilon}$$

Thus inequality

$|x^2-4|<\varepsilon$ hold for all $x\in(\sqrt{4-\varepsilon},\sqrt{4+\varepsilon})$

Step 2 Find a value $\delta>0$ such that $(2-\delta, 2+\delta)$ lie inside $(\sqrt{4-\varepsilon},\sqrt{4+\varepsilon})$



Therefore take δ to be the distance from $x_0=2$ to the nearer end point of $(\sqrt{4-\varepsilon},\sqrt{4+\varepsilon})$. Then it will automatically satisfy,

$$|f(x)-4|<\varepsilon \text{ whenever } 0<|x-2|<\delta.$$

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Aliter:

$$|f(x)-4| = |x^2-4|$$
$$= |x-2|\,|x+2|$$

$|x|-2 \le |x-2| < \delta$

$|x| < 2+\delta < 2+1 = 3$

$|x+2| \le |x|+2 < 3+2$

Suppose $\delta \le 1$ and $|x-2| < \delta$

Then $|x-2| < \delta \le 1$ implies $0 < |x+2| < 5$

therefore $|x-2| < \delta \le 2$ implies

~~then~~ $|x-2|\,|x+2| < 5|x-2|$

Choose $\delta = \varepsilon/5$

therefore

$$|f(x)-4| = |x^2-4|$$
$$= |x-2|\,|x+2|$$
$$< 5|x-2|$$
$$< 5\,\delta = 5\cdot\frac{\varepsilon}{5} \quad \because |x-2| < \delta$$

imply

$|f(x)-4| < \varepsilon$ whenever $0 < |x-2| < \delta$

which is desired.

Example: $\lim\limits_{x\to 3} (4x-7) = 5$

Sol: Let $\varepsilon > 0$,

$$|4x-7-5| = 4|x-3|$$
$$< 4\delta \qquad \text{if } 0<|x-3|<\delta \quad \text{——} (1)$$

Choose $\delta = \varepsilon/4$ (1) becomes

$|f(x)-5| < \varepsilon$ whenever $0 < |x-3| < \delta$.

Assignment Ex 1.3, P-74 T. Finny

BR THEOREM: If $f: A \to R$ and if c is a cluster point of A, then f can have only one limit at c. (Assignment) G.M

BR THEOREM:

Let $f: A \to R$ and let c be a cluster point of A, then $\lim_{x \to c} f = \ell$ iff For any $\varepsilon > 0$ there exist a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - \ell| < \varepsilon$.
(Assignment)

(BR)

Sequential Criterion for limits:

Statement: Let $f: A \to R$ and let c be a clusterpoint of A. Then $\lim_{x \to c} f = \ell$ iff for every sequence $\{x_n\}$ in A that converges to c such that $x_n \neq c$ for all $n \in N$, the sequence $\{f(x_n)\}$ converges to ℓ.

PROOF: Suppose $\lim_{x \to c} f = \ell$ and $\{x_n\}$ be a sequence in A that converges to c, $x_n \neq c$ $\forall n \in N$.
Then for $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, where $x \in A$
then $|f(x) - \ell| < \varepsilon$. —① Moreover by definition of convergent sequence, For $\delta > 0$, there exist a natural number K (depending on δ) such that $|x_n - c| < \delta$ whenever $n > K$
But each x_n is in A, therefore by ①

$$|f(x_n) - \ell| < \varepsilon \quad \text{whenever } n > K$$

Hence the sequence $\{f(x_n)\}$ converges to ℓ.
Since $\{x_n\}$ was arbitrary, so we have proved the

the desired condition.

Conversely suppose $\lim_{n\to\infty} f(x_n) = \ell$ for every sequence $\{x_n\}$ in A with $x_n \neq c$ and $x_n \to c$ as $n \to \infty$.

but $\lim_{x\to c} f \neq \ell$.

Then for atleast one $\varepsilon_0 > 0$ and for every $\delta > 0$, we can find a point x (depending on δ) such that $|f(x) - \ell| \not< \varepsilon_0$ even though $0 < |x - c| < \delta$ and this hold particularly when $\delta = \frac{1}{n}$

It follows that for $n = 1, 2, \dots$ there exist a sequence $\{x_n\}$ for which

$$0 < |x_n - c| < \frac{1}{n} \text{ but } |f(x_n) - \ell| \geq \varepsilon_0 \quad \forall n \in N.$$

Contradicting to the hypothesis. Which complete the proof.

(BR) Devergence Criteria

Let $A \subseteq R$, let $f : A \to R$ and let $c \in R$ be a cluster point of A

(a) If $\ell \in R$, Then f does not have a limit ℓ at c iff there exist a sequence $\{x_n\}$ in A with $x_n \neq c$ for all $n \in N$ such that the sequence $\{x_n\}$ converges to c but the sequence $\{f(x_n)\}$ does not converges to ℓ.

(b) The function f does not have a limit at c iff there exist a sequence $\{x_n\}$ in A with $x_n \neq c$ for all $n \in N$ such that $\{x_n\}$ converges to c but the sequence $\{f(x_n)\}$ does not converges in R.

PROOF: Assignment (use contrapositive statement)

Examples:

① $\lim\limits_{x \to 0} \frac{1}{x}$ does not exist in $\mathbb{R}$.

Let $\varphi(x) = \frac{1}{x}$ for $x > 0$

Take $c = 0$. Take sequence $x_n = \frac{1}{n}$

Then $x_n \neq c \ \forall n$ and $\lim\limits_{n \to \infty} x_n = 0$

But $\varphi(x_n) = \frac{1}{\left(\frac{1}{n}\right)} = n$.

Then $\varphi(x_n)$ is not convergent.

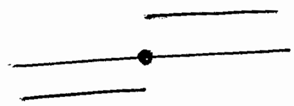
Therefore by divergence criterion

$\lim\limits_{x \to 0} \varphi(x)$ does not exist in $\mathbb{R}$.

② $\lim\limits_{x \to 0} \operatorname{sgn}(x)$ does not exist.

Define $\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0. \end{cases}$

Then $\operatorname{sgn}(x) = \frac{x}{|x|}$ for $x \neq 0$



Let $x_n = \frac{(-1)^n}{n}$ for $n \in \mathbb{N}$

Then $\lim\limits_{n \to \infty} x_n = 0$. But

$$\operatorname{Sgn}(x_n) = \frac{(-1)^n / n}{\left| \frac{(-1)^n}{n} \right|} = (-1)^n \ \forall n \in \mathbb{N}.$$

Therefore $\lim\limits_{n \to \infty} \operatorname{sgn}(x_n) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$

It follows that $\{\operatorname{sgn}(x_n)\}$ does not converge.

Therefore $\lim\limits_{x \to 0} \operatorname{sgn}(x)$ does not exist

③ $\lim\limits_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist in R.

let $g(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$

Def $x_n = \frac{1}{n\pi}$ for $n \in N$

and $y_n = \frac{1}{\frac{1}{2}\pi + 2n\pi}$ for $n \in N$

$\left\{ \begin{array}{l} \because \sin t = 0 \Rightarrow t = n\pi, \; n\in Z \\ \& \sin t = 1 \Rightarrow t = \frac{1}{2}\pi + 2n\pi, \; n \in Z \end{array} \right.$

then $x_n \neq 0$ & $y_n \neq 0$ for all $n \in N$.

More over $\lim\limits_{n\to\infty} x_n = 0 = \lim\limits_{n\to\infty} y_n$.

Now

$\lim\limits_{n\to\infty} g(x_n) = \lim\limits_{n\to\infty} \sin(n\pi) = 0 \;\; \forall n \in N$

$\lim\limits_{n\to\infty} g(y_n) = \lim\limits_{n\to\infty} \sin\left(\frac{\pi}{2} + 2n\pi\right)$

$= \lim\limits_{n\to\infty} \sin\left(\frac{\pi}{2}\right) \;\; \forall n \in N$

$= 1$

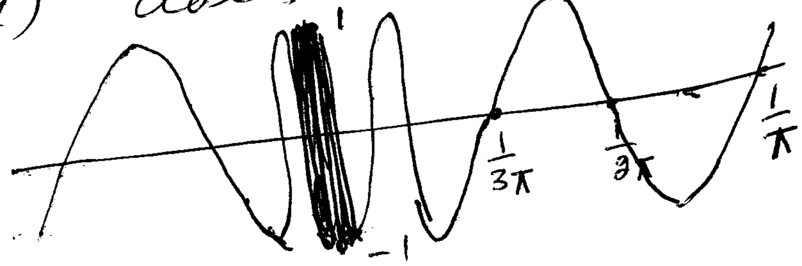
Thus we have two subsequences in R namely $\{x_n\}$ and $\{y_n\}$ which are convergent with $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = 0$

But $\lim\limits_{n\to\infty} g(x_n) \neq \lim\limits_{n\to\infty} g(y_n)$

which shows that $\lim\limits_{n\to\infty} g(a_n)$ is not convergest.

Hence, we conclude

$\lim\limits_{x\to 0} g(x)$ does not exist.



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Bounded Functions:

Def A function $f: A \to B$ is called bounded provided there is a real number $M > 0$ such that $|f(x)| \leq M$ for every $x \in A$.

Remark:

1. f is bounded iff its range $f(A)$ is a bounded set.
2. If $f: A \to B$ is bounded function, then f is bounded on C for every $C \subseteq A$.
3. Every constant function is bounded
4. If f assume finitely many values, then it has a finite range, so it is bounded. $\{-1,1,-1,1,\ldots\}$
5. Identity function is not bounded on R but is bounded on every bounded interval I.

Example $f(x) = x^2$, Show that $f(x)$ is unbounded on R but is bounded on each bounded interval I.

Sol Suppose f is bounded. Then there exist $M > 0$ s.t $|f(x)| \leq M \;\forall x \in R$

But $f(M+1) = (M+1)^2 > M+1$
$> M$, a contradiction.

Now Restrict 'f' to a bounded interval say $[a,b]$. Then $|f(x)| \leq M$ where $M = \max(a^2, b^2)$ for every $x \in [a,b]$. Since $[a,b]$ was arbitrary therefore f is bounded on every bounded interval.

If f and g are each bounded on A and k is any real number then the function $f+g$, kf and $f\cdot g$ are each bounded on

(2P)
Remark

① Since constant function is bounded on R and $f(x)=x$ is bounded on each bounded interval I. Therefore every polynomial function with co-efficient in R is bounded on every bounded interval I.

② If f and g are bounded on A, then f/g may fail to be bounded on A.

e.g $f(x)=1$ & $g(x)=x$ $\forall x \in A=(0,1)$

Then $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{1}{x}$ is unbounded on $(0,1)$.

③ If f is bounded on A and g is "bounded away from zero" on A, that is there is an $\alpha>0$ such that $|g(x)|\geq\alpha$ for every $x\in A$, then $\frac{f}{g}$ is bounded on A.

Since $|f(x)|\leq M$ & $|g(x)|\geq\alpha \Rightarrow \frac{1}{|g(x)|}\leq\frac{1}{\alpha}$.

imply $\left|\left(\frac{f}{g}\right)(x)\right|=\left|\frac{f(x)}{g(x)}\right|=\frac{|f(x)|}{|g(x)|}\leq\frac{M}{\alpha}$

④ A function 'g' is bounded away from zero iff the function $\frac{1}{g}$ is bounded.

⑤ If f is bounded on A, then f is bounded at each $x_0\in A$ but if f is bounded at each $x_0 \in A$, then it may not be bounded on A. e.g $f(x) = \frac{1}{x}$ on $(0,1)$ is not bounded but is bounded at each $x_0 \in (0,1)$.

(6) If f is either even or odd and if f is bounded on $(0,\infty)$, then it is bounded on R.

(7) If f is odd and bounded above then f is necessarily bounded.

$f(x) \leq M \quad \forall x \in R$ $\because$ f is bdd below.

$-f(x) = f(-x) \leq M$ for every $x \in R$.

therefore $-M \leq f(x) \leq M$

$\Rightarrow \quad |f(x)| \leq M. \quad \forall x \in R.$

(8) f is said to be bounded at $x_0 \in A$ provided there is an open interval J containg x_0 such that f is bounded on $J \cap A$.

(2P)
THEOREM: If f is bounded at each point $x_0 \in A$ and A is compact then f is bounded on A.

PROOF: We assume f is bounded at each $x_0 \in A$ then for each $x \in A$ there is an open interval $I_x = (x-\delta_x, x+\delta_x)$ such that f is bounded on $A \cap I_x$. The set of open intervals $\{I_x : x \in A\}$ is an open covering of A, so by Heine-Borel theorem there are finitely many of these open intervals (say) $I_{x_1}, I_{x_2}, \ldots, I_{x_n}$ such that $A \subseteq \bigcup_{k=1}^{n} I_{x_k}$

Since for each $k = 1, 2, 3, \ldots, n$ f is bounded on $A \cap I_{x_k}$ so there are $M_k > 0$ such that $|f(x)| \leq M_k \quad \forall x \in A \cap I_{x_k}$.

let $M = \max\limits_{1 \leq k \leq n} M_k$.

Now, for any $x \in A$ there is a j with $1 \leq j \leq n$ such that $x \in I_{x_j}$. Hence $x \in A \cap I_{x_j}$ and so $|f(x)| \leq M_j \leq M$.

i-e $|f(x)| \leq M \quad \forall x \in A$.

Therefore f is bounded on A.

BR

Definition: let $A \subseteq R$, $f: A \to R$ and let $c \in R$ be a cluster point of A. we say that 'f' is bounded on a neighborhood of c if there exist a δ-neighborhood $V_\delta(c)$ of c and a constant $M > 0$ such that

$$|f(x)| \leq M \text{ for all } x \in A \cap V_\delta(c).$$

BR/2P

THEOREM If $A \subseteq R$ and $f: A \to R$ has a limit at $c \in R$ then f is bounded on some neighborhood of c

OR If $\lim\limits_{x \to c} f(x) = l$ then f is bounded on some deleted neighborhood of $x = c$;

That is there is a $\delta > 0$ such that f is bounded on $N_{\delta^*}(c)$ *(stand for deleted neighborhood of c)

PROOF: choose $\varepsilon = 1$. If $\lim\limits_{x\to c} f(x) = \ell$ then there is a $\delta > 0$ such that

$|f(x) - \ell| < 1$ whenever $0 < |x - c| < \delta$

thus for each $x \in N_{\delta}^{*}(c)$, we have

$$|f(x)| = |f(x) - \ell + \ell|$$
$$\leq |f(x) - \ell| + |\ell|$$
$$< 1 + |\ell| = M \text{ (say)}$$

Hence $|f(x)| < M \quad \forall\, x \in N_{\delta}^{*}(c)$.

Hence f is bounded on $N_{\delta}^{*}(c)$.

(Zp)

THEOREM: Let $A \subseteq R$, let $f, g : A \to R$ and $c \in R$ be a cluster point of A. Further if $b \in R$, then

(a) If $\lim\limits_{x\to c} f = L$ and $\lim\limits_{x\to c} g = M$ then

$\lim\limits_{x\to c} (f \pm g) = L \pm M$

$\lim\limits_{x\to c} fg = LM$ & $\lim\limits_{x\to c} bf = bL$.

(b) If $h : A \to R$, if $h(x) \neq 0 \;\forall\, x \in A$ and $\lim\limits_{x\to c} h = H \neq 0$, then $\lim\limits_{x\to c} \left(\frac{f}{h}\right) = \frac{L}{H}$

Assignment (Give presentation)

Hint @ Zip @ T. Finny

SEE Remark, P-122 (BAR):

 229

Assignment:

① $\lim_{x\to c} P(x) = P(c)$ P-124 (BR)

② $\lim_{x\to c} \frac{P(x)}{q(x)} = \frac{P(c)}{q(c)}$ if $q(c) \neq 0$

(BR/ZP)

THEOREM

If $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = \ell'$ and if $f(x) \le g(x)$ for every x in some deleted neighborhood $N_\delta^*(a)$ of $x=a$ then $\ell \le \ell'$

PROOF: On contrary suppose $\ell' < \ell$ imply $\ell - \ell' > 0$

Now $\lim_{x\to a}[f(x) - g(x)] = \ell - \ell' > 0$

Choose $\varepsilon = \frac{\ell - \ell'}{2}$, then there is an $N_\delta^*(a)$ such that

$$|f(x) - g(x) - (\ell - \ell')| < \varepsilon = \frac{\ell - \ell'}{2}$$

because $\lim_{x\to a} f(x) = \ell$ & $\lim_{x\to a} g(x) = \ell'$

therefore $\frac{1}{2}(\ell - \ell') < f(x) - g(x) < \frac{3}{2}(\ell - \ell')$

$\forall x \in N_\delta^*(a)$

or $f(x) - g(x) > \frac{1}{2}(\ell - \ell') > 0$

$\Rightarrow$ $f(x) > g(x)$ which is a contradiction against the fact $f(x) \le g(x)$ $\forall x \in N_\delta^*(a)$. Hence $\ell \le \ell'$.

(BK)

THEOREM: Let $A \subseteq R$. Let $f: A \to R$ and Let $c \in R$ be a cluster point of A. If $a \le f(x) \le b \ \forall x \in A, x \ne c$ and if $\lim_{x \to c} f$ exist, then $a \le \lim_{x \to c} f \le b$

PROOF: Since $\lim_{x\to c} f$ exists, let it be ℓ

Then for any sequence x_n in A with $x_n \ne c \ \forall n \in N$ if $x_n \to c$ Then $\{f(x_n)\} \to \ell$.

Since $a \le f(x) \le b \quad \forall x \in A$

Therefore $a \le f(x_n) \le b \quad \forall n \in N$

Then $\lim_{n\to\infty} a \le \lim_{n\to\infty} f(x_n) \le \lim_{n\to\infty} b$

or $a \le \ell \le b$.

Which complete the proof.

Sequeeze (Sandwich) THEOREM:

Let $A \subseteq R$, $f, g, h : A \to R$. Let $c \in R$ be a cluster point of A. If

$$f(x) \le g(x) \le h(x) \quad \forall x \in A ; x \ne c$$

and if $\lim_{x\to c} f = \lim_{x\to c} h = \ell$

Then $\lim_{x\to c} h = \ell$.

PROOF: Assignment. Hint use if $f(x) \le g(x) \ \forall x$. Then $\ell \le \ell'$

Examples:

① Since $-\frac{1}{x} \not\to \ell = 1$ & $\frac{1}{x} \not\to \ell = 1$, therefore we will not apply sandwich theorem for $\lim_{x\to 0} \frac{\sin x}{x}$.

② $$-1 \le \sin(\tfrac{1}{x}) \le 1$$
$$\Rightarrow -x \le x \sin\tfrac{1}{x} \le x$$
By sandwich theorem
$$\lim_{x\to 0} x \sin(\tfrac{1}{x}) = 0$$

(BR)
Assignment: Let $A \subseteq R$, let $f: A \to R$ and let $c \in R$ be a cluster point of A. If $\lim_{x\to c} f > 0$ ($\lim_{x\to c} f < 0$) then there exist a neighborhood $V_\delta(c)$ of c such that $f(x) > 0$ ($f(x) < 0$ respectivily) for all $x \in A \cap V_\delta(c)$, $x \neq c$.

SOME EXTENSIONS OF THE LIMIT CONCEPTS

ONE SIDED LIMITS

Def: Let $A \subseteq R$ and $f: A \to R$

(i) If $c \in R$ is a clusterpoint of the set $A \cap (c, \infty) = \{x \in A : x > c\}$ then we say that $\ell \in R$ is a right-hand limit of f at and we write $\ell = \lim_{x\to c^+} f$ if given $\varepsilon > 0$ there exist a δ (depending uppon ε) > 0 such that for all $x \in A$ with $0 < x < c+\delta$ imply $|f(x) - \ell| < \varepsilon$

(ii) If $c \in R$ is a clusterpoint of the set $A \cap (-\infty, c) = \{x \in A : x < c\}$ then we say that $\ell \in R$ is a left-hand limit of f at c and we write $\lim_{x\to c^-} f = \ell$ if given $\varepsilon > 0$, there exist a $\delta > 0$ such that $c-\delta < x < c$ then $|f(x) - \ell| < \varepsilon$

COMPARISON: $0 < |x - c| < \delta$ iff $c - \delta < x < c + \delta$

Def: $\lim\limits_{x \to a^+} f(x) = \ell$ if given any $\varepsilon > 0$ there is a $\delta > 0$ such that if $a < x < a+\delta$ then $|f(x) - \ell| < \varepsilon$

OR $\lim\limits_{x \to a^+} f(x) = \ell$ if given any neighborhood of ℓ, say $N_\varepsilon(\ell)$ there is a $\delta > 0$ such that $x \in (a, a+\delta)$ implies that $f(x) \in N_\varepsilon(\ell)$. If $\lim\limits_{x\to a^+} f(x) = \ell$, we say that the lt of f as x approaches a from the right is ℓ or ℓ is the right-hand limit of f at $x = a$.

Exp: $\lim\limits_{x \to 0^+} \dfrac{x^2}{x + |x|} = 0$

Sol: If $x \in (0, \infty)$, then $|x| = x$

and $f(x) = \dfrac{x^2}{x+|x|} = \dfrac{x^2}{x+x} = \dfrac{x}{2}$.

Let $\varepsilon > 0$ be given, choose $\delta = 2\varepsilon$

~~Then~~ Now, if $0 < x < \delta$ then

$$|f(x) - 0| = \left|\frac{x}{2} - 0\right| = \frac{x}{2} < \frac{\delta}{2} = \varepsilon.$$

i.e $|f(x) - 0| < \varepsilon$ whenever $0 < x < \delta$.

Therefore $\lim\limits_{x \to 0^+} \dfrac{x^2}{x+|x|} = 0$ NOTE $f(0)$ does not exist.

Def $\lim\limits_{x \to a^-} f(x) = \ell$ if given $\varepsilon > 0$ there is a $\delta > 0$ such that $a - \delta < x < a$ then $|f(x) - \ell| < \varepsilon$.

OR $\lim\limits_{x \to a^-} f(x) = \ell$ if given any neighborhood of ℓ, say $N_\varepsilon(\ell)$ there is a $\delta > 0$ such that $x \in (a-\delta, a)$ implies $f(x) \in N_\varepsilon(\ell)$.

If $\lim\limits_{x \to a^-} f(x) = \ell$, we say that limit of f as x approches a from the left is ℓ or ℓ is the left-hand limit of f at $x = a$

Example $\lim_{x \to 0^-} f(x)$; $f(x) = \dfrac{x^2}{x + |x|}$

$= \lim_{x \to 0^-} \dfrac{x^2}{x - x}$; $x \in (-\infty, 0)$

$= \infty.$

Example: $f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ 2x - 3 & \text{if } x > 2 \end{cases}$

Sol Let $\varepsilon > 0$ be given. choose $\delta = \varepsilon/2$

If $2 < x < 2 + \delta$, then

$|f(x) - 1| = |2x - 3 - 1|$

$= 2|x - 2|$

$< 2 \dfrac{\varepsilon}{2}$ whenever $0 < |x - 2| < \delta$

Therefor $|f(x) - 1| < \varepsilon$ whenever $0 < x - 2 < \delta.$
or $2 < x < \delta + 2.$

therefore $\lim_{x \to 2^+} f(x) = 1.$

Again let $\varepsilon > 0$ be given, choose $\delta = \varepsilon$

if $2 - \delta < x < 2$

Then $|f(x) - 3| = |x + 1 - 3|$

$= |x - 2|$

$< \delta = \varepsilon.$

Therefore $\lim_{x \to 2^-} f(x) = 3.$

Note $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x).$

(7P) THEOREM:
$$\lim_{x \to a} f(x) = l \text{ iff } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = l.$$

Proof: If $\lim_{x \to a} f(x) = l$ then given any $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < |x-a| < \delta$ then $|f(x) - l| < \varepsilon$. clearly then $a < x < a+\delta$ (i-e if $a < x < a+\delta$) implies $|f(x) - l| < \varepsilon$ and $a-\delta < x < a$ implies $|f(x) - l| < \varepsilon$, and so $\lim_{x \to a^-} f(x) = l$ and $\lim_{x \to a^+} f(x) = l$.

Conversely, let $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$

let $\varepsilon > 0$ be given. Then there is $\delta_1 > 0$ such that if $a < x < a+\delta_1$ then $|f(x) - l| < \varepsilon$. Also there is $\delta_2 > 0$ such that if $a-\delta_2 < x < a$ then $|f(x) - l| < \varepsilon$. Note that δ_1, δ_2 may be same. Thus in either case let $\delta = \min(\delta_1, \delta_2)$

Now if $0 < |x-a| < \delta$, then either

$x \in (a, a+\delta)$ ~~$(a, a+\delta_1)$~~ $a+\delta < x < a+\delta$

or $x \in (a-\delta, a)$ ~~$(a-\delta_2, a)$~~

Then $a-\delta_2 < a-\delta < x < a+\delta < a+\delta_1$

i-e $a-\delta < x < a+\delta$.

i-e $0 < |x-a| < \delta$

Therefore
$|f(x) - l| < \varepsilon$ wherever $0 < |x-a| < \delta$.

Hence $\lim_{x \to a} f(x) = l$.

MONOTONE FUNCTIONS:

Def A function $f: A \to B$ is said to be monotone increasing on A if $f(x_1) \le f(x_2)$ for all $x_1, x_2 \in A$ with $x_1 < x_2$; $f(x)$ is said to be monotone decreasing on A if $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in A$ with $x_1 < x_2$.

Def A function is called monotone on A if it is either monotone increasing or monotone decreasing on A.

Remark ① Any constant function is both monotone increasing and monotone decreasing.

② If $f: A \to B$ and $I \subseteq A$, it may be that f is monotone I but not monotone on A.

e.g $f(x) = |x|$ is monotone decreasing on $(-\infty, 0]$ and is monotone increasing on $[0, \infty)$ but f is not monotone on R.

Def Strictly increasing and St. decreasing

NOTE: ① A constant function is neither St. increasing nor St. decreasing.

② If $f: A \to B$ is St. monotone on A then it is one-one. i-e $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

In this case $f^{-1}: f(A) \to A$ defined by $f^{-1}(y) = x$ iff $f(x) = y$ exists.

THEOREM:

If 'f' is monotone on (a,b), then for each x_0 in the interval (a,b) limit $\lim\limits_{x\to x_0^+} f(x)$ and $\lim\limits_{x\to x_0^-} f(x)$ both exist.

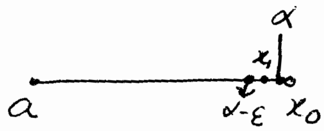
Proof: We assume that 'f' is monotone increasing on (a,b). let $x_0 \in (a,b)$, ~~then~~ for every $x \in (a, x_0)$, we have $f(x) \leq f(x_0)$.

Therefore the set $\{f(x) \mid x \in (a, x_0)\}$ is bounded above and so $\sup\limits_{x} f(x) = \alpha \in R$ for $x \in (a, x_0)$ exist. We will show $\lim\limits_{x\to x_0^-} f(x) = \alpha$. let $\varepsilon > 0$ be given.

Since $\alpha - \varepsilon < \alpha$; $\alpha = \sup f(x)$; $x \in (a, x_0)$

So there exist $x_1 \in (a, x_0)$ ~~with~~ such that

① — $\alpha - \varepsilon < f(x_1)$. let $\delta = x_0 - x_1$



Now if x satisfies $x_0 - \delta < x < x_0$

then $f(x) \geq f(x_1)$ $\quad \because x_1 = x_0 - \delta < x$

i.e $x_1 < x$

and also

$$|f(x) - \alpha| = \alpha - f(x) \qquad \because \alpha \geq f(x) \Rightarrow \alpha - f(x) \geq 0$$

$$\leq \alpha - f(x_1)$$

$$< \varepsilon \quad \text{by ①} \quad \forall x \in (x_0 - \delta, x_0)$$

Therefore

$$\lim_{x\to x_0^-} f(x) = \alpha$$

$$= \sup_{x\in(a,x_0)} f(x)$$

i.e $\lim\limits_{x\to x_0^-} f(x)$ exists.

