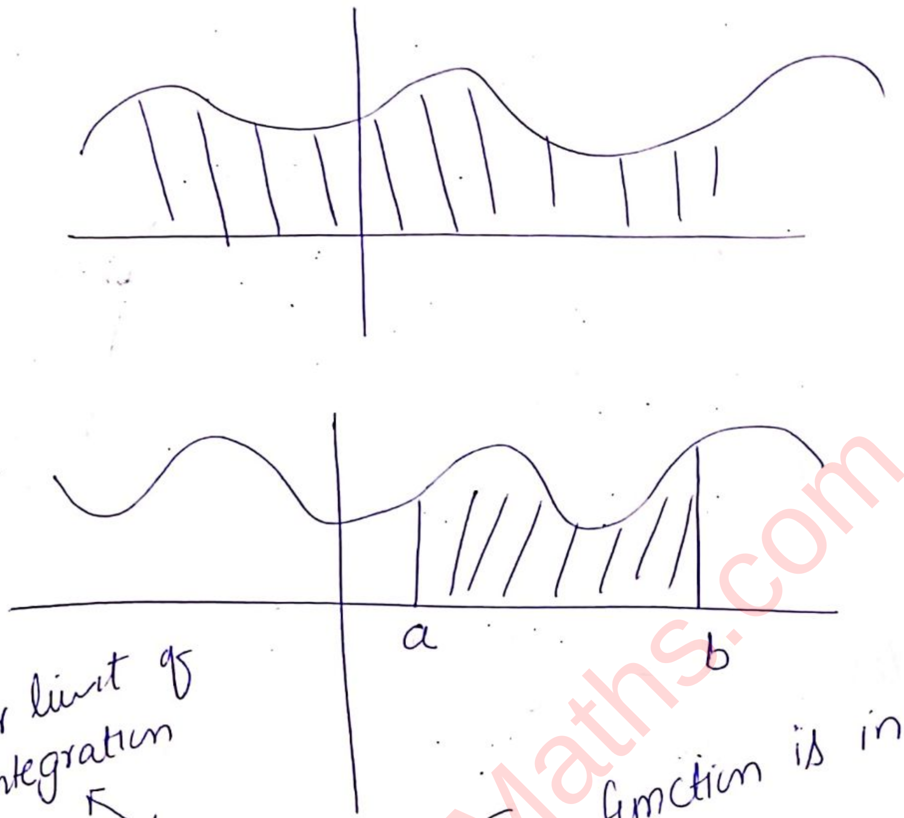
5.3 The Definite Integral:-



upper limit of integration

$$\int_a^b f(x)\,dx$$

integral sign

The function is integrand

variable of integration

lower limit of integration

Integral from a to b.

Properties of Definit Integrals:-

(i) $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

(ii) $$\int_a^a f(x)\,dx = 0$$

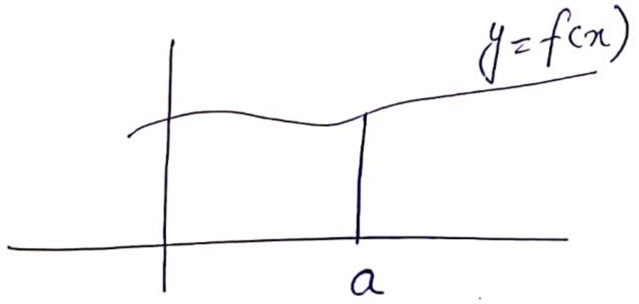
Rules Satisfied by definite Integrals:-

1- order of Integration:

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

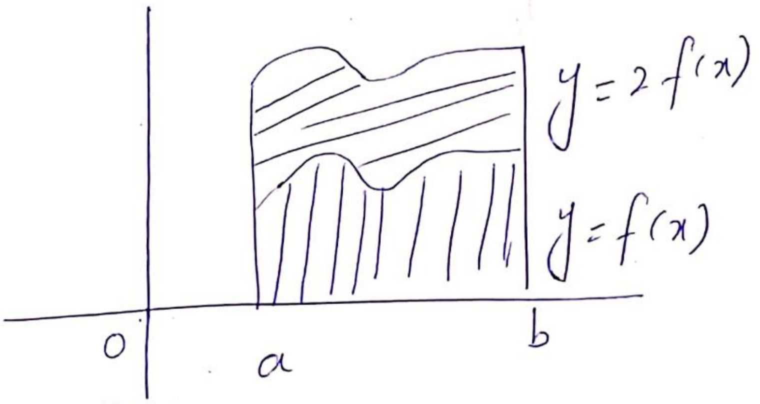
2- Zero Width Interval:

