Course Outline

Limit and continuity;

- → Introduction to functions
- → Introduction to limits
- → Techniques of finding limits
- → Indeterminate form of limits
- → Continuous and discontinuous functions and their applications

Differential Calculus;

- → Concept and idea of differentiation
- → Geometrical and physical meaning of derivatives
- → Rules of differentiation
- → Techniques of differentiation
- → Rates of change, Tangents and Normal lines
- → Chain Rule
- → linear approximation
- → Implicit differentiation
- → Application of differentiation
- → Extreme value functions
- → Mean value theorems.
- → Maxima and minima of functions for single-variable
- → Concavity

Integral Calculus;

- → Concept and idea of integration
- → Indefinite integrals
- → Techniques of integration
- → Riemann sums and Definite integrals
- → Application of definite integrals
- → improper integrals
- → Applications of integration

Set of natural numbers $= N = \{1, 2, 3, 4, \dots\}$

whole " $= W = \{0, 1, 2, 3, \dots\}$

Integers " $= Z = \{0, \pm 1, \pm 2, \pm 3, \dots\}$

Even " $= E = \{0, \pm 2, \pm 4, \dots\}$

Even Numbers: The integers of the form $2n$ are called even numbers. where $n \in Z$.

ODD Numbers: The integers of the form $2n+1$, where $n \in Z$.

$\Gamma = \{\pm 1, \pm 3, \pm 5, \dots\}$

Positive Number: A number n is said to be +ve number if $n > 0$.

Negative Number: A number n is if $n < 0$.

Note: 0 is neither -ve nor +ve.

Prime Number: A number $p > 1$ is said to be prime if 1 and p are only its divisor.

$P = \{2, 3, 5, 7, 11, 13, \dots\}$

Note: 2 is the only even prime number all other prime number are odd.

Non-negative integers $= \{0, 1, 2, 3, \dots\}$

non-positive $= \{0, -1, -2, \dots\}$

Rational number:

A number of the form p/q wher $p, q \in Z$ and $q \neq 0$

e.g $1 = \frac{1}{1}$, $2 = \frac{2}{1}$, $3/2$.

$0 = \frac{0}{1}$ $\therefore$ 0 is rational

$\frac{1}{0}$ not rational

Irrational number:

The ~~union~~ of numbers which not be written in the form of p/q wher $p, q \in Z$ and $q \neq 0$

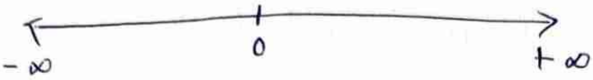
e.g: $\sqrt{2}$, $\sqrt{3}$, $\dots$, π

Real Number:

The union of rational and irrational numbers is called set of real numbers.

$$R = Q \cup Q'$$
$$= \{0, \pm 1, \pm\sqrt{2}, \ldots\}$$

Note: Real line denote the set of real numbers.



Absolute value of a real number:

Let $x \in R$.

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Theorem:

Let $x, y \in R$, then

i- $|x| = 0$ iff $x = 0$

ii- $|-x| = |x|$ for all $x \in R$.

iii- $|xy| = |x| \cdot |y|$ $\forall\ x, y \in R$.

iv- if $a \geq 0$ then $|x| \leq a$ iff $-a \leq x \leq a$

v- $|x + y| \leq |x| + |y|$ $\forall\ x, y \in R$.

Intervals:

open Interval: Let $a, b \in R$ and $a < b$. The set

$$(a, b) \text{ or }]a, b[= \{x \in R : a < x < b\}$$

is called open interval. determined by a and b.

Closed Interval:

$$[a, b] = \{x \in R : a \leq x \leq b\}$$

Half Open / Half Closed:

$$]a, b] = \{x \in R : a < x \leq b\}$$
$$[a, b[= \{x \in R : a \leq x < b\}$$

Prove that $||a|-|b|| \leq |a-b|$ for every $a,b \in R$.

Consider $|a| = |a-b+b|$

$\leq |a-b| + |b|$

$|a|-|b| \leq |a-b| \longrightarrow ①$

now

$|b| = |b-a+a|$

$\leq |b-a| + |a|$

$-|a|+|b| \leq |-(a-b)|$ *

$-|a|+|b| \leq |a-b|$

Multiplying both sides by $(-)$

$|a|-|b| \geq -|a-b| \longrightarrow ②$

By combining (1) and ②

$-|a-b| \leq |a|-|b| \leq |a-b| \quad \forall a,b \in R.$

$|x| \leq a$ iff $(\because -a \leq x \leq a)$

Q Express $3<x<7$ in modulus notation.

Subtracting 5

$3-5 < x-5 < 7-5$

$-2 < x-5 < 2$

$|x-5| < 2.$

Q If $\delta > 0$ and $a \in R$. Show that $a-\delta < x < a+\delta$ iff $|x-a| < \delta$

Consider $a-\delta < x < a+\delta$

Subtracting a.

