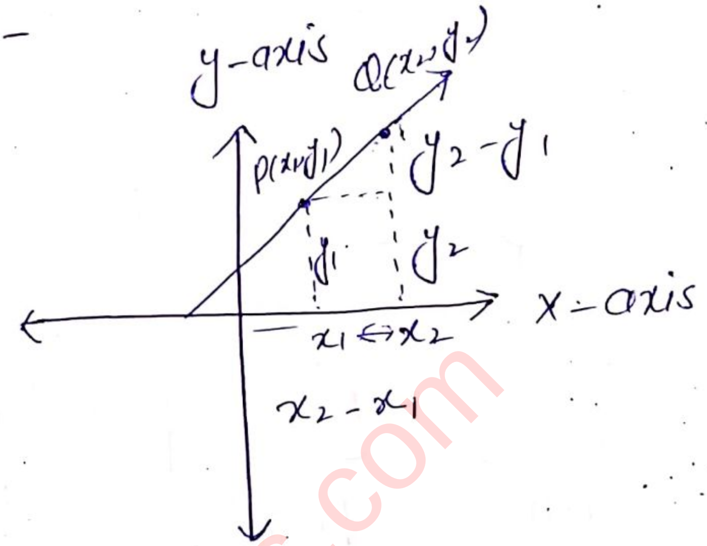
Lecture # 10

Derivative

Slop of a line :-

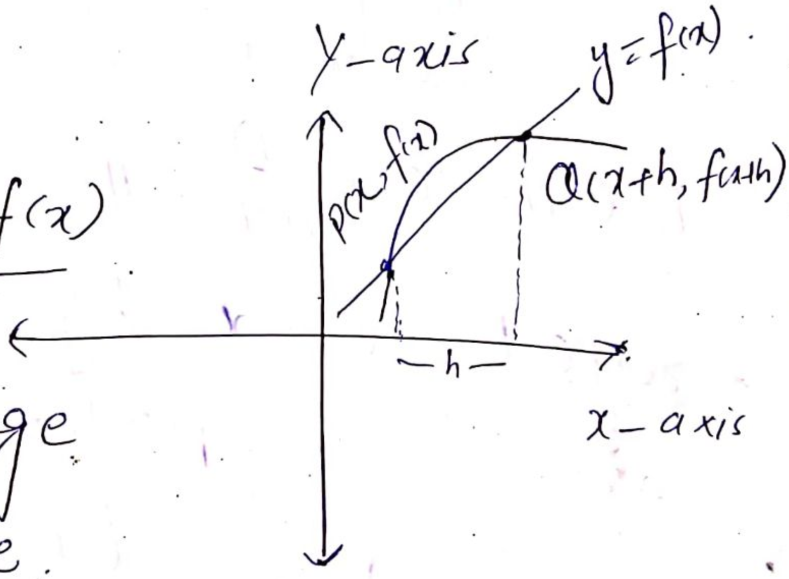
$$m = \frac{rise}{run}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



Secant line:-

A line joining any two points of a curve is called a Secant line

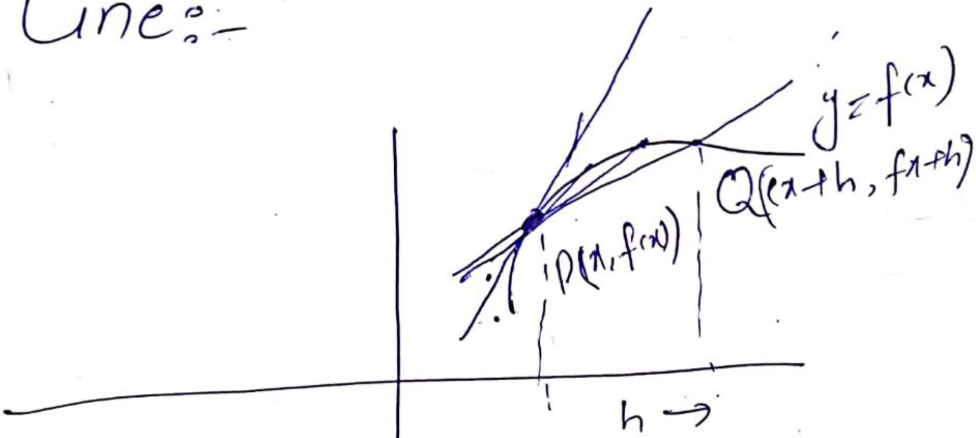


$$m_{sec} = \frac{f(x+h) - f(x)}{h}$$

also called average rate of change.

