4.1 Extreme values of Functions:-

Definitions:-

Let f be a function with domain D Then f has absolute maximum value at a point c if

$$f(x) \leq f(c) \text{ for all } x \text{ in } D$$

and an absolute minnimum value on D at c if

$$f(x) \geq f(c) \text{ for all } x \text{ in } D.$$

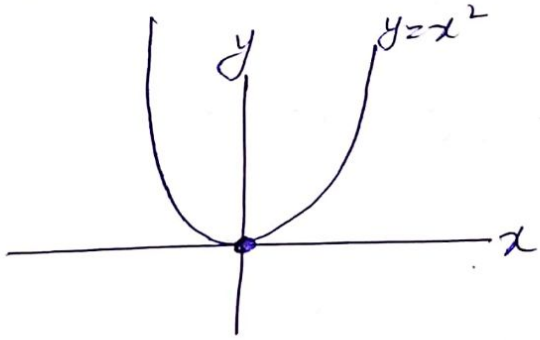
* Minimum and maximum value are called extreme values of functions.

* Absolute maxima or minima are also reffered to as global maxima or minima

Examples:- (1)

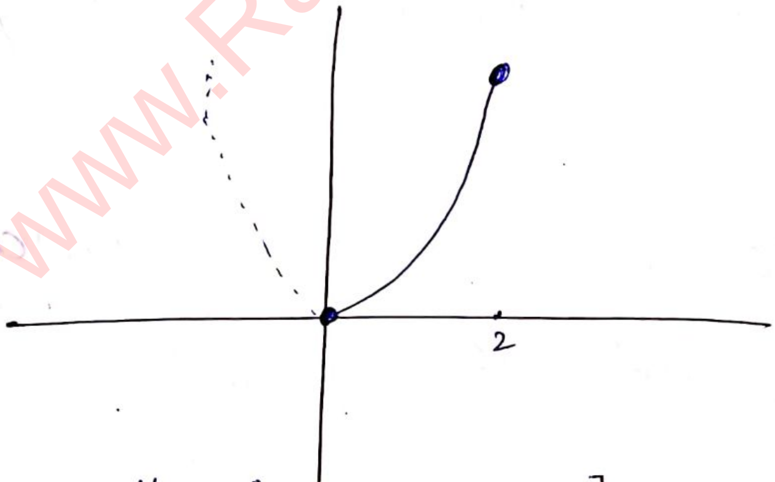
(a) $y = x^2$ $(-\infty, \infty)$

* No absolute maximum
* Absolute minimum of 0 at $x=0$



(b) $y = x^2$ $[0,2]$

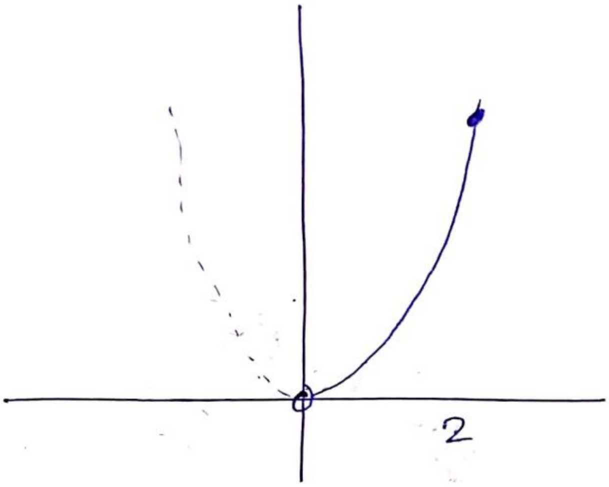
* Absolute maximum of 4 at $x=2$
* Absolute minimum of 0 at $x=0$