$a-\delta-a < x-a < a+\delta-a$

$-\delta < x-a < \delta$

Conversly

$$|x - a| < \delta$$

$$-\delta < x - a < \delta$$

Adding a.

$$a - \delta < x - a + a < \delta + a$$

$$a - \delta < x < \delta + a$$

Working Rules for Solution of inequalities:

Step I
Convert the inequality into an eq. that is called associated eq.

Step II
Solve the associated eq. these solution is called boundary numbers of inequality.

Step III
Locate boundary numbers on the real line and real line divided into distinct region.

Step IV
Now check these regions by using orbitrary pt. (test pts) from the region.
The region whose test pts Satisfy the inequality are in the solution set.

Step V
Union of all those regions which belong to solution set makes the solution set of inequality.

3) Solve the inequalities. (Solutions should be in form of intervals)

Q $|2x+5| > |2-5x| \longrightarrow ①$

the associated eq.

$|2x+5| = |2-5x|$

$\pm(2x+5) = (2-5x)$

$2x+5 = 2-5x$; $-(2x+5) = 2-5x$

$2x+5x = 2-5$; $-2x-5 = 2-5x$

$7x = -3$; $-2x+5x = 2+5$

$\boxed{x = -3/7}$ $= -0.4$; $3x = 7$

$\boxed{x = 7/3}$

$-\infty$ $-3/7$ $7/3$ $+\infty$

Here possible intervals

$(-\infty, -3/7)$ $(-3/7, 7/3)$ $(7/3, +\infty)$

Check $(-\infty, -\frac{3}{7})$

Put $x = -1$ in eq ①

$|2(-1)+5| > |2-5(-1)|$

$|-2+5| > |2+5|$

$3 > 7$ false

Check $(-3/7, 7/3)$

Put $x = 0$ in eq ①

$|2(0)+5| > |2-5(0)|$

$|5| > |2|$ true.

$5 > 2$

Check $(7/3, +\infty)$

Put $x = 3$ in eq ①

$|2(3)+5| > |2-5(3)|$

$|11| > |-13|$ false.

$11 > 13$.

$S.S = (-3/7, 7/3)$

$\left|\frac{x+8}{12}\right| < \frac{x-1}{10}$.

the associated eq is.

$\left|\frac{x+8}{12}\right| = \frac{x-1}{10}$

$\pm\left(\frac{x+8}{12}\right) = \frac{x-1}{10}$

$\frac{x+8}{12} = \frac{x-1}{10}$

$10(x+8) = 12(x-1)$

$10x+80 = 12x-12$

$80+12 = 12x-10x$

$92 = 2x$

$x = 46$

Now

$-\left(\frac{x+8}{12}\right) = \frac{x-1}{10}$

$-10(x+8) = 12(x-1)$

$-10x-80 = 12x-12$

$-10x-12x = -12+80$

$-22x = 68$

$x = -\frac{68}{22}$

$x = -\frac{34}{11}$ $= -3.\ldots$

$(-\infty, -\frac{34}{11})$ $-34/11$ $+46$ $+\infty$

$(-\infty, -\frac{34}{11})$ $(-\frac{34}{11}, 46)$ $(46, +\infty)$

$S.S = (46, +\infty)$

$|x| + |x-1| > 1$

the associated eq, is.

$|x| + |x-1| = 1$

$\pm(x) \pm (x-1) = 1$

Case I $\quad x + x - 1 = 1$

$2x = 2 \Rightarrow x = 1$

Case II

$-x - (x-1) = 1$

$-2x + 1 = 1 \Rightarrow x = 0$

Case III

$+x - (x-1) = 1$

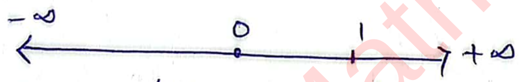
$x - x + 1 = 1$

$1 = 1$

Case IV

$-x + x - 1 = 1$

$-1 = 1$ not possible



Here possible intervals are

$(-\infty, 0)\ (0, 1)\ (1, \infty)$

Check. $(-\infty, 0)$

put $x = -1$ in eq ①

$|-1| + |-1-1| > 1$

$1 + 2 > 1$

$3 > 1$ true

Check $(0, 1)$

put $x = 0.5$ in eq ①

$|0.5| + |0.5 - 1| > 1$

$0.5 + 0.5 > 1$

$1 > 1$ false.

Check $(1, \infty)$

Put $x = 2$.

$|2| + |2-1| > 1$

$2 + 1 > 1 \Rightarrow 3 > 1$ true.

$S.S = (-\infty, 0) \cup (1, \infty)$.

Completeness property of R.

Upper Bound:

Let S be a non-empty subset of real numbers.
An element $M \in R$ is called an upper bound of S if $x \leq M$ for all $x \in S$.

Lower Bound:

An element $m \in R$ is called lower bound of S if $m \leq x$ for all $x \in S$.