N

Tangent Line:-



$Q \to P$ i.e $h \to 0$

$$m_{tan} = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$\downarrow$

$$\frac{dy}{dx}, \; f'(x), \; \frac{d\,f(x)}{dx}, \; y'(x)$$

$$\left.\frac{df}{dx}\right|_{x=a}$$

Example:- ① Find the derivatives with respect to x, by using definition.

$$f(x) = x^2$$

$$f'(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = (x+h)^2$$

$$f(x) = x^2$$

putting in (1)

$$f'(x) = \underset{h\to 0}{Lt} \frac{(x+h)^2 - x^2}{h}$$

$$= \underset{h\to 0}{Lt} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \underset{h\to 0}{Lt} \frac{\cancel{h}(h+2x)}{\cancel{h}}$$

$$= 2x$$

(2) $f(x) = \sqrt{x}$

$$f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h\to 0} \frac{\cancel{x} + \cancel{h} - \cancel{x}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$f'(x) = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

(3) consider the particle, whose

$$S(t) = f(t) = 1 + 5t - 2t^2.$$

Find instantaneous velocity as

$$V(t)_{ins} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \to \frac{\text{change in position}}{\text{time elapsed}}$$

$$= \lim_{h \to 0} \frac{\{1 + 5(t+h) - 2(t+h)^2\} - [1 + 5t - 2t^2]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{1} + \cancel{5t} + 5h - \cancel{2t^2} - 2h^2 - 4th - \cancel{1} - \cancel{5t} + \cancel{2t^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(5 - 2h - 4t)}{\cancel{h}}$$

$$V(t)_{inst} = 5 - 4t \quad \text{Ans.}$$

Tangent line $y=f(x)$ at point x

step 1 : Evaluate $f(x_1)$; the point of tengency is (x_1, y_1)

Step2: Find $f'(x)$ and evaluate $f'(x_1)$ which is slope.

step03: Substitute the value of slope m and the point (x_1, y_1) into the point slope form of the line

$$y - y_1 = m(x - x_1)$$

$$y = y_1 + m(x - x_1)$$

Example:-

Find the tangent line of function $f(x) = 2x^2 - 3$ at $(2,5)$

(1) $f(2) = 2(2)^2 - 3$

$$= 2(4) - 3$$

$$= 8 - 3$$

$$= 5$$

(2) $f'(x) = ?$

$f(x+h) = 2(x+h)^2 - 3$

$f(x) = 2x^2 - 3$

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 - 3 - 2x^2 + 3}{h}$$

$$= \lim_{h \to 0} \frac{2(x^2 + h^2 + 2xh) - 3 - 2x^2 + 3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^2} + 2h^2 + 4xh \cancel{- 3} \cancel{- 2x^2} \cancel{+ 3}}{h}$$

$$= \lim_{h \to 0} \frac{2\cancel{h}(h + 2x)}{\cancel{h}}$$

$$= 2(0 + 2x)$$

$$= 4x$$

(3) $f'(x) = 4x$

$f'(x_1) = 4(2) = 8$

$y - 5 = 8(x - 2)$

$y = 5 + 8(x - 2)$

32 (16)

