ADVANCED TOPOLOGY - I

INSTRUCTOR :- Prof. Dr. Moiz-ud-Din Khan.

This course was established in 1963 by an American mathematician Norman Levine (N. Levine)

★ **First Research Paper:-**

Title :- Semi open sets and semi continuity in Topology spaces.

This paper was published by American Mathematical Monthly Vol. 70 N-I (Jan 1963): Pages (36 – 41)

***Topology**:- Let X be a non-empty set and τ be a collection of subsets of X. Then τ is called topology if

(i) ϕ and X belongs to τ

(ii) The intersection of any two sets in τ belongs to τ

(iii) The union of any number of sets in τ belongs to τ.

The members of τ are then called τ-open sets or simply open sets (and compliment of open sets is called a closed set). X together with τ i.e (X, τ) is called a topological space.

The set X is called its ground set and the elements of X are called its points.

* ϕ and X are always open as well as closed (clopen).

* Neighbourhood of a point $x \in X$ is a set N s.t $x \in O \subseteq N$, where O is an open set.

* An open set is neighbourhood of each of its points.

* Each point of a topological space has atleast one neighbourhood and that is X.

A point of a topological space may have more than one neighbourhoods

Example:- Let $X = \{a, b, c, d\}$

$P(X) = \phi, X, \{a\}, \{b\}, \{c,d\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}$

$\tau_1 = \{\phi, X, \{a\}, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

$\tau_2 = \{\phi, X, \{b\}, \{d\}, \{b,c\}, \{b,d\}, \{b,c,d\}\}$

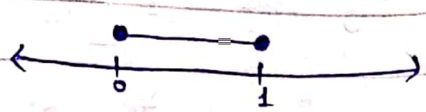
τ_1 and τ_2 satisfy all the conditions of a topological space so τ_1 and τ_2 are topological spaces.

⇒ **Interior of a Set:-** Let (X, τ) be a topological space and A is a non-empty subset of X. A point $x \in A$ is an interior point of A if there exist an open neighbourhood O s.t $x \in O \subseteq A$

Example:- Let $X = \mathbb{R}$

τ is the collection of all possible open intervals of $\mathbb{R}$ and and ϕ. Then τ is a topology on $\mathbb{R}$. This topology is called usual topology on $\mathbb{R}$ or standard topology on $\mathbb{R}$.

$A = [0, 1]$



$x = 0 \in A$.

Here $0 \in A$ but not interior point of A.

$1 \in A$ but not interior point of A.

All other points of A are interior points of A.

$B = (0,1) \Rightarrow$ every point of B is interior point of B.

Note:- * Every point of an open set is an interior point of that set.

* Interior of a set is a collection of all interior points of that set and is denoted by $Int(A)$.

* A set 'A' is open if and only if $Int(A) = A$

* $Int(A) \subseteq A$.

⇒ **Limit point of a set:-** Let (X, τ) be a topological space and A is a subset of X. A point $x \in X$ is called a limit point of A if every open neighbour of x contains a point of A other than x i.e $\forall\ U \in N(x);\ A \cap U - \{x\} \neq \phi$

* Limit point of a set may not be member of that set.

* A set is closed if it contains all of its limit points.

* Collection of all limit points of A is called derived set of A and is usually denoted by A^d.

⇒ **Closure of a set:-** Let (X, τ) be a topological space and $A \subseteq X$. Then closure of A is denoted by $Cl(A)$ and is defined by $Cl(A) = A \cup A^d$

★ A is closed iff $A = Cl(A)$

★ $A \subseteq Cl(A)$

⇒ **Exterior Point:-** Let (X, τ) be a topological space and $A \subseteq X$ then $x \in X$ is said to be an exterior point of A if x is an interior point of A'. i.e x is said to be exterior point of A if there exist some open set U such that $x \in U \subseteq A'$

OR x is exterior point of A if there exist open set U containing x such that
$$U \cap A = \phi$$

⇒ **Boundry Point:-** Let (X, τ) be a topological space and $A \subseteq X$ then $x \in X$ is said to be boundry point or frontier point of A if x is neither the interior point of A nor the interior point of A'. In other words $x \in X$ is said to be boundry point of $A \subseteq X$ if for every open set U containing x
$$U \cap A \neq \phi \quad \text{and} \quad U \cap A' \neq \phi$$

⇒ **Isolated Point:-** Let (X, τ) be a topological space and $A \subseteq X$. Then a point $x \in A$ is said to be isolated point of A if x is not the limit point of A. i.e there exist an open set U containing x such that $U \cap A / \{x\} = \phi$

The set of all isolated points is denoted by A^* i.e

$A^* = \{x : x \in A$ and x is not a limit point$\}$

★ Let (X, τ) be a topological space then any closed subset A of X is disjoint union of the set of isolated points of A and the set of limit points of A.

⇒ **Dense Set**:- Let (X, τ) be a topological space and $A \subseteq X$ then A is called dense in X if $\bar{A} = X$

Example:- Let $X = \{1, 2, 3, 4, 5\}$, $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$

Let $A = \{1, 2\}$

Closed sets are X, ϕ, $\{2,3,4,5\}$, $\{1,3,4,5\}$, $\{3,4,5\}$

closed superset of A is X only.

Therefor $\bar{A} = X$

$\Rightarrow$ A is dense in X.

⇒ **Semi-open Sets** :- (In Topological spaces) - Let (X, τ) be a topological space a subset U of X is said to be semi-open in X if there exist an open set O in X such that $O \subseteq U \subseteq cl(O)$

Example:- $X = \{a, b, c, d\}$

$\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$

Let $A = \{a, c\}$

Here closed sets are $\phi, X, \{b,c,d\}, \{a,c,d\}$

$cl(a) = \{a, c, d\}$, $cl(b) = \{b, c, d\}$

$cl\{a, b\} = X$

As $\{a\}$ is open set and

$\{a\} \subseteq \{a, c\} \subseteq \{a, c, d\} = cl(a)$

$\Rightarrow \{a\} \subseteq \{a, c\} \subseteq cl(a)$

$\Rightarrow \{a, c\}$ is semi-open set.

★ Every open set is also a semi-open set.

★ A semi-open set may not be open.

Equivalently; a subset U of X is semi-open in X iff $U \subseteq cl[Int(U)]$

Proof) Let U be a semi-open in X

Put $Int(U) = O \longrightarrow (1)$

and $Int(U) \subseteq U \Rightarrow O \subseteq U \subseteq cl(O)$ by def

$\Rightarrow U \subseteq cl[Int(U)]$ by (1)

$\boxed{\text{collection of Interior points is a open set}}$

Conversely:- Let $U \subseteq cl[Int(U)]$

Since $Int(U) \subseteq U$

$\Rightarrow Int(U) \subseteq U \subseteq cl[Int(U)]$

i.e $V \subseteq U \subseteq cl(V)$, where V is open in X.

$\Rightarrow U$ is semi-open in X.

Note:- Collection of all semi-open sets in X is denoted by $SO(X)$

$V \in SO(X) \Rightarrow V$ is semi-open in X.

★ The compliment of a semi-open set is called a semi-closed set.

★ Collection of all semi-closed sets in X is denoted by $SC(X)$.

Example:- Let $X = \mathbb{R}$ with the usual topology on $\mathbb{R}$.

Let $E = (0,1)$ then $cl(E) = [0,1]$

If $A = [0,1)$, $B = (0,1]$, $C = [0,1]$ Then each A, B and C are semi-open in X.

Note:- $C = [0,1]$ is a closed set which is semi-open as well. This mean closed set can be semi-open set as well. [but open sets are always sem-open]

open set $\leftarrow (0,1) \subseteq C = [0,1] \subseteq cl(0,1) = [0,1]$

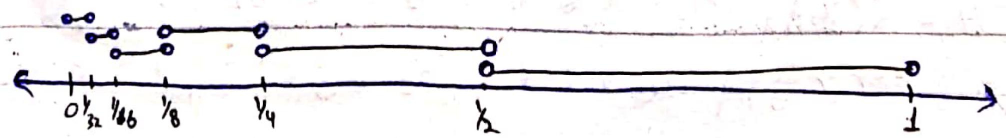
Thats why C is semi-open.

Example:- Let $X = \mathbb{R}$ with usual topology and

Let $A = (\frac{1}{2}, 1) \cup (\frac{1}{4}, \frac{1}{2}) \cup \dots \cup (\frac{1}{2^{n+1}}, \frac{1}{2^n}) \cup \dots$

and $B = \{0\} \cup (\frac{1}{2}, 1) \cup (\frac{1}{4}, \frac{1}{2}) \cup \dots \cup (\frac{1}{2^{n+1}}, \frac{1}{2^n}) \cup \dots$

$\Rightarrow$ A is open set. Since A is union of open intervals and every open interval is a open set and union of any number of open sets is a open set.



Here $A = (0,1)$ & $cl(A) = [0,1]$

& $B = [0,1)$

$\Rightarrow A \subseteq B \subseteq cl(A)$

$\Rightarrow B$ is semi-open set.

A is open so is semi-open.

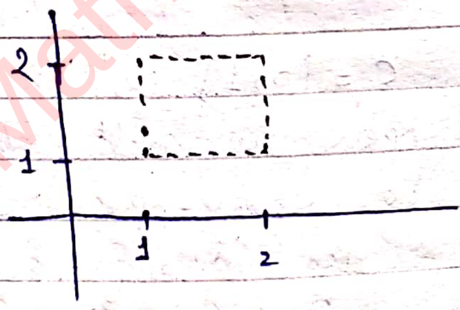
In this case B is neither open nor closed (but is semi-open set)

Example:- Let X be the Euclidean plan $\mathbb{R}^2$ with usual topology

Union of all open discs is called usual topology on $\mathbb{R}^2$

Let E be the set s.t

$$E = \left\{(x,y) \,\middle|\, \begin{matrix} a_1 < x < a_2; \\ b_1 < y < b_2 \end{matrix}\right\}$$



Then

$$cl(E) = \left\{(x,y) \,\middle|\, \begin{matrix} a_1 \leq x \leq a_2; \\ b_1 \leq y \leq b_2 \end{matrix}\right\}$$

Then semi-open sets are $A = \{(x,y) \mid a_1 < x \leq a_2;\ b_1 \leq y \leq b_2\}$,

$B = \{(x,y) \mid a_1 \leq x < a_2;\ b_1 \leq y \leq b_2\}$,

$C = \{(x,y) \mid a_1 \leq x \leq a_2;\ b_1 < y \leq b_2\}$,

$D = \{(x,y) \mid a_1 \leq x \leq a_2;\ b_1 \leq y < b_2\}$,

$F = \{(x,y) \mid a_1 < x < a_2;\ b_1 \leq y \leq b_2\}$,

$G = \{(x,y) \mid a_1 \leq x \leq a_2;\ b_1 < y < b_2\}$;

$H = \{(x,y) \mid a_1 < x \leq a_2;\ b_1 < y \leq b_2\}$,

$I = \{(x,y) \mid a_1 \leq x < a_2;\ b_1 \leq y < b_2\}$,

and so on (so many semi-open sets are available)

⇒ **Theorem 2:-** Let (X,τ) be a topological space and $\{A_\alpha : \alpha \in \nabla\}$ be any collection of semi-open sets in X. Then $\bigcup_{\alpha\in\nabla} A_\alpha$ is semi open in X. (i.e union of any number of semi-open sets is semi open in X)

Proof) Since A_α is semi-open in X $\forall \alpha \in \nabla$

Therefore there exist an open set O_α in X s.t $O_\alpha \subseteq A_\alpha \subseteq cl(O_\alpha)$ $\forall \alpha \in \nabla$

$$\Rightarrow \bigcup_{\alpha\in\nabla} O_\alpha \subseteq \bigcup_{\alpha\in\nabla} A_\alpha \subseteq \bigcup_{\alpha\in\nabla} cl(O_\alpha) = cl \bigcup_{\alpha\in\nabla} O_\alpha$$

$$\Rightarrow O \subseteq \bigcup_{\alpha\in\nabla} A_\alpha \subseteq cl(O) \quad \left\{ \because \bigcup_{\alpha\in\nabla} O_\alpha = O \text{ \& } O \text{ is open set} \right.$$

$$\Rightarrow \bigcup_{\alpha\in\nabla} A_\alpha \text{ is semi-open in } X.$$

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Theorem 3:- Let (X,τ) be a topological space and A is a semi-open sub set of X. suppose $A \subseteq B \subseteq cl(A)$ then prove that B is also semi-open in X.

Proof) Since A is semi-open in X,

Therefore there exist an open set O in X s.t $O \subseteq A \subseteq cl(O)$

Now $O \subseteq A \subseteq B \longrightarrow ①$ $\left\{ \begin{array}{l} \text{by supposition} \\ \underline{A \subseteq B} \subseteq cl(A) \end{array} \right.$

Now $A \subseteq cl(O)$ $\{O \subseteq \underbrace{A \subseteq cl(O)}$

$$\Rightarrow cl(A) \subseteq cl[cl(O)] = cl(O)$$

$$\Rightarrow cl(A) \subseteq cl(O) \longrightarrow ②$$

Again $B \subseteq cl(A)$ $\quad \because A \subseteq \underbrace{B \subseteq cl(A)}$ by given

$\Rightarrow cl(B) \subseteq cl[cl(A)]$

$\Rightarrow cl(B) \subseteq cl(A) \longrightarrow ③$

By relation ①, ② & ③ we get

$O \subseteq A \subseteq B \subseteq cl(B) \subseteq cl(A) \subseteq cl(O)$

$\boxed{\because B \subseteq cl(B) \text{ always true}}$

$\Rightarrow O \subseteq B \subseteq cl(O)$

This proves that B is semi-open in X.

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Note:- Every open set is semi-open but a semi-open set may not be open.

Theorem 5:- Let $\beta = \{B_\alpha : \alpha \in \nabla\}$ be a collection of sets in X s.t

① $\tau \in \beta$ ② If $B \in \beta$ and $B \subseteq D \subseteq cl(B)$ then $D \in \beta$, Then $SO(X) \subseteq \beta$

Proof)

Let $A \in SO(X)$

Then by definition there exist an open set $O \in \tau$ such that

$O \subseteq A \subseteq cl(O)$

Then by condition ① $O \in \beta$

So by ② $A \in \beta$

$\Rightarrow SO(X) \subseteq \beta$ proved.

Statement continued:- Further $SO(X)$ is the smallest class of sets in X

satisfying ① and ②

Suppose $GO(X)$ be another class of sets satisfying ① and ② s.t $GO(X) \subseteq SO(X) \longrightarrow$ ③

Let $A^* \in SO(X)$ Then $\exists$ $O^* \in \tau$ s.t

$O^* \subseteq A^* \subseteq cl(O^*) \longrightarrow$ ⓐ

$\because GO(X) \subseteq \beta$ and satisfying ① & ②

by ② $O^* \in GO(X)$ and $O^* \subseteq A^* \subseteq cl(O^*)$

$\Rightarrow A^* \in GO(X)$ by ② $\Rightarrow SO(X) \subseteq GO(X) \longrightarrow$ ④

So $GO(X) = SO(X)$

Hence $SO(X)$ is the smallest class of sets satisfying ① & ②

Theorem 4:- Let (X, t) be a topological space then ① $\tau \subseteq SO(X)$ (Just by definition)

② for $A \in SO(X)$ and $A \subseteq B \subseteq cl(A)$ then $B \in SO(X)$ (Already proved)

⇒ Relative Topology or subspace Topology:-

Let (X, τ) be a topological space and Y be a subspace of X. Then the collection $\tau_y = \{U \cap Y : U \in \tau\}$ is a topology on Y and is called relative topology

Note:- If τ_y is relative topology on Y then (Y, τ_Y) is subspace of (X, τ)

Example:- $X = \{1, 2, 3\}$

$\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$, Let $Y = \{2, 3\}$

Then $\tau_y = \{\phi \cap Y, X \cap Y, \{1\} \cap Y, \{2\} \cap Y, \{1,2\} \cap Y\}$

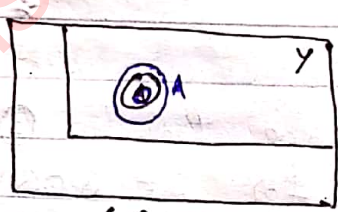
$= \{\phi, Y, \{2\}\}$,$\{1\}$

Then τ_Y is a topology on Y

Note:- $\tau_1 = \{\phi, Y\}$ is a topology on Y (Indiscrete)
$\tau_2 = P(Y)$ is also a topology on Y (Discrete)
$\tau_3 = \{\phi, Y, \{3\}\}$ " " " " "
But τ_1, τ_2 & τ_3 are not relative topologies.

Theorem 6:- Let (X, τ) be a topological space and $A \subseteq Y \subseteq X$, where Y is a subspace of X. Let $A \in SO(X)$ then prove that $A \in SO(Y)$.

Proof) Since $A \in SO(X)$, Therefore there exist an open set O in X such that



(X, τ)

$$O \subseteq A \subseteq cl_X(O)$$

$$\Rightarrow O \cap Y \subseteq A \cap Y \subseteq Y \cap cl_X(O)$$

$$\Rightarrow O \subseteq A \subseteq cl_Y(O)$$, where O is open in Y

$\Rightarrow A$ is semi-open in Y
i.e $A \in SO(Y)$

** ———— **

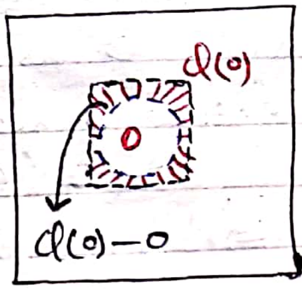
Lemma 1:- Let (X, τ) be a topological space and O is open in X. Prove that $cl(O) - O$ is nowhere dense in X.

Proof) We have to prove
$Int[cl(cl(O) - O)] = \phi$

$E \subseteq (X, \tau)$ is nowhere dense in X if $Int[cl(E)] = \phi$

$E^c = X - E$
$cl(X - E) = X - Int(E)$

$\Rightarrow Int[cl(cl\,O \cap (X-O))] \subseteq$
$Int[cl(cl\,O) \cap cl(X-O)]$
$\because cl\,O - O = cl\,O \cap (X-O)$ and
$cl(A \cap B) \subseteq cl(A) \cap cl(B)$



(X,τ)

$\Rightarrow Int[cl(cl\,O \cap (X-O))] \subseteq Int[cl\,O \cap (X-O)]$
$= Int[cl\,O] \cap Int(X-O)$
$= Int(cl\,O) \cap (X - cl\,O)$
$= \phi$

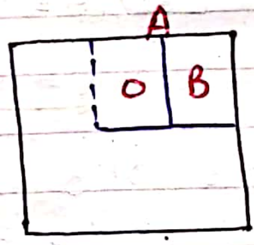
O اوپن ہے اس کا $X-O$ ہے
کلوز ہو گا اور $cl(X-O) =$
$(X-O)$ ہی ہو گا

$\because Int(X-O) = X - cl(O)$

$\Rightarrow Int[cl(cl\,O - O)] = \phi$ proved.

★★———————————★★

Theorem 7:- Let (X,τ) be a topological space and $A \in SO(X)$. Then $A = O \cup B$, where
① $O \in \tau$ ② $O \cap B = \phi$ and
③ B is no where dense.



Proof) Given A is semi open in X. Then by definition there exist an open set O in X s.t
$$O \subseteq A \subseteq cl(O)$$
But $A = O \cup (A - O)$
Let $B = A \setminus O$
Then clearly $A = O \cup B$ where
① $O \in \tau$ ② $O \cap B = \phi$
The only thing we need to prove is that B is no where dense set.

Now $B = A \setminus O \subseteq cl(O) \setminus O$ $\quad \because A \subseteq cl(O)$
$\Rightarrow Int[cl\,B] \subseteq Int[cl(cl\,O - O)]$

Since O is open, therefore $cl O - O$ is nowhere dense and hence

$$Int[cl(cl O - O)] = \phi$$

$\Rightarrow Int[cl\ B] \subseteq \phi \Rightarrow Int(cl B) = \phi$

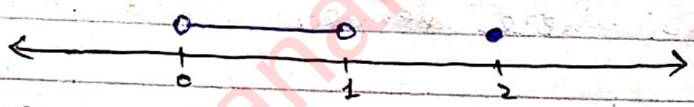
$\Rightarrow B$ is nowhere dense in X.

** —————— **

Remark 3.- The converse of theorem 7 is not true in general i.e

In a topological space (X, τ) a set 'A' written as $A = O \cup B$, where O is open, B is nowhere dense and $O \cap B = \phi$ Then A may not be a semi-open set.

Example 2.- Imp Let $X = \mathbb{R}$ with usual topology. Let $A = \{x \in \mathbb{R} \mid 0 < x < 1\} \cup \{2\}$. Then



① $A = O \cup B$, where $O = (0,1) \in \tau$

and ② $B = \{2\}$ ③ $O \cap B = \phi$

Now we show that B is nowhere dense.

Consider $Int[cl B] = Int[cl\{2\}] = Int\{2\} = \phi$

$\Rightarrow B$ is no where dense.

Now if we let $O = (0,1)$

Then $O \subseteq A$ but $A \not\subset cl(O)$

Hence we can not find an open set satisfying the relation.

$$O \subseteq A \subseteq cl(O)$$

$$\Rightarrow A \notin SO(X)$$

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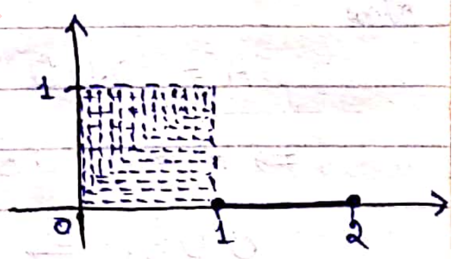
Remark 4:- The converse of theorem 7 is false even when connectedness is imposed upon A.

Disconnected Set:- In a topological space (X,τ), a subset A of X is disconnected if it can be expressed as union of two non-empty disjoint open sets.

Example 3:- Let $X = \mathbb{R}^2$ with the usual topology (open discs or open sets or open rectangles whose sides are parallel to coordinate axis form basis for τ)

Let $A = \{(x,y) \mid 0 < x < 1, \text{ and } 0 < y < 1\} \cup \{(x,0) \mid 1 \leq x \leq 2\}$



We note that $A = O \cup B$, where $O = \{(x,y) \mid 0 < x < 1 \text{ and } 0 < y < 1\} \in \tau$

and $B = \{(x,0) \mid 1 \leq x \leq 2\}$ and

$$O \cap B = \phi$$

and B is nowhere dense because

$\text{Int}[cl[1,2]] = \phi$.

