# The method of Trigonometric substitution:-

| Expression in the Integrand | Substitution | Restriction on $\theta$ | Simplification |
| --- | --- | --- | --- |
| $\sqrt{a^2 - x^2}$ | $x = a\sin\theta$ | $-\pi/2 \le \theta \le \pi/2$ | $a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta$. |
| $\sqrt{a^2 + x^2}$ | $x = a\tan\theta$ | $-\pi/2 \le \theta \le \pi/2$ | $a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$. |
| $\sqrt{x^2 - a^2}$ | $x = a\sec\theta$ | $\begin{cases} 0 \le \theta < \pi/2 & \text{if } x \geq a \\ \pi/2 < \theta \le \pi & \text{if } x \le -a \end{cases}$ | $x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$. |

Examples:-

(1)

$$\int \frac{dx}{x^2\sqrt{4-x^2}}$$

$x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$.

$$= \int \frac{dx}{x^2\sqrt{4-x^2}}$$

$$= \int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}$$

$$= \int \frac{2\cancel{\cos\theta}\, d\theta}{4\sin^2\theta \; 2\cancel{\cos\theta}}$$

$$= \frac{1}{4}\int \frac{1}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \operatorname{cosec}^2\theta\, d\theta$$

$$= \frac{1}{4}\cot\theta + C$$

$$\sin\theta = \frac{x}{2}$$

$$\cos\theta = \sqrt{1-\frac{x^2}{4}} = \frac{\sqrt{4-x^2}}{2}$$

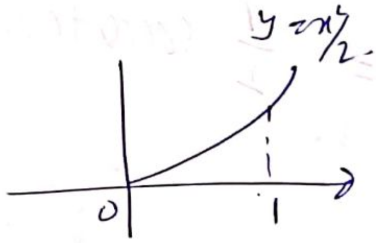
$$\tan\theta = \frac{\frac{x}{\cancel{2}}}{\frac{\sqrt{4-x^2}}{\cancel{2}}}$$

$$\tan\theta = \frac{x}{\sqrt{4-x^2}}$$

$$\cot\theta = \frac{\sqrt{4-x^2}}{x}$$

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \frac{1}{4}\,\frac{\sqrt{4-x^2}}{x} + C$$

(2) Find the arc length of the curve $y = x^2/2$ from $x=0$ to $x=1$




Sol:

$$L = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^1 \sqrt{1+x^2}\,dx$$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$.

$x = 0, \quad \tan^{-1}(0) = \theta.$

$\theta = 0$

$x = 1; \quad \theta = \tan^{-1}(1) = \pi/4$

$$= \int_0^{\pi/4} \sqrt{1+\tan^2\theta} \cdot \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \sqrt{\sec^2\theta} \cdot \sec^2\theta\, d\theta.$$

$$= \int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta.$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\sqrt{2} + \ln(\sqrt{2}+1)\right].$$

$= 1.148$

(3) Evaluate $\int \frac{\sqrt{x^2-25}}{x}\,dx \quad x \geq 5$ (3)

$x = 5\sec\theta.$

$\frac{dx}{d\theta} = 5\sec\theta\tan\theta$

$dx = 5\sec\theta\tan\theta\,d\theta.$

$= \int \frac{\sqrt{25\sec^2\theta - 25}}{5\cancel{x}\sec\theta}\,5\sec\theta\tan\theta\,d\theta.$

$= \int \frac{5\tan\theta}{\cancel{5\sec\theta}}\,5\cancel{\sec\theta}\tan\theta\,d\theta.$

$= 5\int \tan^2\theta\,d\theta.$

$= 5\int(\sec^2\theta - 1)\,d\theta.$

$= 5\tan\theta - 5\theta + c.$

$= 5\left(\frac{x^2-25}{5}\right) - 5\sec^{-1}\frac{x}{5} + c.$

# Integral Involving $ax^2+bx+c$ :-

Example:-

$$\int \frac{x}{x^2-4x+8}\,dx$$

$$= \int \frac{x}{(x-2)^2+4}\,dx$$

$u = x-2$

$du = \cancel{x\,dx}\ 1\,dx.$

$$= \int \frac{u+2}{u^2+4}\,du$$

$$= \int \frac{u}{u^2+4}\,du + 2\int \frac{du}{u^2+4}$$

$$= \frac{1}{2}\ln(u^2+4) + 2\,\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$