Note:

→ If S is bounded above, then an upper bound M of S is called least upper bound. or Supremum. if it is less than any other upper bound of S. $M = \sup S$

→ If S is bounded below, then a lower bound m of S is called greatest lower bound. or infimum.

$$m = \inf S.$$

Example:

$S = \{1, 2, 3, \ldots, 20\}$

$M \geq 20$ is an upper bound of S.

$m \leq 1$ is a lower bound of S.

$S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\}$

upper bound is 1

Lower bound is 0.

Bounded set:

Let $S: N \to R$ be defined by $S_n = (-1)^n + (1)^n$ then S is bounded, since $\text{Rng}\, S = \{0, 2\}$ is bounded.

Unbounded set:

Let $f: R \to R$ be defined by $f(x) = x^2$

f is not bounded.

$\text{Rng}\, f$ has no upper bound so its unbounded set.

Binary Relation:

Let $A = \{2, 4, 6\}$ $B = \{1, 3\}$

Then Cartesian product $A \times B$ of A and B is

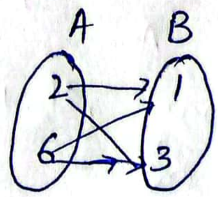
$A \times B = \{(2,1), (2,3), (4,1), (4,3), (6,1), (6,3)\}$

Then any subset of $A \times B$ is called B.R of $A \times B$.

e.g.

$R_1 = \{(2,1)(2,3), (6,1)(6,3)\}$

This is B.R of $A \times B$.



Function :

A function is defined as a relation between a set of inputs having one output each. (each input is related to exactly one output).

Every function has a domain and codomain or range.

A function is denoted by $f(x)$. where x is input.

General representation.

$$y = f(x).$$

Domain :

The set of elements of A That occurs in the first position of member of f.

Range :

The set of element of B.

Into function : Let $f: A \to B$ be a function Such that Rng $f = B$. then f is called onto or surjective function.

If Rng $f \neq B$. then f is Called into function.

one-one function :(injective)

when there is mapping for a range for each domain between two sets

Bijective Function :

A function $f: A \to B$ which is both one-to-one and onto is called bijective function. or one-to-one correspondence.

Let A and B be any two non-empty sets. f is a binary relation from set A to B then f is Called function from A to B if

i- Dom f = A

ii- In binary relation f every element of set A is attached only one

Identify Domain and Range of given functions.

Function	Domain	Range
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{1-x^2}$	$[-1, 1]$	$[0, 1]$

$y = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range is $[0, \infty)$ because square of any real number is non-negative.

⇒ $y = 1/x$ gives a real y-value for any real number x except $x = 0$. We cannot divide any number by zero. The range of $y = 1/x$, the set of all non-zero real number.

⇒ The formula $y = \sqrt{1-x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Outside this domain, $1-x^2$ is negative and its square root is not a real number. The values of $1-x^2$ vary from 0 to 1. on the given domain.
So the range of $\sqrt{1-x^2}$ is $[0, 1]$.

Definition:

Let $f: A \to B$ and $g: B \to C$ be two given functions. The composite function $g \circ f$ (or the composition $g \circ f$) is defined by the rule

$$(g \circ f)x = g(f(x))$$

The domain of $g \circ f$ is the set of all x in the domain of f for which $f(x)$ is in the domain of g.

Example:

Let f, g be functions on R to R. defined by

$f(x) = 2x, \quad g(x) = 3x^2 - 1$, then

$$(g \circ f)x = g(f(x))$$
$$= 3(2x)^2 - 1 = 3(4x^2) - 1 = 12x^2 - 1$$

$$(f \circ g)x = f(g(x))$$
$$= f(3x^2 - 1) = 2(3x^2 - 1) = 6x^2 - 2$$

thus $f \circ g \neq g \circ f$

Further note that

$$(fg)(x) = f(x)\,g(x)$$
$$= 2x(3x^2 - 1) = 6x^3 - 2x$$

$$(gf)(x) = g(x)\,f(x)$$
$$= (3x^2 - 1)(2x) = 6x^3 - 2x$$

Thus $fg = gf$.

Monotone Sequences and functions:

A sequence $\{S_n\}$ of real number is said to be

i) nondecreasing if $S_n \leq S_{n+1}$

ii) non increasing if $S_n \geq S_{n+1}$

iii) increasing if $S_n < S_{n+1}$

iv) decreasing if $S_n > S_{n+1}$

for all $n \in N$.

These sequences are called monotone (or monotic).

A function $f: R \to R$ is said to be

nondecreasing if $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$

non increasing if $x_1 > x_2$ implies $f(x_1) \geq f(x_2)$

for all $x_1, x_2 \in \text{Dom} f$.

increasing if $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

decreasing if $x_1 > x_2$ implies $f(x_1) > f(x_2)$.

for all $x_1, x_2 \in \text{Dom} f$.