$$\int_a^a f(x)\,dx = 0$$



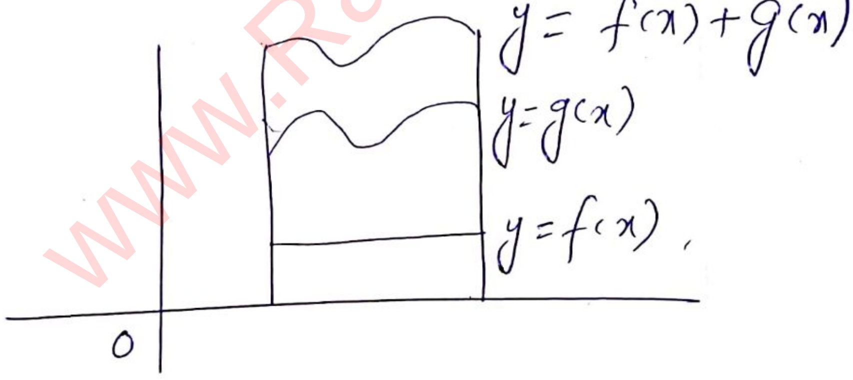
Constant Multiple:-

$$\int_a^b K f(x)\,dx = K\int_a^b f(x)\,dx$$



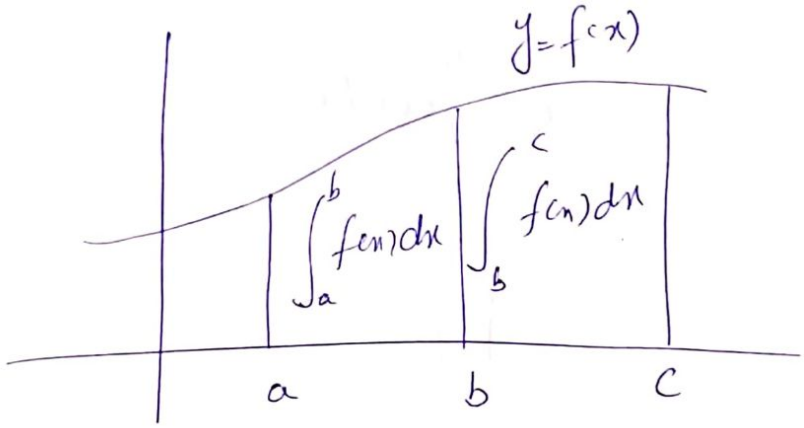
4- Sum and Difference:

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$



5. Additivity:-

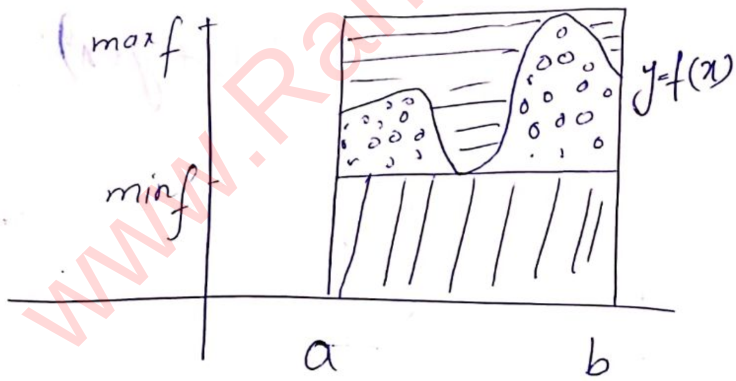
$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



6. Max-Min Inequality:

If f has maximum value $\max f$ and minimum value $\min f$ on $[a,b]$, then

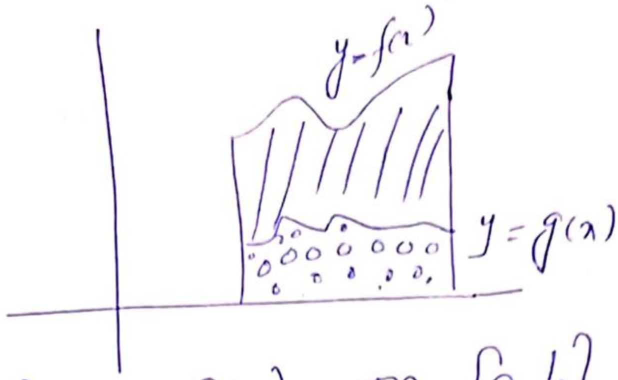
$$\min f\,(b-a) \leq \int_a^b f(x)\,dx \leq \max f\cdot(b-a)$$



7. Domination:

$$f(x) \geq g(x) \text{ on } [a,b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

$$f(x) \geq 0 \text{ on } [a,b] \Rightarrow \int_a^b f(x)\,dx \geq 0$$



$f(x) \geq g(x)$ on $[a,b]$

$$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

Examples:-

$$\int_{-1}^{1} f(x)\,dx = 5, \int_{1}^{4} f(x)\,dx = -2$$

and $\int_{-1}^{1} h(x)\,dx$.

