5.5 Indefinite Integrals and Substitution Method:

$$\int f(x)\,dx = F(x) + C$$

Properties of Indefinite Integrals:-

Theorem:-

Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$ respectively, and that c is constant, then;

(a) A constant factor can be moved through an integral sign, that is

$$\int c f(x)\,dx = c F(x) + C$$

(b) An antiderivative of a sum is the sum of the antiderivatives, that is

$$\int [f(x) + g(x)]\,dx = F(x) + G(x) + C$$

(C) $\int [f(x) - g(x)]\,dx = F(x) - G(x) + C$

Examples:- ①

$$\int 4\cos x\,dx$$

$$= 4\int \cos x\,dx$$

$$= 4\sin x + C$$

② $\int (x + x^2)\,dx$

$$= \int x\,dx + \int x^2\,dx$$

$$= \frac{x^2}{2} + \frac{x^3}{3} + C$$

③ $\int \frac{\cos x}{\sin^2 x}\,dx$

$$= \int \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}\,dx$$

$$= \int \operatorname{cosec} x \cdot \cot x\,dx$$

$$= -\operatorname{cosec} x + C \quad \underline{\text{Ans}}$$

(4) $$\int \frac{t^2 - 2t^4}{t^4}\, dt$$

$$= \int \left(\frac{1}{t^2} - 2\right) dt$$

$$= \int (t^{-2} - 2)\, dt$$

$$= \frac{t^{-1}}{-1} - 2t + C$$

$$= -\frac{1}{t} - 2t + C$$

(5) $$\int \frac{x^2}{x^2+1}\, dx$$

$$= \int \left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right) dx$$

$$= \int \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= x - \tan^{-1}(x) + C$$

Substitution:-

Examples:- (1)

$$\int (x^2+1)^{50} \cdot 2x \, dx$$

$u = x^2+1$

$du = 2x$

$$= \int u^{50} du.$$

$$= \frac{u^{50+1}}{50+1} + C$$

$$= \frac{u^{51}}{51} + C = \frac{(x^2+1)^{51}}{51} + C$$

Guidelines for substitutions:-

1 $u = g(x)$, $du = g'(x) dx$

2- If you successful, then try to evaluate the resulting integral in terms of u.

3. If you succeed in step 2. then u by $g(x)$ to express your final answer.

Examples:-

(2)

$$\int \sin(x+9)\,dx$$

$u = x+9$.

$\Rightarrow \dfrac{du}{dx} = 1$, $du = 1\,dx$

$= \int \sin(u)\,du$

$= -\cos u + C$

$= -\cos(x+9)+C$

(3) $$\int (x-8)^{23}\,dx$$

$u = x-8$

$du = 1\,dx$

$$\int u^{23}\,du$$

$= \dfrac{u^{24}}{24} + C$

$= \dfrac{(x-8)^{24}}{24} + C$

(4)

$$\int \frac{dx}{(\frac{1}{3}x-8)^5}$$

$$u = \frac{1}{3}x - 8$$

$$du = \frac{1}{3}dx$$

$$dx = 3du$$

$$\Rightarrow \quad = \int \frac{3du}{u^5}$$

$$= 3\int \frac{1}{u^5}du$$

$$= 3\int u^{-5}du$$

$$= -3\frac{u^{-4}}{4} + C$$

$$= -\frac{3}{4}\left(\frac{1}{3}x - 8\right)^{-4} + C$$

(5)

$$\int \left(\frac{1}{x} + \sec^2 \pi x\right) dx$$

$$= \int \frac{dx}{x} + \int \sec^2 \pi x \, dx$$

$$= \ln|x| + \int \sec^2 \pi x \, dx$$

$u = \pi x$
$du = \pi dx$
$dx = \frac{1}{\pi} du$

$$= \ln|x| + \frac{1}{\pi} \int \sec^2 u \, du$$

$$= \ln|x| + \frac{1}{\pi} \tan u + C$$

$$= \ln|x| + \frac{1}{\pi} \tan u + C$$

$$= \ln|x| + \frac{1}{\pi} \tan \pi x + C$$

(6)

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx.$$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} dx$ or $2du = \frac{1}{\sqrt{x}} dx$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int e^u 2 du$$

$$= 2 \int e^u du$$

$$= 2 e^u + C$$

$$= 2 e^{\sqrt{x}} + C$$

Different Substitution:-

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}, \qquad u = z^2+1$$
$$du = 2z\,dz$$

$$= \int \frac{du}{u^{1/3}}$$

$$= \int u^{-1/3}\,du$$

$$= \frac{u^{-1/3+1}}{-1/3+1} + C$$

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C$$

5.6

Definite Integral Substitutions and Area Between Curves:-

5

Examples:- 1

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

$u = x^3+1$

$du = 3x^2$

$x=-1,\ u=0$

$x=1,\ u=2$

$$= \int_0^2 \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2}\Big|_0^2$$

$$= \frac{2}{3}\left[2^{3/2}\right]$$

$$= \frac{2}{3}\left[2\sqrt{2}\right]$$

$$= \frac{4\sqrt{2}}{3}$$

2

$$\int_0^2 x(x^2+1)^3\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$x=0,\ u=1$

