Lecture # 26.

1- $\int k\,dx = kx + c$

2- $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$

3- $\int \dfrac{dx}{x} = \ln|x| + c$

4- $\int e^x dx = e^x + c$

5- $\int a^x dx = \dfrac{a^x}{\ln a} + c, \quad a > 0, a \neq 1$

6- $\int \sin x\,dx = -\cos x + c$

7- $\int \cos x\,dx = \sin x + c$

8- $\int \sec^2 x\,dx = \tan x + c$

9- $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

10- $\int \sec x \tan x\,dx = \sec x + c$

11- $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

12- $\tan x\,dx = \ln|\sec x| + c$

13- $\int \cot x\,dx = \ln|\sin x| + c$

14- $\int \sec x = \ln|\sec x + \tan x| + c$

15- $\int \operatorname{cosec} x\,dx = -\ln|\operatorname{cosec} x + \cot x| + c$

16- $\int \sinh x\,dx = \cosh x + c$

17- $\int \cosh x\,dx = \sinh x + c$

18- $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + c$

19- $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + c$

20- $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{x}{a}\right| + c$

21- $\int \dfrac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\dfrac{x}{a}\right) + c$

22- $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\dfrac{x}{a}\right) + c.$

Techniques of Integration:-

Examples:- (1)
$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx$$

$$u = x^2-3x+1$$
$$du = (2x-3)\,dx.$$

$u=1$, when $x=3$

$u=11$, when $x=5$

$$= \int_1^{11} \frac{du}{\sqrt{u}}$$

$$= \int_1^{11} u^{-1/2}\,du.$$

$$= 2\sqrt{u}\Big|_1^{11}$$

$$= 2(\sqrt{11}-1) \approx 4.63$$

Examples :- (2)

$$\int (\cos x \sin 2x + \sin x \cos 2x)\,dx$$

$$= \int \sin(x+2x)\,dx$$

$$= \int \sin 3x\,dx$$

$$= -\frac{1}{3}\cos 3x + C.$$

(3)

$$\int_0^{\pi/4} \frac{dx}{1-\sin x}$$

$$= \int_0^{\pi/4} \frac{1}{1-\sin x} \times \frac{1+\sin x}{1+\sin x}\,dx$$

$$= \int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\,dx$$

$$= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\,dx$$

$$= \tan x + \sec x \Big|_0^{\pi/4}$$

$$= (1+\sqrt{2}) - (0+1)$$

$$= \sqrt{2}$$

(4)

$$\int \frac{3x^2-7x}{3x+2}\,dx.$$

$$\begin{array}{r|l} & x-3 \\ \hline 3x+2 & 3x^2-7x \\ & -(3x^2+2x) \\ \hline & -9x \\ & -(-9x-6) \\ \hline & 6 \end{array}$$

$$= \int x-3+\frac{6}{3x+2}\,dx.$$

$$= \frac{x^2}{2}-3x+2\ln(3x+2)+C.$$

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5.

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx$$

$$= 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}}$$

\# $u = \sqrt{1-x^2}$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$$3\frac{x\,dx}{\sqrt{1-x^2}} = 3\int \frac{-\frac{1}{2}du}{\sqrt{u}}$$

$$= -\frac{3}{2}\int u^{-1/2}du$$

$$= \frac{-3}{2}\cdot\frac{u^{1/2}}{1/2} + C_1$$

$$= -3\sqrt{1-x^2} + C_1.$$

$$\int \frac{3x+2}{\sqrt{1-x^2}}dx = -3\sqrt{1-x^2} + 2\sin^{-1}x + C.$$

Integration By Parts:-

(LIATE)
(Herbet Kasube)

$$\int u\,dv = uv - \int v\,du \quad (1)$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

