Vector Space

24-02-2014

Def:- Let "V" be a non-empty set with two operations:

1):- Vector Addition:- This assigns to any $u, v \in V$, a sum $u+v \in V$.

2):- Scalar Multiplication:- This assigns to any $u \in V$, $k \in K$ (scalar field), a product $ku \in V$

*(this is scalar multiplication).

Then V is called a vector space (over the field K) if the following axioms hold for any vectors u, v, w in V.

A(I):- $(u+v)+w = u+(v+w)$

A II:- There is a vector in V denoted by "0" and called the **zero vector** such that for any $u \in V$.

$$u+0 = 0+u = u$$

A III):- For each $u \in V$ there is a vector in V denoted by $-u$ and called the **negative of u** such that,

$$u+(-u) = -u+u = 0$$

A IV):- $u+v = v+u$ (commutative under "+")

M i):- $k(u+v) = ku + kv$, for any scalar $k \in K$

M ii):- $(k_1+k_2)u = k_1u + k_2u$, for any scalars $k_1, k_2 \in K$

M III):– $(b_1 b_2)u = b_1(b_2 u)$, for any scalars $b_1, b_2 \in K$

M IV):– $1u = u$, for scalar $1 \in K$

The above axioms naturly splits into two sets. The first four are only concerned with the additive structure of "V" and can be summarized by saying "V" is a commutative group under addition. On the other hand the remaining four axioms are concerned with the action of the field K of scalars on the vector space V. Using these additional axioms we prove the following simple properties of a vector space:–

Theorem:– Let "V" be a vector space over a field K

(i): For any scalar $b \in K$ and $0 \in V$, $0b = b0 = 0$

(ii):– For $0 \in K$ & any vector $u \in V$ then $0u = 0$

(iii):– if $bu = 0$ where $b \in K$ & $u \in V$ then either $b = 0$ or $u = 0$

(IV):– For any $b \in K$ and any $u \in V$, $(-b)u = b(-u) = -bu$

Examples of vector space:–

(i):– Space K^n

Linear Combinations 28-02-2014

Let "V" be a vector space over a field K. A vector v in V is a linear combination of vectors $u_1, u_2, \dots, u_m$ in V if $\exists$ scalars $k_1, k_2, \dots, k_m$ in K such that

$$v = k_1 u_1 + k_2 u_2 + \dots + k_m u_m$$

Alternatively v is a linear combination of $u_1, u_2, \dots, u_m$ if there is solution to the vector eq. $V = x_1 u_1 + x_2 u_2 + \dots + x_m u_m$ where $x_1, x_2, \dots, x_m$ are unknown scalars.

Example:– (Linear Combination in $\mathbb{R}^n$)

Suppose we want to express $v = (3, 7, -4) \in \mathbb{R}^3$ as a linear combination of the vectors

$$u_1 = (1, 2, 3)$$
$$u_2 = (2, 3, 7)$$
$$u_3 = (3, 5, 6)$$

We seek scalars x, y, z such that

$$V = x u_1 + y u_2 + z u_3$$

that is

$$\begin{bmatrix} 3 \\ 7 \\ -4 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} + z \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

$$x + 2y + 3z = 3$$
$$2x + 3y + 5z = 7$$
$$3x + 7y + 6z = -4$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 5 & 7 \\ 3 & 7 & 6 & -4 \end{array}\right]$$

$R_2 - 2R_1$, $R_3 - 3R_1$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & -3 & -13 \end{array}\right]$$

$R_3 + R_2$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -4 & -12 \end{array}\right]$$

$-R_2$, $-\frac{1}{4}R_3$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$x + 2y + 3z = 3 \longrightarrow (1)$

$y + z = -1 \longrightarrow (2)$

$\boxed{z = 3} \longrightarrow (3)$

$(2) \Rightarrow \quad y + 3 = -1 \Rightarrow \boxed{y = -4}$

$(1) \Rightarrow \quad x + 2(-4) + 3(3) = 3$

$x - 8 + 9 = 3$

$x + 1 = 3$

$\boxed{x = 2}$

$\Rightarrow \quad V = 2u_1 - 4u_2 + 3u_3$

Linear Combination In P(t)

denote the set of all real polynomials of the form

Polynomial space is also a vector space.

$p(t) \in P(t)$

$\Rightarrow p(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n$

where $b_0, b_1, b_2, \ldots, b_n \in K$ (real field).

$(n = 1, 2, \ldots\ldots)$

Example: Suppose we want to express the polynomial $v = 3t^2 + 5t - 5$ as a linear combination of the polynomials

$$p_1 = t^2 + 2t + 1$$

$$p_2 = 2t^2 + 5t + 4$$

$$p_3 = t^2 + 3t + 6$$

we seek scalars x, y, z such that

$$v = xp_1 + yp_2 + zp_3$$

$$3t^2 + 5t - 5 = x(t^2 + 2t + 1) + y(2t^2 + 5t + 4) + z(t^2 + 3t + 6)$$

$$3t^2 + 5t - 5 = t^2(x + 2y + z) + t(2x + 5y + 3z) + (x + 4y + 6z)$$

By comparing the coefficients of $t^2, t, 1$.

$$x + 2y + z = 3$$

$$2x + 5y + 3z = 5$$

$$x + 4y + 6z = -5$$

$$\left|\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 2 & 5 & 3 & 5 \\ 1 & 4 & 6 & -5 \end{array}\right|$$

$R_2 - 2R_1,\ R_3 - R_1$

$$\underset{R}{\sim}\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 5 & -8 \end{array}\right]$$

$R_3 - 2R_2$

$$\underset{R}{\sim}\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -6 \end{array}\right]$$

$\frac{1}{3}R_3$

$$\underset{R}{\sim}\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$x + 2y + z = 3 \longrightarrow (1)$$

$$y + z = -1 \longrightarrow (2)$$

$$\boxed{z = -2} \longrightarrow (3)$$

$$(2) \Rightarrow y - 2 = -1$$

$$\boxed{y = 1}$$

$$(1) \Rightarrow x + \cancel{2(1)} \cancel{-2} = 3$$

$$\Rightarrow \boxed{x = 3}$$

$$\Rightarrow v = 3P_1 + P_2 - 2P_3$$

Spanning Set 03-03-2014

The vectors $u_1, u_2, \ldots, u_m$ in V are said to span V or to form a spanning set of V if every v in V is a linear combination of the vectors $u_1, u_2, \ldots, u_m$ in V.

i.e if $\exists$ scalars $b_1, b_2, \ldots, b_m$ in K such that

$$v = b_1 u_1 + b_2 u_2 + \cdots + b_m u_m$$

Remark:- i) Suppose $u_1, u_2, \ldots, u_m$ span V then for any vector 'w' then the set $w, u_1, u_2, \ldots, u_m$ also spans V.

II) Suppose $u_1, u_2, \ldots, u_m$ spans V & suppose u_k is linear combination of some of the vectors u_i's. Then the u_i's without the u_k also spans V.

$$u_k = k_1 u_1 + \cdots + b_n u_n + b_{n+1} u_{n+1} + \cdots + b_m u_m$$

$$\Rightarrow u_k = b_1 u_1 + b_2 u_2 + \cdots + b_n u_n$$

(III):- Suppose $u_1, u_2, \ldots, u_m$ spans V and suppose one of the u_i's is the zero vector then the u_i's without the zero vector also spans V.

$$v = u_k = b_1 u_1 + b_2 u_2 + \cdots + b_{k-1} u_{k-1}$$

Example:- Consider the vector space
$V = \mathbb{R}^3$

(i). We claim that the following vectors form a spaning set of $\mathbb{R}^3$

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

Specifically if $v = (b_1, b_2, b_3)$ in any vector $\mathbb{R}^3$, then

$\Rightarrow v = b_1 e_1 + b_2 e_2 + b_3 e_3$

For example

$v = (5, -6, 2)$

$v = 5e_1 - 6e_2 + 2e_3$

(ii):- We claim that the following vectors also form a spaning set of $\mathbb{R}^3$

$w_1 = (1,1,1)$, $w_2 = (1,1,0)$, $w_3 = (1,0,0)$

specifically if $v = (b_1, b_2, b_3)$ is any vector in $\mathbb{R}^3$ then

$v = (b_1, b_2, b_3) = b_3 w_1 + (b_2 - b_3) w_2 + (b_1 - b_2) w_3$

$v = (5, -6, 2) = 2w_1 - 8w_2 + 11w_3$

(III):- One can show that

$v = (2, 7, 8)$

$u_1 = (1,2,3)$, $u_2 = (1,3,5)$, $u_3 = (1,5,9)$

Accordinly u_1, u_2, u_3 do not span $\mathbb{R}^3$.

because $v = b_1 u_1 + b_2 u_2 + b_3 u_3$

$$R.H.S = 2(1,2,3)+7(1,3,5)+8(1,5,9)$$
$$= (2+7+8,\ 4+21+40,\ 6+35+72)$$
$$= (17, 65, 113)$$
$$\neq L.H.S$$

Subspace 6-03-14

Let V be a vector space over a field K and let W be a subset of V. Then W is a subspace of V if W is itself a vector space over K w.r.t the operations of vector addition and scalar multiplication on V.

Theorem:- (Subspace)

Suppose W is a subset of a vector space V. Then W is a subspace of V if the following two conditions hold.

a):- The zero vector 0 belongs to W.

b):- For every $u, v \in W$, $k \in K$;

(i):- The sum
$$u+v \in W$$

(ii):- The multiple $ku \in W$.

Ch#6 V.S. structure 13-3-2014

Proposition:—

Let $V(F)$ be a vector space. If 0_v & 0_F be the additive identities of the group V & field F respectively. Then,

(i):— $a0_v = 0_v$; $\forall a \in F$

(ii):— $0_F v = 0_v$

(iii):— $av = 0_v \Rightarrow$ either $a = 0_F$ or $v = 0_v$

(iv):— $(-a)v = a(-v) = -(av)$, $\forall a \in F$ & $\forall v \in V$

(v):— $a(v_1 - v_2) = av_1 - av_2$, $\forall a \in F$, $\forall v_1, v_2 \in V$

(vi):— $av = v \Leftrightarrow a = 1_F \in F$

Proof:— (i):— For $a \in F$ & $v \in V$, $av \in V$ and $av = a(0_v + v)$ $\because v = 0_v + v$

i.e $av = a0_v + av$

$\Rightarrow av + 0_v = a0_v + av$ $\because av = 0_v + av$

$\Rightarrow av + 0_v = av + a0_v$

$\Rightarrow 0_v = a0_v$, $\forall a \in F$ by cancellation law under "+".

(II):— For $a \in F$ & $v \in V$, $av \in V$

and $av = (0_F + a)v$ $\because a = 0_F + a$

$\Rightarrow av + 0_v = 0_F v + av$ $\because av = av + 0_v$

$\Rightarrow av + 0_v = av + 0_F v$

$\Rightarrow 0_v = 0_F v$

14-3-14

(III):– Suppose that $a \neq 0_F$ & $v \neq 0_v$

Let $av = 0_v$

then $a^{-1} \in F$ and hence

$$a^{-1}(av) = a^{-1}(0_v) \quad \because av = 0_v$$

$$= 0_v \quad \text{by part (I).}$$

$$\Rightarrow a^{-1}(av) = 0_v$$

$$\Rightarrow (a^{-1}a)v = 0_v$$

$$\Rightarrow 1_F v = 0_v$$

$$\Rightarrow v = 0_v$$

which is contradiction to our supposition that $v \neq 0_v$. Hence $a \neq 0_F$ is wrong, which gives that $a = 0_F$.

This proof can be completed by the argument supposing, that $0_v \neq v$ and $av = 0_v \longrightarrow (A)$ then

$$0_v = av = (a + 1_F)v \quad \because a = a + 1_F$$

$$\Rightarrow 0_v = av + 1_F v$$

$$\Rightarrow 0_v = 0_v + v \quad \because av = 0_v$$

$$\Rightarrow 0_v = v$$

which contradicts to the supposition that $v \neq 0_v$. Hence $a = 0_F$

(IV):- $0_F V = 0_V$ by part (II)

$\Rightarrow [a+(-a)]V = 0_V$

$\Rightarrow aV + (-a)V = 0_V$

$\Rightarrow -(aV)+aV+(-a)V = -(aV)+0_V$ Taking additive inverse of aV on B.S.

$\Rightarrow (-a)V = -(aV)$

Similarly. $0_V = a0_V$ by part (I).

$\Rightarrow 0_V = a[V+(-V)]$

$\Rightarrow 0_V = aV + a(-V)$

$\Rightarrow -(aV) + 0_V = -(aV)+aV+a(-V)$

$\Rightarrow -(aV) = a(-V)$ $\because -aV = 0_V+(-aV)$ & $a(-V) = 0_V + a(-V)$

(V):- $a(V_1 - V_2) = a[V_1 + (-V_2)]$

$= aV_1 + a(-V_2)$

$= aV_1 - aV_2$ by part (IV).

(VI):- Suppose that $0_F \neq a$ & $aV = V$ for $V \neq 0_V$ then

$aV = V$

$\Rightarrow aV - V = V - V$

$\Rightarrow aV + (-V) = 0_V$

$\Rightarrow aV + (-1_F)V = 0_V$

$\Rightarrow [a + (-1_F)]V = 0_V$

$\Rightarrow [a - 1_F]V = 0_V$

Using (III).

either $a-1_F=0_F$ or $v=0_v$

Since $v\neq 0_v$ So, $a-1_F=0_F$

$$\Rightarrow a-1_F+1_F=0_F+1_F$$

$$\Rightarrow a=1_F$$

Thus $av=v$ iff $a=1_F$

where $v\neq 0_v$.

Theorem:- 6.1.5: 17-3-14

Let $V(F)$ be a vector space. Then the following cancellation laws hold within $V(F)$;

(l_1):- $\alpha v=\beta v \Rightarrow \alpha=\beta$, $\forall \alpha,\beta\in F$, $v(\neq 0_v)\in V$

(l_2):- $\alpha v_1=\alpha v_2 \Rightarrow v_1=v_2$, $\forall \alpha\in F$, $\forall v_1, v_2\neq 0_v$

Proof:- (l_1):- $\alpha v=\beta v$

$$\Rightarrow \alpha v-\beta v=0_v$$

$$\Rightarrow \alpha v+(-\beta)v=0_v$$

$$\Rightarrow [\alpha+(-\beta)]v=0_v \quad \text{By part III}$$

$\Rightarrow$ either $[\alpha+(-\beta)]=0_F$ or $v=0_v$

But $v\neq 0_v$ So $\alpha+(-\beta)=0_F$

$$\Rightarrow \alpha=\beta, \text{ in } F.$$

(2):- $\alpha \neq 0_F$ & $\alpha v_1 = \alpha v_2$

$\Rightarrow \alpha v_1 - \alpha v_2 = 0_V$

$\Rightarrow \alpha v_1 + \alpha(-v_2) = 0_V$ By last theorem part 4.