Again for every $x \in (x_0, b)$, $f(x) \geq f(x_0)$ $\because f$ is increasing

thus the set $\{f(x) : x \in (x_0, b)\}$ is bounded below and so $\inf f(x) = \beta \in R$, $x \in (x_0, b)$ exists.

We will show $\lim\limits_{x \to x_0} f(x) = \beta$. Let $\varepsilon > 0$ be given.

Since $\beta + \varepsilon > \beta$; $\beta = \inf f(x)$, $x \in (x_0, b)$

So there exist $x_2 \in (x_0, b)$ s.t

② — $\beta + \varepsilon > f(x_2)$. Let $\delta = x_2 - x_0$

Now if x satisfies $x_0 < x < x_0 + \delta$

then $f(x) \leq f(x_2)$ $\because$ $x_2 = x_0 + \delta > x$
$x_2 > x$. & f is increasing

and so

$$|f(x) - \beta| = f(x) - \beta$$
$$\leq f(x_2) - \beta$$
$$< \varepsilon \quad \text{by ②}$$

Therefore $\lim\limits_{x \to x_0^+} f(x) = \beta$
$= \inf f(x)$; $x \in (x_0, b)$

Hence $\lim\limits_{x \to x_0^+} f(x)$ exists.

BR

THEOREM: Let $I \subseteq R$ be an interval and let $f : I \to R$ be increasing on I. Suppose $c \in I$ is not an end point of I. Then

(i) $\lim\limits_{x \to c^-} f = \sup\{f(x) : x \in I , x < c\}$

(ii) $\lim\limits_{x \to c^+} f = \inf\{f(x) : x \in I , x > c\}$

Assignment.

Corollary: If f is monotone increasing on (a,b) then for each $x_0 \in (a,b)$

$$\lim_{x\to x_0^-} f(x) = \sup_{x\in(a,x_0)} f(x) \le f(x_0) \le \inf_{x\in(x_0,b)} f(x) = \lim_{x\to x_0^+} f(x)$$

Lemma: If 'f' is monotone increasing on (a,b) and if $a < x_1 < x_2 < b$ then $\lim_{x\to x_1^+} f(x) \le \lim_{x\to x_2^-} f(x)$

Proof let $x_0 \in (x_1, x_2)$

Then $x_0 \in (x_1, b)$ $\because a < x_1 < x_2 < b$

Since $x_0 \in (x_1, b)$ and f is increasing therefore

$$f(x_0) \ge \inf_{x\in(x_1,b)} f(x) \quad \text{--- (1)}$$

Similarly, Since $x_0 \in (a, x_2)$ and 'f' is increasing, therefore

$$f(x_0) \le \sup_{x\in(a,x_2)} f(x) \quad \text{--- (2)}$$

From (1) & (2)

$$\inf_{x\in(x_1,b)} f(x) \le \sup_{x\in(a,x_2)} f(x)$$

Then $\lim_{x\to x_1^+} f(x) \le \lim_{x\to x_2^-} f(x)$

By last theorem. Desired.

TH Let f be monotone on D. If $b, c \in D$ with $b<c$, then fore every a s.t $b<a<c$ the one-sided limits as x tends to a of f exist.

Take $D=[x,y]$, then $x<b<c<y$.

Assignment.

Example: Let $f(x) = 2^{\frac{1}{x-1}}$ Show that $\lim_{x\to 1} f(x)$ does not exist. We will show $\lim_{x\to 1^+} 2^{\frac{1}{x-1}}$ does not exist and $\lim_{x\to 1^-} 2^{\frac{1}{x-1}} = 0$

Given $\varepsilon>0$, choose n_0 so large that $\frac{1}{2^{n_0}}<\varepsilon$.

Take $\delta = \frac{1}{n_0}$. Let $x\in(1-\delta, 1)$.

Then $1-\delta<x<1 \Rightarrow -\delta<x-1<0$

$\Rightarrow \frac{1}{x-1} < -\frac{1}{\delta} < 0$ $\because -\delta<0$ $\therefore -\frac{1}{\delta}<0$

Thus $|2^{\frac{1}{x-1}} - 0| = 2^{\frac{1}{x-1}} < 2^{-\frac{1}{\delta}} = 2^{-n_0} = \frac{1}{2^{n_0}} < \varepsilon$.

Therefore $\lim_{x\to 1^-} 2^{\frac{1}{x-1}} = 0$

Next consider x to the right of 1.

Let $\delta>0$ be arbitrary and choose the natural number N such that $\frac{1}{N}<\delta$. Then if $n>N$

Then $1+\frac{1}{n} \in (1, 1+\delta)$. So

$$2^{\frac{1}{(1+\frac{1}{n})-1}} = 2^n \quad ; \quad n>N$$

Thus $2^{\frac{1}{-1+x}}$ is unbounded in $(1, 1+\delta)$

Therefore $\lim_{x\to 1^+} 2^{\frac{1}{x-1}}$ does not exist.

Consequentely $\lim_{x\to 1} 2^{\frac{1}{x-1}}$ does not exist.

P.T.O

CONTINUOUS FUNCTIONS

Def: Let 'f' be defined on D, we say that f is continuous on D provided that for every $y \in D$ and every positive ε, there is a positive δ such that for all $x \in D$ $0<|x-y|<\delta$ implies $|f(x)-f(y)|<\varepsilon$

Further, if for a particular $a \in D$, we know that for every positive ε, there is a positive δ such that for all $x \in D$, $0<|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon$ we will say that 'f' is continuous at a.

OR

(2P) 'f' is said to be continuous at 'a' if $\lim_{x\to a} f(x) = f(a)$. i.e given any $\varepsilon>0$ there is a $\delta>0$ such that

$|f(x)-f(a)|<\varepsilon$ whenever $|x-a|<\delta$

Explaination

(i) a is in the domain of f; i.e f(a) exists.

(ii) $\lim_{x\to a} f(x)$ exists (i.e $\lim_{x\to a^-} f = \lim_{x\to a^+} f$

(iii) f(a) and $\lim_{x\to a} f(x)$ are equal.

(2P)
REMARK:

① If any of the above three conditions fails, then 'f' is not continuous at $x=a$. we say that f is discontinuous at $x=a$ or that f has a discontinuity at $x=a$

② If condition ② holds but 1 (and hence 3) fails OR if ① and ② hold but ③ fails then 'f' is said to have a removable discontinuity at $x=a$

③ If both one sided limits exists but not equal then it is called jump at $x=a$ and we call this discontinuity as jump discontinuity.

④ If 'f' is unbounded at a or f is oscillate near a, then f is discontinuous.
e.g $\frac{1}{x}$ and $\sin(\frac{1}{x})$

Def The function f has a simple (or a discontinuity of the first kind) at $x=a$ if the discontinuity is either removable or a jump discontinuity. Every other discontinuity is called a discontinuity of the second kind.

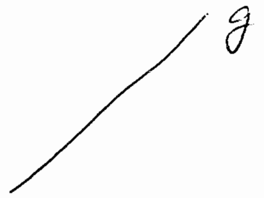
Examples

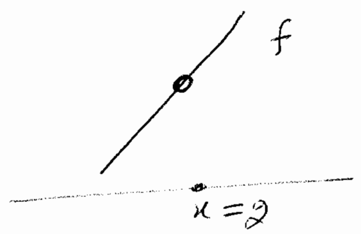
① let $f(x) = \dfrac{x^2-4}{x-2}$; $x \neq 2$.

Clearly $\lim\limits_{x\to 2} f(x) = 4$ But $f(2)$ does not exist. Hence f has a removable discontinuity at $x=2$.

Def $g(x) = \begin{cases} f(x) & ; x \neq 2 \\ 4 & ; x = 2 \end{cases}$

Then $g(x)$ is continuous at $x=2$





② $f(x) = \operatorname{sgn}|x|$; $x \neq 0$ and $f(0)=0$

Then $\lim\limits_{x\to 0} f = 1 \neq f(0)$. Thus f has a removable discontinuity at $x=0$

Define $g(x) = \begin{cases} \operatorname{sgn}|x| & ; x \neq 0 \\ 1 & ; x=0 \end{cases}$

The g is continuous at $x=0$

③ $f(x) = [x]$

Then $\lim_{x\to 1} f(x)$ does not exist because

$\lim_{x\to 1^+} f = 1$ and $\lim_{x\to 1^-} f = 0$

Therefore we have jump discontinuty.

" It is not removable. This function is discontinuous at every integer but is ~~every~~ continuous at every non-integer real number."

④ Let $f(x) = \begin{cases} 1 ; & x \in Q \\ 0 ; & x \in \bar{Q} \end{cases}$

Let a be an arbitrary real number

The 'f' is Oscilleate near a.

Therefore $\lim_{x\to a} f(x)$ does not exist.

Hence 'f' has discontinuity of second kind at every real number.

(ZP) THEOREM: Assume that f is defined in some neighborhood of x_0. Then function f is continuous at x_0 iff $f(x_n) \to f(x_0)$ for every sequence $\{x_n\}$ in the domain of f with $x_n \to x_0$.

(BR) Discontinuity Criterion:

Let $A \subseteq R$, let $f: A \to R$ and $x_0 \in A$ Then f is discontinuous at x_0 iff there exist a sequence $\{x_n\}$ in A such that $\{x_n\}$ converges to x_0 but the sequence $\{f(x_n)\}$ does not converges to $f(x_0)$.

Examples ①

$$f(x) = \begin{cases} 1 & x \in Q \\ 0 & x \notin Q \text{ ; } x \text{ is irrational.} \end{cases}$$

Sol Let x_0 be a rational number.
Then $f(x_0) = 1$. Let x_n be a sequence of irrational numbers that converges to x_0
(By corollary of condensation property of R)
Since $f(x_n) = 0$ for all $n \in \mathbb{N}$. Then $\lim_{x_n \to x_0} f(x_n) = 0$
But $f(x_0) = 1$. Therefore 'f' is not continuous at $x = x_0$. Similarly. Let x' be an irrational number. Then $f(x') = 0$ Let $\{y_n\}$ be a sequence of rational numbers that converges to x'
(By condensation property of R).
Since $f(y_n) = 1$ for all $n \in \mathbb{N}$. So $\lim_{y_n \to x'} f(y_n) = 1$
But $f(x') = 0$. Hence f is discontinuous at $x = x'$

.... 1

.... 0

② Let $f(x) = \sin(\frac{1}{x})$ for $x \neq 0$
Then 'f' does not have limit at $x = 0$
Thus f is discontinuous at $x = 0$
On the other hand
$g(x) = x \sin(\frac{1}{x})$ for $x \neq 0$
is continuous at $x = 0$
(See Bartle)

③ Let $A = \{x \in R : x > 0\}$. For any irrational $x > 0$, Define $h(x) = 0$ and for a rational number in A of the form $\frac{m}{n}$; $(m, n) = 1$; $m, n \in N$
Define $h(\frac{m}{n}) = \frac{1}{n}$. Then h is ~~dis~~continuous at every irrational number in A and is discontinuous at every rational number in A. This function is known as Thomae's function. See Kosmala.

THEOREM Let $A \subseteq R$, let $f, g : A \to R$ and let $b \in R$. Suppose that $c \in A$ and that f and g are continuous at c, then
(a) $f+g$, $f-g$, fg and bf are continuous at c.
(b) If $h : A \to R$ is continuous at $c \in A$ and if ~~h~~ $h(x) \neq 0$ for all $x \in A$, then the quotient f/g is continuous at c.
(c) Let $|f|$ be defined for $x \in A$ by $|f|(x) = |f(x)|$, then $|f|$ is continuous at c.
(d) Let $f(x) \geq 0$ for all $x \in A$ and $\sqrt{f}$ be defined $(\sqrt{f})x = \sqrt{f(x)}$ for all $x \in A$. Then $\sqrt{f}$ is continuous at c.

PROOF: Since f and g are continuous at $x = c$. For given $\varepsilon > 0$, there exist a $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon/2 \quad \text{whenever } |x-c| < \delta$$

and $$|g(x) - g(c)| < \varepsilon/2 \quad \text{whenever } |x-c| < \delta.$$

Then

$$|f(x) + g(x) - (f(c) + g(c))|$$
$$\leq |f(x) - f(c)| + |g(x) - g(c)|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \text{whenever } |x-c| < \delta.$$

Then '$f+g$' is continuous.

Alternative: $\lim_{x\to c} f = f(c)$ & $\lim_{x \to c} g = g(c)$

By def $(f+g)(c) = f(c) + g(c)$
$$= \lim_{x\to c} f + \lim_{x\to c} g$$
$$= \lim_{x\to c} (f+g).$$ Which complete the proof.

(c) Since f is continuous at $x = c$,
For $\varepsilon > 0$, there exist a $\delta > 0$ s.t
$|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$.

We know
$$||a| - |b|| \leq |a - b|$$
Therefore
$$||f|(x) - |f(c)||$$
$$= ||f(x)| - |f(c)||$$
$$\leq |f(x) - f(c)|$$
$$< \varepsilon \quad \text{whenever } |x - c| < \delta.$$
Hence $|f|$ is continuous.

(d) Consider
$$|\sqrt{f(x)} - \sqrt{f(c)}| = \frac{|f(x) - f(c)|}{|f(x) + f(c)|}$$
$$< |f(x) - f(c)| \quad \because f(x) \geq 0 \ \forall x \Rightarrow |f(x) + f(c)| > 0$$
$$< \varepsilon \quad \text{whenever } |x - c| < \delta. \quad x \neq c$$
Hence $\sqrt{f}$ is continuous.

Remark ① Above theorem is true for all $x \in A$.
Since c was arbitrary point of A.
Therefore all parts are continuous on A.
A function is continuous on A if it is continuous at each point of A.

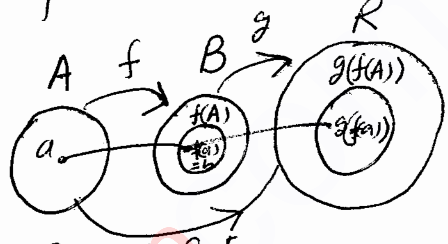
② A polynomial function is continuous on R. See P-147, 148 BAR

Composition of continuous functions

KOS

Statement:

Let $f : A \to R$ and $g : B \to R$ with $A, B \subseteq R$ such that $f(A) \subseteq B$. If f is continuous at some $x = a \in A$ and g is continuous at some $b = f(a) \in B$, then the function $g \circ f$ is continuous at $x = a$



PROOF: We have to show $\lim_{x \to a} g(f(x)) = g(f(a))$ $g \circ f$

For this we need to find a $\delta > 0$ for a given $\varepsilon > 0$ such that

$|g(f(x)) - g(f(a))| < \varepsilon$ provided $|x - a| < \delta$

and x is in the domain of the function $g \circ f$.

Since g is continuous (say) at $t = b$, i.e

$\lim_{t \to b} g(t) = g(b)$. Therefore,

For $\varepsilon > 0$, there exists $\delta_1 > 0$, such that

$$|g(t) - g(b)| < \varepsilon, \text{ if } |t - b| < \delta_1 \quad \text{--- (1)}$$

and t is in the domain of g.

If $t = f(x)$, then (1) gives

$$|g(f(x)) - g(f(a))| < \varepsilon \text{ whenever } |f(x) - f(a)| < \delta_1 \quad \text{--- (2)}$$

$\because \boxed{b = f(a)}$ & $t = b$ $\Rightarrow$ $t = f(a)$ i.e $f(x) = t$

Note that $f(A) \subseteq B$ and that f is continuous at $x = a$, thus for $\delta_1 > 0$ there exists $\delta_2 > 0$ such that

$$|f(x) - f(a)| < \delta_1 \text{ if } |x - a| < \delta_2 \quad \text{--- (3)}$$

Now combining (2) & (3) when $\delta = \delta_2$, we get

$|g(f(x)) - g(f(a))| < \varepsilon$ whenever $|x - a| < \delta$, which was desired.

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Examples:

① let $g(x) = |x| \ \forall x \in R$

Then $||x| - |c|| \leq |x - c|$

or $|g(x) - g(c)| < \varepsilon$ whenever $|x - c| < \delta$

Hence $g(x)$ is continuous $\forall c \in R$. Choose $\delta = \varepsilon$. If $f: A \to R$ is any function that is continuous on A, then by composition of continuous functions

$g \circ f = g(f) = |f|$ is continuous.

Thus $|f|$ is continuous whenever f is continuous on A (Alternative proof)

② let $g(x) = \sqrt{x}$ for $x \geq 0$

Then g is continuous at any number $c \geq 0$ and if let $f: A \to R$ is continuous on A and if $f(x) \geq 0 \ \forall x \in A$. Then by composition of continuous function

$g \circ f = g(f) = \sqrt{f}$ is continuous on A.

③ let $g(x) = \sin x$, then

$|\sin x - \sin c| \leq |x - c|$

Therefore $g(x)$ is continuous for all $x \in R$. If $f: A \to R$ is continuous on A. Then $g \circ f = g(f) = \sin(f(x))$ is continuous on A.

In particular if $f(x) = \frac{1}{x}$; $x \neq 0$ then $g(f) = \sin(\frac{1}{x})$ is continuous at every $c \neq 0$.

PROPERTIES OF CONTINUOUS FUNCTIONS

(ZP) Lemma 1: If 'f' is continuous at x_0 then there exist a $\delta>0$ such that 'f' is bounded on $(x_0-\delta, x_0+\delta)$; that is f is bounded at x_0

Proof: Suppose f is continuous at x_0. Then for $\varepsilon = 1 > 0$, there exist a $\delta > 0$ such that $|f(x) - f(x_0)| < 1$ whenever $|x - x_0| < \delta$

i-e $|f(x)| < 1 + |f(x_0)| = M$ (say)

for each $x \in (x_0-\delta, x_0+\delta)$

or $|f(x)| < M$ for each $x \in (x_0-\delta, x_0+\delta)$

and so f is bounded on $(x_0-\delta, x_0+\delta)$

(ZP) Lemma 2: If 'f' is right-continuous at x_0 then there exist a $\delta>0$ such that 'f' is bounded on $[x_0, x_0+\delta)$; Similarly if f is left continuous at x_0 then there exist a $\delta>0$ such that 'f' is bounded on $(x_0-\delta, x_0]$

(BR ZP) THEOREM: If 'f' is continuous on the closed bounded interval $[a,b]$, Then 'f' is bounded on $[a,b]$.

OR let $I = [a,b]$ be a closed bounded interval and let $f: I \to R$ be continuous on I. Then 'f' is bounded on I.

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PROOF: On contrary, Suppose f is not bounded on I. Then for any $n \in N$, there is a number $x_n \in I$ such that $|f(x_n)| > n$. —① Since I is bounded, the sequence $\{x_n\}$ is bounded. Then By Bolzano-Weierstrass theorem there is a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ that converges to a number x. Since $a \le x_n \le b$ for all n. —② Then $a \le \text{Lt}\, x_{n_r} \le b$ by ②

i-e $x \in I$. Since $x \in I$, so 'f' is continuous at x. Therefore $\{f(x_{n_r})\}$ converges to $f(x)$.

Hence $\{f(x_{n_r})\}$ is bounded because a convergent sequence is bounded

But by ①

$$|f(x_{n_r})| > n_r \geq r \quad \text{for } r \in N$$

which is a contradiction against the boundedness of $\{f(x_{n_r})\}$.

Hence our supposition that f is not bounded on I is wrong.

Consequently 'f' is bounded on I

(BR) Def: Let $A \subseteq R$ and let $f : A \to R$. We say that 'f' has an absolute maximum on A if there is a point $x^* \in A$ such that

$$f(x^*) \geq f(x) \quad \forall\, x \in A.$$

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Also, we say that f has an absolute minimum on A if there is a point $x_* \in A$ such that $f(x_*) \leq f(x) \;\; \forall x \in A$

We say that x^* is an absolute maximum point for f on A and x_* is an absolute minimum point for f on A if they exist.

Let M and m be the absolute maximum and absolute minimum values, then

$$M = f(x^*) = \sup_{x \in A} f(x)$$

$$m = f(x_*) = \inf_{x \in A} f(x)$$

Remark

① $f(x) = x^2$ on $(0,2)$

Then f is continuous and bounded on $(0,2)$. Further $M = 4$, $m = 0$ but there are no points $x_1, x_2 \in (0,2)$ such that $f(x_1) = 4$ and $f(x_2) = 0$

To find $x_1, x_2 \in A$, we have the following theorem.

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(7,i)

Extreme-Value theorem:

Statement: If f is continuous on $[a,b]$ then there exists points $x_1, x_2 \in [a,b]$ such that $f(x_2) \le f(x) \le f(x_1)$ $\forall x \in [a,b]$; that is $f(x_1) = M$ and $f(x_2) = m$.

PROOF: Suppose f is continuous on $[a,b]$; Then 'f' is bounded on $[a,b]$ (If f is cont on closed bdd, then it is bdd) and so $M = \sup\limits_{x\in[a,b]} f(x)$ and

$m = \inf\limits_{x\in[a,b]} f(x)$ existas real numbers.

Suppose that the value M is not assumed (i-e not attained). Then

$$f(x) < M \quad \forall x \in [a,b]$$

Define $g(x) = \dfrac{1}{M - f(x)}$ on $[a,b]$

Clearly $g(x) > 0$ for every $x \in [a,b]$.

Since $g(x) > 0$ for every $x \in [a,b]$

Therfore g is continuous on $[a,b]$.

and therefore g is bounded on $[a,b]$.

Then there is a real number $K > 0$ such that

$g(x) \le K$ for every $x \in [a,b]$.

Now for every $x \in [a,b]$, we have

$$K \ge g(x) = \frac{1}{M - f(x)}$$

$$\Rightarrow \quad M - f(x) \ge \frac{1}{K} > 0$$

$$\Rightarrow \quad f(x) < M - \frac{1}{K} \quad \text{on } [a,b]$$

But $M = \sup_{x \in [a,b]} f(x)$, a contradiction.

Therefore the value M must be attained. Then there must exist some $x_1 \in [a,b]$ s.t $f(x_1) = M$, similary $f(x_2) = m$.

BAR Altornative Statement:

let $I = [a,b]$ be closed bounded interval and let $f: I \to R$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I.

Assignment (Location of Roots Theorem)

BR Statement:

let I be an interval and let $f: I \to R$ be continuous on I. If $\alpha < \beta$ are numbers in I such that $f(\alpha) < 0 < f(\beta)$ (or $f(\alpha) > 0 > f(\beta)$). then there exist a number $c \in (\alpha, \beta)$ such that $f(c) = 0$

G.M

OR Suppose $f \in C[a,b]$, class of all continuous functions on $[a,b]$

(i) If $f(a) < 0$ and $f(b) > 0$ then there is a point c, $a < c < b$ such that $f(c) = 0$

(ii) If $f(a) > 0$ and $f(b) < 0$ then there is a point c, $a < c < b$ such that $f(c) = 0$

Bolzano's Intermediate Value Theorem:

G.M Statement: Suppose $f \in C[a,b]$ and that $f(a) \neq f(b)$. Then for a given number K that lies between $f(a)$ and $f(b)$, there exists a point c, $a<c<b$ with $f(c)=K$.

Proof: Let $f(a) < f(b)$ and $f(a) < K < f(b)$

Define $g(x) = f(x) - K$ —— ①

Then $g(a) < 0$ and $g(b) > 0$

By location of roots theorem, there exists c, $a<c<b$ such that $g(c)=0$

put in ① imply $f(c) = K$.

Let $f(a) > f(b)$ and $f(a) > K > f(b)$

Define $g(x) = f(x) - K$. —— ①

Then $g(a) > 0$ and $g(b) < 0$

Again by location of roots theorem there exists a c, $a<c<b$ such that $g(c) = 0$ put in ①

$$0 = f(c) - K$$

i-e $f(c) = K$.

Thus in either case, we have the theorem.

Remark① Bolzano's gave idea through above theorem. A function is continuous if the entire graph can be traced without lifting the pencil off the paper.

Corollary ① Let $I = [a, b]$ be a closed and bounded interval and let $f: I \to R$ be continuous on I. If $K \in R$ is any number satisfying $\inf f(I) \leq K \leq \sup f(I)$ then there is a point $c\ I$ such that $f(c) = K$.

2D

② If 'f' is a non-constant, continuous function on $[a, b]$ then the range of f is the closed bounded interval $[m, M]$ where $m = \sup\limits_{x \in [a,b]} f(x)$, $m = \inf\limits_{x \in [a,b]} f(x)$

③ Any polynomial of odd degree has atleast one real root. (KOS)

Assignment.

Remark 2 By Bolzano's intermediate value theorem. Any continuous function possesses the intermediate value property. However, there are certain functions that are not continuous and yet satisfy the intermediate value property.

G.M

TH If f is continuous on an interval I, then $f(I)$ is also an interval. If I is closed and bounded, so is $f(I)$

i.e Continuous function maps compact intervals onto compact interval.

Proof: Suppose that y_1 and y_2 are in $f(I)$ and 'f' is continuous on I, so there exist points a, b in I such that

$$y_1 = f(a) \quad \& \quad y_2 = f(b)$$

Take K, $y_1 < K < y_2$. Without any loss of generality, let $a < b$. Then f is continuous on $[a, b]$ becas $[a, b] \subseteq I$ and

f is continuous on I.
Therefore by Bolzano's intermediate value theorem, there exist a point c, $a<c<b$ such that $f(c)=K$ where $y_1<K<y_2$.
Since K was arbitrary, therefore by def of interval $f(I)$ is an interval.
(See page 158, Bartle, Lemma 5.3.9)

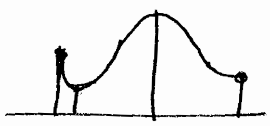
Next, suppose $[a,b]=I$, a closed and bounded interval, and 'f' is continuous. Therefore $f(I)$ is bounded. (Self)
Moreover by extreme value theorem there exist $c, d \in [a,b]$

s.t. $m = \inf f = f(c)$ &
$M = \sup f = f(d)$

Since $f(I)$ is bounded on $[a,b]$
Therefore $[m, M]$ is a closed and bounded interval.

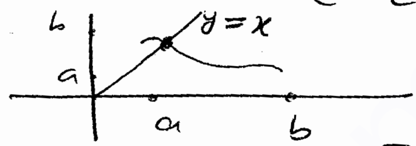
BR Preservation of Intervals theorem:
Let I be an interval and let $f: I \to R$ be continuous on I, Then $f(I)$ is an interval.

Remark If f is continuous on $[a,b]$ then it is not the case always $f(a)=\inf f(x)$ & $f(b)=\sup f(x)$



Brouwer's Fixed point Theorem (1881–1966)

Statement: If a function $f:[a,b]\to[a,b]$, not necessarily a surjection, is continuous then f has atleast one fixed point; i-e there exist atleast one real $x_0\in[a,b]$ such that $f(x_0)=x_0$



Proof: Suppose f is continuous on $[a,b]$ and $f(x)\in[a,b]$, for every $x\in[a,b]$

If $f(a)=a$ or $f(b)=b$ then we are done.

