ADVANCED PARTIAL DIFFERENTIAL EQUATIONS-I

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⇒ **PDE :-** A PDE is an equation involving unknown functions of two or more variables and certain of its partial derivatives.
A general linear PDE of order 1 in two space dimension has the form

$$a u_x + b u_y + c u + d = 0,$$

where $u_x = \partial u / \partial x$ and $u_y = \partial u / \partial y$ are partial derivatives and a, b, c, d are known co-efficients which can also depend on (x, y)

⇒ **Homogeneous PDE :-** A PDE is said to be homogeneous if its all terms contains the unknown function u or its derivatives.
Check:- plug in $u=0$ into the PDE if it satisfies the PDE, Then the PDE is homogeneous e.g

$$u_{xx} + x u_{xy} + y u^2 = 0 \quad \text{(Homogeneous)}$$

$$3u_x + u u_y = f(x,y) \quad \text{(Non-Homogeneous)}$$

⇒ **Solution of a PDE:-** Function $u = u(x, y, \dots)$ that satisfy a PDE, $F(x, y, \dots, u, u_x, u_y, \dots, u_{xx}, \dots) = 0$ ⊛ in a suitable domain D is called a solution of PDE ⊛.
Example:- $u(x,y) = (x+y)^3$ & $u(x,y) = \sin(x-y)$ are solutions of the PDE $u_{xx} - u_{yy} = 0$

* If u_1 and u_2 are two solutions of a linear homogeneous PDE, Then their combination $c_1 u_1 + c_2 u_2$ is also a solution of that PDE. This is known as principle of superposition.

★ If u_1 and u_2 are two solutions of a non-homogeneous PDE. Then $u_1 - u_2$ is also a solution of corresponding homogeneous PDE.

To see this consider the general form of a non-homogeneous PDE.

$$au + \sum_{i=1}^{\infty} b_i u x_i + \cdots = f(x) \quad , x = x(x_1, x_2, \ldots) \longrightarrow (*)$$

If u_1 and u_2 are solutions of $(*)$ then

$$\left.\begin{aligned} &a u_1 + \sum_{i=1}^{n} b_i (u_1) x_i + \cdots = f(x) \\ \& \; &a u_2 + \sum_{i=1}^{n} b_i (u_2) x_i + \cdots = f(x) \end{aligned}\right\} \text{satisfies}$$

$$\Rightarrow a(u_1 - u_2) + \sum_{i=1}^{n} b_i (u_1 - u_2) x_i + \cdots = 0$$

$\Rightarrow u_1 - u_2$ is a solution of corresponding homogeneous PDE.

★ The number of independent solutions of PDE are infinite.

★ The boundries of the region of the independent variable over which we desire to solve the PDE are not discrete points (as in one dimentional case) but are continuous curves or surfaces.

★ Thus the complete formation of a PDE of physical system in terms of region careful attention

⇒ not only the equations that govern the system but also to the correct formulation of the boundry conditions.

* Furthermore, most differential equations that we encounter in applications are mathematical expression of physical laws (for example the heat equation is the expression of the law of energy conservation)

* Therefor, in order to obtain a unique solution we must specify the initial conditions in addition to the boundry conditions.

⇒ **Initial Conditions:-** Conditions at an initial time $t = t_0$ from which a given set of mathematical equations or physical system evolves are known as initial conditions. A system with initial conditions specified is known as the initial value problem.

Alternatively, if the equations relates one value of the independent variable such as $u(x_0) = A$ and $u_x(x_0) = B$ then these conditions are called initial conditions and x_0 is called the initial point.

Example 1:- As a simple example, we suppose that our unknown function u is dependent on one variable x. Thus the following problem is known as initial value problem $u_{xx}+u_x-2u=0$, $u(0)=3$ & $u_x(0)=7$

Example 2:- Now we suppose that our un-known function u is dependent on two variables t, x. Then we have the following initial value problem.

$$u_{xx}+u_{tt}-2u=0, \quad u(0,x)=3x, \quad u_t(0,x)=\sin x$$

⇒ **Boundry Conditions:-** The set of conditions specified for the behaviour of the solution to the set of differential equations or partial differential equations at the boundry of its domain are known as Boundry Conditions. A system with boundry conditions is known as the boundry value problem.

Alternatively, The problem of finding the solution of a differential equation such that all the associated conditions relate to two different values of the independent variable is called a boundry value problem.

Example:- If $u(x,t)$ is the displacement of a vibrating string and its ends are fixed ~~at~~ $x=a$ and $x=b$, then the conditions

$$u(a,t)=0 \quad , \text{ and } \quad u(b,t)=0$$

are boundry conditions.

⇒ Solution of a Bondry Value Problem

By a solution to a boundry value problem on an open region D, we mean a function u that satisfies the differential equation on D and its is continuous on $D \cup \partial D$, and satisfies the specified boundry conditions on ∂D.

⇒ Linear Boundry Conditions:-

The boundry conditions are linear if they express a linear relationship between u and its partial derivatives (up to appropriate order) on ∂D. (In other words, a boundry condition is linear if it is expressed as a linear equation between u and its derivative on ∂D.)

⇒ Classification of Boundry Conditions:-

1) **Drichlet Conditions:-** The boundry conditions satisfy the values of the unknow function u on the boundry. This type of boundry conditions is called the Drichlet condition.

2) **Neumann Conditions:-** The boundry conditions specify the derivatives of the unknown function u in the direction normal to the boundry, which is write as $\partial u / \partial n$. This type of boundry condition is called Neumann condition.

Remark:- The normal derivative on the boundry $\partial u/\partial n$ is defined as

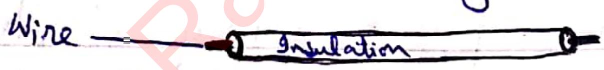
$$\frac{\partial u}{\partial n} = \text{grad}\, u \cdot n = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n}\right) \cdot n,$$

where n is the outward normal to ∂D.

3. Mixed Conditions OR Robin's Boundry Conditions:- The boundry conditions specify a linear relationship between u and its normal derivative on the boundry. These are reffered to as mixed boundry conditions are Robin's Boundry Conditions. The general form of such a boundry condition is

$$\left[\alpha u + \beta \frac{\partial u}{\partial n}\right]\Big|_{\partial D} = f(x)\Big|_{\partial D} \; ; \; \alpha, \beta \text{ are constants.}$$

Example:- Consider the problem of heat conduction in a laterally insulated thin wire.

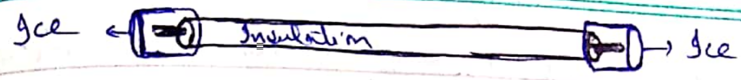


$u(x,t)$ is the temperature in the wire; the constant k is the diffusivity, which indicates the rate of diffusion of heat along the wire, and the length of the wire is L.
The initial condition is $u(x,0) = F(x)$, where $F(x)$ is the initial temperature distribution in the wire.

The three major types of boundry conditions are as follows.

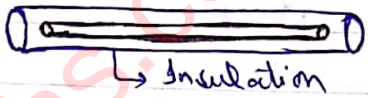
1):- Immerse the wire in melting ice ($0°c$) at each end point and let $u(x,t)$ be measured in $°C$

$$u(0,t) = u(L,t) = 0 \quad , \quad \text{for } t > 0$$



These are drichlet or fixed boundry conditions. Alternatevely, prescribed the temperature of each end point to be $p(t)$ and $q(t)$, respectively: $u(0,t) = p(t)$, $u(L,t) = q(t)$, for $t > 0$. These also are Drichlet, or fixed, boundry conditions.

2- Insulate each end point, Thus the wire is totally insulated.



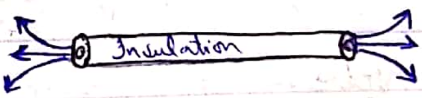
$u_x(0,t) = u_x(L,t) = 0$, $t > 0$

These are Neumann, or free boundry conditions.

Alternatively, prescribe the the flow of heat at each end point to be $p(t)$ and $q(t)$, respectively: $u_x(0,t) = -\frac{p(t)}{K}$, and $u_x(L,t) = -\frac{q(t)}{K}$, for $t > 0$

$K > 0$ is the thermal conductivity. These also are Neumann, or free, boundry conditions.