A is connected because it is not disconnected

More over $A \notin SO(\mathbb{R}^2)$

$\because O \subseteq A \nsubseteq cl(O)$

کیونکہ $\text{int}[cl[1,2]] = \phi$ ہے
$cl[1,2]$ میں کوئی بھی اوپن Rectangle
contain نہیں ہوتی
A کو ہم دو اوپن Rectangles
میں نہیں لکھ سکتے اس لیے
$\mathbb{R}^2$ میں A Connected ہے

Theorem 8:- Let (X,τ) be a topological space and $A = O \cup B$, where ① $O \neq \phi$ is open ② A is connected and ③ $B^d = \phi$, where B^d is derived set of B. Then prove that $A \in SO(X)$.

Proof

$A = O \cup B$

$\Rightarrow O \subseteq A$

We have to prove $O \subseteq A \subseteq cl(O)$

The only thing we need to show is that $A \subseteq cl(O)$

or $O \cup B \subseteq cl(O)$

or we need to show $B \subseteq cl(O)$ $\because O \subseteq cl(O)$ obvious

Assume contrary $B \nsubseteq cl(O)$

Let $B = B_1 \cup B_2$, where

$B_1 \subseteq cl(O)$

but $B_2 \subset X - cl(O)$ $\because B \neq \phi$

Now

$A = O \cup B = O \cup (B_1 \cup B_2)$

$\Rightarrow A = (O \cup B_1) \cup B_2$ and

$O \cup B_1 \neq \phi$ $\because O \neq \phi$,

$B_2 \neq \phi$ $\because B_2 \nsubseteq cl(O)$

and $O \cup B_1 \subset cl(O)$

and $B_2 \subset B_2$; a closed set

$B_2 \cap cl(O) = \phi$

$\Rightarrow O \cup B_1$ and B_2 constitute a partition for A.

$\Rightarrow$ A is disconnected.

If U & V form partition then $cl(U) \cap V = \phi$ and $U \cap cl(V) = \phi$

Which is not true

So our supposition is wrong and hence

$B \subseteq cl(O) \Rightarrow O \cup B \subseteq cl(O)$

$\Rightarrow A \Rightarrow cl(O)$

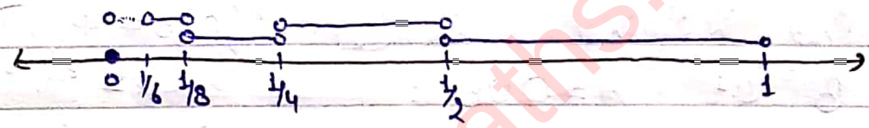
$\Rightarrow O \subseteq A \subseteq cl(O)$

$\Rightarrow A \in SO(X)$ $\quad \because O$ is open.

Remark It is not true that components of a semi-open sets are semi-open.

Example 4:- Let $X = \mathbb{R}$, and $A = \{0\} \cup (\frac{1}{2}, 1) \cup (\frac{1}{4}, \frac{1}{2}) \cup (\frac{1}{8}, \frac{1}{4}) \cup \cdots \cup (\frac{1}{2^{n+1}}, \frac{1}{2^n}) \cup \cdots$

Then A is semi-open and $\{0\}$ is a component of A, but $\{0\}$ is not semi-open in X.

$A =$ 

$$\underbrace{A - \{0\}}_{\text{open set}} \subseteq A \subseteq cl[A - \{0\}]$$

$A - \{0\}$ is open set $\because A - \{0\}$ is union of open sets.

$\Rightarrow A$ is semi-open $\because$ open set $\subseteq A \subseteq cl(\text{open set})$

$\{0\}$ is a component of A but $\{0\}$ is neither open nor semi-open.

Remark 5:- ① In general the compliment of a semi-open set may not be semi-open

② Intersection of two semi-open sets may not be semi-open.

Example:- ② Let $X = \mathbb{R}$ with usual topology. we consider $A = [0,1] \in SO(X)$.

$B = [1,2] \in SO(X)$

$A \cap B = \{1\} \notin SO(X)$

① Let $X = [0,1]$

$A = (\frac{1}{2}, 1) \cup (\frac{1}{4}, \frac{1}{2}) \cup \cdots \cup (\frac{1}{2^{n+1}}, \frac{1}{2^n}) \cup \cdots$

$\Rightarrow A \in SO(X)$

& $A' = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\} \notin SO(X)$

★★———★★

⇒Theorem 9:- Let (X, τ_x) and (Y, τ_Y) be topological spaces. Let $f: X \to Y$ be continuous and open mapping. Let $A \in SO(X)$. Prove that $f(A) \in SO(Y)$

Proof Since $A \in SO(X)$, Therefore There exist an open set O and nowhere dense set B s.t $A = O \cup B$:

$O \cap B = \phi$ & $B \subseteq cl(O) - O$

$\boxed{cl(O) - O \subseteq cl(O)}$

Now $O \subseteq A = O \cup B$

$\Rightarrow f(O) \subseteq f(A) = f(O \cup B)$

$= f(O) \cup f(B)$

$\subseteq f(O) \cup f[cl(O)]$ $\because B \subseteq cl(O)$

$= cl[f(O)]$ $\because f$ is conti. $\because f[cl(O)] = cl\, f(O)$ & $f(O) \subseteq cl(f(O))$ $\Rightarrow f(O) \cup f[cl(O)] = cl\, f(O)$

$\Rightarrow f(O) \subseteq f(A) \subseteq cl[f(O)]$

Since f is open, Therefore $f(O)$ is open in Y and hence $f(A) \in SO(Y)$

① If $A \in SO(X)$ then $A = O \cup B$, where O is open, B is nowhere dense & $O \cap B = \phi$, Then $B \subseteq cl(O) - O$

② If f is continuous then $f[cl A] = cl\, f(A)$ $\forall A \subseteq X$

③ If f is open/closed $cl\, f^{-1}(A) \subseteq f^{-1}[cl(A)]$

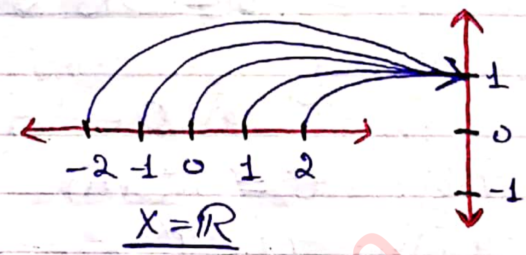
★★———★★

Remark 6:- f must be open in theorem 9, otherwise for $A \in SO(X)$; $f(A)$ may not be semi open in Y.

Example 5:- Let $X = Y = \mathbb{R}$ with usual topology. Let $f: X \to Y$ be defined by $f(x) = 1 \ \forall x \in X$ then X is semi-open in X but $f(X)$ is not semi-open in Y.

Solution

① Since $f(x) = 1$ $\forall x \in X$



Therefore f is a constant function and every constant function is continuous therefore f is a continuous function.

② Let U be any open set in X, Then $f(U) = \{1\} \notin \tau_Y$. This gives that f is not an open function.

Now X is open and hence semi-open but $f(X) = 1$. Since $\{1\}$ contains no open set therefore $\{1\}$ can not be semi-open in Y.

⇒ **Definition:-** Let (X, τ) be a topological space and $\beta = \{B_\alpha\}$ be a collection of subsets. Then $Int(\beta)$ will denote $\{Int\, B_\alpha\}$ i.e $Int(\beta) = \{Int\, B_\alpha\}$

⇒ **Lemma 2:-** Let τ be the class of open sets in the topological space X. Then prove that $\tau = Int\ S.O(X)$

Proof) Let $O \in \tau$

Therefore O is an open set

$\Rightarrow O \in S.O(X) \quad \because O$ is open and since $O = \text{Int}(O)$ $\because O$ is open

$\Rightarrow O \in \text{Int}\ \mathcal{S}.O(X)$

$O = \text{Int}(O) \in \text{Int}.\mathcal{S}O(X)$
$\Rightarrow O \in \text{Int}\ \mathcal{S}.O(X)$

$\Rightarrow \tau \subseteq \text{Int}\ \mathcal{S}.O.(X) \longrightarrow (1)$

Conversely
Let $O \in \text{Int}\ \mathcal{S}.O.(X)$

Then $O = \text{Int}(A)$ for some $A \in \mathcal{S}.O(X)$
& Thus $O \in \tau$ $\because$ Int of any set is open

$\Rightarrow \text{Int}\ \mathcal{S}.O(X) \subseteq \tau \longrightarrow (2)$

from (1) & (2)

$$\tau = \text{Int}\ \mathcal{S}.O(X)$$

⇒ Theorem 10:- Let τ and τ^* be two topologies for X. Suppose $\mathcal{S}.O(X,\tau) \subset \mathcal{S}.O(X,\tau^*)$. Then $\tau \subset \tau^*$

Proof

$$\mathcal{S}.O(X,\tau) \subset \mathcal{S}.O(X,\tau^*)$$

$$\Rightarrow \text{Int}[\mathcal{S}.O(X,\tau)] \subset \text{Int}[\mathcal{S}.O(X,\tau^*)]$$

$\Rightarrow \tau \subset \tau^*$ $\because \text{Int}[\mathcal{S}.O(X,\tau)]$ are open sets in τ &
$\text{Int}[\mathcal{S}.O(X,\tau^*)]$ " " " " τ^*

Corollary 1:- Let τ and τ^* be two topologies for X. Suppose $\mathcal{S}.O(X,\tau) = \mathcal{S}.O(X,\tau^*)$
Then $\tau = \tau^*$

Proof Given
$$\mathcal{S}.O(X,\tau) = \mathcal{S}.O(X,\tau^*)$$

$$\Rightarrow \text{Int}[\mathcal{S}.O(X,\tau)] = \text{Int}[\mathcal{S}.O(X,\tau^*)]$$

$$\Rightarrow \tau = \tau^*$$

Remark 7:- It is interesting to note that converse of Theorem 10 is false.

Example 6:- Let $X = \mathbb{R}$,

$$\tau = \{(x,y) \mid x < y\}$$

& $$\tau^* = \{[x,y) \mid x < y\},$$

Then $\tau \subseteq \tau^*$ But

$$S.O(X,\tau) \nsubseteq S.O(X,\tau^*)$$

$\because (x,y] \in S.O(X,\tau)$ but $(x,y] \notin S.O(X,\tau^*)$

$S.O(X,\tau)$
$(x,y), [x,y), (x,y]$
$[x,y] \in \tau$
& $[x,y), [x,y] \in \tau^*$
i.e $S.O(X,\tau^*)$

Basis

$\forall\, x \in X \ \exists\ B \in \beta$ s.t $x \in B$	B_1 and $B_2 \in \beta$, $x \in B_1 \cap B_2$ Then $\exists\ B_3$ s.t $x \in B_3 \subseteq B_1 \cap B_2$	$\cup B_i = X$

★ Let β, γ are two basis s.t β is basis for (X, τ_X) and γ is a basis for (Y, τ_Y) Then

$$\beta \times \gamma = \{B \times C \mid B \in \beta, C \in \gamma\}$$

★ There can be construct more than one basis corresponds to each topology but there is only one topology corresponds to each base.

Example:- Let $X = \{a, b, c, d\}$

$$\tau_1 = \{\phi, X, \{a\}, \{c\}, \{a,c\}, \{a,c,d\}\}$$

$\beta_1 = \{\{a\}, \{c\}, \{a, c, d\}, X\}$

$\beta_2 = \{\{a\}, \{c\}, \{a, c\}, \{a, c, d\}, X\}$

But if $\gamma_1 = \{\{a\}, \{b\}, \{c, d\}\}$ Then only topology is

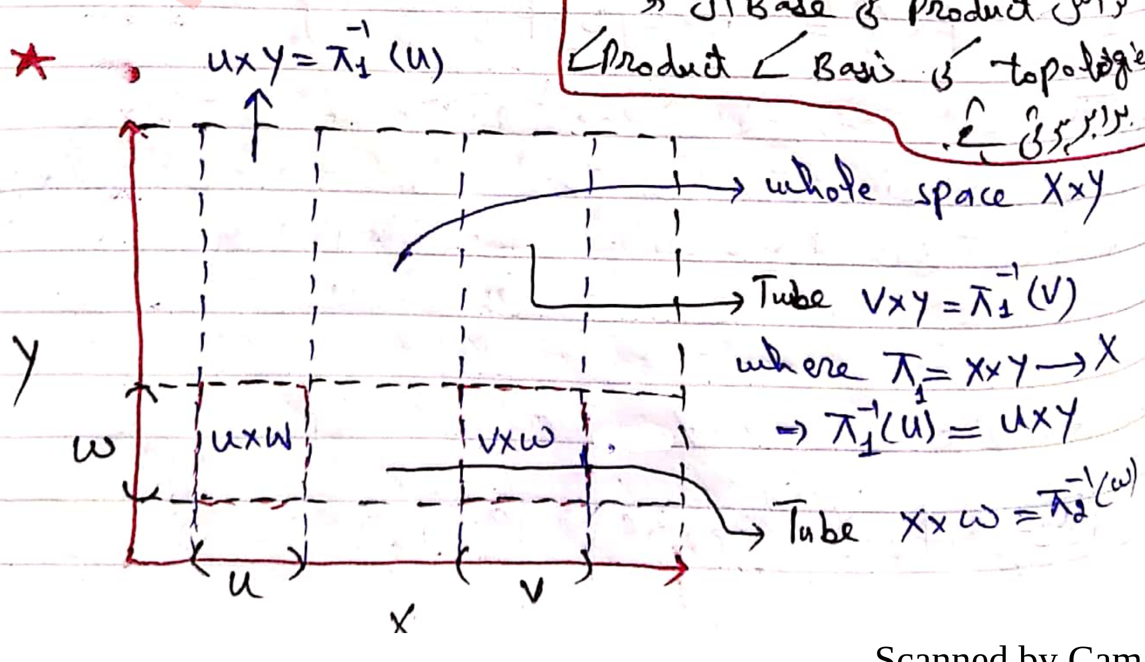
$\tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$

$\tau_1 \times \tau_2 \Rightarrow \beta = \beta_1 \times \gamma_1$

$= \{\{a\}\times\{a\}, \{a\}\times\{b\}, \{a\}\times\{c, d\}, \{c\}\times\{a\}, \{c\}\times\{b\}, \{c\}\times\{c, d\}, \{a, c, d\}\times\{a\}, \{a, c, d\}\times\{b\}, \{a, c, d\}\times\{c, d\}, X\times\{a\}, X\times\{b\}, X\times\{c, d\}\}$

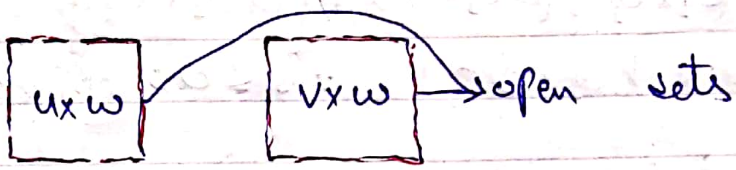
$= \{\{(a, a)\}, \{(a, b)\}, \{(a, c), (a, d)\}, \{(c, a)\}, \{(c, b)\}, \ldots\}$

مطلب کہ دو Topologies کا پروڈکٹ لیتے ہیں تو اس Product کی Base اور ان topologies کی Basis کے Product کے برابر ہوتی ہے۔



where $\pi_2 = X \times Y \to Y$

$\Rightarrow \pi_2^{-1}(Y) = X \times Y$



★ If $O \in \tau_1 \times \tau_2$
Then $O = O_1 \times O_2$ s.t $O_1 \in \tau_1$ and $O_2 \in \tau_2$ always true.

Subspace Topology:- Let (X, τ) be a topological space and Y be a non-empty subset of X. Then collection

$$\tau_Y = \{U \cap Y : U \in \tau\}$$ is a topology on Y and is called relative topology.

Note:- If τ_Y is relative topology on Y, then (Y, τ_Y) is subspace of (X, τ).

Theorem 11:- Let (X, τ_1) and (X, τ_2) be topological spaces and $X = X_1 \times X_2$ be the topological product. Let $A_1 \in SO(X_1)$ and $A_2 \in SO(X_2)$. Then prove that $A_1 \times A_2 \in SO(X_1 \times X_2)$

Proof: We have $A_i = O_i \cup B_i$; $i = 1, 2$

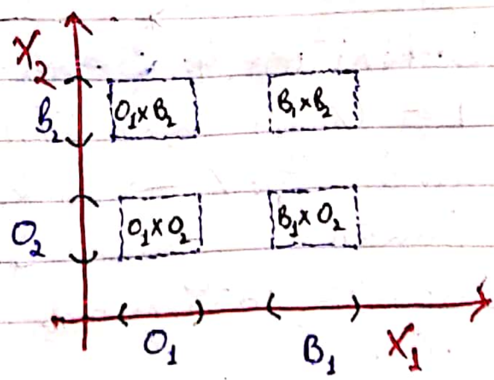
where O_i is open in X_i ; $i = 1, 2$

and B_i is nowhere dense in X_i for $i = 1, 2$

and $O_i \cap B_i = \phi \quad \forall i = 1, 2$

Further

$B_i \subseteq cl(O_i) - O_i$; $i = 1, 2$



Now

$A_1 \times A_2 = (O_1 \cup B_1) \times (O_2 \cup B_2)$

$$\Rightarrow A_1 \times A_2 = (O_1 \times O_2) \cup (O_1 \times B_2) \cup (B_1 \times O_2) \cup (B_1 \times B_2) \quad \to \circledast$$

$$\subseteq (O_1 \times O_2) \cup [cl\, O_1 \times cl\, O_2] \cup [cl\, O_1 \times O_2]$$

$$\cup [cl\, O_1 \times cl\, O_2] \quad \because B_1 \subseteq cl\, O_1,\ B_2 \subseteq cl\, O_2$$

$$= cl\, O_1 \times cl\, O_2$$

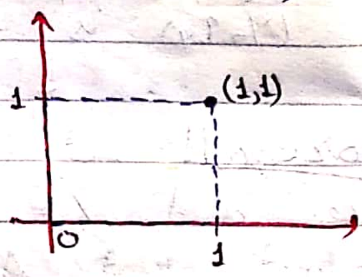
$$\Rightarrow \underbrace{O_1 \times O_2 \subseteq A_1 \times A_2}_{\text{from } \circledast} \subseteq cl\, O_1 \times cl\, O_2 = cl(O_1 \times O_2)$$

Since $O_1 \times O_2$ is open in the product space, therefore $A_1 \times A_2 \in SO(X_1 \times X_2)$

★ Remark 8:- If $A \in SO(X_1 \times X_2)$ then in general we can not write $A = A_1 \times A_2$, where $A_1 \in SO(X_1)$ and $A_2 \in SO(X_2)$

Example 7:- Let $X = \mathbb{R}^2$ with usual topology.

Let $A = \{(x,y) \mid 0 < x < 1, 0 < y < 1\} \cup (1,1)$



Then A is semi open in $\mathbb{R} \times \mathbb{R}$.

A is semi-open because $\{(x,y) \mid {0<x<1 \atop 0<y<1}\} \subseteq A \subseteq cl\{(x,y) \mid {0<x<1 \atop 0<y<1}\}$

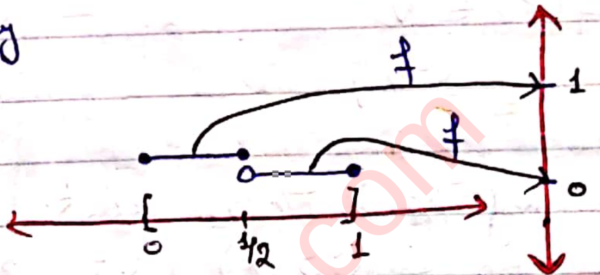
But we can not find two sets A_1 & A_2 s.t $A = A_1 \times A_2$ and $A_1 \in SO(\mathbb{R})$ & $A_2 \in SO(\mathbb{R})$

Definition 4:- Semi-Continuous Function:- Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f: X \to Y$ be a single valued function then f is said to be semi-continuous if and only if, for each open set V in Y. $f^{-1}(V)$ is semi-open in X.

Remark 9:- Every continuous function is semi-continuous as well but a semi-continuous function may not be continuous.

Example 8:- Let $X = Y = [0,1]$ with usual topology $f: X \to Y$ defined by

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x \le 1 \end{cases}$$



$[0,a)$, (b,c), $(d,1]$, $[0,1]$ open sets in Y

This is a semi-continuous function but not a continuous function. Let V is open set in Y

$$V = \begin{cases} 1 \in V,\ 0 \notin V \Rightarrow f^{-1}(V) = [0, \tfrac{1}{2}] \in SO(X_1) \\ 0 \in V,\ 1 \notin V \Rightarrow f^{-1}(V) = (\tfrac{1}{2}, 1] \in \tau_X \text{ i.e open} \\ 0 \notin V,\ 1 \notin V \Rightarrow f^{-1}(V) = \phi \in \tau_X \\ 0 \in V,\ 1 \in V \Rightarrow f^{-1}(V) = [0,1] \in \tau_X \end{cases}$$

** ———————— **

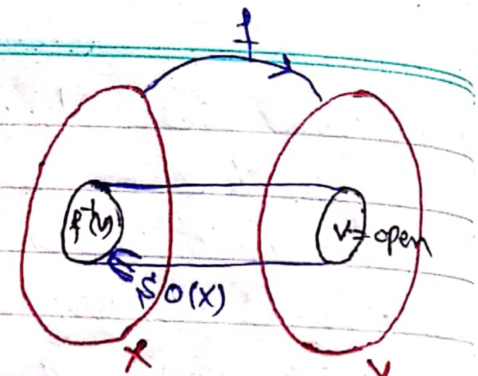
Theorem 12:- Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f: X \to Y$ be a single valued function, then f is semi-continuous if and only if for $f(p) \in V$, there exist an $A \in SO(X)$ s.t $p \in A$ and $f(A) \subseteq V$.

