Lecture 04

Limit of a Function:-

Let f be a function. If as x approaches "a" from both left and right sides of "a" f(x) approaches to a special number "L". Then "L" is called limit of f(x) as "x" approaches "a" written as

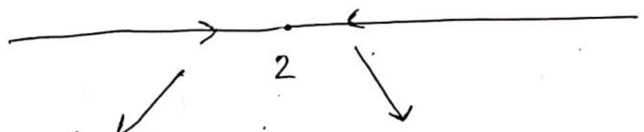
$$\lim_{x \to a} f(x) = L$$

Example:-

$$\lim_{x \to 2} f(x) = \frac{x^2-4}{x-2}$$

$f(1.999) = 3.99$

$f(2.0001) = 4.0001$



L.H. Lt $\qquad\qquad$ R.H. Lt

$$\lim_{x \to a^-} f(x) \qquad\qquad \lim_{x \to a^+} f(x)$$

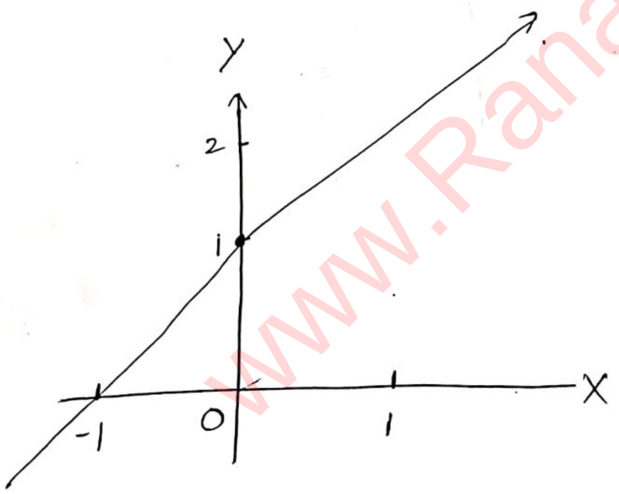
Example:- $f(x) = \dfrac{x^2-1}{x-1}$, how does function behave near $x=1$?

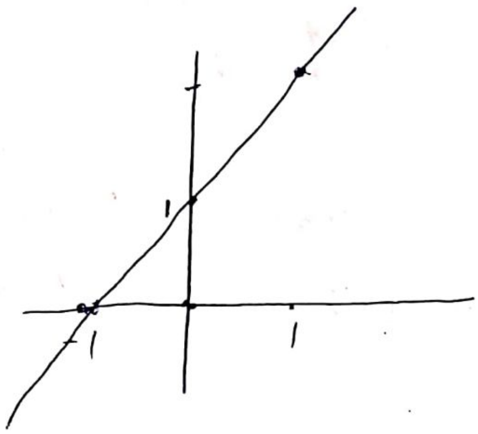
Solution:-

The function defines f for all real numbers x except $x=1$.

For $x \neq 1$, we can simplify the formula by factoring the numerator and canceling the common factor.

$$f(x) = \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}} = x+1 \quad \text{for } x \neq 1$$