3- Each end point is exposed and radiates heat into the surroundings medium which has a temperature of $T(t)$:



$\beta u_x(0,t) = \alpha[u(0,t) - T(t)]$,

and $\gamma u_x(L,t) = -\delta[u_x(L,t) - T(t)]$, for $t > 0$

which specify to

$\alpha u(0,t) - \beta u_x(0,t) = \alpha T(t)$ and

$\delta u(L,t) + \gamma u_x(L,t) = \delta T(t)$, for $t > 0$

where α, β, γ are positive constants. These are Robin, or mixed, boundry conditions. If the surrounding medium has a temperature of $0°C$ (i.e $T(t) = 0$), and $u(x,t)$ is measured in °C, then we have

$$\alpha u(0,t) - \beta u_x(0,t) = 0 \quad \text{and}$$

$$\gamma u(L,t) + \delta u_x(L,t) = 0 \quad , \text{for } t>0$$

⇒ Superposition Principle:-

Superposition Principle for Linear Boundry Conditions:-

Theorem:- If u_1 and u_1 are solutions of a linear homogeneous partial differential equation with linear boundry conditions

$$\left[\alpha u_1(x) + \beta \frac{\partial u_1(x)}{\partial n}\right]\Big|_{\partial D} = f(x)\Big|_{\partial D}$$

$$\left[\alpha u_2(x) + \beta \frac{\partial u_2(x)}{\partial n}\right]\Big|_{\partial D} = g(x)\Big|_{\partial D}$$

where α, β are constants, then $\omega = u_1 + u_2$ is a solution of the partial differential equation that satisfy the boundry conditions

$$\left[\alpha \omega(x) + \beta \frac{\partial \omega(x)}{\partial n}\right]\Big|_{\partial D} = (f(x) + g(x))\Big|_{\partial D}$$

Note:- The above result is particularly useful in applications in which the boundry conditions are complex.

Example:- Consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

In rectangle with the following

linear boundry conditions

$u(x,0) = f_1(x)$, $\quad u(0,y) = g_1(y)$

$u(x,b) = f_2(x)$, $\quad u(a,y) = g_2(y)$

We splip the problem in two parts

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0 \qquad \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

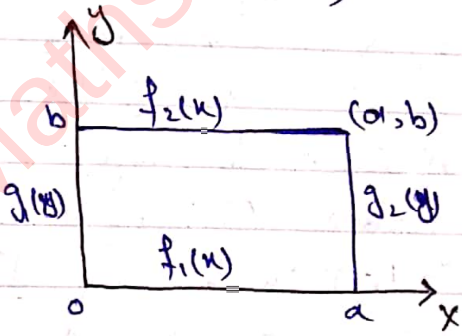
$u_1(x,0) = f_1(x)$ $\qquad u_2(x,0) = 0$

$u_1(x,b) = f_2(x)$ $\qquad u_2(x,b) = 0$

$u_1(0,y) = 0$ $\qquad u_2(0,y) = g_1(y)$

$u_1(a,y) = 0$ $\qquad u_2(a,y) = g_2(y)$

obviously, if we solve u_1, u_2, then $u_1 + u_2$ is a solution of Laplace equation, which satisfy all the boundry conditions.



Note:- Neumann boundry conditions usually do not specify the unique solution of a. boundry value problem.

Example:- Consider the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with neumann boundry conditions

$\frac{\partial u}{\partial y}(x,0) = f_1(x)$, $\qquad \frac{\partial u}{\partial y}(x,b) = f_2(x)$

$\frac{\partial u}{\partial x}(0,y) = g_1(y)$ $\qquad \frac{\partial u}{\partial x}(a,y) = g_2(y)$

It is obvious that if u is the solution of this boundry value problem, then $\omega = u + c$ (c is constant) is also a solution of above boundry value problem. Thus Neumann boundry conditions determine the solution of this boundry value problem up to a constant.

⇒ Formation of Partial Differential Equation:-

Suppose u, v are two given functions of x, y and z. Let F be an arbitrary function of u and v of the form $F(u,v)$, or $F[u(x,y,z), v(x,y,z)]$

A differential equation can be formulated by eliminating the arbitrary function F. Taking partial derivatives of $F(u,v)$ with respect to x and y and taking z is a function of x and y we obtain

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right] + \frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right] = 0$$

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\,\frac{\partial z}{\partial y}\right] + \frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right] = 0$$

eliminating $\frac{\partial F}{\partial u}$ and $\partial F/\partial v$ from above equation, we obtain

$$Pp + Qq = R,$$

where $P = \frac{\partial(u,v)}{\partial(y,z)}$, $Q = \frac{\partial(u,v)}{\partial(z,x)}$, $R = \frac{\partial(u,v)}{\partial(x,y)}$

and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

★ The first order Partial Differential equ. $Pp + Qq = R$, is called Lagrang's PD equation of first order.

Assignment

Form the partial Differential equation by eliminating the arbitrary function from

1. $z = f(x+it) + g(x-it)$, where $i = \sqrt{-1}$
2. $f(x+y+z, x^2+y^2+z^2) = 0$
3. $z = xy + f(x^2+y^2)$
4. $z = f(xy/z)$ (5) $z = ax+by+ab$

Solutions:- Q.2:- $f(x+y+z, x^2+y^2+z^2) = 0$

Partial Differential equation is

$$Pp + Qq = R, \longrightarrow (*)$$

where $P = \dfrac{\partial(u,v)}{\partial(y,z)}$, $Q = \dfrac{\partial(u,v)}{\partial(x,z)}$

$R = \dfrac{\partial(u,v)}{\partial(x,y)}$, $p = \dfrac{\partial z}{\partial x}$ & $q = \dfrac{\partial z}{\partial y}$

Here $u = x+y+z$ & $v = x^2+y^2+z^2$

$$P = \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial}{\partial y}(x+y+z) & \frac{\partial}{\partial z}(x+y+z) \\ \frac{\partial}{\partial y}(x^2+y^2+z^2) & \frac{\partial}{\partial z}(x^2+y^2+z^2) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 2y & 2z \end{vmatrix} = (2z - 2y)$$

$$Q = \frac{\partial(u,v)}{\partial(x,z)} = \begin{vmatrix} \frac{\partial}{\partial x}(x+y+z) & \frac{\partial}{\partial z}(x+y+z) \\ \frac{\partial}{\partial x}(x^2+y^2+z^2) & \frac{\partial}{\partial z}(x^2+y^2+z^2) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 2x & 2y \end{vmatrix} = (2y - 2x)$$

similarly $R = (2y-2x)$ Put all in ⊛

$$\Rightarrow (2z-2y)p + (2z-2x)q = (2y-2x)$$

$$\Rightarrow 2[(z-y)p + (z-x)q] = 2(y-x)$$

$$\Rightarrow \underline{(z-y)p + (z-x)q = y-x}$$ Ans.

⇒ Classification of 2nd-Order Linear Equations:-

"When we have good understanding of the problem, we are able to clear it of all auxiliary notions and to reduce it to simplest element."

René Descartes

"The first process... in effectual study of sciences must be one of simplification and reduction of the results of previous investigations to a form in which the mind can grasp them. James Clerk Maxwell

⇒ **Second Order Equation in Two Independent variables**:- The general linear second order partial differential equation in one dependent variable u may be written as

$$\sum_{i,j=1}^{n} A_{ij}\, u_{x_i x_j} + \sum_{i=1}^{n} B_i\, u_{x_i} + Fu = G \longrightarrow (1)$$

In which we assume $A_{ij} = A_{ji}$ and A_{ij}, B_i, F and G are real valued functions defined in some region of the space $(x_1, x_2, \dots, x_n)$

Here we shall be concerned with second order equations in the dependent variable u and the independent variables x, y. Hence equ (1) can be put in the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \longrightarrow (2)$$

where the coefficients are functions of

x and y, or constants, and do not vanish simultaneously. We shall assume that the function u and the coefficients are twice continuously differentiable in R^2.

The classification of partial differential equation is suggested by the classification of the quadratic equation of conic section in analytic geometry. The equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents hyperbola, parabola or ellipse accordingly as $B^2 - 4ac$ is positive, zero or negative.

The classification of second order equation is based upon the possibility of reducing eqn ② by coordinate transformation to canonical or standard form at a point. An equation is said to be hyperbolic, parabolic or elliptic at a point (x_0, y_0) accordingly as

$$B^2(x_0, y_0) - 4A(x_0, y_0)\,C(x_0, y_0) \longrightarrow ③$$

is +ve, zero, or -ve. If this is true for all points, then the equation is said to be hyperbolic, parabolic or elliptic in a domain.

In the case of two independent variables, a transformation can always be found to reduce the given equation to canonical form in a given domain. However, In the case of several independent variables, it is not, in general, possible to find such a transformation.