Proof:

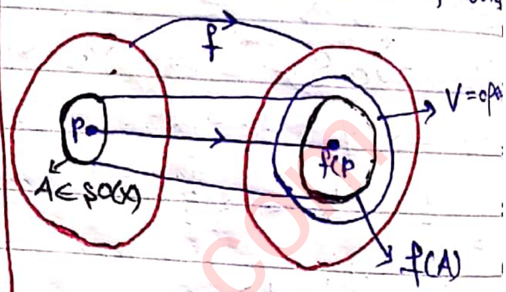
Let $f(P) \in V \in \tau_Y$.

$\Rightarrow$ There exist an $A_p \in SO(X)$

s.t $p \in A_p$ & $f(A_p) \subseteq V$



Semi-continuous function



We have to prove that f is semi-continuous.
For this we show that

$$f^{-1}(V) \in SO(X)$$

Now

$$f(P) \in V$$

$$\Rightarrow P \in f^{-1}(V)$$

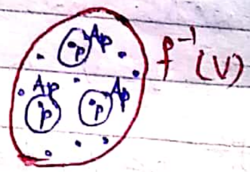
By hypothesis there exist an $A_p \in SO(X)$ s.t $P \in A_p$ and $f(A_p) \subseteq V$

$$\Rightarrow \quad P \in A_p \subseteq f^{-1}f(A_p) \subseteq f^{-1}(V)$$

$\because A \subseteq f^{-1}f(A)$; $f f^{-1}(A) \subseteq A$

$$\Rightarrow P \in A_p \subseteq f^{-1}(V)$$

Thus $\quad f^{-1}(V) = \bigcup_{P \in f^{-1}(V)} A_p$



Since arbitrary union of semi-open sets is semi-open therefor

$$f^{-1}(V) = \bigcup_{P \in f^{-1}(V)} A_p \text{ is semi-open}$$

$$\Rightarrow f \text{ is semi-continuous}$$

Conversely:- Let $f : X \to Y$ be semi-continuous
Let $f(P) \in V \in \tau_Y$

$$\Rightarrow P \in f^{-1}(V) \in SO(X)$$

$\because f$ is semi continuous $V \in \tau_Y$; $f^{-1}(V) \in SO(X)$

$$\text{Let } f^{-1}(V) = A$$

i.e $p \in A$ and $f(A) = f\,f^{-1}(V) \subseteq V$

$\Rightarrow p \in A$ and $f(A) \subseteq V$

This completes the proof.

★★ ★★

⇒ **Theorem 13:-** Let (X, τ_x) and (Y, τ_y) be topological spaces. Let $f: X \to Y$ be semi-continuous and Y be a 2nd axiom space. Let P be the set of discontinuities of f. Then prove that P is of 1st category.

***2nd Axiom Space:-** A topological space (X, τ) is said to be 2nd axiom space if it has countable basis.

*** First Category:-** A set is of 1st category if it is countable union of nowhere dense sets.

Proof: Given:- ① f is semi-conti.
② Y, τ_y is 2nd axiom space
③ $P \subseteq X$; P is set of discontinuities of f.

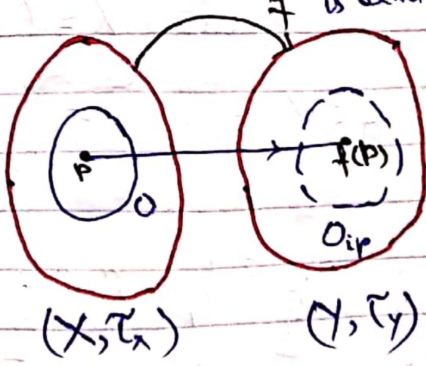
We have to prove P is of 1st category.

$\Rightarrow P = \bigcup_{\text{countable}} G_\alpha$ & $Int[cl\, G_\alpha] = \phi$

Let $p \in P$, let $f(p) \in O_{ip} \subseteq (Y, \tau_y)$

where O_{ip} is the countable union of basic open sets because (Y, τ_y) is a 2nd axiom space.

f is semi-conti.



Now if O is open in X s.t $p \in O$, then $f(o) \notin O_{ip}$ because f is discontin at $p \in P$.

Now, since f is semi-continuous, therefore there exist

$$A_{ip} \in SO(X, p) \text{ s.t } p \in A_{ip} \text{ \& }$$

$$f(A_{ip}) \subseteq O_{ip}$$

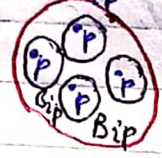
As A_{ip} is semi-open in X, therefore, there exist U_{ip} and B_{ip} s.t

$$A_{ip} = B_{ip} \cup U_{ip}$$, where U_{ip} is open in X and B_{ip} is nowhere dense in X.

Moreover $B_{ip} \subseteq cl(U_{ip}) - U_{ip}$

Thus $p \in B_{ip}$ a nowhere dense set $\because p \notin$ open set i.e U_{ip}

$$\Rightarrow P \subset \bigcup_{p \in P} B_{ip}$$



P

$\Rightarrow P$ is of first category.

Remark 10:- The converse of theorem 13 is false in general.

Example 9:-

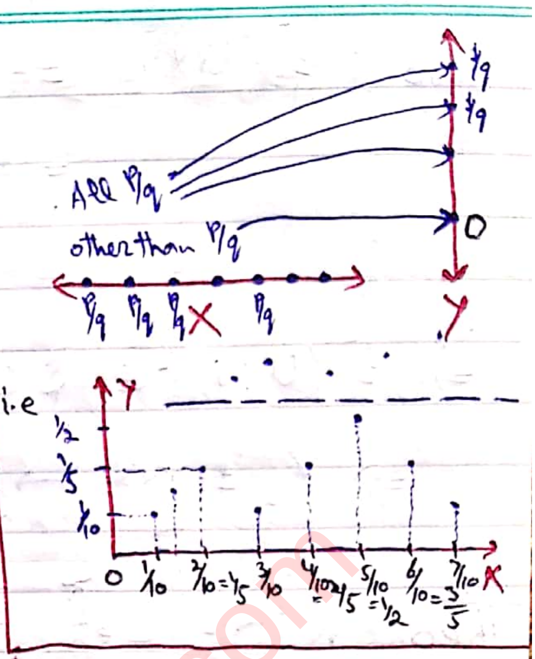
Let $X = (0, 1]$ and $X^* = [0, 1]$.

Let
$$f : X \to X^* = \begin{cases} 0 & x \text{ is irrational} \\ 1/q & x = \text{rational} = Q, \end{cases}$$

where $Q = \{ p/q ; p, q \in \mathbb{Z}, q \neq 0, (p, q) = 1 \}$

Then f is continuous at irrational and discontinuous at the rationals

Hence the set of discontinuities is of the 1st category [∵ set of rational numbers is countable set]



Consider $u = (\frac{1}{2}, 1] \in X^*$ is open

as $0 \notin u$

$\Rightarrow f^{-1}(u) = f^{-1}\left(\frac{1}{2}, 1\right] =$ sub set of rational numbers b/w $(\frac{1}{2}, 1]$

& we can not find an open set O in X s.t

$O \subseteq$ subset of rationals b/w $(\frac{1}{2},1]$ $\subseteq cl(O)$ $\Rightarrow$ f is not semi-con

Theorem 14:- Let $f_i : X_i \to X_i^*$ be semi-continuous. Let $f : X_1 \times X_2 \longrightarrow X_1^* \times X_2^*$ be defined as $f((x_1, x_2)) = (f_1(x_1), f_2(x_2))$. Then prove that $f : X_1 \times X_2 \longrightarrow X_1^* \times X_2^*$ is semi-continuous.

Proof) Given $f_1 : X_1 \to X_1^*$ and $f_2 : X_2 \to X_2^*$ are semi-continuous functions.

Let u and v are open sets s.t $u \subseteq X_1^*$ and $v \subseteq X_2^*$.

As f_1 and f_2 are semi-continuous,

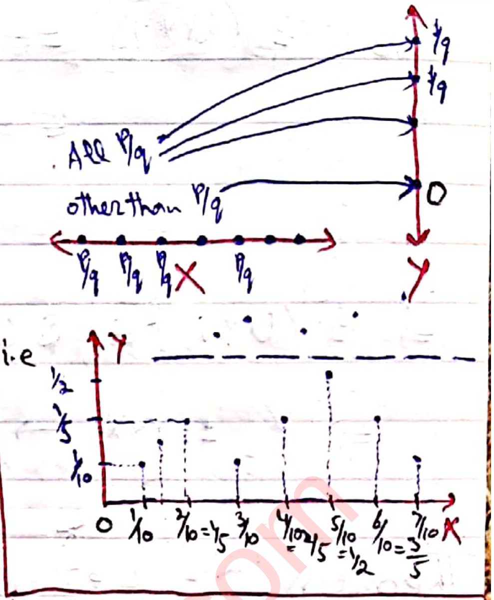
Therefore $f_1^{-1}(u) \in SO(X_1)$ & $f_2^{-1}(v) \in SO(X_2)$

[i.e inverse image of open sets is semi open]

Now Let $u \times v \subseteq X_1^* \times X_2^*$

We have to prove $f^{-1}(u \times v) \in SO(X_1 \times X_2)$

Hence the set of discontinuities is of the 1st category [∵ set of rational numbers is countable set]



Consider $U = (\frac{1}{2}, 1] \in X^*$ is open

as $0 \notin U$

$\Rightarrow f^{-1}(U) = f^{-1}(\frac{1}{2}, 1] =$ sub set of rational numbers b/w $(\frac{1}{2}, 1]$

& we can not find an open set O in X

s.t $O \subseteq$ subset of rationals b/w $(\frac{1}{2},1]$ $\subseteq cl(O)$ $\Rightarrow$ f is not semi-con

Theorem 14: Let $f_i : X_i \to X_i^*$ be semi-continuous. Let $f : X_1 \times X_2 \longrightarrow X_1^* \times X_2^*$ be defined as $f((x_1, x_2)) = (f_1(x_1), f_2(x_2))$. Then prove that $f : X_1 \times X_2 \longrightarrow X_1^* \times X_2^*$ is semi-continuous.

Proof) Given $f_1 : X_1 \to X_1^*$ and $f_2 : X_2 \to X_2^*$ are semi-continuous functions.

Let U and V are open sets s.t $U \subseteq X_1^*$ and $V \subseteq X_2^*$.

As f_1 and f_2 are semi-continuous,

Therefore $f_1^{-1}(U) \in SO(X_1)$ & $f_2^{-1}(V) \in SO(X_2)$

[i.e inverse image of open sets is semi open]

Now Let $U \times V \subseteq X_1^* \times X_2^*$

We have to prove $f^{-1}(U \times V) \in SO(X_1 \times X_2)$

Now $f^{-1}(U \times V) = f_1^{-1}(U) \times f_2^{-1}(V)$

$\in SO(X_1) \times SO(X_2)$

$\subset SO(X_1 \times X_2)$ by Theorem 11

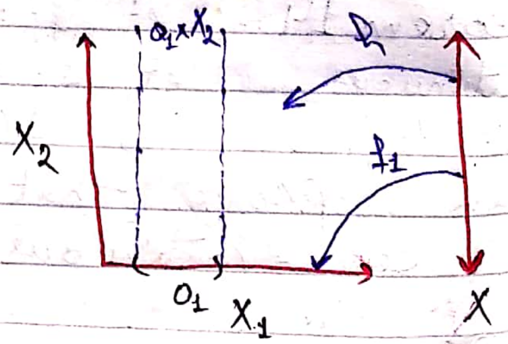
$\Rightarrow f^{-1}(U \times V) \in SO(X_1 \times X_2)$

$\Rightarrow \therefore f : X \times X_2 \to X_1^* \times X_2^*$ is semi continuous

Theorem 15:- Let $h : X \to X_1 \times X_2$ be semi-continuous, where X, X_1 and X_2 are topological spaces. Let $f_i : X \to X_i$ be defined as follows. for $x \in X$; $h(x) = (x_1, x_2)$. Let $f_i(x) = x_i$. Then $f_i : X \to X_i$ is semi-continuous for $i = 1, 2$.

Proof:

$f : X \to X_1$ semi-continuous.



Let O_1 be open in X_1. Then $O_1 \times X_2$ is open in $X_1 \times X_2$ and hence $h^{-1}(O_1 \times X_2)$ is semi open in X.

But $f_1^{-1}(O_1) = h^{-1}(O_1 \times X_2) \in SO(X)$

$\because h^{-1}(x_1, x_2) = x$ & $f_1^{-1}(x_1) = x$

$\Rightarrow f_1$ is semi continuous

Similarly for f_2.

Remark 11:- The converse of theorem 15 is generally false.

Example 10:- Let $X = X_1 = X_2 = [0,1]$.

$$f_1 : X \to X_1 = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

$$f_2 : X \to X_2 = \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x < 1 \end{cases}$$

Then $f_i : X \to X_i$ is semi-continuous but $h(x) = [f_1(x), f_2(x)] : X \to X_1 \times X_2$ is not semi-continuous.

Remark 12:- Composition of two semi-continuous functions is not a semi-continuous function.

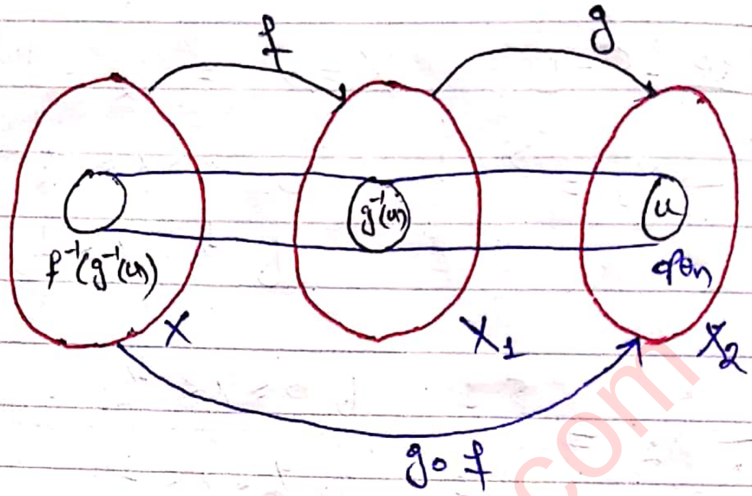
f is said to be continuous at $x = x_0$ if $\forall\, \varepsilon > 0 \; \exists$ a $\delta > 0$ s.t $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$

Example 11:-

Let $X = X_1 = X_2 = [0,1]$

$$f_1 : X \to X_1 = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

$$f_2(x) : X_1 \to X_2 = \begin{cases} 0 & \text{if } 0 \le x < 1/2 \\ 1 & \text{if } 1/2 \le x \le 1 \end{cases}$$



$$(f_2 \circ f_1)^{-1}(x) = (f_1^{-1} \circ f_2^{-1})(x)$$

Let $u \in X_2$; $0 \in u$, $1 \notin u$

$\Rightarrow f_2^{-1}(u) = [0, 1/2]$

$(f_1^{-1} \circ f_2^{-1})\, u = f_1^{-1} f_2^{-1}(u) = f_1^{-1}([0, \frac{1}{2}])$

$= X$ open

Now $0 \notin u$, $1 \in u$

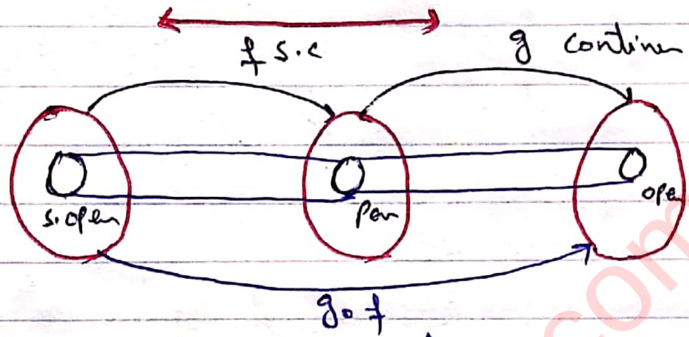
$\Rightarrow f_2^{-1}(u) = [\frac{1}{2}, 1]$

$f_1^{-1} f_2^{-1}(u) = f_1^{-1}[\frac{1}{2}, 1] = \{\frac{1}{2}, 0\} \notin \Im(X)$

$\Rightarrow$ Composition of two semi-continuous functions is not a semi-continuous.

★ A function is semi-continuous and 2nd function is of what kind that the composition is semi-continuous.

Ans:- that function is called irresolute.



In this case gof is also semi-continuous

⇒ **Remark 13:-** The algebraic sum and product of two semi-continuous functions are not in general semi-continuous.

Theorem 16:- Let $f_n : M \to M^*$, where M and M^* are metric spaces with metrics d and d^*, be s.c for $n = 1, 2, \dots$, and let $f_0 : M \to M^*$ be the uniform limit of $\{f_n\}$. Then $f_0 : M \to M^*$ is semi-continuous.

f is semi-conti iff $f(P) \in V \in \mathcal{T}_Y$ $\exists\ A \in S\,O(X) : P \in A$ and $f(A) \subseteq V$

Proof: Let O^* be open in M^* and $f_0(x) \in O^*$.

As (M^*, d^*) be metric space Then $\exists\ \eta > 0$ s.t

$$f_0(x) \in S^*_\eta(f_0(x)) \subseteq O^*$$

As $f_0 : M \to M^*$ is uniform limit of $\{f_n\}$ then for $\varepsilon = \eta/2$ $\exists\ n^*$ s.t

$$d^*(f^*_n(x), f_0(x)) < \frac{\eta}{2} \quad \forall x \in M$$

$$\Rightarrow f^*_n(x) \in S^*_{\frac{\eta}{2}}(f_0(x)) \subseteq O^*$$

As f^*_n is semi-continuous, Then by a well know theorem $\exists\ A \in S\,O(M)$ s.t $x \in A$ and $f^*_n(A) \subset S^*_{\frac{\eta}{2}}(f_0(x))$

Theorem will be prove if we show

$$f_0(A) \subseteq O^*$$

Let $y \in A$, Then

$$d^*[f_0(y), f_0(x)] \le d^*[f_0(y), f^*_n(y)] + d[f^*_n(y) + f_0(x)]$$

$$< \frac{\eta}{2} + \frac{\eta}{2} = \eta$$

$$\Rightarrow f_0(A) \subseteq S^*_\eta(f_0(x)) \subseteq O^*$$

$$\rightarrow f_0 \text{ is semi-continuous}$$

** ——— **

*3rd Research Paper:-

This course was established in (1973) by "S. Gene Crossley and S.K Hildebrand"

* This course was published by "Taxas Journal Math (1973)

SEMI-TOPOLOGICAL PROPERTIES

Introduction:- In [1] Norman Lewine defined a semi-open set in a topological space as a set A such that there exist an open set O so that $O \subseteq A \subseteq \overline{O}$, where $\overline{O}$ denotes closure in the topological spaces. He also defined a function to be semi-continuous if and only if the inverse of open sets are semi-open. Also in [1], among others, the following two results were established.

⇒ **Theorem 0.1:-** Let (X,τ) be a topological space, Then: ① $\tau \subseteq SO(X)$ where $SO(X)$ denotes the class of semi-open sets in (X,τ)
② for $A \in SO(X,\tau)$ and $A \subseteq B \subseteq \overline{A}$, Then $B \in SO(X,\tau)$

⇒ **Theorem 0.2:-** Let $f: X \to Y$ be a continuous and open, where X and Y are topological spaces. Let $A \in SO(X)$, then $f(A) \in SO(Y)$.

In [2] the author defined a set to be semi-closed if and only if its compliment is semi-open. Semi-closure and semi-interior were defined in a manner analogous to closure and interior. Also in [2], among others, the following four results were established.

⇒ **Theorem 0.3:-** In a topological space all non-void semi-open sets must contain non-void open sets.

Proof

Let (X, τ) be a topological space and $A \in SO(X)$ be a semi-open set. s.t $A \neq \phi$.

Then there exist an open set O in X s.t

$$O \subseteq A \subseteq \bar{O}$$

Then O must non empty i.e $O \neq \phi$

Because if $O = \phi$

$$\Rightarrow \bar{O} = \phi \quad \because \bar{\phi} = \phi$$

and in this case $A \nsubseteq \bar{O}$ $\because A \neq \phi$

$$\Rightarrow O \neq \phi$$

Hence $A \neq \phi$ is semi-open must contain a non empty open set

⇒ **Semi-Interior of a set:-** Let (X, τ) be a topological space and $A \neq \phi$ is a subset of X. Then semi-interior of A is denoted by $sInt(A)$ or A_o and is the union of all semi-open sets contained in A.

Note:- sInt(A) is a semi-open set

② sInt(A) is the largest semi-open set contained in A.

⇒ **Semi-Interior Point:-** Let (X,τ) be a topological space and $A \subseteq X$. A point $x \in A$ is called semi-interior point of A if there exist a semi-open set U in X s.t $x \in U \subseteq A$.

Note:- Collection of all semi-interior points of A is called s.Int(A).

Note- If $A \in SO(X)$, Then every point of A is semi-interior point of A. Because $\forall\ x \in A$, $x \in A \subseteq A$.

⇒ **Semi-Closure of a set:-** Let (X,τ) be a topological space and A is a non-void subset of X. Then semi closure of A is denoted by $sCl(A)$. OR $\underline{A}$ and is the intersection of all semi-closed sets containing A.

Note:- $sCl(A)$ is a semi closed set

② $sCl(A)$ is the smallest semi-closed sets containing A.

③ $Int(A) \subseteq sInt(A) \subseteq A \subseteq sCl(A) \subseteq Cl(A)$

⇒ **Semi-Limit Point:-** Let (X,τ) be a topological space and A is a subset of X. A point $x \in X$ is called semi-limit point of A if for each semi-open set U containing x, we have $U \cap A \neq \phi$, $\boxed{U \cap (A - \{x\}) \neq \phi}$.

Note:- A is semi-closed if A contains all semi-limit points.

⇒ **Theorem 0.4:**- ① A is semi-open iff $A_0 = A$
② A is semi-closed iff $\underline{A} = A$.

Proof: ① Let A be a semi-open in X,
Then $A \subseteq A_0$ But $A_0 \subseteq A$ (always)
$$\Rightarrow A = A_0$$
Conversely
Let $A = A_0$ (semi-open)
Since A_0 is semi-open, therefore A is semi-open

②
Let A be a semi-closed, Then
$\underline{A} \subseteq A$ But $A \subseteq \underline{A}$ (always)
$$\Rightarrow A = \underline{A}$$
Conversely:- Let $A = \underline{A}$
Since $\underline{A}$ is semi closed, Therefore A is semi-closed.