$$y = \frac{x+3}{1-x}, \quad x = -2$$

(1) $y(x) = \dfrac{-2+3}{1+2} = \dfrac{1}{3}$

(2) $y'(x) = ?$

$$y(x+h) = \frac{x+h+3}{1-(x+h)}$$

$$y'(x) = \lim_{h\to 0} \frac{\dfrac{x+h+3}{1-(x+h)} - \dfrac{x+3}{1-x}}{h}$$

$$= \lim_{h\to 0} \frac{(1-x)(x+h+3) - (x+3)(1-x-h)}{h(1-x-h)(1-x)}$$

$$y'(x) = \lim_{h\to 0} \frac{x+h+3-x^2-xh-3x-(x-x^2-xh+3-3x-3h)}{h(1-x-h)(1-x)}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h+\cancel{3}-\cancel{x^2}-\cancel{xh}-\cancel{3x}-\cancel{x}+\cancel{x^2}+\cancel{xh}-\cancel{3}+\cancel{3x}+3h}{h(1-x-h)(1-x)}$$

$$= \lim_{h\to 0} \frac{4\cancel{h}}{\cancel{h}(1-x-h)(1-x)}$$

$$= \lim_{h\to 0} \frac{4}{(1-x-h)(1-x)} = \frac{4}{(1-x-0)(1-x)}$$

$$= \frac{4}{(1-x)^2}$$

at $x=2$

(3)

$$f'(-2) = \frac{4}{(1+2)^2} = \frac{4}{9}$$

$$y - y_1 = \frac{4}{9}(x - x_1)$$

$y - 1/3 = 4/9(x+2)$ Tangent line

Differentiation Rules:-

* Derivative of constant function: The derivative of constant function is zero.

$$\frac{d}{dx}(c) = 0 \quad , \quad \frac{d}{dx}(3) = 0$$

* Derivative of a positive integer power:

if n is positive integer

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

for all x where powers x^n and x^{n-1} defined.

Example:- (i) $y = x^{\pi}$

$$\frac{d}{dx}(y) = \frac{d}{dx}x^{\pi}$$

$$= \pi x^{\pi - 1}.$$

(ii) $y = x^{1/3}$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{1/3})$$

$$= \frac{1}{3}x^{1/3 - 1}$$

$$= \frac{1}{3}x^{-2/3}$$

* Derivative Constant Multiple Rule:

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Example:-

$$y = 4x^8$$

$$\frac{d}{dx}(y) = \frac{d}{dx}(4x^8)$$

$$= 4\frac{d}{dx}(x^8)$$

$$= 4.(8x^{8-1})$$

$$= 32x^7$$

* Derivative Sum Rule:-

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

Example:-

$$y = 2x^6 + x^{-9}$$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^6 + x^{-9})$$

$$= \frac{d}{dx}(2x^6) + \frac{d}{dx}(x^{-9})$$

$$\frac{dy}{dx} = 2\cdot 6x^5 + -9x^{-10}$$

$$= 12x^5 - 9x^{-10}$$

* Derivative Product Rule:

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example:-

$$y = (x^2+1)(x^3+3)$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2+1)(x^3+3)$$

$$= (x^2+1)(3x^2) + (x^3+3)(2x)$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$

$$= 5x^4 + 3x^2 + 6x$$

Derivative Quotient Rule:-

$v(x) \neq 0$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example:-

$$y = \frac{x^3+2x^2-1}{x+5}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^3+2x^2-1}{x+5}\right)$$

$$= \frac{(x+5)\frac{d}{dx}(x^3+2x^2-1) - (x^3+2x^2-1)\frac{d}{dx}(x+5)}{(x+5)^2}$$

$$= \frac{(x+5)(3x^2+4x) - (x^3+2x^2-1)(1)}{(x+5)^2}$$

$$= \frac{3x^3+4x^2+15x^2+20x - x^3 - 2x^2+1}{(x+5)^2}$$

$$= \frac{2x^3+17x^2+20x+1}{(x+5)^2}$$

Second- and Higher-order Derivatives:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx}$$

$$= y'' = D^2 f(x) = D_x^2 f(x)$$

Example:-

$$y = x^3 - 3x^2 + 2$$

$$y' = 3x^2 - 6x + 0$$

$$y'' = 6x - 6$$

$$y''' = 6$$

$$y^{(4)} = 0.$$

Derivative of Trigonometric Functions:- 3.5 (1-62)