To transform equ ② to a canonical form we make a change of independent variables. Let the new variable be

$$\xi = \xi(x,y) \quad , \quad \eta = \eta(x,y) \longrightarrow ④$$

Assuming that ξ and η are twice continuously differentiable and that the jacobian

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \longrightarrow ⑤$$

is non-zero in the region under consideration, then x and y can be determined uniquely from the system. Let x and y be twice continuously differentiable function of ξ and η. Then we have.

$$u_x = u_\xi \xi_x + u_\eta \eta_x \quad , \quad u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}(\xi_x\eta_x) + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy} \longrightarrow ⑥$$

Substituting these values in equ ① we obtain

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^* \longrightarrow ⑦$$

where $A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$

$$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

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$$\left.\begin{array}{l} C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\ D^* = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\ E^* = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\ F^* = F \quad \& \quad G^* = G \end{array}\right] \longrightarrow (8)$$

The resulting equation (7) is the same form as the original equation under the general transformation (4). The nature of the equation remains invariant under such a transformation if the Jacobian does not vanish. This can be seen from the fact that the sign of the discreminant does not alter under the transformation i.e

$$B^{*2} - 4A^*C^* = J^2(B^2 - 4AC) \longrightarrow (9)$$

Which can be easily varified. It should be noted here that the equation can be a different points of the domain, but for our purpose we shall assume that the equation under considration is of the single type in a given domain.

The classification of equation (2) depends on the coefficients $A(x,y)$, $B(x,y)$ and $C(x,y)$ at a given point (x,y). we shall therefore rewrite equ (2) as

$$Au_{xx} + Bu_{xy} + Cu_{yy} = H(x,y,u,u_x,u_y) \longrightarrow (10)$$

and eqn (7) as

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} = H^*(\xi,\eta,u,u_\xi,u_\eta) \longrightarrow (11)$$

⇒ **Canonical Forms:-** In this section we shall consider the problem of reducing equ (10) to canonical form.

We suppose first that none of A, B, C is zero. Let ξ and η be new variable such that the coefficients A^* & C^* in equation (11) vanish. Thus from (8) we have

$$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

$$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$$

These two equations are of the same type and hence we may write them in the form

$$A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2 = 0 \longrightarrow (4.2.1)$$

in which ζ stand for either of the functions ξ or η. Dividing through by ζ_y^2, equation (4.2.1) becomes

$$A\left(\frac{\zeta_x}{\zeta_y}\right)^2 + B\left(\frac{\zeta_x}{\zeta_y}\right) + C = 0 \longrightarrow (4.2.2)$$

Along the curve ζ = constant, we have

$$d\zeta = \zeta_x dx + \zeta_y dy = 0$$

Thus

$$\frac{dy}{dx} = -\frac{\zeta_x}{\zeta_y} \longrightarrow (4.2.3)$$

and therefore, equ (4.2.2) may be written in the form

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \longrightarrow (4.2.4)$$

The roots of which are

$$\frac{dy}{dx} = \left(B + \sqrt{B^2 - 4AC}\right)/2A \longrightarrow (4.2.5)$$

$$\frac{dy}{dx} = \left(B - \sqrt{B^2 - 4AC}\right)/2A \longrightarrow (4.2.6)$$

These equations, which are known as the characteristic equations, or ordinary differential equations for families of curves in the xy-plane along which ξ = constant and η = constant. The integral of equation (4.2.5) and (4.2.6) are called the characteristic curves. Since the equations are first order ordinary differential equs, the solution may be written as

$$\phi_1(x,y) = c_1 \quad , \quad c_1 = \text{constant}$$

$$\phi_2(x,y) = c_2 \quad , \quad c_2 = \text{constant}$$

Hence the transformation

$$\xi = \phi_1(x,y) \quad , \quad \eta = \phi_2(x,y)$$

will transform equ 10 to a canonical form.

(A) Hyperbolic Type:- If $B^2 - 4AC > 0$, Then integration of equ (4.2.5) & (4.2.6) yield two real & distinct families of characristics. Equation 11 reduces to

$$u_{\xi\eta} = H_1 \longrightarrow (4.2.7)$$

where $H_1 = H^*/B^*$. It can be easily shown that $B^* \neq 0$. This form is called the first canonical form of the hyperbolic equation. Now if new independent variables

$$\alpha = \xi + \eta \quad , \beta = \xi - \eta \longrightarrow (4.2.8)$$

are introduced, Then equ (4.2.7) is transformed into

$$u_{\alpha\alpha} - u_{\beta\beta} = H_2(\alpha, \beta, u, u_\alpha, u_\beta) \longrightarrow (4.2.9)$$

This form is called the 2nd canonical form of the hyperbolic equation.

(B) Parabolic Type:- In this case we have $B^2 - 4AC = 0$, and equation (4.2.5) & (4.2.6) coincide. Thus there exist one real family of characteristics, and we obtain only a single integral $\xi = \text{constant}$ (or $\eta = \text{constant}$) Since $B^2 = 4AC$ and $A^* = 0$, we find that

$$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = (\sqrt{A}\,\xi_x + \sqrt{C}\,\xi_y)^2 = 0$$

From this it is follows that

$$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$= 2(\sqrt{A}\,\xi_x + \sqrt{C}\,\xi_y)(\sqrt{A}\,\eta_x + \sqrt{C}\,\eta_y) = 0$$

for abitrary value of $\eta(x,y)$ which is functionally independent of $\xi(x,y)$; for instance if $\eta = y$. the Jacobian does not vanish in the domain of parabolicity.

Division of equation (11) by C^* yields

$$u_{\eta\eta} = H_3(\xi, \eta, u, u_\xi + u_\eta) \;,\; C^* \neq 0 \longrightarrow (4.2.10)$$

This is called the canonical form of the parabolic equation. Eqn (11) may also assume the form $u_{\xi\xi} = H_3^*(\xi, \eta, u, u_\xi, u_\eta)$ $\longrightarrow$ (4.2.11)

If we choose η = constant as the integral of equation (4.2.5)

(c) **Elliptic Type:-** For an equation of elliptic type, we have $B^2 - 4AC < 0$. Consequently the quadratic equation (4.2.4) has no real solutions, but it has two complex conjugate solutions which are continuous complex-valued functions of the real variable x and y. Thus, in this case, there are no real characteristic curves. However if the co-efficients A, B and C are analytic functions of x and y, then one can consider equation (4.2.4) for complex x and y. A function of two real variables x and y is said to be analytic in a certain domain if in some neighbourhood of every point (x_0, y_0) of this domain the function can be represented as a taylor series in the variable $(x - x_0)$ and $(y - y_0)$.

Since ξ and η are complex, we introduce new real variables

$$\alpha = \frac{1}{2}(\xi + \eta), \quad \beta = \frac{1}{2i}(\xi - \eta) \longrightarrow (4.2.12)$$

so that $\xi = \alpha + i\beta$, $\eta = \alpha - i\beta$ $\longrightarrow$ (4.2.13)

First, we transform eqn (10), we then have

$$A^{**}(\alpha,\beta)u_{\alpha\alpha}+B^{**}(\alpha,\beta)u_{\alpha\beta}+C^{**}(\alpha,\beta)u_{\beta\beta}=H_4(\alpha,\beta,u,u_\alpha,u_\beta) \longrightarrow (4.2.14)$$

In which the coefficients assume the same form as the coefficients in equ (11). With the use of (4.2.13), the equs $A^* = C^* = 0$ become

$$[A\alpha_x^2+B\alpha_x\alpha_y+C\alpha_y^2]-[A\beta_x^2+B\beta_x\beta_y+C\beta_y^2]$$
$$+i[2A\alpha_x\beta_x+B(\alpha_x\beta_y+\alpha_y\beta_x)+2C\alpha_y\beta_y]=0$$

$$(A\alpha_x^2+B\alpha_x\alpha_y+C\alpha_y^2)-(A\beta_x^2+B\beta_x\beta_y+C\beta_y^2)$$
$$-i[2A\alpha_x\beta_x+B(\alpha_x\beta_y+\alpha_y\beta_x)+2C\alpha_y\beta_y]=0$$

or

$$(A^{**}-C^{**})+iB^{**}=0, \quad (A^{**}-C^{**})-iB^{**}=0$$

these equations are satisfied if

$$A^{**}=C^{**} \quad \& \quad B^{**}=0$$

Hence equ (4.2.14) transforms into the form

$$A^{**}u_{\alpha\alpha}+A^{**}u_{\beta\beta}=H_4(\alpha,\beta,u,u_\alpha,u_\beta)$$

Dividing through by A^{**}, we obtain

$$u_{\alpha\alpha}+u_{\beta\beta}=H_5(\alpha,\beta,u,u_\alpha,u_\beta) \longrightarrow (4.2.15)$$

Here $H_5 = (H_4/A^{**})$ this is called the canonical form of the elliptic equation.

