Lecture 27:-

Integrating Products of Sines and Cosines:-

$\int \sin^m x \cos^n x \, dx$	Procedure	Relevant Identities
n odd	• split factor of $\cos x$ • Apply the relevent identity • Make the substitution $u = \sin x$	$\cos^2 x = 1 - \sin^2 x$
m odd	• $\sin x$ • " • $u = \cos x$	$\sin^2 x = 1 - \cos^2 x$
m even, n even	• Use the relevant identities to reduce the powers on $\sin x$ and $\cos x$.	$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1 + \cos 2x}{2}$

Examples:-

(a) $\int \sin^4 x \cos^5 x \, dx$

$n=5$ is odd

$$\int \sin^4 x \cos^5 x\,dx = \int \sin^4 x \cos^4 x \cos x\,dx$$

$$= \int \sin^4 x \left(1-\sin^2 x\right)^2 \cos x\,dx.$$

$u = \sin x$.
$du = \cos x\,dx$.

$$= \int u^4 (1-u^2)^2\,du$$

$$= \int u^4 (1+u^4-2u^2)\,du.$$

$$= \int (u^4+u^8-2u^6)\,du$$

$$= \frac{u^5}{5}+\frac{u^9}{9}-\frac{2}{7}u^7+C.$$

$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C.$$

(b) $\int \sin^4 x \cos^4 x \, dx$

$n = m = 4$

$$= \int (\sin^2 x)^2 (\cos^2 x)^2 dx$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{16} \int (1-\cos^2 2x)^2 dx$$

$$= \frac{1}{16} \int (\sin^2 2x)^2 dx$$

$$= \frac{1}{16} \int \sin^4 2x \, dx$$

$u = 2x$

$du = 2dx$ or $dx = \frac{1}{2} du$

$$= \frac{1}{16} \frac{1}{2} \int \sin^4 u \, du$$

$$= \frac{1}{32} \int \sin^4 u \, du$$

As we know that

$$\begin{cases} \int \sin^4 x\,dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c \\ \int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c \end{cases}$$

So

$$= \frac{1}{32}\left[\frac{3}{8}u - \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u\right] + c$$

$$= \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + c$$

Eliminating Square Roots:-

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx.$$

Sol:-

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \quad \text{or} \quad 1+\cos 2\theta = 2\cos^2\theta.$$

$$\theta = 2x.$$

$$1+\cos 4x = 2\cos^2 2x.$$

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx$$

$$= \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx.$$

$$= \sqrt{2}\ \left.\frac{\sin 2x}{2}\right|_0^{\pi/4}$$

$$= \frac{\sqrt{2}}{2}(1-0)$$

$$= \frac{\sqrt{2}}{2}.$$

Integral Powers of $\tan x$ and $\sec x$:-

$$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx$$

$$\int \sec^n x\, dx = \frac{\sec^{n-2} \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} dx$$

$$\Rightarrow \int \tan^2 x\, dx = \tan x - x + C$$

$$\int \sec^2 x\, dx = \tan x + C.$$

Examples:-

(a) $\int \tan^4 x \, dx$.

$$= \int \tan^2 x \cdot \tan^2 x \, dx$$

$$= \int \tan^2 x \, (\sec^2 x - 1) \, dx$$

$$= \int (\tan^2 x \sec^2 x - \tan^2 x) \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx$$

$\therefore$ $u = \tan x$.

$du = \sec^2 x \, dx$

$$= \int u^2 \, du - \int du + \int dx$$

$$= \frac{u^3}{3} - u + x + C$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

Integrating Products of tangents and Secants:-

$\int \tan^m x \sec^n x\,dx$	Procedure	Relevant Identities
n even	* splits factor of $\sec^2 x$ * Apply relevant Identity * $u = \tan x$	$\sec^2 x = \tan^2 x + 1$
m odd	* $\sec x \tan x$ * // * $u = \sec x$	$\tan^2 x = \sec^2 x - 1$
m even, n odd	* use the relevant identities to reduces power of $\sec x$. * reduction formula for powers of $\sec x$	$\tan^2 x = \sec^2 x - 1$

Examples:-

(a)

$$\int \tan^2 x \sec^4 x\,dx$$

$$= \int \tan^2 x \sec^4 x\,dx$$

$$= \int \tan^2 x \sec^2 x \cdot \sec^2 x\,dx.$$

$$= \int \tan^2 x (\tan^2 x + 1) \sec^2 x \, dx.$$

$u = \tan x \quad du = \sec^2 x \, dx.$

$$= \int u^2 (u^2 + 1) \, du$$

$$= \int (u^4 + u^2) \, du.$$

$$= \frac{u^5}{5} + u^3/3 + c.$$

$$= \frac{1}{5} \tan^5 + \frac{1}{3} \tan^3 x + c.$$

(b) $\int \tan^2 x \sec x \, dx$

$$= \int (\sec^2 x - 1) \sec x \, dx$$

$$= \int (\sec^3 x - \sec x) \, dx$$

$$= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x) - \ln|\sec x + \tan x| + C$$

* $\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x - \ln|\sec x| + c$

* $\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + c$

Products of Sines and Cosines:

- $$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right].$$

$$\cos mx \cos mx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right].$$

Example:

$$\int \sin 3x \cos 5x \, dx$$

$m=3,\ n=5$

$$= \frac{1}{2}\int \left[\sin(-2x) + \sin(8x)\right] dx$$

$$= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx$$

$$= \frac{1}{2}\left[-\frac{\cos 8x}{8} + \frac{\cos 2x}{2}\right] + c$$

$$= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + c.$$