Assume $f(a)>a$ and $f(b)<b$ and

Define $g(x)=f(x)-x$ for every $x\in[a,b]$ — (1)

Clearly $g(x)$ is continuous on $[a,b]$

More over $g(a)>0$ and $g(b)<0$

Hence by Intermediate value theorem there exist an $x_0\in[a,b]$ such that

$g(x_0)=0$ $\because$ 0 is the intermediate value for g on $[a,b]$.

put in (1)

$0=f(x_0)-x_0$

or $f(x_0)=x_0$, which complete the proof.

2p

THEOREM If f is one-one and continuous on $[a,b]$ then f is strictly monotone on $[a,b]$. Assignment.

Continuous Inverse theorem:

Statement: If 'f' is continuous and strictly monotone on an interval I, then f^{-1} is continuous on f(I).

PROOF: We consider the case that f is st. increasing. Other case is similar.

Since f is continuous on I and I is an interval, By preservation of intervals theorem, f(I) = J (say) is an interval (First show J = f(I) is an interval)
Since f is st. increasing on I, ~~then~~ so f is one-one on I as it is continuous on I. Therefore $g: J \to R$, inverse to f exists. We claim that g is st. increasing.
For, if $y_1, y_2 \in J$ with $y_1 < y_2$
Then $y_1 = f(x_1)$ and $y_2 = f(x_2)$ for some $x_1, x_2 \in I$. We must have $x_1 < x_2$ otherwise $x_1 \geq x_2$ implies that $y_1 = f(x_1) \geq f(x_2) = y_2$, contrary to the hypothesis that $y_1 < y_2$.
Therefore we have

$$g(y_1) = x_1 < x_2 = g(y_2)$$

i-e $g(y_1) < g(y_2)$ whenever $y_1 < y_2$
Since y_1, y_2 are arbitrary elements of J with $y_1 < y_2$, we conclude that

g is st. increasing on J.

Now we will show that g is continuous on $J = f(I)$.

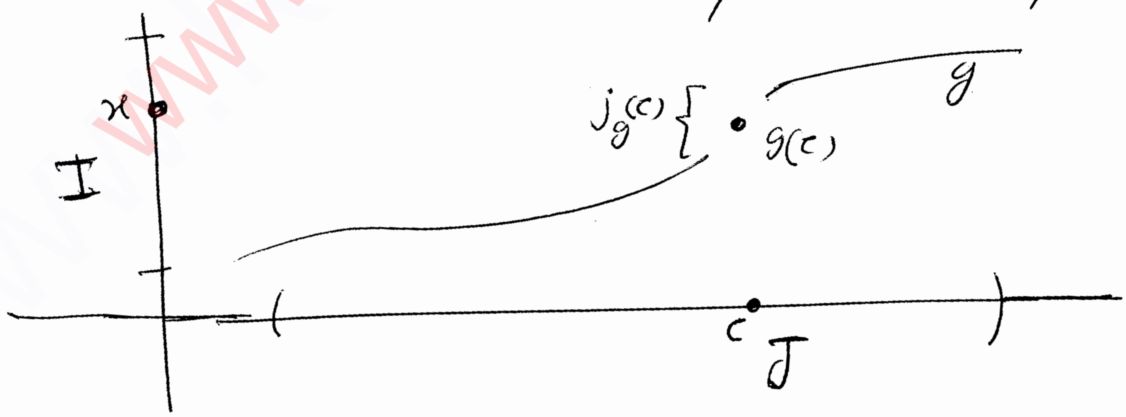
Now, if g is discontinuous at a point $c \in J$, then the jump of g at c is non-zero so that

$$\lim_{y \to c^-} g < \lim_{y \to c^+} g$$

If we choose any number $x \neq g(c)$ satisfying $\lim_{y \to c^-} g < x < \lim_{y \to c^+} g$

then $x \neq g(y)$ for any $y \in J$.

Therefore $x \notin I$, which is a contradiction to the fact that I is not an interval. Therefore we conclude that g is continuous on $J = f(I)$. Which complete the proof.



KHALID MAHMOOD

① Let $f(x)=\begin{cases} x\cos x & , x\neq 0 \\ 0 & , x=0 \end{cases}$

For $\varepsilon>0$, choose $\delta=\varepsilon$,

then

$$|f(x)-f(0)| = |x\cos x| \le |x| < \delta = \varepsilon.$$

Thus

$$|f(x)-f(0)|<\varepsilon \text{ whenever } |x-0|<\delta$$

Therefore $f(x)$ is continuous at $x=0$

② $f(x)=\sqrt{x}$ is continuous on $[0,\infty)$

Let $c\in[0,\infty)$

then For $\varepsilon>0$, choose $\delta=\varepsilon\sqrt{c}$ s.t

$$|f(x)-f(c)| = |\sqrt{x}-\sqrt{c}|$$

$$= \frac{|x-c|}{\sqrt{x}+c} < \frac{|x-c|}{\sqrt{c}} < \frac{\delta}{\sqrt{c}} = \varepsilon.$$

Therefore

$$|f(x)-f(c)|<\varepsilon \text{ whenever } |x-c|<\delta.$$

Since c was arbitrary, therefore $f(x)$ is continuous on $[0,\infty)$.

③ Let $g(x)=\begin{cases} x^n\sin x, & x\neq 0, n\geq 1 \\ 0 & x\neq 0 \end{cases}$

Discuss the continuity of $g(x)$ at $x=0$.

Uniform Continuity:

G Def A real-valued function f is said to be uniformly continuous on a set X of real numbers, if for a given real number $\varepsilon>0$ there exist a positive real number δ (depending upon ε only) such that

$$|f(x)-f(y)|<\varepsilon \text{ whenever } |x-y|<\delta \quad \forall x,y\in X$$

Remark:

① The number $\delta>0$, to be determined, depends upon both ε and x_1 in the case f is continuous on X, and δ is independent of the choice of $x_1\in X$ when f is uniformly continuous on X.

② δ may be different for different points in X if f is continuous on X, and δ is identical for all points in X in the case f is uniformly continuous on X.

Examples: $f(x)=x^2$ is not uniformly continuous on R but is uniformly continuous on $[0,b]$ for $b\in R$.

Sol Let x_0 be a fixed point in R.

For given $\varepsilon>0$,

$$\begin{aligned}|f(x)-f(x_0)| &= |x^2-x_0^2|\\ &= |x+x_0|\,|x-x_0|\\ &\le (|x|+|x_0|)\,|x-x_0|\end{aligned}$$

Take $\delta \le 1$, Then $|x - x_0| < \delta \le 1$

i.e $|x - x_0| < 1$

implies $|x| < 1 + |x_0|$

or $|x| + |x_0| < 1 + 2|x_0|$

Therefore

$$|f(x) - f(x_0)| \le (1 + 2|x_0|)\,|x - x_0|$$

If $\delta \le \dfrac{\varepsilon}{1 + 2|x_0|}$, then

$|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \varepsilon$.

If we choose $\delta > 0$ such that

$$\delta \le \min\left(1, \frac{\varepsilon}{1 + 2|x_0|}\right)$$

Then clearly δ- depends ε and also on x_0.

Now if $x_1 = \frac{1}{\delta}$, $x_2 = \frac{1}{\delta} + \frac{\delta}{2}$

Then $|x_1 - x_2| = \left|\frac{1}{\delta} - \frac{1}{\delta} - \frac{\delta}{2}\right|$

$= \delta/2 < \delta.$

But

$$|f(x_1) - f(x_2)| = 1 + \frac{\delta^2}{4} > 1$$

Thus for $\varepsilon = 1$, there is no $\delta > 0$ which always give $|f(x_1) - f(x_2)| < 1$.

Hence 'f' is not uniformly continuous on R.

For $f(x) = x^2$ on $[0, b]$

For $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{2b}$

Then
$$|f(x_1) - f(x_2)| = |x_1 + x_2|\,|x_1 - x_2|$$
$$\leq |x_1| + |x_2|\,|x_1 - x_2|$$
$$\leq (b+b)\,|x_1 - x_2| = 2b\,|x_1 - x_2|$$
$$< 2b\delta = \varepsilon \text{ whenever } |x_1 - x_2| < \delta$$

Therefore

$|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$.
$\forall\, x_1, x_2 \in [0, b]$

Hence 'f' is uniformly continuous on $[0, b]$.

Note that if $\delta = \frac{\varepsilon}{2b}$, then δ depends only on ε but not on x_1, x_2.

(2) $f(x) = 3x + 2$ on R is uniformly continuous.

For $\varepsilon > 0$, choose $\delta = \varepsilon/3$

Then

$$|f(x_1) - f(x_2)| = 3\,|x_1 - x_2|$$
$$< 3\delta = \varepsilon \quad \text{whenever } |x_1 - x_2| < \delta$$
$$\forall\, x_1, x_2 \in R.$$

Hence 'f' is uniformly continuous.

THEOREM:

If 'f' is uniformly continuous on I, then f is continuous on I. But converse may not be true. Give a counter example.

Proof Since 'f' is uniformly continuous on I For $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ s.t

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever } |x - y| < \delta \quad \forall\, x, y \in I$$

which shows that 'f' is continuous at $x = y$. Since $y \in I$ is an arbitrary point, Therefore 'f' is continuous on I.

Counter Example:

$$f(x) = \frac{1}{x} \; ; \quad x \in (0,1)$$

Note that 'f' is continuous on $R \setminus \{0\}$, Therefore 'f' is continuous on $(0,1)$.

Let $\varepsilon = \frac{1}{2}$ and suppose $\delta > 0$ is arbitrary.

Choose natural number n s.t $\frac{1}{n} < \delta$

Let $x_1 = \frac{1}{n}$ and $x_2 = \frac{1}{n+1}$, Then $x_1, x_2 \in (0,1)$

and $$|x_1 - x_2| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \left|\frac{n+1-n}{n(n+1)}\right|$$

$$= \left|\frac{1}{n(n+1)}\right| < \frac{1}{n} < \delta$$

But $|f(x_1) - f(x_2)| = |n - (n+1)| = 1 > \varepsilon$.

Hence for $\varepsilon = \frac{1}{2}$, There is no $\delta > 0$ (Since δ was arbitray) which satisfy the def of uniform continuity. Thus $\frac{1}{x}$ is not uniformly continuous on $(0,1)$

Example: $f(x) = \frac{1}{x}$ on (a, ∞), $a > 0$ is uniformly continuous.

Sol let $\varepsilon > 0$ be given, choose $\delta = \varepsilon a^2$

if $x_1, x_2 \in (a, \infty)$ and $|x_1 - x_2| < \delta$

we have

$$|f(x_1) - f(x_2)| = \left| \frac{x_1 - x_2}{x_1 x_2} \right|$$

$$\leq \frac{1}{a^2} |x_1 - x_2| < \frac{\delta}{a^2} = \varepsilon.$$

Hence $f(x)$ is uniformly continuous on (a, ∞)

GPS

Example: let $f(x) = \sqrt{x}$, $x \in [0, \infty)$

Then f is uniformly continuous.

Sol let $\varepsilon > 0$ be given, choose $\delta = \varepsilon^2$

let $x, y \in [0, \infty)$ s.t $x \neq y$

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}|$$

$$= \frac{|x - y|}{\sqrt{x} + \sqrt{y}}$$

$$\leq \frac{|x-y|}{\cancel{\quad}} \quad ; \quad \cancel{M = \max_{x \in [0,\infty)} f(x)}$$

$$< \delta = \varepsilon \qquad \because x \neq y \ \& \ x, y \in [0, \infty)$$

Hence

$|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$.

Therefore 'f' is uniformly continuous on $[0, \infty)$.

Example: $f(x) = x^n$; $n \geq 0$ is uniformly continuous on the closed interval $[0,1]$

Sol Let $x_1, x_2 \in [0,1]$, we have

$$|f(x_1) - f(x_2)| = |x_1^n - x_2^n|$$
$$= |x_1^{n-1} + n\, x_1^{n-2} x_2 + \cdots + x_2^{n-1}|\,|x_1 - x_2|$$
$$< n\,|x_1 - x_2|$$

For given $\varepsilon > 0$, choose $\delta = \varepsilon/n$ independent of x_1 and x_2 such that

$$|f(x_1) - f(x_2)| < n|x_1 - x_2|$$
$$< n\delta = \varepsilon$$

Thus

$$|f(x_1) - f(x_2)| < \varepsilon \text{ whenever } |x_1 - x_2| < \delta.$$

Hence f is uniformly continuous on $[0,1]$.

G.M

TH ** Give Def of negation first.

If f is continuous on a closed and bounded interval $[a,b]$, then f is uniformly continuous.

KOS

OR (Uniform continuity theorem)

If $D \subset R$ is a closed and bounded set, i.e compact. and a function $f: D \to R$ is continuous, then f is uniformly continuous.

~~Proof~~: Def: Negation of the def of uniform continuity:

A function f is not uniformly continuous on a domain D if there exist an $\varepsilon_0 > 0$ such that for every $\delta > 0$, there exist $x, y \in D$ s.t

$$|x - y| < \delta \text{ and } |f(x) - f(y)| \geq \varepsilon_0$$

(ii) There exist an $\varepsilon_0 > 0$ and two
sequences $\{x_n\}$ and $\{y_n\}$ in D such that
$\lim (x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0 \ \forall n \in \mathbb{N}$.

PROOF: let $I = [a,b]$, If f is not uniformly continuous on I then there exist $\varepsilon_0 > 0$ and two sequences $\{x_n\}$ and $\{y_n\}$ in I such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$ $\forall n \in \mathbb{N}$.

Since I is bounded, the sequence $\{x_n\}$ is bounded and By Bolzano-Weierstrass theorem there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to z. Since I is closed therefore $z \in I$. Moreover

$$|y_{n_k} - z| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - z|$$
$$< \frac{1}{n} + \varepsilon = \varepsilon'$$

Therefore $\{y_{n_k}\}$ also converges to z.

Now ~~Since~~ if 'f' is continuous at z, then both of the sequences $\{f(x_{n_k})\}$ and $\{f(y_{n_k})\}$ must converges to $f(z)$. But this is not possible since

$$|f(x_n) - f(y_n)| \geq \varepsilon_0 \quad \forall n \in \mathbb{N}.$$

Thus the hypothesis that f is not uniformly continuous on the closed bounded interval I implies that f is not continuous at some point $z \in I$, a contradiction. Hence f is uniformly continuous on I.

Example: $f(x) = \sin(\frac{1}{x})$, $x \neq 0$ is not uniformly continuous on the closed interval $[0,1]$

Sol Suppose f is uniformly continuous on $[0,1]$. Then for $\varepsilon = 1$, there is $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < 1 \text{ when } |x_1 - x_2| < \delta \quad \text{—①}$$

Take $x_1 = \frac{1}{(n-\frac{1}{2})\pi}$, $x_2 = \frac{1}{3} x_1$, $n \geq 1$

then $x_1, x_2 \in [0,1]$,

Therefore $|x_1 - x_2| = \left| \frac{1}{(n-\frac{1}{2})\pi} - \frac{1}{3(n-\frac{1}{2})\pi} \right|$

$$= \frac{2}{3(n-\frac{1}{2})\pi} < \delta.$$

But

$$|f(x_1) - f(x_2)| = \left| \sin((n-\tfrac{1}{2})\pi) - \sin(3(n-\tfrac{1}{2})\pi) \right|$$

$$= 2 > \varepsilon$$

Which is a contradiction.

Hence 'f' is not uniformly continuous.

Example ① $f(x) = \sin(\frac{1}{x})$ on $(0,1)$ is not uniformly continuous.

Q's

② $f(x) = \frac{1}{x}$ on $[a,1]$, $a > 0$ is uniformly continuous.

Hint: Use uniform continuity theorem.

Assignment

① Let f be defined on $D \subseteq R$. Then f is uniformly continuous on D iff for every pair of sequences $\{a_n\}$ and $\{b_n\}$ of elements of D, $\lim(a_n - b_n) = 0$ imply $\lim(f(a_n) - f(b_n)) = 0$

KOS ② If a function $f : [a, \infty) \to R$ is continuous with $\lim\limits_{x\to\infty} f(x)$ finite, then f is uniformly continuous on $[a, \infty)$.

Lipschitz Functions: (Rudolf Otto Lipschitz (1832–1903)) German

KOS **Def** A function $f : D \to R$ with $D \subseteq R$ is Lipschitzian iff there exists $K > 0$, called Lipschitz constant, such that

$$|f(x) - f(y)| \le K|x - y| \quad \forall\, x, y \in D \quad \text{— ①}$$

Often, a Lipschitzian function is referred to as a Lipschitz function or a function that satisfies Lipschitz condition. Furthermore, if $K \in (0,1)$ then 'f' is a contraction, sometime called a contractive function.

The condition that a function $f : D \to R$ on D is a Lipschitz function can be interpreted geometrically as follows.

If we write inequality in ① as

$$\left| \frac{f(x) - f(y)}{x - y} \right| \le K \,; \quad x \ne y$$

Where $\frac{f(x) - f(y)}{x - y}$ represent the slope of line segment joining the points $(x, f(x))$ & $(y, f(y))$.

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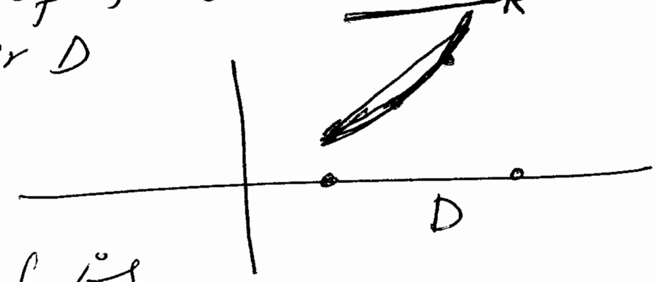
Thus a function f satisfies a Lipschitz condition iff slopes of all lines joining two points on the graph of f over D is K bounded by some number D



BR

THEOREM: If $f: A \to R$ is a Lipshitz function, then f is uniformly continuous.

Proof: Suppose f is Lipschitzian function

Then $|f(x) - f(t)| \leq K|x - t| \quad \forall\ x, t \in A$; $(x \neq t)$

Then for $\varepsilon > 0$ given, there exist $\delta = \frac{\varepsilon}{K}$

s.t
$$|f(x) - f(t)| \leq K\delta = K \cdot \varepsilon/K \text{ whenever } |x - t| < \delta.$$

Therefore
$$|f(x) - f(t)| < \varepsilon \text{ whenever } |x - t| < \delta.$$

Hence 'f' is uniformly continuous.

BR

Example: Converse of above theorem may not be true. i.e Not every Uniformly continuous function is a Lipschitzian function.

let $g(x) = \sqrt{x}$; $x \in [0, 2]$

Since g is continuous on $[0, 2]$ and $[0, 2]$ is closed and bounded therefore by uniform continuity theorem 'g' is uniformly continuous on $[0, 2]$.

On contrary suppose, there exist $K > 0$

s.t $|g(x)| \leq K|x| \Rightarrow |\sqrt{x}| \leq K|x|$
$$\Rightarrow \frac{1}{x} \leq K$$

which is absurd for all $x \in [0, 2]$

Thus there exist no $K>0$ s.t 26

$$|g(x)| \le K|x| \quad \text{or in particular}$$

$$|g(x)-g(0)| \le K|x-0|$$

Hence 'g' is not Lipschitzian.

② $f(x)=x^2$ on $A=[0,b]$ where b is positive constant.

Then $|f(x)-f(t)| = |x+t|\,|x-t|$

$$\le 2b\,|x-t| \quad \forall\ x,t\in[0,b]$$

Thus 'f' is Lipschitzian with $K=2b$ on $[0,b]$, and therefore it is uniformly continuous on A.

③ $f(x)=\sqrt{x}$ ~~ans~~ on $[a,\infty)$; $a>0$

then 'f' is uniformly continuous on $[a,\infty)$

$$|f(x)-f(t)| = \frac{|x-t|}{\sqrt{x}+\sqrt{t}} \le \frac{1}{2a}|x-t|$$

Take $M=\frac{1}{2a}$ $\quad\forall\ x,t\in[a,\infty)$

which shows that f is Lipschitzian on $[a,\infty)$

Therefore 'f' is uniformly continuous.

KQs ④ Let I be an interval and $f: I\to\mathbb{R}$. If $S=\{f(x)-f(t)/x-t\ ;\ x,t\in I,\ x\ne t\}$ is bounded, prove that f is uniformly continuous on I. Assignment.

THEOREM: If $f: A \to R$ is uniformly continuous on a Subset A of R and if $\{x_n\}$ is a Cauchy sequence in A, then $\{f(x_n)\}$ is a cauch sequence in R.

Proof: Since f is uniformly continuous on A, For $\varepsilon > 0$, given, we choose $\delta > 0$ such that

$$|f(x) - f(t)| < \varepsilon \text{ whenever } |x - t| < \delta \quad \forall x, t \in A. \quad \text{—— (1)}$$

let x_n be a Cauchy sequence in A.

then for $\delta > 0$, there exist a natural number N (depending on δ) such that

$|x_n - x_m| < \delta \quad \forall n, m > N$. Then by the choice of δ and using (1), we have

$$|f(x_n) - f(x_m)| < \varepsilon \quad \forall n, m > N.$$

Therefore $\{f(x_n)\}$ is a cauchy sequence in R.

Example: let $f(x) = \frac{1}{x}$, Then 'f' is not uniformly continuous on $(0,1)$.

let $x_n = \frac{1}{n}$ in $(0,1)$ is a Cauchy sequence but $\{f(x_n)\} = \{n\}$ is not a cauch sequence.

Hence 'f' is not uniformly continuous on $(0,1)$.

- Continuous Extension Theorem:

Statement: A function f is uniformly continuous on the interval (a,b) iff it can be defined at the end points a and b such that the extended function is continuous on $[a,b]$.

PROOF: Suppose extended function is continuous on $[a,b]$, then obviously it is uniformly continuous on $[a,b]$. Further, since f is uniformly continuous on $[a,b]$, therefore it is uniformly continuous on (a,b).

Conversely, suppose f is uniformly continuous on (a,b). We shall show $\lim_{x\to a} f(x) = l$ and we will define $f(a)=l$, then 'f' will be continuous on a. Similarly it will be for b. Let $\{x_n\}$ be a sequence in (a,b) with $\lim_{n\to\infty} x_n = a$, then $\{x_n\}$ is a cauchy sequence and 'f' is uniformly continuous on (a,b), therefore $\{f(x_n)\}$ is a cauchy sequence in R. Thus by Cauchy Criterion $\{f(x_n)\}$ is convergent. That is $\lim_{n\to\infty} f(x_n) = l$ exists. Let $\{t_n\}$ be any other sequence in (a,b) such that $\lim_{n\to\infty} t_n = a$

Then $\lim_{n\to\infty}(x_n - t_n) = a - a = 0$ and 'f' is uniformly continuous on (a,b), therefore

$$\lim_{n\to\infty}\left(f(x_n) - f(t_n)\right) = 0$$

Then

$$\lim_{n\to\infty} f(t_n) = \lim_{n\to\infty}\left(f(t_n) - f(x_n)\right) + \lim_{n\to\infty} f(x_n)$$
$$= 0 + l.$$

Since $\{t_n\}$ was arbitrary such that $t_n \to a$ and $\{f(t_n)\} \to \ell$. Hence by sequential criterion for limits, we infer $\lim_{x \to a} f = \ell$. If we define $f(a) = \ell$. Then 'f' is continuous at $x = a$, Similarly 'f' is continuous at $x = b$. Hence 'f' is continuous on $[a,b]$ as it is already continuous on (a,b) being uniformly continuous on (a,b).

Remark: Let $f(x) = \sin\left(\frac{1}{x}\right)$

Since $\lim_{x \to 0} f(x)$ does not exist

Hence by continuous extension theorem 'f' is not uniformly continuous on $(0,b]$ for any $b > 0$. But $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$

Hence $f(x) = x \sin\left(\frac{1}{x}\right)$ is uniformly continuous on $(0,b]$ $\forall$ ~~all~~ $b > 0$.

Observe

$$\left| x \sin\frac{1}{x} - t \sin\left(\frac{1}{t}\right) \right|$$

$$\leq M\,|x - t| \quad ; \quad M = \max_{x \in A} \sin\left(\tfrac{1}{x}\right)$$

$$< M\delta = M \cdot \frac{\varepsilon}{M} \text{ whenever } |x - t| < \delta$$

choose $\delta = \varepsilon/M$

$|f(x) - f(t)| < \varepsilon$ whenever $|x - t| < \delta$.

G.M Assignment:

Let $f: [a,b] \to [a,b]$ be a contraction mapping, then there is a unique point c in $[a,b]$ for which $f(c) = c$.

BR **Step Function:**

Let $I \subseteq R$ be an interval and let $f: I \to R$. Then f is called a step function if it has only a finite number of distinct values, each value being assumed on one or more intervals in I.

GIas let $f(x) = K_i \geq 0$ if $x \in [x_{i-1}, x_i)$;
$i = 1, 2, \dots n$ and $f(b) = K_n$

e.g $f: [-2,4] \to R$ defined by

$$f(x) = \begin{cases} 0 & ; \ -2 \leq x < -1 \\ 1 & ; \ -1 \leq x \leq 0 \\ \frac{1}{2} & ; \ 0 < x < \frac{1}{2} \\ 3 & ; \ \frac{1}{2} \leq x < 1 \\ -2 & ; \ 1 \leq x \leq 3 \\ 2 & ; \ 3 < x \leq 4 \end{cases}$$

BR
THEOREM: Let I be a closed bounded interval and let $f: I \to R$ be continuous on I. If $\varepsilon > 0$, then there exist a step function $S_\varepsilon(x): I \to R$ such that $|f(x) - S_\varepsilon(x)| < \varepsilon \ \forall x \in I$

PROOF: Since I is closed and bounded and f is continuous on I, By uniform continuity theorem 'f' is uniformly continuous on I. So for $\varepsilon > 0$ there is a number $\delta > 0$ such that if $x, y \in I$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Let $I = [a,b]$ and $m \in \mathbb{N}$ be sufficiently large so that $h = \frac{b-a}{m} < \delta$. Divide $[a,b]$ into m disjoint interval of lengh h; namely

$$I_1 = [a, a+h],\quad I_2 = (a+h, a+2h], \ldots$$

$$I_K = (a+(K-1)h,\ a+Kh] \text{ for } K = 2, \ldots, m.$$

Since $\ell(I_K) = h < \delta \quad \forall\ K = 1,2,3,\ldots,m$

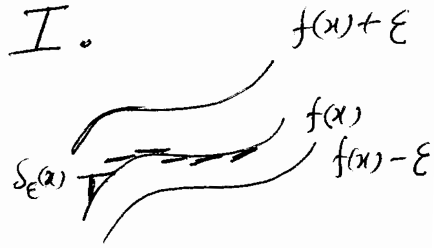
Therefore the difference between any two values of ~~of~~ 'f' in I_K is less than ε. (Since f is uniformly continuous)

Define $S_\varepsilon(x) = f(a+Kh) \quad \forall\ x \in I_K$
$K = 1,2,\ldots,m$

So that $S_\varepsilon(x)$ is constant on each sub-interval I_K. Now if $x \in I_K$

then $|f(x) - S_\varepsilon(x)| = |f(x) - f(a+Kh)|$
$< \varepsilon.$

Thus $|f(x) - S_\varepsilon(x)| < \varepsilon \quad \forall\ x \in I.$



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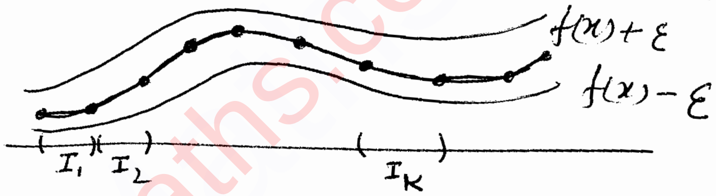
Corollary: Let $I = [a,b]$ be a closed bounded interval and let $f: I \to R$ be continuous on I. If $\varepsilon > 0$ there exists a natural number m such that if we divide I into m disjoint intervals I_K having length $h = \frac{b-a}{m}$, then the step function S_ε; $S_\varepsilon(x) = f(a+Kh)$ for $x \in I_K$; $K = 1,2,\ldots,m$ satisfies $|f(x) - S_\varepsilon(x)| < \varepsilon \quad \forall\ x \in I.$

Piecewise linear function:

Let $I = [a,b]$ be an interval. Then a function $g : I \to R$ is said to be piecewise linear on I if I is the union of a finite number of disjoint intervals $I_1, I_2, \ldots, I_m$ such that the restriction of g to each interval I_K is a linear function.