⇒ **Theorem 0.5:**- If A is open and S is semi-open, Then $A \cap S$ is semi-open.

Proof: Let S be semi open in X, Then

There exist an open set $O \in X$ s.t

$$O \subseteq S \subseteq \bar{O}$$

$$\Rightarrow \underbrace{O \cap A}_{\text{open}} \subseteq S \cap A \subseteq \bar{O} \cap A \subseteq \overline{(O \cap A)}$$

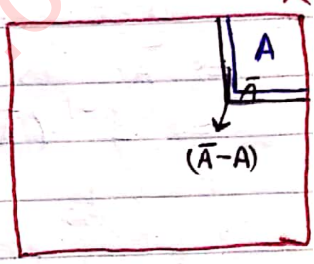
Since $O \cap A$ is open in X and

$$O \cap A \subseteq S \cap A \subseteq \overline{(O \cap A)}$$

$\Rightarrow$ $S \cap A$ is semi-open in X.

$\Rightarrow$ **Theorem 0.6:-** Let (X, τ) be a topological space and $A \subseteq X$, Then prove that

$$\underline{[X - (\bar{A} - A)]} = X$$



Proof

L.H.S

$\bar{A} - A$ contains no semi-interior points.

$\Rightarrow s\text{Int}(\bar{A} - A) = \phi$

$\Rightarrow X - s\text{Int}(\bar{A} - A) = X$

$\Rightarrow s\text{Cl}[X - (\bar{A} - A)] = X$

or $\underline{[X - (\bar{A} - A)]} = X$

$\because X - \text{Int}(A) = \text{Cl}(X - A)$
$X - \text{Cl}(A) = \text{Int}(X - A)$

$\Rightarrow$

⇒ **Irresolute function:-** Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f: X \to Y$ is called irresolute if $f^{-1}(B)$ is semi-open in X for every semi-open set B in Y.

⇒ **Theorem 1.1:-** Let $f:(X,\tau_X) \to (Y,\tau_Y)$ be continuous and open. Then

$$f^{-1}(\bar{A}) = \overline{f^{-1}(A)}$$

Proof

$f: X \to Y$ is continuous & open

Let A by any sub set of Y

$\Rightarrow \bar{A}$ is a closed set of Y

$\Rightarrow f^{-1}(\bar{A})$ is a closed subset of X

As $A \subseteq \bar{A}$

$\Rightarrow f^{-1}(A) \subseteq f^{-1}(\bar{A})$ $\quad \because f$ is continuous

$\Rightarrow \overline{[f^{-1}(A)]} \subseteq \overline{[f^{-1}(\bar{A})]} = f^{-1}(\bar{A})$ $\quad \because f^{-1}(\bar{A})$ is closed

$\Rightarrow \overline{[f^{-1}(A)]} \subseteq f^{-1}(\bar{A}) \longrightarrow (1)$

As f is open

$\Rightarrow$ image of every open set is open under f

$\Rightarrow f^{-1}$ is a continuous function

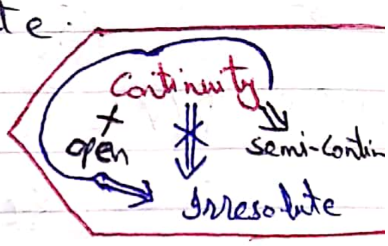
Then by a well known theorem fore every $A \subseteq Y$

Let (X,τ_X) & (Y,τ_Y) be two topological spaces. A function $f: X \to Y$ is continuous iff for every subset A of X $f(\bar{A}) \subseteq \overline{f(A)}$

$$f^{-1}(\bar{A}) \subseteq \overline{f^{-1}(A)} \longrightarrow (2)$$

$$\Rightarrow f^{-1}(\bar{A}) = \overline{f^{-1}(A)} \quad \text{Proved.}$$

Theorem 1.2:- Let $f:(X,\tau_X)\to(Y,\tau_Y)$ be continuous and open then f is irresolute.



Proof)

Let $A \in SO(Y)$

By definition $\exists\ O \in \tau_Y$ s.t

$$O \subseteq A \subseteq cl(O)$$

$$\Rightarrow f^{-1}(O) \subseteq f^{-1}(A) \subseteq f^{-1}(\bar{O}) = \overline{f^{-1}(O)}$$ $\because f$ is conti & open

As O is open $\Rightarrow f^{-1}(O)$ is open because f is Continuous

$$\Rightarrow f^{-1}(O) \subseteq f^{-1}(A) \subseteq \overline{(f^{-1}(O))}$$

$$\Rightarrow f^{-1}(A) \in SO(X)$$

$\Rightarrow f$ is irresolute function.

Example 1.1:- A continuous irresolute function need not be open.

Proof)

Let $X = \{a,b,c\}$

$\tau = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$

$\tau^* = \{\phi, \{a\}, \{a,b\}, X\}$

Let $f:(X,\tau) \to (X,\tau^*)$ be defined by $f(x) = x\ \forall\ x \in X$

Then this function is continuous & irresolute but not an open function.

See

$f^{-1}(\phi) = \phi \in \tau \Rightarrow f^{-1}(\phi)$ is open

$f^{-1}(\{a\}) = \{a\}$ open in (X,τ)

$f^{-1}(\{a,b\}) = \{a,b\}$ open in (X,τ)

$f^{-1}(X) = X$ open

As inverse image of every open set is open $\Rightarrow$ f is continuous.

Now $P(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\},$
$\{c,b\}, \{a,b,c\}\}$

Closed sets of (X, τ) are

$X, \{b,c\}, \{c\}, \{b\}, \phi$

Now closed supersets of ϕ are
$\{b\}, \{c\}, \{b,c\}, \{X\}$
Intersection of all closed supersets of
$\phi = \phi \Rightarrow \overline{\phi} = \phi$
similarly
$\overline{\{a\}} = X$, $\overline{X} = X$
$\overline{\{a,b\}} = X$, $\overline{\{a,c\}} = X$
$\phi, X \in SO(X, \tau)$
$\{a\}, \{a,b\}, \{a,c\} \in \tau$ Therefore $\in SO(X, \tau)$
Now
$\Rightarrow SO(X, \tau) = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$

Now Closed sets of (X, τ^*) are
$\phi, X, \{b,c\}, \{c\}$

Now $\overline{\phi} = \phi$, $\overline{X} = X$
$\overline{\{a,b\}} = X$, $\overline{\{a\}} = X$

As $\phi, X, \{a,b\}, \{a\} \in (X, \tau^*)$

$\rightarrow \phi, X, \{a,b\}, \{a\} \in SO(X, \tau^*)$

& also $\{a\} \subseteq \{a,c\} \subseteq \overline{\{a\}} = X$

$\Rightarrow \{a,c\} \in SO(X,\tau^*)$

$\Rightarrow SO(X,\tau^*) = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$

And
$f^{-1}(\phi) = \phi \in SO(X,\tau)$
$f^{-1}(\{a\}) = \{a\} \in SO(X,\tau)$
$f^{-1}(\{a,b\}) = \{a,b\} \in SO(X,\tau)$
$f^{-1}(\{a,c\}) = \{a,c\} \in SO(X,\tau)$
$f^{-1}(X) = X \in SO(X,\tau)$

As inverse image of every semi-open set is semi open $\Rightarrow f$ is irresolute.

Now as $\{a,c\}$ is open is (X,τ)

$\Rightarrow f(\{a,c\}) = \{a,c\} \notin (X,\tau^*)$

$\Rightarrow$ image of every open set is not open

$\Rightarrow f$ is not open.

$\Rightarrow$ **Theorem 1.3:-** Let $C(X,Y)$, $SC(X,Y)$ and $I(X,Y)$ denote respectively, the classes of continuous, semi continuous and irresolute functions from $X \to Y$, where X & Y are topological spaces. Then $C(X,Y) \subseteq SC(X,Y)$ and $I(X,Y) \subseteq SC(X,Y)$

Proof) ① Let $f \in C(X,Y)$

$\Rightarrow f$ is continuous function

$\Rightarrow$ Inverse image of every open set (say A) of Y is open in X

$\Rightarrow f^{-1}(A)$ is open in X

As every open set is also semi-open

$\Rightarrow f^{-1}(A)$ is semi-open in X

This implies inverse image of every open set of Y is semi open in X

$\Rightarrow f \in SC(X,Y)$ $\boxed{f \in \text{semi-continous function}: X \to Y}$

$\Rightarrow C(X,Y) \subseteq SC(X,Y)$

② Let $g \in I(X,Y)$

$\Rightarrow g$ is irresolute function from X to Y. Therefore inverse image of every semi-open set of Y under g is semi open in X

As All open sets of $Y \subseteq$ Semi open sets of Y

$\Rightarrow$ Inverse image of every open set of Y under g is semi-open in X

$\Rightarrow g$ is semi-continuous

$\Rightarrow g \in SC(X,Y)$

$\Rightarrow I(X,Y) \subseteq SC(X,Y)$

⇒ **Theorem 1.4:-** A function $f:(X,\tau_X)\to(Y,\tau_Y)$ is irresolute iff, for every semi-closed subset H of Y, $f^{-1}(H)$ is semi closed in X.

Proof)
Let $f: X\to Y$ be irresolute.
Let $H \in SC(Y)$, Then $Y-H$ is semi open in Y
or $f^{-1}(Y-H) = f^{-1}(Y) - f^{-1}(H) = X - f^{-1}(H)$ is semi open $\because f^{-1}(Y)=X$

$\Rightarrow X - f^{-1}(H)$ is semi open in X $\because f$ is irresolute

$\Rightarrow f^{-1}(H)$ is semi-closed in X

Conversely
Let $f^{-1}(H)$ is semi closed in X for every semi-closed set H in Y
We have to prove that f is irresolute

As $B \in SO(Y)$

$\Rightarrow (Y-B) \in SC(Y)$

$\Rightarrow f^{-1}(Y-B) \in SC(X)$ $\because f^{-1}(H) \in SC(X)$ for every $H \in SC(Y)$

$\Rightarrow f^{-1}(Y) - f^{-1}(B) \in SC(X)$

$\Rightarrow X - f^{-1}(B) \in SC(X)$

$\Rightarrow f^{-1}(B) \in SO(X)$

$\Rightarrow f$ is irresolute

⇒ **Theorem 1.5:-** A function $f: S\to T$, where S and T are topological spaces is irresolute iff for every subset A of S $f(\underline{A}) \subseteq \underline{f(A)}$

Proof Let $f: S \to T$ be irresolute function.

Let $A \subseteq S$

Then $\underline{f(A)} \in SC(T)$

$\Rightarrow f^{-1}[\underline{f(A)}]$ is semi closed in S [$\because f$ is irreso

Now

$$A \subseteq f^{-1}f(A) \subseteq f^{-1}[\underline{f(A)}] \quad \because f(A) \subseteq \underline{f(A)}$$

$$\Rightarrow \underline{A} \subseteq s\,cl\, f^{-1}[\underline{f(A)}] = f^{-1}\underline{f(A)} \quad \because f^{-1}(\underline{f(A)}) \text{ is s.cl}$$

$$\Rightarrow f(\underline{A}) \subseteq f[f^{-1}\underline{f(A)}] \subseteq \underline{f(A)}$$

$$\Rightarrow f(\underline{A}) \subseteq \underline{f(A)}$$

Conversely

Assume that

$$f(\underline{A}) \subseteq \underline{f(A)}$$

We have to prove that f is irresolute

Let $H \in SC(T)$

Then $f[\underline{f^{-1}(H)}] \subseteq \underline{f f^{-1}(H)} \subseteq \underline{H} = H \quad \because H$ is s.cl

Now

$$\Rightarrow \underline{f^{-1}(H)} \subseteq f^{-1}f[\underline{f^{-1}(H)}] \subseteq f^{-1}(\underline{H}) = f^{-1}(H)$$

$$\Rightarrow \underline{f^{-1}(H)} \subseteq f^{-1}(H)$$

But $f^{-1}(H) \subseteq \underline{f^{-1}(H)}$ (always)

$$\Rightarrow f^{-1}(H) = \underline{f^{-1}(H)}$$

$\Rightarrow f^{-1}(H) \in SC(S) \Rightarrow f$ is irresolute.

Theorem 1.6:- Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f: X \to Y$ is irresolute if and only if for all $B \subseteq Y$,

$$\underline{f^{-1}(B)} \subseteq f^{-1}(\underline{B})$$

Proof: Assume that f is irresolute.

Let B be any subset of Y.

Then $\underline{B} \in SC(Y)$, hence

$$f^{-1}(\underline{B}) \in SC(X)$$

But we know $B \subseteq \underline{B}$

$$\Rightarrow f^{-1}(B) \subseteq f^{-1}(\underline{B})$$

$$\Rightarrow SC(f^{-1}(B)) \subseteq SC(f^{-1}(\underline{B})) = f^{-1}(\underline{B})$$

$$\Rightarrow \underline{f^{-1}(B)} \subseteq f^{-1}(\underline{B})$$

Conversely, Let

$\underline{f^{-1}(B)} \subseteq f^{-1}(\underline{B})$ for every subset B of Y.

We will prove that f is irresolute.

For this we will show that the inverse image of semi-closed set is semi-closed.

Let $B \in SC(Y)$ Then $\underline{B} = B$

By Hypothesis $\underline{(f^{-1}(B))} \subseteq f^{-1}(\underline{B}) = f^{-1}(B)$

& $f^{-1}(B) \subseteq \underline{(f^{-1}(B))} \subseteq f^{-1}(\underline{B}) = f^{-1}(B)$

$$\Rightarrow f^{-1}(B) \subseteq \underline{(f^{-1}(B))} \subseteq f^{-1}(B)$$

$$\Rightarrow f^{-1}(B) = \underline{f^{-1}(B)}$$

$\Rightarrow f^{-1}(B) \in SC(X)$

$\Rightarrow f$ is irresolute.

Theorem 1.7:- Let $(X, \tilde{\tau}_X)$ and $(Y, \tilde{\tau}_Y)$ and $(Z, \tilde{\tau}_Z)$ be topological spaces. If $f: X \to Y$ and $g: Y \to Z$ are both irresolute then $g \circ f: X \to Z$ is irresolute.

Proof: Let $B \in SO(Z)$

$\Rightarrow g^{-1}(B)$ is semi open in Y because g is irresolute from $Y \to Z$.

Now as $g^{-1}(B) \in SO(Y)$ and f is irresolute from $X \to Y$

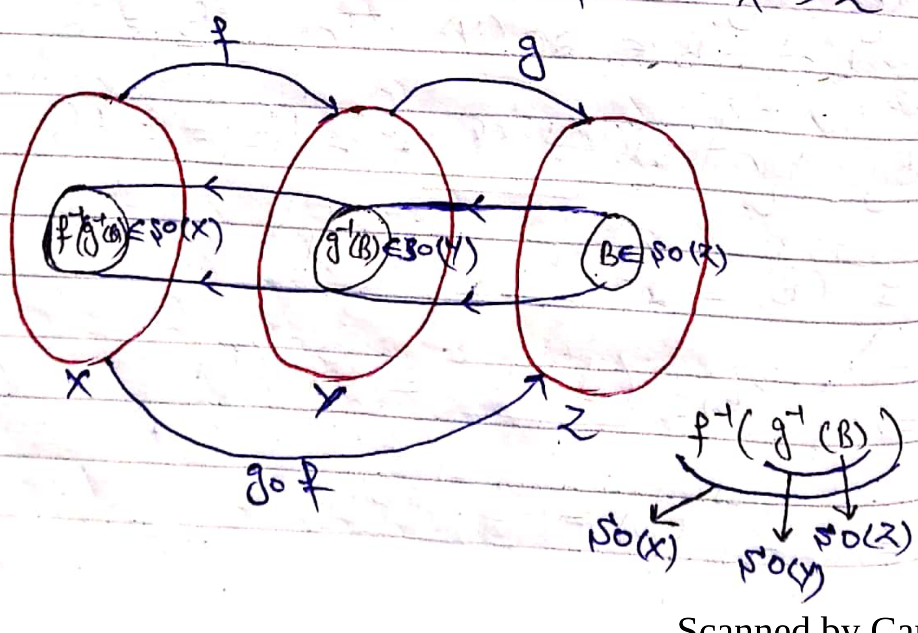
$\Rightarrow f^{-1}(g^{-1}(B)) \in SO(X)$

$$f^{-1}(g^{-1}(B)) = (f^{-1} \circ g^{-1})(B) = (g \circ f)^{-1}(B)$$

$\Rightarrow f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B) \in SO(X)$

Now as $B \in SO(Z)$ and $(g \circ f)^{-1}(B) \in SO(X)$

$\Rightarrow g \circ f$ is irresolute from $X \to Z$



§1.2

⇒ **Pre-Semi-Open Function:-** Let X and Y be topological spaces, a function $f: X \longrightarrow Y$ is said to be pre-semi-open if and only if, for all $A \in SO(X)$, $f(A) \in SO(Y)$

Theorem 1.8:- Let (X, τ_X) and (Y, τ_Y) be topological if $f: X \to Y$ is continuous and open, Then f is irresolute and pre-semi-open.

Proof) Let $f: X \to Y$ be continuous and open mapping.

To prove f is irresolute.

Consider a semi-open set B in Y

Then there exist an open set U in Y s.t $U \subseteq B \subseteq Cl(U)$

$$\Rightarrow f^{-1}(U) \subseteq f^{-1}(B) \subseteq f^{-1}(Cl(U)) = Cl[f^{-1}(U)]$$

$\because f$ is continuous & open

Since f is continuous,

Therefore $f^{-1}(U)$ is open in X and

$$f^{-1}(U) \subseteq f^{-1}(B) \subseteq Cl[f^{-1}(U)]$$

$$\Rightarrow f^{-1}(B) \in SO(X)$$

$\Rightarrow f$ is irresolute.

Now we prove that f is pre-semi-open

Let $A \in SO(X)$

$\Rightarrow$ There exist an open set O in X s.t

$$O \subseteq A \subseteq Cl(O) \Rightarrow f(O) \subseteq f(A) \subseteq f(Cl(O)) \subseteq Cl[f(O)]$$

$\because f$ is continuous

$\Rightarrow f(O) \subseteq f(A) \subseteq cl[f(O)]$

Since f is an open mapping
Therefore $f(O)$ is open in Y and
hence $f(A) \in SO(Y)$

$\Rightarrow f$ is pre-semi-open.

Def 1.3:- Semi-Homeomorphism:- Let (X, τ_X) and (Y, τ_Y) be topological spaces. X and Y are said to be semi-homeomorphism if and only if there exist a function $f: X \to Y$ s.t

① f is bijective ② f is irresolute
③ f is pre-semi-open.

Theorem 1.9:- Let (X, τ_X) and (Y, τ_Y) be topological spaces. If $f: X \to Y$ is homeomorphism then f is semi-homeomorphism.

Proof)

Let $f: X \to Y$ be homeomorphism
Then ① f is bijective ② f is continuous
③ f is open.

Since f is continuous and open bijection, Therefore it is irresolute and pre-semi open bijection.
Hence f is semi-homeomorphism.

Example 1.2:- A semi-homeomorphism need not be homeomorphism.

Proof)

Let $X = \{a, b, c\}$

$\tau = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$

$\tau^* = \{\phi, \{a\}, \{a,b\}, X\}$

Let $f : (X,\tau) \longrightarrow (X,\tau^*)$ be defined by
$f(x) = x \quad \forall\, x \in X$

Then f is semi-homeomorphism but not homeomorphism.

Closed sets of $\tau^* = \phi, X, \{b,c\}, \{c\}$

$\overline{\{a\}} = X, \quad \overline{\{a,b\}} = X$

$\Rightarrow$ Semi-open sets of $\tau^* = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$

Closed sets of $\tau = \{X, \{b,c\}, \{c\}, \{b\}, \phi\}$

$\overline{\{a\}} = X, \quad \overline{\{a,b\}} = X, \quad \overline{\{a,c\}} = X$

$\Rightarrow$ Semi-open sets of $\tau = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$

$\Rightarrow SO(X,\tau) = SO(X,\tau^*)$

$f^{-1}(u) \in \tau \quad \forall\, u \in \tau^* \Rightarrow f$ is continuous

① $\because f(x) = x \quad \forall\, x \in X \Rightarrow f$ is bijective

② $f^{-1}(u) \in SO(X,\tau) \quad \forall\, u \in SO(X,\tau^*)$

$\Rightarrow f$ is irresolute

③ $f(V) \in SO(X,\tau^*) \quad \forall\, V \in SO(X,\tau)$

$\Rightarrow f$ is pre-semi-open

$\Rightarrow f$ is semi-homeomorphism

Now $\{a,c\}$ is open in (X,τ) but

$f(\{a,c\}) = \{a,c\} \notin \tau^*$

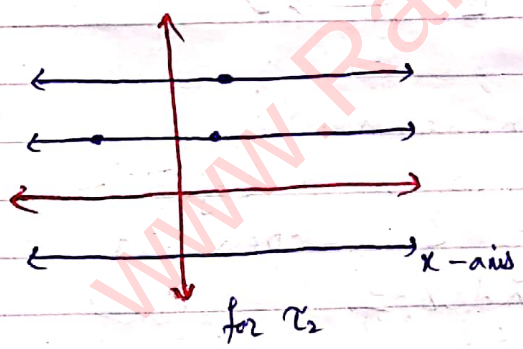
$\Rightarrow f$ is not open $\Rightarrow f$ is not homeomorphism

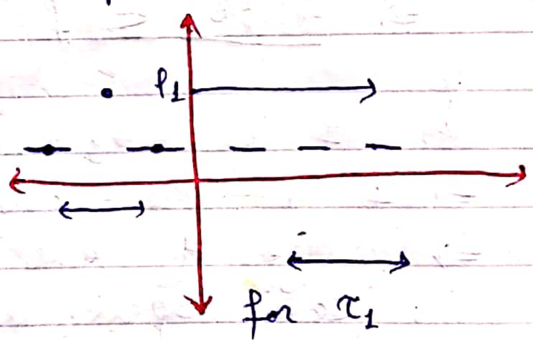
★ Remark 1.2:- The image of a T_1-space under a semi-homeomorphism is not necessarily a T_1-space.