$\Rightarrow \alpha[v_1 + (-v_2)] = 0_V$

as $\alpha \neq 0_F$ So,

$v_1 + (-v_2) = 0_V$

$\Rightarrow v_1 = v_2$

It is important to note that;

☆ $S \subseteq L(S)$

☆ $L(S) = 0_V$ if $S = \{0_V\}$

☆ $L(S) = \{\alpha v : \forall \alpha \in F\}$, if $S = \{v\}$ is singleton set.

☆ $L(v_i) \subseteq L(S)$, for each $v_i \in S$ where $i = 1, 2, \dots, k$ $(k \leq n)$

☆ $L(S) = L(v_1) + L(v_2) + \dots + L(v_n)$, if $S = \{v_1, v_2, \dots, v_n\}$

☆ If S and T are subsets of $V(F)$ then $S+T = \{s+t : s \in S, t \in T\}$ is a subset of $V(F)$.

Lemma:- Let $V(F)$ be a vector space and S, T be two subsets of $V(F)$. Then

(1):- $S \subseteq T \Rightarrow L(S) \subseteq L(T)$

(2):- $L(S \cup T) = L(S) + L(T)$

(3):- $L(L(S)) = L(S)$

Proof: (1):– Let $S \subseteq T$. If

$T = \{v_1, v_2, \dots, v_n\}$ and $S = \{v_1, v_2, \dots, v_s\}$, where $s < n$. Take $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_s v_s \in L(S)$

Then, $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_s v_s$

$= \alpha_1 v_1 + \dots + \alpha_s v_s + 0_F v_{s+1} + \dots + 0_F v_n \in L(T)$,

For all $v \in L(S)$

$\Rightarrow v \in L(T)$

$\Rightarrow L(S) \subseteq L(T)$

2): If $S = \{v_1, v_2, \dots, v_m\}$ and $T = \{w_1, \dots, w_s\}$, then, $S \cup T = \{v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_s\}$

Take, $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m + \beta_1 w_1 + \beta_2 w_2 + \dots + \beta_s w_s \in L(S \cup T)$,

Where $\alpha_i, \beta_i \in F$. Then

$v = (\alpha_1 v_1 + \dots + \alpha_m v_m) + (\beta_1 w_1 + \dots + \beta_n v_n) \in L(S) + L(T)$

$\Rightarrow L(S \cup T) \subseteq L(S) + L(T) \longrightarrow (a)$

If $\alpha_1 v_1 + \dots + \alpha_m v_m \in L(S)$

& $\beta_1 w_1 + \beta_2 w_2 + \dots + \beta_s w_s \in L(T)$

then $v = (\alpha_1 v_1 + \dots + \alpha_m v_m) + (\beta_1 w_1 + \beta_2 w_2 + \dots + \beta_s w_s) \in L(S) + L(T)$

$\Rightarrow v = \alpha_1 v_1 + \dots + \alpha_m v_m + \beta_1 w_1 + \dots + \beta_s w_s \in L(S \cup T)$

$\Rightarrow L(S) + L(T) \subseteq L(S \cup T) \longrightarrow (b)$

From (a) & (b)

$L(S \cup T) = L(S) + L(T)$

3):- Let

$$V=\sum_{i=1}^{m}\alpha_i v_i=\alpha_1 v_1+\alpha_2 v_2+\cdots+\alpha_m v_m \in L(S),$$

$$\bar{V}=\sum_{i=1}^{m}\beta_i v_i=\beta_1 v_1+\cdots+\beta_m v_m \in L(S)$$

&

$$\bar{\bar{V}}=\sum_{i=1}^{m}\gamma_i v_i=\gamma_1 v_1+\cdots+\gamma_m v_m \in L(S) \text{ ..etc}$$

Then $\mu V+\lambda\bar{V}+\xi\bar{\bar{V}}+\cdots$

$$\Rightarrow \sum_{i=1}^{m}(\mu\alpha_i)v_i+\sum_{i=1}^{m}(\lambda\beta_i)v_i+\sum_{i=1}^{m}(\xi\gamma_i)v_i\cdots \in L(L(S))$$

$$=\sum_{i=1}^{m}(\mu\alpha_i+\lambda\beta_i+\xi\gamma_i)v_i \in L(S)$$

$$\Rightarrow L(L(S)) \subseteq L(S) \longrightarrow (a)$$

Since $S \subseteq L(S)$

$$\Rightarrow L(S) \subseteq L(L(S)) \longrightarrow (b)$$

From (a) & (b).

$$L(L(S))=L(S)$$

Linearly Dependent 20-3-2014

Let $S=\{v_1, v_2, \ldots, v_n\}$ be a subset of a vector space $V(F)$. S is said to be linearly dependent if, there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$, not all zero, such that $\alpha_1 v_1+\alpha_2 v_2+\cdots+\alpha_n v_n=0_V \longrightarrow (1)$

It may be understood that there exists a non-zero solution $(\alpha_1, \alpha_2, \ldots, \alpha_n) \neq (0, 0, \ldots 0)$ in F^n of the eq.(1) i.e, if $\alpha_i \neq 0_F$ for some i, than

$$\alpha_i V_i = -\alpha_1 V_1 - \alpha_2 V_2 - \cdots - \alpha_{i-1} V_{i-1} - \alpha_{i+1} V_{i+1} - \cdots - \alpha_n V_n$$

giving

$$V_i = -\left(\frac{\alpha_1}{\alpha_i}\right) V_1 - \left(\frac{\alpha_2}{\alpha_i}\right) V_2 - \cdots - \frac{\alpha_n}{\alpha_i} V_n$$

as a linear combination of the remaining vectors $\{V_1, V_2, \ldots, V_{i-1}, V_{i+1}, \ldots, V_n\}$ of S.

6.2.6 Important Observations

It is important to note that:

1):– The zero vector 0_V of $V(F)$ is always linear dependent because $\alpha 0_V = 0_V$ for non-zero $\alpha (\neq 0_F)$.

2):– Any finite subset S of $V(F)$, which contains zero vector is L.D.

3):– Let $S = \{V_1, V_2, \ldots, V_n\}$ be a finite subset of $V(F)$. S is L.D in $V(F)$ if, $\exists$ at least one vector $V_i \in S$ such that $V_i \in L(S - \{V_i\})$.

4):– A singleton subset $\{V \neq 0_V\}$ of $V(F)$ is not L.D because $\exists$ no $0_F \neq \alpha \in F$ such that $\alpha V = 0_V$

5):- A subset $=\{v_1, v_2\}$ containing two non-zero vectors of vector space $V(F)$ is L.D iff, $\exists$ two non-zero scalars $\alpha_1, \alpha_2 \in F$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 = 0_V$$

6):- Let $S=\{v_1, v_2, \ldots, v_n\}$ be a finite subset of a vector space $V(F)$ which is L.D. Then the extended set

$$S = \{v_1, v_2, \ldots, v_n, v\}, \; v \neq 0_V, \text{ whithin}$$

$V(F)$ remains L.D.

7):- Any subset S_1 of a L.D subset S of vector space $V(F)$ is not essentially L.D in $V(F)$.

8):- If $v_1, v_2, \ldots, v_m \in V(F)$ and $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m \in F$ such that the set $S=\{v_2+\alpha_2 v_1, v_3+\alpha_3 v_1, \ldots, v_n+\alpha_n v_1\}$ is L.D in $V(F)$ then the set $S_1=\{v_1, v_2, \ldots v_n\}$ is also L.D in $V(F)$

Theorem:- In a vector space $V(F)$ if the subset $S=\{v_1, v_2, \ldots, v_m\}$ of vectors (non-zero) of $V(F)$ $(m \geq 2)$ is L.D then, $\exists$ an integer k, $2 \leq k \leq m$ such that v_k is the linear combination of the preceding vectors $\{v_1, v_2, \ldots, v_{k-1}\}$ and conversely.

Proof:- Let the subset $S = \{v_1, v_2, \ldots, v_m\}$ $(m \geq 2)$ of vector space $V(F)$ be L.D. Suppose that v_k, $2 \leq k \leq m$ is the first vector of S from the right for which the set $\{v_1, v_2, \ldots, v_k\}$ is L.D. Then $\exists$ scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in F$ such that $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = O_v$, where $\alpha_k \neq O_F$. Then

$$\alpha_k v_k = -\alpha_1 v_1 - \alpha_2 v_2 - \cdots - \alpha_{k-1} v_{k-1}$$

$$\text{or} \quad v_k = \frac{-\alpha_1}{\alpha_k} v_1 - \frac{\alpha_2}{\alpha_k} v_2 - \cdots - \frac{\alpha_{k-1}}{\alpha_k} v_{k-1}$$

showing v_k as a linear combination of the preceding vectors $v_1, v_2, \ldots, v_{k-1}$ in S.

Conversely:- If

$$v_k = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{k-1} v_{k-1} \text{ then}$$

$$v_k = \alpha_1 v_1 + \cdots + \alpha_{k-1} v_{k-1} + O_F v_{k+1} + \cdots + O_F v_m$$

and then the set $\{v_1, v_2, \ldots, I_F v_k, v_{k+1}, \ldots, v_m\}$ is L.D. because of $I_F \neq O_F \in F$.

Note:- If $k=1$ then we have $\alpha_1 v_1 = O_v$, $(v_1 \neq O_v) \Rightarrow v_1 = O_v$, which provide the contradiction to the supposition of non-zero vectors.

Linearly Independent 21-03-2014

In a vector space $V(F)$, a finite subset $S = \{v_1, v_2, \ldots, v_m\}$ of $V(F)$ is called linearly independent in $V(F)$ if; there exist scalars, $\alpha_1, \alpha_2, \ldots, \alpha_m \in F$ such that the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = O_V$$

has only it's zero solution and no nonzero solution exists at all.

Immediate Observations form Def-

In a vector space $V(F)$,

1:- If a finite subset S of $V(F)$ is not L.D, then S is L.I.

2:- The empty subset ϕ of $V(F)$ is always taken, by definition, a L.I subset.

3:- Any non-zero singleton subset of $V(F)$ is L.I.

4:- A finite subset $T = \{v_1, v_2, \ldots, v_n\}$ of a vector space $V(F)$ is L.I, if none of the vectors of T is a linear combination of the remaining vectors.

5:- Every subset of a L.I set S of a vector space $V(F)$ is L.I.

6:- Every finite subset S of non-zero vectors of vector space $V(F)$, which is

L.D contains a subset which is L.I in $V(F)$.

7:- Every L.I subset S of any subset T of $V(F)$, remains L.I in $V(F)$.

8:- If $S=\{v_1, v_2, \ldots, v_k\}$ is a L.I subset of a vector space $V(F)$, then each vector $\forall v$ of $V(F)$ which is contained in the linear span $L(S)$ of S, is uniquely expressed as a linear combination of the vectors of S.

Proof:- Suppose that $v \in L(S)$ and S is a L.I subset of $V(F)$. Then there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in F$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k \longrightarrow (a)$$

If (a) is not unique, then suppose that v is expressed as a L.C of the vectors of S in another way,

i.e there exist $\beta_1, \beta_2, \ldots, \beta_k \in F$, not all $\alpha_i = \beta_i$ for each i, such that

$$v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k \longrightarrow (b)$$

From (a) and (b).

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k$$

$$\Rightarrow (\alpha_1 - \beta_1) v_1 + (\alpha_2 - \beta_2) v_2 + \cdots + (\alpha_k - \beta_k) v_k = 0 \rightarrow (c)$$

As the vectors $v_1, v_2, \ldots, v_k$ are L.I the eq. © has only the zero solution in scalars.
i.e $(\alpha_1-\beta_1)=0=(\alpha_2-\beta_2)=\cdots=(\alpha_k-\beta_k)$,
which gives $\alpha_i=\beta_i$ for all $i=1,2,3,\ldots,k$
which contradicts to the fact that $\alpha_i \neq \beta_i$, for all i. Hence, one conclude that v has a unique expression as a linear combination of L.I vectors $v_1, v_2, \ldots, v_k$ of $V(F)$.

Theorem:

Let $V(F)$ be a vector space over the field F. Let $S=\{v_1, v_2, \ldots, v_k\}$ be a subset of $V(F)$ which contains $v_1, v_2, \ldots, v_r$ as L.I vectors of maximum $r<k$. Then,

$$L\{v_1, v_2, \ldots, v_r\}=L\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_k\}$$

Proof:- Let $u \in L\{v_1, v_2, \ldots, v_r\}$. Then, there exist $\alpha_1, \alpha_2, \ldots, \alpha_r \in F$ such that

$$u=\alpha_1 v_1+\alpha_2 v_2+\cdots+\alpha_r v_r$$

Also $u=\alpha_1 v_1+\alpha_2 v_2+\cdots+\alpha_r v_r+0_F v_{r+1}+0_F v_{r+2}+\cdots+0_F v_k$
which implies that

$u \in L\{v_1, v_2, \ldots, v_r, \ldots, v_k\}$ and Hence

$$L\{v_1, v_2, \ldots, v_r\} \subseteq L\{v_1, v_2, \ldots, v_r, \ldots, v_k\} \longrightarrow ⓐ$$

Now, suppose that

$$u_1 \in L\{v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_k\}$$ Then, $\exists$

$\alpha_1, \alpha_2, \dots, \alpha_r, \dots, \alpha_k \in F$, such that

$$u_1 = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r + \dots + \alpha_k v_k$$

Since the set $\{v_1, v_2, \dots, v_r, v_{r+1}\}$ is L.D

Therefore v_{r+1} is a linear combination of the L.I vectors $\{v_1, v_2, \dots, v_r\}$ and

Hence $v_{r+1} \in L\{v_1, v_2, \dots, v_r\}$ and similarly $v_{r+2}, \dots, v_k \in L\{v_1, v_2, \dots, v_r\}$.

Hence $u_1 \in L\{v_1, v_2, \dots, v_r\}$ and consequently

$$L(S) \subseteq L\{v_1, v_2, \dots, v_r\} \longrightarrow (b)$$

from (a) & (b).

$$L\{v_1, \dots, v_r\} = L\{v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_k\}$$

Basis of a Vector Space

6.2.13 Let $V(F)$ be a vector space over a field F. A subset B of $V(F)$ is said to form a basis of the vector space $V(F)$, if

(1):– $L(B) = V(F)$

i.e Linear span of B is the whole of the space $V(F)$ and

2):– The set B is a L.I subset of $V(F)$.

6.2.14:-

A vector space V(F) with basis B ($\subseteq V(F)$) is called m-dimensional vector space if number of vectors in the Basis B are finite in number equal to m.

Otherwise if there exists no such finite m, V(F) is understood infinite dimensional.

Observations

1):- If a finite number of vectors span vector space V(F) then vector space V(F) is said to be called a finite dimensional vector space.

2):- Each subset S of a vector space V(F) which spans V(F) cannot be a basis of V(F) unless S is L.I subset of V(F).