Solution:-

rule 1

$$\int_4^1 f(x)\,dx = -2$$

$$\Rightarrow \int_1^4 f(x)\,dx = -\int_4^1 f(x)\,dx = -(-2) = 2$$

$$\int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = \int_{-1}^{4} f(x)\,dx$$

$5 \quad -2 \quad = \quad -$

$3 \quad = \quad -$

Example:-

Show that $\int_0^1 \sqrt{1+\cos x}\, dx$ is less then or equal to $\sqrt{2}$.

Sol:- As we know that

$$\min f\cdot(b-a) \le \int_a^b f(x)\,dx \le \max f\cdot b-a$$

$$\int_a^b f(x)\,dx \le \max f(b-a)$$

The maximum value of $\sqrt{1+\cos x}$ on $[0,1]$ is

$$\sqrt{1+\cos 0}$$

$$= \sqrt{2}$$

So

$$\int_0^1 \sqrt{1+\cos x} \le \sqrt{2}(1-0)$$

$$\int_0^1 \sqrt{1+\cos x} \le \sqrt{2}$$

Average value of Continuous function:-

$$\text{average} = \frac{1}{b-a}\int_a^b f(x)\,dx \quad \text{(mean)}$$

Exercise 5.3

Q=10

$$\int_1^9 f(x)\,dx = -1 \quad , \quad \int_7^9 f(x)\,dx = 5,$$

$$\int_7^9 h(x)\,dx = 4$$

(a) $\displaystyle\int_1^9 -2f(x)\,dx$

$$= -2\int_1^9 f(x)\,dx$$

$$= -2(-1)$$

$$= 2$$

(b) $\displaystyle\int_7^9 [f(x) + h(x)]\,dx$

$$= 5 + 4 = 9$$

(e) $\displaystyle\int_1^7 f(x)\,dx$

$$\int_1^7 f(x)\,dx = \int_1^9 f(x)\,dx + \int_9^7 f(x)\,dx$$

$$= -1 - 5$$

$$= -6$$

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$$\int_3^1 7\,dx$$

$$= \int_3^1 7\,dx$$

$$= 7x\Big|_3^1$$

$$= 7(1-3) = -14$$

55-

$$f(x) = x^2 - 1 \qquad 0, \sqrt{3}$$

$$av(f) = \frac{1}{\sqrt{3}-0}\int_0^{\sqrt{3}} x^2 - 1\,dx$$

$$= \frac{1}{\sqrt{3}} \; \frac{x^3}{3} - x\Big|_0^{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}\left[\left(\frac{\cancel{3}\sqrt{3}}{\cancel{3}} - \sqrt{3}\right)\right] - 0$$

$$= \frac{1}{\sqrt{3}}$$

$$= 0$$

71. What values of a and b maximize the value of

$$\int_a^b (x - x^2)\,dx \; ?$$

$x - x^2 \geq 0$

$x(1-x) \geq 0$

$x = 0, \; x = 1$. if $0 < x - x^2$

$a = 0, \; b = 1$ maximize the Integral

72- Minimize

$$\int_a^b (x^4 - 2x^2)\,dx.$$

$x^4 - 2x^2 \leq 0$

$x^2(x^2 - 2) \leq 0$

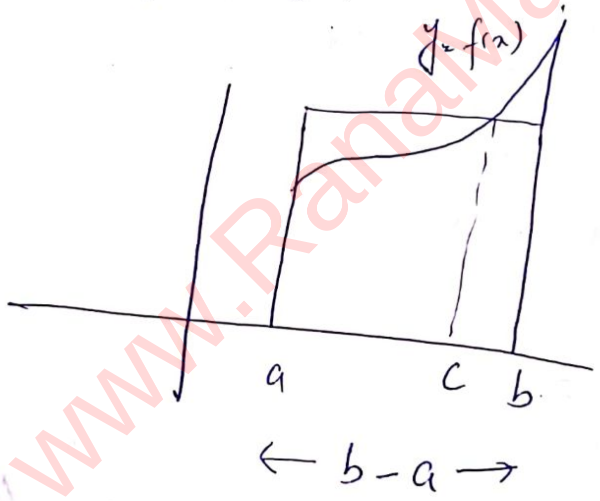
$x = 0, \; \pm\sqrt{2}$

$a = -\sqrt{2}, \; b = \sqrt{2}$ minimize the value.