Example:

The function $f: [0, \infty[\to R$ given by $f(x) = x^2$ is increasing on $[0, \infty[$.

while $f:]-\infty, 0] \to R$ defined by $f(x) = x^2$ is decreasing on $]-\infty, 0]$.

Limit : (Finite Limit at finite point).

Let a be any real number and let f be a function from R to R which is defined for all values of x near a with the possible exception of the point $x=a$. The function f is said to have limit l as x approaches a if for every $\varepsilon > 0$, there exist a positive real number δ (which usually depends on ε) such that

$$|f(x) - l| < \varepsilon \quad \text{whenever} \quad 0 < |x-a| < \delta$$

In this case, we write

$$\lim_{x \to a} f(x) = l.$$

and say that the function f has the limit l (or $f(x)$ converges to l). as x approaches a. or $f(x) \to l$ as $x \to a$.

Right Hand Limit:

A function f is said to be right hand limit l_1 as x tends to a through values greater than a (i.e $x \to a^+$) if for every $\varepsilon > 0$, there exist a $\delta > 0$ such that

$$|f(x) - l_1| < \varepsilon \quad \text{whenever} \quad 0 < x \leq a + \delta$$

in this case we write $\lim\limits_{x \to a^+} f(x) = l_1$

Left Hand Limit:

A function f is said to have the left hand limit l_2 as x tends to a through values less than a (i.e $x \to a^-$) if for every $\varepsilon > 0$, there exist a $\delta > 0$ such that

$$|f(x) - l_2| < \varepsilon \quad \text{whenever} \quad a - \delta \leq x < a$$

and we write $\lim\limits_{x \to a^-} f(x) = l_2$

Prove that $\lim_{x \to 4} \frac{1}{2}(3x-1) = \frac{11}{2}$

Sol// Let $f(x) = \frac{1}{2}(3x-1)$, $a=4$ and $L = \frac{11}{2}$

According to definition, we must show that for every $\varepsilon > 0$, there exist a $\delta > 0$ such that

if $0 < |x-4| < \delta$, then $\left|\frac{1}{2}(3x-1) - \frac{11}{2}\right| < \varepsilon$

δ can be found by examining last inequality involving ε. The following is a list of equivalent inequalities.

$$\left|\frac{1}{2}(3x-1) - \frac{11}{2}\right| < \varepsilon$$

$$\frac{1}{2}|(3x-1) - 11| < \varepsilon$$

$$|3x-1-11| < 2\varepsilon$$

$$|3x-12| < 2\varepsilon$$

$$3|x-4| < 2\varepsilon$$

$$|x-4| < \frac{2}{3}\varepsilon$$

If we let $\delta = \frac{2}{3}\varepsilon$, then if $0 < |x-4| < \delta$. the last inequality in the list is true.

So by definition $\lim_{x \to 4} \frac{1}{2}(3x-1) = \frac{11}{2}$.

Prove by definition that $\lim_{x \to 3} 2x = 6$

Sol// for each $\varepsilon > 0$, we want to find a $\delta > 0$. such that $|2x-6| < \varepsilon$ whenever. $0 < |x-3| < \delta$.

we have $|2x-6| = 2|x-3|$ which leads us to chose $\delta = \frac{\varepsilon}{2}$.

thus for every $\varepsilon > 0$. if we chose $\delta = \varepsilon/2$ then

$$|2x-6| = 2|x-3| < 2\delta = \varepsilon$$

whenever $0 < |x-3| < \delta$.

Example: $\lim\limits_{x\to 3} \dfrac{x^3-27}{x^2-9}$

write L.H.L and R.H.L.

Sol// L.H.L

$$\lim_{x\to 3-0} \frac{x^3-27}{x^2-9} = \lim_{h\to 0} \frac{(3-h)^3-27}{(3-h)^2-9}$$

$$= \lim_{h\to 0} \frac{\cancel{(3-h-3)}\left((3-h)^2+3(3-h)+9\right)}{\cancel{(3-h-3)}(3-h+3)}$$

$$= \lim_{h\to 0} \frac{9+h^2-6h+9-3h+9}{6-h} = \lim_{h\to 0} \frac{27+h^2-9h}{6-h}$$

$$= \frac{27}{6} = \frac{9}{2} \longrightarrow (1)$$

R.H.L

$$\lim_{x\to 3+0} \frac{x^3-27}{x^2-9} = \lim_{h\to 0} \frac{(3+h)^3-27}{(3+h)^2-9}$$

$$= \lim_{h\to 0} \frac{\cancel{(3+h-3)}\left((3+h)^2+3(3+h)+(3)^2\right)}{\cancel{(3+h-3)}(3+h+3)}$$

$$= \lim_{h\to 0} \frac{9+h^2+6h+9+3h+9}{6+h} \quad \lim_{h\to 0} \frac{27+h^2+9h}{6+h} = \frac{27}{6}$$

$$= \frac{9}{2} \longrightarrow (2)$$

From (1) and (2)

L.H.L = R.H.L

$$\lim_{x\to 3} \frac{x^3-27}{x^2-9} = \frac{9}{2} \quad \text{Ans}$$

Theorems on Limits:

Limit of constant function:

If $f(x) = c$ for all $x \in R$, where c is a fixed real number, then

$$\lim_{x \to a} f(x) = c \text{ for every real number } a.$$

Identity Function:

$$\lim_{x \to a} f(x) = a. \text{ for every } a \in R.$$

Sum (Difference).

