Linearization and Differentials:-

Linearization:-

If f is differentiable at $x=a$ then the approximating function

$$L(x) = f(a) + f'(a)(x-a)$$

is the linearization of f at a.
The approximation

$$f(x) \simeq L(x)$$

of f by L is the standard linear approximation of f at a. The point $x=a$ is the centre of the approximation.

Examples:- (1)

(a) Find the local linearization of $f(x) = \sqrt{x}$ at $x=1$

(b) use linearization in (a) to approximate $\sqrt{1.1}$ and compare your approximation to the result produced directly by calculating.

Solution:-

(a) $f'(x) = \frac{d}{dx}(x^{1/2})$

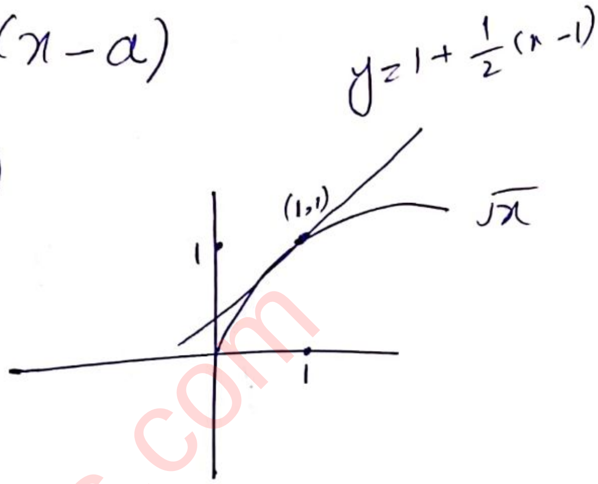
$= \frac{1}{2\sqrt{x}} \Rightarrow f'(a) = \frac{1}{2\sqrt{a}}$

$L(x) = f(a) + f'(a)(x-a)$

$= \sqrt{a} + \frac{1}{2\sqrt{a}}(x-a)$

$= 1 + \frac{1}{2}(x-1)$

$L(x) = 1 + \frac{1}{2}(x-1)$



(b)

$\sqrt{x} = 1 + \frac{1}{2}(x-1)$

$\sqrt{1.1} = 1 + \frac{1}{2}(1.1-1)$

$\sqrt{1.1} = 1 + \frac{1}{2}(.1)$

$1.049 \approx 1.05$

(2) (a) $f(x) = \sin x$, $a = 0$

(b) $\sin 2^\circ$?

Solution:-

(a)

$$L(x) = f(a) + f'(a)(x-a)$$

$$f(0) = \sin(0) = 0$$

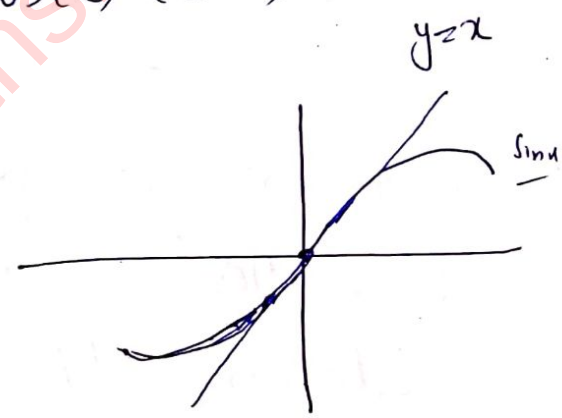
$$f'(x) = \cos x$$

$$f'(0) = \cos(0) = 1$$

$$L(x) = \sin(a) + \cos(a)(x-a)$$

$$= \sin(0) + \cos(0)(x-0)$$

$$L(x) = x$$



(b)

$$\sin 2^\circ \approx .0348 \quad -(1)$$

$$L(x) = 2^\circ = 2 \times \frac{\pi}{180} \approx .0349$$

Differentials:-

Let $y=f(x)$ be a differentiable function. The differential dx is an ~~indede~~ independent variable. The differential dy is

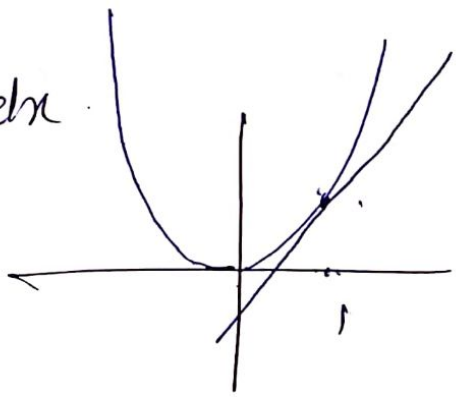
$$dy = f'(x)\,dx$$

Examples:-

① Express the derivative with respect to x of $y=x^2$ in differential form and discuss relationship between dy and dx at $x=1$

Solution:-

$$dy = 2x\,dx$$



$x=1$

$$dy = 2dx$$

(2) $y=\sqrt{x}$

(a) Find Δy and dy

(b) Evaluate Δy and dy at $x=4$ with $dx=\Delta x=3$.