3):- Each L.I subset S of V(F) does not form a basis of vector space V(F), unless S spans V(F).

4):- There can be more than one basis of a vector space V(F).

5):- If V(F) is a finite dimensional vector space then each basis has the same finite number of vectors of V(F).

6):- Every subset of a basis of a vector space $V(F)$ is L.I subset of $V(F)$ which spans a vector space within $V(F)$.

7):- If $V(F)$ is an n-dimensional vector space, then

(i):- Any subset of $(n+1)$ vectors of $V(F)$ is L.D.

(ii):- No set of $(n-1)$ vectors can span the whole space $V(F)$ but does span a vector space within $V(F)$.

(iii):- Any subset of a set of n L.I vectors of $V(F)$ forms a basis of a vector space properly contained within $V(F)$.

8):- Important note:-

(i):- The null subspace $\{0_v\}$ has zero dimension.

(ii):- The empty subset ϕ of $V(F)$ forms a basis of the null subspace.

(iii):- The space R^n is finite dimensional of dimension n.

(iv):- The vector space $M_{m\times n}(F)$ of all $m\times n$ matrices is of dimension $m.n$ and the vector space $M_{n\times n}(F)$ is of dimension n^2 (The no. of entries in the matrix).

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(v):- Every field F over itself is a vector space of dimension one.

Theorem:- (Basis Theorem)

Any finite dimensional vector space $V(F)$ contains a finite basis.

Proof:- Let $S=\{v_1,v_2,\dots,v_n\}$ be a subset of $V(F)$ containing n vectors such that $L(S)=V(F)$

If $S=\{v_1,v_2,\dots,v_n\}$ is a L.I subset of $V(F)$, then S forms a basis of $V(F)$ and hence $V(F)$ contains a finite basis.

If $S=\{v_1,v_2,\dots,v_i,\dots,v_n\}$ is L.D, then there exists at least one vector, say v_i, of S, which is expressed as a linear combination of it's preceding vectors.

~~we~~ Eliminating such a vector v_i from S and relabelling the remaining vectors we obtain the set $S_1=\{v_1,v_2,\dots,v_{n-1}\}$ of $V(F)$ containing $(n-1)$ vectors. Also $L(S_1)=V(F)$.

If S_1 is L.I, then S forms a basis of $V(F)$. If S_1 is again L.D, then there exists a vector say v_j, in S_1 which is a linear combination of it's preceding vectors. Eliminating v_j again from S_1 and relabelling the remaining $n-2$ vectors we form a set S_2 of $(n-2)$ vectors which spans $V(F)$. If S_2 is L.I then S_2 forms a basis of $V(F)$. If S_2 is again L.D, we continue the process of elimination of dependent vectors till we arrive at a subset $S_r = \{v_1, v_2, \dots, v_r\}$, $1 \le r \le n$, which is L.I and spans $V(F)$ which is a finite basis of $V(F)$

Cor:- The dimension of a finite dimensional vector space cannot exceed the number of elements in a spanning set of that space.

Theorem:-

The dimension of a finite dimensional vector space is unique.

Proof:- Let us suppose that $B_1 = \{v_1, v_2, \dots, v_m\}$ and $B_2 = \{u_1, u_2, \dots, u_n\}$ be two basis of a vector space $V(F)$, where $n \ne m$.

Since B_1 spans $V(F)$ and B_2 is a basis of $V(F)$, therefore $n \le m$ — (a)

Similarly if, B_2 spans $V(F)$, and B_1 forms a basis of $V(F)$ then $m \leq n \longrightarrow (b)$

From (a) & (b)

$\Rightarrow m = n$

Hence all the basis of a vector space have the same number of vectors of $V(F)$ and uniqueness of the dimension of a vector space follows.

Cor:-

Let U be a proper subset of a finite dimensional vector space $V(F)$ which spans a vector space within $V(F)$. Then $\dim(U) \leq \dim V$

i.e The dimension of the ~~vector~~ linear span of U as a vector space can not exceed the dimension of the vector space. $\dim(U) \not> \dim V(F)$

Theorem:- (Extension of Basis)

Every L.I subset of a finite dimensional vector space $V(F)$ can be extend to be a basis of $V(F)$.

OR

Let $V(F)$ be a vector space of dimension $n>1$. If $S = \{v_1, v_2, \dots, v_r\}$

$1 \le r < n$ is a subset of $V(F)$, which is L.I, then there exist vectors $\{v_{r+1}, v_{r+2}, \dots, v_n\}$ of $V(F)$ such that the extended set $S = \{v_1, v_2, \dots, v_r, \dots, v_n\}$ forms a basis of $V(F)$.

Proof:- Let $V(F)$ be a n-dimensional space. Since $r < n$, therefore the subset $S = \{v_1, v_2, \dots, v_r\}$ of $V(F)$ does not form a basis of $V(F)$ and hence cannot generate the whole of the vector space $V(F)$. Then there exists a vector $v_{r+1} \in V(F)$ such that $v_{r+1} \notin L(S)$.

Including v_{r+1} in S, we get a subset $S_1 = \{v_1, v_2, \dots, v_r, v_{r+1}\}$ which is L.I in $V(F)$.

If $r+1 < n$, we repeat this argument and continue combining one vector in S_1 in each repetition, untill we obtain the set $\overline{S} = \{v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_n\}$ of $V(F)$ which is L.I and spans $V(F)$.

Hence $\overline{S}$ forms a basis of $V(F)$.

Cor:-

Every L.I subset of V(F) is contained in a basis of V(F). Hence there are as many basis of V(F), as many choices of basis vectors are there.

Examples: 1 — 6. (Self)

27-3-2014

Sub-Space of A Vector Space V(F)

Definition:- Let V(F) be a vector space over a field F. A subset U of V(F) is called a subspace of V(F), if

(i):- U is subgroup of V(F).

& (ii):- U is closed under the field multiplication. i.e $\alpha . u \in U, \forall \alpha \in F, \forall u \in U$

★ Noted that the other axioms of definition of vector space are hereditary from V(F) to U.

Definition:-

A subset U of a vector space V(F) over the field F, is said to form a subspace of V(F), if

(i):- $u_1 - u_2 \in U$, $\forall u_1, u_2 \in U$ (Subgroup Criterion)

& (ii):- $\alpha u \in U$, $\forall \alpha \in F$, $\forall u \in U$
(Closure of scalar multiplication)

Sub-Space Criterion

A subset U of a vector space $V(F)$ is called a subspace of $V(F)$ if, $\forall \alpha, \beta \in F$ and $\forall u, v \in U$,

$$\alpha u + \beta v \in U$$

Immediate Deductions from The Definition

D_1 If $V(F)$ is a vector space, then each subgroup U of abelian group $(V, +)$ is a subspace of $V(F)$, which is invariant under the scalar multiplication by the elements of F.

D_2 Subspace must be a vector space over the same field as that of its super space V.

D_3 The vector space $V(F)$ itself is a subspace of it's own vector space.

D_4 The zero-vector of $V(F)$ is contained in each subspace of $V(F)$.

D_5 The subset $\{0_v\}$ of $V(F)$ forms a subspace of $V(F)$. It is the only subspace containing one vector of $V(F)$.

It is called the zero ~~sub~~space or null space.

D_6: The subspaces $\{0_V\}$ and the space $V(F)$ are called trivial (or improper) subspaces of $V(F)$ and other subspaces, if they exist, are called proper subspace or non-trivial subspaces.

D_7 If $S = \{u\}$ is a singleton subset of a non-zero vector u of $V(F)$ then $L(u) = \{\alpha u : \alpha \in F\}$ is a subspace of $V(F)$, generated (or spanned) by u and denoted by $\langle u \rangle$.

i.e $\langle u \rangle = L(u) = \{\alpha u : \alpha \in F\}$

D_8 If $u(\neq 0) \in V(F)$ then $L(u) \cong F$ the field over which $L(u)$ is a vector space.

D_9 $L(1) \cong \mathbb{R}$, over the field $\mathbb{R}$ and $L(1) \cong \mathbb{C}$, over the field $\mathbb{C}$.

D_{10} **Lemma**

If U_1 & U_2 are two subspaces of a vector space $V(F)$, then $U_1 \cap U_2$ is also a subspace of $V(F)$.

Proof: Take $u, v \in U_1 \cap U_2$.

Then $u, v \in U_1$ & $u, v \in U_2$

If $u, v \in U_1$ & U_1 is a subspace of $V(F)$ then for all $\alpha, \beta \in F$,

$$\alpha u + \beta v \in U_1 \longrightarrow (a)$$

If $u, v \in U_2$ and U_2 is a subspace of $V(F)$, then for $\alpha, \beta \in F$,

$$\alpha u + \beta v \in U_2 \longrightarrow (b)$$

From (a) & (b)

$$\alpha u + \beta v \in U_1 \cap U_2 \quad , \forall \alpha, \beta \in F$$

and $\forall u, v \in U_1 \cap U_2$.

Hence $U_1 \cap U_2$ form a subspace of $V(F)$. $U_1 \cap U_2 \cap U_3 \cap U_4 \cdots$

D_{11}: The union of two subspaces is not always a subgroup of $V(F)$ therefore Union of subspaces is not a subspace of $V(F)$, in general.

Theorem: 28-3-2014

Let U_1 and U_2 be two subspaces of a vector space $V(F)$. Then the subset,

$$U_1 + U_2 = \{u_1 + u_2 : u_1 \in U_1, u_2 \in U_2\}$$

forms a subspace of $V(F)$.

Proof:- Let

$x = u_1 + u_2$ & $y = \bar{u}_1 + \bar{u}_2$ be two elements of $U_1 + U_2$, where $u_1, \bar{u} \in U_1$ & $u_2, \bar{u}_2 \in U_2$.

For arbitrary $\alpha, \beta \in F$,

$$\alpha x + \beta y = \alpha(u_1 + u_2) + \beta(\bar{u}_1 + \bar{u}_2)$$
$$= \alpha u_1 + \alpha u_2 + \beta \bar{u}_1 + \beta \bar{u}_2$$
$$= (\alpha u_1 + \beta \bar{u}_1) + (\alpha u_2 + \beta \bar{u}_2)$$

, By the commutative property of addition.

Since U_1 is subspace, therefore $\alpha u_1 + \beta \bar{u}_1 \in U_1$.

Similarly $\alpha u_2 + \beta \bar{u}_2 \in U_2$.

Hence $\alpha x + \beta y \in U_1 + U_2$, for all $x, y \in U_1 + U_2$ and thus $U_1 + U_2$ forms a subspace of $V(F)$, which is understood as sum of the subspaces of U_1 & U_2 of $V(F)$.

Not V.I Remarks

1:- The subspace $U_1 + U_2$ contains the subsets U_1 & U_2 both and consequently $U_1 \cup U_2$

2:- $U_1 + U_2$ is the smallest subspace which contains $U_1 \cup U_2$.

3:- Infact $L(U_1 \cup U_2) = U_1 + U_2$

4:- For a finite family $W_1, W_2, \ldots, W_n$ of subspaces of a vector space $V(F)$,

$$W_1 + W_2 + \cdots + W_n = \sum_{i=1}^{n} W_i$$ is a subspace of $V(F)$, which is sum of the subspaces W_i $(i = 1, 2, \ldots, n)$ of $V(F)$.

5:- $U + U = U$, for each subspace U of vector space $V(F)$. i.e for every $u \in U$, & $\alpha \in F$,

$$\alpha u + (1-\alpha) u = u = \beta u + (1-\beta) u$$

& $$\alpha u + \beta u = (\alpha + \beta) u$$

Lemma

Let S be a non-empty subset of a vector space $V(F)$. Then the linear span $L(S)$ in $V(F)$ is a subspace of $V(F)$.

Proof:- If $S = \{s_1, s_2, \ldots, s_n\}$ then

$$L(S) = \{\alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_n s_n : \alpha_i \in F\}$$

$$= \left\{\sum_{i=1}^{n} \alpha_i s_i : \alpha_i \in F\right\}_n$$

For arbitrary $u = \sum_{i=1}^{n} \alpha_i s_i$ and

$$v = \sum_{i=1}^{n} \beta_i s_i \quad , \quad \because \gamma, \delta \in F,$$

$$\gamma u + \delta v = \gamma \sum_{i=1}^{n} \alpha_i s_i + \delta \sum_{i=1}^{n} \beta_i s_i$$

$$= \sum_{i=1}^{n} (\gamma \alpha_i) s_i + \sum_{i=1}^{n} (\delta \beta_i) s_i$$

$$= \sum_{i=1}^{n} (\gamma \alpha_i + \delta \beta_i) s_i \in L(S)$$

$\Rightarrow L(S)$ is a subspace of $V(F)$, for each $S \subseteq V(F)$

Cor: 1:—

$L(S_1) = L(S_2) = \cdots = L(S_n) \subseteq F$, for each non-zero singleton $\{s_i\} \in V(F)$

Cor: 2:—

$L(S_i)$ is a subspace of $V(F)$ of dimension 1 for each i, $\because$ basis $\{s_i\}$

Cor-3:—

$L(S) = L(s_1) \oplus L(s_2) \oplus \cdots \oplus L(s_n)$, if $S = \{s_1, s_2, \ldots, s_n\}$ is L.I.

Cor-4:—

$L(s_i)$ is a line through the origin for each i.

Internal Direct Sum

Let $W = S + T$, where S and T are subspaces of the vector space $W(F)$. If every element $w \in W$ is expressed uniquely as a sum of the elements of S and T (or $S \cap T = \{0\}$) then, the space W is called the internal direct sum of the subspaces S and T of $W(F)$.

It shall be denoted by

$$W = S \oplus T$$

If $W = W_1 \oplus W_2 \oplus \cdots \oplus W_n$, then W is called the Internal direct sum of a family of subspaces $W_1, W_2, \ldots, W_n$ of vector space $V(F)$.

Note that,

$$\text{Dim}(W) = \dim(W_1) + \dim(W_2) + \cdots + \dim(W_n).$$

Theorem:- Let $V(F)$ be a vector space and $U(F)$ be a subspace of $V(F)$. A relation R is defined on the elements of $V(F)$ by, "v_1 in $V(F)$, is in relation R to an element v_2 in $V(F)$, if $v_1 - v_2 \in U(F)$". Then R is an equivalence relation on $V(F)$

Proof:- We know that $0_v \in U(F)$

and $v_1 - v_1 = 0_v \in U(F)$. Then v_1 is related to itself by the relation $\mathbb{R}$.

i.e, $v_1 \mathbb{R} v_1$, $\forall v_1 \in V(F)$. Thus the relation $\mathbb{R}$ is reflexive on $V(F)$.

In addition, if $v_1 \mathbb{R} v_2$ then $v_1 - v_2 \in U(F)$, and $-(-(v_2 - v_1)) = v_2 - v_1 \in U(F)$. which implies that $v_2 \mathbb{R} v_1$; It shows that the relation $\mathbb{R}$ is symmetric on $V(F)$.