Let $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$

then

$$\lim_{x \to a} [f(x) \pm g(x)] = l \pm m$$
$$= \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

Product:

Let $\lim_{x \to a} f(x) = l$, $\lim_{x \to a} g(x) = m$

Then

$$\lim_{x \to a} (f(x) \cdot g(x)) = lm$$
$$= \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

Quotient:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}.$$

Inequalities:

If $f(x) \le g(x)$ $\forall x$ in some interval a, then $l \le m$.

Sandwitch Theorem:

Suppose that f, g and h are functions defined on $0 < |x - a| < K$. If $f(x) \le g(x) \le h(x)$ on this domain.

and if $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$

then $\lim_{x \to a} g(x) = l$

Some important Formulas:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$$

$$\lim_{x\to 0} \frac{x^n - a^n}{x-a} = n a^{n-1}$$

Q Find the limits:-

$$\lim_{x\to 1} \frac{x^3-1}{x-1}$$

$\lim_{x\to 1} \frac{x^3-1}{x-1}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1}$$

$$= \lim_{x\to 1} (x^2+x+1)$$

$$= 1^2+1+1$$

$$= 3.$$

$$\lim_{y\to x} \frac{y^{2/3} - x^{2/3}}{y-x}$$

$$= \lim_{y\to x} \frac{(y^{1/3})^2 - (x^{1/3})^2}{(y^{1/3})^3 - (x^{1/3})^3}$$

$$= \lim_{y\to x} \frac{(y^{1/3} + x^{1/3})(y^{1/3} - x^{1/3})}{(y^{1/3} - x^{1/3})((y^{1/3})^2 + y^{1/3}x^{1/3} + (x^{1/3})^2)}$$

$$= \lim_{y\to x} \frac{y^{1/3} + x^{1/3}}{y^{2/3} + y^{1/3}x^{1/3} + x^{2/3}}$$

$$= \frac{x^{1/3} + x^{1/3}}{x^{2/3} + x^{1/3}x^{1/3} + x^{2/3}}$$

$$= \frac{2x^{1/3}}{x^{2/3} + x^{2/3} + x^{2/3}}$$

$$= \frac{2}{3} \frac{x^{1/3}}{x^{2/3}}$$

$$= \frac{2}{3} \frac{1}{x^{2/3 - 1/3}}$$

$$= \frac{2}{3x^{1/3}} \quad \text{Ans}$$

Q $\lim_{x\to 0} \frac{\csc x - \cot x}{x}$

$$= \lim_{x\to 0} \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x}$$

$$= \lim_{x\to 0} \frac{1}{x}\left[\frac{1-\cos x}{\sin x}\right]$$

$$= \lim_{x\to 0} \frac{1}{x}\left[\frac{1-\cos x}{\sin x} \times \frac{1+\cos x}{1+\cos x}\right]$$

$$= \lim_{x\to 0} \frac{1}{x}\left[\frac{1-\cos^2 x}{\sin x(1+\cos x)}\right]$$

$$= \lim_{x\to 0} \frac{1}{x}\left[\frac{\sin^2 x}{\sin x(1+\cos x)}\right]$$

$$= \lim_{x\to 0} \frac{1}{x}\left[\frac{\sin x}{1+\cos x}\right]$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{1+\cos x}\right)$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{1+\cos x}$$

$$= (1) \cdot \frac{1}{1+1} = \frac{1}{2}. \quad \text{Ans}$$

$$\lim_{x \to \pi} \frac{\tan(\sin x)}{\sin x}$$

Let $\sin x = \theta$

When $x \to \pi$, $\theta \to 0$

$$= \lim_{\theta \to 0} \frac{\tan\theta}{\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta \to 0} \frac{1}{\cos\theta}$$