$x=2\quad u=5$

$$= \frac{1}{2}\int_0^2 (x^2+1)^3 2x\,dx$$

$$= \frac{1}{2}\int_1^5 u^3\,du$$

$$= \frac{1}{2}\left.\frac{u^4}{4}\right|_1^5$$

$$= \frac{1}{8}(625-1)$$

$$= \frac{1}{8}(624)$$

$$= 78$$

(3) $\displaystyle\int_0^{\frac{\pi}{8}} \sin^5 2x \cos 2x\,dx$

$u = \sin 2x$

$du = \cos 2x \cdot 2\,dx$

$du = 2\cos 2x\,dx$

$u = 0$ when $x = 0$

when $x = \frac{\pi}{8}$, $u = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$= \frac{1}{2}\int_0^{\frac{\pi}{8}} \sin^5(2x)\, 2\cos 2x\, dx$$

$$= \frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}} u^5\, du$$

$$= \frac{1}{2}\, \frac{u^6}{6}\Big|_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{12}\left[\left(\frac{1}{\sqrt{2}}\right)^6\right.$$

$$= \frac{1}{12}\left(\frac{1}{8}\right) = \frac{1}{96}.$$

Theorem:- Let f be continuous on symmetric interval $[-a, a]$

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$

Examples:-

(1)

$$\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx$$

$$f(-x) = f(x)$$

So

$$\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx = 2\int_{0}^{2} (x^4 - 4x^2 + 6)\,dx$$

$$= 2\left(\frac{x^5}{5} - \frac{4x^3}{3} + 6x\right)\Big|_0^2$$

$$= \frac{232}{15}$$

(2) Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$

General:-

$$A = \int_a^b [f(x) - g(x)]\,dx$$

$2-x^2 = -x$

$x^2 - x - 2 = 0$

$x^2 - 2x + x - 2 = 0$

$x(x-2) + 1(x-2) = 0$

$(x+1)(x-2) = 0$

$x+1=0, \quad x-2=0$

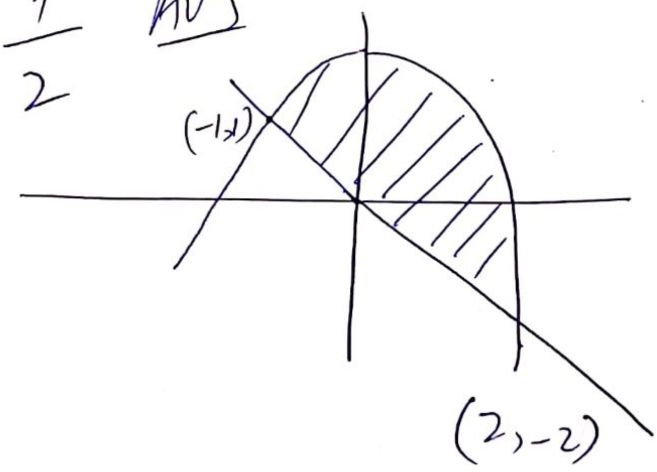
$\boxed{x=-1} \qquad \boxed{x=2}$

$$A = \int_{-1}^{2} (2-x^2) - (-x)\,dx$$

$$= \int_{-1}^{2} (2+x-x^2)\,dx$$

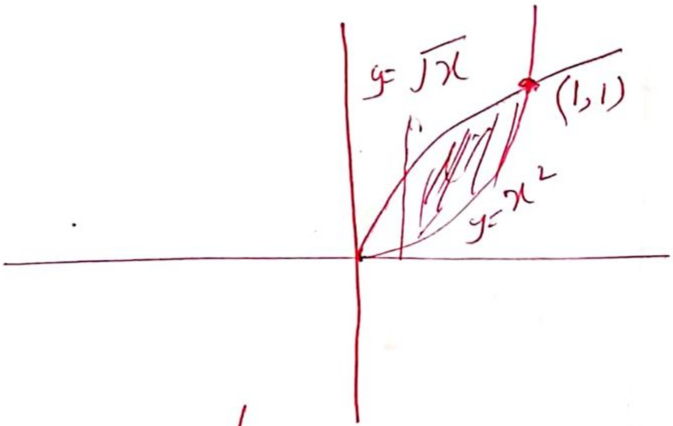
$$= 2x + \frac{x^2}{2} - \frac{x^3}{3} \Big|_{-1}^{2}$$