Examples:-

(1) $\int x\cos x\,dx$

$\cancel{u = \cos x}$ $\cancel{du = -\sin x\,dx}$
$\cancel{dv = x\,dx}$ $\cancel{v = \frac{x^2}{2}}$

$$\int x \cos x \, dx$$

$$= x(-\sin x) - \int -\sin x \cdot 1 \, dx$$

$$= -x\sin x + \int \sin x \, dx$$

$$= -x\sin x + \cos x + C$$

$u = x$
$dv = \cos x \, dx$

(2) $$\int x e^x \, dx$$

$$= x e^x - \int e^x \cdot 1 \, dx$$

$$= x e^x - e^x + C$$

$$= e^x (x-1) + C$$

(3) $$\int \ln x \, dx$$

$$= \ln x \cdot x - \int x \cdot \frac{1}{x}$$

$$= x \ln x - x + C$$

$$\int x^2 e^{-x} dx$$

$$u = x^2 \quad dv = e^{-x} dx$$

$$\Rightarrow \quad du = 2x \ , \ v = -e^{-x}$$

$$\int x^2 e^{-x} dx = \int u\,dv = uv - \int v\,du$$

$$= x^2(-e^{-x}) - \int -e^{-x} 2x\,dx$$

$$= -x^2 e^{-x} + 2\int e^{-x} x\,dx$$

$$= -x^2 e^{-x} + 2\left\{ x(e^{-x}) \cdot 1 - \int -e^{-x} 1\,dx \right.$$

$$= -x^2 e^{-x} - 2x e^{-x} + 2\int e^{-x} dx$$

$$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$$

(5)

$$\int e^x \cos x \, dx$$

$$= \cos x \, e^x - \int e^x (-\sin x) \, dx$$

$$= e^x \cos x + \int e^x \sin x \, dx .$$

$$\int e^x \cos x \, dx = \cancel{e^x \cos x + e^x(-\cos x) - \int (-\cos x)}$$

$$= e^x \cos x + \sin x \, e^x - \int e^x \cos x \, dx .$$

$$\int e^x \cos x \, dx = e^x \cos x + \sin x \, e^x - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x \cos x + \sin x \, e^x .$$

$$e^x \cos x \, dx = \frac{1}{2} \left(e^x \cos x + \sin x \, e^x \right) + C .$$

A Tabular Method:

Example:- 1

$$\int x^2 e^x dx$$

f(x) derivative		g(x) Integral
x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

$$\int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + C.$$

2

$$\int x^2 \sqrt{x-1}$$

f(x)		g(x)
x^2	$+$	$(x-1)^{1/2}$
$2x$	$-$	$\frac{2}{3}(x-1)^{3/2}$
2	$+$	$\frac{4}{15}(x-1)^{5/3}$
0	$-$	$\frac{8}{105}(x-1)^{7/2}$

$$\int x^2\sqrt{(x-1)}\,dx = \frac{2}{3}x^2(x-1)^{3/2} - \frac{8}{15}(x-1)^{5/2} + \frac{16}{105}(x-1)^{7/2} + C$$

Integration By Parts For Definite Integrals:-

$$\int_0^1 \tan^{-1}x\,dx$$

$$= \tan^{-1}x \cdot x\Big|_0^1 - \int_0^1 x\,\frac{1}{1+x^2}\,dx.$$

$$= \left(\tan^{-1}(1) - \tan^{-1}(0)\right) - \frac{1}{2}\ln(1+x^2)\Big|_0^1$$

$$= \left(\frac{\pi}{4} - 0\right) - \frac{1}{2}(\ln 2)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

$$= \frac{\pi}{4} - \ln\sqrt{2}$$

Reduction Formulas:-

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$$

Prove:-

$$\int \cos^n x\,dx$$

$$u = \cos^{n-1}x \quad , \quad dv = \cos x\,dx.$$

$$du = (n-1)\cos^{n-2}(x)(-\sin x)\,dx \,, \quad v = \sin x.$$

$$\int \cos^n x\,dx = uv - \int v\,du.$$

$$= \cos^{n-1}x\sin x - \int (n-1)\cos^{n-1}(x)(-\sin x)\sin x$$

$$= \cos^{n-1}x\sin x + (n-1)\int \sin^2 x\cos^{n-2}x\,dx.$$

$$= \cos^{n-1}(x)\sin x + (n-1)\int (1-\cos^2)\cos^{n-2}x\,dx.$$

$$\int \cos^n x\,dx = \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2} x\,dx$$

$$- (n-1)\int \cos^n x\,dx.$$

$$\cancel{\int \cos^n x\,dx} + n\int \cos^n x\,dx - \cancel{\int \cos^n x\,dx}$$

$$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2} x\,dx.$$

$$n\int \cos^n x\,dx = \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2} x\,dx$$

Example:-

$$\int \cos^4 x\,dx$$

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x\,dx.$$

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left[\frac{1}{2}\cos x \sin x + \frac{1}{2}\int 1\,dx\right.$$

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C.$$