Example 1:- Consider the equation

$$y^2u_{xx}-x^2u_{yy}=0$$

Here $A=y^2$, $B=0$, $C=-x^2$

Thus $B^2-4AC=4x^2y^2>0$

This equation is hyperbolic everywhere except on the coordinate axis $x=0$ and $y=0$. From the characteristic equation (4.2.5) and (4.2.6), we have

$$\frac{dy}{dx} = \frac{x}{y}, \quad \frac{dy}{dx} = \frac{-x}{y} \qquad \because \frac{dy}{dx} = \frac{B \pm \sqrt{B^2-4AC}}{2A}$$

After integrating of these equations, we obtain

$$\frac{1}{2}y^2 - \frac{1}{2}x^2 = c_1, \quad \frac{1}{2}y^2 + \frac{1}{2}x^2 = c_2$$

The first of these curves is a family of hyperbolas. $\frac{1}{2}y^2 - \frac{1}{2}x^2 = c_1$

And the second is a family of circles

$$\frac{1}{2}y^2 + \frac{1}{2}x^2 = c_2$$

To transform the given equation to canonical form, we consider

$$\xi = \frac{1}{2}y^2 - \frac{1}{2}x^2, \quad \eta = \frac{1}{2}y^2 + \frac{1}{2}x^2$$

from relation (B) we have

$$u_x = u_\xi \xi_x + u_\eta \eta_x = -x u_\xi + x u_\eta$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = y u_\xi + y u_\eta$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$= x^2 u_{\xi\xi} - 2x^2 u_{\xi\eta} + x^2 u_{\eta\eta} - u_\xi + u_\eta$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$$

$$= y^2 u_{\xi\xi} + 2y^2 u_{\xi\eta} + y^2 u_{\eta\eta} + u_\xi + u_\eta$$

Thus the given equation assumes the canonical form

$$u_{\xi\eta} = \frac{\eta}{2(\xi^2-\eta^2)}u_\xi - \frac{\xi}{2(\xi^2-\eta^2)}u_\eta$$

Example:-2 Consider the partial differential equation $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=0$

In this case the discriminent is

$$B^2-4AC = 4x^2y^2-4x^2y^2 = 0$$

The equation is therefore parabolic everywhere, the characteristic equation is

$$\frac{dy}{dx} = y/x$$

and hence the characteristics are $y/x = c$, which is the equation of a family of straight lines.

Consider the transformations

$$\xi = y/x, \eta = y$$

η ایسا لیں کہ Jacobian
zero نہ ہو (η ذرا لیں).

where η is chosen arbitrarily.

the given equation is then reduces to the canononical form

$$y^2u_{\eta\eta} = 0$$

Thus $u_{\eta\eta} = 0$ for $y \neq 0$

Example:-3 The equation $u_{xx}+x^2u_{yy}=0$

is elliptic everywhere except on the coordinate axis $x=0$ because

$$B^2-4AC = -4x^2 < 0 \quad , x \neq 0$$

The characteristic equations are

$$\frac{dy}{dx} = ix \quad , \quad \frac{dy}{dx} = -ix$$

Integration yields

$$2y - ix^2 = c_1 \quad , \quad 2y + ix^2 = c_2$$

Thus if we write

$$\xi = 2y - ix^2 \quad , \quad \eta = 2y + ix^2$$

and hence

$$\alpha = \frac{1}{2}(\xi + \eta) = 2y$$

$$\beta = \frac{1}{2i}(\xi - \eta) = -x^2$$

we obtain the canonical form

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2\beta} u_\beta$$

It should be remarked here that a given PDE may be of a different type in a different domain. Thus, for example, Tricomi's equation $u_{xx} + x u_{yy} = 0 \longrightarrow (4.2.16)$ is elliptic for $x > 0$ and hyperbolic for $x < 0$, since $B^2 - 4AC = -4x$.

Explanation:-

In eq (4.2.7) $\quad u_{\xi\eta} = H_1 \quad , \quad H_1 = \dfrac{H^*}{B^*}$

where $\quad H^* = G^* - D^* u_\xi - E^* u_\eta - F^* u$

$$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\eta_y\xi_y$$

$$\Rightarrow H_1 = \frac{H^*}{B^*} = \frac{G^* - D^* u_\xi - E^* u_\eta - F^* u}{2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y}$$

$$\Rightarrow H_1 = \frac{G - (A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y)u_\xi - (A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y)u_\eta - Fu}{2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y}$$

Now In Example 1 (By * Method)

$$y^2 u_{xx} - x^2 u_{yy} = 0$$

Transformations

$$\xi = \frac{1}{2}y^2 - \frac{1}{2}x^2 \rightarrow (1) \quad , \quad \eta = \frac{1}{2}y^2 + \frac{1}{2}x^2 \rightarrow (2)$$

$\xi_x = -x$, $\xi_{xx} = -1$ | $\eta_x = x$, $\eta_{xx} = 1$

$\xi_y = y$, $\xi_{yy} = 1$ | $\eta_y = y$, $\eta_{yy} = 1$

$\xi_{xy} = 0$ | $\eta_{xy} = 0$

Here $A = y^2$, $C = -x^2$

Adding (1) & (2) $\Rightarrow$

$$\xi + \eta = \frac{1}{2}y^2 - \cancel{\frac{1}{2}x^2} + \frac{1}{2}y^2 + \cancel{\frac{1}{2}x^2} = y^2 = A$$

$$\Rightarrow A = \xi + \eta$$

(1) − (2) $\Rightarrow$

$$\xi - \eta = \cancel{\frac{1}{2}y^2} - \frac{1}{2}x^2 - \cancel{\frac{1}{2}y^2} - \frac{1}{2}x^2 = -x^2 = C$$

$$\Rightarrow C = \xi - \eta$$

Canonical form is

$$u_{\xi\eta} = H_1 = \frac{G - (A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y)u_\xi - (A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y)u_\eta - Fu}{2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y}$$

$$\Rightarrow u_{\xi\eta} = \frac{\cancel{0-((\xi+\eta)(-1)+0+(\xi-\eta)(1)+0+0)u_\xi} \; -[(\xi+\eta)(1)+0+(\xi-\eta)(1)+0+0)u_\eta - 0}{2(\xi+\eta)(-x)(x)+0+2(\xi-\eta)(y)(y)}$$

$$= \frac{[(\xi+\eta)-(\xi-\eta)]u_\xi - [(\xi+\eta)+(\xi-\eta)]u_\eta}{2(\xi+\eta)(\xi-\eta)+2(\xi-\eta)(\xi+\eta)} \qquad \because x^2=\xi-\eta, \; y^2=\xi+\eta$$

$$= \frac{2 u_\xi - 2\xi u_\eta}{2(\xi^2-\eta^2)+2(\xi^2-\eta^2)} = \frac{2\eta u_\xi - 2\xi u_\eta}{4(\xi^2-\eta^2)}$$

$$= \frac{2\eta}{4(\xi^2-\eta^2)}u_\xi - \frac{2\xi}{4(\xi^2-\eta^2)}u_\eta$$

$$= \frac{\eta}{2(\xi^2-\eta^2)}u_\xi - \frac{\xi}{2(\xi^2-\eta^2)}u_\eta \quad \text{Ans.}$$

2nd Method

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$= x^2 u_{\xi\xi} + (-2x^2)u_{\xi\eta} + x^2 u_{\eta\eta} - u_\xi + u_\eta$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$$

$$= y^2 u_{\xi\xi} + 2y^2 u_{\xi\eta} + y^2 u_{\eta\eta} + u_\xi + u_\eta$$

Given $y^2 u_{xx} - x^2 u_{yy} = 0$

$$\Rightarrow y^2[x^2 u_{\xi\xi} - 2x^2 u_{\xi\eta} + x^2 u_{\eta\eta} - u_\xi + u_\eta]$$
$$- x^2[y^2 u_{\xi\xi} + 2y^2 u_{\xi\eta} + y^2 u_{\eta\eta} + u_\xi + u_\eta] = 0$$

$$\Rightarrow \cancel{x^2y^2u_{\xi\xi}} - 2x^2y^2u_{\xi\eta} + \cancel{x^2y^2u_{\eta\eta}} - y^2u_{\xi} + y^2u_{\eta}$$

$$- \cancel{x^2y^2u_{\xi\xi}} - 2x^2y^2u_{\xi\eta} - \cancel{x^2y^2u_{\eta\eta}} - x^2u_{\xi} - x^2u_{\eta} = 0$$

$$\Rightarrow -4x^2y^2u_{\xi\eta} - y^2u_{\xi} + y^2u_{\eta} - x^2u_{\xi} - x^2u_{\eta} = 0$$

$$\Rightarrow u_{\xi\eta} = \frac{y^2u_{\xi} - y^2u_{\eta} + x^2u_{\xi} + x^2u_{\eta}}{-4x^2y^2}$$