Example 1.4:- Let $X = (\mathbb{R} \times \mathbb{R})$, where $\mathbb{R}$ denote the set of real numbers and let $\tau_1 = \{\phi$, together with all subsets of X whose compliments are subsets of a finite number of lines parallel to the x-axis$\}$

Note that $SO(X, \tau_1) = \tau_1$;

let $\tau_2 = \{\phi$, together with all subsets of X whose compliments are a finite number of lines parallel to x-axis$\}$





Note that $SO(X, \tau_2) = SO(X, \tau_1)$ $\boxed{\text{But } \tau_2 \neq \tau_1}$

Further more defining $f : (X, \tau_1) \longrightarrow (X, \tau_2)$ by $f(p) = p$ for $p \in X$, we see that f is a semi-homeomorphism.
Observe that (X, τ_1) is a T_1 space where (X, τ_2) is not.

Theorem 1.10:- If $f: X \to Y$ is a semi-homeomorphism, then $\underline{f^{-1}(B)} = f^{-1}(\underline{B})$ for all $B \subseteq Y$.

Proof)

$f: X \to Y$ is semi-homeomorphism

$\Rightarrow f$ is ① Bijective ② irresolute ③ pre-semi-open

Let B be any subset of Y

Then $\underline{B} \in SC(Y)$, hence

$$f^{-1}(\underline{B}) \in SC(X)$$

As we know that $B \subseteq \underline{B}$

$$\Rightarrow f^{-1}(B) \subseteq f^{-1}(\underline{B})$$

$$\Rightarrow sCl(f^{-1}(B)) \subseteq sCl(f^{-1}(\underline{B})) = f^{-1}(\underline{B}) \quad \because f^{-1}(\underline{B}) \in SC($$

$$\Rightarrow \underline{f^{-1}(B)} \subseteq f^{-1}(\underline{B}) \longrightarrow ①$$

As f is semi-homeomorphism

$\Rightarrow f$ is pre-semi-open & bijective

$\Rightarrow$ image of every semi-open set is semi open under f

$\Rightarrow f^{-1}$ is irresolute

Then by Theorem 1.5 for every $B \subseteq Y$

$$f^{-1}(\underline{B}) \subseteq \underline{f^{-1}(B)} \longrightarrow ②$$

from ① & ②

$$f^{-1}(\underline{B}) = \underline{f^{-1}(B)}$$

* **Corollary 1.1:-** If $f: X \to Y$ is semi-homeomorphism, Then $\underline{f(B)} = f(\underline{B})$ for all $B \subseteq X$.

Proof) $f: X \to Y$ is semi-homeomorphism.

$\Rightarrow$ f is ① bijective ② irresolute ③ pre-semi-open

Let $B \subseteq X$

Then $\underline{f(B)} \in SC(Y)$

$\Rightarrow f^{-1}[\underline{f(B)}]$ is semi closed in X [$\because f$ is irresolute

Now $B \subseteq f^{-1}(f(B)) \subseteq f^{-1}[\underline{f(B)}]$ $\quad \because f(B) \subseteq \underline{f(B)}$

$\Rightarrow \underline{B} \subseteq s\,cl[f^{-1}(\underline{f(B)})] = f^{-1}(\underline{f(B)})$ $\quad \because f^{-1}(\underline{f(B)})$ is s. closed

$\Rightarrow f(\underline{B}) \subseteq f[f^{-1}(\underline{f(B)})] \subseteq \underline{f(B)}$.

$\Rightarrow f(\underline{B}) \subseteq \underline{f(B)} \longrightarrow ①$

Since f is bijective & irresolute (Preserve mapping)

$\Rightarrow f^{-1}$ is exist & also irresolute

Then By theorem 1.6

for every $B \in X$

$\underline{f(B)} \subseteq f(\underline{B}) \longrightarrow ②$

from ① & ②

$\underline{f(B)} = f(\underline{B})$

* **Corollary 1.2:-** If $f: X \to Y$ is semi-homeomorphism Then $f(B_\circ) = (f(B))_\circ$ for all $B \subseteq X$

Proof)

$$B_o = (X - (\underline{X - B}))$$

Thus,

$$f(B_o) = f[X - (\underline{X - B})]$$

$$= [Y - f(\underline{X - B})]$$

$$= [Y - \underline{f[X - B]}] \quad \because f \text{ is irresolute}$$

$$= [Y - [\underline{Y - f(B)}]]$$

$$\Rightarrow f(B_o) = [f(B)]_o$$

★ Corollary 1.3:- If $f : X \longrightarrow Y$ is semi-homeomorphism, then $f^{-1}(B_o) = (f^{-1}(B))_o$ for all $B \subseteq Y$.

Proof) As $f : X \to Y$ is semi-homeomorphi
$\Rightarrow f^{-1} : Y - X$ is irresolute (bijective & pre

Let $B \subseteq Y$

$$B_o = (Y - (\underline{Y - B}))$$

Thus $$f^{-1}(B_o) = f^{-1}[Y - (\underline{Y - B})]$$

$$= [X - f^{-1}(\underline{Y - B})]$$

$$= [X - [\underline{f^{-1}(Y - B)}]] \quad \because f^{-1} \text{ is irresol}$$

$$= [X - [\underline{X - f^{-1}(B)}]]$$

$$= [f^{-1}(B)]_o \qquad \text{Proved.}$$

Theorem 1.11:- $(\underline{A})_\circ = \phi$ iff A is nowhere dense set.

Proof: Let A is no where dense set.
As we know

$$A^\circ \subseteq A_\circ \subseteq A \subseteq \underline{A} \subseteq \bar{A} \longrightarrow (1)$$

As A is nowhere dense set

$\Rightarrow (\bar{A})^\circ = \phi$

$\Rightarrow \bar{A}$ contain no open set

$\Rightarrow \underline{A}$ contain no open set $\because \underline{A} \subseteq \bar{A}$

$\Rightarrow \underline{A}$ contain no semi-open set

$\Rightarrow (\underline{A})_\circ = \phi$

Conversely:- Let $(\underline{A})_\circ = \phi$

We know by a well known theorem (theorem 0.7)

$$(\bar{A})^\circ \subseteq (\underline{A})_\circ$$

Since $(\underline{A})_\circ = \phi \Rightarrow (\bar{A})^\circ \subseteq \phi$

$\Rightarrow (\bar{A})^\circ = \phi$

$\Rightarrow A$ is no where dense set.

Theorem 1.12:- If $f: X \to Y$ is a semi homeomorphism and $A \subseteq X$ is nowhere dense in X. Then $f(A)$ is nowhere dense in Y.

Proof:- As A is nowhere dense in X. Then by Theorem 1.11

$$(\underline{A})_{\circ} = \phi$$

We have to show

$$(\underline{f(A)})_{\circ} = \phi$$

As $f: X \to Y$ is semi-homeomorphism

$$\Rightarrow \underline{f(A)} = f(\underline{A})$$

$$\Rightarrow [\underline{f(A)}]_{\circ} = [f(\underline{A})]_{\circ} = f(\underline{A})_{\circ} \quad \because \text{Corollary } 1.2$$

$$= f(\phi) = \phi$$

$$\Rightarrow [\underline{f(A)}]_{\circ} = \phi$$

$\Rightarrow f(A)$ is nowhere dense set.

Def 1.4:- Semi-Topological Property:- A property which is preserved under semi-homeomorphism is said to be a semi-topological property.

★ Example 1.3 & 1.4 show that T_0 & T_1 are not semi-topological properties.

★ If $f: X \to Y$ is continuous function then for

$$f(\bar{A}) \subseteq \overline{f(A)} \quad \text{for every } A \subseteq X$$

$$\overline{f^{-1}(A)} \subseteq f^{-1}(\bar{A}) \quad \text{for every } A \subseteq Y$$

★ If $f: X \to Y$ is open and continuous function then for

$$f(\bar{A}) = \overline{f(A)} \quad \text{for every } A \subseteq X$$

$$f^{-1}(\bar{A}) = \overline{f^{-1}(A)} \quad \text{for every } A \subseteq Y$$

★ If $f: X \to Y$ is irresolute function then for every

$$f(\underline{A}) \subseteq \overline{f(A)} \quad \text{for every } A \subseteq X$$

$$\underline{f^{-1}(B)} \subseteq f^{-1}(\underline{B}) \quad \text{for every } B \subseteq Y$$

★ If $f: X \to Y$ is semi homeomorphism then

$$\underline{f(B)} = f(\underline{B}) \quad \text{for every } B \subseteq X$$

$$\underline{f^{-1}(B)} = f^{-1}(\underline{B}) \quad \text{for every } B \subseteq Y$$

★ Let (X, τ_X) & (Y, τ_Y) be topological spaces. A function $f: X \to Y$ is said to be continuous iff inverse image of each open set is open.

★ A function $f: X \to Y$ is said to be continuous iff inverse image of each closed set is closed.

★ If $f: X \to Y$ is semi-homeomorphism then $f(B_o) = [f(B)]_o$, where X and Y are topological spaces.

★ If $f: X \to Y$ is semi-homeomorphism then $f^{-1}(B_o) = [f^{-1}(B)]_o$

★ $\overline{A \cup B} = \bar{A} \cup \bar{B}$, $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

★ $(A \cap B)^\circ = A^\circ \cap B^\circ$, $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

★ **T_0-Space:-** A topological space (X, τ) is said to be T_0-space if for each $x, y \in X$ s.t $x \neq y$ either there exist an open set U s.t $x \in U$, $y \notin U$ or $\exists$ an open set V s.t $y \in V$, $x \notin V$.

★ **T_1-Space:-** A topological space (X, τ) is said to be T_1-space if for every $x, y \in X$ s.t $x \neq y$ there exist two open sets U and V s.t $x \in U$, $y \notin U$ & $y \in V$, $x \notin V$.

★ **T_2-Space:-** A topological space (X, τ) is said to be T_2-space if for each $x, y \in X$ s.t $x \neq y$ then there exit two open sets U & V s.t $x \in U$, $y \in V$ & $U \cap V = \phi$

* 4th Research Paper :-

This course was established by "M. Ganster (AUS), T. Noiri (Japanes) and I. L. Reilly".

WEAK AND STRONG FORMS OF θ-IRRESOLUTE FUNCTIONS**

* $SO(X,x)$ = Collection of all semi-open sets in X containing x.

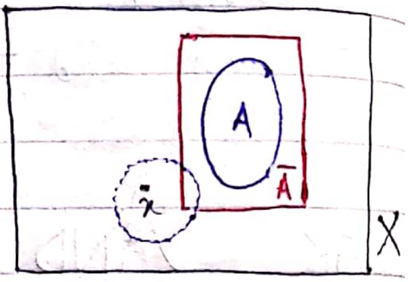
* $RO(X)$ = Collection of all regular open sets in X
  $A = Int[Cl(A)]$

* $RC(X)$ = Collection of all regular closed sets in X
  $A = Cl[Int(A)]$

* **Semi-Closure Point:-** Let (X,τ) be a topological space and $x \in X$. x is called semi closure point of $A \subseteq X$ if for each $U \in SO(X,x)$;
$$U \cap A \neq \phi$$
and is denoted by $x \in sCl(A)$.

* **Semi-θ Closure Point:-** Let (X,τ) be a topological space and $A \subseteq X$. A point $x \in X$ is called semi

θ- closure point of A if
$A \cap Cl(U) \neq \phi$ for every
$U \in SO(X, \tau)$.
It is denoted by
$x \in \theta\text{-}sCl(A)$ or
$x \in sCl_\theta(A)$



Note:- $x \in sCl_\theta(A)$ iff
every regular closed set containing
x intersects A.

★ Every regular closed / regular open set is semi-regular.

★ A set is semi-regular if it is semi-open as well as semi closed

★ Regular Closed Set:-

$A = Cl[Int(A)]$

Closed $\Rightarrow$ semi closed $\Rightarrow$ semi-open

$Int(A) \subseteq Cl[Int(A)] = A$

$\Rightarrow Int(A) \subseteq A = Cl[Int(A)]$

$\Rightarrow Int(A) \subseteq A \subseteq Cl[Int(A)]$

Regular Closed set $\Rightarrow$ semi-regular set
[Closed + semi-open]

Regular open set $\Rightarrow$ Semi-regular set
[open + semi-closed]

$$A \in SO(X)$$

$$O \subseteq A \subseteq cl(O)$$

$A \in SC(X)$, closed V s.t

$$Int(V) \subseteq A \subseteq V$$

Now

$$O \subseteq U \subseteq cl(O)$$

$$X - cl(O) \subseteq X - U \subseteq X - O$$

$$Int(X - O) \subseteq X - U \subseteq X - O$$

MUHAMMAD TAHIR WATTOO ***

M.S. MATHEMATICS CIIT ISLAMABAD *

0344-8563284

FA15-RMT-007

* 2nd Research Paper:-

This course was established in 1973 by "Nota di Takashi Noire"

This course was published by "Accademia Nazionale Dei Lincei"

SEMI-CONTINUOUS MAPPING

*** Introduction:-** In 1963 N-Levine defined a subset A of topological space X to be semi-open if there exist an open set U in X such that $U \subseteq A \subseteq Cl(U)$, where $Cl(U)$ denotes the closure of U. He also define a mapping f of a topological space X into a topological space Y to be semi-continuous if for any open set V in Y, $f^{-1}(V)$ is a semi-open set in X. The purpose of present note is to give a generalization of the following two theorems in [3] and to investigate some properties of semi-open sets and semi-continuous mappings.

*** Theorem A:-** Let X_1 and X_2 be topological spaces. If A_i is a semi-open set in X_i for $i=1,2$, Then $A_1 \times A_2$ is a semi-open set in the product space $X_1 \times X_2$.

★ **Theorem B:-** Let X_i and Y_i be topological spaces and $f_i : X_i \to Y_i$ be a semi-continuous mapping for $i=1,2$. Then a mapping $f : X_1 \times X_2 \to Y_1 \times Y_2$ defined by putting $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$ is semi-continuous.

Semi-Open Sets

★ The intersection of two semi-open sets is not always semi-open [3, Remark 5], however we have the following lemma.

★ **Lemma 1:-** If U is open and A is semi open, then $U \cap A$ is semi-open.

Proof: As $A \in SO(X)$

Then there exist an open set O in X s.t $O \subseteq A \subseteq Cl(O)$

$\Rightarrow U \cap O \subseteq U \cap A \subseteq U \cap Cl(O) \subseteq Cl(U \cap O)$

Since $U \cap O$ is open in X and

$U \cap O \subseteq U \cap A \subseteq \overline{(U \cap O)}$

$\Rightarrow U \cap A \in SO(X)$

Theorem 1:- Let A and X_0 be subsets of X s.t $A \subseteq X_0$ and $X_0 \in SO(X)$, Then $A \in SO(X)$ iff $A \in SO(X_0)$

Proof: As $A \subset X_0$ and $X_0 \in SO(X)$

So X_0 is a subspace of X by a well known theorem.

$\because A \subset X_0 \in SO(X)$
$\Rightarrow A \in SO(X_0)$

Hence $A \in SO(X_0)$

So we need only to prove that
$A \in SO(X)$

Let $A \in SO(X_0)$,
Then by definition there exist an open set U_0 in X_0 s.t
$$U_0 \subseteq A \subseteq Cl(U_0)$$
Since U_0 in X_0, Then there exist an open set U in X s.t $U_0 = U \cap X_0$
$$\Rightarrow U \cap X_0 \subseteq A \subseteq Cl(U \cap X_0)$$
Since U is open and X_0 is semi-open so $U \cap X_0$ is semi-open in X
$$\Rightarrow A \in SO(X)$$
**

★ **Lemma 2:-** A is semi-open iff
$Cl(A) = Cl(Int(A))$

Proof) Suppose A is semi open then by well known theorem
$$A \subseteq Cl[Int(A)]$$
$$\Rightarrow Cl(A) \subseteq Cl[Cl(Int A)] = Cl[Int(A)]$$
$$\Rightarrow Cl(A) \subseteq Cl(Int(A)) \longrightarrow ①$$
As $Int(A) \subseteq A$
$$\Rightarrow Cl(Int(A)) \subseteq Cl(A) \longrightarrow ②$$
by ① & ② $Cl(A) = Cl(Int(A))$

Conversely Let, $Cl(A) = Cl[Int(A)]$
To prove A is semi-open

As $\text{Int}(A) \subseteq A \subseteq \text{Cl}(A)$

$\Rightarrow \text{Int}(A) \subseteq A \subseteq \text{Cl}[\text{Int}(A)]$ $\because \text{Cl}(A) = \text{Cl}(\text{Int}(A))$

As $\text{Int}(A)$ is open set and

$\text{Int}(A) \subseteq A \subseteq \text{Cl}[\text{Int}(A)]$

$\Rightarrow A$ is semi-open set.

★ **Lemma 3:-** Let $\{X_\alpha \mid \alpha \in \beta\}$ be any family of topological spaces and $\prod A_\alpha$ is a subset of $\prod X_\alpha$ denotes the product space, Then

① $\text{Int} \prod A_\alpha = \prod \text{Int} A_\alpha$ if $A_\alpha = X_\alpha$ except for finite $\alpha \in \beta$ and $\prod \text{Int} A_\alpha \neq \phi$

② $\text{Cl} \prod A_\alpha = \prod \text{Cl} A_\alpha$

Proof ① As $A_\alpha = X_\alpha$ except for finite $\alpha \in \beta$

So the result is obvious for all $A_\alpha = X_\alpha$

So we prove this lema just for finite case

As $\text{Int} A_\alpha$ is open in X_α $\forall \alpha = 1, 2, \dots, n$

So $\prod_{\alpha=1}^{n} \text{Int} A_\alpha$ is open in $\prod_{\alpha=1}^{n} X_\alpha$

Also $\prod_{\alpha=1}^{n} \text{Int} A_\alpha \subseteq \prod_{\alpha=1}^{n} A_\alpha$

$\Rightarrow \prod_{\alpha=1}^{n} \text{Int} A_\alpha \subseteq \text{Int} \prod_{\alpha=1}^{n} A_\alpha \longrightarrow ①$

Now let $(x_1, x_2, \dots, x_n) \in \text{Int} \prod_{\alpha=1}^{n} A_\alpha$

As $\text{Int} \prod_{\alpha=1}^{n} A_\alpha$ is open in $\prod_{\alpha=1}^{n} X_\alpha$

$\Rightarrow \exists$ open set U_α in X_α $\forall \alpha = 1, 2, \cdots, n$
s.t $(x_1, x_2, \cdots, x_n) \in \prod_{\alpha=1}^{n} U_\alpha \subseteq \text{Int} \prod_{\alpha=1}^{n} A_\alpha \subseteq \prod_{\alpha=1}^{n} X_\alpha$

$\therefore U_\alpha \in A_\alpha \quad \forall \alpha = 1, 2, \cdots, n$

It follows that $x_\alpha \in \text{Int} A_\alpha \quad \forall \alpha = 1, 2, \cdots, n$

$$\Rightarrow (x_1, x_2, \cdots, x_n) \in \prod_{\alpha=1}^{n} \text{Int} A_\alpha$$

$$\Rightarrow \text{Int} \prod_{\alpha=1}^{n} A_\alpha \subseteq \prod_{\alpha=1}^{n} \text{Int} A_\alpha \longrightarrow (2)$$

from (1) & (2)

$$\text{Int} \prod_{\alpha=1}^{n} A_\alpha = \prod_{\alpha=1}^{n} \text{Int} A_\alpha$$

proof (2) As $A_\alpha = X_\alpha$ except for finite $\alpha \in \beta$ and X_α are topological spaces, so result holds obviously for all $A_\alpha = X_\alpha$

So we prove this lemma just for finite case

As $\quad A_\alpha \subseteq \text{cl} A_\alpha \quad \forall \alpha = 1, 2, 3, \cdots, n$

$$\Rightarrow \prod_{\alpha=1}^{n} A_\alpha \subseteq \prod_{\alpha=1}^{n} \text{cl}(A_\alpha)$$

Also $\left(\prod_{\alpha=1}^{n} X_\alpha\right) \setminus \prod_{\alpha=1}^{n} \text{cl}(A_\alpha) = \bigcup_{\alpha=1}^{n} \left(X_\alpha \times (X_\omega \setminus \overline{A_\omega})\right)$
$\alpha \neq \omega, \ 1 \leq \alpha, \omega \leq n$

which is open in $\prod_{\alpha=1}^{n} X_\alpha$

$\Rightarrow \prod_{\alpha=1}^{n} \text{cl} A_\alpha$ is closed and so

$$\text{Cl}\prod_{\alpha=1}^{n} A_\alpha \subseteq \prod_{\alpha=1}^{n} \text{Cl}\, A_\alpha \longrightarrow (1)$$

$\because \prod_{\alpha=1}^{n} A_\alpha \subseteq \prod_{\alpha=1}^{n} \text{Cl}\, A_\alpha$

$\Rightarrow \text{Cl}\left[\prod_{\alpha=1}^{n} A_\alpha\right] \subseteq \text{Cl}\left[\prod_{\alpha=1}^{n} \text{Cl}\, A_\alpha\right]$

$\Rightarrow \text{Cl}\prod_{\alpha=1}^{n} A_\alpha \subseteq \prod_{\alpha=1}^{n} \text{Cl}\, A_\alpha$

Now let

$(x_1, x_2, \dots, x_n) \in \prod_{\alpha=1}^{n} \text{Cl}\, A_\alpha$

let ω be a neighbourhood of $(x_1, x_2, \dots, x_n)$ in $\prod_{\alpha=1}^{n} X_\alpha$