(c) $y = x^2$ $(0,2]$

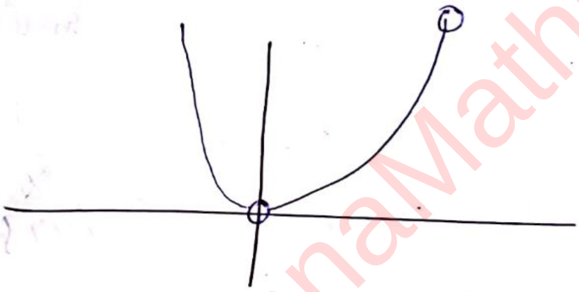
$(0,2]$

* No absolute minimum
* Absolute maximum is 4 at $x=2$



(d) $y = x^2 \cdot (0,2)$

* NO
* NO

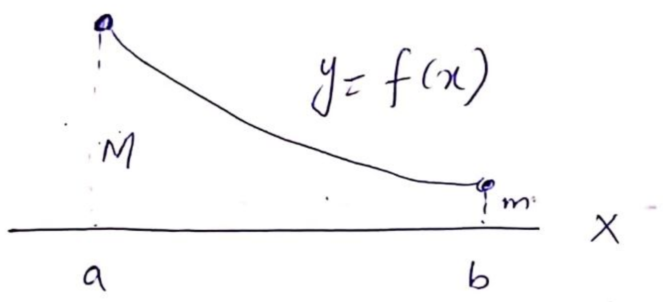


Theorem:-

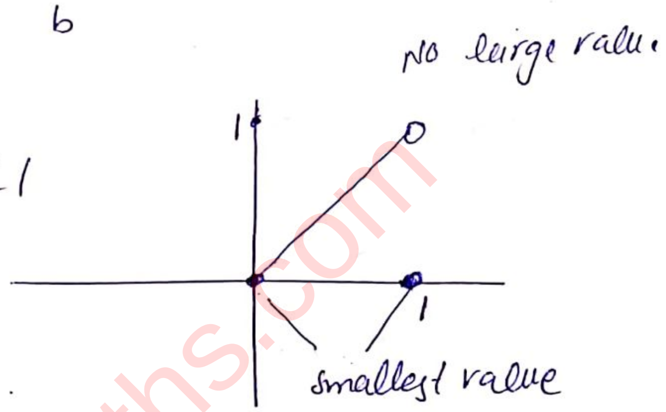
The Extreme Value Theorem:-

If f is continuous on a closed interval $[a,b]$, then f attains both an absolute maximum value M and an absolute minimum value m at $[a,b]$. that is there are numbers x_1 and x_2 in $[a,b]$ with $f(x_1) = m$, $f(x_2) = M$ and

$m \le f(x) \le M$ for every other x in $[a, b]$.

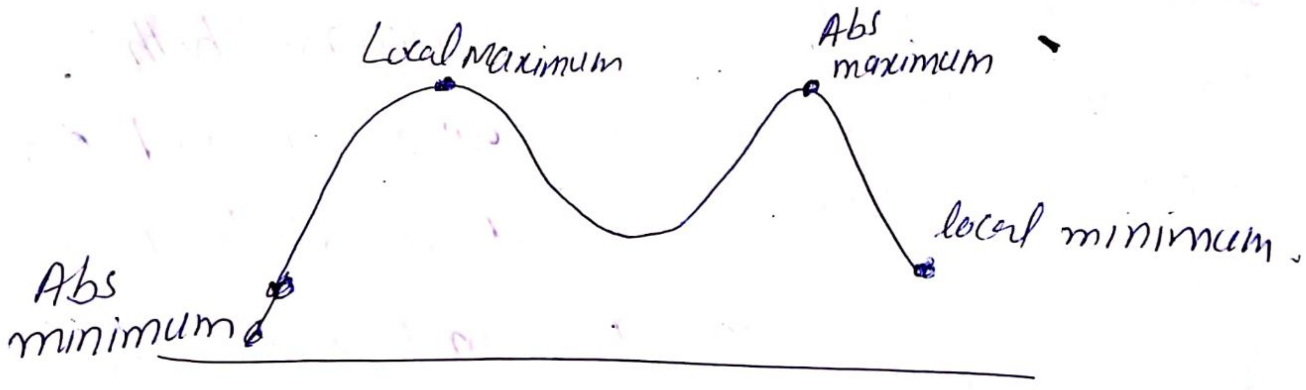


$$y = \begin{cases} x & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$



Definition:- A function f has local maximum value at a point c with its domain D if $f(x) \le f(c)$ for all $x \in D$ lying open interval containing c.

$$f(x) \ge f(c)$$



Theorem 2.

The First Derivative Theorem for local Extreme values:-

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0$$

Critical Point:-

An interior point of the domain of a function f where f' is zero or undefined is a critical point of f.