$$= \frac{(\xi+\eta)u_{\xi} - (\xi+\eta)u_{\eta} + (\eta-\xi)u_{\xi} + (\eta-\xi)u_{\eta}}{-4(\xi+\eta)(\eta-\xi)}$$

$$= \frac{(\xi+\eta+\eta-\xi)u_{\xi} - (\xi+\eta-\eta+\xi)u_{\eta}}{4(\xi+\eta)(\xi-\eta)}$$

$$= \frac{2\eta u_{\xi} - 2\xi u_{\eta}}{4(\xi^2-\eta^2)}$$

$$= \frac{\eta}{2(\xi^2-\eta^2)}u_{\xi} - \frac{\xi}{2(\xi^2-\eta^2)}u_{\eta}$$

———

In Parabolic Form:-

ξ = constant , η = arbitrary

$$\Rightarrow u_{\eta\eta} = H_3, \text{ where } H_3 = \frac{H^*}{C^*}, \quad A^* = 0$$

$$\Rightarrow u_{\eta\eta} = \frac{G^* - D^*u_{\xi} - E^*u_{\eta} - F^*u}{A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2}$$

or if $\eta = \text{constant}$ & $\xi = \text{arbitrary}$

Then $u_{\xi\xi} = \dfrac{H^*}{A^*} = H_3^*$ $\quad \because c^* = 0$

Now In Example 2 :-

(By * Method)

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0$$

$$\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \ln y = \ln x + \ln c$$

$$\Rightarrow \ln y - \ln(x) = \ln c \Rightarrow \ln\left(\frac{y}{x}\right) = \ln c$$

$$\Rightarrow \frac{y}{x} = c$$

Consider the Transformations.

$\xi = y/x \longrightarrow ①$, $\eta = y \longrightarrow ②$

η is arbitrary & so chosen that Jacobian does not vanish

$\xi_x = \dfrac{-y}{x^2}$, $\xi_y = \dfrac{1}{x}$ $\quad$ $\eta_x = 0$, $\eta_y = 1$

$\xi_{xx} = \dfrac{2y}{x^3}$, $\xi_{yy} = 0$ $\quad$ $\eta_{xx} = 0$, $\eta_{yy} = 0$,

$\xi_{xy} = -1/x^2$ $\quad$ $\eta_{xy} = 0$

diving eq ② by ① $\Rightarrow$

$$\frac{\xi}{\eta} = \frac{y/x}{y} \Rightarrow \xi/\eta = \frac{y}{x}\cdot\frac{1}{y} \Rightarrow \boxed{x = \frac{\eta}{\xi}}$$

Put in ①

$$\xi = \frac{y}{\eta/\xi} \Rightarrow y = \xi \cdot \eta/\xi \Rightarrow \boxed{y = \eta}$$

Here $A = x^2 \Rightarrow A = \eta^2/\xi^2$

$B = 2xy = 2(\eta/\xi)\eta = 2\eta^2/\xi$, $C = y^2 \Rightarrow C = \eta^2$

Canonical form

$$u_{\eta\eta} = H_3 = \frac{H^*}{C^*} = \frac{G^* - D^* u_\xi - E^* u_\eta - F^* u}{A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2}$$

$$\Rightarrow u_{\eta\eta} = \frac{G - [A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y]u_\xi - [A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y]u_\eta - Fu}{A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2}$$

$$= \frac{0 - [\eta^2/\xi^2 (\frac{2y}{x^3}) + (2\eta^2/\xi)(\frac{-1}{x^2}) + \eta^2(0) + 0 + 0]u_\xi - [\eta^2/\xi^2(0) + 0 + 0 + 0 + 0]u_\eta - Fu}{\eta^2/\xi^2 (-y/x^2)^2 + (2\eta^2/\xi)(0) + \eta^2(1)}$$

$$= \frac{[-\eta^2/\xi^2(\frac{2\eta}{\eta^3}\cdot\xi^3) + \frac{2\eta^2}{\xi}\cdot\frac{\xi^2}{\eta^2}]u_\xi + 0 - 0}{\eta^2/\xi^2 \cdot \frac{\eta^2}{\eta^4}\xi^4 + 0 + 0}$$

$$= \frac{(-\xi + \xi)u_\xi}{\xi^2} = \frac{0}{\xi^2}$$

$$\Rightarrow u_{\eta\eta} = 0$$

2nd Method

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}(\xi_x\eta_x) + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx}$$

$$= u_{\xi\xi}\left(\frac{y^2}{x^4}\right) + 2u_{\xi\eta}\left(-\frac{y}{x^2}\cdot 0\right) + u_{\eta\eta}(0) + u_\xi\left(\frac{2y}{x^3}\right) + u_\eta(0)$$

$$= \left(\frac{\eta^2}{\eta^4}\cdot\xi^4\right)u_{\xi\xi}$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}$$

$$= u_{\xi\xi}\left(\frac{-y}{x^2}\right)\left(\frac{1}{x}\right) + u_{\xi\eta}\left[\frac{-y}{x^2}\cdot 1 + \frac{1}{2}\cdot 0\right] + u_{\eta\eta}(0) + u_\xi\left(\frac{-1}{x^2}\right) + u_\eta(0)$$

$$= \left(\frac{-\eta}{\eta^3}\cdot\xi^3\right)u_{\xi\xi} + \left[-\frac{\eta}{\eta^2}\cdot\xi^2\right]u_{\xi\eta} - \frac{\xi^2}{\eta^2}u_\xi$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}$$

$$= u_{\xi\xi}\cdot\frac{1}{x^2} + 2u_{\xi\eta}\cdot\frac{1}{x} + u_{\eta\eta} + u_\xi(0) + u_\eta(0)$$

Given $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$

$$\Rightarrow x^2\left[\xi^4/\eta^2\right]u_{\xi\xi} + 2xy\left\{\left[-\frac{\xi^3}{\eta^2}\right]u_{\xi\xi} - \frac{\xi^2}{\eta}u_{\xi\eta} - \frac{\xi^2}{\eta^2}u_\xi\right]\Big\}$$

$$+ y^2\left\{\left[\xi^2/\eta^2\right]u_{\xi\xi} + 2\left(\frac{\xi}{\eta}\right)u_{\xi\eta} + u_{\eta\eta}\right\} = 0$$

$$\Rightarrow \frac{\cancel{\eta^2}}{\xi^2}\cdot\frac{\xi^4}{\cancel{\eta^2}}u_{\xi\xi} + 2\frac{\cancel{\eta}}{\cancel{\xi}}\cdot\eta\left(\frac{-\xi^3}{\eta^2}\right)u_{\xi\xi} - \frac{2\eta^2}{\xi}\frac{\xi^2}{\eta}u_{\xi\eta} - \xi^2/\cancel{\eta^2}\,u_\xi\cdot\frac{2\cancel{\eta^2}}{\xi}$$

$$+ \cancel{\eta^2}\cdot\xi^2/\cancel{\eta^2}\,u_{\xi\xi} + 2\frac{\xi\eta^2}{\cancel{\eta}}u_{\xi\eta} + u_{\eta\eta}y^2 = 0$$

$$\Rightarrow \xi^2\cancel{u_{\xi\xi}} - 2\xi^2\cancel{u_{\xi\xi}} - 2\eta u_{\xi\eta} - 2\xi u_\xi + \xi^2\cancel{u_{\xi\xi}}$$

$$+ 2\xi\eta u_{\xi\eta} + u_{\eta\eta}y^2 = 0$$

اسے گڑبڑ ہے۔ برحل طریقہ یہی ہے۔

End Result will be

$$y^2u_{\eta\eta} = 0$$

In Elliptic form:-

$$u_{xx} = u_{\alpha\alpha}\alpha_x^2 + 2u_{\alpha\beta}\alpha_x\beta_x + u_{\beta\beta}\beta_x^2 + u_\alpha\alpha_{xx} + u_\beta\beta_{xx}$$

$$u_{yy} = u_{\alpha\alpha}\alpha_y^2 + 2u_{\alpha\beta}\alpha_y\beta_y + u_{\beta\beta}\beta_y^2 + u_\alpha\alpha_{yy} + u_\beta\beta_{yy}$$

$$u_{xy} = u_{\alpha\alpha}\alpha_x\alpha_y + u_{\alpha\beta}(\alpha_x\beta_y + \alpha_y\beta_x) + u_{\beta\beta}\beta_x\beta_y + u_\alpha\alpha_{xy} + u_\beta\beta_{xy}$$

$$A^{**} = A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2$$

$$B^{**} = 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y$$

$$C^{**} = A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2$$

$$D^{**} = A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y$$

$$E^{**} = A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y$$

$$F^{**} = F \quad , \quad G^{**} = G$$

$$H_4 = G^{**} - D^{**}u_\alpha - E^{**}u_\beta - F^{**}u$$

Canonical form

$$u_{\alpha\alpha} + u_{\beta\beta} = H_5 \quad , \text{where } H_5 = H_4/A^{**}$$

