Lecture #20

- Second Derivative Test for local Extrema:-

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$
2. If $f'(c) = 0$, then f has a local minimum if $f''(c) > 0$.
3. If $f'(c) = 0$ and $f''(c) = 0$ then test fails.

Example:-

$$f(x) = 3x^5 - 5x^3$$

Solution:-

$$f'(x) = 15x^4 - 15x^2$$

$$f''(x) = 60x^3 - 30x$$

$$f'(x) = 0$$

$$15x^4 - 15x^2 = 0 \Rightarrow 15x^2(x^2 - 1)$$

~~$30x(2x^2-1)=0$~~

$$15x^2(x+1)(x-1)=0$$

$x=0$, $x=1$, $x=-1$

stationary P	$30x(2x^2-1)$	$f''(x)$	Test
$x=-1$	-30	$-$	maximum
$x=0$	0	0	inconclusiv
$x=1$	30	$+$	minimum

At $x=0$ we try first derivative test

$$f'(x)=15x^2(x+1)(x-1)$$

4.5 Indeterminate Forms and L'Hospital's Rule:-

(L'Hôpital Rule for Form 0/0):-

$$\lim_{x \to a} f(x) = 0$$

and

$$\lim_{x \to 0} g(x) = 0$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}, \quad g'(x) \neq 0$$

Examples:- Find the limit

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

using L'Hospital rule, and check the result by factoring.

Sol:- Indeterminate form of type 0/0.

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{2x}{1}$$

$$= 2(2)$$

$$= 4$$

Now we check

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(\cancel{x-2})(x+2)}{(\cancel{x-2})}$$

$$= \lim_{x \to 2} (x+2)$$

$$= 4.$$

(2) $$\lim_{x \to 0} \frac{\sin 2x}{x}$$

Sol:-

$$\lim_{x \to 0} \frac{2\cos 2x}{1}$$

$$= 2(1)$$

$$= 2$$

(3) (3)

$$\lim_{x\to \pi/2} \frac{1-\sin x}{\cos x}$$

$$= \lim_{x\to \pi/2} \frac{1-\cos x}{-\sin x}$$

$$= \frac{0}{-1} = 0$$

(4)

$$\lim_{x\to 0} \frac{e^x - 1}{x^3}$$

$$= \lim_{x\to 0} \frac{e^x}{3x^2}$$

$$= +\infty$$

(5)

$$\lim_{x\to 0^-} \frac{\tan x}{x^2}$$

$$= \lim_{x\to 0^-} \frac{\sec^2 x}{2x} = \frac{1}{0^-} = -\infty$$

Indeterminate Forms of type ∞/∞.

$\lim_{x \to a^+} f(x) = \infty$ means $\lim_{x \to a^+} f(x) = \infty$ or $\lim_{x \to a^+} = -\infty$.

$\lim_{x \to \infty} f(x) = \infty$ means $\lim_{x \to +\infty} f(x) = +\infty$ or $\lim_{x \to +\infty} f(x) = -\infty$

$\lim_{x \to a} f(x) = \infty$ means $\lim_{x \to a^+} f(x) = \pm\infty$ $\lim_{x \to a^-} = \pm\infty$

$\Rightarrow$

$$\lim_{x \to a} f(x) = \infty \; , \; \lim_{x \to a} g(x) = +\infty$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Examples:

① $\lim_{x \to +\infty} \frac{x}{e^x}$

$$= \lim_{x \to +\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$$

$$\lim_{x\to 0^+} \frac{\ln x}{\operatorname{cosec} x}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\operatorname{cosec} x \cot x}$$

$$= -\lim_{x\to 0^+} \frac{\sin x}{x} \tan x$$

$$= -(1)(0)$$

$$= 0$$

(3) Type $0.\infty$

(a)

$$\lim_{x\to 0^+} x \ln x$$

$$= \lim_{x\to 0} \frac{\ln x}{1/x}$$

$$= \lim_{x\to 0} \frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0} (-x) = 0$$

④ Type $(\infty, -\infty)$

Example:-

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$

$$= \lim_{x \to 0^+} \left(\frac{\sin x - x}{x \sin x} \right)$$

$$= \lim_{x \to 0^+} \frac{\cos x - 1}{\sin x + x \cos x}$$

$$= \lim_{x \to 0^+} \frac{-\sin x}{\cos x + \cos x - x \sin x}$$

$$= \frac{0}{2} = 0$$

⑤ Forms $(0^0, \infty^0, 1^\infty)$

Example:-

Find $\lim_{x \to 0} (1 + \sin x)^{\frac{1}{x}}$

$$y = (1+\sin x)^{1/x}$$

$$\ln y = \frac{1}{x}\ln(1+\sin x)$$

$$\lim_{x\to 0}(\ln y) = \lim_{x\to 0}\frac{\ln(1+\sin x)}{x}$$

$$\lim_{x\to 0}(\ln y) = \lim_{x\to 0}\frac{\cos x}{1+\sin x}$$

$$\boxed{\lim_{x\to 0}(\ln y) = 1}$$

$$\ln y = 1$$

$$e^{\ln y} = e^{1}$$

$$y = e$$

$$\boxed{(1+\sin x)^{\frac{1}{x}} = e}$$