$$= (1) \frac{1}{\cos 0} = 1 \times \frac{1}{1}$$

$= 1$ Ans

$$\lim_{x \to 1^-} \frac{\sqrt{1-x^2}}{1-x}$$

$$= \lim_{x \to 1^-} \frac{\sqrt{(1-x)(1+x)}}{1-x}$$

$$= \lim_{x \to 1^-} \frac{\sqrt{1-x}\,\sqrt{1+x}}{1-x}$$

$$= \lim_{x \to 1^-} \frac{\sqrt{1+x}}{\sqrt{1-x}}$$

Put $x = 1-h$

$h \to 0$, $x \to 1$

$$= \lim_{h \to 0} \frac{\sqrt{1+1-h}}{\sqrt{1-1+h}}$$

$$= \lim_{h \to 0} \frac{\sqrt{2-h}}{\sqrt{h}}$$

$$= \frac{2}{0} = \infty$$

$$\lim_{x \to -2^-} \frac{x^2+2x-8}{x^2-4}$$

$$= \lim_{x \to -2^-} \frac{x^2+4x-2x-8}{(x+2)(x-2)}$$

$$= \lim_{x \to -2^-} \frac{x(x+4)-2(x+4)}{(x+2)(x-2)}$$

$$= \lim_{x \to -2^-} \frac{(x-2)(x+4)}{(x+2)(x-2)}$$

$$= \lim_{x \to -2^-} \frac{(x+4)}{(x+2)}$$

Put $x = -2-h$

$h \to 0$, $x \to -2$

$$= \lim_{h \to 0} \frac{-2-h+4}{-2-h+2}$$

$$= \lim_{h \to 0} \frac{2-h}{-h}$$

$$= \lim_{h \to 0} \frac{2}{-h} + 1$$

$$= \infty + 1 = \infty.$$

Theorem:

Limit of polynomials can be found by substitution.

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \dots + a_0$$

Theorem:

If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x).$$

Limit as x approaches ∞ or $-\infty$

we say that $f(x)$ has the limit L as x approaches infinity. and we write.

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exist a corresponding number M such that for all x.

$$x > M \Rightarrow |f(x) - L| < \epsilon$$

We say that $f(x)$ has the limit L as x approaches minus infinity and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exist a corresponding number N such that for all x.

$$x < N \Rightarrow |f(x) - L| < \epsilon.$$

Continuity:

A function f is said to be continuous at a point $a \in Dom f$ if

i) The point a lies in an open interval contained in $Dom f$

ii) $\lim_{x \to a} f(x) = f(a)$.

or

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a).$$

Continuity seems to be almost equivalent to that of the existence of $\lim_{x \to a} f(x)$ with the exception that for the limit the point a may not belong to $Dom f$, but for continuity it is essential that the function must be defined at a.

A function which is not continuous is called discontinuous fn.

Example:

Discuss the continuity of f defined by

$$f(x) = \begin{cases} \dfrac{x-4}{\sqrt{x}-2} & \text{if} \quad x \geq 0 \quad \text{and } x \neq 4 \\ 4 & \text{if} \quad x = 4 \end{cases}$$

Sol

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x \to 4} \frac{(\sqrt{x})^2-(2)^2}{\sqrt{x}-2}$$

$$= \lim_{x \to 4} (\sqrt{x}+2)$$

$$= 2+2$$

$$= 4$$

Thus f, is continuous at $x=4$.

Limit of Piecewise Function

Q

$$f(x) = \begin{cases} 3 & \text{if} \quad x \le -2 \\ -\frac{1}{2}x^2 & \text{if} \quad -2 < x < 2 \\ 3 & \text{if} \quad x \ge 2 \end{cases}$$

Find $\lim\limits_{x \to \pm 2^+} f(x)$ and $\lim\limits_{x \to \pm 2^-} f(x)$.

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} (3) = 3$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2} \left(-\frac{1}{2}x^2\right) = -\frac{1}{2}(2)^2 = -2$$

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2} \left(-\frac{1}{2}x^2\right) = -2$$

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2} 3 = 3$$

Q

$$f(x) = \begin{cases} x+2 & \text{if} \quad x \le -1 \\ ax^2 & \text{if} \quad x > -1 \end{cases}$$

Find a so that $\lim\limits_{x \to -1} f(x)$ exist.

Because given that $\lim\limits_{x \to -1} f(x)$ exist

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x)$$

$$\lim_{x \to -1} (ax^2) = \lim_{x \to -1} (x+2)$$

$$a(-1)^2 = (-1+2)$$

$$+a = 1$$

$$a = 1$$

Examples

Let $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Examine the continuity of f at $x=0$.

Sol:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1 - 1/e^{1/x}}{1 + 1/e^{1/x}} = \frac{1-0}{1+0} = 1$$