Now In Example 3 (By * Method)

$$\xi = 2y - ix^2 \qquad , \eta = 2y + ix^2$$

and hence

$$\alpha = \frac{1}{2}(\xi + \eta) = 2y \qquad \beta = \frac{1}{2i}(\xi - \eta) = -x^2$$

$$\Rightarrow \alpha = 2y \qquad \Rightarrow \beta = -x^2$$

Given Equation is $u_{xx} + x^2 u_{yy} = 0$

Here $\boxed{A=1}$, $B=0$, $C=x^2 \Rightarrow \boxed{C=-\beta}$

$\alpha_x = 0$, $\alpha_{xx} = 0$ $\quad$ $\beta_x = -2x$, $\beta_{xx} = -2$

$\alpha_y = 2$, $\alpha_{yy} = 0$ $\quad$ $\beta_y = 0$, $\beta_{yy} = 0$

$\alpha_{xy} = 0$ $\quad$ $\beta_{xy} = 0$

Canonical form is

$$u_{\alpha\alpha} + u_{\beta\beta} = \frac{G^{**} - D^{**}u_\alpha - E^{**}u_\beta - F^* u}{A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2}$$

$$= \frac{G - [A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y]u_\alpha - [A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y]u_\beta - Fu}{A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2}$$

$$= \frac{0 - [0 + 0 + (-\beta)(0) + 0 + 0]u_\alpha - [(-2) + 0 + (-\beta)(0) + 0 + 0]u_\beta - 0}{1(0) + 0 + -\beta(2)^2}$$

$$= \frac{2u_\beta}{-4\beta} = -\frac{u_\beta}{2\beta}$$

$$\Rightarrow u_{\alpha\alpha} + u_{\beta\beta} = -\frac{u_\beta}{2\beta} \quad \text{Ans.}$$

Other Method:-

Given Equation $u_{xx} + x^2 u_{yy} = 0$

Now

$$u_{xx} = u_{\alpha\alpha}\alpha_x^2 + 2u_{\alpha\beta}\alpha_x\beta_x + u_{\beta\beta}\beta_x^2 + u_\alpha\alpha_{xx} + u_\beta\beta_{xx}$$

$$= (0)^2 u_{\alpha\alpha} + 2(0)(-2x)u_{\alpha\beta} + (-2x)^2 u_{\beta\beta} + 0u_\alpha + (-2)u_\beta$$

$$\Rightarrow u_{xx} = 4x^2 u_{\beta\beta} - 2u_\beta$$

&

$$u_{yy} = u_{\alpha\alpha}\alpha_y^2 + 2u_{\alpha\beta}\alpha_y\beta_y + u_{\beta\beta}\beta_y^2 + u_\alpha\alpha_{yy} + u_\beta\beta_{yy}$$

$$= 2^2 u_{\alpha\alpha} + 2(2)(0)u_{\alpha\beta} + 0u_{\beta\beta} + 0 + 0$$

$$u_{yy} = 4u_{\alpha\alpha}$$

$$\Rightarrow (4x^2 u_{\beta\beta} - 2u_\beta) + x^2(4u_{\alpha\alpha}) = 0$$

$$\Rightarrow 4x^2 u_{\beta\beta} + 4x^2 u_{\alpha\alpha} = 2u_\beta$$

$$\Rightarrow 4x^2(u_{\alpha\alpha} + u_{\beta\beta}) = 2u_\beta$$

$$\Rightarrow u_{\alpha\alpha} + u_{\beta\beta} = \frac{u_\beta}{2x^2}$$

$$\Rightarrow u_{\alpha\alpha} + u_{\beta\beta} = -\frac{u_\beta}{2\beta} \quad \because x^2 = -\beta$$

⇒ Separation of Variable:- The conduction of heat is prescribed by the equation

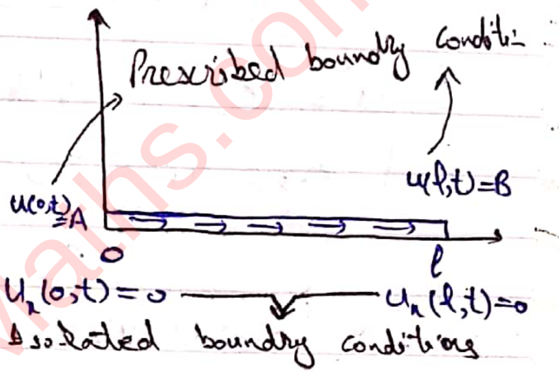
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \longrightarrow (1) \quad \Rightarrow \boxed{u_t = k u_{xx}}$$

with initial condition $u(x,0) = f(x) \longrightarrow (2)$
and boundry conditions
$u(0,t) = T_0 \longrightarrow (3)$ & $u(a,t) = T_1 \longrightarrow (4)$
Here boundry conditions are non-homogeneous

The boundry conditions are said to be homogeneous if $T_0 = 0$ and $T_1 = 0$.

To solve the partial differential equations by using the standard method of seperation of variable, 1st boundry conditions are made homogeneous if they are not homogeneous.

★ To solve the PDE by seperation of variable, PDE must also be Homogeneous.



★ Find Eigen values and eigen functions of the regular SL equation $u'' + \lambda u = 0$ with $u(0) = 0$ and $u(l) = 0$

Solution

Given system is $u'' + \lambda u = 0$;

$u(0) = 0$ & $u(l) = 0$

Now corresponding characteristic equation is

$$D^2 + \lambda = 0 \Rightarrow D^2 = -\lambda$$

$$\Rightarrow D = \pm i\sqrt{\lambda}$$

$$\Rightarrow u(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$$

Now $u(0) = 0 \Rightarrow A\cos(0) + B\sin(0) = 0$

$$\Rightarrow A(1) + B(0) = 0 \Rightarrow \boxed{A = 0}$$

$$\Rightarrow u(x) = B\sin\sqrt{\lambda}\,x$$

Next $u(l) = 0 \Rightarrow B\sin\sqrt{\lambda}\,l = 0$

$\Rightarrow B = 0$ or $\sin\sqrt{\lambda}\,l = 0$

But $B \neq 0$ $\because$ If $B = 0$ then $u(x) = 0$, which is not true. So $B \neq 0$

then $\sin\sqrt{\lambda}\,l = 0$

$\Rightarrow \sqrt{\lambda}\,l = n\pi$, where $n = 0, 1, 2, \ldots$

But note that if $n = 0$, then $\lambda = 0$ then

$$u'' + 0 \cdot u = 0 \quad \Rightarrow u'' = 0$$

$\Rightarrow u' = Ac_1 \quad \Rightarrow u = c_1 x + c_2$

and then $u(0) = 0 \Rightarrow c_1(0) + c_2 = 0$

$$\Rightarrow c_2 = 0$$

$$\Rightarrow u(x) = c_1(x)$$

Now $u(l) = 0 \quad \Rightarrow c_1(l) = 0 \quad \Rightarrow c_1 = 0$

$$\Rightarrow u \equiv 0$$

which is not possible

$\because u(0)$ is trivial solution & we are not interested in trivial solution

So $n \neq 0$

$\Rightarrow \sqrt{\lambda} = \frac{n\pi}{l}$, $n = 1, 2, 3, \ldots$

$\Rightarrow \lambda = \left(\frac{n\pi}{l}\right)^2$, $n = 1, 2, 3, \ldots$

$\Rightarrow \lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2$, $n = 1, 2, 3, \ldots$

are eigen values of the system. And $u_n(x) = B_n \sin\left(\frac{n\pi}{l}x\right)$, $n = 1, 2, 3, \ldots$ are corresponding eigen functions or eigen solutions of the system.