5.4 The Fundamental Theorem of Calculus:-

Theorem: The Mean value theorem for definite Integrals:- If f is continuous on $[a,b]$, then at some point c in $[a,b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$



Theorem:- Part(a) If f is continuous on $[a,b]$, $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b)

∴ and its derivative is $f(x)$.

∴

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Example:-

① $$y = \int_a^x (t^3+1)\,dx$$

$$\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3+1)\,dx$$

$$= x^3+1$$

② $$y = \int_x^5 3t\sin t\,dt$$

$$\frac{dy}{dx} = \frac{d}{dx}\int_x^5 3t\sin t\,dt$$

$$= -\frac{d}{dx}\int_5^x 3t\sin t\,dt$$

$$= -3x\sin x$$

Theorem: (continued)

If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Example:-

(a) $$\int_0^{\pi} \cos x\,dx = \sin x \Big|_0^{\pi}$$

$$= 0$$

(b) $$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx$$

$$= x^{3/2} + \frac{4}{x}\Big]_1^4$$

$$= [8 + 1] - (1 + 4)$$

$$= 9 - 5$$

$$= 4$$

Theorem:- The net Change theorem:-

- The net change in a differentiable function $F(x)$ over an interval $a \le x \le b$ is the integral of its rate of change

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

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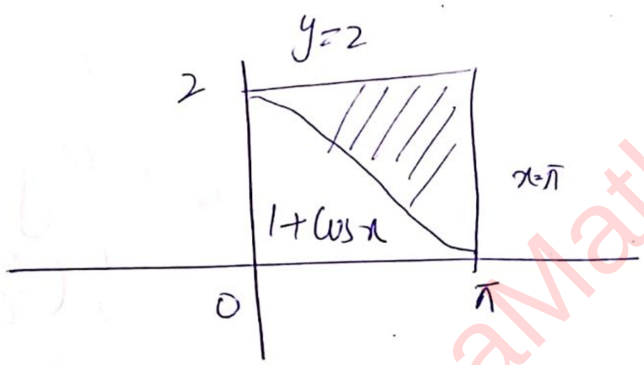


$x = \pi,\ x = 0$
$y = 2,\ y = 0$

Area of rectangle $= 2\pi$

Area under the curve is

$$= \int_0^{\pi} (1 + \cos x)\,dx$$

$$= x + \sin x \Big|_0^{\pi}$$

$$= \pi + \sin\pi - (0 + 0)$$

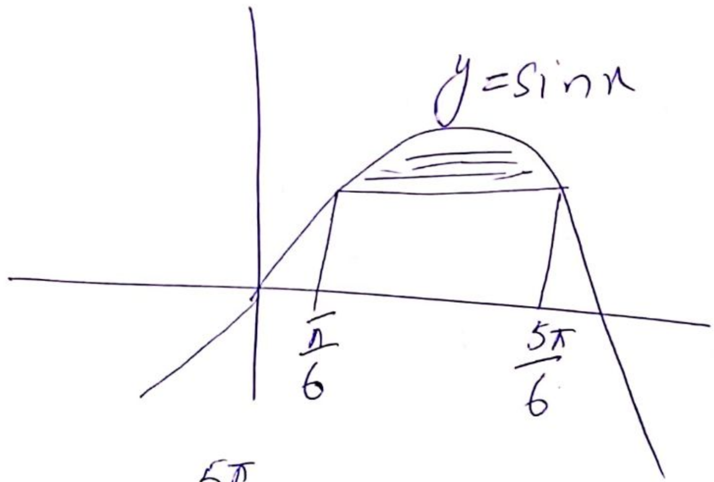
$$= \pi$$

Area of shaded region

$$= 2\pi - \pi$$

$$= \pi$$

52.



$$= \int_{\pi/6}^{5\pi/6} \sin x \, dx$$

$$= \int_{\pi/6}^{5\pi/6} + \sin x \, dx$$

$$= - \cos x \Big|_{\pi/6}^{5\pi/6}$$

$$= -\left(\cos\frac{5\pi}{6} - \cos\frac{\pi}{6} \right)$$

$$= \sqrt{3}$$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$.

$y = \sin\frac{\pi}{6} = \frac{1}{2}$.

Area of rectangle

$$= \frac{1}{2}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{3}$$