Remark: A piecewise linear function g is continuous on I if the line segments that form the graph of g ~~to each~~ meet at end points of adjacent subintervals I_K, I_{K+1}; $K = 1, \ldots, m$.



THEOREM:

Let I be a closed bounded interval and let $f : I \to R$ be continuous on I. If $\varepsilon > 0$, then there exist a continuous piecewise linear function $g_\varepsilon : I \to R$ such that $|f(x) - g_\varepsilon(x)| < \varepsilon \ \forall x \in I$.

PROOF: Obviously f is uniformly continuous on I since it is continuous on I and I is a closed and bounded interval. Then for $\varepsilon > 0$, there is a number $\delta(\varepsilon) > 0$ such that if $x, y \in I$ and $|x - y| < \delta(\varepsilon)$, then $|f(x) - f(y)| < \varepsilon$. —— (1)

Let $m \in N$ be sufficiently large so that $h = \frac{b-a}{m} < \delta(\varepsilon)$. Divide $I = [a,b]$ into m disjoint intervals of length h, namely $I_1 = [a, a+h]$, $I_K = (a + (K-1)h, a+Kh]$ for $K = 2, 3, \ldots, m$. On each interval I_K, we define g to be the linear function joining $(a + \overline{K-1}\,h, f(a + \overline{K-1}\,h))$ and $(a+Kh, f(a+Kh))$. Then g is continuous piecewise l. function. Since for each $x, y \in I_K$, the values differ by less than ε by (1), therefore
$|f(x) - g_\varepsilon(x)| < \varepsilon \ \forall x \in I_K$ and so hold for all $x \in I$.

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BR

DIFFERENTIATION

Def: Let $I \subseteq R$ be an interval. Let $f: I \to R$, $c \in I$. We say that a real number l is the derivative of f at c if for any given number $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that for any $x \in I$ with $0 < |x-c| < \delta(\varepsilon)$, then $\left|\frac{f(x)-f(c)}{x-c} - l\right| < \varepsilon$.

In this case, we say that f is differentiable at c and we write $l = f'(c)$.

In other words, the derivative of f at c is given by the limit

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

If $x = c+h$

Then $f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$.

Example: Let $f(x) = x|x| \quad \forall x \in R$

Then $f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$

If $x_0 > 0$, then

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h \to 0} \frac{(x_0+h)^2 - x_0^2}{h}$$

$$= \lim_{h \to 0} \frac{2x_0 h + h^2}{h} = 2x_0$$

If $x_0 < 0$, $f'(x_0) = -2x_0$

Finally

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h|h|}{h} = 0$$

BR we conclude $f'(x) = 2|x|$ for every real x.

THEOREM:

If $f: I \to R$ has a derivative at $c \in I$ then f is continuous at c.

PROOF: For all $x \in I$, $x \neq c$, we have

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \cdot x - c$$

Since $f'(c)$ exists, and $\lim_{x\to c}(x-c) = 0$

therefore

$$\lim_{x\to c} [f(x) - f(c)] = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x\to c} (x - c)$$

$$= f'(c) \cdot 0$$

$$= 0$$

Therefore $\lim_{x\to c} f(x) = f(c)$, so that f is continuous at c.

Counter example: Converse of above theorem may not be true. $f(x) = |x|$, $x = 0$

Corollary:

If f is discontinuous at $c \in D_f$ then $f'(c)$ does not exist.

Use contrapositive.

Right and Left Derivatives:

Let $f: X \to Y$; $X, Y \subseteq R$

(a) f is said to have a right derivative at c if

(i) For some $\delta > 0$, f is defined on $(c, c+\delta)$ in X and

(ii) $\lim\limits_{h \to 0^+} \dfrac{f(c+h) - f(c)}{h}$ exists.

If this limit exists, it is denoted by f'_+ or $D_+ f(c)$

(b) f is said to have left derivative at c if

(i) For some $\delta > 0$, f is defined on $(c-\delta, c)$ in X and

(ii) $\lim\limits_{h \to 0^-} \dfrac{f(c) - f(c-h)}{h}$ exists.

If this limit exists, it is denoted by f'_- or $D_- f(c)$

Remark: f is differentiable at c iff

(i) f is continuous at c

(ii) $D_+ f(c)$ and $D_- f(c)$ exist and $D_+ f(c) = D_- f(c)$. In this case $D_+ f(c) = D_- f(c) = D f(c)$.

Examples:

① $f(x) = \begin{cases} x^2 ; & x > 1 \\ x^3 ; & x \leq 1 \end{cases}$

f is continuous at $x=1$ But $D_+ f \neq D_- f$.

② $f(x) = \begin{cases} x^2 ; & x > 0 \\ x^2+1 ; & x < 0 \end{cases}$

$D_+ f = D_- f$ But Df ~~does~~ not exist bcs f is not continuous at $x=0$

BR

Assignment:

Let $I \subseteq R$ be an interval and let $c \in I$. Let $f: I \to R$ and $g: I \to R$ be functions which are differentiable at c. Then

(a) If $\alpha \in R$, then αf is diff at c and

$$(\alpha f)'(c) = \alpha f'(c)$$

(b) $f \pm g$ is differentiable at c, and

$$(f \pm g)'(c) = f'(c) \pm g'(c)$$

(c) The function fg is differentiable at c and $(fg)'(c) = f'(c)\, g(c) + f(c)\, g'(c)$

(d) If $g(c) \neq 0$, then the function $\frac{f}{g}$ is differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)\, f'(c) - f(c)\, g'(c)}{[g'(c)]^2}$$

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The Chain Rule:

Statement: Suppose f is differentiable at the point c in the domain of f and that g is differentiable at $t = f(c)$, then the composite function $\psi = g \circ f$ is differentiable at c and $\psi'(c) = g'(f(c))\, f'(c)$

PROOF: Define φ on the domain of g by

$$\varphi(y) = \begin{cases} \dfrac{g(y) - g(t)}{y - t} - g'(t), & y \neq t \\ 0, & y = t. \end{cases}$$

Since $g'(t)$ exists, so $\lim\limits_{y \to t} \varphi(y)$

$$= g'(t) - g'(t)$$
$$= 0$$
$$= \varphi(t)$$

It follows that φ is continuous at t. Furthermore, since f is differentiable and so continuous at c, therefore $\varphi \circ f$ is continuous at c. So and

$$\lim_{x \to c} (\varphi \circ f)(x) = \lim_{x \to c} \varphi(f(x))$$

$$= \varphi(f(c)) = \varphi(t) = 0$$

Now, if x is in the neighborhood of c, consider

$$\frac{\varphi(x)-\varphi(c)}{x-c} = \frac{g(f(x))-g(f(c))}{f(x)-f(c)} \cdot \frac{f(x)-f(c)}{x-c}$$

$$= \{\varphi(f(x)) + g'(t)\} \frac{f(x)-f(c)}{x-c}$$

Letting $x \to c$, it follows that

$$\varphi'(c) = g'(t) \cdot \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

$$= g'(t) \cdot f'(c)$$

$$= g'(f(c)) f'(c).$$

or $(g\circ f)'(x) = g'(f(c)) f'(c).$

Remark let $y = f(x)$ and $u = g(x)$

Then chain rule can be stated as

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}.$$

Example ① $y = (x+1)^2$

② $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

$$\left|\frac{f(x)-f(0)}{x-0} - 0\right| = |x \sin(\tfrac{1}{x})| \leq |x| \; ; x \neq 0$$

$< \delta = \varepsilon$ where $|x-0| < \delta$.

Thus $f'(0) = 0$

③ Discuss the differentiability of

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & ; \; x \neq 0 \\ 0 & ; \; x = 0 \end{cases}$$

$$f'(x) = \sin\left(\frac{1}{x}\right) - x \cos\left(\frac{1}{x}\right) \cdot \left(\frac{-1}{x^2}\right)$$

$$f'(x) = \sin\left(\frac{1}{x}\right) + \frac{1}{x} \cos\left(\frac{1}{x}\right)$$

thus f' exist for all values of $x \neq 0$.

But $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$

$$= \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

$$= a \in [-1, 1]$$

Thus $f'(0)$ does not exist.

On the other hand.

$$\left| \frac{f(x) - f(0)}{x - 0} \right| = \left| \sin \frac{1}{x} \right|$$

$$\underline{\leq} \; \underline{1} \not< \varepsilon \quad \text{where } \varepsilon = \frac{1}{N}.$$

Hence f is not diff at $x = 0$.

Inverse Function:

The following theorem relate the derivative of a function with the derivative of its inverse when inverse exists.

G.o.M **THEOREM:**
Let $f: I \to Y$; $I, Y \subseteq R$. Let f be one-one and continuous on an interval I. Let $c \in I$ and $d \in Y$ such that $c = f^{-1}(d)$ — (1). Suppose f is differentiable at c such that $f'(c) \neq 0$ then f^{-1} is differentiable at d and

$$(f^{-1})'(d) = \frac{1}{f'(c)}.$$

PROOF: Since f is one to one and continuous on I, it is st. monotonic. Therefore f^{-1} exists and is continuous.

We know that

$$(f\circ f^{-1})(d) = f(f^{-1}(d)) = f(c) \quad \text{by (1)}$$

If $(f\circ f^{-1})'(d)$ exists, then by Chain Rule

$$(f\circ f^{-1})'(d) = f'(f^{-1}(d))\,(f^{-1})'(d) = f'(c)\,(f^{-1})'(d) \quad \text{--- (2)}$$

Since $(f\circ f^{-1})(y) = y$ for all $y \in f(I)$ therefore $(f\circ f^{-1})'$ exists on $f(I)$ and

$(f\circ f^{-1})'(d) = 1$ put in (2), we get

$$(f^{-1})'(d) = 1/f'(c) \quad \text{Required.}$$

PREZENTATIONS:

① Mean Value theorem
② Rolle's theorem
③ Generalized M.V.T

and Geometrical interpretations.

RU THEOREM: Suppose f is differentiable in (a,b)

(a) If $f'(x) \geq 0 \ \forall \ x \in (a,b)$, then f is monotonically increasing

(b) If $f'(x) = 0 \ \forall \ x \in (a,b)$, then f is constant

(c) If $f'(x) \leq 0 \ \forall x \in (a,b)$, then f is monotonically decreasing.

PROOF: Let $x_1, x_2 \in (a,b)$ such that $x_2 > x_1$

Then

$f(x_2) - f(x_1) = (x_2 - x_1) f'(x)$ which is ② valid for each x_1, x_2 in (a,b)

If $x \in (a,b)$, i.e $a < x < b$,

then $f'(x) \geq 0$. and ② gives

$f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$
$\forall x_1, x_2 \in (a,b)$

Hence f is monotonically increasing.

(c) Same. (b) If $f'(x) = 0$, then $f(x_2) = f(x_1) \ \forall x_1, x_2 \in (a,b)$
Hence f is constant.

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THEOREM: Suppose $f: G \to Y$; $G, Y \subseteq \mathbb{R}$. G is an open interval containing c and f has a local extremum at c. If f is differentiable at c, then $f'(c) = 0$

PROOF: Suppose that f has local minimum at c. Then there exists $\delta > 0$ such that

$$f(c+h) \geq f(c) \text{ for all } h,\ 0 < |h| < \delta.$$

Since $\frac{1}{h}\{f(c+h) - f(c)\} \geq 0$ for all $h > 0$

it follows that

$$f'_+(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \geq 0 \quad \text{— (1)}$$

Also $\frac{1}{h}\{-f(c-h) + f(c)\} \leq 0$ for all $h < 0$

it follows that

$$f'_-(c) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} \leq 0 \quad \text{— (2)}$$

Since $f'(c)$ exist, therefore

$$f'(c) = f'_+(c) = f'_-(c)$$

Hence (1) & (2) imply $f'(c) = 0$.

2nd case (self)

Interior Extremum Theorem: Let c be an interior point of the interval I at which $f: I \to \mathbb{R}$ has a relative extremum. If the derivative of f at c exists, then $f'(c) = 0$.

Corollary: Let $f: I \to \mathbb{R}$ be continuous on an interval I and suppose that f has a relative extremum at an interior point c of I. Then either $f'(c)$ does not exist or it is equal to zero.

Lemma 6.2.11 page 203 Self

(BR)

Darboux's Theorem:

If f is differentiable on $I = [a,b]$ and if k is a number between $f'(a)$ and $f'(b)$, then there exist atleast one point $c \in (a,b)$ such that $f'(c) = k$.

Proof: Suppose $f'(a) < k < f'(b)$. Define g on I by $g(x) = kx - f(x)$. Since f is diff on I therefore it is continuous on I, and so $g(x)$ is continuous on I, ~~therefore~~ it attains a maximum value on I. Since $g'(a) = k - f'(a) > 0$, it follows that the maximum of g does not occur at $x = a$.

Similarly, since $g'(b) = k - f(b) < 0$, it follows ~~from lemma~~ that ~~the~~ the maximum does not occur at $x = b$. therefore g attains its maximum at some c in (a,b). therefore

$$g'(c) = 0 \quad \text{or} \quad f'(c) = k.$$

Assignment: $f'(a) > k > f'(b)$.

Ex ① Let f be defined for all real x, and suppose that $|f(x) - f(y)| \le (x-y)^2$ for all x and y. Prove that f is constant.

Sol

$$\left|\frac{f(x) - f(y)}{x-y}\right| \le |x-y|$$

$$\lim_{y \to x} \left|\frac{f(x) - f(y)}{x-y}\right| \le 0$$

or $|f'(x)| \le 0$ But $|f'(x)| \ge 0$

imply $f'(x) = 0 \quad \forall x$.

therefore $f(x) = \text{constant}$.

(2) If $c_0 + \frac{c_1}{2} + \cdots + \frac{c_{n-1}}{n} + \frac{c_n}{n+1} = 0$ where $c_0, c_1, \ldots, c_n \in \mathbb{R}$ are all constants, prove that the equation $c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + c_n x^n = 0$ has atleast one real root between 0 and 1.

Sol Let $f(x) = c_0 x + c_1 \frac{x^2}{2!} + \cdots + c_{n-1}\frac{x^n}{n} + c_n \frac{x^{n+1}}{n+1}$

is continuous ~~and~~ ~~diff~~ on $[0,1]$ and diff on $(0,1)$. Further $f(0)=0$, $f(1)=0$ (given)

By Rolle's theorem, there exist atleast one $c \in (0,1)$ s.t $f'(c) = 0$

i.e $c_0 + c_1 c + \cdots + c_{n-1}c^{n-1} + c_n c^n = 0$

which shows that c is a root of $c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + c_n x^n = 0$.

(1) L. Hospithal Rule Assignment.
(2) Taylor's Theorem
(3) Machlaurin's Series.

Convex Functions:

Def: Let $I \subseteq R$ be an interval. A function $f: I \to R$ is said to be convex on I if for any t satisfies $0 \le t \le 1$ and any points x_1, x_2 in I, we have

$$f((1-t)x_1 + t x_2) \le (1-t) f(x_1) + t f(x_2)$$

(BR)
THEOREM: Let I be an open interval and suppose that $f: I \to R$ has a 2nd derivative on I. Then f is convex function on I if and only if $f''(x) \geq 0 \; \forall x \in I$

PROOF: Since $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$

therefore $f''(a) = \lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$

for each $a \in I$

Let h be such that $a+h$ and $a-h$ belong to I.
Then $a = \frac{1}{2}\{(a+h)+(a-h)\}$ and since f is convex on I
we have $f(a) \leq \frac{1}{2} f(a+h) + \frac{1}{2} f(a-h)$ $\quad \because 1-\frac{1}{2} = \frac{1}{2}$.

Therefore $f(a+h) - 2f(a) + f(a-h) \geq 0$. Since $h^2 > 0$ for all $h \neq 0$

Then $\lim\limits_{h \to 0} f(a+h) - 2f(a) + f(a-h) / h^2 \geq 0$

i.e $f''(a) \geq 0$ for any $a \in I$.

Conversely let x_1, x_2 be two points in I, let $0 < t < 1$
let $x_0 = (1-t)x_1 + t x_2$.
Applying Taylor's theorem to f at x_0 we obtain a point c_1 between x_0 and x_1 such that

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + \frac{1}{2} f''(c_1)(x_1 - x_0)^2$$

and a point c_2 between x_0 and x_2 such that

$$f(x_2) = f(x_0) + f'(x_0)(x_2 - x_0) + \frac{1}{2} f''(c_2)(x_2 - x_0)^2$$

If f'' is non-negative on I, then the term

$$R = \frac{1}{2}(1-t) f''(c_1)(x_1 - x_0)^2 + \frac{1}{2} t f''(c_2)(x_2 - x_0)^2$$

is also non-negative. Thus we obtain

$$(1-t)f(x_1) + t f(x_2)$$

$$= (1-t)\left[f(x_0) + f'(x_0)(x_1 - x_0) + \frac{1}{2} f''(c_1)(x_1 - x_0)^2\right]$$
$$+ t\left[f(x_0) + f'(x_0)(x_2 - x_0) + \frac{1}{2} f''(c_2)(x_2 - x_0)^2\right]$$

$$= f(x_0) + f'(x_0)\{(1-t)(x_1 - x_0) + t(x_2 - x_0)\}$$
$$+ \frac{1}{2}(1-t) f''(c_1)(x_1 - x_0)^2 + \frac{1}{2} t f''(c_2)(x_2 - x_0)^2$$

$$= f(x_0) + f'(x_0)\{\underbrace{(1-t)x_1 + t x_2} - x_0 + t x_0 - t x_0\} + R$$

$$= f(x_0) + f'(x_0)\{x_0 - x_0\} + R$$

$$= f(x_0) + R$$

$$\geq f(x_0) = f((1-t)x_1 + t x_2)$$

Hence, we say that f is convex function on I.

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Differentiability in R^n

(G.M) Let $f: V \to R$ where $V \subset R^n$, is a neighborhood of $a \in R^n$. Then the directional derivative $D_\beta(f)$ at a in the direction $\beta \in R^n$ is defined by the limit, if it exists,

$$D_\beta(f) = \lim_{h \to 0} \frac{f(a+h\beta) - f(a)}{h} \quad \beta \text{ is unit}$$

In particular if $\beta = (b_1, b_2)$; $b_1^2 + b_2^2 = 1$

at $\underline{a} = (x_0, y_0)$, Then

$$D_\beta(f) = \lim_{h \to 0} \frac{f(x_0 + hb_1, y_0 + hb_2) - f(x_0, y_0)}{h}$$

(ZP) Example: Find the directional derivative of $f(x,y) = x^2 - xy + 3$ at $(1,1)$ in the direction of $\beta = (\frac{3}{5}, -\frac{4}{5})$

Sol $D_\beta f(1,1) = \lim_{h \to 0} \frac{f(1+\frac{3}{5}h, 1-\frac{4}{5}h) - f(1,1)}{h}$

$$= \lim_{h \to 0} \frac{(1+\frac{3}{5}h)^2 - (1+\frac{3}{5}h)(1-\frac{4}{5}h) + 3 - 3}{h}$$

$$= \lim_{h \to 0} (1 - \frac{3}{25}h) = 7/5$$

G.M

Definition: The directional derivative of $f(x_1, x_2, \ldots, x_i, \ldots, x_n)$ at $\underline{a} = (a_1, a_2, \ldots, a_i, \ldots, a_n)$ in the direction of unit vector $(0, 0, \ldots, 1, 0, 0, \ldots, 0)$ is called the partial derivative of f at $\underline{a}$

with respect to ith component x_i and is denoted by $D_i f(a)$ or $\frac{\partial f(a)}{\partial x_i}$ or $f_{x_i}(a)$

where

$$\frac{\partial f(a)}{\partial x_i} = \lim_{h\to 0} \frac{f(a_1, a_2, \ldots, a_i+h, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{h}$$

In particular if $\beta = (1, 0)$, $\underline{a} = (x_0, y_0)$

Then $f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$

Similarly

$$f_y(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

Example

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Find $D_\beta f(0,0)$ for arbitrary β. (AND are the partial derivatives)

Let $\beta = (b_1, b_2)$

Then $D_\beta \{f(0,0)\}$

$$D_\beta f(0,0) = \lim_{h\to 0} \frac{b_1 b_2}{b_1^2+b_2^2} \cdot \frac{1}{h}$$

exists only when $\beta = (1,0)$ or $\beta = (0,1)$

Remark Above function is not continuous at $(0,0)$. For this

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \quad \begin{cases} \text{if } x \to 0, y \neq 0 \text{ or} \\ \text{if } y \to 0, x \neq 0 \end{cases}$$

But if $y = x$ $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{2}$. Thus limit does not exist.

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Example: let $f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$

Show that f has directional derivative at $(0,0)$ in any direction $\beta=(b_1,b_2)$ but f is discontinuous at $(0,0)$.

Sol: $D_\beta f(0,0)=\lim\limits_{h\to0}\dfrac{f(hb_1,hb_2)-f(0,0)}{h}$

$$=\begin{cases}\dfrac{b_1^2}{b_2} & \text{if } b_2\neq0\\ 0 & \text{if } b_2=0\end{cases}$$

For discontinuity (along the path)

$\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$ when $y=0$

and $\lim\limits_{(x,y)\to(0,0)} f(x,y)=\dfrac{1}{2}$ along the path $y=x^2$

Hence $\lim\limits_{(x,y)\to(0,0)} f(x)$ does not exist. Thus f is discontinuous at $(0,0)$.

GPM

Assignment: ① let $f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^4+y^4} & ; x^4+y^4\neq0\\ 0 & ;(x,y)=0\end{cases}$

Show that directional derivative of f at $(0,0)$ exist and are the partial derivatives f_x and f_y.

② Show by counter examply that

$$\lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right)\neq\lim_{(x,y)\to(a,b)} f(x,y)$$

③ Exercise T. Firmy

Criterion for integrability (some called Cauchy):

Statement:

$f \in R(\alpha)$ on $[a,b]$ iff for every $\varepsilon > 0$ there exists a partition P such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon.$$

PROOF: Suppose that for given $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \quad \text{--- (1)}$$

Now,

For every partition P, we have

$$L(P,f,\alpha) \leq \underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha \leq U(P,f,\alpha)$$

which implies

$$0 \leq \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \varepsilon. \text{ using (1)}$$

Since above inequality is satisfied for every $\varepsilon > 0$, we have

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$

that is $f \in R(\alpha)$.

Conversely, Suppose $f \in R(\alpha)$ and let $\varepsilon > 0$ be given. Then there exist partitions P_1 and P_2 such that

$$\left.\begin{aligned} U(P_2,f,\alpha) - \int f\,d\alpha &< \varepsilon/2 \\ \text{and } \int f\,d\alpha - L(P_1,f,\alpha) &< \varepsilon/2 \end{aligned}\right\} \text{ --- (2)}$$

Choose $P = P_1 \cup P_2$, i-e P is the common refinement of P_1 and P_2.

Then

$$U(P, f, \alpha) \le U(P_2, f, \alpha)$$
$$< \int f\, d\alpha + \varepsilon/2$$
$$< L(P_1, f, \alpha) + \varepsilon/2 + \varepsilon/2$$
$$\le L(P, f, \alpha) + \varepsilon.$$

Therefore $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

THEOREM:

(a) If $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ —— ① holds for some P and some ε, then ① holds (with the same ε) for every refinement of P.

(b) If ① holds for some partition P and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then $\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon$.

(c) If $f \in R(\alpha)$ and the hypothesis of (b) hold, then

$$\left| \sum_{i=1}^{n} f(t_i) \Delta\alpha_i - \int_a^b f\, d\alpha \right| < \varepsilon.$$

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PROOF:

(a) Suppose $U(P,f,\alpha)-L(P,f,\alpha)$ holds for ① some P and some ε.

Let P^* be a refinement of P, then

$$U(P^*,f,\alpha)\le U(P,f,\alpha) \quad \text{—(i)}$$

$$L(P,f,\alpha)\le L(P^*,f,\alpha) \quad \text{—(ii)}$$

$$\Rightarrow\ -L(P^*,f,\alpha)\le -L(P,f,\alpha) \quad \text{—(iii)}$$

(i) and (iii) implies that

$$U(P^*,f,\alpha)-L(P^*,f,\alpha)<\varepsilon.$$

Since P^* was arbitrary, therefore ① holds for every refinement of P.

(b) Let m_i and M_i be the bounds (sups and inf) of f in $[x_{i-1},x_i]$ and s_i, t_i be two arbitrary points in $[x_{i-1},x_i]$

then $|f(s_i)-f(t_i)|\le M_i-m_i$

imply $\displaystyle\sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i \le \sum_{i=1}^{n}(M_i-m_i)\Delta\alpha_i$

$= U(P,f,\alpha)-L(P,f,\alpha)$

$<\varepsilon$. Required.

According to (c)

THEOREM: Let $f\in B(a,b)$ and α be increasing on $[a,b]$. Then $f\in RS_\alpha[a,b]$ iff for a real number $\varepsilon>0$, there exist a real $\delta>0$ (depending on ε) such that

$$\left|\sum_{K=1}^{n} f(t_K)\Delta\alpha_K-\int_a^b f\right|<\varepsilon \quad \text{for every partition}$$

P with $\|P\|<\delta$ and for every choice of point $t_K\in[x_{K-1},x_K]$.

PROOF: Suppose $f \in RS_\alpha[a,b]$. Let m_K and M_K be the bounds of f in $[x_{K-1}, x_K]$.

Now for every partition P of $[a,b]$ and for every choice of a point $t_K \in [x_{K-1}, x_K]$, we have

$$m_K < f(t_K) < M_K$$

imply $$L(P,f,\alpha) < \sum_{K=1}^{n} f(t_K)\Delta\alpha_K < U(P,f,\alpha) \quad \text{—} ①$$

Since $f \in RS_\alpha[a,b]$, there exist a partition P of $[a,b]$ s.t $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$. — ②

Furthermore $$\int_a^b f\,d\alpha = \inf U(P,f,\alpha)$$

Then $$\int_a^b f + \varepsilon > U(P,f,\alpha) \quad \text{—} ③$$ by the prop of infimum.

Similarly $$\int_a^b f - \varepsilon < L(P,f,\alpha) \quad \text{—} ④$$ by the prop of sup.