Examples:- Find all critical points of $f(x) = x^3 - 3x + 1$

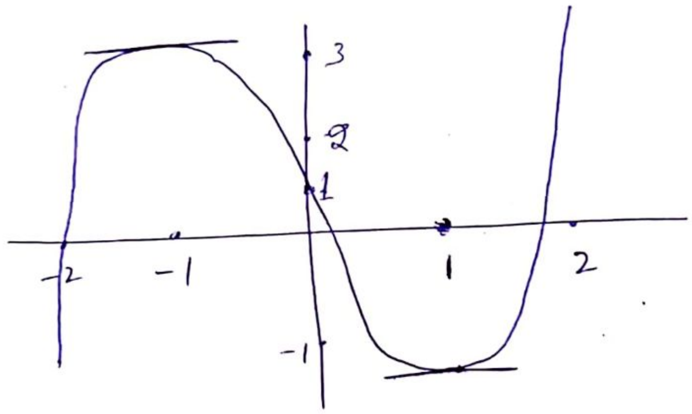
$$f'(x) = 0$$

$$3x^2 - 3 = 0$$

$$3(x+1)(x-1) = 0$$

$x = 1$

$x = -1$



② Find all critical points of

$$f(x) = 3x^{5/3} - 15x^{2/3}$$

$$f'(x) = \cancel{3}\cdot\frac{5}{\cancel{3}}x^{2/3} - \cancel{15}\cdot\frac{2}{\cancel{3}}x^{-1/3}$$

$$= 5x^{2/3} - 10x^{-1/3}$$

$$= 5x^{-1/3}(x-2)$$

$$= \frac{5(x-2)}{x^{1/3}}$$

$f'(x) = 0$ if $x = 2$ and

$f'(x)$ is undefined if $x = 0$.

Thus $x = 0$ and $x = 2$ are

critical points.

Roll's Theorem:-

Suppose that $y = f(x)$ is continuous over the closed interval $[a,b]$ and differentiable at every point of its interior (a,b).

If $f(a) = f(b)$, then there is one number c in (a,b) at which $f'(c) = 0$

(i) $f(x)$ is continuous $[a,b]$

(ii) $f(x)$ is differentiable in $[a,b]$

(iii) $f(a) = f(b)$

$c \in (a,b)$

$f'(c) = 0$

Example:-

$f(x) = x^2 - 5x + 6$

$f'(x) = 2x - 5$

$f(a) = f(b)$

$4 - 10 + 6 = 9 - 15 + 6$

$0 = 0$

a b

$(2,3)$

$2x - 5 = 0$

$\boxed{x = 5/2}$

$\boxed{c = 5/2}$

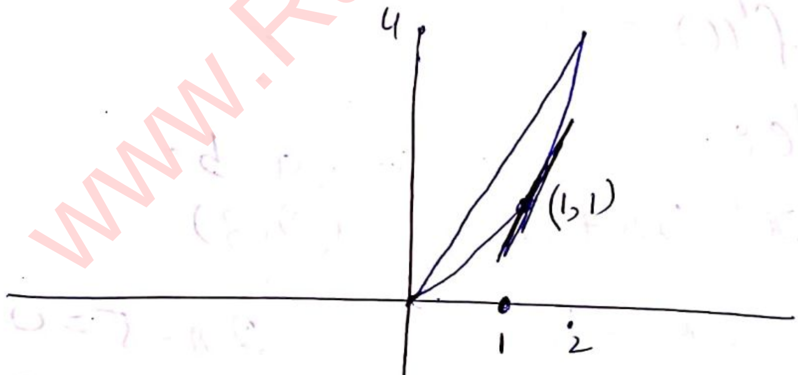
1.2

Mean Value Theorem :-

Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the intervals (a, b) Then their is at least on point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Example:-



$$f'(c) = \frac{4-0}{2-0} = 2$$

$f'(x) = 2x$

$f'(c) = 2c$

$$2c = 2$$

$$\boxed{c = 1}$$

Monotonic Functions and First derivative test:- ①

COROLLARY:-

Suppose that f is continuous on $[a,b]$ and differentiable on (a,b)

if $f'(x) > 0$ at each point $x \in (a,b)$, then f is increasing on $[a,b]$

If $f'(x) < 0$ at each point $x \in (a,b)$, then f is decreasing on $[a,b]$

Example 1:- Find the intervals on which $f(x) = x^2 - 4x + 3$ is increasing and the Interval on which decreasing