$y = \dfrac{x^2-1}{x-1}$

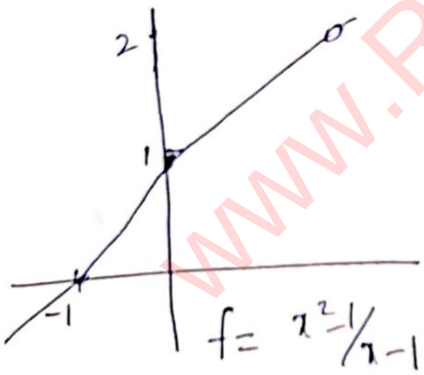
$y = x+1$

An Informal Description of the limit of a Function:-

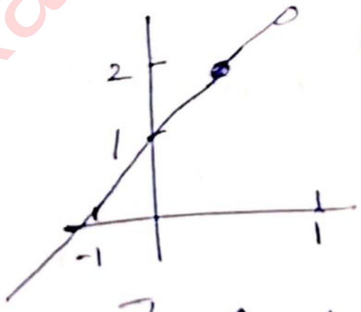
$$f(x) = \frac{x^2-1}{x-1}$$

x	$f(x) = x^2-1/x-1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

Example:-



$f = x^2-1/x-1$

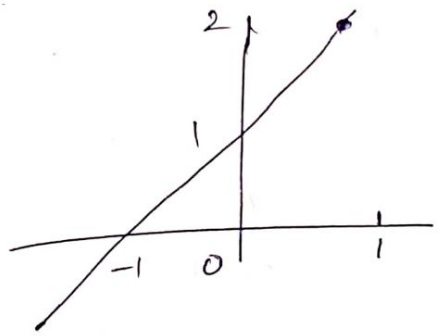


$$g(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 0 \\ 1 & x = 1 \end{cases}$$

The function f has limit 2 as $x \to 1$ even though f is not defined at $x = 1$

The function g has limit 2 as $x \to 1$ even though $2 \neq g(1)$

$h(x) = x+1$



The function h is only one of three functions whose limit as $x \to 1$ equals it value at $x=1$.

For h we have

$$\lim_{x \to 1} h(x) = h(1).$$

The equality of limit and function value has important meaning.

Example 3:- we find the limits of the identity function and of a constant function as x approaches $x=c$.

(a) If f is the identity function $f(x)=x$ then for any value of c

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c$$

(b) If f is the constant function $f(x) = k$ then for any value of c

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k.$$

* $\lim_{x \to 3} x = 3.$

* $\lim_{x \to -7} (4) = \lim_{x \to 2} (4) = 4.$ ⇒ whether c is -7 or 2, the result will be same.

* A function may not have limit at particular point

Example:- Discuss the behaviour of following functions, explain why they have no limit as $x \to 0$

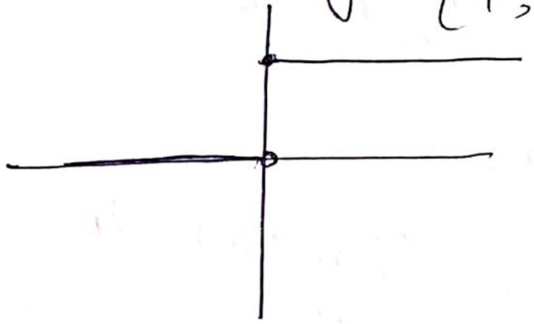
(a) $U(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} 1/x & x \neq 0 \\ 0 , & x = 0 \end{cases}$

(C) $\begin{cases} 0 & x \le 0 \\ \sin\frac{1}{x} & x > 0 \end{cases}$

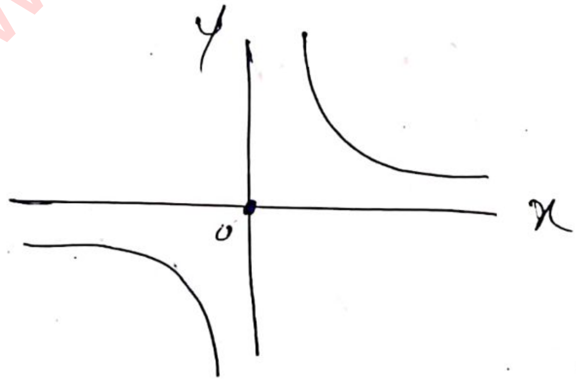
$$y = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

(a)



(a) The function jumps; the unit step function $U(x)$ has no limit as $x \to 0$ because its value jumps at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no single value L approached by $U(x)$ as $x \to 0$.

(b)



The function grows too "large" to have a limit.

$g(x)$ has no limit as $x \to 0$, because the value of g grow arbitrarily large in absolute value as $x \to 0$, therefor do not stay close to any real number. We say function is not bounded.