①, ②, ③ and ④, we get

$$\int_a^b f - \varepsilon < L(P,f,\alpha) < \sum_{K=1}^{n} f(t_K)\Delta\alpha_K < U(P,f,\alpha) < \int_a^b f + \varepsilon.$$

$$\Rightarrow \quad -\varepsilon < \sum_{K=1}^{n} f(t_K)\Delta\alpha_K - \int_a^b f < \varepsilon.$$

Therefore $$\left| \sum_{K=1}^{n} f(t_K)\Delta\alpha_K - \int_a^b f \right| < \varepsilon.$$

Conversely, let $\varepsilon > 0$, then there exist $\delta > 0$ such that for every partition P with $\|P\| < \delta$ and for all choice of t_K and t_K' in $[x_{K-1}, x_K]$

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We have

$$\left| \sum_{K=1}^{n} f(t_K)\Delta\alpha_K - \int_a^b f \right| < \varepsilon/4 \quad \text{and}$$

$$\left| \sum_{K=1}^{n} f(t_K')\Delta\alpha_K - \int_a^b f \right| < \varepsilon/4.$$

Since $M_K - m_K = \sup\{f(t) - f(t') \;;\; t, t' \in [x_{K-1}, x_K]\}$

Then for $\varepsilon > 0$, we can find points t_K, t_K' in $[x_{K-1}, x_K]$ such that

$$(M_K - m_K) - \frac{\varepsilon}{2(\alpha(b)-\alpha(a))} < f(t_K) - f(t_K') \quad \text{prop of sup.}$$

or $M_K - m_K < f(t_K) - f(t_K') + \varepsilon/2(\alpha(b)-\alpha(a))$

Consider

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{K=1}^{n} (M_K - m_K)\Delta\alpha_K$$

$$< \sum_{K=1}^{n} \left[f(t_K) - f(t_K') + \frac{\varepsilon}{2(\alpha(b)-\alpha(a))} \right] \Delta\alpha_K.$$

$$= \sum_{K=1}^{n} f(t_K)\Delta\alpha_K - \int_a^b f + \int_a^b f - \sum_{K=1}^{\infty} f(t_K')\Delta\alpha_K + \frac{\varepsilon}{2}.$$

$$\leq \left| \sum_{K=1}^{n} f(t_K)\Delta\alpha_K - \int_a^b f \right| + \left| \int_a^b f - \sum_{K=1}^{\infty} f(t_K')\Delta\alpha_K \right| + \varepsilon/2$$

$$< \varepsilon/4 + \varepsilon/4 + \varepsilon/2 = \varepsilon.$$

Hence $f \in RS_\alpha[a,b]$.

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THEOREM: If f is continuous on $[a,b]$ then $f \in R(\alpha)$ on $[a,b]$.

Proof: Let $\varepsilon > 0$ be given. Choose $\eta > 0$ so that $[\alpha(b) - \alpha(a)]\eta < \varepsilon$.

Since f is continuous on $[a,b]$ (a closed and bounded interval), so it is uniformly continuous on $[a,b]$. So for $\eta > 0$, there exists $\delta > 0$ (depending on η) such that

① — $|f(x) - f(t)| < \eta$ whenever $|x-t| < \delta$ $\forall x, y \in [a,b]$.

Now, if P is a partition of $[a,b]$ such that $\Delta x_i < \delta$ for all i, i.e the length of each interval less than δ which implies that $M_i - m_i < \eta$ by ①

Where M_i and m_i are the bounds of f in $[x_{i-1}, x_i]$, and therefore

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$< \sum_{i=1}^{n} \eta \Delta\alpha_i = \eta(\alpha(b) - \alpha(a))$$

$$< \varepsilon.$$

Hence $f \in R(\alpha)$.

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G.M

Example: (Assignment)

If f is continuous on $[a,b]$ and α is defined by

$$\alpha(x) = \begin{cases} \lambda, & a \le x < c, \\ \mu, & c \le x < b \end{cases}$$

where $\lambda < \mu$ and $a < c < b$.

Then $f \in RS_\alpha[a,b]$ and

$$\int_a^b f \, d\alpha(x) = f(c)(\mu - \lambda).$$

THEOREM:

If f is monotonic on $[a,b]$ and if α is continuous on $[a,b]$, then $f \in R(\alpha)$.

PROOF: Let $\varepsilon > 0$ be given. For any positive integer n, choose a partition such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n} \; ; \; i = 1, 2, \ldots, n$$

This is possible since α is continuous. (I.M.V.T)

Next, we suppose that α is monotonically increasing. Then $M_i = f(x_i)$, $m_i = f(x_{i-1})$ $\forall i = 1, 2, \ldots n$.

So that

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \cdot [f(b) - f(a)] < \varepsilon$$ if n is taken large enough ...

RU

Sol

Assignment:

TH Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity on $[a,b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathcal{R}(\alpha)$.

THEOREM Suppose $f \in \mathcal{R}(\alpha)$ on $[a,b]$, $m \leq f \leq M$, Φ is continuous on $[m,M]$, and $h(x) = \Phi(f(x))$ on $[a,b]$. Then $h \in \mathcal{R}(\alpha)$ on $[a,b]$.

PROOF: choose $\varepsilon > 0$. Since Φ is continuous on $[m,M]$, so it is uniformly continuous on $[m,M]$ (a closed and bounded interval), there exist $\delta > 0$ such that $\delta < \varepsilon$ and

① — $|\Phi(s) - \Phi(t)| < \varepsilon$ if $|s-t| \leq \delta$ $\forall\ s,t \in [m,M]$

Since $f \in \mathcal{R}(\alpha)$, there ~~exist~~ is a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \delta^2 \quad \text{—} ②$$

Let M_i, m_i and M_i^*, m_i^* be the bounds of f and h respectively in $[x_{i-1}, x_i]$.

Divide the numbers $1, 2, \dots, n$ into two classes

$i \in A$ if $M_i - m_i < \delta$ —③, $i \in B$ if $M_i - m_i \geq \delta$ —④

For $i \in A$, our chose of δ shows that

$$M_i^* - m_i^* < \varepsilon \quad \text{—} ⑤$$

For $i \in B$, $M_i^* - m_i^* \leq 2K$ where $K = \sup|\Phi(t)|$, $t \in [m,M]$ — ⑥

$\because$ ($|a-b| \leq |a| + |b|$)

From (2)

$$\delta^2 > U(P,f,\alpha) - L(P,f,\alpha)$$

$$> \sum_{i\in B}(M_i - m_i)\Delta\alpha_i \quad \because i\in A \text{ is neglected.}$$

$$\geq \sum_{i\in B}\delta\,\Delta\alpha_i$$

imply $\sum_{i\in B}\Delta\alpha_i < \delta$ ——— (7) $\because \delta>0$

Therefore

$$U(P,h,\alpha) - L(P,h,\alpha) = \sum_{i\in A}(M_i^* - m_i^*)\Delta\alpha_i + \sum(M_i^* - m_i^*)\Delta\alpha_i$$

$$< \sum_{i\in A}\varepsilon\Delta\alpha_i + \sum_{i\in B}2K\,\Delta\alpha_i$$

$$< \varepsilon[\alpha(b)-\alpha(a)] + 2K\delta$$

$$< \varepsilon\,[\alpha(b)-\alpha(a)] + 2K\varepsilon$$

$$= \varepsilon\,[\alpha(b) - \alpha(a) + 2K]$$

Since ε was arbitrary, therefore

$h \in R(\alpha)$.

Khalil
2/4/2004

IMPROPER INTEGRALS:

IMPORTANCE:

The concept of Reimann integral requires that the range of integration is finite (closed and bounded) and the integrand remains bounded in the domain. Now we extend the definition of the integral to a bounded function on an infinite interval and to an unbounded function on a bounded interval. That is if either (or both) of the assumptions defined above is not satisfied, e.g

(i) If the integrand f becomes infinite in the interval $a \le x \le b$, i.e f has points of infinite discontinuity (singular points) OR

(ii) The limits of integration a or b (or both) becomes infinite.

Def Then the symbol $\int_a^b f\,dx$ is called an improper (or infinite or generalized) integral.

Example $\int_1^\infty \frac{dx}{x^2}$, $\int_{-\infty}^\infty \frac{dx}{1+x^2}$, $\int_0^1 \frac{dx}{x(1-x)}$, $\int_{-1}^\infty \frac{dx}{x}$

all are improper integrals.

Def The integral which are not improper are called proper integrals.

e.g $\int_1^2 x^2\,dx$.

Def ① Suppose that the function f is bounded in $[a, d]$ for all $d > a$. If

$\lim_{d\to\infty} \int_a^d f(x)\,dx$ exists, then we say that the infinite integral

$\int_a^\infty f(x)\,dx$ converges and $\int_a^\infty f(x)\,dx = \lim_{d\to\infty} \int_a^d f\,dx$.

If the limit does not exist then the integral is said to be divergent.

② Suppose that the function f is bounded on $[-\mu, b]$ for $b > -\mu$.

If $\lim_{\mu\to\infty} \int_{-\mu}^b f(x)\,dx$ exists then we say that

$\int_{-\infty}^b f(x)\,dx$ converges and $\int_{-\infty}^b f(x)\,dx = \lim_{\mu\to\infty} \int_{-\mu}^b f\,dx$.

If the limit does not exist then the integral is said to be divergent.

③ Suppose f is bounded on the set of all real numbers ~~except at x=b~~.

We define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx, \quad -\infty < b < \infty.$$

Remark For any value c, $a<c<b$

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ and

$\int_c^b f(x)dx$ is a proper integral. Then the two integrals $\int_a^b f(x)dx$ & $\int_a^c f(x)dx$ converges and diverges together. Thus while testing $\int_a^b f(x)dx$ for convergence at a, it may be replaced $\int_a^c f$ for any convenient c such that $a<c<b$.

Definition $\int_a^b f(x)dx$ diverges if it is unbounded.

e.g $\int_0^\infty x\,dx$ is unbounded, so it is divergent.

Examples: ① $\int_1^\infty (1+x)^{-\frac{3}{2}}dx$ is convergent.

$$\int_1^\infty (1+x)^{-3/2}dx = \lim_{d\to\infty} \int_1^d (1+x)^{-3/2}dx$$
$$= \sqrt{2}.$$

② $\int_{-\infty}^\infty \frac{dx}{1+x^2}dx$ is convergent.

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \lim_{d\to\infty}\left[\int_{-d}^0 \frac{dx}{1+x^2} + \int_0^d \frac{dx}{1+x^2}\right]$$
$$= 2\pi$$

③ $\int_0^1 \frac{dx}{x^2}$, $\int_0^1 \frac{dx}{x^2} = \lim_{d\to 0^+} \int_d^1 \frac{dx}{x^2}$, $0<d<1$

$= \infty$, diverges

④ $\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{\mu\to 1^-} \int_0^{\mu} \frac{dx}{\sqrt{1-x}}$, $0<\mu<1$

$= 2$, converges.

⑤ $\int_0^2 \frac{dx}{2x-x^2} = \int_0^1 \frac{dx}{2x-x^2} + \int_1^2 \frac{dx}{2x-x^2}$

$= \lim_{d\to 0^+} \int_d^1 \frac{dx}{2x-x^2} + \lim_{\mu\to 2^-} \int_1^{\mu} \frac{dx}{2x-x^2}$; $0<d<1$, $1<\mu<2$.

$= \lim_{d\to 0^+} \ln\left(\frac{2-d}{d}\right) + \lim_{\mu\to 2^-} \ln\left(\frac{2-\mu}{\mu}\right)$

$= \left(\frac{1}{2}+\frac{1}{2}\right)\ln 0 \quad = \infty$. diverges.

<u>REMARK</u> By using the definition of limit, we can establish

" $\int_a^{\infty} f$ exists iff for every real $\varepsilon>0$, there is a num K, however large, such that

$$\left| \int_a^{\infty} f(x)dx - \int_a^{d} f(x)dx \right| < \varepsilon \quad \forall\ d > K \text{"}.$$

That is $\int_a^{\infty} f(x)\,dx = \lim_{d\to\infty} \int_a^{d} f(x)\,dx$ ∵ $d>K$ & K is large.

G.M

THEOREM: Cauchy General principle of convergence.

~~PROOF~~: Suppose f is bounded on $[a, d]$ for all d. Then $\int_a^\infty f$ exists iff for a real number $\varepsilon > 0$, there exists a real number $K > a$ however large, such that

$$\left| \int_{d'}^{d''} f(x)\,dx \right| < \varepsilon \quad \text{for all } d'' > d' > K.$$

PROOF: Suppose $\int_a^\infty f$ exists. Then by definition of limit. For every real number $\varepsilon > 0$ there is a real K such that

$$\left| \int_a^\infty f(x)\,dx - \int_a^d f(x)\,dx \right| < \varepsilon/2 \quad \forall\, d > K.$$

So that

$$\left| \int_{d'}^{d''} f(x)\,dx \right| = \left| \left(\int_a^\infty f(x)\,dx - \int_a^{d'} f(x)\,dx \right) - \left(\int_a^\infty f(x)\,dx - \int_a^{d''} f(x)\,dx \right) \right|$$

$$\le \left| \int_a^\infty f(x)\,dx - \int_a^{d'} f(x)\,dx \right| + \left| \int_a^\infty f(x)\,dx - \int_a^{d''} f(x)\,dx \right|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \text{for } d'' > d' > K.$$

Therefore

$$\left| \int_{d'}^{d''} f(x)\,dx \right| < \varepsilon \quad \text{for } d'' > d' > K.$$

Conversely, suppose that for every real $\varepsilon > 0$ there exists a real $K > 0$ (however larg) s.t

$$\left| \int_{\alpha'}^{\alpha''} f(x)\,dx \right| < \varepsilon/2 \text{ for all } \alpha'' > \alpha' > K \quad \text{—— (1)}$$

Let $S_n = \int_a^{\alpha+n} f(x)\,dx$. Then

$$|S_n - S_m| = \left| \int_a^{\alpha+n} f(x)\,dx - \int_a^{\alpha+m} f(x)\,dx \right|$$

$$= \left| \int_{\alpha+m}^{\alpha+n} f(x)\,dx \right|$$

$$< \varepsilon/2 \text{ for all } n > m > K.$$

Hence by def $\{S_n\}$ is a Cauchy sequence of real number, so it is convergent. Let $\lim_{n \to \infty} S_n = \ell$.

Then for $\varepsilon > 0$, there exist a K_1 such that

$$\left| \int_a^{\alpha+n} f(x)\,dx - \ell \right| = |S_n - \ell| < \varepsilon/2$$

for all $n > K_1$ —— (2)

By (1) & (2), there exist a number K s.t

$$\left| \int_a^{\alpha} f(x)\,dx - \ell \right| = \left| \left(\int_a^{\alpha+n} f\,dx - \ell \right) - \int_{\alpha+n}^{\alpha} f(x)\,dx \right|$$

$$\leq \left| \int_a^{\mu+n} f(x)\,dx - \ell \right| + \left| \int_{\mu+n}^{d} f(x)\,dx \right|$$

$< \epsilon/_2 + \epsilon/_2 = \epsilon$ whenever $d > \mu+n > K$.

Hence $\lim_{d\to\infty} \int_a^d f(x)\,dx$ exists.

Example $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent.

Solution consider $d' < d < d''$, then

$$\int_{d'}^{d''} \frac{\sin x}{x}\,dx = \int_{d'}^{d} \frac{\sin x}{x}\,dx + \int_{d}^{d''} \frac{\sin x}{x}\,dx \quad \text{——} (1)$$

Then there exists number c_1 & c_2 such that

$0 < d' < c_1 < c_2 < d''$

$$\int_{d'}^{d} \frac{\sin x}{x} = \frac{1}{c_1} \int_{d'}^{d} \sin x \qquad \because \int_a^b fg\,dx = f(c) \int_a^b g\,dx.$$

$$\& \int_{d}^{d''} \frac{\sin x}{x} = \frac{1}{c_2} \int_{d}^{d''} \sin x \; ; \; f(x) = \frac{1}{x}.$$

Then by (1), we have

$$\left| \int_{d'}^{d''} \frac{\sin x}{x}\,dx \right| = \left| \frac{1}{c_1} \int_{d'}^{d} \sin x\,dx + \frac{1}{c_2} \int_{d'}^{d} \sin x\,dx \right|$$

$$\le \left|\frac{1}{c_1}\int_{d'}^{d} \sin x\,dx\right| + \left|\frac{1}{c_2}\int_{d}^{d''} \sin x\,dx\right|$$

$$= \frac{1}{c_1}\left|\cos d - \cos d'\right| + \frac{1}{c_2}\left|\cos d'' - \cos d\right|$$

$$= \frac{1}{c_1}\,2\left|\sin\left(\frac{d+d'}{2}\right)\sin\left(\frac{d'-d}{2}\right)\right| + \frac{1}{c_2}\,2\left|\sin\left(\frac{d''+d}{2}\right)\cdot\sin\left(\frac{d-d''}{2}\right)\right|$$

$$\le \frac{1}{c_2}2 + \frac{1}{c_2}2 \qquad \because |\sin x| \le 1$$

$$= 2\left(\frac{1}{d'} + \frac{1}{d'}\right) \qquad \because d' < c_1 \ \& \ d' < c_2$$

$$\Rightarrow \frac{1}{d'} > \frac{1}{c_1} \ \& \ \frac{1}{d'} > \frac{1}{c_2}$$

$$= \frac{4}{d'}$$

$$< \frac{4}{K} \quad ; \ d' > K \ \& \ K \text{ is large.} \quad \text{—— (3)}$$

For any real number $\varepsilon > 0$, choose $K = \frac{4}{\varepsilon}$.

Then (3) gives

$$\left|\int_{d'}^{d''} \frac{\sin x}{x}\,dx\right| < \varepsilon \quad \text{for } d'' > d' > K = \frac{4}{\varepsilon}.$$

Hence by Cauchy General principle of convergence

$$\int_0^\infty \frac{\sin x}{x}\,dx \text{ is convergent.}$$

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THEOREM: A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)\,dx$ at a, where f is positive in $[a,b]$, is that there exists a positive M, independent of λ such that $\int_{a+\lambda}^b f(x)\,dx < M$, $0<\lambda<b-a$

$(a<a+\lambda<b)$

$a\to a^+$ if $\lambda\to 0^+$

Proof We know that the improper integral $\int_a^b f(x)\,dx$ converges at 'a' if for $0<\lambda<b-a$, $\int_{a+\lambda}^b f(x)\,dx$ tends to a finite limit as $\lambda\to 0^+$.

Since f is positive in $[a,b]$, the positive function of λ, $\int_{a+\lambda}^b f(x)\,dx$ is mononotonically increasing as λ decreases ($\int_{1^+}^4 2x\,dx > \int_{2^+}^4 2x\,dx$) and therefore tends to a finite limit iff it is bounded above, i.e there exist a positive number M independent of λ such that

$$\int_{a+\lambda}^b f(x)\,dx < M, \quad 0<\lambda<b-a.$$

Which complete the proof.

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NOTE: If no such number M exists, the monotonic increasing function $\int_{a+\lambda}^{b} f(x)\,dx$ is not bounded above, and therefore tends to $+\infty$ as $\lambda \to 0^+$ and so the improper integral diverges to $+\infty$.

COMPARISON TEST

If f and g be two positive functions such that $f(x) \le g(x)$, for all x in $[a,b]$, then

(i) $\int_a^b f(x)\,dx$ converges, if $\int_a^b g(x)\,dx$ converges

and

(ii) $\int_a^b g(x)\,dx$ diverges, if $\int_a^b f(x)\,dx$ diverges.

PROOF: let f and g be both bounded and integrable in $[a+\lambda, b]$, $0 < \lambda < b-a$ and a is the only point of infinite discontinuity in $[a,b]$. Since f and g are positive

and $f(x) \le g(x) \quad \forall\, x \in [a,b]$.

Therefore
$$\int_{a+\lambda}^{b} f(x)\,dx \le \int_{a+\lambda}^{b} g(x)\,dx. \quad \text{——} (1)$$

(i) let $\int_a^b g\,dx$ be convergent, so there exists a positive number M such that

$$\int_{a+\lambda}^{b} g\,dx < M, \quad 0 < \lambda < b-a$$

Thus from (1) $\int_{a+\lambda}^{b} f\,dx < M$ for $0 < \lambda < b-a$

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Hence $\int_a^b f\,dx$ is convergent at a.

(ii) Again, if $\int_a^b f\,dx$ is divergent at a, then the positive function $\int_{a+\lambda}^b f\,dx$ is not bounded above and therefore by ① $\int_{a+\lambda}^b g\,dx$ is also not bounded above.

Hence $\int_a^b g\,dx$ is divergent at a.

COMPARISON TEST (LIMIT FORM):

If f and g be two positive functions in $[a,b]$ such that $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \ell$, where ℓ is a non-zero finite number, then the two integrals

$$\int_a^b f\,dx \text{ and } \int_a^b g\,dx \text{ converges}$$

and diverge together at a.

PROOF: Evidently $\ell > 0$.

Let ε be a positive number such that $\ell - \varepsilon > 0$. Since $\lim\limits_{x\to a+}\dfrac{f}{g} = \ell$, therefore there exists a neighborhood $]a,c[$, $a<c<b$ such that for all $x \in]a,c[$

$$\left|\frac{f(x)}{g(x)} - \ell\right| < \varepsilon \quad \text{or}$$

$$(\ell-\varepsilon)\,g(x) < f(x) < (\ell+\varepsilon)\,g(x) \quad \forall x \in]a,c[.$$

Next apply comparison. Assignment.

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REMARK: (i) If $\frac{f}{g} \to 0$ and $\int_a^b g\,dx$ converges then $\int_a^b f(x)dx$ also converges.

Since $\frac{f}{g} \to 0$, for $\varepsilon > 0$

$\frac{f}{g} < \varepsilon \quad \forall\, x \in \,]a,c[$

$\Rightarrow \quad f(x) < \varepsilon\, g(x) \quad \forall\, x \in \,]a,c[$

Apply comparison.

(ii) If $\frac{f}{g} \to \infty$ and $\int_a^b g\,dx$ diverges then $\int_a^b f(x)\,dx$ also diverges.

if $\frac{f}{g} \to \infty$, then there exist

$M > 0$ s.t

$\frac{f}{g} > M \quad \forall\, x \in \,]a,c[$

$\Rightarrow \quad f > Mg.$

$\Rightarrow \quad Mg < f \quad \forall\, x \in \,]a,c[$

Apply given condition to show the required result.

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PROBLEM: $\int_a^b \frac{dx}{(x-a)^n}$ converges iff $n<1$.

Sol: It is a proper integral if $n \le 0$ and improper for other values of n; a being the only point of infinite discontinuity.

Now if $n \ne 1$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\delta \to 0^+} \int_{a+\delta}^b \frac{dx}{(x-a)^n} \; ; \quad 0<\delta<b-a.$$

$$= \lim_{\delta \to 0^+} \frac{1}{1-n}\left[\frac{1}{(b-a)^{n-1}} - \frac{1}{\delta^{n-1}}\right]$$

$$= \begin{cases} \frac{1}{1-n} \cdot \frac{1}{(b-a)^{n-1}} & ; \text{ if } n<1 \\ \infty & ; \text{ if } n>1 \end{cases}$$

Again, For $n=1$

$$\int_a^b \frac{dx}{(x-a)^n} = \int_a^b \frac{dx}{x-a}$$

$$= \lim_{\delta \to 0^+} \int_{a+\delta}^b \frac{dx}{x-a}$$

$$= \lim_{\delta \to 0^+} \left(\log(b-a) - \log \delta\right) = \infty.$$

Thus $\int_a^b \frac{dx}{(x-a)^n}$ converges only for $n<1$

Example: Test the Convergence of the

(i) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ (ii) $\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx$

Solution: Here 0 is the point of infinite discontinuity. Now

$$\frac{\sin x}{x^p} = \frac{1}{x^{p-1}} \cdot \frac{\sin x}{x}$$

But $\frac{\sin x}{x}$ is bounded and $\frac{\sin x}{x} \le 1$

therefore $\frac{\sin x}{x^p} \le \frac{1}{x^{p-1}}$ But

$\int_0^{\pi/2} \frac{dx}{x^{p-1}}$ converges only if $p-1<1$ or $p<2$.

Therefore by comparison test

$\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx$ converges for $p<2$

and diverges for $p \ge 2$. (i) Assignment.

THEOREM (α-Test)

If $\lim_{x \to a^+} [(x-a)^\alpha f(x)]$ exists and is non-zero finite, then the integral $\int_a^b f(x)\,dx$ converges iff $\alpha < 1$ (SELF)

PROOF: We know $\int_a^b \frac{dx}{(x-a)^\alpha}$ converges iff $\alpha<1$

let $g(x) = \frac{1}{(x-a)^\alpha}$.

Then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} [(x-a)^\alpha f(x)] = l \ (\ne 0)$ (say) as given

Therefor $\int^b f(x)dx$ is convergent iff $\alpha<1$ (as $\int_a^b g\,dx$ is convergent iff $\alpha<1$)

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REMARK: If f is positive in a nbd of a, then the integral $\int_a^b f\,dx$ converges at a if there exists a positive number n less than 1 and a fixed positive number M such that $f(x) \leq \frac{M}{(x-a)^n}$ $\forall x$ in $]a,b]$

Also $\int_a^b f\,dx$ diverges if there exists a number $n \geq 1$ and a fixed positive number G such that $f(x) \geq \frac{G}{(x-a)^n}$ in $]a,b]$

Example: Find the values of m and n for which the following integerals converge.

(i) $\int_0^1 e^{-mx} x^n\,dx$ (ii) $\int_0^1 \left(\log \frac{1}{x}\right)^m dx$

Solution: (i) Let K be a number greater than 1 and e^{-m}, for all m.

Now in $[0,1]$, $e^{-mx} x^n \leq K x^n$ for all m.

and $\int_0^1 x^n\,dx = \int_0^1 \frac{dx}{x^{-n}}$ converges for $-n < 1$ or $n > -1$ only.

Thus $\int_0^1 e^{-mx} x^n\,dx$ converges only for $n > -1$ irrespective of the values of m.

(ii) For $\int_0^1 \left(\log \frac{1}{x}\right)^m dx$, 0 and 1 $(m>0)$ $(m<0)$ are the points of infinite discontinuity of the integrals on the right.

Put $\int_0^1 (\log \frac{1}{x})^m dx = \int_0^{\frac{1}{2}} (\log \frac{1}{x})^m dx + \int_{\frac{1}{2}}^{1} (\log \frac{1}{x})^m dx$

Convergence at 0.

$\int_0^{\frac{1}{2}} (\log \frac{1}{x})^m dx$ is a proper integral if $m \le 0$

Thus 0 is the only point of discontinuity if $m > 0$.

For $m > 0$, take $g(x) = \frac{1}{x^p}$, $0 < p < 1$

So that $\frac{f(x)}{g(x)} = x^p (\log \frac{1}{x})^m \to 0$ as $x \to 0$

Also $\int_0^{\frac{1}{2}} g\, dx$ converges, therefore $\int_0^{\frac{1}{2}} (\log \frac{1}{x})^m dx$ converges. Thus $\int_0^{\frac{1}{2}} (\log \frac{1}{x})^m dx$ converges for all m.

Convergence at 1.

$\int_{\frac{1}{2}}^{1} (\log \frac{1}{x})^m dx$ is a proper if $m \ge 0$ and 1 is the only point of infinite discontinuity if $m < 0$.

For $m < 0$. Let $g(x) = \frac{1}{(1-x)^{-m}}$ so that

$$\frac{f(x)}{g(x)} = \frac{(\log \frac{1}{x})^m}{(1-x)^m} \to 1 \text{ as } x \to 1^+$$

Hence the integrals $\int_{1/2}^{1} f\, dx$ and $\int_{1/2}^{1} g\, dx$ behave alike.

Now $\int_{1/2}^{1} \frac{dx}{(1-x)^{-m}}$ converges if $-m < 1$ or $m > -1$

therefore $\int^{1/2} (\log \frac{1}{x})^m dx$ also converges if $0 > m > -1$

Consequently $\int_0^1 (\log \frac{1}{x})^m dx$ is convergent when $0 > m > -1$.

S.A
Assigment: Show that (i) $\int_0^1 \frac{\log x}{\sqrt{x}} dx$ is convergent, but (ii) $\int_1^2 \frac{\sqrt{x}}{\log x} dx$ is divergent

(iii) Show that $\int_0^{\pi/2} \frac{\sin^m x}{x^n} dx$ exists iff $n < m+1$

BETA FUNCTION

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, \quad m,n > 0.$$

Example: Show that $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ exists iff m, n are both positive.