If $v_1 \mathbb{R} v_2$ and $v_2 \mathbb{R} v_3$, for $v_1, v_2, v_3 \in U(F)$, then $v_1 - v_2 \in U(F)$ and $v_2 - v_3 \in U(F)$ and consequently their addition, $v_1 - v_2 + v_2 - v_3 = v_1 - v_3 \in U(F)$.

It shows that the relation $\mathbb{R}$ is transitive on $V(F)$. i.e $v_1 \mathbb{R} v_3$.

By the reflexive, symmetric, and transitive properties of $\mathbb{R}$ on $V(F)$, $\mathbb{R}$ is an equivalence relation on $V(F)$.

Cor-1:- The relation $\mathbb{R}$ on $V(F)$ subdivides the vector space $V(F)$ into subsets which are mutually disjoint and their union produces the whole of the vector space $V(F)$. The subdivided subsets of $V(F)$ formed by an equivalence relation $\mathbb{R}$ on $V(F)$, are called the equivalence classes by $\mathbb{R}$ on $V(F)$.

Cor-2:- The relation $\mathbb{R}$ on $V(F)$ defined by ($v_1 \mathbb{R} v_2$ if $v_1 - v_2 \in U(F)$) is understood by

$v_1 \equiv v_2$ (modulo U), the congruence relation $\equiv$ under which, the subsets

$$v_1 + U = v_2 + U \Leftrightarrow v_1 \mathbb{R} v_2 \text{ in } V(F)$$

or $v_1 \in v_2 + U$ and $v_2 \in v_1 + U$

Cor-3: The equivalence classes formed by the relation $\mathbb{R}$ are subset $v_i + U = \{v_i + u : u \in U\}$, which contain respective v_i, for each i and are mutually disjoint. The classes are called cosets of $V(F)$ by $U(F)$.

Cor-4:- The coset $v_i + U = U$, if $v_i \in U$. Thus U itself is one of the cosets of $V(F)$ by $U(F)$ if $v_i \in U$.

Cor-5:- The set of all cosets of $V(F)$ by a subspace $U(F)$ is called the quotient set of $V(F)$ by $U(F)$

and is denoted by the quotient set,

$$\frac{V}{U} = \{v_1+U, v_2+U, \dots, v_n+U, \dots\}$$

$$= \{[v_1], [v_2], \dots, [v_n], \dots\}$$

which is finite if V is finite and its index

$$[V:U] = \left|\frac{V}{U}\right| = \frac{|V|}{|U|}$$ of cosets of

$\frac{V(F)}{U(F)}$ is finite.

Cor. 6:- If any element of $V(F)$ is common in any two of the cosets of V by U then the cosets coincide.

Proof:- Let $v_1+U, v_2+U \in \frac{V}{U}$ and $x \in (v_1+U) \cap (v_2+U)$. If $x \in v_1+U$, then $x = v_1+u_1$ and if $x \in v_2+U$, then $x = v_2+u_2$, where $u_1, u_2 \in U$. Thus

$v_1+u_1 = v_2+u_2 = x$ which gives that $v_1+U \subseteq v_2+U$. Similarly $v_2+U \subseteq v_1+U$ and consequently

$$v_1+U = v_2+U.$$

Self

Theorem:-

Let $V(F)$ be a vector space and W, W_1, W_2 be subspaces of $V(F)$. Then,

a):- $v \in u+W$ iff $v+W = u+W$

b):- $v \in W$ iff $v+W = W$

c):- If $u+W_1 \subseteq u+W_2$ then $W_1 \subseteq W_2$

d):- If $u+W_1 \subset u+W_2$ then $W_1 \subset W_2$

e):- If $u+W_1 = u+W_2$, then $W_1 = W_2$

Theorem:-

Let W be a subspace of a vector space $V(F)$. Then the quotient set $\frac{V}{W} = \{[v_i] : v_i + W : v_i \in V\}$ of all cosets of $V(F)$ by W, forms a vector space over the same field F as that of $V(F)$.

Proof:- The addition of cosets $[v_i]$ of $V(F)$ by W is defined by $[v_1]+[v_2] = [v_1+v_2] \in \frac{V}{W}$ and the scalar multiplication of coset $[v_i] = v_i + W$, by scalar $\alpha \in F$ is defined by $\alpha[v_i] = [\alpha v_i] \in \frac{V}{W}$.

(i) The addition of cosets is associative, since,

$v_1, v_2, v_3 \in V$

$$([v_1]+[v_2])+[v_3] = [v_1+v_2]+[v_3]$$
$$= [(v_1+v_2)+v_3]$$
$$= [v_1+(v_2+v_3)]$$
$$= [v_1]+[v_2+v_3]$$
$$= [v_1]+([v_2]+[v_3])$$

(ii):– $0_v + W = [0_v] = W$ is the additive identity since,

$$[0_v]+[v_i] = [0_v+v_i] = [v_i] = [v_i]+[0_v]$$
$$\forall \; v_i \in V$$

(iii):– $[-v_i] = -[v_i]$, the inverse of $[v_i]$, because,

$$-[v_i]+[v_i] = [-v_i]+[v_i] = [-v_i+v_i] = [0_v]$$

(iv):– $[v_1]+[v_2] = [v_1+v_2]$
$$= [v_2+v_1]$$
$$= [v_2]+[v_1]$$

Thus the quotient set $(V/W, +)$ is an abelian group.

The scalar multiplication to the cosets by the field elements is defined by $\alpha[v] = [\alpha v] \in \frac{V}{W}$

which observes the following properties

1):- $\alpha([v_1]+[v_2]) = \alpha[v_1+v_2]$, $\alpha \in F$

$= [\alpha(v_1+v_2)] = [\alpha v_1 + \alpha v_2]$

$= [\alpha v_1] + [\alpha_2 v_2]$

$= \alpha[v_1] + \alpha[v_2]$, $\forall [v_1],[v_2] \in \frac{V}{W}$

2):- $(\alpha+\beta)[v_1] = [(\alpha+\beta)v_1]$, $\alpha,\beta \in F$

$= [\alpha v_1 + \beta v_1] = [\alpha v_1] + [\beta v_1]$

$= \alpha[v_1] + \beta[v_1]$, $\forall [v_1],[v_2] \in \frac{V}{W}$

3):- $(\alpha\beta)[v_1] = [\alpha\beta v_1]$, $\alpha,\beta \in F$, $\forall [v_1] \in \frac{V}{W}$

$= [\alpha(\beta v_1)]$

$= \alpha[\beta v_1]$

$= (\beta\alpha)[v_1]$

$= \beta(\alpha[v_1])$

4):- $1_F[v_1] = [1_F v_1] = [v_1]$, $\forall [v_1] \in \frac{V}{W}$

Thus the quotient set $\frac{V}{W}$ forms a vector space over the same field F, over which V is a vector space.

Remarks:- 1):- For each subspace W of $V(F)$, there is a quotient space $\frac{V}{W}$ over the same field F.

2):- $\frac{V}{V} = 0_v + V = [0_v]$, the zero quotient space.

3):- $\frac{V}{[0_V]} = V$, the quotient space of V by it's zero subspace.

Lemma:— 4):-

$$\frac{V}{W_1 \cap W_2} = \frac{V}{W_1} \cap \frac{V}{W_2},$$

If W_1 and W_2 are two subspace of V over the field F.

Proof:— Let $\bar{v} \in \frac{V}{W_1 \cap W_2}$

Then

$$\bar{v} \in v_i + (W_1 \cap W_2)$$

$$\Rightarrow \bar{v} - v_i \in W_1 \cap W_2$$

$$\Rightarrow \bar{v} - v_i \in W_1 \text{ and } \bar{v} - v_i \in W_2$$

if $\bar{v} - v_i \in W_1$, then $[\bar{v}] \in \frac{V}{W_1}$

if $\bar{v} - v_i \in W_2$, then $[\bar{v}] \in \frac{V}{W_2}$

Hence $[\bar{v}] \in \frac{V}{W_1} \cap \frac{V}{W_2}$

$$\Rightarrow \frac{V}{W_1 \cap W_2} \subseteq \frac{V}{W_1} \cap \frac{V}{W_2} \longrightarrow (a)$$

On the other hand, if

$[\bar{v}] \in \frac{V}{W_1} \cap \frac{V}{W_2}$ then

$[\bar{v}] \in \frac{V}{W_1}$ and $[\bar{v}] \in \frac{V}{W_2}$

$\Rightarrow \bar{v} - v_i \in W_1$ & $\bar{v} - v_i \in W_2$, for $v_i \in V$

$\Rightarrow \bar{v} - v_i \in W_1 \cap W_2$

$\Rightarrow [\bar{v}] \in \dfrac{V}{W_1 \cap W_2}$

$\Rightarrow \dfrac{V}{W_1} \cap \dfrac{V}{W_2} \subseteq \dfrac{V}{W_1 \cap W_2} \longrightarrow (b)$

From (a) and (b).

$$\frac{V}{W_1 \cap W_2} = \frac{V}{W_1} \cap \frac{V}{W_2}$$

Theorem:- Let $V(F)$ be a finite dimensional vector space and S be a subspace of $V(F)$. Let V/S forms a vector space over the same field F, called the quotient space of $V(F)$ by S. Then

$$\dim\left(\frac{V}{S}\right) = \dim V - \dim S$$

Proof:- Since $V(F)$ is finite dimensional vector space and S is a subspace of $V(F)$ therefore S is also finite dimensional and $\dim S < \dim V$. If $B = \{v_1, v_2, \ldots, v_m\}$ is a basis of S with m vectors then $L(B) = S$ and B can be extended to be a basis

$\bar{B} = \{v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_n\}$ of $V(F)$

containing $m+r$ vectors.

The vectors of $\frac{V}{S} = \{v_i + S : v_i \in V\}$ are the r cosets of $V(F)$ by S, whose union $\bigcup_{i=1}^{r} \{v_i + S\} = V$ and

$$L(\{v_i + S : v_i \in V\}) = \frac{V}{S}$$

Suppose that, for $\alpha_1, \alpha_2, \ldots, \alpha_r \in F$

$\alpha_1 [W_1] + \alpha_2 [W_2] + \cdots + \alpha_r [W_r] = \underset{0_v + S}{\overset{\uparrow}{S}}$ the zero of $\frac{V}{S}$. Then

$$[\alpha_1 W_1] + [\alpha_2 W_2] + \cdots + [\alpha_r W_r] = S$$

$$\Rightarrow [\alpha_1 W_1 + \alpha_2 W_2 + \cdots + \alpha_r W_r] = S$$

$$\Rightarrow (\alpha_1 W_1 + \alpha_2 W_2 + \cdots + \alpha_r W_r) + S = S$$

$$\Rightarrow (\alpha_1 W_1 + \alpha_2 W_2 + \cdots + \alpha_r W_r) \in S$$

$$\Rightarrow \alpha_1 W_1 + \alpha_2 W_2 + \cdots + \alpha_r W_r = \beta_1 V_1 + \beta_2 V_2 + \cdots + \beta_m V_m \; \& \; \beta_i \in F$$

$$\Rightarrow \alpha_1 W_1 + \alpha_2 W_2 + \cdots + \alpha_r W_r - \beta_1 V_1 - \beta_2 V_2 - \cdots - \beta_m V_m = 0_V$$

$$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_r = 0_F = \beta_1 = \beta_2 = \cdots = \beta_m$$

Since $\bar{B} = \{v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_r\}$ is a basis of $V(F)$

Hence $[W_1], [W_2], \ldots, [W_r]$ are L.I vectors of V/S which spans $\frac{V}{S}$.

Hence $\dim\left(\frac{V}{S}\right) = r = (m+r) - m = \dim V - \dim S$

Cor.

The set of all cosets of $V(F)$ by S forms a basis of V/S and $\dim V/S = [V:S]$, the index of V by S.

Theorem:-

Let S and T be two subspaces of a finite dimensional vector space $V(F)$. Then

$$\dim(S+T) = \dim(S) + \dim(T) - \dim(S \cap T)$$

Proof:- Since S and T are two subspaces of a finite dimensional vector space $V(F)$ then $S \cap T$ is also a subspace of $V(F)$ and is a finite dimensional subspace. Let $\{u_1, u_2, \dots, u_r\}$ be a basis of $S \cap T$ which contains r vectors.

Since $S \cap T$ is a subspace of both of its super subspaces S and T, therefore, the basis $\{u_1, u_2, \dots, u_r\}$ of $S \cap T$ can be extended to form bases of S and T both by

$B_1 = \{u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_s\}$ and

$B_2 = \{u_1, u_2, \dots, u_r, w_1, w_2, \dots, w_t\}$ containing $(r+s)$ and $(r+t)$ vectors respectively.

Since S and T are subspaces of the further super subspace $(S+T)$ of $V(F)$ therefore, the bases B_1 and B_2 of S and T respectively can be extended to a basis of $S+T$. Let

$B_3 = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_t\}$ be a basis of a subspace of $V(F)$ containing $(r+s+t)$ vectors, which span $(S+T)$. Thus $(S+T) \leq (r+s+t)$

Suppose that there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_r, b_1, b_2, \ldots, b_s, c_1, c_2, \ldots, c_t \in F$ such that:

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r + b_1 v_1 + b_2 v_2 + \cdots + b_s v_s + c_1 w_1 + c_2 w_2 + \cdots + c_t w_t = O_V$$

then

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r + b_1 v_1 + b_2 v_2 + \cdots + b_s v_s = -c_1 w_1 - c_2 w_2 - \cdots - c_t w_t \quad \text{(i)}$$

then $x \in S \cap T$ and hence

$$x = d_1 u_1 + d_2 u_2 + \cdots + d_r u_r, \text{ for } d_i \in F \quad \text{(ii)}$$

subtracting (ii) from (i), we get that

$$O_V = x - x = (a_1 - d_1)u_1 + (a_2 - d_2)u_2 + \cdots + (a_r - d_r)u_r + b_1 v_1 + b_2 v_2 + \cdots + b_s v_s$$

As B_1 is a basis of S, therefore B_1 is L.I and hence,

$$a_1 - d_1 = 0_F = a_2 - d_2 = \cdots = a_r - d_r = b_1 = b_2 = \cdots = b_s$$

$\Rightarrow a_i = d_i$, $i = 1, 2, \ldots, r$ and $b_1 = b_2 = \cdots = b_s = 0$

But $d_i = 0$, for $i = 1, 2, 3, \ldots, s$

Hence $a_i = 0$ for $i = 1, 2, \ldots, r$

Thus the set $B_3 = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s, w_1, \ldots, w_t\}$ is L.I and hence is a basis of $(S+T)$ and consequently

$$\dim(S+T) = r + s + t$$
$$= (r+s) + (t+r) - r$$
$$= \dim(S) + \dim(T) - \dim(S \cap T)$$

Linear Transformation 24-04-14

(Homomorphism)

Let U and V be two vector spaces. $T : V \longrightarrow U$

(i):- $T(x+y) = Tx + Ty \quad x,y \notin V$

(ii):- $T(\alpha x) = \alpha Tx \quad , \quad \alpha \in F$

OR

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty \quad , \alpha, \beta \in F$$
$$x, y \in V$$

Trivial Homo.