$$= \frac{9}{2}$$ Ans



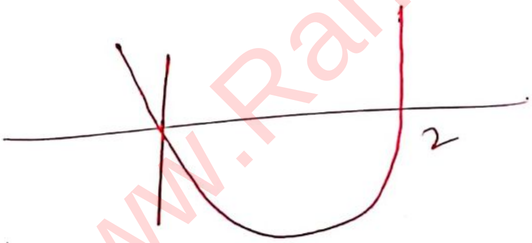
Area Between the Curves:- ①

① $y = x^2$, $y = \sqrt{x}$, $x = \frac{1}{4}$, $x = 1$



$$A = \int_{1/4}^{1} (\sqrt{x} - x^2)\, dx$$

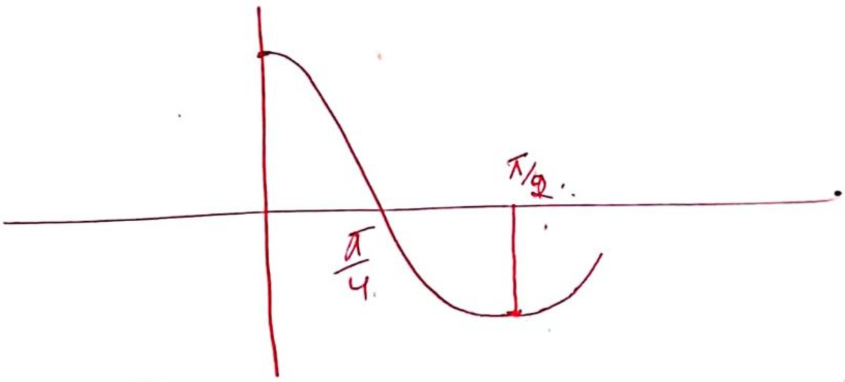
② $y = x^3 - 4x$, $y = 0$, $x = 0$, $x = 2$



$$A = \int_{0}^{2} 0 - (x^3 - 4x)\, dx.$$

③

$y = \cos 2x, \; y = 0, \; x = \frac{\pi}{4}, \; x = \pi/2.$



$$A = \int_{\pi/4}^{\pi/2} (0 - \cos 2x)\,dx.$$

$$= -\int_{\pi/4}^{\pi/2} \cos 2x\,dx.$$

$$= \; ?$$

Best of luck

Q#3 (i) Find $\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}}$ (ii) $\lim_{x\to 0} x\ln x$

Q#2 Sketch the graph of $y = 2 + x - x^2$.

Q#1 Find the local maximum and minimum of function $f(x) = 2x^3 + 3x^2 - 12x$.

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Q- Find the area bounded by the curve $x^2+4y-8=0$ and the $x=2y$?

Y-intercept

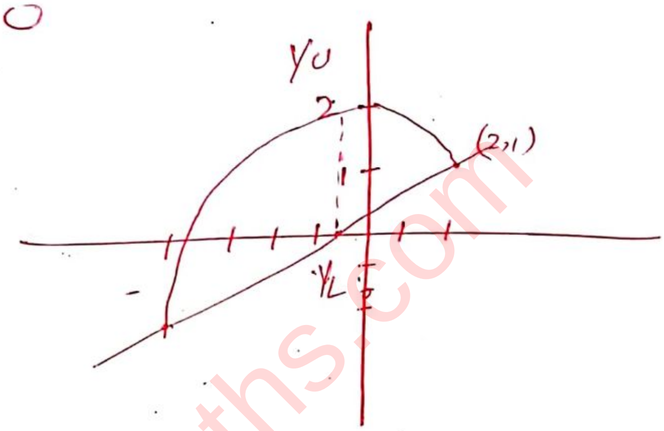
$x=0$

$0+4y-8=0$

$4y=8$

$y=2$

$(0,2)$



intersection

$(2y)^2+4y-8=0$

$4y^2+4y-8=0$

$y^2+y-2=0$

$y^2+2y-y-2=0$

$y(y+2)-1(y+2)=0$

$y-1=0, \quad y=-2$

$y=1, \quad y=-2$

$x=2 \quad x=-4$

$(2,1)$ and $(-4,-2)$

y_u.

$$x^2 + 4y - 8 = 0$$

$$4y = -x^2 + 8$$

$$y = -\frac{1}{4}x^2 + 2$$

y_L

$$y = \frac{1}{2}x$$

Area

$$\int_a^b (y_u - y_L)\,dx$$

$$= \int_{-4}^{2} \left(-\frac{1}{4}x^2 + 2 - \frac{1}{2}x\right)dx$$

$$\Rightarrow -\frac{1}{4}\int_{-4}^{2} x^2\,dx + 2\int dx - \frac{1}{2}\int x\,dx$$

$$\Rightarrow$$

— — — —

$$= 9$$

Best of luck

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