Then there exist open sets U_α in X_α $\forall\, \alpha = 1, 2, 3, \dots, n$ s.t

$$(x_1, x_2, \dots, x_n) \in \prod_{\alpha=1}^{n} U_\alpha \subseteq \omega$$

Then $x_\alpha \in U_\alpha$ $\quad \forall\, \alpha = 1, 2, \dots, n$

But $x_\alpha \in \bar{A}_\alpha$ $\quad \forall\, \alpha = 1, 2, \dots, n$

and $U_\alpha \cap A_\alpha \neq \phi$ $\forall\, \alpha = 1, 2, \dots, n$

Since $\prod_{\alpha=1}^{n} U_\alpha \subseteq \omega$

we know that

$$\omega \cap \prod_{\alpha=1}^{n} A_\alpha \neq \phi$$

$\Rightarrow (x_1, x_2, \dots, x_n) \in \text{Cl}\prod_{\alpha=1}^{n} A_\alpha$

$\Rightarrow \prod_{\alpha=1}^{n} \text{Cl}\, A_\alpha \subseteq \text{Cl}\prod_{\alpha=1}^{n} A_\alpha \longrightarrow (2)$

from (1) & (2)

$$\prod_{\alpha=1}^{n} \text{Cl}\, A_\alpha = \text{Cl}\prod_{\alpha=1}^{n} A_\alpha$$

★ **Lemma 4:-** If A is a non-empty semi-open set, then $\text{Int}\, A \neq \phi$

Proof:

Since A is semi-open

Then $Cl(A) = Cl(Int\, A)$

Suppose $Int(A) = \phi$

then $Cl(A) = \phi$

$\Rightarrow A = \phi$

which is contradiction

Hence $Int(A) \neq \phi$

$Cl(A) = Cl(\phi) \quad \because Int\, A = \phi$

$\Rightarrow Cl(A) = \phi \quad \because \bar{\phi} = \phi$

$\Rightarrow A = \phi \quad \because \bar{\phi} = \phi$

★ **Theorem 2:-** Let $\{X_\alpha \mid \alpha \in \beta\}$ be any family of topological spaces, $X = \prod X_\alpha$ the product space and $A = \prod_{j=1}^{n} A\alpha_j \times \prod_{\alpha \neq \alpha_j} X_\alpha$, a not empty subset of X, where n is a +ve integer. Then $A\alpha_j \in SO(X\alpha_j)$ for each $j\,(1 \leq j \leq n)$ if and only if $A \in SO(X)$

Proof) Suppose $A\alpha_j \in SO(X\alpha_j)$ for each $j\,(1 \leq j \leq n)$

Since $A \neq \phi \Rightarrow A\alpha_j \neq \phi$ for each $j\,(1 \leq j \leq n)$

As $A\alpha_j \in SO(X\alpha_j)$

so $Int\, A\alpha_j \neq \phi \quad (\because A\alpha_j \neq \phi)$

Thus $\prod_{j=1}^{n} Int\, A\alpha_j \times \prod_{\alpha \neq \alpha_j} X_\alpha \neq \phi$

Now $Cl(Int(A)) = \prod_{j=1}^{n} Cl\, Int\, A\alpha_j \times \prod_{\alpha \neq \alpha_j} X_\alpha$

$= \prod_{j=1}^{n} Cl\, A\alpha_j \times \prod_{\alpha \neq \alpha_j} X_\alpha \quad \because A\alpha_j \in SO(X\alpha_j)$

$\Rightarrow Cl(Int(A)) = Cl(A)$

[$\because A\alpha_j$ is semi-open for each $j\,(1 \leq j \leq n)$

so by well known theorem

$\Rightarrow A \in SO(X)$

Conversely, Let $A \in SO(X)$

Then $Int(A) \neq \phi$ $\because A \neq \phi$

As $Int(A) \subset \prod_{j=1}^{n} Int A\alpha_j \times \prod_{\alpha \neq \alpha_j} X_\alpha$

So $\prod_{j=1}^{n} Int A\alpha_j \times \prod_{\alpha \neq \alpha_j} X_\alpha \neq \phi$

Since $A \in SO(X)$
So by a well known theorem

$\prod_{j=1}^{n} Cl\, Int\, A\alpha_j \times \prod_{\alpha \neq \alpha_j} X_\alpha = Cl\, Int(A)$

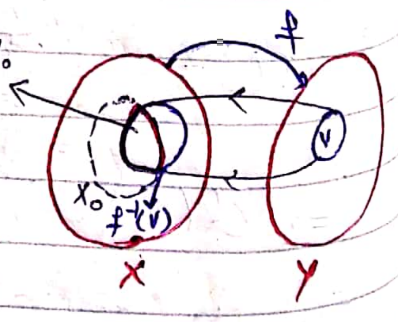
$= Cl(A) = \prod_{j=1}^{n} Cl\, A\alpha_j \times \prod_{\alpha \neq \alpha_j} X_\alpha$

$\Rightarrow Cl\, Int\, A\alpha_j = Cl\, A\alpha_j$ for each $j (1 \leq j \leq n)$

$\Rightarrow A\alpha_j \in SO(X_\alpha)$ for each $j (1 \leq j \leq n)$

Semi-Continuous Mapping

★ Theorem 3:- If $f: X \to Y$ is a semi-continuous mapping and X_0 is an open set in X, then restriction $f|X_0 : X_0 \to Y$ is semi-continuous.



Proof) Since f is a semi-continuous mapping.
$\Rightarrow$ for any open set V in Y $f^{-1}(V)$ is semi open in X.
Since X_0 is open. So $f^{-1}(V) \cap X_0$ is semi-open in X.

Therefore $(f|X_0)^{-1}(V) = f^{-1}(V) \cap X_0$ is semi open in X_0

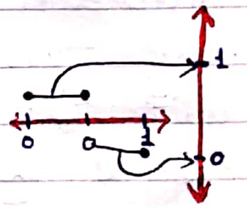
$\Rightarrow f|X_0$ is semi-continuous.

Remark:- In above theorem if $X_0 \in SO(X)$ then $f|X_0$ is not always semi-continuous.

Example:- Let $X = Y = [0,1]$ with usual topology and $X_0 = [1/2, 1]$

Let $f: X \to Y$ be mapping as follows

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1/2 \\ 0 & \text{if } 1/2 < x \leq 1 \end{cases}$$



Then f is semi-continuous.

However $(\frac{1}{2}, 1]$ is open in Y and $f^{-1}((\frac{1}{2},1]) \cap X_0 = \{\frac{1}{2}\} \notin SO(X_0)$

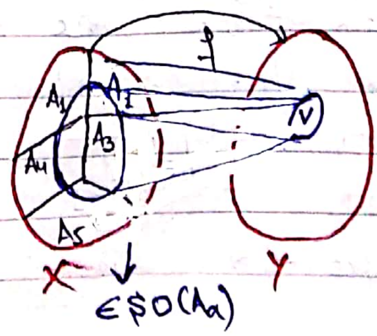
Therefore, $f|X_0$ is not semi-conti–

Theorem 4:- Let $f: X \to Y$ be a mapping and $\{A_\alpha | \alpha \in \beta\}$ a semi-open cover for X i.e $A_\alpha \in SO(X)$ for each $\alpha \in \beta$ and $\bigcup_{\alpha \in \beta} A_\alpha = X$. If the restriction

$$f|A_\alpha : A_\alpha \to Y \text{ is semi-continuous}$$

for each $\alpha \in \beta$, then f is semi-continuous.

Proof: Suppose V is an arbitrary open set in Y, Then for each $\alpha \in \beta$ we have



$$(f|A_\alpha)^{-1}(V) = f^{-1}(V) \cap A_\alpha \in SO(A_\alpha)$$

because $f|A_\alpha$ is semi-continuous

and also $\bigcup_{\alpha\in\beta} A_\alpha = X$

Hence by well known theorem

$$f^{-1}(V) \cap A_\alpha \in SO(X) \text{ for each } \alpha\in\beta$$

As union of any number of semi-open sets is semi open so

$$\bigcup_{\alpha\in\beta}\left[f^{-1}(V)\cap A_\alpha\right] = f^{-1}(V) \in SO(X)$$

$\Rightarrow f$ is semi-continuous

⇒ **Theorem 5:-** Let $\{X_\alpha | \alpha\in\beta\}$ & $\{Y_\alpha | \alpha\in\beta\}$ be any two families of topological spaces with the same index set β.
For each $\alpha\in\beta$, Let $f_\alpha : X_\alpha \to Y_\alpha$ be a mapping. Then a mapping $f: \prod X_\alpha \to \prod Y_\alpha$ defined by $f((x_\alpha)) = (f_\alpha(x_\alpha))$ is semi-continuous iff f_α is semi-continuous for each $\alpha\in\beta$.

Proof: Let f_α is semi continuous for each $\alpha\in\beta$.
Suppose V is basic open set of the topology of $\prod Y_\alpha$.
Then there are $\alpha_j \in\beta$ $(1\le j\le n)$ and open sets V_{α_j} in Y_{α_j} s.t

$$V = \prod_{j=1}^{n} V_{\alpha_j} \times \prod_{\alpha\neq\alpha_j} Y_\alpha$$

Since f_{α_j} is semi-continuous

So $f_{\alpha_j}^{-1}(V_{\alpha_j})$ is semi-open in X_{α_j} for each j $(1 \leq j \leq n)$

If there exist α_j s.t $f_{\alpha_j}^{-1}(V_{\alpha_j}) = \phi$

Then $f^{-1}(V) = \prod_{j=1}^{n} f_{\alpha_j}^{-1}(V_{\alpha_j}) \times \prod_{\alpha \neq \alpha_j} X_\alpha = \phi$

Hence $f^{-1}(V)$ is semi open in $\prod X_\alpha$

If $f_{\alpha_j}^{-1}(V_{\alpha_j}) \neq \phi$ fore each $j (1 \leq j \leq n)$

Then $f^{-1}(V) = \prod_{j=1}^{n} f_{\alpha_j}^{-1}(V_{\alpha_j}) \times \prod_{\alpha \neq \alpha_j} X_\alpha \neq \phi$

Hence by well known theorem $f^{-1}(V)$ is semi-open in $\prod X_\alpha$

Now for any open set ω in Y $\exists$ a family $\{V_\lambda \mid \lambda \in \Delta\}$ of basic open sets s.t

$$\omega = \bigcup_{\lambda \in \Delta} V_\lambda$$

Hence by well known theorem

$$f^{-1}(\omega) = \bigcup_{\lambda \in \Delta} f^{-1}(V_\lambda) \text{ is semi-open in } \prod X_\alpha$$

$\Rightarrow f$ is semi-continuous.

Conversely :- Let f is semi-continuous

Let for each fixed $\alpha \in \beta$,

Let $P_\alpha : \prod Y_\lambda \to Y_\alpha$ be the projection.

Suppose V_α is an arbitrary open set in Y_α, Then $P_\alpha^{-1}(V_\alpha) = V_\alpha \times \prod_{\lambda \neq \alpha} Y_\lambda$ is open in $\prod Y_\lambda$

Since f is semi-continuous then

$f^{-1}[P_\alpha^{-1}(V_\alpha)] = f_\alpha^{-1}(V_\alpha) \times \prod_{\lambda \neq \alpha} X_\lambda$ is semi-continuous in $\prod X_\lambda$

If $f_\alpha^{-1}(V_\alpha) = \phi$

Then it is obvious that f_α is semi-continuous.

$$\text{If } f_\alpha^{-1}(V_\alpha) \neq \phi$$

Then $f_\alpha^{-1}(V_\alpha) \times \prod_{r \neq \alpha} X_r \neq \phi$

Hence by well known theorem,

$$f_\alpha^{-1}(V_\alpha) \text{ is semi-open in } X_\alpha$$

$$\Rightarrow f_\alpha \text{ is semi-continuous } \forall \alpha \in \beta$$

Theorem 6:- Let $\{X_\alpha \mid \alpha \in \beta\}$ be any family of topological spaces. If $f: X \to \prod X_\alpha$ is semi continuous mapping, then $P_\alpha \circ f: X \to X_\alpha$ is semi-continuous, where P_α is projection of $\prod X_r$ onto X_α.

Proof) Let for a fixed $\alpha \in \beta$

Suppose U_α is an arbitrary open set in X_α then $P_\alpha^{-1}(U_\alpha)$ is open in $\prod X_\alpha$, ~~then~~

Since f is semi-continuous, we have

$$f^{-1}[P_\alpha^{-1}(U_\alpha)] = (P_\alpha \circ f)^{-1}(U_\alpha) \in SO(X)$$

$$\Rightarrow P_\alpha \circ f \text{ is semi-continuous.}$$

Theorem 7:- If $f: X \to Y$ is an open and semi-continuous mapping, Then $f^{-1}(B) \in SO(X)$ for every $B \in SO(Y)$

Proof) For an arbitrary $B \in SO(Y)$

there exict an open set V in Y s.t

$$V \subseteq B \subseteq cl(V)$$

since f is open and continuous

$$\Rightarrow f^{-1}(V) \subseteq f^{-1}(B) \subseteq f^{-1}(\bar{V}) \subseteq cl\, f^{-1}(V)$$

Since f is semi-continuous and V is an open in $Y \Rightarrow f^{-1}(V) \in SO(X)$

Hence

$f^{-1}(B)$ is semi-open in X.

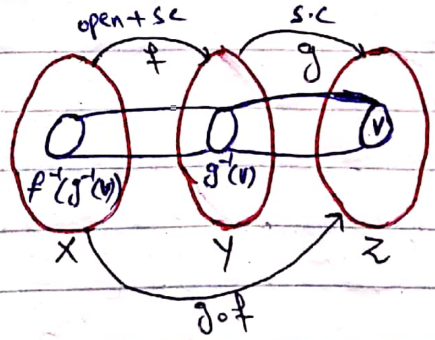
★ The composition mapping of two semi-continuous mappings is not always semi-continuous.

★ **Corollary:-** Let X, Y and Z be three topological spaces. If $f: X \to Y$ is an open and semi-continuous mapping and $g: Y \to Z$ is semi-continuous mapping, then $g \circ f: X \to Z$ is semi-continuous.

Proof:

$\because g: Y \to Z$ is semi continuous then for any open set $V \in Z$

$g^{-1}(V) \in SO(Y)$



And since f is open & semi-continuous then by theorem 7

$$f^{-1}(g^{-1}(V)) \in SO(X) \Rightarrow (f^{-1} \circ g^{-1})V \in SO(X)$$

$$\Rightarrow (g \circ f)^{-1}(V) \in SO(X)$$

$\Rightarrow g \circ f$ is semi-continuous

* 5th Research Paper:-

This course was established in 1985 by "T. Noiri and B. Ahmad"

This course was published by "Kyungpook Math Journal Vol. 25, No. 2 Page 123–126

SEMI WEAKLY CONTINUOUS MAPPINGS **

* **Weakly Continuous Function:-** Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f: X \to Y$ is said to be weakly continuous at X if for each $x \in X$ and for each open set V containg $f(x)$, there exist $U \in SO(X, x)$ s.t $f(U) \subseteq Cl(V)$

* **Almost Continuous Function:-** Let (X, τ_X) and (Y, τ_Y) be topological spaces, a function $f: X \to Y$ is said to be almost continuous if for each $x \in X$ and for each open set V containing $f(x)$, there exist a semi-open set U in X containing x s.t $f(U) \subseteq Int[Cl(V)]$

Note:- Almost continuous function is also weakly continuous function ($\because Int[Cl(V)] \subseteq Cl(V)$)

but converse is not true in general

★ **Semi-Weakly Continuous Function:-** Let (X, τ_X) and (Y, τ_Y) are topological spaces. A function $f: X \to Y$ is said to be semi-weakly continuous (s.w.c) at x if for each $x \in X$ and for each open set V containing $f(x)$ there exist $U \in SO(X, x)$ s.t $f(U) \subseteq sCl(V)$

Note:- Semi-continuous $\Rightarrow$ semi-weakly continuous $\Rightarrow$ Weakly continuous

★ Almost continuous $\Rightarrow$ Weakly continuous

Example:- Let $X = Y = \mathbb{R}$

Let τ be the usual topology on X and σ be the countable topology on Y. Then the identity mapping $f: X \to Y$ is semi-weakly continuous but not semi-continuous.

★ **Theorem 1:-** Let (X, τ_X) and (Y, τ_Y) be topological spaces. A mapping $f: X \to Y$ is semi-weakly continuous iff for every open set V in Y

$$f^{-1}(V) \subseteq s\,Int\,[f^{-1}(sCl(V))]$$

Proof:-

Let $x \in X$ and V be an open set containing $f(x)$, satisfying the relation

$$f^{-1}(V) \subseteq s\,Int\,[f^{-1}\,sCl(V)]$$

We will prove that f is semi-weakly continuous.

Put $U = sInt[f^{-1}(sCl(V))]$, Then

$x \in U \in SO(X, \tau)$

$\Rightarrow U = sInt[f^{-1} sCl(V)] \subseteq f^{-1}(sCl(V))$

$\Rightarrow f(U) \subseteq f f^{-1}[sCl(V)] \subseteq sCl(V)$

$\Rightarrow f(U) \subseteq sCl(V)$

$\Rightarrow f$ is semi-weakly continuous

$f(x) \in V \subseteq sCl(V)$
$\Rightarrow x \in f^{-1}(V) \subseteq sInt[f^{-1}sCl(V)]$
$\Rightarrow x \in U$

Conversely Let $f: X \to Y$ be semi-weakly continuous.

Let $x \in X$ and V be an open set containig $f(x)$.

$\Rightarrow x \in f^{-1}(V)$

By hypothesis (f is s.w.c), there exist a semi-open set U in X containg x s.t $f(U) \subseteq sCl(V)$

$\Rightarrow x \in U \subseteq f^{-1}[sCl(V)]$

$\Rightarrow U = sInt(U)$ $\because U$ is semi-open

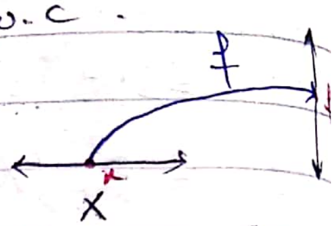
$\subseteq sInt[f^{-1} sCl(V)]$

$\Rightarrow x \in sInt[f^{-1}(sCl(V))]$

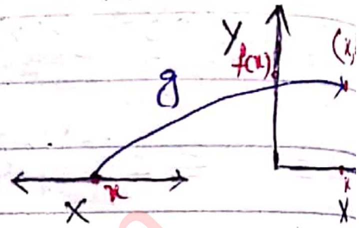
$\Rightarrow f^{-1}(V) \subseteq sInt[f^{-1}(sCl(V))]$

* **Theorem 2:-** Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f: X \longrightarrow Y$ be a function and $g: X \longrightarrow X \times Y$ be the graph mapping of f given by $g(x) = (x, f(x))$

for every $x \in X$. If g is semi-weakly continuous, Then f is s.w.c.



Proof Let $x \in X$ and V be open set containing $f(x)$.



$\Rightarrow X \times V$ containing $(x, f(x)) = g(x)$

Since g is s.w.c, therefor there exist $U \in SO(X, x)$ s.t

$$g(U) \subseteq sCl(X \times V) = sCl(X) \times sCl(V)$$
$$= X \times sCl(V)$$

or $(U, f(U)) \subseteq X \times sCl(V)$ $\quad \because g(x) = (x, f(x)) \Rightarrow g(U) = (U, f(U))$

$\Rightarrow f(U) \subseteq sCl(V)$ $\quad \because g$ is a graph of f

$\Rightarrow f$ is semi-weakly continuous.

★ Theorem 3:- Let (X, τ_x) and (Y, τ_y) be topological spaces and If $f : (X, \tau_x) \to (Y, \tau_y)$ is semi-weakly continuous mapping and Y is housdorff space. Then the graph $G(f)$ is a semi-closed set of $X \times Y$.

Proof $\quad \boxed{G(f) = \{(x, f(x)) \mid x \in X\}}$

Let $(x, y) \notin G(f)$

we will show that (x, y) is not semi-limit point of $G(f)$

Now, Since $(x, y) \notin G(f)$

$\Rightarrow y \neq f(x)$

Since Y is a T_2-space therefore there exist open sets W and V in Y s.t

$f(x) \in W$; $y \in V$ and

$$W \cap V = \phi$$

Since f is semi-weakly continuous therefore there exist a $U \in SO(X,x)$ s.t $f(U) \subseteq s\,cl(W)$

Since $V \cap W = \phi$

$\Rightarrow V \cap s\,cl(W) = \phi$

$\Rightarrow V \cap f(U) = \phi \quad \because f(U) \subseteq s\,cl(W)$

$\Rightarrow (U \times V) \cap G(f) = \phi$

$$\boxed{\begin{aligned} & V \cap W = \phi \\ & \Rightarrow V \subseteq Y - W \\ & \Rightarrow s\,Int(V) = V \subseteq s\,Int(Y-W) \\ & \qquad = Y - s\,cl(W) \\ & \Rightarrow V \subseteq Y - s\,cl(W) \\ & \Rightarrow V \cap s\,cl(W) = \phi \end{aligned}}$$

where $U \times V \in SO(X \times Y, (x,y))$

$\Rightarrow (x,y)$ is not semi-limit point of $G(f)$

$\Rightarrow G(f)$ contains all of its semi-limit points

$\Rightarrow G(f)$ is semi-closed set of $X \times Y$.

——————

* **Semi-Connected Space :-** (S-connected space) A topological space $(X, \mathcal{T}_X)$ is said to be semi-connected space if it can not be expressed as union of two non-empty disjoint semi-open sets.

Note:- Every semi-connected space is connected.

* A connected space may not be semi-connected.

——————

Example:- $X = \{a, b, c\}$

$\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$

It is connected because we can not write it as union of two non-empty disjoint open sets.

Now

$S O(X) = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\}$

and

$\{a\} \cup \{b, c\} = X$

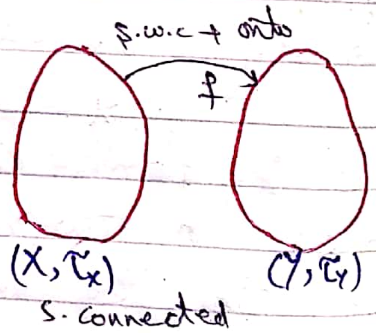
& $\{a\} \cap \{b, c\} = \phi$

$\Rightarrow$ This is semi-disconnected

$\Rightarrow$ This is not semi-connected space.

*** Theorem 4:-** Let (X, τ_X) is an s-connected space and $f:(X, \tau_X) \longrightarrow (Y, \tau_Y)$ is a s.w.c surjection then Y is connected.

Proof Suppose that Y is disconnected.