Question:- Solve by using method of seperation of variables. $u_t = k u_{xx}$; $u(0,t) = 0$, $u(l,t) = 0$ & $u(x,0) = f(x)$

Solution

$$u_t = k u_{xx}$$

Let $u = \phi(x) \cdot G(t)$

$$\Rightarrow u_{xx} = \phi''(x) \cdot G(t) \quad \& \quad u_t = \phi(x) G'(t)$$

$u(0,t) = 0 \Rightarrow \phi(0) \cdot G(t) = 0 \Rightarrow \phi(0) = 0 \quad \because G(t) \neq 0$

$u(l,t) = 0 \Rightarrow \phi(l) \cdot G(t) = 0 \Rightarrow \phi(l) = 0$

$u(x,0) = f(x)$
$\phi(x) G(0) = f(x)$
$G(0) = \dfrac{f(x)}{\phi(x)}$ no use

$$\Rightarrow \phi(x) G'(t) = k \phi''(x) G(t)$$

$$\Rightarrow \frac{G'(t)}{k G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda \quad (\text{constant})$$

$$\Rightarrow \frac{G'(t)}{k G(t)} = -\lambda \Rightarrow \frac{G'(t)}{G(t)} = -k\lambda$$

k کو G کے ساتھ اس لیے رکھتے ہیں کہ k، G کے ساتھ کنسٹنٹ ہو جائے اور ϕ کو نہ چھیڑے تاکہ وہ سٹرم لیوویل پر آسانی سے حل ہو جائے

$$\Rightarrow \int \frac{G'(t)}{G(t)} dt = \int -k\lambda \, dt$$

$$\Rightarrow \ln G = -k\lambda t + \ln c \Rightarrow \ln G - \ln c = -k\lambda t$$

$$\Rightarrow \ln\left(\frac{G}{c}\right) = -k\lambda t \Rightarrow \frac{G}{c} = e^{-k\lambda t}$$

$$\Rightarrow \boxed{G(t) = c\, e^{-k\lambda t}}$$

&

$$\frac{\phi''(x)}{\phi(x)} = -\lambda \Rightarrow \phi'' = -\lambda \phi$$

$$\Rightarrow \phi'' - \lambda \phi = 0$$

Regular S.L equation.
Already solved

with $\phi(0) = 0$ &
$\phi(l) = 0$

$$\Rightarrow \phi_n(x) = B_n \sin\left(\frac{n\pi}{l}x\right)$$

& $$G_n(t) = C e^{-k\left(\frac{n\pi}{l}\right)^2 t}$$

Product Solution is

$$u_n(x,t) = \phi_n(x)\, G_n(t)$$

$$\Rightarrow u_n(x,t) = B_n \sin\left(\frac{n\pi}{l}\right)x \cdot C e^{-k\left(\frac{n\pi}{l}\right)^2 t}$$

$$= A_n \sin\left(\frac{n\pi}{l}x\right) e^{-k\left(\frac{n\pi}{l}\right)^2 t} \quad , n = 1, 2, 3, \ldots$$

Now by principle of superposition

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right) e^{-k\left(\frac{n\pi}{l}\right)^2 t}$$

Now using the initial value condition

$$u(x,0) = f(x)$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right)$$

$$\Rightarrow \int_0^l f(x) \sin\left(\frac{m\pi}{l}x\right) dx = \sum_{n=1}^{\infty} A_n \int_0^l \sin\left(\frac{n\pi}{l}x\right) \sin\left(\frac{m\pi}{l}\right) dx$$

$$= A_n \left(\frac{l}{2}\right) \quad \text{for } m = n$$

$$\because \int_0^l \sin\left(\frac{m\pi}{l}x\right) \sin\left(\frac{n\pi}{l}x\right) dx = \begin{cases} 0 & \text{for } m \neq n \\ \frac{l}{2} & \text{for } m = n \end{cases}$$

Hence $A_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi}{\ell}x\right)dx$

So general solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi}{\ell}x\right)dx\right]\sin\left(\frac{n\pi}{\ell}x\right)e^{-k\left(\frac{n\pi}{\ell}\right)^2 t}$$

Question Solve the wave equation $u_{xx} = \frac{1}{c^2}u_{tt}$ with $u(0,t)=0$, $u(\ell,t)=0$, $u(x,0)=f(x)$, $\dot{u}(x,0)=(0)$

Solution

Since the boundry conditions are homogeneous so we use method of seperation of variable. So

$$u(x,t) = \phi(x)\,G(t)$$

then given equation become.

$$\phi''(x)G(t) = \frac{1}{c^2}\phi(x)G''(t)$$

$$\Rightarrow \frac{\phi''(x)}{\phi(x)} = \frac{1}{c^2}\frac{G''(t)}{G(t)} = -\lambda^2$$

$$\Rightarrow \frac{\phi''(x)}{\phi(x)} = -\lambda^2 \Rightarrow \phi'' = -\lambda^2\phi(x)$$

$$\Rightarrow \phi''(x) + \lambda^2\phi(x) = 0$$

$$\Rightarrow \phi(x) = B\sin\left(\frac{m\pi}{\ell}x\right)\ ;\ m = 1,2,3,\cdots$$ (Already solved)

And

$G''(t)$ mean G_{tt}
& $\phi''(x)$ mean ϕ_{xx}

$$\frac{1}{c^2}\frac{G''(t)}{G(t)} = -\lambda^2$$

$$\Rightarrow G''(t) = -\lambda^2c^2G(t)$$

$$\Rightarrow G''(t) + \lambda^2c^2G(t)$$

$$\Rightarrow G(t) = D\cos(\lambda c)t + E\sin(\lambda c)t$$

Now $\dot{u}(x,0) = 0 \Rightarrow \dot{G}(0) = 0$

& $G'(t) = -c\lambda D\sin(c\lambda)t + Ec\lambda\cos(c\lambda)t$

$G'(0) = 0 \Rightarrow E = 0$

So $G(t) = D\cos(c\lambda)t$

$G'(t)$ mean G_t i.e partial derivative of G w.r.t t

$$\Rightarrow G(t) = D\cos\left(\frac{cm\pi}{\ell}\right)t \qquad \because \lambda = \frac{m\pi}{\ell}$$

Hence

$$u_m(x,t) = \left[B\sin\left(\frac{m\pi}{\ell}x\right)\right]\left[D\cos\left(\frac{cm\pi}{\ell}\right)t\right]$$

$$= BD\sin\left(\frac{m\pi}{\ell}x\right)\cos\left(\frac{cm\pi}{\ell}t\right)$$

So

$$u_n(x,t) = BD\sin\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{cn\pi}{\ell}t\right)$$

$$\Rightarrow u_n(x,t) = \lambda_n\sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{cn\pi}{\ell}t\right), \quad n = 1, 2, 3, \dots$$

Now by principle of superposition

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \lambda_n\sin\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{n\pi ct}{\ell}\right)$$

Now

$$u(x,0) = f(x)$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \lambda_n\sin\left(\frac{n\pi}{\ell}x\right)$$

$$\Rightarrow \int_0^l f(x)\sin\left(\frac{m\pi}{l}x\right)dx = \sum_{n=1}^{\infty}\lambda_n\int_0^l \sin\left(\frac{m\pi}{l}x\right)\sin\left(\frac{n\pi}{l}x\right)dx$$

$$= \lambda_n\left(\frac{l}{2}\right) \quad \text{for } m=n$$

$$\Rightarrow \lambda_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)dx$$

So general solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)\right]\sin\left(\frac{n\pi}{l}x\right)\cos\left(\frac{cn\pi}{l}t\right)$$

⇒ **Solution of SL Equation with Neumann Condition:-**

* Find eigen values and eigen function of the regular SL equation $u''+\lambda=0$, with $u'(0)=0$ and $u'(l)=0$

Solution

$$u''+\lambda u=0$$

Auxiliary equation is

$$D^2+\lambda=0 \Rightarrow D=\pm i\lambda$$

$$\Rightarrow u(x) = c_1\cos\sqrt{\lambda}\,x + c_2\sin\sqrt{\lambda}\,x \longrightarrow (*)$$

$$u'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\,x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\,x$$

Now

$$u'(0)=0 \Rightarrow 0+c_2\sqrt{\lambda}=0 \Rightarrow c_2=0$$

$$\Rightarrow u'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\,x$$

$$u'(l)=0 \Rightarrow -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\,l=0$$

$$\Rightarrow \sin\sqrt{\lambda}\,l=0 \Rightarrow \sqrt{\lambda}\,l=n\pi$$

$$\Rightarrow \sqrt{\lambda}=\frac{n\pi}{l}$$

$$\Rightarrow \lambda=\left(\frac{n\pi}{l}\right)^2, \quad n=0,1,2,3,\dots$$

$$\Rightarrow \lambda_n = \frac{n^2\pi^2}{l^2} \; ; \quad n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow U_n(x) = C_n \cos\left(\frac{n\pi}{l} x\right) \quad ; \; n = 0, 1, 2, \ldots$$

** ————————————— **

Question:- Determine eigen values of the system $u'' + \lambda u = 0$ with $u(0) = u(\pi)$ and $u'(0) = 2u'(\pi)$