$$f'(x) = 2x - 4$$

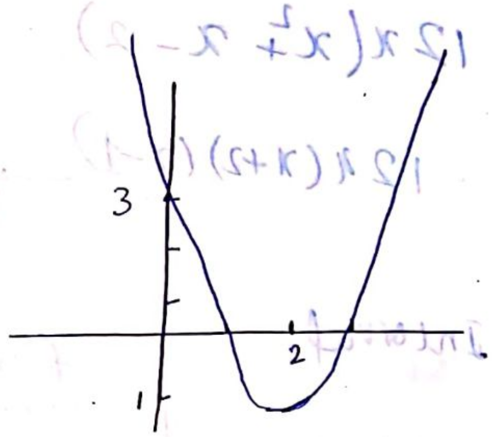
$$= 2(x-2)$$

$f'(x) < 0$ if $x < 2$

$f'(x) > 0$ if $2 < x$

Increasing $[2, +\infty)$

Decreasing $(-\infty, 2]$

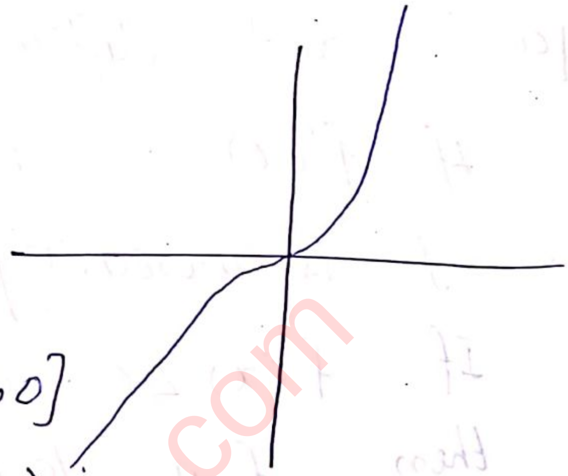


Example 2:- Find the intervals on which $f(x) = x^3$ is increasing and Intervals on which it is decreasing.

Sol:-

$$y = x^3$$

$$y' = 3x^2$$



f is increasing on $(-\infty, 0]$

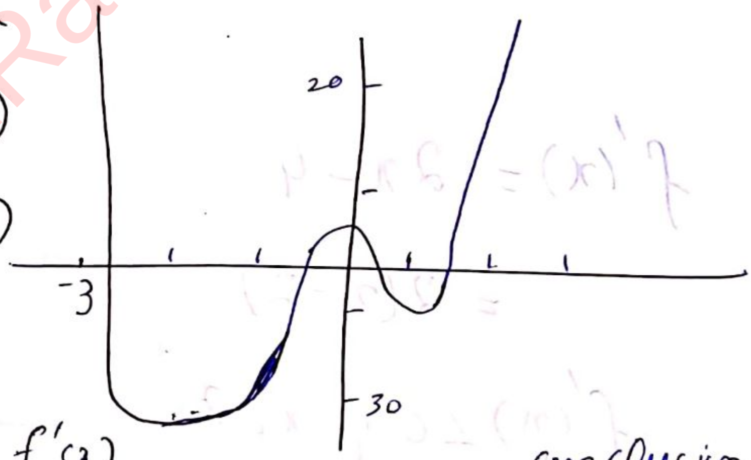
f is increasing $[0, +\infty)$

Example 3:- $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$

$$f'(x) = 12x^3 + 12x^2 - 24x$$

$$= 12x(x^2 + x - 2)$$

$$= 12x(x+2)(x-1)$$



Interval	$f'(x)$ $12x(x+2)(x-2)$	conclusion.
$x < -2$	$(-)(-)(-)$	f is decreasing $(-\infty, -2]$
$-2\ x < 0$	$(-)(+)(-)$	f increasing $[-2, 0]$
$0 < x < 1$	$(+)(+)(-)$	Decreasing $[0, 1]$
$1 < x$	$(+)(+)(+)$	Increasing $[1, \infty)$

First Derivative test for local Extrema:-

1. If f' changes from negative to positive at c, then f has local minimum at c

2. If f' changes from positive to negative at c' then f has a local maximum at c'

3. If f' does not change sign at c, then f has no local maximum at c.

Example:- Find critical points of

$$f(x) = x^{1/3}(x-4)$$
$$= x^{4/3} - 4x^{1/3}$$

Indentify thes open interval on which f is increasing and decreasing. Find the functions local and absolute extreme values?