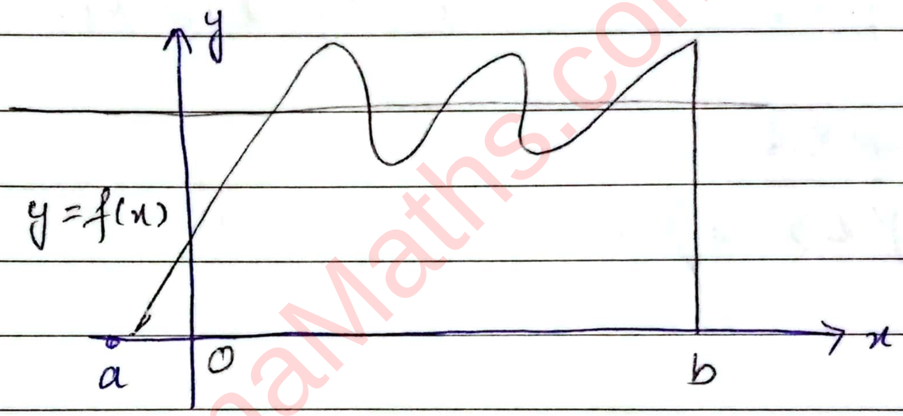
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^{1/x}-1}{e^{1/x}+1} = \frac{0-1}{0+1} = -1$$

Thus $\lim_{x\to 0} f(x)$ does not exist and so f is discontinuous at $x=0$.

Intermediate Value Theorem:

Let f be continuous on $[a, b]$ and $c \in \mathbb{R}$ such that $f(a) < c$ and $f(b) > c$. Then there is atleast one point $x_0 \in [a, b]$ such that $f(x_0) = c$

There may be more than one such points x_0 as is clear from the figure.



Theorem:

If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs, then $f(c_0) = 0$ for at least one $c_0 \in]a, b[$.

Boundedness Theorem:

If f is continuous on $[a, b]$, then it is bounded theorem, (i.e the range of f is bounded.

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Extreme Value Theorem:

If f is continuous on $[a, b]$ and M and m are supremum and infimum of f respectively on this interval, then f assumes each of the values M and m at least once in $[a, b]$ i.e,
there exist $c, d \in [a, b]$ such that
$f(c) = m$ and $f(d) = M$.

Example:

$$g(x) = \begin{cases} x^3 & \text{if } x < 1 \\ -4-x^2 & \text{if } 1 \le x \le 10 \\ 6x^2+46 & \text{if } x > 10 \end{cases}$$

Find discontinuity of fn.

Solution:

At $x = 1$

$$g(1) = -4-(1)^2 = -4-1 = -5$$

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (x^3) = (1)^3 = 1$$

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} -4-x^2 = -4-1 = -5$$

$$\lim_{x \to 1^-} g(x) \ne \lim_{x \to 1^+} g(x).$$

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$g(x)$ is discont at $x = 1$.

At $x = 10$

$$g(10) = -4-(10)^2 = -4-100$$
$$= -104$$

$$\lim_{x \to 10^-} g(x) = \lim_{x \to 10} -4-x^2$$

$$= -4-(10)^2 = -104$$

$$\lim_{x \to 10^+} g(x) = \lim_{x \to 10^+} 6(10)^2+46$$

$$= 6(100)+46$$
$$= 646$$

$$\lim_{x \to 10^+} g(x) \neq \lim_{x \to 10^-} g(x).$$

$g(x)$ is also discontinuous at $x = 10$.

Differentiation

Definition:

Let f be a function defined on an open interval containing point x. The derivative of f at x, is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{provided the limit exists}$$

A function f is differentiable on an open interval $]a,b[$ if it is differentiable at each point of $]a,b[$.

A function f is differentiable on a closed interval $[a,b]$ if i) it is differentiable on the open interval $]a,b[$
ii) its right hand derivative at a and left hand derivative at b exist.

A function f is differentiable if and only if it is differentiable at each point of its domain.

Theorem:

If f is differentiable at a point $a \in \text{Dom} f$, then f is continuous at a.

Proof:

To prove that f is continuous at a, we need to show that

$$\lim_{x \to a} f(x) = f(a)$$

i.e
$$\lim_{x \to a} [f(x) - f(a)] = 0$$

Now.
$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} (x - a)$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a} (x - a)$$

$$= f'(a) \cdot 0 = 0$$

Thus f is continuous at a.

NOTE: The converse of this theorem is false. i.e a continuous function may not be differentiable.

Example:
Differentiate $f(x) = \dfrac{x}{x-1}$ (by definition).

Sol/ $f(x) = \dfrac{x}{x-1}$ and $f(x+h) = \dfrac{(x+h)}{(x+h)-1}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h} = \lim_{h\to 0} \frac{\dfrac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{(x^2 - x + xh - h) - x^2 - xh + x}{(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{-h}{(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)}$$

By applying limit we get

$$= -\frac{1}{(x-1)^2}$$

Notations:

Some common notations of derivatives are

$$f'(x),\quad y',\quad \frac{dy}{dx},\quad \frac{df}{dx},\quad \frac{d}{dx}f(x).$$

$$D(f(x)),\quad D_x f(x).$$

Derivative at specific number.

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a}.$$

Intermediate Value Theorem:

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between f'(a) and f'(b).