(i) $\frac{d}{dx}(\sin x) = \cos x.$

$\frac{d}{dx}(\cos x) = -\sin x$

(iii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(iv) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(v) $\frac{d}{dx}(\sec x) = \sec x \tan x.$

(vi) $\frac{d}{dt}(\cot x) = -\operatorname{cosec}^2 x.$

Some Examples:-

(i) Find dy/dx if $y = x\sin x$

Solution:-

$$y = x\sin x$$

$$\frac{dy}{dx} = \frac{d}{dx}(x\sin x)$$

$$= x\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(x)$$

$$= x\cos x + \sin x$$

(ii) $\frac{dy}{dx}$ if $y = \frac{\sin x}{1+\cos x}.$

$$y = \frac{\sin x}{1+\cos x}.$$

$$\frac{dy}{dx} = \frac{(1+\cos x)\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$\frac{dy}{dx} = \frac{(1+\cos x)(\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$\frac{dy}{dx} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$\frac{dy}{dx} = \frac{\cancel{\cos x + 1}}{(1+\cos x)^{\cancel{2}}} = \left(\frac{1}{1+\cos x}\right)$$

(iii) Find $f''(\pi/4)$ if $f(x) = \sec x$.

$$f'(x) = \sec x \tan x.$$

$$f''(x) = \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x)$$

$$= \sec x . \sec^2 x + \tan x \sec x \tan x$$

$$= \sec^3 x + \sec x \tan^2 x.$$

$$f''(\pi/4) = \sec^3 \pi/4 + \sec\frac{\pi}{4} \tan^2\frac{\pi}{4}$$

$$= (\sqrt{2})^3 + \sqrt{2}\,(1)^2$$

$$= 2\sqrt{2} + \sqrt{2}$$

$$= 3\sqrt{2}$$

3.4 Derivative as a Rate of change:

Some applications of derivatives.

(A)

(i) Position $s(t)$

(ii) velocity $v(t)$

(iii) acceleration $a(t)$

velocity:-

velocity is derivative of position $v(t) = s'(t) = \frac{d}{dt} s(t)$

Acceleration:-

acceleration is the derivative of velocity.

$$a(t) = v'(t) = \frac{d}{dt}(v(t))$$

Example:-

The position of an object in t seconds is given by $s(t) = t^3 - 3t^2 - 45t + 7$. when is it speeding up and when is it speeding down?

$v(t) = s'(t) = 3t^2 - 6t - 45$

$a(t) = 6t - 6$

$3t^2 - 6t - 45 = 0$

$3(t^2 - 2t - 15) = 0$

$t^2 - 2t - 15 = 0$

$t^2 - 5t + 3t - 15 = 0$

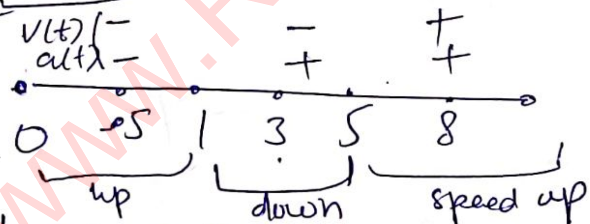
$t(t-5) + 3(t-5) = 0$

$\boxed{t = 5}, \; t = -3 \times$

and

$6t - 6 = 0$

$\boxed{t = 1}$



x	$v(x) = 3x^2 - 6x - 45$	$a(x) = 6x - 6$
.5	$= 3(.5)^2 - 6(.5) - 45$ $= 3(.25) - 3 - 45$ $= .75 - 48$ $= -47.25$	$= -3$
3	$27 - 18 - 45 = -36$	12
8	+	42

speed : $(0,1) \cup (5,\infty)$

slow : $(1,5)$

B) Revenue, cost, profit & marginal changes

1) Marginal change for "one more" Estimate with derivative.