(i):- Identity Operator

$$I : V \longrightarrow V$$
$$I(v) = v \quad , \quad v \in V$$

(ii):- Zero Operator:-

$$O : V \longrightarrow V$$
$$O(V) = 0 \quad , \quad 0 \in V$$

(iii):-

$$T : R^3 \longrightarrow R^2$$

Let $(\alpha, \beta, \gamma) \in R^3$

$(\alpha, \beta) \in R^2$

i.e $T(\alpha, \beta, \gamma) = (\alpha, \beta)$

Proof:- Take two points

$(\alpha_1, \beta_1, \gamma_1)$ & $(\alpha_2, \beta_2, \gamma_2) \in R^3$

(i):- $T((\alpha_1, \beta_1, \gamma_1) + (\alpha_2, \beta_2, \gamma_2)) = T(\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2)$

$= (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$

$= (\alpha_1, \beta_1) + (\alpha_2, \beta_2)$

$$= T(\alpha_1, \beta_1, \gamma_1) + T(\alpha_2, \beta_2, \gamma_2)$$

(ii):- $T(\alpha(\alpha_1, \beta_1, \gamma_1)) = T(\alpha\alpha_1, \alpha\beta_1, \alpha\gamma_1)$

$$= (\alpha\alpha_1, \alpha\beta_1)$$

$$= \alpha(\alpha_1, \beta_1)$$

$$= \alpha T(\alpha_1, \beta_1, \gamma_1)$$

Example:-

$$T: V \longrightarrow V$$

$$T(f(x)) = \frac{d}{dx}(f(x)), \quad f(x) \in V$$

Sol: Take $f, g \in V$

i) $T(f+g) = \frac{d}{dx}(f+g)$

$$= \frac{d}{dx} f + \frac{d}{dx} g$$

$$= T(f) + T(g)$$

ii) $T(\alpha f) = \frac{d}{dx}(\alpha f)$, $\alpha \in F$, $f \in V$

$$= \alpha \frac{d}{dx}(f)$$

$$= \alpha T(f)$$

So, our mapp is linear transformation

Example of non-linear transformation.

$$T: R^3 \longrightarrow R$$

defined by $T(x, y, z) = x^2 + y^2 + z^2$

Take element
$(1,0,0) \in R^3$

$LHS = T((1,0,0)+(1,0,0)) = T(2,0,0)$
$= 4$

But

$R.H.S = T(1,0,0) + T(1,0,0) = 1+1 = 2$

Hence the mapping is not L.T.

✓Exercise:– Q.1 $T: R^2 \longrightarrow R^2$

$T(x_1, x_2) = (1+x_1, x_2)$

Let $(0,3) \in R^2$

(i) $T((0,3)+(0,3)) = T(0,6)$
$= (1,6)$

$T(0,3) + T(0,3) = (1,3) + (1,3)$
$= (2,6)$

As $T(0,3) + T(0,3) \neq T((0,3)+(0,3))$

So, This map is not a linear transformation

Q.2. $T: R^2 \longrightarrow R^2$ Prove by taking

$T(x_1, x_2) = (x_2, x_1)$ $\alpha \in F$ & $(x_1, x_2) \in R^2$

Let $(0,2) \in R^2$ $(y_1, y_2) \in R^2$

(i) $T((0,2)+(0,2)) = T(0,4)$
$= (4,0)$

$T(0,2) + T(0,2) = (2,0) + (2,0)$
$= (4,0)$

اگر کسی Mapping کو L.T دکھانا ہو تو کوئی constants سے Points لے
Linear سے Prove کرنے کے ساتھ ساتھ یہی اُس کو In general سے table
Prove کروانا ضروری ہے۔ non-L.T دکھانے کے لیے constant points سے ہی non-L.T
Prove کر لیا جاتا ہے

Now Let $2 \in F$

(ii) $T(2(0,2)) = T(0,4)$

$= (4,0)$

$2T(0,2) = 2(2,0)$

$= (4,0)$

Hence, this map is linear transformation.

Q.3- $T: R^3 \to R^4$ Prove by taking $\alpha \in F$, $(x_1, x_2, x_3) \in R^3$ & $(y_1, y_2, y_3) \in R^3$

$T(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, x_3)$

Let $(0,1,1) \in R^3$

(i):- $T((0,1,1) + (0,1,1)) = T(0,2,2)$

$= (0,2,4,2)$

$T(0,1,1) + T(0,1,1) = (0,1,2,1) + (0,1,2,1)$

$= (0,2,4,2)$

Let $2 \in F$

(ii):- $T(2(0,1,1)) = T(0,2,2)$

$= (0,2,4,2)$

$2T(0,1,1) = 2(0,1,2,1)$

$= (0,2,4,2)$

Hence, this map is linear transformation.

Q.4:- $T: R \to R^3$

$T(x) = (x, x^2, x^3)$

Let $2 \in R$

(i):- $T(2+2) = T(4) = (4, 16, 64)$

$$T(2) + T(2) = (2,4,8) + (2,4,8)$$
$$= (4,8,16)$$

As

$$T(2+2) \neq T(2) + T(2)$$

So, this map is not linear transformation.

Theorem:- $T: V \longrightarrow U$, T is Linear Transformation

$$T(0) = 0$$
$$T(-v) = -T(v) \quad , v \in V$$

Proof:- $T(0) = T(0+0)$

$$= T(0) + T(0)$$

By cancellation law

$$\Rightarrow T(0) = 0$$

Now $T(0) = T(x + (-x)) = T(x) + T(-x)$

$$T(0) = 0 \Rightarrow T(x) + T(-x) = 0$$
$$T(-x) = -T(x)$$

Null space of L.T OR Kernal of L.T

$$T: V \longrightarrow U$$
$$\text{Ker } T = \{v \in V : T(v) = 0\}$$
$$\text{Ker} T \neq \phi$$
$$T(v) = 0$$

To prove Subspace

Let $x, y \in \text{Ker } T$, $\alpha, \beta \in F$

$\Rightarrow Tx = 0$, $Ty = 0$

$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ $\because T$ is LT.

$= 0$

$\Rightarrow \alpha x + \beta y \in \text{Ker } T$

$\Rightarrow \text{Ker } T$ is a subspace

(Self) Img subspace

Theorem:- $T: V \longrightarrow U$

$\text{Ker } T = \{0\}$ iff T is one-one.

Proof: Suppose $\text{Ker } T = \{0\}$

$T(x) = T(y)$

$\Rightarrow T(x) - T(y) = 0$

$\Rightarrow T(x-y) = 0$ $\because T$ is L.T.

$\Rightarrow x - y \in \text{Ker } T$

$\because \text{Ker } T = \{0\}$

$\Rightarrow x - y = 0$

$\Rightarrow x = y$

Conversely: Suppose T is one-one.

Let $x \in \text{Ker } T$

$\Rightarrow T(x) = 0$

But $T(0) = 0$

$T(x) = T(0)$

$\Rightarrow x = 0$ $(\because T$ is one-one$)$

$\Rightarrow \text{Ker } T = \{0\}$

Sir.
24-04-14

Fundamental Theorem of L.T

Let $T: V(F) \to U(F)$, be a vector space homomorphism from $V(F)$ onto $U(F)$.

Then $\frac{V}{\operatorname{Ker}T} \cong T(V)$

Proof:- Since $\operatorname{Ker}T$ is a subspace of V, therefore $\frac{V}{\operatorname{Ker}T}$ is a vector space over the field F. Of course, $T(V)$ is a subspace of $U(F)$, as an image space of $V(F)$ within $U(F)$, under

$$\theta: \frac{V}{\operatorname{Ker}T} \longrightarrow T(V)$$

defined by

$$\theta(K+x) = T(x) \quad , x \in V$$

Let $K+x, K+y \in \frac{V}{\operatorname{Ker}T}$ Let $\operatorname{Ker}T = K$

1-1 & well defined.

$$K+x = K+y$$
$$\Leftrightarrow K+x-y = K$$
$$\Leftrightarrow x-y \in K$$
$$\Leftrightarrow T(x-y) = 0$$
$$\Leftrightarrow T(x)-T(y) = 0$$
$$\Leftrightarrow T(x) = T(y)$$
$$\Leftrightarrow \theta(K+x) = \theta(K+y)$$

Onto:-

If $T(x) \in T(V)$ then $\exists\, x \in V,\ K+x \in \frac{V}{K}$

Such that

$$\theta(K+x) = T(x)$$

Homo. or L.T.

Let $K+x,\ K+y \in \frac{V}{\text{Ker}T}$

(i):- $\theta\big((K+x)+(K+y)\big) = \theta\big(K+(x+y)\big)$

$= T(x+y)$

$= T(x) + T(y)$ $\quad \because T$ is L.T.

$= \theta(K+x) + \theta(K+y)$

(ii). $\theta(\alpha(K+x)) = \theta(K+\alpha x)$ $\quad$ let $\alpha \in F$ $\because K$ is subspace so $\alpha K = K$

$= T(\alpha x)$

$= \alpha T(x)$ $\quad \because T$ is L.T.

$= \alpha\, \theta(K+x)$

Hence proved

$$\frac{V}{\text{Ker}T} \cong T(V)$$

Cor: If T is linear transformation and onto then 28-04-14

$$\frac{V}{\text{Ker}T} \cong U$$

Theorem:-

Let A & B be two subspaces of a vector space $V(F)$. Then

$$\frac{A+B}{A} \cong \frac{B}{A\cap B} \quad \text{or} \quad \frac{A+B}{B} \cong \frac{A}{A\cap B}$$

Proof:- Define

$$\theta : B \longrightarrow \frac{A+B}{A}$$

$$\Rightarrow \theta(b) = A+b \quad , \quad b \in B$$

Let $b_1, b_2 \in B$ such that

$$\Rightarrow b_1 = b_2$$

$$\Rightarrow A+b_1 = A+b_2$$

$$\Rightarrow \theta(b_1) = \theta(b_2)$$

$\Rightarrow \theta$ is welldefined

For $\alpha, \beta \in F$ & $b_1, b_2 \in B$

$$\theta(\alpha b_1 + \beta b_2) = A + (\alpha b_1 + \beta b_2)$$

$$= (A + \alpha b_1) + (A + \beta b_2)$$

$$= \alpha(A+b_1) + \beta(A+b_2) \qquad \because A \text{ is S.S} \Rightarrow \alpha A = A$$

$$= \alpha\,\theta(b_1) + \beta\,\theta(b_2)$$

$\Rightarrow \theta$ is L.T.

Let $A+x \in \frac{A+B}{A}$

$$x \in A+B$$

$$\Rightarrow x = a+b \; , \; a \in A \; , \; b \in B$$

$$A+x = A+(a+b)$$
$$= (A+a)+(A+b)$$
$$= A+(A+b)$$
$$= A+b$$
$$= \theta(b)$$

$\Rightarrow \theta$ is onto.

Fundamentament theorem of L.T.

$T: V \longrightarrow U$

$$\frac{V}{\ker T} \cong T(V) = U$$

Now $\dfrac{B}{\ker\theta} \cong \dfrac{A+B}{A}$.

Now we have to prove that,

$\ker\theta = A \cap B$

Let $x \in \ker\theta$

$\Leftrightarrow \theta(x) = A$ But $\theta(x) = A+x$

$\Leftrightarrow A+x = A$

$\Leftrightarrow x \in A$

Also $x \in \ker\theta$

As $\ker\theta$ is subspace of B

$\Leftrightarrow \ker\theta \subseteq B$

Hence $x \in \ker\theta \subseteq B \Leftrightarrow x \in B$

$\Longleftrightarrow \quad x \in A \cap B$

$\Longleftrightarrow \quad \ker \theta = A \cap B$

Hence $\dfrac{A+B}{A} \cong \dfrac{B}{A \cap B}$

Remark:

If $A \cap B = \{0\}$

then $\frac{A \oplus B}{B} \cong \frac{A}{\{0\}} = A$

Theorem:

Let W be a subspace of vector space $V(F)$ then $\exists$ an onto Linear transformation.

$\theta : V \longrightarrow \frac{V}{W}$ such that,

$W = Ker\,\theta$.

$\theta(x) = x + W \quad , x \in V$

This map is called natural Homomorphism or Qoutient map.

Proof:— Let $\alpha, \beta \in F$, $x, y \in V$

θ is welldefined clearly.

$\theta(\alpha x + \beta y) = W + (\alpha x + \beta y)$

$= (W + \alpha x) + (W + \beta y)$

$= \alpha(W + x) + \beta(W + y)$ $\because W$ is S.S $\Rightarrow \alpha W = W$

$= \alpha\,\theta(x) + \beta\,\theta(y)$

$\Rightarrow \theta$ is L.T

Onto map:— For every $x + W \in \frac{V}{W}$ $\exists$ $x \in V$ s.t.

$\theta(x) = x + W$

Let $x \in \ker\theta$

$\Leftrightarrow \theta(x) = W$

$\Leftrightarrow W + x = W$

$\Leftrightarrow x \in W$

$\Leftrightarrow \ker\theta = W$

Note:- If $W = \{0\}$
then this θ mapping is also 1-1.

page-45 on this Register.

Theorem:- Let V be a finite dimensional vector space and W is a subspace of V then

$$\dim\left(\frac{V}{W}\right) = \dim V - \dim W.$$

Theorem:- $\frac{V}{\ker T} \cong T(V)$

$$\dim\left(\frac{V}{\ker T}\right) = \dim T(V)$$

We can conclude that

$$\Rightarrow \dim V - \dim \ker T = \dim T(V).$$

Let $T: V \longrightarrow W$

$\dim(\ker T) = \text{nulity}(T) = m$

$\dim(T(V)) = \text{Rank}(T) = n$

$\dim(V) = m + n$

02-05-14

Sylvester's Law

Theorem: Let $T: V \longrightarrow W$ be a linear transformation then

$$\text{Rank}(T) + \text{nulity}(T) = \dim(V)$$

Proof:– Let $\{x_1, x_2, \ldots, x_m\}$ be a basis of $\text{Ker}(T)$.

$\{x_1, x_2, \ldots, x_m\}$ being linearly independent in Ker T, will be linearly independent in V.

Thus it can be extended to form basis of V.

Let $\{x_1, x_2, \ldots, x_m, V_1, V_2, \ldots, V_n\}$ is basis of V. We will show that $\{T(v_1), \ldots, T(V_n)\}$ form a basis for $T(V)$.

Now

$\alpha_1 T(V_1) + \cdots + \alpha_n T(V_n) = 0$ for $\alpha_i \in F$, $i \in \{1, 2, \ldots, n\}$

$T(\alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_n V_n) = 0$ $\because T$ is L.T.