Sol:- (a) $y=f(x)=\sqrt{x}$, $\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{x}}$

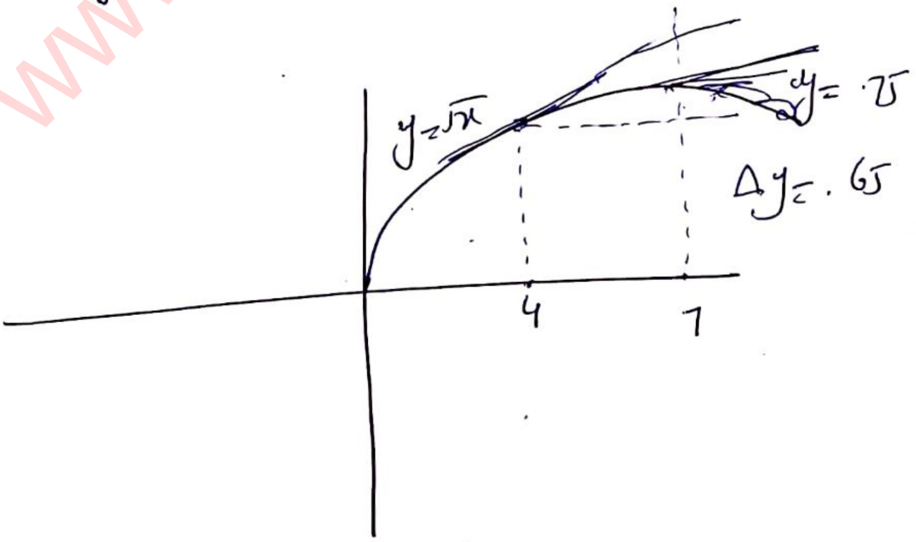
$\Delta y = f(x+\Delta x)-f(x)$ $\quad dy=\frac{1}{2\sqrt{x}}dx$

$= \sqrt{x+\Delta x}-\sqrt{x}$

(b) at $x=4$ and $\Delta x=3$,

$\Delta y = \sqrt{7}-\sqrt{4} \simeq .65$

$dy = \dfrac{1}{2(2)}(3) = \dfrac{3}{4} = .75$



Error In Differential Approximation:-

true change: $\Delta f \underset{x=a}{=} f(a+\Delta x) - f(a)$

The differential estimate: $df = f'(a)\,\Delta x$.

Approximation error $= \Delta f - df$

$$= \left(\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)\right)\Delta x.$$

$$= \epsilon \cdot \Delta x$$

True change $= f'(a)\,\Delta x + \epsilon\,\Delta x$

$\Delta x \to 0, \quad \epsilon \to 0.$

Related Rates:-

The Problem of finding a rate of change from other rate of change is called a related rate problem.

Example

$$V = \frac{4}{3}\pi r^3$$

Pumping in spherical balloon

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

(1) Suppose that the radious r and area $A = \pi r^2$ of a circle are differentiable functions at t. write an equation that relates $\frac{dA}{dt}$ to dr/dt

Sol:-

$$A = \pi r^2$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

(3) Assume that

$y = 5x$ and $\frac{dx}{dt} = 2$. Fin $\frac{dy}{dt}$

Sol:-

$$y = 5x$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$= 5(2)$$

$$= 10$$

(10) If $r + s^2 + v^3 = 12$, $\frac{dr}{dt} = 4$, and

$\frac{ds}{dt} = -3$, find $\frac{dv}{dt}$ when $r = 3$ and

$s = 1$

$\frac{dr}{dt} = 4$, $\frac{ds}{dt} = -3$

$3 + 1 + v^3 = 12$

$v = 2$

$$\frac{dr}{dt} + 2s\frac{ds}{dt} + 3v^2\frac{dv}{dt} = 0$$

$$4 + 2(1)(-3) + 3(2)^2\frac{dv}{dt} = 0$$

$$4 - 6 + 3(4)\frac{dv}{dt} = 0$$

$$12\frac{dv}{dt} = +2$$

$$\boxed{\frac{dv}{dt} = \frac{1}{6}}$$

Q.No.13 The radious r and height h of a right circular cylinder are related to the cylinder's volume V by formula $V = \pi r^2 h$

(a) How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ if r is constant

(b) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ if h is constant

(c) How $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ and $\frac{dh}{dt}$ if: neither r and h are constants.

(a) $V = \pi r^2 h$.

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

(b) $V = \pi r^2 h$

$$\frac{dV}{dt} = 2\pi r h \frac{dr}{dt}$$

(c) $V = \pi r^2 h$

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt}$$