=> cost $C(x)$ & marginal cost $MC(x) = C'(x)$, Revenue $R(x) = xP$ & maginal revenue $= R'(x)$

Profit $P(x) = R(x) - C(x)$ and marginal profit $MP(x) = P'(x)$

(a) The cost to develop a product is $C(x) = 500 + 12x$. What is marginal cost of 101 item

$MC(x) = C'(x) = 12$

(b) The price function for the product is $P = 126 - .16x$. What is marginal revenue for the 101 item.

$$R(x) = x(126 - .16x)$$
$$= 126x - .16x^2$$

$$MR(x) = R'(x)$$
$$= 126 - .32x.$$

$$MR(101) = 126 - .32(101)$$
$$= \$93.68$$

consider the actual number:

$$R(100) = 126(100) - .16(100)^2 = \$11,000$$
$$R(101) = 126(101) - .16(101)^2 = \$11,093.84$$

$$11,093.84 - 11,000 = \$93.84$$

(c) What is the marginal profit for 101^{st} item?

$$P(x) = R(x) - C(x)$$
$$= 126x - .16x^2 - 500 - 12x$$

$$= -.16x^2 + 114x - 500$$

$$MP(x) = P'(x) = -.32x + 114$$

$$= P'(101) = -.32(101) + 114$$

$$= \$81.68$$

(d) Population change

1) If $P(t)$ is population at time t, $P'(t)$ is the rate of change of population.

Ex:- A bacteria growing according to the function $P(x) = x^3 - 18x^2 + 96x + 20$, where t is in hours, when is the population growing and shrinking?

$$P'(x) = 3x^2 - 36x + 96$$

$$0 = 3x^2 - 36x + 96$$

$$x = 8, 4$$

$\begin{cases} P'(1) = 63 > 0 \\ P'(3) = 15 > 0 \end{cases}$

$\begin{cases} P'(5) = -9 < 0 \\ P'(6) = -12 < 0 \end{cases}$

grow: $(0,4) \cup (8, \infty)$

shrink: $(4, 8)$

$\begin{cases} P'(9) = 39 > 0 \end{cases}$

Exercise 3.4

Q.1 $s = t^2 - 3t + 2, \quad 0 \leq t \leq 2$

(a) Find displacement and average velocity for the given interval?

(b) Find the body speed and acceleration at the endpoints of interval

(c) When, if ever, during the interval does the body change direction.?

(a) $s(2) = (2)^2 - 3(2) + 2$

$= 4 - 6 + 2$

$= 0$

$s(0) = 2$

$\Delta s = 0 - 2 = -2m,$

$V_{av} = \frac{\Delta s}{\Delta t} = \frac{-2}{2} = -1 \, m/sec$

$= \frac{s(2) - s(0)}{t_2 - t_1} = \frac{-2}{2} = -1 \, m/sec$

(b) $v = \frac{ds}{dt} = \frac{d}{dt}(t^2 - 3t + 2)$

$= 2t - 3$

$v(0) = -3$ m/sec

$v(2) = 1$ m/sec.

$a = \frac{dv}{dt} = 2.$

$a(0) = 2$ m/sec^2

$a(2) = 2$ m/sec^2

(c) $v = 0$

$2t - 3 = 0$, $t = 3/2$

v is negative $0 < t < 3/2$ and

v is positive when $3/2 < t < 2$.

The body change direction at

$t = 3/2$

Q.7 A time t, the position of a body moving along the s-axis is $s = t^3 - 6t^2 + 9t$ m (12)

(a) Find the body's acceleration each the velocity is zero.

(b) Find body speed when acceleration is zero.