$\Rightarrow \alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_n V_n \in \text{Ker}(T)$

$\alpha_1 V_1 + \cdots + \alpha_n V_n = \beta_1 x_1 + \cdots + \beta_m x_m$

$\beta_j \in F, j \in \{1, 2, \ldots, m\}$

$\alpha_1 V_1 + \cdots + \alpha_n V_n + (-\beta_1)x_1 + \cdots + (-\beta_m)x_m = 0$

$\Rightarrow \alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_m = 0$

Since $\alpha_i = 0 \ \forall i \Rightarrow \{T(v_1), \dots, T(v_n)\}$ is linearly independent.

Now we choose

$$T(v) \in T(V)$$

Since $v \in V$

$$\Rightarrow v = a_1 x_1 + \dots + a_m x_m + b_1 v_1 + \dots + b_n v_n$$

$b_j, a_i \in F$

$$\Rightarrow T(v) = a_1 \underset{0}{\underbrace{T(x_1)}} + \dots + a_m T(x_m) + b_1 T(v_1) + \dots + b_n T(v_n)$$

$\because T$ is L.T.

$x_i \in \operatorname{Ker} T$

$\Rightarrow T(x_i) = 0$

$$T(v) = b_1 T(v_1) + \dots + b_n T(v_n)$$

$\Rightarrow \{T(v_1), \dots, T(v_n)\}$ form basis for $T(V)$.

$\Rightarrow \dim T(V) = \text{Rank } T = n$.

Algebra of linear Transformation.

Let V and W be two vector spaces over field F.

Let $T: V \xrightarrow{L.T} W$,

& $S: V \xrightarrow{L.T} W$ then

$$T + S: V \xrightarrow{L.T} W$$

If $\alpha \in F$ then

$$\alpha T: V \xrightarrow{L.T} W$$

defined

$$(T+S)(v) = T(v) + S(v), \quad v \in V$$

Similarly

$$\alpha T(v) = \alpha T(v)$$

Self Check

$$(T+S)(\alpha v_1 + \beta v_2) = \alpha(T+S)(v_1) + \beta(T+S)(v_2)$$

Hom (V, W)

In this set every map from V to W is linear transformation.

Self Hom (U, W) is V.S.

~~complete is S.S.~~

Note:
$\alpha + T$ have no sense.
$\alpha I + T$ is meaningful

Matrix of Linear Transformation.

Let $U(F)$ and $V(F)$ be two vector spaces over the field F of dimension n and m (respectively).

Suppose

$T: U \longrightarrow V$ and let $\beta = \{u_1, \cdots, u_n\}$, $\beta' = \{v_1, \cdots, v_m\}$ be their ordered basis (resp.).

Suppose $T: U \rightarrow V$ is a linear transformation. Since $T(u_1), \ldots, T(u_n) \in V$ & $\{v_1, \cdots, v_m\}$ spans V.

$\Rightarrow$ each $T(u_i)$ is a linear combination of vectors $\{v_1, v_2, ---, v_m\}$.

$$T(u_1) = a_{11}v_1 + a_{21}v_2 + --- + a_{m1}v_m$$

$$T(u_2) = a_{12}v_1 + ---- + a_{m2}v_m$$

$$\vdots$$

$$T(u_n) = a_{1n}v_1 + --- + a_{mn}v_m$$

$$A = \begin{bmatrix} a_{11} & --- & a_{m1} \\ a_{12} & --- & a_{m2} \\ \vdots & & \\ a_{1n} & --- & a_{mn} \end{bmatrix}$$

A unique matrix corresponding to vector spaces.

Ordered Basis:-

An ordered basis for a vector space $V(F)$ of dimension n is a basis $\{v_1, ---, v_n\}$ together with one-one correspondence between the set $\{v_1, ---, v_n\}$ and $\{1, 2, ---, n\}$

Example:- Let V be a vector space of all polynomials of degree less than or equal to 2. With co-efficients from $\mathbb{R}$.

The set $\{1-x, 1+x, x^2\}$ is a basis $P_2(\mathbb{R})$.

$$a_0 + a_1 x + a_2 x^2 = \left(\frac{a_0 - a_1}{2}\right)(1-x) + \left(\frac{a_0 + a_1}{2}\right)(1+x) + a_2 x^2$$

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$\left(\frac{a_0-a_1}{2}, \frac{a_0+a_1}{2}, a_2\right).$

But for $\{1+x, 1-x, x^2\}$

then $\left(\frac{a_0+a_1}{2}, \frac{a_0-a_1}{2}, a_2\right)$

$\left.\begin{array}{l}(u_1, \ldots, u_n) \\ (u_n, u_1, \ldots, u_{n-1}) \\ (u_2, u_1, u_3, \ldots, u_n)\end{array}\right]$ ordered pair are different.

Hom (U, W) :- Theorem:-

Let $M_{m\times n}(F)$ denote the vector space of matrices (dim n) over F.

Let Hom (U, V) denote the vector space of all linear transformation from U to V over F.

Theorem:- $Hom(U,V) \cong M_{m\times n}(F)$

Proof:-

Self L.T

$\theta : Hom(U,V) \longrightarrow M_{m\times n}(F)$

θ is welldefined.

define $\theta(T) = [T]_{\beta, \beta'}$

Let $S, T \in Hom(U, V)$

$\theta(S) = \theta(T)$

$[S]_{\beta,\beta'} = [T]_{\beta,\beta'}$

$(a_{ij})_{\beta,\beta'} = (b_{ij})_{\beta,\beta'}$ $\quad i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$

$$\Rightarrow S(U_j) = \sum_{i=1}^{n} a_{ij} V_i = \sum_{i=1}^{n} b_{ij} V_i = T(U_j)$$

$$\Rightarrow S = T \Rightarrow \theta \text{ is } 1\text{-}1$$

$A = (a_{ij})_{m \times n} \in M_{m \times n}(F)$

then $\exists$ a linear transformation,

$T \in Hom(U, V)$ such that

$$T(U_j) = \sum_{i=1}^{n} a_{ij} V_i$$

Since $A = [T]_{B,B'} = \theta[T]$

$$\begin{matrix} \| \\ \theta(T) \end{matrix}$$

$\Rightarrow \theta$ is onto.

$$\begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix} = A$$

$$\dim A = mn$$

$$\dim(Hom(U, V)) = mn$$

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Theorem:

Let $T: V \to V$ be a L.T. If $\beta = \{u_1, u_2, \ldots, u_n\}$ and $\beta' = \{v_1, v_2, \ldots, v_n\}$ are ordered basis for V then $\exists$ a non-singular matrix P over F s.that

$$[T]_{\beta'} = P^{-1}[T]_{\beta}P$$

Proof:– Let $S: V \xrightarrow{L.T} V$ such that

$$S(u_i) = v_i \quad \forall \; i \in \{1, 2, \ldots, n\}$$

Let $x \in \ker S$

$\Rightarrow S(x) = 0$;

$x = \alpha_1 u_1 + \cdots + \alpha_n u_n$, $\alpha_i \in F$

$\Rightarrow S(\alpha_1 u_1 + \cdots + \alpha_n u_n) = 0$

$\Rightarrow \alpha_1 S(u_1) + \cdots + \alpha_n S(u_n) = 0$

$\Rightarrow \alpha_1 v_1 + \cdots + \alpha_n v_n = 0$

$\Rightarrow \alpha_i = 0$ for all i

$\Rightarrow x = 0$

$\Rightarrow \ker S = \{0\}$

$\Rightarrow S$ is 1-1 & so, onto obviously.

$\therefore S$ is an isomorphism.

Let $[T]_{\beta} = (a_{ij})$ then

$$T(u_j) = \sum_{i=1}^{n} a_{ij} u_i$$

$$(S T S^{-1})(v_j) = S T(u_j)$$

$$= S\left(\sum_{i=1}^{n} a_{ij} u_i\right)$$

$$= \sum_{i=1}^{n} a_{ij} S(u_i)$$

$$= \sum_{i=1}^{n} a_{ij} v_i$$

$$[S T S^{-1}]_{\beta'} = (a_{ij}) = [T]_{\beta}$$

$$\Rightarrow [S]_{\beta'} [T]_{\beta'} [S^{-1}]_{\beta'} = [T]_{\beta}$$

$$\Rightarrow [S]_{\beta'} [T]_{\beta'} [S]_{\beta'}^{-1} = [T]_{\beta}$$

$$\Rightarrow [T]_{\beta'} = [S]_{\beta'}^{-1} [T]_{\beta} [S]_{\beta'}$$

$$= P^{-1} [T]_{\beta} P \text{ , where } P = [S]_{\beta'}$$

Dual Spaces

Hom(U, V), the set of all linear transformations from vector space U to vector space V over same field F. If we replace V by F then Hom(U, F) is called dual space of U over F and it is denoted by $\hat{U}$.

Theorem:—

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. define

$$\hat{v}_i : V \longrightarrow F$$

$$\hat{v}_i(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_i \quad \forall\ i \in \{1, 2, \ldots, n\}$$

Then $\hat{v}_i$ is a linear transformation $\forall\ i \in \{1, 2, \ldots, n\}$ and $\{\hat{v}_1, \ldots, \hat{v}_n\}$ is a basis for $\hat{V}$.

$$\dim \operatorname{Hom}(U, V) = ?mn$$

$$\hat{V} = \dim \operatorname{Hom}(U, F) = ?$$

Proof:- Let $\{v_1, \ldots, v_n\}$ be a set of basis for $V(F)$. Let $v, v' \in V$ — 9-5-14

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n \qquad \alpha_i, \beta_j \in F$$

&

$$v' = \beta_1 v_1 + \cdots + \beta_n v_n$$

Let $v = v'$

also $\hat{v}_i(v) = \alpha_i = \beta_i = \hat{v}_i(v')$

images are same for all $i \in \{1, 2, \ldots, n\}$

So, this mapp is well defined.

let $v, v' \in V$

$$\hat{v}_i(v + v') = \hat{v}_i(\alpha_1 v_1 + \cdots + \alpha_n v_n + \beta_1 v_1 + \cdots + \beta_n v_n)$$

$$= \hat{v}_i((\alpha_1 + \beta_1) v_1 + \cdots + (\alpha_n + \beta_n) v_n)$$

$$= \alpha_i + \beta_i = \hat{v}_i(v) + \hat{v}_i(v')$$

Now $\hat{v}_i(\alpha v) = \hat{v}_i(\alpha \alpha_1 v_1 + \cdots + \alpha \alpha_n v_n)$

$$= \alpha \alpha_i$$

$= \alpha \hat{V}_i(V)$

$\Rightarrow \hat{V}_i$ is L.T.

By definition, we know that,

$\hat{V}_i(V_j) = \hat{V}_i(0V_1 + \cdots + 0V_{j-1} + V_j + \cdots + 0V_n)$

$$= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$\hat{V}_i(V_j) = \delta_{ij} \quad \forall\, i,j \in \{1,2,\ldots,n\}$

δ is direct delta function.

Now

$\alpha_1 \hat{V}_1 + \cdots + \alpha_n \hat{V}_n = 0 \quad , \alpha_i \in F$

$\alpha_1 \hat{V}_1 + \cdots + \alpha_n \hat{V}_n : V \longrightarrow F$

$O(V_j) = 0$

$(\alpha_1 \hat{V}_1 + \cdots + \alpha_n \hat{V}_n)(V_j) = 0$

$\alpha_1 \underbrace{\hat{V}_1(V_j)}_{= 0 \text{ if } j \neq 1} + \alpha_2 \hat{V}_2(V_j) + \cdots + \alpha_n \hat{V}_n(V_j) = 0$

$\Rightarrow \alpha_i \hat{V}_i(V_j) = 0 \quad \text{if } j = i$

$\Rightarrow \alpha_i = 0 \quad \forall\, i \in \{1,2,\ldots,n\}$

Let $f \in \hat{V}$

$f(V_i) = \alpha_i$ for $i \in \{1,2,\ldots,n\}$

We know that,

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$(\alpha_1 \hat{v}_1 + \cdots + \alpha_n \hat{v}_n)(v_i) = \alpha_i \hat{v}_i(v_i) = \alpha_i$

So, $f = \alpha_1 \hat{v}_1 + \cdots + \alpha_n \hat{v}_n$

therefore

$\{\hat{v}_1, \ldots, \hat{v}_n\}$ span $\hat{V}$.

Hence $\{\hat{v}_1, \ldots, \hat{v}_n\}$ form basis for $\hat{V}$ called dual space.

Corollary:- Let $V(F)$ be a finite dimensional vector space. Let $v \in V$ such that $v \neq 0$ then $\exists$ $f \in \hat{V}$ such that

$$f(v) \neq 0$$

Proof:- Since $v \neq 0$

if $\alpha v = 0$, $\alpha \in F$

$\Rightarrow \alpha = 0$

$\Rightarrow \{v\}$ is L.I

So, it can be extended to form a basis of V.

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. Let

$\{\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n\}$ be a corresponding dual basis. Then

$\hat{v}_i(v_j) = \delta_{ij}$ $\qquad \hat{v}: V \to F$

if $\hat{v}_1(v_1) = 1$

$\Rightarrow v_1 \neq 0$

We have $\hat{v}_1 \neq 0 \in \hat{V}$

Hence $f(v) \neq 0$

Theorem:- Let V be a finite dimensional vector space over field F.

Define $\theta : V \longrightarrow \hat{\hat{V}}$ s.t

$$\theta(v) = T_v \quad , \forall v \in V$$

$$T_v : \hat{V} \longrightarrow F$$

$$T_v(f) = f(v) \, , \; v \in V, \forall f \in \hat{V}$$

Then θ is isomorphism from V to $\hat{\hat{V}}$. i.e $V \cong \hat{\hat{V}}$

Proof:- ~~Let $T_v \in \hat{\hat{V}} = Hom(\hat{V}, F)$~~

Let $f, g \in \hat{V}$

Then

$$T_v(f+g) = (f+g)(v)$$
$$= f(v) + g(v)$$
$$= T_v(f) + T_v(g)$$

Take $\alpha \in F$

$$T_v(\alpha f) = (\alpha f)(v)$$
$$= \alpha f(v)$$
$$= \alpha T_v(f)$$

$\Rightarrow T_v \in \hat{\hat{V}}$

For θ is welldefined,

let $v, v' \in V$

take $v = v'$

$\Rightarrow T_v(f) = f(v)$

$= f(v') \quad \because v = v'$

$= T_{v'}(f) \quad \forall f \in \hat{V}$

$\Rightarrow T_v = T_{v'}$

$\Rightarrow \theta$ is $\boxed{\text{well defined.}}$

$\boxed{L.T} \rightarrow \theta(v+v') = T_{v+v'} = T_v + T_{v'} = \theta(v) + \theta(v')$

$T_{v+v'}(f) = f(v+v')$

$= f(v) + f(v')$

$= T_v(f) + T_{v'}(f)$

$= (T_v + T_{v'})(f), \quad \forall f \in \hat{V}$

$T_{v+v'} = T_v + T_{v'}$

Also, $\theta(\alpha v) = T_{\alpha v}(f)$

$= f(\alpha v)$

$= \alpha f(v) \quad \forall f \in \hat{V}$

$= \alpha T_v(f) \quad$ "

$= \alpha \theta(v)$

Let

$$0 \neq v \in \operatorname{Ker}\theta$$

$$\Rightarrow \theta(v) = 0$$

$$\Rightarrow T_v = 0$$

By con. $\exists\ f \in \hat{V}$ s.t $f(v) \neq 0$

$$\therefore\ T_v(f) \neq 0$$

which is a contradiction of $T_v = 0$

$$\Rightarrow T_v(f) = 0$$

$$\therefore\ \operatorname{Ker}\theta = \{0\}$$

$$\Rightarrow \boxed{\theta \text{ is } 1-1}$$

$$\therefore\ V \cong \theta(V) \subseteq \hat{\hat{V}}$$

$$\Rightarrow \dim \theta(V) = \dim V$$
$$= \dim \hat{V}$$
$$= \dim \hat{\hat{V}}$$

i.e $\theta(V) = \hat{\hat{V}}$

as $\theta(V)$ is a subspace of $\hat{\hat{V}}$

$\therefore\ \theta$ is onto from V to $\hat{\hat{V}}$

Thus θ is an isomorphism.

