Improper Integrals: (Ist kind)

Integrals with infinite limits of Integration are improper integrals of type I

1- If $f(x)$ is continuous on $[a,\infty)$, then

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

2- If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx.$$

3- If $f(x)$ is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx.$$

In each case, if the limits is finite we say that the improper Integral converges, and that the limit is value of improper integral. If limit is fail to exist, the improper Integral diverges.

Examples:- (1) Is the area under the Curve $y = (\ln x)/x^2$ from $x=1$ to $x=\infty$ finite? If so, what is value?

$$\int_1^b \frac{\ln x}{x^2} = \ln x \left(-\frac{1}{x}\right|_1^b - \int_1^b -\frac{1}{x}\cdot\frac{1}{x}\,dx.$$

$$= -\frac{\ln b}{b} + \int_1^b \frac{1}{x^2}\,dx.$$

$$= -\frac{\ln b}{b} - \frac{1}{x}\Big|_1^b$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1$$

as $b \to \infty$.

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx.$$

$$= \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right]$$

$$= -\lim_{b\to\infty} \frac{1/b}{1} + 1 \Rightarrow 0 + 1 = 1$$

(2) $\int_1^{+\infty} \frac{dx}{x^3}$

$$\int_1^{+\infty} \frac{dx}{x^3} = \lim_{a \to +\infty} \int_1^{a} \frac{dx}{x^3}$$

$$= \lim_{a \to +\infty} -\frac{1}{2x^2}\Big|_1^a$$

$$= \lim_{a \to +\infty} \left(-\frac{1}{2a^2} + \frac{1}{2}\right)$$

$$= \frac{1}{2}$$

Since limit is finite, the Integral converges and its value is $\frac{1}{2}$.

(3) $\int_1^{+\infty} \frac{dx}{x}$

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{a \to +\infty} \int_1^{a} \frac{dx}{x}$$

$$= \lim_{a \to +\infty} \ln x \Big|_1^a$$

$$= \lim_{a \to +\infty} (\ln a - 0)$$

$$= +\infty$$

In this case integral diverges.

(4)

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

we choose $c = 0$.

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2} \quad -(1)$$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1+x^2}$$

$$= \lim_{a \to -\infty} \tan^{-1} x \Big|_{a}^{0}$$

$$= \lim_{a \to -\infty} -\tan^{-1}(a)$$

$$= \frac{\pi}{2}$$

$$\int_{0}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi.$$

The Integral $\int_1^\infty \frac{dx}{x^p}$

Example:- For what values of p does the integral $\int_1^\infty dx/x^p$ converge? when the integral does converge, what is its value?

Sol:- If $p \neq 1$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{a\to\infty} \int_1^a \frac{dx}{x^p}$$

$$= \lim_{a\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^a$$

$$= \lim_{a\to\infty} \left(\frac{a^{1-p}}{1-p} - 1\right)$$

$$= \lim_{a\to\infty} \frac{1}{1-p}\left(\frac{1}{a^{p-1}} - 1\right)$$

$$= \begin{cases} \frac{1}{p-1}, & p>1 \\ \infty, & p<1 \end{cases}$$

1 diverge

Improper Integrals :- (Type 2)

1- If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2- If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3- If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

$$= \lim_{d \to c^-} \int_a^d f(x)\,dx + \lim_{e \to c^+} \int_e^b f(x)\,dx$$

Examples:

(1) Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx. \quad \text{continuous } [0,1)$$

$$\lim_{b\to 1^-} \int_0^b \frac{1}{1-x}\,dx.$$

$$= \lim_{b\to 1^-} -\ln(1-x)\Big|_0^b$$

$$= \lim_{b\to 1^-} \left[-\ln(1-b)+0\right]$$

$$= \infty$$

(2) $$\int_0^3 \frac{dx}{(x-1)^{2/3}}$$

Sol: $\int_0^3 \frac{dx}{(x-1)^{2/3}}$ continuous on $[0,1)$

and $(1,3]$.

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{b\to 1^-} 3(x-1)^{1/3}\Big|_0^b$$

$$= \lim_{b\to 1^-}\left[3(b-1)^{1/3} - 3(0-1)^{1/3}\right]$$

$$= \lim_{b\to 1^-}\left[3(b-1)^{1/3} + 3\right]$$

$$= 3$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+}\int_c^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{c\to 1^+} 3(x-1)^{1/3}\Big|_c^3$$

$$= \lim_{c\to 1^+}\left|3\,2^{1/3} - 3(c-1)^{1/3}\right|$$

$$= 3\sqrt[3]{2}$$

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}$$

Limit Comparison Test:-

If the positive functions f and g are continuous on $[a,\infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \quad , 0 < L < \infty$$

the $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$

both converge or both diverge.

Example:-
Show that $\int_1^\infty \frac{dx}{1+x^2}$ converges by comparison with $\int_1^\infty \frac{1}{x^2}\,dx$. Find and compare two Integrals?

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{1+x^2}}$$

$$= \lim_{x\to\infty} \frac{1+x^2}{x^2}$$

$$= \lim_{x\to\infty} \left(\frac{1}{x^2} + 1\right)$$

$$= 1$$

② Investigate the convergence of

$f(x) = \int_1^\infty \frac{1-e^{-x}}{x}\,dx$, $g(x) = \frac{1}{x}$.

$$\lim_{x\to\infty} \frac{1-e^{-x}}{x} \cdot \frac{x}{1}$$

$$= 1.$$

$f(x)$ and $g(x)$ both diverge

$\lim \frac{f(x)}{g(x)} = c$

$\lim \frac{f(x)}{g(x)} = 0$

$\lim \frac{f(x)}{g(x)} = \infty$

Direct Comparison Test:-

Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ $x \geq a$.

1 $\int_a^\infty f(x)dx$ converges if $\int_a^\infty g(x)dx$ converges

2. $\int_a^\infty f(x)\,dx$ diverges if $\int_a^\infty f(x)dx$ diverges

Examples:-

① $f(x) = \int_1^\infty \frac{\sin^2 x}{x^2}$, $g(x) = \frac{1}{x^2}$

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \text{ on } [1, \infty)$$

$f(x)$ converges, $g(x)$ converges.

② $f(x) = \frac{1}{\sqrt{x^2 - 0.1}}$, $g(x) = \frac{1}{x}$.

$$\frac{1}{\sqrt{x^2 - 0.1}} \geq \frac{1}{x}.$$

Both Diverges.