(c) Find total distence traveled by body from $t=1$ to $t=2$

Solution:-

(a)

$$s(t) = t^3 - 6t^2 + 9t$$

$$v = \frac{d}{dt}(t^3 - 6t^2 + 9t)$$

$$v(t) = 3t^2 - 12t + 9$$

$$3t^2 - 12t + 9 = 0$$

$$t^2 - 4t + 3 = 0$$

$$t^2 - 3t - t + 3 = 0$$

$$t(t-3) - 1(t-3) = 0$$

$(t-1)(t-3)=0$

$t=1,\ t=3$

$a(t) = 6t-12$

$a(1) = 6-12$
$= -6\ m/sec^2$

$a(3) = 6(3)-12$
$= 18-12 = 6\ m/sec^2$

(b) $a=0$

$6t-12=0$

$6t=12 \Rightarrow \boxed{t=2}$

$|v(2)| = |12-24+9|$
$= -3$
$= |-3| = 3$

(c) The body moves to righ or forward on $0 \le t < 1$ and left or backward on $1 < t < 2$

Total distence $= |s(1)-s(0)| + |s(2)-s(1)|$
$= |4| + |-2| = 6m$

(30) The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radious.

(a) At what rate (ft^3/ft) does the volume change with respect to the radious when $r = 2ft$?

(b) By Approximately how much does the volume increase when the radious changes from 2 to 2.2ft?

(a)
$$\frac{dV}{dr} = \frac{d}{dr}\frac{4}{3}\pi r^3$$
$$= \frac{4}{\cancel{3}}\pi \cancel{3} r^2$$
$$= 4\pi r^2$$
$\Rightarrow$
$$\left.\frac{dV}{dr}\right|_{r=2} = 4\pi(2)^2$$
$$= 16\pi$$

(b) when $r=2$, $\frac{dV}{dr}=16\pi$

$2.2 \times 2.2 = 4.84$

$r=2.2$, $\frac{dV}{dr}=4\pi(2.2)^2$

$= 4\pi(4.84)$

$= 19.36\pi$

So $V(2.2)-V(2) \approx 19.36\pi - 16\pi$

$= 3.36(\pi)$

$= 10.55$

The chain Rule:-

$(f\circ g)'(x) = f'(g(x)).g'(x)$.

In Leibniz's notation,

if $y=f(u)$, and $u=g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}.$$

where $\frac{dy}{du}$ evaluated at $u=g(x)$.

Derivative of the the absolute value Function.

$$\frac{d}{dx}(|x|) = \frac{x}{|x|}, x \neq 0$$

$$= \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

Examples:- (power chain rule)

1.

$$\frac{d}{dx}(|x|) = \frac{d}{dx}\sqrt{x^2}$$

$$= \frac{1}{2\sqrt{x^2}} \cdot \frac{dx^2}{dx}$$

$$= \frac{1}{\cancel{2}\sqrt{x^2}} \cdot \cancel{2}x$$

$$= \frac{x}{\sqrt{x^2}}$$

$$\frac{d}{dx}|x| = \frac{x}{|x|}$$

Note that not differentiable at

$x = 0$

(2) (1)

$y = 6u - 9,\quad u = \frac{1}{2}x^4$

$\frac{dy}{du} = 6,\quad \frac{du}{dx} = \frac{1}{\cancel{2}}\cancel{4}^{2}x^3$

$= 2x^3$

$dy/dx = \frac{dy}{du} \cdot \frac{du}{dx}$

$= 6 \cdot 2x^3$

$= 12x^3$

(7)

$y = \tan u,\quad u = \pi x^2$

$\frac{dy}{du} = \sec^2 u,\quad \frac{du}{dx} = 2\pi x$

$= \sec^2(\pi x^2)$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \sec^2(\pi x^2) \cdot 2\pi x$

$= 2\pi x \sec^2(\pi x^2)$

(9)

$$y = (2x+1)^5$$

$$u = 2x+1$$

$$y = u^5 \quad , \quad u = 2x+1$$

$$\frac{dy}{dx} = 5u^4 \quad , \quad \frac{du}{dx} = 2.$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5u^4 . 2$$

$$= 10u^4$$

$$= 10(2x+1)^4.$$