16-5-14 "Annihilator" 16-5-14

Let W be a subset of V

define $A(W) = \{f \in \hat{V} \mid f(w) = 0 \;\forall\; w \in W\}$

Then $A(W)$ is a sub-space of $\hat{V}$ as

$\alpha, \beta \in F$, $f, g \in A(W)$

$$\Rightarrow f(\omega) = 0 = g(\omega) \quad \forall\; \omega \in W$$

$$\Rightarrow \alpha f(\omega) + \beta g(\omega) = 0 \quad \forall\; \omega \in W$$

$$\Rightarrow (\alpha f + \beta g)(\omega) = 0 \quad \forall\; \omega \in W$$

$$\Rightarrow \alpha f + \beta g \in A(W)$$

$\Rightarrow A(W)$ is subspace of $\hat{V}$

$A(W)$ is called annihilator of W.

Theorem:- Let V be a finite dimensional vector space and W be a subspace of V.

Then

$$\dim A(W) = \dim V - \dim W$$

Proof:- Let $\{w_1, \dots, w_m\}$ be a basis of W. It can be extended to form a basis of V.

Let $\{w_1, \dots, w_m, v_{m+1}, \dots, v_n\}$ be a basis of V.

Let $\{f_1, \dots, f_m, f_{m+1}, \dots, f_n\}$ be corresponding dual basis. Then

$$f_i(w_j) = 0 \quad \forall\; i \in \{m+1, \dots, n\},$$
$$j \in \{1, \dots, m\}$$

$\therefore f_i \in A(W) \quad \forall\; i \in \{m+1, \dots, n\}$

We show $\{f_{m+1}, \dots, f_n\}$ is a basis of $A(W)$

Let

$$\alpha_{m+1} f_{m+1} + \cdots + \alpha_n f_n = 0$$

$\therefore (\alpha_{m+1} f_{m+1} + \cdots + \alpha_n f_n)(v_k) = 0$

$\forall\, k \in \{m+1, \ldots, n\}$

$\therefore \alpha_k f_k(v_k) = 0$

$\therefore \alpha_k = 0 \quad \forall\, k \in \{m+1, \ldots, n\}$

So, $\{f_{m+1}, \ldots, f_n\}$ is a L.I set.

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Let $f \in A(W)$

$\Rightarrow f(w) = 0 \quad \forall\, w \in W, f \in \hat{V}$

$\Rightarrow f(w_j) = (\beta_1 f_1 + \cdots + \beta_m f_m + \cdots + \beta_n f_n)(w_j)$

$\Rightarrow 0 = f(w_j) = \beta_j f_j(w_j) = \beta_j \quad \forall j \in \{1, \ldots, m\}$

$\Rightarrow f(w_j) = (\beta_{m+1} f_{m+1} + \cdots + \beta_n f_n)(w_j)$

$\Rightarrow f = \beta_{m+1} f_{m+1} + \cdots + \beta_n f_n$

$\Rightarrow \{f_{m+1}, \ldots, f_n\}$ spans $A(W)$

$\therefore \{f_{m+1}, \ldots, f_n\}$ is a basis of $A(W)$

Hence

$$\dim A(W) = n - m$$
$$= \dim V - \dim W$$

★ Cor. 1:- $\dfrac{\hat{V}}{A(W)} \cong \hat{W}$

Proof:- Since

$$\dim \frac{\hat{V}}{A(W)} = \dim \hat{V} - \dim(A(W))$$

$$= \dim V - (\dim V - \dim W) \quad \because \dim \hat{V} = \dim V$$

$$= \cancel{\dim V} - \cancel{\dim V} + \dim W$$

$$= \dim \hat{W} \quad \because \dim \hat{W} = \dim W$$

So, $\frac{\hat{V}}{A(W)} \cong \hat{W}$

Cor-2:- If V is a finite dimensional vector space and W a subspace of V, then

$$A(A(W)) = W$$

Proof:- Define $\theta : W \longrightarrow A(A(W))$ s.t

$$\theta(w) = T_w \quad , w \in W$$

Where $T_w : \hat{W} \longrightarrow F$

$$T_w(f) = f(w)$$

$$T_w \in A(A(W))$$

as $T_w(f) = f(w) = 0 \quad \forall f \in A(W)$.

θ is well defined as

Let $w_1, w_2 \in W$

$$w_1 = w_2$$

$$\Rightarrow T_{w_1}(f) = f(w_1) = f(w_2) \quad \because w_1 = w_2$$

$$= f(w_2) = T_{w_2}(f) \quad \forall f \in \hat{W}$$

θ is a L.T as

(i) $\theta(w_1 + w_2) = T_{(w_1+w_2)}(f)$

$$= f(w_1 + w_2)$$

$$= f(w_1) + f(w_2) \qquad \because f \text{ is Linear}$$

$$= T_{w_1}(f) + T_{w_2}(f)$$

$$= (T_{w_1} + T_{w_2})(f)$$

$$= \theta(w_1) + \theta(w_2)$$

(ii) $\theta(\alpha w_1) = T_{\alpha w_1}(f)$

$$= f(\alpha w_1)$$

$$= \alpha f(w_1)$$

$$= \alpha T_{w_1}(f)$$

$$= \alpha \theta(w_1) \qquad \forall f \in \hat{W}$$

Let $0 \neq w_1 \in \operatorname{Ker} \theta \Rightarrow \theta(w_1) = 0$

$$\Rightarrow T_{w_1} = 0$$

as $w_1 \neq 0$ $\exists$ $f \in \hat{W}$ s.t $f(w_1) \neq 0$

$\therefore$ $T_{w_1}(f) \neq 0$

as contradiction as $T_{w_1} = 0 \Rightarrow T_{w_1}(f) = 0$

$\therefore$ $\operatorname{Ker} \theta = \{0\} \Rightarrow \theta$ is 1·1.

$$W \cong \theta(W) \subseteq A(A(W))$$

Since

$$\dim A(A(W)) = \dim \hat{V} - \dim A(W)$$

$$= \dim V - [\dim V - \dim W]$$

$$= \dim W$$

we know $\dim \theta(W) = \dim W$

$$\Rightarrow A(A(W)) = \theta(W)$$

$\therefore$ θ is onto from W to $A(A(W))$

$$\therefore W \cong A(A(W))$$

For the sake of convenience, we shall write

$$A(A(W)) = W$$

Inner Product Space

In inner product space we take F to the field of real or complex number. In the first case, the vector space is called real vector space and in second complex vector space

Inner product Actually

$$\langle \cdot,\cdot \rangle : V \times V \longrightarrow F$$

Let $w, u, v \in V$ then $\langle u, v \rangle \in F$

I.P.S over real field is called Euclidean Space. and over complex field is called Unitary space.

1st:- $\langle u, v\rangle = \langle \overline{v, u}\rangle$

$\Rightarrow$ But incase of Real $\langle u, v\rangle = \langle v, u\rangle$

2:)- $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0 \Leftrightarrow u = 0$

3):- $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$

Then vector space V is called an inner product space and the function satisfying 1, 2 & 3 is called an Inner product.

Example:- $V = F^n$, $F = \mathbb{C}$

Let $u, v \in V$

$u = (u_1, \ldots, u_n)$ & $v = (v_1, \ldots, v_n)$

$\langle u, v\rangle = u_1\bar{v}_1 + u_2\bar{v}_2 + \cdots + u_n\bar{v}_n$

$\langle \cdot, \cdot\rangle : \mathbb{C}^n \longrightarrow \mathbb{C}$

Prove that it defines an inner product

Sol:-

Norm of a Vector 22-5-14

Let V be an inner product space. Let $v \in V$. Then norm of v (or length of v) is defined as $\sqrt{\langle v, v\rangle}$ and denoted by $\|v\|$.

i.e $\sqrt{\langle v, v\rangle} = \|v\|$

Problem 18:– $\|\alpha v\| = |\alpha| \, \|v\|$

$$\forall \alpha \in F, \, v \in V$$

Solution:– $\|\alpha v\| = \sqrt{\langle \alpha v, \alpha v\rangle}$

$$\|\alpha v\|^2 = \langle \alpha v, \alpha v\rangle$$
$$= \alpha\bar{\alpha}\langle v, v\rangle$$
$$= |\alpha|^2 \|v\|^2$$
$$\Rightarrow \|\alpha v\| = |\alpha| \, \|v\|$$

Cauchy Schwarz Inequality

Theorem 28:– Let V be an inner product space. Then

$$|\langle u, v\rangle| \leq \|u\| \, \|v\| \quad , \forall \, u, v \in V$$

Proof:– Choose $u = 0 \in V$, $v \in V$

then L.H.S $= \langle 0, v\rangle = 0$

R.H.S $= \|v\| \, \|u\| = \|v\| \cdot 0 = 0$

If $v \neq 0$, $u \neq 0$

Let $w = v - \dfrac{\langle v, u\rangle}{\|u\|^2} u$

$$\langle w, w\rangle = \left\langle v - \frac{\langle v, u\rangle}{\|u\|^2} u , \; v - \frac{\langle v, u\rangle}{\|u\|^2} u \right\rangle$$

$$= \langle v, v\rangle - \frac{\overline{\langle v, u\rangle}}{\|u\|^2}\langle v, u\rangle - \frac{\langle v, u\rangle}{\|u\|^2}\langle u, v\rangle$$

$$+ \frac{\langle v, u\rangle}{\|u\|^2}\cdot\frac{\overline{\langle v, u\rangle}}{\|u\|^2}\langle u, u\rangle$$

$$= \|v\|^2 - \frac{|\langle v, u\rangle|^2}{\|u\|^2} - \frac{\overline{\langle u, v\rangle}}{\|u\|^2}\cdot\langle u, v\rangle$$

$$+ \frac{|\langle v, u\rangle|^2}{\|u\|^2\cdot\cancel{\|u\|^2}}\cdot\cancel{\|u\|^2}$$

$$= \|v\|^2 - \cancel{\frac{|\langle v, u\rangle|^2}{\|u\|^2}} - \frac{|\langle u, v\rangle|^2}{\|u\|^2} + \cancel{\frac{|\langle v, u\rangle|^2}{\|u\|^2}}$$

$$= \frac{\|u\|^2\|v\|^2 - |\langle u, v\rangle|^2}{\|u\|^2}$$

Since $\langle w, w\rangle \geq 0$

$$\Rightarrow \|u\|^2\|v\|^2 - |\langle u, v\rangle|^2 \geq 0$$

$$\|u\|^2\|v\|^2 \geq |\langle u, v\rangle|^2$$

$$\|u\|\,\|v\| \geq |\langle u, v\rangle|$$

$$|\langle u, v\rangle| \leq \|u\|\,\|v\|$$

Remarks:-

This inequality become equality iff u & v are linearly dependent.

Proof:- Suppose $|\langle u,v\rangle| = \|u\|\,\|v\|$

Trivial case. If $u=0$, then $u = 0v \Rightarrow u, v$ are linearly dependent.

Let $u \neq 0$. Then by above

$\langle w,w\rangle = 0 \Rightarrow w = 0$ (By 2nd property of IPS)

$$\therefore \quad v - \frac{\langle v,u\rangle}{\|u\|^2} u = 0$$

$$\Rightarrow \quad v = \frac{\langle v,u\rangle}{\|u\|^2} u$$

$$\Rightarrow \quad v = \alpha u, \quad \text{where } \alpha = \frac{\langle v,u\rangle}{\|u\|^2}$$

$\Rightarrow u$ & v are linearly dependent.

Conversely:- Suppose u & v are L.D.

$$\Rightarrow v = \alpha u \;,\; \alpha \in F$$

Then $|\langle u,v\rangle| = |\langle u,\alpha u\rangle|$

$$= |\bar{\alpha}\langle u,u\rangle|$$

$$= |\bar{\alpha}|\,\|u\|^2$$

Now $\|u\|\,\|v\| = \|u\|\,\|\alpha u\|$

$$= \|u\|\,|\alpha|\,\|u\|$$

$$= \|u\|^2 |\alpha|$$

$$= \|u\|^2 |\bar{\alpha}| \qquad \because |\bar{\alpha}| = |\alpha|$$

So,

$$|\langle u,v\rangle| = \|u\|\,\|v\|$$

Book **Theorem:** Let V be I.P.S. Then

(i):- $\|x+y\| \le \|x\| + \|y\|$, $\forall\ x, y \in V$
(Triangle inequality)

(ii):- $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$
(Parallelogram Law)

Proof:- (i).

$$\|x+y\| = \sqrt{\langle x+y, x+y\rangle}$$

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$

$$= \langle x, x\rangle + \langle y, x\rangle + \langle x, y\rangle + \langle y, y\rangle$$

$$= \|x\|^2 + \overline{\langle x, y\rangle} + \langle x, y\rangle + \|y\|^2$$

$$= \|x\|^2 + 2\,Re\langle x, y\rangle + \|y\|^2$$

$$\le \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$$

$$\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

Hence $\|x+y\| \le \|x\| + \|y\|$

(ii) $\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$

$$= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 +$$
$$\|x\|^2 - \langle x, y\rangle - \langle y, x\rangle + \|y\|^2$$

$$= 2(\|x\|^2 + \|y\|^2)$$

Note, $|\cos\theta| = \dfrac{|\langle u, v\rangle|}{\|u\|\|v\|} \le \dfrac{\|u\|\|v\|}{\|u\|\|v\|} = 1$

Orthogonality:-

26-5-14

Let W be a subspace of V.

Define $W^{\perp} = \{v \in V \mid \langle v, w \rangle = 0, \ \forall\, w \in W\}$

($W^{\perp}$ is read as "W perpendicular").