Solution: It is proper integral for $m \geq 1, n \geq 1$. 0 and 1 are the only points of infinite discontinuity when $m < 1$ and $n < 1$ respectively. Therefore, we put

$$\int_0^1 x^{m-1} (1-x)^{n-1} dx = \int_0^{\frac{1}{2}} x^{m-1} (1-x)^{n-1} dx + \int_{\frac{1}{2}}^1 x^{m-1} (1-x)^{n-1} dx$$

Convergence at 0 when $m < 1$.

Let $f(x) = x^{m-1}(1-x)^{n-1} = \frac{(1-x)^{n-1}}{(x)^{1-m}}$

Let $g(x) = \frac{1}{x^{1-m}}$

Then $\frac{f(x)}{g(x)} = (1-x)^{n-1} \to 1$ as $x \to 0^+$.

Also $\int_0^{1/2} g\,dx = \int_0^{1/2} \frac{dx}{x^{1-m}}$ converges at 0 if $1-m<1$ or $m>0$

Thus $\int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx$ converges at 0 iff $m>0$

Convergence at 1, when $n<1$

$$f(x) = x^{m-1}(1-x)^{n-1} = \frac{x^{m-1}}{(1-x)^{1-n}}$$

Let $g(x) = \frac{1}{(1-x)^{1-n}}$, Then $\frac{f(x)}{g(x)} = x^{m-1} \to 1$ as $x \to 1^-$

Also $\int_{1/2}^{1} g\,dx = \int_{1/2}^{1} \frac{dx}{(1-x)^{1-n}}$ converges at 1 iff $1-n<1$ or $n>0$.

Thus $\int_{1/2}^{1} x^{m-1}(1-x)^{n-1}\,dx$ converges at 1 iff $n>0$.

Hence $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ converges (exists) for positive values of m and n only.

Assignment

① For what values of m and n is the integral $\int_0^1 x^{m-1}(1-x)^{n-1}\log x\,dx$ convergent?

② Show that the integral $\int_0^{\pi/2} \log(\sin x)\,dx$ is convergent and hence evaluate it.

G.M ③ $\int_1^{\infty} x^{-2} e^{-x}\,dx$ is convergent

④ $\int_0^{\infty} \frac{\tan^{-1} x}{1+x^2}\,dx$ is convergent.

ABSOLUTE CONVERGENCE

Def: The improper integral $\int_a^b f\,dx$ is said to be absolutely convergent if $\int_a^b |f|\,dx$ is convergent.

THEOREM: Every absolutely convergent integral is convergent. OR $\int_a^b f\,dx$ exists if $\int_a^b |f|\,dx$ exists.

PROOF: Since $\int_a^b |f|\,dx$ exists, therefore by Cauchy's test. For $\varepsilon>0$, there exists a $\delta>0$ such that

$$\left|\int_{a+d_1}^{a+d_2} |f|\,dx\right| < \varepsilon \quad ; \; 0<d_1<d_2<\delta.$$

Further we know

$$\left|\int_{a+d_1}^{a+d_2} f\,dx\right| \le \int_{a+d_1}^{a+d_2} |f|\,dx$$

$$< \varepsilon \quad ; \; 0<d_1<d_2<\delta.$$

$\Rightarrow \int_a^b f\,dx$ exists.

REMARK: ① Since $|f|$ is always positive, therefore comparison tests are applicable for examining the convergence of $\int_a^b |f|\,dx$

i-e absolute convergence of $\int_a^b f\,dx$.

② Every convergent integral is not absolutely convergent. For this reason, a convergent integral which is not absolutely convergent is called a conditionally convergent integral.

Example: $\int_0^1 \frac{\sin(\frac{1}{x})}{x^p}\,dx$, $p>0$ Converges absolutely for $p<1$

Solution: Let $f(x) = \frac{\sin(\frac{1}{x})}{x^p}$; $p>0$

'0' is the only point of infinite discontinuity and f does not keep the same sign in any nbd of 0.

In $[0,1]$, $|f(x)| = \left|\frac{\sin(\frac{1}{x})}{x^p}\right| < \frac{1}{x^p}$

Also $\int_0^1 \frac{dx}{x^p}$ converges iff $p<1$

Hence by comparison test, the integral $\int_0^1 \left|\frac{\sin(\frac{1}{x})}{x^p}\right| dx$ converges, so $\int_0^1 \frac{\sin(\frac{1}{x})}{x^p}\,dx$ converges absolutely iff $p<1$.

Infinite Range of integration (a or b or both).

Examples: Test the convergence of the integrals.

① $\int_0^\infty \sin x\,dx$ ② $\int_{-\infty}^\infty \frac{dx}{1+x^2}$ ③ $\int_2^\infty \frac{2x^2}{x^4-1}\,dx$

④ $\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2}$ ⑤ $\int_0^\infty x^3 e^{-x^2}\,dx$.

⑤ $\int_0^\infty x^3 e^{-x^2}\,dx = \lim_{d\to\infty} \int_0^d x^3 e^{-x^2}\,dx$

$$= \lim_{d\to\infty}\left[\frac{1}{2} - \frac{1}{2}\left(\frac{d^2+1}{e^{d^2}}\right)\right]$$

$$= \frac{1}{2} \text{ converges.}$$

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COMPARISON TEST FOR CONVERGENCE AT ∞.

Assignment

TH ① A necessary and sufficient condition for the convergence of $\int_a^\infty f\,dx$, when f is positive in $[a, \lambda]$ is that there exists a positive number M, independent of λ, such that

$$\int_a^\lambda f\,dx < M, \text{ for every } \lambda \geq a.$$

② Comparison test 1 (Comparison of two integrals)

Statement: If f and g are positive and $f(x) \leq g(x)$ for all x in $[a, \lambda]$, then

(i) $\int_a^\infty f\,dx$ converges if $\int_a^\infty g\,dx$ converges.

(ii) $\int_a^\infty g\,dx$ diverges if $\int_a^\infty f\,dx$ diverges.

③ Comparison test 2 (LIMIT FORM):

Statement If f and g are positive in $[a, x]$ and $\lim_{x\to\infty} \frac{f}{g} = l$ (a non-zero finite number) then the two integrals $\int_a^\infty f\,dx$ and $\int_a^\infty g\,dx$ converges or diverges together. Also if $\frac{f}{g} \to 0$ and $\int_a^\infty g\,dx$ converges then $\int_a^\infty f\,dx$ converges. Further if $\frac{f}{g} \to \infty$ and $\int_a^\infty g\,dx$ diverges then $\int_a^\infty f\,dx$ diverges.

④ α-Test If $\lim_{x\to\infty} x^\alpha f(x)$ exists and is non-zero finite, then the integral $\int_a^\infty f\,dx$ converges iff $\alpha > 1$.

NEXT Exple 11, 12, 13 ([redacted] Gamma Function) • 14 See BOOK Sweeta Arora.

❖ *Implicit Function*

If $F(x,y,z)$ is a given function of x,y & z, then the equation $F(x,y,z)=0$ is a relation which may describe one or several functions z of x & y.

Thus if $x^2+y^2+z^2-1=0$, then

$$z=\sqrt{1-x^2-y^2} \quad \text{or} \quad z=-\sqrt{1-x^2-y^2}$$

Where both functions being defined for $x^2+y^2\le 1$. Either function is said to be implicitly defined by the equation $x^2+y^2+z^2-1=0$.

Similarly, an equation $F(x,y,z,w)=0$ may define one or more implicit functions w of x,y,z. If two such equations are given;

$$F(x,y,z,w)=0 \quad , \quad G(x,y,z,w)=0 \ ,$$

It is in general possible (at least in theory) to reduce the equations by elimination to the form

$$w=f(x,y) \quad , \quad z=g(x,y)$$

i.e. to obtain two functions of two variables. In general, if m equations in n unknown are given $(m<n)$, it is possible to solve for m of the variables in terms of the remaining $n-m$ variables; the number of dependent variables equals the number of equations

❖ *Example*

If $\quad 3x+2y+z+2w=0$

$\quad\quad 2x+3y-z-w=0$

then $w=f(x,y)=-5x-5y \quad$ & $\quad z=g(x,y)=7x+8y$

❖ *Example*

Suppose that the functions $w=f(x,y)$ & $z=g(x,y)$ are implicitly defined by

$$2x^2+y^2+z^2-zw=0$$

$$x^2+y^2+2z^2-8+zw=0$$

Then taking the differentials, we obtain

$$4xdx+2ydy+2zdz-wdz-zdw=0 \quad \ldots\ldots\ldots (i)$$

$$wdz+zdw+2xdx+2ydy+4zdz=0 \quad \ldots\ldots\ldots (ii)$$

Eliminate dw between (i) and (ii) to have

$$6xdx+4ydy+6zdz=0$$

$$\Rightarrow \ dz=-\frac{x}{z}dx-\frac{2y}{3x}dy$$

$$\Rightarrow \ \frac{\partial z}{\partial x}=-\frac{x}{z} \quad , \quad \frac{\partial z}{\partial y}=-\frac{2y}{3z}$$

Eliminating of dz from (i) and (ii) gives

$$6x(2z+w)dx+4y(z+w)dy-6z^2dw = 0$$

$$\Rightarrow \ dw=\frac{x(2z+w)}{z^2}dx+\frac{2y(z+w)}{3z^2}dy$$

$$\frac{\partial w}{\partial x}=\frac{x(2x+w)}{x^2} \quad , \quad \frac{\partial w}{\partial y}=\frac{2y(z+w)}{z^2}$$

❖ Examples

Suppose that the functions $w = f(x,y)$ & $z = g(x,y)$ are implicitly define by

$$F(x,y,z,w) = 0 \quad \text{and} \quad G(x,y,z) = 0\text{, then}$$

$$F_x dx + F_y dy + F_z dz + F_w dw = 0$$

and $$G_x dx + G_y dy + G_z dz + G_w dw = 0$$

$$\Rightarrow F_z dz + F_w dw = -\left[F_x dx + F_y dy\right]$$

and $$G_z dz + G_w dw = -\left[G_x dx + G_y dy\right]$$

Then by crammer rule, we have

$$dz = \frac{-\begin{vmatrix} F_x dx + F_y dy & F_w \\ G_x dx + G_y dy & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} = -\frac{\begin{vmatrix} F_x & F_w \\ G_x & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} dx - \frac{\begin{vmatrix} F_y & F_w \\ G_y & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} dy$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\begin{vmatrix} F_x & F_w \\ G_x & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} \quad , \quad \frac{\partial z}{\partial y} = -\frac{\begin{vmatrix} F_y & F_w \\ G_y & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}}$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,w)}}{\frac{\partial(F,G)}{\partial(z,w)}} \quad , \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(y,w)}}{\frac{\partial(F,G)}{\partial(z,w)}} \qquad \text{provided } \frac{\partial(F,G)}{\partial(z,w)} \neq 0$$

Similarly, we have

$$dw = -\frac{\begin{vmatrix} F_z & F_x dx + F_y dy \\ G_z & G_x dx + G_y dy \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}}$$

and we can find $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ in the same manner.

❖ Particular Cases

i) One equation in 2 unknowns i.e. $F(x,y) = 0$

$$\Rightarrow F_x dx + F_y dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} \quad (F_y \neq 0)$$

ii) One equation in 3 unknowns i.e. $F(x,y,z) = 0$

$$F_x dx + F_y dy + F_z dz = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad , \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \qquad (F_z \neq 0)$$

iii) 2 equations in 3 unknown

$$F(x,y,z) = 0 \quad , \quad G(x,y,z) = 0$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(y,x)}}{\frac{\partial(F,G)}{\partial(y,z)}} \quad , \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(w,y)}}{\frac{\partial(F,G)}{\partial(z,w)}}$$

………………………………

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❖ *Example*

Find the partial derivatives w.r.t x & y, when

$$u+2v-x^2+y^2=0$$
$$2u-v-2xy=0$$

Solution

Take the differentials

$$du+2dv-2x\,dx+2y\,dy=0 \;\ldots\ldots\ldots\ldots (i)$$
$$2du-dv-2x\,dy-2y\,dx=0 \;\ldots\ldots\ldots\ldots (ii)$$

Eliminating dv between (i) and (ii), we have

$$5du-(2x+4y)dx+(2y-4x)dy=0$$
$$\Rightarrow\; du=\frac{1}{5}(2x+4y)dx-\frac{1}{5}(2y-4x)dy$$
$$\Rightarrow\; \frac{\partial u}{\partial x}=\frac{1}{5}(2x+4y) \quad \& \quad \frac{\partial u}{\partial y}=-\frac{1}{5}(2y-4x)$$

Eliminating du between (i) and (ii), we get

$$5dv-(4x-2y)dx+(4y+2x)dy=0$$
$$\Rightarrow\; dv=\frac{1}{5}(4x-2y)dx-\frac{1}{5}(4y+2x)dy$$
$$\Rightarrow\; \frac{\partial v}{\partial x}=\frac{1}{5}(4x-2y) \quad \& \quad \frac{\partial v}{\partial y}=-\frac{1}{5}(4y+2x)$$

❖ *Question*

Give that

$$2x+y-3z-2u=0$$
$$\& \quad x+2y+z+u=0$$

Find $\left(\frac{\partial x}{\partial y}\right)_z, \left(\frac{\partial y}{\partial x}\right)_u, \left(\frac{\partial z}{\partial u}\right)_x, \left(\frac{\partial y}{\partial z}\right)_x$

Solution

Take the differentials

$$2dx+dy-3dz-2du=0 \;\ldots\ldots\ldots\ldots (i)$$
$$dx+2dy+dz+du=0 \;\ldots\ldots\ldots\ldots (ii)$$

Eliminating du between (i) and (ii), we have

$$4dx+5dy-dz=0 \;\ldots\ldots\ldots (iii)$$
$$\Rightarrow\; dx=-\frac{5}{4}dy+\frac{1}{4}dz$$
$$\Rightarrow\; \left(\frac{\partial x}{\partial y}\right)_z=-\frac{5}{4}$$

From (iii), we have

$$5dy=dz-4dx$$
$$\Rightarrow\; dy=\frac{1}{5}dz-\frac{4}{5}dx$$
$$\Rightarrow\; \left(\frac{\partial y}{\partial z}\right)_x=\frac{1}{5}$$

Eliminating dz between (i) & (ii), we get

$$5dx+7dy+du=0$$

$$\Rightarrow dy = -\frac{5}{7}dx - \frac{1}{7}du$$

$$\Rightarrow \left(\frac{\partial y}{\partial x}\right)_u = -\frac{5}{7}$$

Now eliminating dy between (i) & (ii), we get

$$-3dx - 5dz - 3du = 0$$

$$\Rightarrow dz = -\frac{3}{5}dx - \frac{3}{5}du$$

$$\Rightarrow \left(\frac{\partial z}{\partial u}\right)_x = -\frac{3}{5}$$

❖ ***Question***

Given that

$$x^2 + y^2 + z^2 - u^2 + v^2 = 1 \text{ .... } (i)$$
$$x^2 - y^2 + z^2 + u^2 + 2v^2 = 2 \text{ .... } (ii)$$

a) Find du & dv in terms of dx, dy & dz at the point

$x = 1$, $y = 1$, $z = 2$, $u = 3$ & $v = 2$.

b) Find $\left(\frac{\partial u}{\partial x}\right)_{(y,z)}$, $\left(\frac{\partial v}{\partial y}\right)_{(x,z)}$ at the point given above.

c) Find approximately the values of u & v for $x = 1\cdot 1$, $y = 1.2$, $z = 1.8$

Solutions

Differential gives

$$2xdx + 2ydy + 2zdz - 2udu + 2vdv = 0 \text{ .... } (iii)$$
$$2xdx - 2ydy + 2zdz + 2udu + 2vdv = 0 \text{ .... } (iv)$$

a) Putting $x = 1$, $y = 1$, $z = 2$, $u = 3$ & $v = 2$ in (iii) & (iv), we obtain

$$2dx + 2dy + 4dz - 6du + 4dv = 0 \text{ .... } (v)$$

& $$2dx - 2dy + 4dz + 6du + 8dv = 0 \text{ .... } (vi)$$

Adding gives

$$12dv = -(4dx + 8dz)$$

$$\Rightarrow dv = -\frac{1}{3}(dx + 0\cdot dy + 2dz)$$

Similarly eliminating dv between (v) and (vi), we get

$$du = \frac{1}{9}(dx + 3dy + 2dz)$$

b) $\because du = \frac{1}{9}(dx + 3dy + 2dz)$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_{y,z} = \frac{1}{9}$$

& $\because dv = -\frac{1}{3}(dx + 0\cdot dy + 2dz)$

$$\therefore \left(\frac{\partial v}{\partial y}\right)_{x,z} = 0$$

....

❖ Question

Find the transformation of $x = r\cos\theta$, $y = r\sin\theta$ from rectangular to polar coordinates. Verify the relations

a) $dx = \cos\theta\, dr - r\sin\theta\, d\theta$
$dy = \sin\theta\, dr + r\cos\theta\, d\theta$

b) $dr = \cos\theta\, dx + \sin\theta\, dy$

$$d\theta = -\frac{\sin\theta}{r}dx + \frac{\cos\theta}{r}dy$$

c) $\left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$, $\left(\frac{\partial x}{\partial r}\right)_y = \sec\theta$, $\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$

Solutions

Given that $x = r\cos\theta$ & $y = r\sin\theta$

a) Differential gives

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta \quad \ldots\ldots\ldots (i)$$
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta \quad \ldots\ldots\ldots (ii)$$

b) Multiplying (i) by $\cos\theta$ & (ii) by $\sin\theta$ and adding, we get

$$dr = \cos\theta\, dx + \sin\theta\, dy$$

Now multiply (i) by $\sin\theta$ & (ii) by $\cos\theta$ and subtract to obtain

$$d\theta = -\frac{\sin\theta}{r}dx + \frac{\cos\theta}{r}dy$$

c) Given $x = r\cos\theta$

$$\Rightarrow \left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$$

We have already shown that $dr = \cos\theta\, dx + \sin\theta\, dy$

Which can be written as $dx = \frac{dr}{\cos\theta} - \tan\theta\, dy$

$$\Rightarrow \left(\frac{\partial x}{\partial r}\right)_y = \sec\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

and $$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix} = \frac{1}{r}\cos^2\theta + \frac{1}{r}\sin^2\theta = \frac{1}{r}$$

❖ Question

Given that $x^2 - y^2\cos uv + z^2 = 0$
$x^2 + y^2 - \sin uv + 2z^2 = 2$
and $xy - \sin u\cos v + z = 0$

Find $\left(\frac{\partial x}{\partial u}\right)_v$, $\left(\frac{\partial x}{\partial v}\right)_u$ at $x = 1$, $y = 1$, $u = \frac{\pi}{2}$, $v = 0$, $z = 0$

Solution

Differential gives

$$2x\,dx - 2y\cos uv\,dy + y^2\sin uv\cdot u\,dv + y^2\sin uv\cdot v\,du + 2z\,dz = 0 \quad \ldots\ldots (i)$$

$$2x\,dx + 2y\,dy - \cos uv \cdot u\,dv - \cos uv \cdot v\,du + 4z\,dz = 0 \quad \ldots\ldots\ldots (ii)$$

$$\& \quad x\,dy + y\,dx - \cos u \cdot \cos v\,du + \sin u \cdot \sin v\,dv + dz = 0 \quad \ldots\ldots\ldots (iii)$$

At the given point, these equations reduce to

$$2dx - 2dy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (iv)$$

$$2dx + 2dy - \frac{\pi}{2}dv = 0 \quad \ldots\ldots\ldots (v)$$

$$\& \quad dx + dy + dz = 0 \quad \ldots\ldots\ldots\ldots\ldots (vi)$$

Adding (iv) & (v), we have

$$4dx - \frac{\pi}{2}dv = 0$$

$$\Rightarrow \ dx = \frac{\pi}{8}dv + 0 \cdot du \quad \Rightarrow \left(\frac{\partial x}{\partial u}\right)_v = 0 \ , \ \left(\frac{\partial x}{\partial v}\right)_u = \frac{\pi}{8}$$

❖ Question

Find $\left(\dfrac{\partial u}{\partial x}\right)_y$ if $x^2 - y^2 + u^2 + 2v^2 = 1$

$$x^2 + y^2 - u^2 - v^2 = 2$$

Solution

Taking the differentials, we have

$$2x\,dx - 2y\,dy + 2u\,du + 4v\,dv = 0$$

$$2x\,dx + 2y\,dy - 2u\,du - 2v\,dv = 0$$

Eliminating dv, we get

$$6x\,dx + 2y\,dy - 2u\,du = 0$$

$$\Rightarrow \ du = \frac{3x}{u}dx + \frac{y}{u}dy$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)_y = \frac{3x}{u}$$

❖ Question

Given the transformation

$$x = u - 2v$$

$$y = 2u + v$$

a) Write the equations of the inverse transformation

b) Evaluate the Jacobian of the transformation and that of the inverse transformation.

Solution

a) From the equations, we have

$$u = \frac{1}{5}x + \frac{2}{5}y$$

$$v = -\frac{2}{5}x + \frac{1}{5}y$$

which are the equations of the inverse transformation.

b) Jacobian of the given transformation $= \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 5$$

Jacobian of the inverse transformation $=\dfrac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}$

$$=\begin{vmatrix}\dfrac{1}{5} & \dfrac{2}{5}\\ -\dfrac{2}{5} & \dfrac{1}{5}\end{vmatrix}=\frac{1}{5}$$

❖ *Question*

Given the transformation $x=f(u,v)$, $y=g(u,v)$ with Jacobian $J=\dfrac{\partial(x,y)}{\partial(u,v)}$, show that for the inverse transformation one has

$$\frac{\partial u}{\partial x}=\frac{1}{J}\frac{\partial y}{\partial v}\ ,\ \frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial x}{\partial v}\ ,\ \frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial y}{\partial u}\ ,\ \frac{\partial u}{\partial y}=\frac{1}{J}\frac{\partial x}{\partial u}$$

Solution

The given equations are

$$f(u,v)-x=0\ \ldots\ldots\ldots\ (i)$$
$$g(u,v)-y=0\ \ldots\ldots\ldots\ (ii)$$

Differentiating w.r.t. x, we get

$$f_u\frac{\partial u}{\partial x}+f_v\frac{\partial v}{\partial x}-1=0$$
$$g_u\frac{\partial u}{\partial x}+g_v\frac{\partial v}{\partial x}-0=0$$

Solving these equations by Crammer's rule, we have

$$\frac{\partial u}{\partial x}=-\frac{\begin{vmatrix}-1 & f_v\\ 0 & g_v\end{vmatrix}}{\begin{vmatrix}f_u & f_v\\ g_u & g_v\end{vmatrix}}=\frac{g_v}{J}=\frac{1}{J}\frac{\partial y}{\partial v}\qquad\left(\because\ \frac{\partial y}{\partial v}=g_v\right)$$

$$\frac{\partial v}{\partial x}=-\frac{\begin{vmatrix}f_u & -1\\ g_u & 0\end{vmatrix}}{J}=-\frac{g_u}{J}=-\frac{1}{J}\frac{\partial y}{\partial u}$$

Differentiating (i) & (ii) w.r.t. y, we have

$$f_u\frac{\partial u}{\partial y}+f_v\frac{\partial v}{\partial y}-0=0$$
$$g_u\frac{\partial u}{\partial y}+g_v\frac{\partial v}{\partial y}-1=0$$

Solving these equations by Crammer's rule, we get

$$\frac{\partial u}{\partial y}=-\frac{\begin{vmatrix}0 & f_v\\ -1 & g_v\end{vmatrix}}{J}=-\frac{f_v}{J}=-\frac{1}{J}\frac{\partial x}{\partial v}$$

$$\frac{\partial v}{\partial y}=-\frac{\begin{vmatrix}f_u & 0\\ g_u & -1\end{vmatrix}}{J}=\frac{f_u}{J}=\frac{1}{J}\frac{\partial x}{\partial u}$$

❖ Question

Given the transformation

$$x = u^2 - v^2$$
$$y = 2uv$$

a) Compute its Jacobian.

b) Evaluate $\left(\dfrac{\partial u}{\partial x}\right)_y$ & $\left(\dfrac{\partial v}{\partial x}\right)_y$

Solution

The given equations can be written as

$$u^2 - v^2 - x = 0 \quad \ldots\ldots\ldots. \ (i)$$
$$2uv - y = 0 \quad \ldots\ldots\ldots\ldots. \ (ii)$$

Differentiating (*i*) & (*ii*) partially w.r.t. x, we have

$$2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} - 1 = 0 \quad \ldots\ldots\ldots\ldots \ (iii)$$

$$2v\frac{\partial u}{\partial x} + 2u\frac{\partial v}{\partial x} - 0 = 0 \quad \ldots\ldots\ldots\ldots \ (iv)$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2 + v^2)$$

Solving (*iii*) & (*iv*) by Crammer's rule, we have

$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{\begin{vmatrix} -1 & -2v \\ 0 & 2u \end{vmatrix}}{J} = \frac{2u}{4(u^2+v^2)} = \frac{u}{2(u^2+v^2)}$$

$$\left(\frac{\partial v}{\partial x}\right)_y = -\frac{\begin{vmatrix} 2u & -1 \\ 2v & 0 \end{vmatrix}}{J} = \frac{-2v}{4(u^2+v^2)} = \frac{-v}{2(u^2+v^2)}$$

Note

$\left(\dfrac{\partial u}{\partial y}\right)_x$ & $\left(\dfrac{\partial v}{\partial y}\right)_x$ can be determined in the same manner.

❖ Question

Prove that if $F(x,y,z) = 0$, then

$$\left(\frac{\partial z}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial y}\right)_z \cdot \left(\frac{\partial y}{\partial z}\right)_x = -1$$

Solution

$$F(x,y,z) = 0$$

$$\Rightarrow \ F_x\,dx + F_y\,dy + F_z\,dz = 0$$

$$\Rightarrow \ dx = -\frac{F_y}{F_x}dy - \frac{F_z}{F_x}dz \qquad \Rightarrow \left(\frac{\partial x}{\partial y}\right)_z = -\frac{F_y}{F_x}$$

$$\& \quad dy = -\frac{F_x}{F_y}dx - \frac{F_z}{F_y}dz \qquad \Rightarrow \left(\frac{\partial y}{\partial z}\right)_x = -\frac{F_z}{F_y}$$

$$dz = -\frac{F_x}{F_z}dx - \frac{F_y}{F_z}dy \qquad \Rightarrow \left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z}$$

Hence

$$\left(\frac{\partial z}{\partial x}\right)_y \cdot\left(\frac{\partial x}{\partial y}\right)_z \cdot\left(\frac{\partial y}{\partial z}\right)_x = \left(-\frac{F_x}{F_z}\right)\cdot\left(-\frac{F_y}{F_x}\right)\cdot\left(-\frac{F_z}{F_y}\right) = -1$$

❖ *Question*

Prove that, if $x = f(u,v)$, $y = g(u,v)$, then

$$\left(\frac{\partial x}{\partial u}\right)_v\left(\frac{\partial u}{\partial x}\right)_y = \left(\frac{\partial y}{\partial v}\right)_u\left(\frac{\partial v}{\partial y}\right)_x$$

and
$$\left(\frac{\partial x}{\partial v}\right)_u\left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x\left(\frac{\partial y}{\partial u}\right)_v$$

also that $\left(\frac{\partial x}{\partial y}\right)_u\left(\frac{\partial y}{\partial x}\right)_u = 1$

Solution

$\because\ f(u,v) - x = 0$

$g(u,v) - y = 0$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y = \frac{g_v}{J}, \quad \left(\frac{\partial v}{\partial x}\right)_y = -\frac{g_u}{J}$$

$$\left(\frac{\partial u}{\partial y}\right)_x = -\frac{f_v}{J}, \quad \left(\frac{\partial v}{\partial y}\right)_x = \frac{f_u}{J}$$ as already shown

Taking differentials of the given equations, we have

$$f_u\,du + f_v\,dv - dx = 0$$
$$g_u\,du + g_v\,dv - dy = 0$$
$$\Rightarrow dx = f_u\,du + f_v\,dv \quad \ldots\ldots\ldots\ldots (i)$$
$$dy = g_u\,du + g_v\,dv \quad \ldots\ldots\ldots\ldots (ii)$$
$$\Rightarrow \left(\frac{\partial x}{\partial u}\right)_v = f_u, \quad \left(\frac{\partial x}{\partial v}\right)_u = f_v$$
$$\left(\frac{\partial y}{\partial u}\right)_v = g_u, \quad \left(\frac{\partial y}{\partial v}\right)_u = g_v$$

Now
$$\left(\frac{\partial x}{\partial u}\right)_v \cdot\left(\frac{\partial u}{\partial x}\right)_y = \left(\frac{\partial y}{\partial v}\right)_u \cdot\left(\frac{\partial v}{\partial y}\right)_x$$
$$\Rightarrow f_u \cdot \frac{g_v}{J} = g_v \cdot \frac{f_u}{J},$$ which is true

Similarly, we have the second relation.