$\Rightarrow$ There exist open sets U and V in Y s.t

$U \cup V = Y$ and

$U \cap V = \phi$

$\Rightarrow f^{-1}(Y) = f^{-1}(U \cup V)$

$\Rightarrow X = f^{-1}(U) \cup f^{-1}(V) \longrightarrow (A)$

and $U \cap V = \phi \Rightarrow f^{-1}(U \cap V) = f^{-1}(\phi)$

$\Rightarrow f^{-1}(U) \cap f^{-1}(V) = \phi \longrightarrow (B)$

Since f is onto and $U \neq \phi$ and $V \neq \phi$

$\Rightarrow f^{-1}(U) \neq \phi$ and $f^{-1}(V) \neq \phi$

Now, since f is semi-weakly continuous and U, V are open in Y, therefore

$$f^{-1}(U) \subseteq sInt[f^{-1} sCl(U)]$$

$$\& \quad f^{-1}(V) \subseteq sInt[f^{-1} sCl(V)]$$

$$\Rightarrow f^{-1}(U) \subseteq sInt[f^{-1}(U)]$$

$$\& \quad f^{-1}(V) \subseteq sInt[f^{-1}(V)]$$

$U \cap V = \phi$ & $U \cup V = Y$
$\Rightarrow U$ and V are open as well as closed

$$\Rightarrow f^{-1}(U) = sInt[f^{-1}(U)] \text{ and } f^{-1}(V) = sInt[f^{-1}(V)]$$

$\because sInt[f^{-1}(U)] \subseteq f^{-1}(U)$
$\because sInt[f^{-1}(V)] \subseteq f^{-1}(V)$

$\Rightarrow f^{-1}(U)$ and $f^{-1}(V)$ are semi-open sets.
So by (A) and (B) we get that X is semi-disconnected.
A contradiction
Hence the proof.

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* $A \cap B = \phi \Rightarrow A \subseteq X - B$ & $B \subseteq X - A$
* If $A \cap B = \phi$ & $A \cup B = X$ then A, B are open imply A and B are also closed
  $\Rightarrow A$ & B are clopen.

MUHAMMAD TAHIR (Reg # FA15-RMT-007)

M.S. Mathematics **

* 6th Research Paper:-

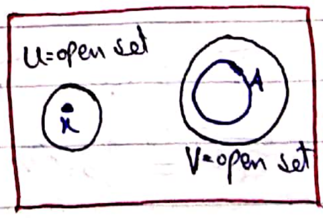
This course was established in 2001 By "M. Khan (Department of Mathematics, Govt. college Multan-Pakistan) and B. Ahmad (BZU Multan - Pakistan)

This course was published by "Journal of Mathematics, Punjab University (ISSN 1016-2526) Vol. XXXIV (2001) PP 107-114

s-CONTINUOUS, s-OPEN & s-CLOSED FUNCTIONS **

*** s-Continuous Function:-** A function $f: X \to Y$ is said to be s-continuous (also called strongly semi-continuous) if the inverse image of every semi-open set is open.

Note:- It is known that an s-continuous function is irresolute, semi-continuous and continuous.



*** Regular Space:-** A topological space (X,τ) is said to be regular if for every $x \in X$ and for any closed sub set A of X s.t $x \notin A$ there exist two open sets U and V in X s.t $x \in U$, $A \subseteq V$ and $U \cap V = \phi$.

*** p-Regular Space:-** A space (X, τ) is said to be p-regular space if for each semi-closed set F and $x \in X - F$, there exist disjoint open sets U and V s.t $x \in U$ and $F \subset V$

*** Semi-Regular Space:-** A space (X, τ) is said to be semi-regular if for each semi-closed set F and $x \in X - F$ there exist disjoint semi-open sets U and V s.t $x \in U$ & $F \subset V$.

* Clearly p-regular space is semi-regular as well as regular but the converse is not true in general.

Example:- Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a,b\}, X\}$. Then X is semi regular but not p-regular.

$$\tau = \{\phi, \{a\}, \{b\}, \{a,b\}, X\}$$

closed sets of $X = \{X, \{b,c\}, \{a,c\}, \{c\}, \phi\}$

$\overline{\phi} = \phi$, $\overline{X} = X$, $\overline{\{a\}} = \{a, c\}$

$\overline{\{b\}} = \{b, c\}$, $\overline{\{a,b\}} = X$

$\Rightarrow SO(X) = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$

$\Rightarrow SC(X) = \{X, \{b,c\}, \{a,c\}, \{c\}, \{b\}, \{a\}, \phi\}$

then for each semi-closed set (say F) of X, & $x \in F^c$ there exist two disjoint semi-open sets (say U & V) s.t

$x \in U$ & $F \subset V$

$\Rightarrow (X, \tau)$ is semi-regular.

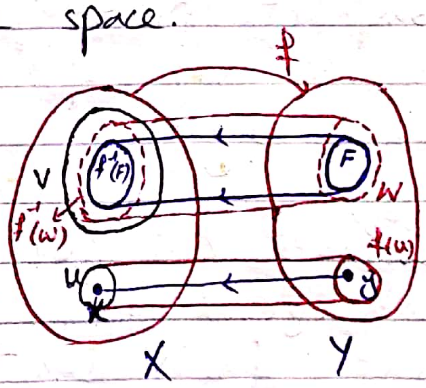
Now for $\{b,c\} \in SC(X)$ and $a \in X - \{b,c\}$ we can not find two open sets U and V in X s.t

$a \in U$ & $\{b,c\} \subset V$

$\Rightarrow (X,\tau)$ is not p-regular.

Similarly (X,τ) is not regular space.

★ **Theorem 1:-** The image of a regular space under a clopen and s-continuous surjection is p-regular space.



Proof Let $F \in SC(Y)$ and $y \in Y - F$

Let $x \in f^{-1}(y)$

Since f is s-continuous therefore by a well known theorem $f^{-1}(F)$ is closed in X. and $x \in X - f^{-1}(F)$

Since X is regular therefore there exist open sets U and V in X s.t

$x \in U$ & $f^{-1}(F) \subset V$

and $U \cap V = \phi$

Since f is closed, therefore by a well known theorem there exist an open set W of Y s.t

$F \subset W$ & $f^{-1}(W) \subset V$

therefore $U \cap f^{-1}(W) = \phi$ $\because U \cap V = \phi$ & $f^{-1}(W) \subset V$

& hence $f(U) \cap W = \phi$

Since f is open $\Rightarrow f(U)$ is open in Y

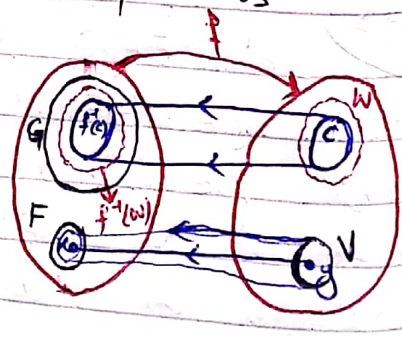
and $y \in f(U)$ $\because f(x) = y$ & $x \in U \Rightarrow f(x) \in f(U)$

i.e there exist two open sets $f(U)$ and W in Y s.t

$F \subset W$ & $y \in f(U)$

$\Rightarrow Y$ is p-regular.

★ **Theorem 2:-** Let $f: X \to Y$ be s-continuous and semi-closed surjection with compact point inverses and X is a regular space, then Y is semi-regular.



Proof) Let $C \in SC(Y)$ and $y \in Y - C$

Since f is s-continuous therefore by a well known theorem $f^{-1}(C)$ is closed in X.

More over, the compact sets $f^{-1}(y)$ and $f^{-1}(C)$ are disjoint in a regular space.

As X is regular space,

therefore there exist two disjoint open sets F and G in X s.t

$$f^{-1}(y) \subseteq F \quad \text{and} \quad f^{-1}(c) \subset G$$

Since f is semi-closed then by a well known theorem there exist two semi open sets V and W containing y and c respectively s.t

$$f^{-1}(V) \subset F \;\&\; f^{-1}(W) \subset G$$

since $F \cap G = \phi$

$$\Rightarrow f^{-1}(V) \cap f^{-1}(W) = \phi$$

$$\Rightarrow V \cap W = \phi$$

i.e for $c \in SC(Y)$ & $y \in Y - c$ there exist two semi open sets V & W in Y s.t $y \in V$ & $c \subseteq W$ and $V \cap W = \phi$

$\Rightarrow$ Y is a semi-regular space.

* **Corollary**:- Let $f: X \to Y$ be s-continuous and closed surjection with compact point inverses. then Y is p-regular if X is regular.

Proof) Let $c \in SC(Y)$ and $y \in Y - c$

Since f is s-continuous, therefor by a well known theorem $f^{-1}(c)$ is closed in X. Moreover the compact sets $f^{-1}(y)$ and $f^{-1}(c)$

are disjoint in a regular space. As X is regular space, therefore there exist two disjoint open sets F and G in X s.t

$$f^{-1}(y) \subseteq F \quad \& \quad f^{-1}(C) \subseteq G$$

Since f is closed surjection, therefore by a well known theorem, $\exists$ two open sets V and W in Y containing y and C respectively, s.t

$$f^{-1}(V) \subset F \quad \text{and} \quad f^{-1}(W) \subseteq G$$

Since $F \cap G = \phi$

$$\Rightarrow f^{-1}(V) \cap f^{-1}(W) = \phi$$

& hence $V \cap W = \phi$

i.e for $C \in SC(Y)$ & $y \in Y - C$, $\exists$ two open sets V & W in Y s.t

$y \in V$ & $C \subseteq W$ &

$$V \cap W = \phi$$

$\Rightarrow Y$ is a p-regular space.

***Open Function:-** A function f is said to be an open function if image of each open set is open.

***Semi-Open Function:-** A function $f: X \to Y$ is said to be semi-open function if image of every open set of X is semi-open in Y.

★ **Pre-Semi-Open Function:-** Let X and Y be topological spaces, a function $f: X \to Y$ is said to be pre-semi-open iff for all $A \in SO(X)$, $f(A) \in SO(Y)$

★ **s-Open Function:-** A function $f: X \to Y$ is said to be s-open if the image of every semi-open set is open.

★ It is known that every s-open function is open, semi-open and pre-semi open.

Theorem 3:- For a function $f: X \to Y$, the following are equivalent

(1) f is s-open

(2) $f[sInt(A)] \subset Int\, f(A)$, for each $A \subset X$

(3) $sInt[f^{-1}(B)] \subset f^{-1}(Int\, B)$, $\forall\ B \subset Y$

(4) $f^{-1}(cl\, B) \subset scl\, f^{-1}(B)$, for each $B \subset Y$

(5) $f^{-1}(Bd\, B) \subset sBd[f^{-1}(B)]$, for each $B \subset Y$

Proof:- ① ⇒ ② obviously $f[sInt(A)] \subset f(A)$

Now $sInt(A)$ is semi open in X

$\Rightarrow f[sInt(A)]$ is open in Y $\quad \because f$ is s-open

$\Rightarrow f[sInt(A)]$ is open subset of $f(A)$ in Y

But $Int\, f(A)$ is the largest open set contained in $f(A)$

$\Rightarrow f[sInt(A)] \subset Int\, f(A)$

② ⇒ ③ For any $B \subset Y$

Put $f^{-1}(B) = A \subset X$

then by ②

$$f[s\,Int\, f^{-1}(B)] \subseteq Int\, f f^{-1}(B) \subseteq Int\,(B).$$

$$\Rightarrow f[s\,Int\, f^{-1}(B)] \subseteq Int(B)$$

$$\Rightarrow s\,Int\, f^{-1}(B) \subseteq f^{-1}[Int(B)]$$

③ $\Rightarrow$ ④ By ③ $s\,Int\, f^{-1}(B) \subseteq f^{-1}(Int(B))$

$$\Rightarrow X - f^{-1}[Int(B)] \subseteq X - s\,Int\, f^{-1}(B)$$

$$= s\,cl\,[X - f^{-1}(B)]$$

$$\Rightarrow f^{-1}(Y) - f^{-1}[Int(B)] \subseteq s\,cl\,[f^{-1}(Y) - f^{-1}(B)] \quad \because X = f^{-1}(Y)$$

$$\Rightarrow f^{-1}[Y - Int\, B] \subseteq s\,cl\, f^{-1}[Y - B]$$

$$\Rightarrow f^{-1}cl\,[Y - B] \subseteq s\,cl\, f^{-1}[Y - B]$$

$$\Rightarrow f^{-1}cl\,(C) \subseteq s\,cl\, f^{-1}(C) \quad \text{, where } Y - B = C \in Y$$

④ $\Rightarrow$ ⑤ For $B \subset Y$

$Bd\, B = cl\, B \cap cl\,(Y - B)$ is closed set in Y

Now $f^{-1}(Bd\, B) = f^{-1}[cl\, B] \cap f^{-1} cl\,(Y - B)$

$$\subseteq s\,cl\, f^{-1}(B) \cap s\,cl\, f^{-1}(Y - B) \quad \text{by ④}$$

$$= s\,cl\, f^{-1}(B) \cap [s\,cl\, f^{-1}(Y) - s\,cl\, f^{-1}(B)]$$

$$= s\,cl\, f^{-1}(B) \cap [s\,cl\, X - s\,cl\, f^{-1}(B)]$$

$$\Rightarrow f^{-1}(Bd\, B) \subseteq s\,cl\, f^{-1}(B) \cap s\,cl\,[X - f^{-1}(B)]$$

$$= s\,Bd\, f^{-1}(B)$$

using $U \cap sBd\, f^{-1}(B) = \phi$ ① becomes

$f(U) \cap BdB \subset \phi$

$\Rightarrow f(U) \cap BdB = \phi$

$\Rightarrow BdB \subseteq Y - f(U) = B$

$\Rightarrow$ B contains all of its boundry Points $\Rightarrow$ B is closed

$\Rightarrow f(U)$ is open in Y

This proves that f is s-open function.

★ **Theorem 4:-** For any function $f: X \to Y$ and $g: Y \to Z$, we have.

(1) $g \circ f$ is s-open if f is s-open & g is open

(2) $g \circ f$ is s-open if f is pre-semi open & g is s-open

(3) $g \circ f$ is open if f is semi-open & g is s-open.

(4) $g \circ f$ is pre semi-open if f is s-open and g is semi-open.

Proof ① We have to show that $g \circ f$ is s-open.

Let $U \in SO(X)$

Since $f: X \to Y$ is s-open

$\Rightarrow f(U)$ is open in Y.

Now, sing $g: Y \to Z$ is open and $f(U)$ is open subset of Y

$\Rightarrow g(f(U))$ is open in Z

$\Rightarrow$ for $U \in SO(X)$, $g(f(U))$ is open in Z $\Rightarrow g(f(U))$ is s-open

$\Rightarrow g \circ f$ is s-open.

(2) We have to show that $g \circ f$ is s-open.

Since $f : X \to Y$ is pre-semi open.

Let $U \in SO(X)$

$\Rightarrow f(U) \in SO(Y)$

Now, since $g : Y \to Z$ is is s-open

and $f(U) \in SO(Y)$

$\Rightarrow g(f(U)) \in$ open set of Z

$\Rightarrow$ for $U \in SO(X)$, $g(f(U)) \in$ open in Z

$\Rightarrow g(f(U))$ is s-open

$\Rightarrow g \circ f$ is s-open.

(3) We have to prove that $g \circ f$ is open.

Let U is open set of X.

As we have $f : X \to Y$ is semi-open

$\Rightarrow f(U) \in SO(Y)$

Again we have $g : Y \to Z$ is s-open

& $f(U) \in SO(Y)$

$\Rightarrow g(f(U))$ is open in Z

$\Rightarrow$ for U is open in X, $g(f(U))$ is open in Z $\Rightarrow g(f(U))$ is open

$\Rightarrow g \circ f$ is open.

(4) We have to show that $g \circ f$ is pre-semi-open.

Let $U \in SO(X)$

Since $f : X \to Y$ is s-open

$\Rightarrow f(U) \in$ open set of Y.

And also we have $g : Y \to Z$.

is semi-open

$\Rightarrow g(f(U)) \in SO(Z)$

$\Rightarrow g(f(U))$ is pre semi-open

$\Rightarrow g \circ f$ is pre semi-open

★ **s-Closed Function:-** A function $f: X \to Y$ is said to be s-closed if the image of every semi-closed set is closed.

★ **Theorem 5:-** A function $f: X \to Y$ is s-closed iff $cl\, f(A) \subset f(scl(A))$, for each $A \subset X$.

Proof) Let f is s-closed

Obviously $f(A) \subset f[scl(A)]$

Now $scl(A)$ is semi-closed in X

$\Rightarrow f[scl(A)]$ is closed in Y $\because f$ is s-closed

$\Rightarrow f[scl(A)]$ is closed superset of A

But $cl\, f(A)$ is the smallest closed set containing $f(A)$

$\Rightarrow cl\, f(A) \subseteq f[scl(A)]$

Conversely)

Let $A \in SC(X)$

we show that $f(A)$ is closed in Y.

By hypothesis

$cl\, f(A) \subseteq f[scl(A)] = f(A) \quad \because A \in SC(X)$

$\Rightarrow cl\, f(A) \subseteq f(A) \longrightarrow ①$

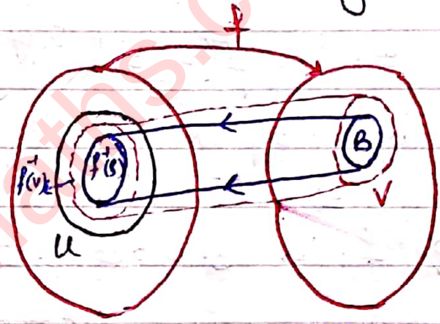
But $f(A) \subseteq cl\, f(A) \longrightarrow ②$ always by ① & ② $f(A) = cl\, f(A)$

$\Rightarrow f(A)$ is closed

$\Rightarrow f$ is s-closed.

This completes the proof.

Theorem 6:- A surjective function $f: X \to Y$ is s-closed iff for each subset B in Y and each semi-open set U in X containing $f^{-1}(B)$, there exist an open set V in Y containing B s.t $f^{-1}(V) \subset U$.



Proof Let U be an arbitrary semi-open set in X containing $f^{-1}(B)$ where $B \subset Y$

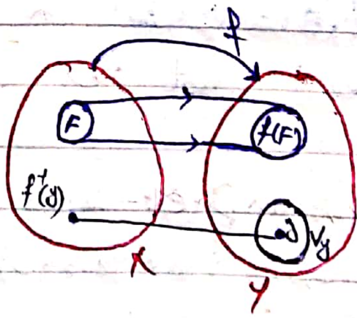
clearly $Y - f(X - U) = V$ (say) is open in Y

Since $f^{-1}(B) \subseteq U$ and f is onto then simple calculation gives $B \subset V$, Moreover we have $f^{-1}(V) \subseteq X - f^{-1}(f(X-U)) \subseteq U$

$\Rightarrow f^{-1}(V) \subseteq U$

Conversely

Let F be an arbitrary semi-closed set in X and $y \in Y - f(F)$



Then $f^{-1}(y) \subseteq f^{-1}(Y - f(F))$

$$\Rightarrow f^{-1}(y) \subseteq X - f^{-1}f(F)$$

$$\subseteq X - F \quad \because \left\{ \begin{array}{l} F \subseteq f^{-1}f(F) \\ \Rightarrow (f^{-1}f(F))' \subseteq F' \end{array} \right.$$

$$\Rightarrow f^{-1}(y) \subseteq X - F$$

Since $X - F$ is semi-open, therefore there exist an open set V_y containing y s.t

$$f^{-1}(V_y) \subset X - F$$

$$\Rightarrow y \in V_y \subseteq Y - f(F)$$

$$\Rightarrow Y - f(F) = \bigcup \{V_y : y \in Y - f(F)\}$$

is open in Y

$\Rightarrow f(F)$ is closed in Y

$\Rightarrow f$ is s-closed. this completes the proof.

→ **Remark 1:-** If $f : X \to Y$ is s-continuous and closed and closed (or irresolute[5] and s-closed) surjection, then using theorem 2.2 (iii) [2], one can easily see that the classes $SC(X)$ and $C(X)$ (closed sets of X) coincide.

⇒ **Remark 2:-** In general, an s-open function need not be s-closed.

Example:- Let $X = \{a, b, c\}$,
$\tau_X = \{\phi, \{a\}, \{b\}, \{a,b\}, X\}$ and $Y = \{a, b, c, d\}$,
$\tau_Y = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, Y\}$
Let $f : X \to Y$ be an identity function.
then f is s-open but not s-closed.

Now $C(X) = \{\phi, \{b,c\}, \{a,c\}, \{c\}, X\}$

$C(Y) = \{\phi, \{b,c,d\}, \{a,c,d\}, \{c,d\}, \{d\}, Y\}$

From X :- $\overline{\{a\}} = \{a,c\}$, $\overline{\{b\}} = \{b,c\}$,

$\overline{\{a,b\}} = X$, $\overline{\phi} = \phi$, $\overline{X} = X$

From Y :- $\overline{\phi} = \phi$, $\overline{\{a\}} = \{a,c,d\}$, $\overline{\{b\}} = \{b,c,d\}$

$\overline{\{a,b\}} = \{Y\}$, $\overline{\{a,c,d\}} = Y$, $\overline{Y} = Y$

$P(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$

$P(Y) = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\},$
$\{b,d\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}, \{a,b,d\}$
$, Y\}$

Now $SO(X) = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$

$SC(X) = \{\phi, \{b,c\}, \{a,c\}, \{c\}, \{b\}, \{a\}, X\}$

$SO(Y) = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,c\}, \{a,d\}, \{b,c\},$
$\{b,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}, \{a,b,d\}, Y\}$

$SC(Y) = \{\phi, \{b,c,d\}, \{a,c,d\}, \{c,d\}, \{d\}, \{b,d\}, \{b,c\},$
$\{a,d\}, \{a,c\}, \{d\}, \{b\}, \{a\}, \{c\}, Y\}$

⇒ **Remark 3:-** However, for bijection, it is easily seen that the notations of s-open and s-closed coincide. Moreover f is s-open iff f^{-1} is s-continuous.

Proof Let $f: X \to Y$ is s-open
⇒ Image of every semi-open set of X is open in Y
As image of every semi-open set is open under f ⇒ By a well known theorem f^{-1} is s-continuous [since f is s-continuous if inverse image of every semi open set is open]

Conversely
Let f^{-1} is s-continuous
⇒ image of every semi-open set of X is open in Y under f
⇒ f is s-open.