Then $W^{\perp}$ is called orthogonal subspace of V. As $0 \in W^{\perp} \Rightarrow W^{\perp} \neq \phi$

Proof: Let $V_1, V_2 \in W^{\perp}$, $\alpha, \beta \in F$

$$\langle \alpha V_1 + \beta V_2, w \rangle = \alpha \langle V_1, w \rangle + \beta \langle V_2, w \rangle$$
$$= \alpha(0) + \beta(0)$$
$$= 0$$

$$\Rightarrow \alpha V_1 + \beta V_2 \in W^{\perp}$$

$W^{\perp}$ is called orthogonal complement of W.

Problem:- Let V be an inner product space. Let $x, y \in V$ s.t $x \perp y$. Then show that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 \quad \text{(Pythagoras Theorem)}$$

Proof:- $L.H.S = \|x+y\|^2 = \langle x+y, x+y \rangle$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$
$$= \|x\|^2 + 0 + 0 + \|y\|^2 \quad \because x \perp y.$$
$$= \|x\|^2 + \|y\|^2$$
$$= R.H.S$$

Orthonormal Set:-

A set $\{u_i\}_i$ of vectors in an inner product space V is said to be orthogonal if $\langle u_i, u_j\rangle = 0$, $\forall\, i \neq j$. If further, $\langle u_i, u_i\rangle = 1$ $\forall\, i$, then the set $\{u_i\}$ is called an orthonormal set.

Example:- Let V be the real vector space of real polynomials of degree less than or equal to n. Define an inner product on V by

$$\left\langle \sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{n} b_j x^j \right\rangle = \sum_{0}^{n} a_i b_j$$

Then $\{1, x, \dots, x^n\}$ is an orthonormal subset of V.

$$\langle a_0 + a_1 x + \dots + a_n x^n, b_0 + b_1 x + \dots + b_n x^n\rangle = \langle a_0, b_0 + b_1 x + \dots + b_n x^n\rangle$$
$$+ \langle a_1 x, b_0 + b_1 x + \dots + b_n x^n\rangle + \dots + \langle a_n x^n, b_0 + b_1 x + \dots + b_n x^n\rangle$$

We know $\langle x^i, x^j\rangle = 1$ $\forall\, i = j$

& $\langle x^i, x^j\rangle = 0$ $\forall\, i \neq j$

$$= a_0 b_0 + \langle a_1 x + b_1 x\rangle + \langle a_2 x^2, b_2 x^2\rangle + \dots + \langle a_n x^n, b_n x^n\rangle$$

$$= a_0 b_0 + a_1 b_1 + \dots + a_n b_n$$

$$= \sum_{i=0}^{n} a_i b_i, \quad i \in \{1, \dots, n\}$$

29-05-14

Theorem:- Let S be an orthogonal set of non zero elements or vectors in an inner product space V. Then S is a linearly independent set.

Proof:- To show S is linearly independent, we have to show that every finite subset of S is linearly independent.

Let $\{v_1, \ldots, v_n\}$ be a finite subset of S.

Let $\alpha_i \in F$, $\forall i \in \{1, 2, \ldots, n\}$

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$$

$$\Rightarrow \langle \alpha_1 v_1 + \cdots + \alpha_n v_n, \alpha_1 v_1 + \cdots + \alpha_n v_n \rangle = 0$$

$$\Rightarrow |\alpha_1|^2 \|v_1\|^2 + \cdots + |\alpha_n|^2 \|v_n\|^2 = 0$$

$$\because v_i \neq 0 \Rightarrow |\alpha_i|^2 = 0$$

$$\Rightarrow |\alpha_i| = 0$$

$$\Rightarrow \alpha_i = 0 \quad \forall i \in \{1, 2, \ldots, n\}$$

$\Rightarrow S$ is linearly independent.

Theorem:- (Gram-Schmidt Orthogonalisation Process) 30-5-14

Let V be a non-zero inner product space of dimension n. Then V has an orthonormal basis.

Proof:- Let $S \subseteq V$ be an orthogonal set. Then $T = \{\frac{x}{\|x\|} \mid x \in S\}$ is an orthonormal set.

Let $\{v_1, \dots, v_n\}$ be a basis of V.

Let $w_1 = v_1$. Define

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \longrightarrow (i)$$

$$= v_2 - \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

Then

$$\langle w_2, w_1 \rangle = \langle w_2, v_1 \rangle = \langle v_2 - \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1, v_1 \rangle$$

$$= \langle v_2, v_1 \rangle - \frac{\langle v_2, v_1 \rangle}{\cancel{\langle v_1, v_1 \rangle}} \cdot \cancel{\langle v_1, v_1 \rangle}$$

$$= 0$$

$$\Rightarrow w_1 \perp w_2$$

Also from (i) $\quad v_2 = \alpha_1 v_1 + w_2$

$$= \alpha_1 w_1 + w_2 \quad \text{where } \alpha_1 = \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle} \in F$$

(Note v_1, is L.I

$\Rightarrow v_1 \neq 0 \Rightarrow \langle v_1, v_1 \rangle \neq 0$)

*Define $\quad w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$

Define

$$W_3 = V_3 - \frac{\langle V_3, W_2\rangle}{\langle W_2, W_2\rangle} W_2 - \frac{\langle V_3, W_1\rangle}{\langle W_1, W_1\rangle} W_1$$

* For nth

$$W_n = V_n - \frac{\langle V_n, W_{n-1}\rangle}{\langle W_{n-1}, W_{n-1}\rangle} W_{n-1} - \cdots - \frac{\langle V_n, W_1\rangle}{\langle W_1, W_1\rangle} W_1$$

$$\langle W_3, W_1\rangle = \left\langle V_3 - \frac{\langle V_3, W_2\rangle}{\langle W_2, W_2\rangle} W_2 - \frac{\langle V_3, W_1\rangle}{\langle W_1, W_1\rangle} W_1, W_1 \right\rangle$$

$$= \langle V_3, W_1\rangle - \frac{\langle V_3, W_2\rangle}{\langle W_2, W_2\rangle}\underbrace{\langle W_2, W_1\rangle}_{0} - \frac{\langle V_3, W_1\rangle}{\cancel{\langle W_1, W_1\rangle}}\cancel{\langle W_1, W_1\rangle}$$

$$= \langle V_3, W_1\rangle - \langle V_3, W_1\rangle$$

$$= 0$$

$$\Rightarrow W_3 \perp W_1$$

Now

$$\langle W_3, W_2\rangle = \left\langle V_3 - \frac{\langle V_3, W_2\rangle}{\langle W_2, W_2\rangle} W_2 - \frac{\langle V_3, W_1\rangle}{\langle W_1, W_1\rangle} W_1, W_2 \right\rangle$$

$$= \langle V_3, W_2\rangle - \frac{\langle V_3, W_2\rangle}{\cancel{\langle W_2, W_2\rangle}}\cancel{\langle W_2, W_2\rangle} - \frac{\langle V_3, W_1\rangle}{\langle W_1, W_1\rangle}\underbrace{\langle W_1, W_2\rangle}_{0}$$

$$= \langle V_3, W_2\rangle - \langle V_3, W_2\rangle$$

$$= 0$$

$$\Rightarrow W_3 \perp W_2$$

We can write $V_3 = \alpha_1 W_1 + \alpha_2 W_2 + \alpha_3 W_3$

Similarly we can write,

$$V_n = \alpha_1 W_1 + \cdots + \alpha_n W_n, \quad \alpha_i \in F$$

We have checked that $\langle W_i, W_j\rangle = 0$ $\forall\ i \neq j$

$\left\{\frac{w_1}{\|w_1\|}, \dots, \frac{w_n}{\|w_n\|}\right\}$ is orthonormal.

where $\|w_i\| \neq 0$

Similar Matrix

Let S be a square $(n\times n)$ matrix. M is $n\times n$, non-singular $(|M|\neq 0)$ matrix, Then

$$T = M S M^{-1}$$

T is similar to matrix S.

Theorem :- 2-6-14

Let M_n be the set of all non-singular square matrices. Then the relation of similarity on the set of M_n is an equivalence relation, which decomposes M_n into mutually disjoint similarity classes.

Proof:- We have to show that three properties; (i) Reflexive (ii) Symmetric (iii) Transitive.

(i) Reflexive

Let $S \in M_n$

$$S = I_n S I_n^{-1} = I_n (S I_n^{-1}) = I_n S = S$$

$\Rightarrow S$ is similar to S.

(ii) Symmetric:- Let $M \in M_n$

$T = MSM^{-1}$ $\quad$ T is similar to S.

$M^{-1}TM = M^{-1}MSM^{-1}M$

$M^{-1}TM = I_n S I_n = S$

(iii) Transitive:-

Suppose

$T = MSM^{-1}$ $\quad$ T is similar to S.

& $S = M_1 U M_1^{-1}$ $\quad$ S is similar to U.

$\Rightarrow T = M(M_1 U M_1^{-1})M^{-1}$

$= MM_1 U (MM_1)^{-1}$

$\Rightarrow T$ is similar to U,

with nonsingular matrix $MM_1 \in M_n$.

Cor: 1:- The eigen values of the matrices are preserved under the similarity transformations of square matrices.

Proof:- Suppose λ be an eigen value of S. Then

$$Sx = \lambda x$$

$$T = MSM^{-1}$$

$$\Rightarrow Tx = (MSM^{-1})x$$

$$= MS(M^{-1}x)$$

$$= M(\lambda M^{-1}x)$$

$$= \lambda MM^{-1}x$$

$$= \lambda (I_n x)$$
$$= \lambda x$$

i.e $Tx = \lambda x$

$\Rightarrow \lambda$ is eigen value of T.

Cor: 2:– If λ is a eigen value of S then $1/\lambda$ is an eigen value of S^{-1}.

(S is a non-singular matrix & $\lambda \neq 0$)

Proof:– $Sx = \lambda x$

$$S^{-1}(Sx) = S^{-1}(\lambda x)$$
$$\Rightarrow (S^{-1}S)(x) = \lambda S^{-1}(x)$$
$$\Rightarrow I_n(x) = \lambda S^{-1}(x)$$
$$\Rightarrow x = \lambda S^{-1}(x)$$
$$\Rightarrow \frac{1}{\lambda}(x) = S^{-1}(x)$$

Hence $1/\lambda$ is an eigen value of S^{-1}.

Cor: 3:– Let λ_1 & λ_2 are eigen values of M_1 & M_2 respectively then $\lambda_1 \lambda_2$ & $(\lambda_1 \pm \lambda_2)$ are eigen values of $M_1 M_2$ & $(M_1 \pm M_2)$.

Proof:– $M_1 x = \lambda_1 x$ & $M_2 x = \lambda_2 x$

then $(M_1 M_2)x = M_1(M_2 x)$

$$= M_1(\lambda_2 x)$$
$$= \lambda_1\lambda_2 x$$

$\Rightarrow \lambda_1\lambda_2$ is an eigen value of M_1M_2.

$$(M_1 \pm M_2)(x) = \lambda_1 x \pm \lambda_2 x$$
$$= (\lambda_1 \pm \lambda_2)x$$

$\Rightarrow \lambda_1 \pm \lambda_2$ are eigen values of $(M_1 \pm M_2)$.

Diagonalization of Matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

From this diagonal matrix we can easily find eigen value.

Example:- $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \lambda = 2, 3$$

For $\lambda = 2$

$$(A - 2I) = \begin{bmatrix} 2-2 & 0 \\ 0 & 3-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(A - \lambda I)(x) = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_2 = 0$

x_1 is arbitrary.

So, $\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \text{Span} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Now for $\lambda = 3$

$$(A - 3I) = \begin{bmatrix} 2-3 & 0 \\ 0 & 3-3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 = 0 \Rightarrow x_1 = 0$

x_2 is arbitrary.

So, $\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \text{Span} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$A_{n \times n}$

$\lambda_1, \ldots, \lambda_n$ are eigen values.

$v_1, \ldots, v_n$ are eigen vectors.

With the help of eigen

vectors, we form,

$$C = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

Let A be real (or complex) $n \times n$ matrix. Let $(\lambda_1, \ldots, \lambda_n)$ be set of eigen values and let $v_1, \ldots, v_n$ be the set of n vectors in $\mathbb{R}^n$ (or $\mathbb{C}^n$) and C be a $n \times n$ matrix formed by using v_j for j-th coulumns. Let D be a diagonal matrix whose diagonal intries are $\lambda_1, \ldots, \lambda_n$. Then

$$AC = CD$$

$$\text{or} \quad A = CDC^{-1}$$

$$\text{or} \quad D = C^{-1}AC$$

Example:- $A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$

Find the matrix that diagonalize A.

Sol:- $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 6 \\ 0 & -1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (2-\lambda)(-1-\lambda) = 0$$

$$\Rightarrow \lambda = -1, 2$$

Vector Projection

5-6-14

The vector projection of a vector 'a' on a non-zero vector 'b' is the orthogonal projection of 'a' onto a straight line parallel to 'b' defined as

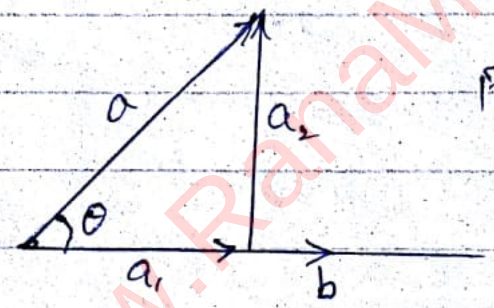
$$\underbrace{a_1}_{\text{is a scalar called scalar projection of 'a' onto 'b'.}} = a\hat{b}$$

$\hat{b}$ is unit vector in the direction of b.

is a scalar called scalar projection of 'a' onto 'b'.

$$a_1 = \underbrace{|\vec{a}|}_{\text{Length of a}} \cos\underbrace{\theta}_{\text{angle b/w 'a' \& 'b'}} = \vec{a}\cdot\hat{b} \longrightarrow ①$$

Length of a — angle b/w 'a' & 'b'



Projection of 'a' from 'b'.

$$a_2 = a - a_1$$

Vector projection ⇒ vector component or vector resolute of 'a' in the direction of 'b'.

$$\vec{a_1} = a_1\hat{b} = |\vec{a}|\cos\theta\,\hat{b} \longrightarrow ②$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|a|\,|b|}$$

Put in ② ⇒ $a_1 = |\vec{a}|\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\hat{b}$

$\Rightarrow a_1 = (\vec{a}\cdot\hat{b})\,\hat{b}$ or $a_1 = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$

Properties:-

$$a_1 = |\vec{a_1}| \quad \text{if} \quad 0 \leq \theta \leq 90$$

$$a_1 = -|\vec{a_1}| \quad \text{if} \quad 90 \leq \theta \leq 180$$

vector projection:-

$\vec{a_1} = 0$ if $\theta = 90$, a_1 & b have same direction if $0 \leq \theta \leq 90$, have oposite direction if $90 \leq \theta \leq 180$