(C)

$$y = \begin{cases} 0 & , x \leq 0 \\ \sin \frac{1}{x} & x > 0 \end{cases}$$

The function oscillates too much to have a limit.

$f(x)$ has no limit as $x \to 0$ because the function's value oscillate between $+1$ and -1. in every open interval containing 0. The value do not stay close to any single number as $x \to 0$.

The Limit laws:-

Theorem 1- Limit Laws

If L, M, c and k are real numbers and

$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

1- Sum rule:

$$\lim_{x \to c} \left(f(x) + g(x) \right) = L + M$$

2- Difference Rule:

$$\lim_{x \to c} \left(f(x) - g(x) \right) = L - M$$

3- Constant Multiple Rule:

$$\lim_{x \to c} \left(k \cdot f(x) \right) = k \cdot L$$

4- Product Rule:

$$\lim_{x \to c} \left(f(x) \cdot g(x) \right) = L \cdot M$$

5. Quotient Rule:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$

6- Root Rule:-

$$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \; n \text{ is positive integer}$$

7- Power rule

$$\lim_{x \to c} [f(x)]^n = L^n, \; n \text{ a positive Integer}$$

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c)

Example:-

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3)$

$$= \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$$

$$= c^3 + 4c^2 - 3$$

(b) $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$

$$= \frac{\lim_{x \to c} (x^4 + x^2 - 1)}{\lim_{x \to c} (x^2 + 5)}$$

$$= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5}$$

$$= \frac{c^4 + c^2 - 1}{c^2 + 5}$$

(c) $\lim_{x \to -2} \sqrt{4x^2 - 3}$

$$= \sqrt{\lim_{x \to -2} (4x^2 - 3)}$$

$$= \sqrt{\lim_{x \to -2} (4x^2) - \lim_{x \to -2} (3)}$$

$$= \sqrt{4(-2)^2 - 3}$$

$$= \sqrt{4(4) - 3}$$

$$= \sqrt{16 - 3}$$

$$= \sqrt{13}$$

Theorem 2: Limits of Polynomials:

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

Theorem 3: Limits of rational Functions:

: If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

Example:-

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5}$$

$$= \frac{-1 + 4 - 3}{6} = 0$$

Eliminating common Factors from zero Denominators:-

Example.

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+2)}{x\cancel{(x-1)}}$$

$$= \lim_{x \to 1} \frac{x+2}{x}.$$

Now Applying limit

$$= \lim_{x \to 1} \frac{x+2}{x}$$

$$= 3.$$

using calculaters and computers to Estimate Limits:-

Example:-

$$\lim_{x \to 0} \frac{\sqrt{x^2+100} - 10}{x^2}$$

near $x = 0$.

x	$f(x)$
± 1	.049876
± 0.5	.049969
$\pm .1$	.049999
$\pm .01$	.050000
$\pm .0005$	0.050000
$\pm .0001$	0.000000
$\pm .00001$	0.000000
$\pm .000001$	0.000000

approaches 0.05? (for ± 1 to $\pm .01$)

approaches 0? (for $\pm .0005$ to $\pm .000001$)

Example:-

$$\lim_{x \to 0} \frac{\sqrt{x^2+100}-10}{x^2}$$

$$= \frac{\sqrt{x^2+100}-10}{x^2} \times \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$$= \frac{\cancel{x^2}+\cancel{100}-\cancel{100}}{\cancel{x^2}\left(\sqrt{x^2+100}+10\right)}$$

$$= \frac{1}{\sqrt{x^2+100}+10}$$

$$\lim_{x \to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x \to 0} \frac{1}{\sqrt{x^2+100}+10}$$

$$= \frac{1}{10+10}$$

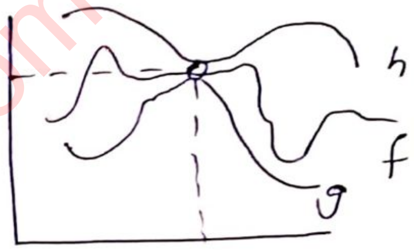
$$= \frac{1}{20} = 0.05$$

Theorem 4

The sandwich Theorem:

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x=c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$



Then $$\lim_{x \to c} f(x) = L$$

Squeeze or Pinching Theorem

Example:-

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \quad \text{for } x \ne 0$$

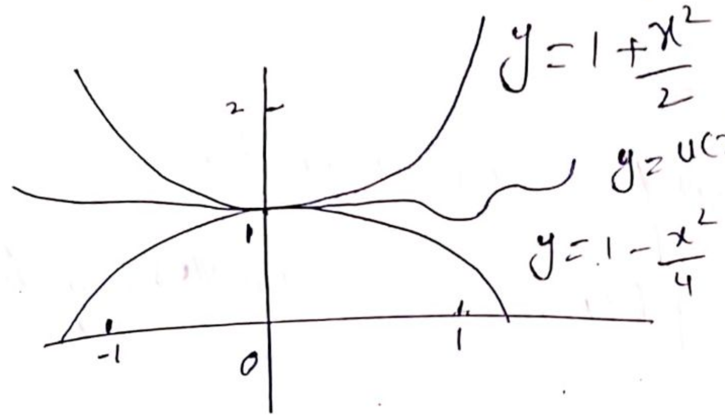
Find $\lim_{x \to 0} u(x) = ?$

Solution: Since

$$\lim_{x \to 0}\left(1 - \left(\frac{x^2}{4}\right)\right) = 1 \text{ and } \lim_{x \to 0}\left(1 + \left(\frac{x^2}{2}\right)\right) = 1$$

So by sandwich theorem

$$\lim_{x \to 0} u(x) = 1$$



Example:-

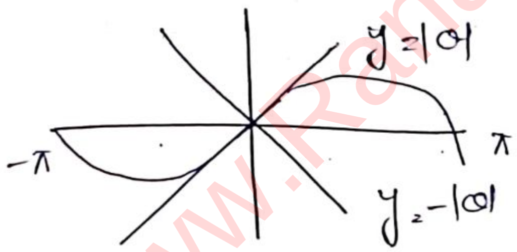
(a) $\lim_{\theta \to 0} \sin\theta = 0$

Solution

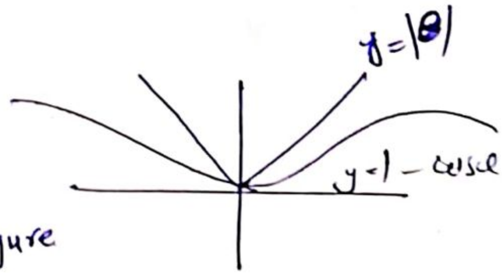
$-|\theta| \le \sin\theta \le |\theta|$ for all θ.

since $\lim_{\theta \to 0} (-|\theta|) = \lim_{\theta \to 0} |\theta| = 0$

So we have $\lim_{\theta \to 0} \sin\theta = 0$.



(b) $\lim_{\theta \to 0} \cos\theta = 1$



Sol:. We can write, from figure

$0 \le 1 - \cos\theta \le |\theta|$ for all θ.

$\lim_{\theta \to 0} 1 - (1 - \cos\theta) = 1 - \lim_{\theta \to} (1 - \cos\theta)$

$= 1 - 0$

$\lim_{\theta \to 0} \cos\theta = 1$

(C)

For any function f,

$\lim_{x \to c} |f(x)| = 0$ implies $\lim_{x \to 0} f(x) = 0$

solution

$-|f(x)| \le f(x) \le |f(x)|$

and $-|f(x)|$ and $|f(x)|$ have limits 0 as $x \to c$. it follows that

$\lim_{x \to c} f(x) = 0$