* **s-closed Space:-** A space X is said to be s-closed if for every semi-open cover of X, there exist a finite subfamily such that the union of their semi-closures cover X.

* **Compact Space:-** A topological space (X, τ) is said to be compact if every open cover for X has a finite sub cover.

* **Lindelof Space**:- A topological space (X,τ) is said to be lindelof space if for every open cover has a countable subcover.

* **Semi-Compact Space**:- A topological space (X,τ) is said to be semi-compact, if for every semi-open cover of X, there exist a finite subfamily s.t their union cover X.

* **Almost Compact space**:- A topological space (X,τ) is said to be almost-compact if for every open cover of X, $\exists$ a finite subfamily s.t union of their closures cover X.

Note:- Every compact space is almost compact, as well as semi-compact.

* Moreover, every semi-compact space is s-closed.

* **Theorem 7**:- The inverse image of an almost compact space under s-open bijection is s-closed.

Proof:- Let $\{V_\alpha : \alpha \in I\}$ be semi open cover for X.

$$\Rightarrow \bigcup_{\alpha \in I} V_\alpha = X$$

As V_α are semi-open in X and $f: X \to Y$ is s-open

$\Rightarrow f(V_\alpha) : \alpha \in I$ are open in Y

V_α are semi-open $\Rightarrow \bigcup_{\alpha \in I} V_\alpha$ is semi-

$\Rightarrow f(\bigcup_{\alpha \in I} V_\alpha)$ is open $\because f$ is s-open

$\Rightarrow \bigcup_{\alpha \in I} f(V_\alpha) = f(X)$

$\Rightarrow \bigcup_{\alpha \in I} f(V_\alpha) = Y \qquad \because f(X) = Y$

$\Rightarrow f(V_\alpha)$ is an open cover for Y. As Y is almost compact, therefore there exist finite subfamily of $\bigcup_{\alpha \in I} f(V_\alpha)$ s.t the union of their closures cover Y

$\Rightarrow \bigcup_{i=1}^{N} cl\, f(V_\alpha) = Y$

$\Rightarrow Y = \bigcup_{i=1}^{N} cl\, f(V_\alpha)$

$\Rightarrow f^{-1}(Y) = f^{-1}\left[\bigcup_{i=1}^{N} cl\, f(V_\alpha)\right]$

$\Rightarrow X = f^{-1}\left[\bigcup_{i=1}^{N} cl\, f(V_\alpha)\right] \subseteq f^{-1}\left[\bigcup_{i=1}^{N} f\, scl(V_\alpha)\right] \quad \because f$ is $\Rightarrow cl\, f(A) \subseteq$

$\Rightarrow X \subseteq f^{-1} f \bigcup_{i=1}^{N} scl(V_\alpha) \subseteq \bigcup_{i=1}^{N} scl\, V_\alpha$

$\Rightarrow X \subseteq \bigcup_{i=1}^{N} scl\, V_\alpha .$

As $\bigcup_{\alpha \in I} V_\alpha$ is semi-open cover for X and we have find a finite subfamily s.t union of theire semi-closures cover X.

$\Rightarrow$ X is s-closed.

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★ **s.Regular Space:-** A topological space (X,τ) is said to be s-regular space if for each closed set F and $x \in X-F$, there exist semi-open sets U and V in X s.t $x \in U$, $F \subseteq V$ & $U \cap V = \phi$

★ Every regular space is s-regular.
★ Every semi-regular space is s-regular.

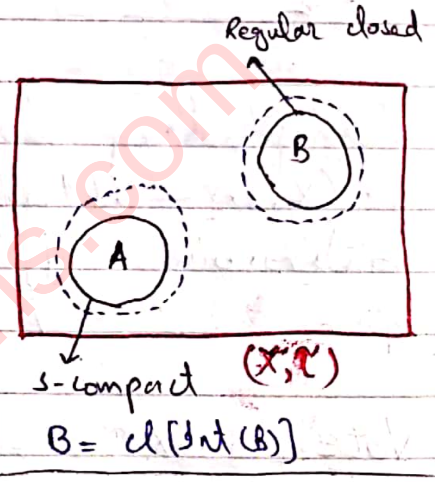
★ **Almost regular Space:-** A topological space (X,τ) is called almost regular space if for each regular closed set F and $x \in X-F$, there exist open sets U and V s.t $x \in U$, $F \subseteq V$ and
$$U \cap V = \phi$$

★ F is regular closed in (X,τ) if
$$F = cl[Int(F)]$$
★ F is regular open in (X,τ) if
$$F = Int[cl(F)]$$
★ Every regular closed set is closed and semi-open.
★ A set which is semi-closed as well as semi open is called semi-regular set.

★ **Semi Compact / s-Compact Space:-** A topological space (X,τ) is called s-compact if for every cover

$\{U_\alpha : \alpha \in \nabla\}$ of X by sets $U_\alpha \in SO(X)$ there exist a finite subset ∇_0 of ∇ s.t $X = \bigcup_{\alpha \in \nabla_0} U_\alpha$

Theorem:- Let (X, τ) be a topological space, Prove that an s-compact set A and disjoint regular closed set B in an s-regular space can be separated by semi-open sets.



Proof) Let $a \in A$

Since X is s-regular and B is a regular closed set s.t $a \in X - B$, therefore there exist semi-open sets G_a and H_a s.t

$$a \in G_a \ ; \ B \subseteq H_a \text{ and } G_a \cap H_a = \phi$$

Clearly $\{G_a : a \in A\}$ is a $\boxed{\text{semi-open}}$ cover of A by semi open sets of X. Since A is s-compact, therefore there exist a finite sub collection (say) $G_{a_1}, G_{a_2}, G_{a_3}, \ldots\ldots, G_{a_n}$ s.t

$$A \subseteq \bigcup_{i=1}^{n} G_{a_i} = G \in SO(X)$$

Now corresponding to these a_i ; $i = 1, 2, \ldots, n$ we have H_{a_i} s.t $B \subseteq H_{a_i}$ for each $i = 1, 2, 3, \ldots, n$

$$\Rightarrow B \subseteq H_{a_1} \cap H_{a_2} \cap \ldots \cap H_{a_n}$$

$\Rightarrow B = s\text{Int}(B) \subseteq s\text{Int}[H_{a_1} \cap H_{a_2} \cap \cdots \cap H_{a_n}]$ $\because B$ is semi open

$= H$

$\Rightarrow B \subseteq H \in SO(X)$; H is semi-open

Consequently, G and H are required disjoint semi open set

$G_{a_1} \cap H_{a_1} = \phi$ & $G_{a_2} \cap H_{a_2} = \phi$

$\Rightarrow G_{a_1} \cap (H_{a_1} \cap H_{a_2}) = \phi$

$G_{a_2} \cap (H_{a_1} \cap H_{a_2}) = \phi$

$\Rightarrow (G_{a_1} \cup G_{a_2}) \cap (H_{a_1} \cap H_{a_2}) = \phi$

$\Rightarrow \cup G_{a_i} \cap (\cap H_{a_i}) = \phi$

★———★

★ Completely Continuous Function:- A function $f : (X, \tau_x) \longrightarrow (Y, \tau_y)$ is said to be completely continuous if $f^{-1}(V) \in RO(X)$ for each open set V in Y.

Note:- $f : (X, \tau_x) \rightarrow (Y, \tau_y)$ is completly continuous iff $f^{-1}(V) \in RC(X)$ for each closed set B in Y.

★———★

★ Theorem:- Let $f : (X, \tau_x) \rightarrow (Y, \tau_y)$ be a completly continuous and semi-closed preserving surjection with s-compact point inverses. If X is s-regular then Y is s-regular.

Proof Let X be s-regular.

Let F be a closed set and $y \in Y - F$, then $f^{-1}(F) \in RC(X)$ and $f^{-1}(y)$ is s-compact.

Clearly $f^{-1}(y) \notin f^{-1}(F)$

Since X is s-regular, therefore,

there exist semi-open sets U_y and U_F in X s.t $f^{-1}(y) \in U_y$ & $f^{-1}(F) \subseteq U_F$ and $U_y \cap U_F = \phi$

Since f is semi closed preserving therefore there exist semi-open sets V_y and V_F s.t

$$y \in V_y \;\; \& \;\; F \subseteq V_F$$

and

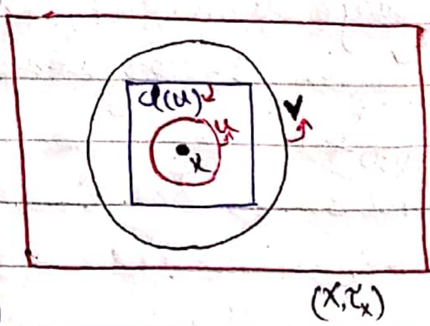
$$f^{-1}(V_y) \subseteq U_y \text{ and } f^{-1}(V_F) \subseteq U_F \text{ and}$$

$$U_y \cap U_F = \phi \text{ gives } V_y \cap V_F = \phi$$

This proves that Y is s-regular.

★ **Theorem:-** Let (X, τ_x) be a topological space then X is s-regular if and only if for each open set V containing $x \in X$, there exist a semi open set U containing x s.t $x \in U \subseteq scl(U) \subseteq V$

Proof) Let (X, τ_x) be s-regular space and V is an open set containing x i.e $x \in V$



$\Rightarrow x \notin X - V$ (closed set)

Since space is s-regular, therefore there exist $U, L \in SO(X)$ s.t

$$x \in U \;\;,\;\; X - V \subseteq L$$

$$\Rightarrow X - L \subseteq V$$

$$\& \;\; U \cap L = \phi$$

$$\Rightarrow U \subseteq X - L \text{ (semi closed)}$$

$\Rightarrow scl(U) \subseteq X-L$ ($\because X-L$ is semi closed)

Thus $x \in U \subseteq scl(U) \subseteq X-L \subseteq V$

$\Rightarrow x \in U \subseteq scl(U) \subseteq V$ Proved.

Conversely:- We prove that X is s-regular.

Let F be a closed subset of X and $x \notin F \Rightarrow x \in X-F$,

where $X-F$ is open in X.

By hypothesis, there exist a semi-open set U in X containing x s.t

$x \in U \subseteq scl(U) \subseteq X-F$

$\Rightarrow x \in U$ and $F \subseteq X-scl(U)$ [semi-open set]

Let $V = X-scl(U)$, then

$x \in U$, $F \subseteq V$ & $U \cap V = \phi$

$\Rightarrow X$ is s-regular.

★ **Theorem:-** Let $f:(X,\tau_X) \to (Y,\tau_Y)$ be a continuous and semi-closed preserving surjection. If X is s-regular then Y is s-regular.

Proof) Let X be s-regular.

Let U be an open set in Y s.t $y \in U$.

Let $x \in f^{-1}(y)$. Now $f^{-1}(U)$ is open in X and $x \in f^{-1}(U)$

Since X is s-regular, therefore there exist $V \in SO(X,x)$ s.t

$x \in V \subseteq scl(V) \subseteq f^{-1}(U)$

$\Rightarrow f(x) \in f(V) \subseteq f\,scl(V) \subseteq ff^{-1}(U) \subseteq U$

where $f(V)$ is semi open and

$$scl[f(V)] \subseteq f[scl(V)] \rightarrow [\text{Learn by Heart}]$$

Thus

$$y \in f(V) \subseteq scl[f(V)] \subseteq f[scl(V)] \subseteq U$$

$$\Rightarrow y \in f(V) \subseteq scl\, f(V) \subseteq U$$

This proves that Y is s-regular.

* ——— *

Prove that:- $sBd[sBd(sBd(A))] = sBd[sBd(A)]$

Proof

We know that

- $scl(A) = A$ iff A is semi-closed
- $sBd(A) = scl(A) \cap scl(X-A)$ is semi closed

Now

$$sBd[sBd(sBd(A))]$$

$$= scl[sBd(sBd(A))] \cap scl[X - sBd(sBd(A))]$$

$$= sBd[sBd(A)] \cap scl[X - sBd(sBd(A))] \longrightarrow (1)$$

Consider

$$X - sBd(sBd(A)) = X - [scl(sBd(A)) \cap scl(X - sBd(A))]$$

$$= X - [sBd(A) \cap scl(X - sBd(A))]$$

$\because sBd(A)$ is semi-closed

$$= [X - sBd(A)] \cup [X - scl(X - sBd(A))]$$

Now

$$scl[X - sBd(sBd(A))] = scl\{[X - sBd(A)] \cup [X - scl(X - sBd(A))]\}$$

$$\supseteq \underbrace{scl[X - sBd(A)]}_{D} \cup scl[X - \underbrace{scl(X - sBd(A))}_{D}]$$

$$= D \cup scl(X - D) = X$$

$\Rightarrow$

$$scl[X - sBd(sBd(A))] = X \longrightarrow (2)$$

By equations ① & ②

$sBd[sBd(sBd(A))] = sBd[sBd(A)] \cap X$

$\Rightarrow sBd[sBd(sBd(A))] = sBd[sBd(A)]$ Proved

★ s-Closed Space:- A topological space (X, τ) is said to be s-closed if for every cover $\{V_\alpha : \alpha \in \nabla\}$ of X by sets V_α semi open in X for each $\alpha \in \nabla$, there exist a finite subset ∇_0 of ∇ s.t

$$X = \bigcup_{\alpha \in \nabla_0} sCl(V_\alpha)$$

★ S-Closed Space:- A topological space (X, τ) is said to be S-closed if for each covering $\{V_\alpha : \alpha \in \nabla\}$ of X by semi-open sets of X, there exist a finite subset ∇_0 of ∇ s.t

$$X = \bigcup_{\alpha \in \nabla_0} Cl V_\alpha$$

Note:- Every S-closed space is s-closed and every s-closed space is s-compact and every s-compact space is compact.

★ s-Regular Space:- (Already Defined)

★ **Theorem:-** A topological space (X,τ) is s-closed if and only if every proper semi-regular subset of X is s-closed relative to X.

Proof) Let (X,τ) be s-closed space.
And $G \subseteq X$ be a proper semi-regular subset of X.
We prove that G is s-closed relative to X.
Let $\{V_\alpha : \alpha \in \nabla\}$ be a cover for G, where $V_\alpha \in SO(X)$ $\forall \alpha \in \nabla$

$$\Rightarrow G \subseteq \bigcup_{\alpha \in \nabla} V_\alpha$$

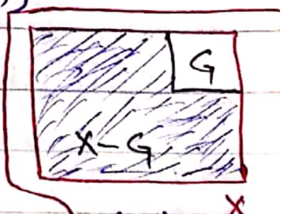
$$\Rightarrow X = \bigcup_{\alpha \in \nabla} V_\alpha \cup (X-G)$$

$\because G$ is semi-regular
$\Rightarrow G$ is semi open as well as semi closed

where $X - G \in SO(X)$
Since X is s-closed, therefore there exist a finite subset ∇_0 of ∇_0 s.t

$$X = \bigcup_{\alpha \in \nabla_0} scl(V_\alpha) \cup scl(X-G)$$

$$\Rightarrow G \cap X = G \cap \left[\bigcup_{\alpha \in \nabla_0} scl(V_\alpha) \cup (X-G)\right]$$



$(X-G) \cup \{\} = X$
$\Rightarrow G \subseteq \{\}$

$$\text{or} \quad G \subseteq \bigcup_{\alpha \in \nabla_0} scl(V_\alpha)$$

$\Rightarrow G$ is s-closed relative to X.

Conversely:- Let every proper semi-regular subset of X be s-closed relative to X.

We prove that X is s-closed.

Let $\{V_\alpha : \alpha \in \nabla\}$ be a cover of X by sets semi-open in X.

For some $\beta \in \nabla$, $scl(V_\beta) \in sR(X)$

Let $G = scl(V_\beta) \in sR(X)$

$$\Rightarrow X - G \in sR(X)$$

By Hypothesis, $X - G$ is s-closed relative to X.

Since, $X - G \subseteq \bigcup\{V_\alpha : \alpha \in \nabla\}$

Hypothesis $\Rightarrow$ $X - G = \bigcup_{\alpha \in \nabla_0} scl V_\alpha$ for some finite set ∇_0 of ∇

$$\Rightarrow X = \bigcup_{\alpha \in \nabla_0} scl V_\alpha \cup scl V_\beta$$

$$= \bigcup_{\alpha \in \nabla_0 \cup \{\beta\}} scl V_\alpha$$

This proves that X is s-closed space.

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★ **Exercise:-** Let A and B be subsets of a topological space (X,τ) s.t $A \subset B \subset X$ & $B \in SO(X)$. If A is s-closed relative to X then prove that A is s-closed relative to B.

Proof

Let $\{V_\alpha : \alpha \in \nabla\}$ be a cover for A where $V_\alpha \in SO(B)$ $\forall \alpha \in \nabla$

$$\Rightarrow A \subseteq \bigcup_{\alpha \in \nabla} V_\alpha$$

As $B \in SO(X)$

$$\Rightarrow A \subseteq \bigcup_{\alpha \in \nabla} V_\alpha \quad \text{s.t } V_\alpha \in SO(X) \ \forall \alpha \in \nabla$$

As A is s-closed relative to X, therefore there exist a finite subset ∇_0 of ∇ s.t

$$A = \bigcup_{\alpha \in \nabla_0} scl(V_\alpha)$$

$$\Rightarrow A \cap B = \bigcup_{\alpha \in \nabla_0} scl(V_\alpha) \cap B$$

$$\Rightarrow A = \bigcup_{\alpha \in \nabla_0} scl_B(V_\alpha),$$

where $V_\alpha \in SO(B)$

$\Rightarrow A$ is s-closed relative to B.

* **Almost Open Mapping:-** A mapping $f : (X, \tau_X) \to (Y, \tau_Y)$ is said to be almost open if for every open set U of Y,

$$f^{-1}(Cl(U)) \subseteq Cl(f^{-1}(U))$$

⇒ Note that every open mapping is almost open mapping. The converse is not true in general.

⇒ Note that the composition of two almost open mappings is not almost mapping in general.

Example:- $X = Y = Z = \{a, b, c\}$

$\tau_X = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$

$\tau_Y = \{\phi, \{a\}, \{a,b\}, Y\}$

$\tau_Z = \{\phi, \{c\}, Z\}$

$f : X \to Y$ be identity mapping

$g : Y \to Z$ be defined by

$g(a) = b$, $g(b) = c$, $g(c) = c$

then f and g are almost open mappings but $g \circ f$ is not almost open.

* **Almost Closed Mapping:-** A mapping $f : (X, \tau_X) \to (Y, \tau_Y)$ is said to be almost closed if for every closed set V of Y

$$Int[f^{-1}(V)] \subseteq f^{-1}[Int(V)]$$

⇒ **Theorem:-** Let $f:(X,\tau_X) \longrightarrow (Y,\tau_Y)$ and $g:(Y,\tau_Y) \longrightarrow (Z,\tau_Z)$ be almost open mappings, Prove that $g\circ f$ is almost open if g is continuous.

Proof

Let U be an open set of Z

As $g: Y \to Z$ is continuous so
$g^{-1}(U)$ is open in Y

Now as $f: X \to Y$ is almost open mapping and $g^{-1}(U)$ is open in Y

$$\Rightarrow f^{-1}[cl\, g^{-1}(U)] \subseteq cl[f^{-1}(g^{-1}(U))] \longrightarrow (*)$$

Since $g: Y \to Z$ is almost open mapping and U is open in Z

$$\Rightarrow \quad g^{-1}[cl(U)] \subseteq cl[g^{-1}(U)]$$

$$\Rightarrow f^{-1}\{g^{-1}\, cl(U)\} \subseteq f^{-1}[cl\, g^{-1}(U)]$$

Put in equ $(*)$ implies

$$f^{-1}\{g^{-1}\, cl(U)\} \subseteq f^{-1}[cl\, g^{-1}(U)] \subseteq cl[f^{-1} g^{-1}(U)]$$

$$\Rightarrow f^{-1}[g^{-1}\, cl(U)] \subseteq cl[f^{-1}(g^{-1}(U))]$$

$$\Rightarrow (f^{-1}\circ g^{-1})\, cl(U) \subseteq cl(f^{-1}\circ g^{-1})(U)$$

$$\Rightarrow (g\circ f)^{-1}\, cl(U) \subseteq cl(g\circ f)^{-1}(U)$$

Now as U is open set in Z and
$$(g\circ f)^{-1}[cl(U)] \subseteq cl[(g\circ f)^{-1}(U)]$$

⇒ $g\circ f$ is an almost open mapping.

* **s-Normal Space:-** A space (X,τ) is said to be s-normal if for every pair of disjoint closed sets A and B of X, there exist disjoint semi open sets U and V s.t

$$A \subset U, \quad B \subset V$$

Note:- $A \subseteq X$ is semi-closed in X iff $Int[Cl(A)] = Int(A)$

Theorem:- Let $f: X \to Y$ be a continuous semi-closed function. If X is normal then Y is s-normal.

Proof

Let F_1 and F_2 be disjoint closed sets of Y. Since f is continuous therefore $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are disjoint closed sets of X.

As X is normal, therefore there exist disjoint open sets U_1 and U_2 in X s.t

$$f^{-1}(F_1) \subseteq U_1 \quad \text{and} \quad f^{-1}(F_2) \subseteq U_2$$

$$\& \quad U_1 \cap U_2 = \phi$$

Since f is semi-closed, therefore there exist two semi open sets V_1 and V_2 in Y containing F_1 and F_2 respectively s.t

$$f^{-1}(V_1) \subseteq U_1 \quad \& \quad f^{-1}(V_2) \subseteq U_2$$

since $U_1 \cap U_2 = \phi$

$\Rightarrow f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$

$\Rightarrow V_1 \cap V_2 = \phi$

i.e for two disjoint closed sets F_1 and F_2 of Y there exist two semi open sets V_1 and V_2 in Y s.t

$F_1 \subseteq V_1$ & $F_2 \subseteq V_2$

and $V_1 \cap V_2 = \phi$

$\Rightarrow Y$ is s-normal Proved.

* **Semi-T_2 Space:** A space (X, τ_X) is said to be Semi-T_2 space if for $x_1, x_2 \in X$ s.t $x_1 \neq x_2$ there exist U and $V \in SO(X)$ s.t $x_1 \in U$, $x_2 \in V$

and $U \cap V = \phi$