Sol:-

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$f'(x) = \frac{4}{3}x^{-2/3}(x-1)$$

$$= \frac{4(x-1)}{x^{2/3}}$$

$x = 1, \quad x = 0$

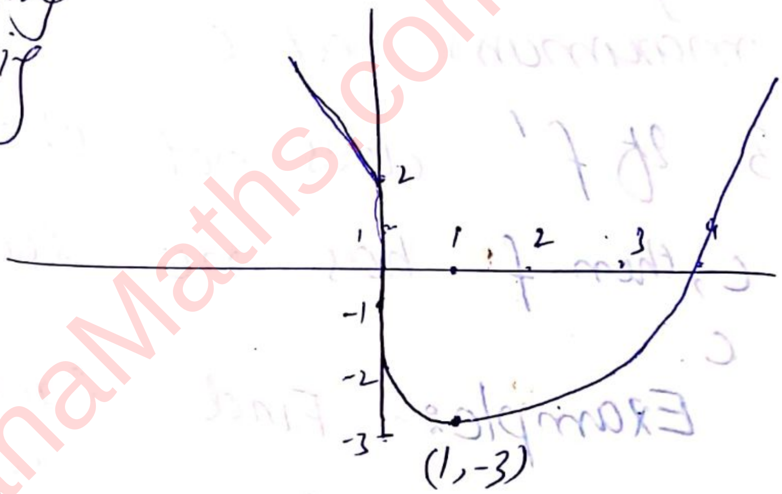
There are no endpoints, so it has Extreme ~~maximum~~ values at $x = 0, 1$

$x < 0$ Decreasing

$0 < x < 1$ Decreasing

$x > 1$ Increasing

~~$f(x) =$~~



Exercise Questions:

Q.1 $f'(x) = x(x-1)$

(a) critical points?

(b) on what open intervals is f increasing or decreasing?

(c) At what points, if any, does f assume local maximum and minimum values?

(a) $x = 0, x = 1$

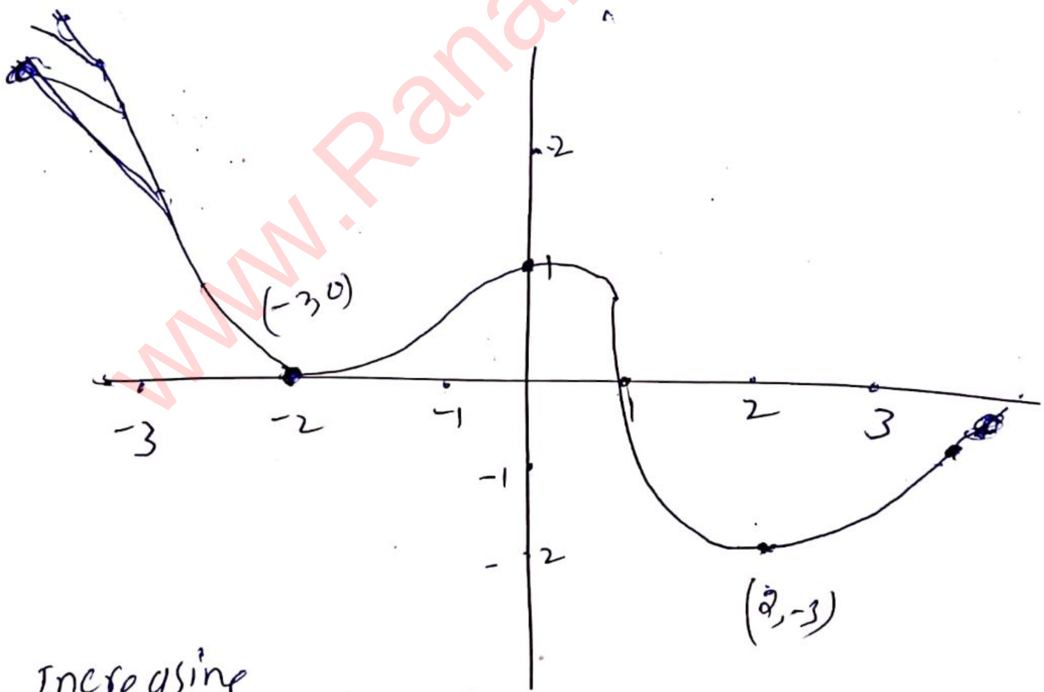
(b) $+ + + \underset{0}{|} - - - \underset{1}{|} + + +$

increasing on $(-\infty, 0)$ and $(1, \infty)$

decreasing on $(0, 1)$

(c) Local maximum at $x = 0$ and local minimum at $x = 1$

Q.15 (a)



Increasing on $(-2, 0)$ $(2, 4)$

Decreasing $(-4, -2)$ and $(0, 2)$

(b) Absolut maximum $(-4, 2)$

Absolut minimum $(2, -3)$

local maximum $(0, 1)$, $(4, -1)$

minimum $(-2, 0)$