Eliminating dv between (i) & (ii), we get

$$(f_u \cdot g_v - f_v \cdot g_u)du - g_v\,dx + f_v\,dy = 0$$
$$\Rightarrow dx = \frac{f_u g_v - f_v g_u}{g_v}\cdot du + \frac{f_v}{g_v}\,dy$$

and
$$dy = \frac{g_v}{f_v}dx - \frac{f_u g_v - f_v g_u}{f_v}\,du$$
$$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_u = \frac{f_v}{g_v} \quad \& \quad \left(\frac{\partial y}{\partial x}\right)_u = \frac{g_v}{f_v}$$
$$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_u \cdot\left(\frac{\partial y}{\partial x}\right)_u = \frac{f_v}{g_v}\cdot\frac{g_v}{f_v} = 1$$

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❖ Question

Given that $x=f(u,v,w)$, $y=g(u,v,w)$, $z=h(u,v,w)$ with the Jacobian $J=\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$, show that for the inverse transformation one has

i) $\dfrac{\partial u}{\partial x}=\dfrac{1}{J}\dfrac{\partial(y,z)}{\partial(v,w)}$, $\dfrac{\partial u}{\partial y}=\dfrac{1}{J}\dfrac{\partial(z,x)}{\partial(v,w)}$, $\dfrac{\partial u}{\partial z}=\dfrac{1}{J}\dfrac{\partial(x,y)}{\partial(v,w)}$

ii) $\dfrac{\partial v}{\partial x}=\dfrac{1}{J}\dfrac{\partial(y,z)}{\partial(w,u)}$, $\dfrac{\partial v}{\partial y}=\dfrac{1}{J}\dfrac{\partial(z,x)}{\partial(w,u)}$, $\dfrac{\partial v}{\partial z}=\dfrac{1}{J}\dfrac{\partial(x,y)}{\partial(w,u)}$

iii) $\dfrac{\partial w}{\partial x}=\dfrac{1}{J}\dfrac{\partial(y,z)}{\partial(u,v)}$, $\dfrac{\partial w}{\partial y}=\dfrac{1}{J}\dfrac{\partial(z,x)}{\partial(u,v)}$, $\dfrac{\partial w}{\partial z}=\dfrac{1}{J}\dfrac{\partial(x,y)}{\partial(u,v)}$

Solution

We have
$$f(u,v,w)-x=0$$
$$g(u,v,w)-y=0$$
$$h(u,v,w)-z=0$$

Differentiating w.r.t. to x, we get
$$f_u\frac{\partial u}{\partial x}+f_v\frac{\partial v}{\partial x}+f_w\frac{\partial w}{\partial x}-1=0$$
$$g_u\frac{\partial u}{\partial x}+g_v\frac{\partial v}{\partial x}+g_w\frac{\partial w}{\partial x}-0=0$$
$$h_u\frac{\partial u}{\partial x}+h_v\frac{\partial v}{\partial x}+h_w\frac{\partial w}{\partial x}-0=0$$

By Crammer's rule, we have
$$\frac{\partial u}{\partial x}=-\frac{\begin{vmatrix}-1 & f_v & f_w\\ 0 & g_v & g_w\\ 0 & h_v & h_w\end{vmatrix}}{J}=\frac{\begin{vmatrix}g_v & g_w\\ h_v & h_w\end{vmatrix}}{J}=\frac{1}{J}\frac{\partial(g,h)}{\partial(v,w)}=\frac{1}{J}\frac{\partial(y,z)}{\partial(v,w)}$$
$$\frac{\partial v}{\partial x}=-\frac{\begin{vmatrix}f_u & -1 & f_w\\ g_u & 0 & g_w\\ h_u & 0 & h_w\end{vmatrix}}{J}=-\frac{\begin{vmatrix}g_u & g_w\\ h_u & h_w\end{vmatrix}}{J}=\frac{1}{J}\frac{\partial(g,h)}{\partial(w,u)}=\frac{1}{J}\frac{\partial(y,z)}{\partial(w,u)}$$
$$\frac{\partial w}{\partial x}=-\frac{\begin{vmatrix}f_u & f_v & -1\\ g_u & g_v & 0\\ h_u & h_v & 0\end{vmatrix}}{J}=\frac{\begin{vmatrix}g_u & g_v\\ h_u & h_v\end{vmatrix}}{J}=\frac{1}{J}\frac{\partial(g,h)}{\partial(u,v)}=\frac{1}{J}\frac{\partial(y,z)}{\partial(u,v)}$$

We can find the other relations in the same way by differentiating given relation w.r.t. y and w.r.t. z respectively.

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❖ Bi-Harmonic Equations

Another important combination of derivatives occurs in the equation

$$\frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} = 0$$

which is known to be the Bi-harmonic equation. This combination can be expressed in terms of Laplacian as

$$\nabla^2\left(\nabla^2 z\right) = \nabla^4 z = 0$$

The solutions of $\nabla^4 z = 0$ are termed as Pri-harmonic functions.

❖ Higher Derivatives of Functions of Functions

(1) Let $z = f(x,y)$ and $x = g(t)$, $y = h(t)$ so that z can be expressed in terms of t alone. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \quad \ldots\ldots\ldots\ldots (i)$$

$$\frac{d^2 z}{dt^2} = \frac{d}{dt}\left(\frac{dz}{dt}\right) = \frac{\partial z}{\partial x}\frac{d^2 x}{dt^2} + \frac{dx}{dt}\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial y}\frac{d^2 y}{dt^2} + \frac{dy}{dt}\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right) \quad \ldots\ldots\ldots (ii)$$

Using (i), we have

$$\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y \partial x}\frac{dy}{dt}$$

& $$\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial z}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}$$

Putting these values in (ii), we have

$$\frac{d^2 z}{dt^2} = \frac{\partial z}{\partial x}\frac{d^2 x}{dt^2} + \frac{\partial^2 z}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial^2 z}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial z}{\partial y}\frac{d^2 y}{dt^2}$$

(2) If $z = f(x,y)$ and $x = g(u,v)$, $y = h(u,v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad \ldots\ldots\ldots\ldots (iii)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \quad \ldots\ldots\ldots\ldots (iv)$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial z}{\partial x}\cdot\frac{\partial^2 x}{\partial u^2} + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\left(\frac{\partial^2 y}{\partial u^2}\right) + \frac{\partial y}{\partial u}\cdot\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right) \quad \ldots\ldots\ldots\ldots\ldots (iv)$$

Using (iii), we have

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 z}{\partial y \partial x}\cdot\frac{\partial y}{\partial u}$$

and $$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial u}$$

Putting these values in (iv), we get

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial u} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2 + \frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial u^2}$$

We can find the values of $\dfrac{\partial^2 z}{\partial u \partial v}$ & $\dfrac{\partial^2 z}{\partial v^2}$ in the same manner.

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❖ Partial Derivative of Higher Order

Let a function $z = f(x,y)$ be given. Then its two partial derivatives $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ are themselves functions of x & y.

i.e. $\frac{\partial z}{\partial x} = f_x(x,y)$, $\frac{\partial z}{\partial y} = f_y(x,y)$

Hence each can be differentiable w.r.t. x & y.

Thus, we obtain four partial derivatives

$$\frac{\partial^2 z}{\partial x^2} = f_{xx}(x,y) \quad , \quad \frac{\partial^2 z}{\partial x\,\partial y} = f_{xy}(x,y)$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = f_{yx}(x,y) \quad , \quad \frac{\partial^2 z}{\partial y^2} = f_{yy}(x,y)$$

$\frac{\partial^2 z}{\partial x^2}$ is the result of differentiating $\frac{\partial z}{\partial x}$ w.r.t. x, where $\frac{\partial^2 z}{\partial y\,\partial x}$ is the result of differentiating $\frac{\partial z}{\partial x}$ w.r.t. y. If all the derivatives concerned are continuous in the domain considered, then $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$ i.e. order of differentiation is immaterial.

Third and higher order partial derivatives are defined in the same manner and under appropriate assumptions of continuity the order of differentiation does not matter.

❖ Laplacian of z

If $z = f(x,y)$, then the Laplacian of z is denoted by $\nabla^2 z$ is the expression

$$\nabla^2 z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

if $w = f(x,y,z)$, the Laplacian of w is the expression

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

The symbol "∇" is a vector differential operator define as

$$\nabla = \frac{\partial}{\partial x}\underline{\hat{i}} + \frac{\partial}{\partial y}\underline{\hat{j}} + \frac{\partial}{\partial x}\underline{\hat{k}}$$

We then have symbolically

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

❖ Harmonic Function

If $z = f(x,y)$ has continuous second order derivatives in a domain D and $\nabla^2 z = 0$ in D, then z is said to be Harmonic in D. The same term is used for the function of three variables which has continuous 2^{nd} derivatives in a domain D in space and whose Laplacian is zero in D. The two equations for harmonic functions

$$\nabla^2 z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

are known as the Laplace equations in two and three dimensions respectively.

❖ *The Laplacian in Polar, Cylindrical and Spherical Co-ordinate*

We consider first the two-dimensional Laplacian

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

and its expression in terms of polar co-ordinates r & θ.

Thus we are given $w = f(x,y)$ and $x = r\cos\theta$, $y = r\sin\theta$ and we wish to express $\nabla^2 w$ in terms of r, θ and derivatives of w with respect to r and θ. The solution is as follows. One has

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial w}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} \quad \text{by chain rule}$$

To evaluate $\frac{\partial r}{\partial x}, \frac{\partial \theta}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial y}$, we use the equations

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

These can be solved for dr and $d\theta$ by determinants or by elimination to give

$$dr = \cos\theta\, dx + \sin\theta\, dy$$

$$d\theta = -\frac{\sin\theta}{r}dx + \frac{\cos\theta}{r}dy$$

Hence $\frac{\partial r}{\partial x} = \cos\theta$, $\frac{\partial r}{\partial y} = \sin\theta$, $\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$ and $\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$

Putting these values above in expressions of $\frac{\partial w}{\partial x}$ & $\frac{\partial w}{\partial y}$, we have

$$\left.\begin{aligned} \frac{\partial w}{\partial x} &= \cos\theta\frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta} \\ \frac{\partial w}{\partial y} &= \sin\theta\frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta} \end{aligned}\right\} \ldots\ldots\ldots\ldots (i)$$

These equations provide general rules for expressing derivatives w.r.t. x or y in terms of derivatives w.r.t. r and θ. By applying the first equation to the function $\frac{\partial w}{\partial x}$, one finds that

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial x}\right)$$

By (i) this can be written as follows:

$$\frac{\partial^2 w}{\partial x^2} = \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}\right)$$

The rule for differentiation of a product gives finally

$$\frac{\partial^2 w}{\partial x^2} = \cos^2\theta\cdot\frac{\partial^2 w}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\cdot\frac{\partial^2 w}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\cdot\frac{\partial^2 w}{\partial \theta^2}$$

$$+\frac{\sin^2\theta}{r}\cdot\frac{\partial w}{\partial \theta} + \frac{2\sin\theta\cos\theta}{r^2}\cdot\frac{\partial w}{\partial \theta} \;\ldots\ldots\; (ii)$$

In the same manner one finds

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta}\right)$$

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$$= \sin^2\theta\cdot\frac{\partial^2 w}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\cdot\frac{\partial^2 w}{\partial r\,\partial\theta} + \frac{\cos^2\theta}{r^2}\cdot\frac{\partial^2 w}{\partial\theta^2}$$

$$+\frac{\cos^2\theta}{r}\cdot\frac{\partial w}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\cdot\frac{\partial w}{\partial\theta} \quad \ldots\ldots (iii)$$

Adding (*ii*) & (*iii*), we conclude

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} \quad \ldots\ldots\ldots (iv)$$

This is the desired result.

Equation (*iv*) at once permits one to write the expression for the 3-demensional Laplacian in cylindrical co-ordinates for the transformation of coordinates

$$x = r\cos\theta \ , \quad y = r\sin\theta \ , \quad z = z$$

involves only x & y. In the same way as above, we have

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

$$= \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}$$

❖ *Laplacian in Spherical Polar Coordinates*

The transformation form rectangular to spherical polar coordinates is

$$x = \rho\sin\varphi\cos\theta \ , \quad y = \rho\sin\varphi\sin\theta \ , \quad z = \rho\cos\varphi$$

Writing $r = \rho\sin\varphi$, we have

$$x = r\cos\theta \ , \quad y = r\sin\theta \ , \quad z = z$$

Which can be considered as a transformation from rectangular to cylindrical coordinates (r,θ,z)

We have

$$\nabla^2 w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \quad \ldots\ldots\ldots\ldots (i)$$

where $\left.\begin{array}{l} z = \rho\cos\varphi \\ r = \rho\sin\varphi \end{array}\right\} \ldots\ldots\ldots (ii)$

We have transformation from (x, y) to (r,θ) as

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial\theta}$$

Now if we take transformation from (z,r) to (ρ,φ), then

$$\Rightarrow \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial r^2} = \frac{\partial^2 w}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial\varphi^2} + \frac{1}{\rho}\frac{\partial w}{\partial\varphi}$$

Also $\displaystyle \frac{\partial w}{\partial r} = \frac{\partial w}{\partial\rho}\cdot\frac{\partial\rho}{\partial r} + \frac{\partial w}{\partial\varphi}\cdot\frac{\partial\varphi}{\partial r}$

Where $\rho^2 = z^2 + r^2 \ , \quad \tan\varphi = \dfrac{r}{z}$

$$\Rightarrow 2\rho\frac{\partial\rho}{\partial r} = 2r \ \Rightarrow \frac{\partial\rho}{\partial r} = \frac{r}{\rho} = \frac{\rho\sin\varphi}{\rho} = \sin\varphi$$

$$\& \quad \sec^2\varphi\cdot\frac{\partial\varphi}{\partial r} = \frac{1}{z} \quad \Rightarrow \frac{\partial\varphi}{\partial r} = \frac{\cos^2\varphi}{z} = \frac{\cos^2\varphi}{\rho\cos\varphi} = \frac{\cos\varphi}{\rho}$$

$$\Rightarrow \frac{\partial w}{\partial r} = \frac{\partial w}{\partial\rho}\cdot\sin\varphi + \frac{\partial w}{\partial\varphi}\cdot\frac{\cos\varphi}{\rho} \quad \ldots\ldots\ldots\ldots (iv)$$

Substituting (*iii*) & (*iv*) in (*i*), we have

$$\nabla^2 w=\left(\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 w}{\partial r^2}\right)+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}+\frac{1}{r}\frac{\partial w}{\partial r}$$

$$=\frac{\partial^2 w}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2 w}{\partial\varphi^2}+\frac{1}{\rho}\frac{\partial w}{\partial\rho}+\frac{1}{\rho^2\sin^2\varphi}\cdot\frac{\partial^2 w}{\partial\theta^2}+\frac{1}{\rho\sin\varphi}\left(\frac{\partial w}{\partial\rho}\sin\varphi+\frac{\partial w}{\partial\varphi}\frac{\cos\varphi}{\rho}\right)$$

$$=\frac{\partial^2 w}{\partial\rho^2}+\frac{1}{\rho^2}\cdot\frac{\partial^2 w}{\partial\varphi^2}+\frac{1}{\rho}\cdot\frac{\partial w}{\partial\rho}+\frac{1}{\rho^2\sin^2\varphi}\cdot\frac{\partial^2 w}{\partial\theta^2}+\frac{1}{\rho}\cdot\frac{\partial w}{\partial\rho}+\frac{\cot\varphi}{\rho^2}\cdot\frac{\partial w}{\partial\varphi}$$

$$=\frac{\partial^2 w}{\partial\rho^2}+\frac{1}{\rho^2}\cdot\frac{\partial^2 w}{\partial\varphi^2}+\frac{2}{\rho}\cdot\frac{\partial w}{\partial\rho}+\frac{1}{\rho^2\sin^2\varphi}\cdot\frac{\partial^2 w}{\partial\theta^2}+\frac{\cot\varphi}{\rho^2}\cdot\frac{\partial w}{\partial\varphi}$$

❖ ***Question***

If u & v are functions of x & y defined by the equations

$$xy+uv=1\,,\quad xu+yv=1$$

then find $\dfrac{\partial^2 u}{\partial x^2}$.

Solution

$$y\,dx+x\,dy+v\,du+u\,dv=0\ \ldots\ldots\ (i)$$
$$u\,dx+v\,dy+x\,du+y\,dv=0\ \ldots\ldots\ (ii)$$

Eliminating dv between (i) & (ii)

$$\left(y^2-u^2\right)dx+\left(xy-uv\right)dy+\left(vy-ux\right)du=0$$

$$\Rightarrow\ du=\frac{u^2-y^2}{vy-ux}dx+\frac{uv-xy}{vy-ux}dy$$

$$\Rightarrow\ \frac{\partial u}{\partial x}=\frac{u^2-y^2}{vy-ux}=\frac{u^2-y^2}{1-2ux}\qquad(\text{ using given eq. })$$

$$\Rightarrow\ \frac{\partial^2 u}{\partial x^2}=\frac{(1-2ux)\cdot 2u\cdot\frac{\partial u}{\partial x}-(u^2-y^2)\left[(-2u)-2x\frac{\partial u}{\partial x}\right]}{(1-2ux)^2}$$

❖ ***Question***

Find $\dfrac{\partial^2 w}{\partial x^2}$, $\dfrac{\partial^2 w}{\partial y^2}$ when

i) $w=\dfrac{1}{\sqrt{x^2+y^2}}$

ii) $w=\tan^{-1}\dfrac{y}{x}$

iii) $w=e^{x^2-y^2}$

❖ ***Question***

Show that the following functions are harmonic in x & y

i) $e^x\cos y$

ii) x^3-3xy^2

iii) $\log\sqrt{x^2+y^2}$

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❖ *Sufficient Condition for the Validity of Reversal in the Order of Derivation*

We now prove two theorems which lay sufficient conditions for the equality of f_{xy} and f_{yx}.

❖ *Schawarz's Theorem*

If (a,b) be a point of the domain of a function $f(x,y)$ such that

i) $f_x(x,y)$ exists in a certain nhood of (a,b).

ii) $f_{xy}(x,y)$ is continuous at (a,b).

then $f_{yx}(a,b)$ exists and is equal to $f_{xy}(a,b)$.

Proof

The given conditions imply that there exists a certain nhood of (a,b) at every point (x,y) of which $f_x(x,y)$, $f_y(x,y)$ and $f_{xy}(x,y)$ exist. Let $(a+h,b+k)$ be any point of this nhood. We write

$$\phi(h,k)=f(a+h,b+k)-f(a+h,b)-f(a,b+k)+f(a,b)$$

$$g(y)=f(a+h,y)-f(a,y)$$

so that $\phi(h,k)=g(b+k)-g(b)$ .... (i)

$\because$ f_y exists in a nhood of (a,b), the function $g(y)$ in derivable in $[b,b+k]$, and, therefore, by applying the M.V. theorem to the expression on R.H.S of (i), we have

$$\phi(h,k)=kg'(b+\theta k) \qquad (0<\theta<1)$$

$$=k\left(f_y(a+h,b+\theta k)-f_y(a,b+\theta k)\right) \text{ .... } (ii)$$

Again since f_{xy} exists in a nhood of (a,b), the function $f_y(x,b+\theta k)$ of x is derivable w.r.t. x in interval $(a,a+h)$ and, therefore, by applying the M.V. theorem to the right of (ii), we have

$$\phi(h,k)=hk\,f_{xy}(a+\theta' h,b+\theta k) \qquad (0<\theta'<1)$$

or $$\frac{1}{k}\left(\frac{f(a+h,b+k)-f(a,b+k)}{h}-\frac{f(a+h,b)-f(a,b)}{h}\right)=f_{xy}(a+\theta' h,b+\theta k)$$

Since $f_x(x,y)$ exists in a nhood of (a,b), this gives when $h\to 0$,

$$\frac{f_x(a,b+k)-f_x(a,b)}{k}=\lim_{h\to 0} f_{xy}(a+\theta' h,b+\theta k)$$

Let, now, $k\to 0$. Since $f_{xy}(x,y)$ is continuous at (a,b), we obtain

$$f_{yx}(a,b)=\lim_{k\to 0}\lim_{h\to 0} f_{xy}(a+\theta' h,b+\theta k)=f_{xy}(a,b)$$

❖ *Young's Theorem*

If (a,b) be a point of the domain of definition of a function $f(x,y)$ such that $f_x(x,y)$ and $f_y(x,y)$ are both differentiable at (a,b), then

$$f_{xy}(a,b)=f_{yx}(a,b)$$

Proof

The differentiability of f_x and f_y at (a,b) implies that they exist in a certain nhood of (a,b) and that f_{xx}, f_{yx}, f_{xy}, f_{yy} exist at (a,b).

Let $(a+h,b+h)$ be a point of this nhood. We write

$$\phi(h,h)=f(a+h,b+h)-f(a+h,b)-f(a,b+h)+f(a,b)$$

$$g(y)=f(a+h,y)-f(a,y)$$

so that $\phi(h,h)=g(b+h)-g(b)$ .... (i)

Since f_y exists in a nhood of (a,b), the function $g(y)$ is derivable in $(b,b+h)$, and, therefore, by applying the M.V. theorem to the expression on the right of (i), we have

$$\phi(h,h) = h\,g'(b+\theta h) \qquad (0<\theta<1)$$
$$= h\big(f_y(a+h,b+\theta h) - f_y(a,b+\theta h)\big) \text{ .... } (ii)$$

Since $f_y(x,y)$ is differentiable at (a,b), we have, by definition,

$$f_y(a+h,b+\theta h) - f_y(a,b) = h\,f_{xy}(a,b) + \theta h\,f_{yy}(a,b)$$
$$+h\varphi_1(h,h) + \theta h\psi_1(h,h) \text{ ...... } (iii)$$

and $f_y(a,b+\theta h) - f_y(a,b) = \theta h\,f_{yy}(a,b) + \theta h\psi_2(h,h)$ .... (iv)

where φ_1, ψ_1, ψ_2 all $\to 0$ as $h \to 0$

From (ii), (iii) and (iv), we obtain

$$\frac{\phi(h,h)}{h^2} = f_{xy}(a,b) + \phi_1(h,h) + \theta\psi_1(h,h) - \theta\psi_2(h,h) \text{ .... } (v)$$

By a similar argument and on considering

$g(x) = f(x,b+k) - f(x,b)$

We can show that

$$\frac{\phi(h,h)}{h^2} = f_{yx}(a,b) + \psi_3(h,h) + \theta'\varphi_2(h,h) - \theta'\varphi_3(h,h) \text{ .... } (vi)$$

where φ_2, φ_3, ψ_3 all $\to 0$ as $h \to 0$

Equating the right hand side of (v) and (vi) and making $h \to 0$, we obtain

$$f_{xy}(a,b) = f_{yx}(a,b)$$

....

❖ *Maxima and Minima for Functions of Two Variables*

Let (x_0, y_0) be the point of the domain of a function $f(x,y)$, then $f(x_0,y_0)$ said to an extreme value of the function $f(x,y)$, if the expression

$$\Delta f = f(x_0+h, y_0+h) - f(x_0,y_0)$$

preserves its sign for all h and k.

The extreme value of $f(x_0,y_0)$ being called a maximum or a minimum value according as this difference is positive or negative respectively.

Necessary Condition

The Necessary Condition for $f(x_0,y_0)$ to be an extreme value of function $f(x,y)$ is that $f_x(x_0,y_0) = 0 = f_y(x_0,y_0)$, provided that these partial derivatives exist.

It is to be noted that it is impossible to determine the nature of a critical point by studying the function $f(x,y_0)$ and $f(x_0,y)$.

e.g. Let $f(x,y) = 1 + x^2 - y^2$

then $f(0,y) = 1 - y^2 \quad \Rightarrow \quad f'(0,y) = -2y = 0 \quad \Rightarrow \quad (0,0)$ is a turning point.

Now $f''(0,y) = -2 \quad \Rightarrow \quad (0,0)$ is a point of maximum value.

But $f(x,0) = 1 + x^2$

$\Rightarrow \quad f'(x,0) = 2x = 0 \quad \Rightarrow \quad x = 0 \quad \Rightarrow \quad (0,0)$ is the critical point

$\Rightarrow \quad f''(x,0) = 2 > 0 \quad \Rightarrow \quad (0,0)$ is the maximum value

Hence we fail to decide the nature of the critical point in this way.

Sufficient Condition

Let $z = f(x,y)$ be defined and have continuous 1st and 2nd order partial derivatives in a domain D. Suppose (x_0,y_0) is a point of D for which f_x and f_y are both zero.

Let $A = f_{xx}(x_0,y_0)$, $B = f_{xy}(x_0,y_0)$, $C = f_{yy}(x_0,y_0)$,

then we have the following cases

i) $B^2 - AC < 0$ and $A + C < 0 \quad \Rightarrow$ relative maximum at (x_0,y_0).

ii) $B^2 - AC < 0$ and $A + C > 0 \quad \Rightarrow$ relative minimum at (x_0,y_0)

iii) $B^2 - AC > 0 \quad \Rightarrow$ saddle point at (x_0,y_0)

iv) $B^2 - AC = 0 \quad \Rightarrow$ nature of the critical point is undetermined

Proof

By the application of M.V. theorem for function of two variables we have

$$\Delta f = hf_x(x_0+\theta h, y_0+\theta k) + kf_y(x_0+\theta h, y_0+\theta k) \qquad (0<\theta<1)$$

$$= h\left[f_x(x_0+\theta h, y_0+\theta k) - f_x(x_0,y_0)\right] + k\left[f_y(x_0+\theta h, y_0+\theta k) - f_y(x_0,y_0)\right]$$

(it is because $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$, a turning point)

$$= h\left[\theta h f_{xx}(x_0,y_0) + \theta k f_{yx}(x_0,y_0) + \varepsilon_1\theta h + \varepsilon_2\theta k\right]$$

$$+k\left[\theta h f_{xy}(x_0,y_0) + \theta k f_{yy}(x_0,y_0) + \varepsilon_3\theta h + \varepsilon_4\theta k\right]$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ & $\varepsilon_4 \to 0$ as $h,k \to 0$

$$\Delta f = h^2 f_{xx}(x_0,y_0) + 2hk\, f_{xy}(x_0,y_0) + k^2 f_{yy}(x_0,y_0) + \varepsilon_1 h^2 + (\varepsilon_2+\varepsilon_3)hk + \varepsilon_4 k^2$$

$$\Rightarrow \quad \Delta f = h^2 A + 2hkB + k^2 C + \varepsilon_1 h^2 + (\varepsilon_2+\varepsilon_3)hk + \varepsilon_4 k^2$$

The sign of Δf depends upon the quadratic $d^2 f = h^2 A + 2hk B + k^2 C$

i & ii) Let $B^2 - AC < 0$, $(A \neq 0)$

$$\Rightarrow \quad d^2 f = \frac{1}{A}\left(h^2 A + 2hk\, AB + k^2 AC\right)$$

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$$=\frac{1}{A}\left(h^2A^2+2hk\,AB+k^2B^2+(k^2AC-k^2B^2)\right)$$

$$=\frac{1}{A}\left((hA+kB)^2+k^2(AC-B^2)\right)$$

Since $(hA+kB)^2$ is positive and $AC-B^2$ (supposed) is +ive, therefore the sign of d^2f depends upon the sign of A.