Q:

If $f(x) = 4 - x^2$; find $f'(-3)$, $f'(0)$, $f'(1)$

$$f(x) = 4 - x^2$$

$$f(x+h) = 4 - (x+h)^2$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{4 - (x+h)^2 - (4 - x^2)}{h}$$

$$= \lim_{h\to 0} \frac{4 - (x^2 + 2xh + h^2) - 4 + x^2}{h}$$

$$= \lim_{h\to 0} \frac{4 - \cancel{x^2} - 2xh - h^2 - \cancel{4} + \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{-2xh - h^2}{h^2} = \lim_{h\to 0} \frac{h(-2x - h)}{h}$$

$$= \lim_{h\to 0} (-2x - h) = -2x. \quad \text{(By applying limit)}$$

$f'(-3) = -2(-3) = 6$

$f'(0) = -2(0) = 0$

$f'(1) = -2(1) = -2.$

Q $f(s) = \sqrt{2s+1}$ find $f'(0), f'(1), f'(1/2)$

Sol

$f(s) = \sqrt{2s+1}$

$f(s+h) = \sqrt{2(s+h)+1}$

$$f'(s) = \lim_{h \to 0} \frac{f(s+h) - f(s)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2(s+h)+1} - \sqrt{2s+1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2(s+h)+1} - \sqrt{2s+1}}{h} \times \frac{\sqrt{2s+2h+1} + \sqrt{2s+1}}{\sqrt{2s+2h+1} + \sqrt{2s+1}}$$

$$= \lim_{h \to 0} \frac{(\sqrt{2s+2h+1})^2 - (\sqrt{2s+1})^2}{h(\sqrt{2s+2h+1} + \sqrt{2s+1})}$$

$$= \lim_{h \to 0} \frac{2s+2h+1 - 2s - 1}{h(\sqrt{2s+2h+1} + \sqrt{2s+1})}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{2s+2h+1}+\sqrt{2s+1}}$$

Applying limit

$$= \frac{2}{\sqrt{2s+1}+\sqrt{2s+1}} = \frac{2}{2(\sqrt{2s+1})}$$

$$f'(s) = \frac{1}{\sqrt{2s+1}}$$

$$f'(0) = \frac{1}{\sqrt{2(0)+1}} = \frac{1}{1} = 1$$

$$f'(1) = \frac{1}{\sqrt{2(1)+1}} = \frac{1}{\sqrt{3}}$$

$$f'(1/2) = \frac{1}{\sqrt{2(1/2)+1}} = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$$

Q Let $f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2x-1 & \text{if } 1 < x \le 2 \end{cases}$

Discuss the continuity and differentiability of f at $x=1$.

Sol: first we will discuss the continuity of f

Here $f(1) = 1$

$$f(1^-) = \lim_{x\to 1^-} f(x)$$

$$= \lim_{x\to 1^-} (x)$$

$$f(1^-) = 1$$

$$f(1^+) = \lim_{x\to 1^+} f(x)$$

$$= \lim_{x\to 1^+} (2x-1)$$

$$= 2(1)-1$$

$$= 1$$

Since

$$f(1^-) = f(1^+) = f(1)$$

So f is continuous at $x=1$.

As

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

So $$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$Lf'(1) = \lim_{h\to 0^-} \frac{1+h-1}{h} = \frac{h}{h} = 1$$

$$Rf'(1) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0^+} \frac{[2(1+h)-1]-1}{h} = \lim_{h\to 0^+} \frac{[2+2h-1]-1}{h}$$

$$= \lim_{h\to 0^+} \frac{2h}{h} = 2.$$

Since

$Lf'(1) \neq Rf'(1)$.

So f is not differentiable at $x=1$.

Q If $f(x) = \begin{cases} x^2 \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$

Show that $f(x)$ is continuous and differentiable at $x=0$.

Sol Here $f(0) = 0$

Now

$$f(0^-) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 \sin 1/x$$

$$= \lim_{h\to 0} (-h)^2 \sin\left(-1/h\right) = \lim_{h\to 0} -h^2 \sin 1/h$$

$$= -(0)^2 \underbrace{\sin(1/0)}_{\text{Some no. in } [-1,1]} = 0$$

$$f(0^-) = 0.$$

$$f(0^+) = \lim_{x\to 0^+}$$

$$= 0.$$

Since $f(0^-) = f(0^+) = f(0)$.
So f is continuous at $x = 0$.

As $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} \Rightarrow f'(0) = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h}$

Now.

$$Lf'(0) = \lim_{h \to 0^-} \frac{h^2 \sin 1/h - 0}{h}$$

$$= \lim_{h \to 0^-} (h \sin 1/h)$$

$$= 0 (\text{some number})$$

$$= 0.$$

$$Rf'(0) =$$

$$= 0$$

Since $Lf'(0) = Rf'(0)$.
So f is derivable at $x = 0$.

Q

$$f(x) = \begin{cases} (x-a)\sin\left(\frac{1}{x-a}\right) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$$

Is this fn continuous and differentiable at $x = a$?