$\Rightarrow\ \Delta f>0$ if $A>0$ & $\Delta f<0$ if $A<0$

Again, since $B^2-AC<0 \Rightarrow B^2<AC \Rightarrow AC>0$

$\Rightarrow$ A and C are either both +ive or both −ive.

If $A>0$, $C>0$ then $A+C>0$ and if $A<0$, $C<0$ then $A+C<0$.

Hence we have the following result

a) $\Delta f>0$ when $A+C>0 \Rightarrow (x_0,y_0)$ is a point of minimum value.

b) $\Delta f<0$ when $A+C<0 \Rightarrow (x_0,y_0)$ is a point of maximum value.

iii) Let $B^2-AC>0$, then

$$d^2f=\frac{1}{A}\left((hA+kB)^2+k^2(AC-B^2)\right)$$

$$=\frac{1}{A}\left((hA+kB)^2-k^2(B^2-AC)\right)$$

which may be +ive or −ive for certain value of h & k, therefore (x_0,y_0) is a saddle point.

iv) Let $B^2-AC=0$, $A\neq 0$

$$\Rightarrow d^2f=\frac{1}{A}(hA+kB)^2$$

which may vanish for certain values of h and k, implies that nature of the point remain undetermined.

❖ Question

Test for maxima and minima

$$z=1-x^2-y^2$$

Solution

$$\frac{\partial z}{\partial x}=-2x=0 \Rightarrow x=0$$

$$\frac{\partial z}{\partial y}=-2y=0 \Rightarrow y=0$$

$\Rightarrow$ $(0,0)$ is the only critical point.

$$A=\frac{\partial^2 z}{\partial x^2}=-2\ ,\ B=\frac{\partial^2 z}{\partial x\partial y}=0\ ,\ C=\frac{\partial^2 z}{\partial y^2}=-2$$

$B^2-AC=0-4=-4<0$ and $A+C=-2-2=-4<0$

$\Rightarrow$ the function has maximum value at $(0,0)$.

❖ Question

Test for maxima and minima

$$z=x^3-3xy^2$$

Solution

$$\frac{\partial z}{\partial y}=3x^2-3y^2=0 \Rightarrow x=-y\ \&\ x=y$$

$$\frac{\partial z}{\partial y}=-6xy=0 \Rightarrow xy=0$$

$\Rightarrow$ $(0,0)$ is the critical point.

$$A=\frac{\partial^2 z}{\partial x^2}=6x=0 \quad \text{at } (0,0)$$

$$B=\frac{\partial^2 z}{\partial x\partial y}=-6y=0 \quad \text{at } (0,0)$$

$$C=\frac{\partial^2 z}{\partial y^2}=-6x=0 \quad \text{at } (0,0)$$

$B^2-4AC=0$ also $A+C=0$

Therefore we need further consideration for the nature of point

$$\begin{aligned}\Delta z &= z(0+h,0+k)-z(0,0)\\ &= z(h,k)-z(0,0)\\ &= h^3-2hk^2\end{aligned}$$

For $h=k$

$$\Delta z = h^3-3h^3=-2h^3$$

$$\Rightarrow \Delta z>0 \text{ if } h<0 \quad \& \quad \Delta z<0 \text{ if } h>0$$

Hence $(0,0)$ is a saddle point.

❖ ***Question***

Examine the function

$$z=f(x,y)=x^2y^2$$

Solution

$f_x=0 \Rightarrow 2xy^2=0$

$f_y=0 \Rightarrow 2yx^2=0$

implies that $(0,0)$ is the critical point

$A=f_{xx}=2y^2=0$ at $(0,0)$

$B=f_{xy}=-4xy=0$ at $(0,0)$

$C=f_{yy}=2x^2=0$ at $(0,0)$

Since $B^2-4AC=0$ and also $A+C=0$

Therefore we need further consideration for the nature of point.

$$\begin{aligned}\Delta f &= f(h,k)-f(0,0)\\ &= h^2k^2\end{aligned}$$

$$\Delta f>0 \quad \text{for all } h \;\&\; k$$

Hence $(0,0)$ is the point where function has minimum value.

………………………………………..

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❖ *Lagrange's Multiplier*
(Maxima & Minima for Function with Side Condition)

A problem of considerable importance for application is that of maximizing and minimizing of function (optimization) of several variables where the variables are related by one or more equations, which are turned as side condition. e.g. the problem of finding the radius of largest sphere inscribable in the ellipsoid $x^2+2y^2+3z^2=6$ is equivalent to minimizing the function $w=x^2+y^2+z^2$ with the side condition $x^2+2y^2+z^2=6$.

To handle such problem, we can, if possible, eliminate some of the variables by using the side conditions and reduce the problem to an ordinary maximum and minimum problem such as that consider previously.
This procedure is not always feasible and following procedure often is more convenient which treat the variable in more symmetrical manner, so that various simplifications may be possible.

Consider the problem of finding the extreme values of the function $f(x_1,x_2,\dots,x_n)$ when the variable are restricted by a certain number of side conditions say

$$\begin{aligned} g_1(x_1,x_2,\dots,x_n) &= 0.\\ g_2(x_1,x_2,\dots,x_n) &= 0\\ &\dots\dots\dots\\ &\dots\dots\dots\\ &\dots\dots\dots\\ g_m(x_1,x_2,\dots,x_n) &= 0 \end{aligned}$$

We then form the linear combination

$$\varphi(x_1,\dots,x_n)=f(x_1,\dots,x_n)+\lambda_1 g_1(x_1,\dots,x_n)+\lambda_2 g_2(x_2,\dots,x_n)+\dots\dots+\lambda_m g_m(x_1,\dots,x_n)$$

where $\lambda_1,\lambda_2,\dots,\lambda_m$ are m constants.

We then differentiate φ w.r.t. each coordinate and consider the following system of $n+m$ equations.

$$D_r\varphi(x_1,x_2,\dots,x_n)=0\quad,\quad r=1,2,\dots,n$$
$$g_k(x_1,x_2,\dots,x_n)=0\quad,\quad k=1,2,\dots,m$$

Lagrange discovered that if the point $(x_1,x_2,\dots,x_n)$ is a solution of the extreme problem then it will also satisfy the system of $n+m$ equation.

In practise, we attempt to solve this system for $n+m$ unknowns, which are $\lambda_1,\lambda_2,\dots,\lambda_m$ & $x_1,x_2,\dots,x_n$

The point so obtain must then be tested to determine whether they yield a maximum, a minimum or neither.

The numbers $\lambda_1,\lambda_2,\dots,\lambda_m$, which are introduced only to help to solve the system for $x_1,x_2,\dots,x_n$ are known as Lagrange's multiplier. One multiplier is introduced for each side condition.

❖ *Question*

Find the critical points of $w=xyz$, subject to condition $x^2+y^2+z^2=1$.

Solution

We form the function

$$\varphi=xyz+\lambda(x^2+y^2+z^2-1)$$

then

$$\frac{\partial\varphi}{\partial x}=yz+2\lambda x=0$$
$$\frac{\partial\varphi}{\partial y}=xz+2\lambda y=0$$

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$$\frac{\partial\varphi}{\partial z} = xy + 2\lambda z = 0$$

$$\& \quad x^2 + y^2 + z^2 - 1 = 0$$

Multiplying the first three equations by x, y & z respectively, adding and using the fourth equation, we find

$$\lambda = -\frac{3xyz}{2}$$

using this relation we find that $(0,0,\pm1)$, $(0,\pm1,0)$, $(\pm1,0,0)$ and $\left(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\right)$ are the critical points.

❖ ***Question***

Find the critical points of $w = xyz$, where $x^2 + y^2 = 1$ & $x - z = 0$. Also test for maxima and minima.

Solution

Consider $F = xyz + \lambda_1\left(x^2 + y^2 + 1\right) + \lambda_2(x - z)$

For the critical points, we have

$$F_x = yz + 2\lambda_1 x + \lambda_2 = 0 \quad \ldots\ldots\ldots\ (i)$$

$$F_y = xz + 2\lambda_1 y = 0 \quad \ldots\ldots\ldots\ (ii)$$

$$F_z = xy - \lambda_2 = 0 \quad \ldots\ldots\ldots\ (iii)$$

and

$$x^2 + y^2 = 1 \quad \ldots\ldots\ldots\ (iv)$$

$$x - z = 0 \quad \ldots\ldots\ldots\ (v)$$

From (iii), $\lambda_2 = xy$ & from (ii) $\lambda_1 = -\frac{xz}{2y}$

Use these values in equation (i) to have

$$yz - \frac{x^2 z}{y} + xy = 0$$

$$\Rightarrow\ y^2 z - x^2 z + xy^2 = 0$$

$\because\ x = z$ from (v)

$\therefore\ y^2 x - x^3 + xy^2 = 0 \quad \Rightarrow\ 2xy^2 - x^3 = 0$

But $y^2 = 1 - x^2$, from (iv)

$\therefore\ 2x\left(1 - x^2\right) - x^3 = 0 \quad \Rightarrow\ 2x - 3x^3 = 0 \quad \Rightarrow\ x = 0\ ,\ \pm\sqrt{\frac{2}{3}}$

This implies the critical points are $\left(\pm\sqrt{\frac{2}{3}},\frac{1}{\sqrt{3}},\pm\sqrt{\frac{2}{3}}\right)$, $\left(\pm\sqrt{\frac{2}{3}},-\frac{1}{\sqrt{3}},\pm\sqrt{\frac{2}{3}}\right)$, $(0,1,0)$, $(0,-1,0)$

$$A = F_{xx} = 2\lambda_1$$

$$B = F_{xy} = z$$

$$C = F_{yy} = 2\lambda_1$$

$$B^2 - AC = z^2 - 4\lambda_1^2$$

$$= z^2 - 4\frac{x^2 z^2}{4y^2} = \frac{z^2\left(y^2 - x^2\right)}{y^2}$$

(i) At $\left(\pm\sqrt{\frac{2}{3}},\frac{1}{\sqrt{3}},\pm\sqrt{\frac{2}{3}}\right)$, we have

$$B^2 - AC = \frac{\frac{2}{3}\left(\frac{1}{3} - \frac{2}{3}\right)}{1/3} < 0$$

$$\& \quad A = F_{xx} = 2\lambda_1 = -\frac{xz}{y} = -\left(\frac{2/3}{1/\sqrt{3}}\right) < 0$$

$\Rightarrow$ function has maximum value at $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right)$

Similarly, we can show that F is also maximum at $(0,-1,0)$ and is minimum at remaining points. (Check yourself)

❖ Question

Find the point of the curve

$$x^2 - xy + y^2 - z^2 = 1 \ , \ x^2 + y^2 = 1$$

which is nearest to the origin.

Solution

Let a point on a given curve be (x,y,z)

Implies that we are to minimize the function

$$f = d^2 = x^2 + y^2 + z^2$$

subject to the conditions

$$x^2 - xy + y^2 - z^2 = 1$$
$$x^2 + y^2 = 1$$

Consider

$$F = x^2 + y^2 + z^2 + \lambda_1\left(x^2 - xy + y^2 - z^2 - 1\right) + \lambda_2\left(x^2 + y^2 - 1\right)$$

For the critical points

$$F_x = 2x(1 + \lambda_1 + \lambda_2) - \lambda_1 y = 0 \ \ldots\ldots\ldots\ldots \ (i)$$
$$F_y = 2y(1 + \lambda_1 + \lambda_2) - \lambda_1 x = 0 \ \ldots\ldots\ldots\ldots \ (ii)$$
$$F_z = 2z(1 - \lambda_1) = 0 \ \ldots\ldots\ldots\ldots \ (iii)$$
$$x^2 - xy + y^2 - z^2 = 1 \ \ldots\ldots\ldots\ldots \ (iv)$$
$$x^2 + y^2 = 1 \ \ldots\ldots\ldots\ldots \ (v)$$

From equation (iii), we have

$$z = 0 \ \text{ and } \ \lambda_1 = 1$$

Put $z = 0$ in equation (iv), gives

$$x^2 - xy + y^2 - 1 = 0$$
$$\Rightarrow xy = x^2 + y^2 - 1$$
$$\Rightarrow xy = 0 \ \text{ by } \ (v)$$
$$\Rightarrow x = 0 \ \text{ or } \ y = 0 \ \text{ or both are zero.}$$

$z = 0$, $x = 0$ in (v) gives, $y^2 = 1 \quad \Rightarrow \quad y = \pm 1$

$\Rightarrow$ $(0,\pm 1,0)$ are the critical points.

$z = 0$, $y = 0 \quad \Rightarrow \quad x = \pm 1 \quad \Rightarrow \quad (\pm 1,0,0)$ are the critical points.

We can not take $x = 0$, $y = 0$ at the same time, because it gives $(0,0,0)$ which is origin itself as a critical point.

$\because$ $d^2 = 1$ at all these four points.

$\therefore$ these are the required point at which function is nearest to origin.

❖ ***Question***

Find the point on the curve

$$x^2 + y^2 + z^2 = 1$$

which is farthest from the point $(1,2,3)$

Solution

We are the maximize the function

$$f = (x-1)^2 + (y-2)^2 + (z-3)^2$$

subject to the condition

$$x^2 + y^2 + z^2 = 1$$

Let

$$F = (x-1)^2 + (y-2)^2 + (z-3)^2 + \lambda\left(x^2 + y^2 + z^2 - 1\right)$$

For the critical points, we have

$$x - 1 + \lambda x = 0 \quad \ldots\ldots\ldots\ldots \; (i)$$

$$y - 2 + \lambda y = 0 \quad \ldots\ldots\ldots\ldots \; (ii)$$

$$z - 3 + \lambda z = 0 \quad \ldots\ldots\ldots\ldots \; (iii)$$

$$\& \quad x^2 + y^2 + z^2 = 1 \quad \ldots\ldots\ldots\ldots \; (iv)$$

$$\Rightarrow \; x = \frac{1}{1+\lambda}, \quad y = \frac{2}{1+\lambda}, \quad z = \frac{1}{3+\lambda}$$

Putting in (iv)

$$\left(\frac{1}{1+\lambda}\right)^2 (1+4+9) = 1 \quad \Rightarrow \quad (1+\lambda)^2 = 14 \quad \Rightarrow \quad \lambda = -1 \pm \sqrt{14}$$

$$\Rightarrow \; x = \frac{1}{\pm\sqrt{14}}, \quad y = \frac{2}{\pm\sqrt{14}}, \quad z = \frac{3}{\pm\sqrt{14}}$$

$\Rightarrow$ critical points are

$$\left(\pm\frac{1}{\sqrt{14}}, \pm\frac{2}{\sqrt{14}}, \pm\frac{3}{\sqrt{14}}\right)$$

Its clear that the required point which is farthest from the point $(1,2,3)$ is

$$\left(-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right)$$

....

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❖ Directional Derivative

i) Let $f:V\to\mathbb{R}$, where $V\subset\mathbb{R}^n$, is nhood of $\underline{a}\in\mathbb{R}^n$. Then the directional derivative $D_\beta f$ at $\underline{a}$ in the direction of $\underline{\beta}\in\mathbb{R}^n$, is defined by the limit, if it exists,

$$D_\beta f(\underline{a})=\lim_{h\to 0}\frac{f(\underline{a}+h\underline{\beta})-f(\underline{a})}{h}$$

ii) The directional derivative of $f(x_1,x_2,\dots,x_i,\dots,x_n)$ at $\underline{a}=(a_1,a_2,\dots,a_i,\dots,a_n)$ in the direction of the unit vector $(0,0,\dots,1,0,0,\dots,0)$ is called partial derivative of f at $\underline{a}$ w.r.t. the ith component x_i and is denoted by

$$D_if(\underline{a})\quad\text{or}\quad\frac{\partial f(\underline{a})}{\partial x}\quad\text{or}\quad f_{x_i}(a)$$

where $D_if(\underline{a})=\lim\limits_{h\to 0}\dfrac{f(a_1,a_2,\dots,a_i+h,\dots,a_n)-f(a_1,a_2,\dots,a_i,\dots,a_n)}{h}$

❖ Example

Let $f(x,y)=x^2+y^2+x+y$, then f has a directional derivative in every direction and at every point in $\mathbb{R}^2$.

Since, if $\beta=(a,b)\in\mathbb{R}^2$, we have

$$\begin{aligned}D_\beta f(x,y)&=\lim_{h\to 0}\frac{(x+ha)^2+(y+hb)^2+(x+ha)+(y+hb)-x^2-y^2-x-y}{h}\\&=\lim_{h\to 0}\left(2ax+2by+ha^2+hb^2+a+b\right)\\&=2(ax+by)+a+b\end{aligned}$$

❖ Exercise

Let $f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^4+y^4} & ;\quad x^4+y^4\neq 0\\ 0 & ;\quad (x,y)\neq(0,0)\end{cases}$

Note that if $\beta=(a,b)\in\mathbb{R}^2$,

$$\begin{aligned}D_\beta f(0,0)&=\lim_{h\to 0}\frac{(0+ah)(0+bh)\left[(0+ah)^2-(0+bh)^2\right]}{h\left[(0+ah)^4+(0+bh)^4\right]}\\&=\lim_{h\to 0}\frac{ab(a^2-b^2)}{h(a^4+b^4)}\end{aligned}$$

This limit obviously exists only if $\beta=(1,0)$ or $(0,1)$. Hence the directional derivatives of f at $(0,0)$ that exists are the partial derivatives f_x and f_y given by $f_x=0$, $f_y=0$.

❖ Example

Let

$$f(x,y)=\begin{cases}\dfrac{xy^2}{x^4+y^4} & ;\quad (x,y)=(0,0)\\ 0 & ;\quad (x,y)\neq(0,0)\end{cases}$$

It is discontinuous at $(0,0)$. To see it, note that

$$\lim_{(x,y)\to(0,0)}f(x,y)\ \text{ is zero along }\ y=0\ \text{ and is }\ \frac{1}{2}\ \text{ along }\ y^2=x.$$

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However, if $\beta=(a,b)$, then

$$f_\beta(0,0)=\lim_{h\to 0}\frac{(0+ah)(0+bh)^2}{h\left[(0+ah)^2+(0+bh)^4\right]}$$

$$=\lim_{h\to 0}\frac{ah\cdot b^2h^2}{h\left[a^2h^2+b^4h^4\right]}=\lim_{h\to 0}\frac{ab^2}{a^2+h^2b^4}$$

$$=\begin{cases} b^2/a & , a\neq 0 \\ 0 & , a=0\end{cases}$$

Hence the directional derivative of f at $(0,0)$ exists in every direction.

❖ Question

Let $z=f(x,y)$, $x=u^2-v^2$, $y=2uv$. Then show that

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\frac{1}{4\left(u^2+v^2\right)}\left\{\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right\}$$

Solution

We have

$$\frac{\partial x}{\partial u}=2u \quad , \quad \frac{\partial x}{\partial v}=-2v \quad , \quad \frac{\partial y}{\partial u}=2v \quad , \quad \frac{\partial y}{\partial v}=2v$$

Also

$$1=2u\frac{\partial u}{\partial x}-2v\frac{\partial v}{\partial x} \quad , \quad 0=2u\frac{\partial u}{\partial y}-2v\frac{\partial v}{\partial y}$$

and $$0=2v\frac{\partial u}{\partial x}+2u\frac{\partial v}{\partial x} \quad , \quad 1=2v\frac{\partial u}{\partial y}+2u\frac{\partial v}{\partial y}$$

Solving these four equations for $\frac{\partial u}{\partial x},\frac{\partial v}{\partial x},\frac{\partial u}{\partial y}$ & $\frac{\partial v}{\partial y}$, we get

$$\frac{\partial u}{\partial x}=\frac{u}{2\left(u^2+v^2\right)} \quad , \quad \frac{\partial v}{\partial x}=\frac{-v}{2\left(u^2+v^2\right)}$$

$$\frac{\partial u}{\partial y}=\frac{v}{2\left(u^2+v^2\right)} \quad , \quad \frac{\partial v}{\partial y}=\frac{u}{2\left(u^2+v^2\right)}$$

And

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=\frac{1}{2\left(u^2+v^2\right)}\left[u\cdot\frac{\partial z}{\partial u}-v\cdot\frac{\partial z}{\partial v}\right]$$

& $$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=\frac{1}{2\left(u^2+v^2\right)}\left[v\cdot\frac{\partial z}{\partial u}+u\cdot\frac{\partial z}{\partial v}\right]$$

Hence

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\frac{1}{4\left(u^2+v^2\right)}\left\{\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right\}$$

....

❖ *Question*

Let $f:\mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y)=\begin{cases} \dfrac{xy}{x^2+y^2} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$$

Show that f_x, f_y exist at $(0,0)$ but f is discontinuous at $(0,0)$.

Solution

$$f_\beta(0,0)=\lim_{h\to 0}\frac{(ah)(bh)}{h\left[(ah)^2+(bh)^2\right]} \qquad \text{where } \beta=(a,b)$$

$$=\lim_{h\to 0}\frac{ab}{h(a^2+b^2)}$$

Which exists only when $\beta=(1,0)$ or $(0,1)$.

$\Rightarrow$ f_x & f_y exist at $(0,0)$

Now

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}$$

Let $y=mx$, then

$$\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}=\lim_{x\to 0}\frac{mx^2}{x^2+m^2x^2}$$

$$=\lim_{x\to 0}\frac{m}{1+m^2}$$

Which is different for different m.

$\Rightarrow$ $f(x,y)$ is discontinuous at $(0,0)$.

❖ *Question*

Let $f:\mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y)=\begin{cases} \dfrac{x^2y}{x^4+y^2} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$$

Show that f_x, f_y exist at $(0,0)$ but f is discontinuous at $(0,0)$.

Solution

$$f_\beta(0,0)=\lim_{h\to 0}\frac{(a^2h^2)(bh)}{h\left[a^4h^4+b^2h^2\right]} \quad , \quad \beta=(a,b)$$

$$=\lim_{h\to 0}\frac{a^2b}{a^4h^2+b^2}$$

$$=\begin{cases} \dfrac{a^2}{b} & , \ b\neq 0 \\ 0 & , \ b=0 \end{cases}$$

Now $\lim_{(x,y)\to(0,0)} f(x,y)$ is zero along $x=0$ and is $\frac{1}{2}$ along $y=x^2$

$\Rightarrow$ it is discontinuous at $(0,0)$.

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❖ *Question*

Find the greatest volume of the box contained in the ellipsoid $3x^2+2y^2+z^2=18$, when each of its edges is parallel to one of the coordinate axes.

Solution

$V=$ volume of the box $=(12x)(2y)(2z)=8xyz$

We need to find maximum of V subject to $3x^2+2y^2+z^2-18=0$

Consider $\varphi(x,y,z)=8xyz+\lambda(3x^2+2y^2+z^2-18)=0$

Then

$$\varphi_x=8yz+6\lambda x=0$$
$$\varphi_y=8xz+4\lambda y=0$$
$$\varphi_z=8xy+2\lambda z=0$$
$$\Rightarrow\ 4xyz+3\lambda x^2=0$$
$$2xyz+\lambda y^2=0$$
$$4xyz+\lambda z^2=0$$
$$\Rightarrow\ \lambda(3x^2-2y^2)=0$$
$$\lambda(3x^2-z^2)=0$$
$$\Rightarrow\ x^2=\frac{2y^2}{3}=\frac{z^2}{3}$$

Substituting these values in

$$3x^2+2y^2+z^2-18=0$$

We get

$$3x^2+3x^2+3x^2=18\quad\Rightarrow\ 9x^2=18$$
$$\Rightarrow\ x=\sqrt{2}\ ,\quad y=\sqrt{3}\ \text{ and }\ z=\sqrt{6}$$

Which gives

$$f(x,y,z)=8xyz=48$$

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❖ Definition

If $f:\mathbb{R}^n \to \mathbb{R}$, $\underline{a} \in \mathbb{R}^n$ then

$$\nabla f(\underline{a}) = \sum_{k=1}^{n} \frac{\partial f(\underline{a})}{\partial x_k} = \frac{\partial f(\underline{a})}{\partial x_1} + \frac{\partial f(\underline{a})}{\partial x_2} + \dots + \frac{\partial f(\underline{a})}{\partial x_n}$$

❖ Definition

Let $f:G \to \mathbb{R}$, G is an open set in $\mathbb{R}^n$.

i) f is said to have a local maximum at $\underline{a} \in G$, if there is a nhood $V_\varepsilon(\underline{a})$ such that $f(\underline{x}) \le f(\underline{a}) \ \forall \ \underline{x} \in V_\varepsilon$.

ii) f is said to have a local minimum at $\underline{a} \in G$, if there is a nhood $V_\varepsilon(\underline{a})$ such that $f(\underline{x}) \ge f(\underline{a}) \ \forall \ \underline{x} \in V_\varepsilon$.

❖ Theorem

Let $f:G \to \mathbb{R}$, G is an open set in $\mathbb{R}^n$. If f has a local extremum at $\underline{a} \in G$, then $\nabla f(\underline{a}) = 0$.

Proof

It is clear that $\nabla f(\underline{a}) = 0$ iff $\dfrac{\partial(\underline{a})}{\partial x_i} = 0$, $i = 1,2,3,\dots,n$

Write $f(x_i + t) = f(x_1, x_2, \dots, x_i + t, \dots, x_n) = f(\underline{x})$

If f has a local maximum at $\underline{a}$, then

$$\frac{f(a_i + t) - f(a_i)}{t} \le 0 \quad \text{if} \quad t > 0$$

$$\Rightarrow \lim_{t \to 0} \frac{f(a_i + t) - f(a_i)}{t} \le 0 \quad \text{if} \quad t > 0$$

So that $\dfrac{\partial f(\underline{a})}{\partial x_i} \le 0$

Similarly,

$$\lim_{t \to 0} \frac{f(a_i + t) - f(a_i)}{t} \ge 0 \quad \text{if} \quad t < 0$$

So that $\dfrac{\partial f(\underline{a})}{\partial x_i} \ge 0$

Hence $\dfrac{\partial f(\underline{a})}{\partial x_i} = 0$, $i = 1,2,3,\dots,n$

$$\Rightarrow \nabla f(\underline{a}) = 0$$

Note

There are situations when $\nabla f(\underline{a}) = 0$ but f has no local maximum or minimum at $\underline{a}$. If so and if the sign of $f(\underline{x}) - f(\underline{a})$ depends upon the direction of $\underline{x}$ and $\underline{a}$, f is said to have a saddle point at a.

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