Solution

Given system is $u'' + \lambda u = 0$

Auxiliary equation is

$$D^2 + \lambda = 0 \Rightarrow D^2 = -\lambda \Rightarrow D = \pm i\sqrt{\lambda}$$

$$\Rightarrow u(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$$

Now $\quad u(0) = u(\pi)$

$$\Rightarrow A\cos(0) + B\sin(0) = A\cos\sqrt{\lambda}\pi + B\sin\sqrt{\lambda}\pi$$

$$\Rightarrow A = A\cos\sqrt{\lambda}\pi + B\sin\sqrt{\lambda}\pi$$

$$\Rightarrow (\cos\sqrt{\lambda}\pi - 1)A + (\sin\sqrt{\lambda}\pi)B = 0 \longrightarrow (1)$$

Now

$$u'(0) = 2u'(\pi)$$

$$\Rightarrow -A\sqrt{\lambda}\sin\sqrt{\lambda}(0) + B\sqrt{\lambda}\cos\sqrt{\lambda}(0) = -2A\sqrt{\lambda}\sin\sqrt{\lambda}\pi + 2B\sqrt{\lambda}\cos\sqrt{\lambda}\pi$$

$$\Rightarrow B\sqrt{\lambda} = -2A\sqrt{\lambda}\sin\sqrt{\lambda}\pi + 2B\sqrt{\lambda}\cos\sqrt{\lambda}\pi$$

$$\Rightarrow B = -2A\sin\sqrt{\lambda}\pi + 2B\cos\sqrt{\lambda}\pi$$

$$\Rightarrow (2\sin\sqrt{\lambda}\pi)A + (1 - 2\cos\sqrt{\lambda}\pi)B = 0 \longrightarrow (2)$$

for non trivial solution of A & B

$$\begin{vmatrix} \cos\sqrt{\lambda}\pi - 1 & \sin\sqrt{\lambda}\pi \\ 2\sin\sqrt{\lambda}\pi & 1 - 2\cos\sqrt{\lambda}\pi \end{vmatrix} = 0$$

$\Rightarrow (\cos\sqrt{\lambda}\pi - 1)(1 - 2\cos\sqrt{\lambda}\pi) - 2\sin^2\sqrt{\lambda}\pi = 0$

$\Rightarrow \cos\sqrt{\lambda}\pi - 2\cos^2\sqrt{\lambda}\pi - 1 + 2\cos\sqrt{\lambda}\pi - 2\sin^2\sqrt{\lambda} = 0$

$\Rightarrow 3\cos\sqrt{\lambda}\pi - 1 - 2[\cos^2\sqrt{\lambda}\pi + \sin^2\sqrt{\lambda}\pi] = 0$

$\Rightarrow 3\cos\sqrt{\lambda}\pi - 1 - 2(1) = 0$

$\Rightarrow 3\cos\sqrt{\lambda}\pi = 3 \Rightarrow \cos\sqrt{\lambda}\pi = 1$

$\Rightarrow \sqrt{\lambda} = 2n, \quad n = 0, 1, 2, \ldots$ $\boxed{\because \sqrt{\lambda}\pi = 2n\pi}$

$\Rightarrow \lambda = \lambda_n = 4n^2, \; n = 0, 1, 2, \ldots$ are eigen values &

or if we replace π with l in B.C then $\sqrt{\lambda}\, l = 2n\pi$

$\Rightarrow \sqrt{\lambda} = \left(\frac{2n\pi}{l}\right)$

$u_n(x) = A_n\cos(2nx) + B_n\sin(2nx)$ are corresponding eigen functions.

i.e $u_n(x) = A\cos\left(\frac{2n\pi}{l}x\right) + B\sin\left(\frac{2n\pi}{l}x\right)$

** **

Question:- Find eigen values and eigen functions of the SL system $u'' + \lambda u = 0$, with $u'(0) = 0 = u(l)$

Solution

Given $u'' + \lambda u = 0$

$\Rightarrow u = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$

$u' = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$u'(0) = 0 \Rightarrow \sqrt{\lambda}c_2 = 0$

$\Rightarrow c_2 = 0 \quad \because \sqrt{\lambda} \neq 0$

$\because$ If $\sqrt{\lambda} = 0$, then solution is trivial

So $u = c_1\cos\sqrt{\lambda}x$

$u(l) = 0 \Rightarrow \cos\sqrt{\lambda}l = 0 \quad \because$ for non trivial sol $c_1 \neq 0$

$$\Rightarrow \sqrt{\lambda}\,l = (2n+1)\frac{\pi}{2} \quad ; n \in Z$$

$$\Rightarrow \lambda_n = (2n+1)^2 \frac{\pi^2}{4l^2} \quad ; n = 0, 1, 2, \ldots$$

which are required eigen values.
The corresponding eigen functions are

$$U_n = C_n \cos\left(\frac{(2n+1)\pi}{2l}x\right) ; n = 0, 1, 2, \ldots$$

★★ ——————————— ★★

⇒ Higher Dimensional PDE's :-

★ **Separation of Time Variable:-** We will show that similar methods can be applied to a variety of problems. We will begin by discussing the vibration of a membrane of any shape, and follow that with some analysis for the conduction of heat in any two or three dimentional region.

⇒ Vibrating Membrane: Any Shape:-

Let us consider the displacement u of a vibrating membrane of any shape. The displacement $u(x, y, t)$ satisfies the two dimensional wave equation.

$$u_{tt} = c^2(u_{xx} + u_{yy}) \rightarrow ①$$

Separation of ... سے PDE ... میں تبدیل ہو جاتی

The initial conditions will be

$$u(x, y, 0) = \alpha(x, y)$$

$$\frac{\partial u}{\partial t}(x, y, 0) = \beta(x, y)$$

Assume $u(x,y,t) = \phi(x,y)\, R(t)$ $\longrightarrow$ ②

$\Rightarrow u_{xx} = \phi_{xx} R(t)$, $u_{yy} = \phi_{yy} R(t)$

$u_{tt} = \phi R''(t)$

u_{xx} اور u_{yy} ابھی بنی PDE's میں
اس لیے ان پہ پرابلم نہیں لگے گا اور u_{tt}
ODE بن چکی ہے اس لیے R'' لکھوا جائے گا

In eqn ② $\phi(x,y)$ is an as yet unknown function of two variables x and y.
We do not (at this time) specify further $\phi(x,y)$ since we might expect different results in different geometries or with different boundry conditions. Later, we will show that for rectangular membranes $\phi(x,y) = F(x)\, G(y)$, while for circular membranes $\phi(x,y) = F(r)\, G(\theta)$; i.e, the form of further separation depends on the geometry.
Now PDE becomes

$$\phi(x,y) R''(t) = c^2 [\phi_{xx} R(t) + \phi_{yy} R(t)]$$

Divide both sides by $c^2 \phi(x,y) R(t)$

$$\Rightarrow \underbrace{\frac{R''(t)}{c^2 R(t)}}_{\text{(function of } t \text{ only)}} = \underbrace{\frac{\phi_{xx} + \phi_{yy}}{\phi(x,y)}}_{\text{(function of } x \text{ any } y)}$$

$$\Rightarrow \frac{R''}{c^2 R} = \frac{\phi_{xx} + \phi_{yy}}{\phi} = -\lambda \text{ (constant)} \longrightarrow ③$$

$$\Rightarrow \frac{R''}{c^2 R} = -\lambda \Rightarrow R'' + c^2 \lambda R = 0 \longrightarrow ④$$

(Equation for t (ODE))

Solution of this equation is

$$R(t) = A \cos(c\sqrt{\lambda}\, t) + B \sin(c\sqrt{\lambda}\, t)$$

Now
$$\frac{\phi_{xx}+\phi_{yy}}{\phi} = -\lambda$$

$$\Rightarrow \phi_{xx}+\phi_{yy} = -\lambda\phi \quad \text{—— (5)}$$

$\Rightarrow$ PDE (2) is $\phi_{xx}+\phi_{yy}+\lambda\phi=0$

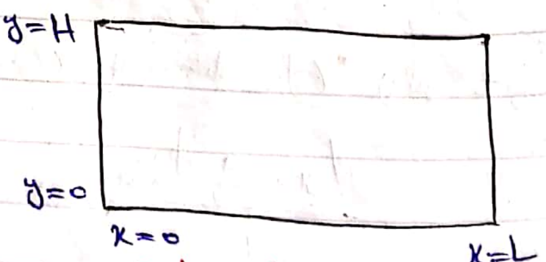
The notation $-\lambda$ for the separation constant was chosen because the time dependent differential equation (4) has oscillatory solutions if $\lambda>0$. if $\lambda>0$, then h is a linear combination of $\sin c\sqrt{\lambda}t + \cos c\sqrt{\lambda}t$. it oscillates with frequency $c\sqrt{\lambda}$. The value of λ determine the natural frequencies of oscillation of a vibrating membrane.

*Vibrating Rectangular Membrane:-

The vertical displacement $u(x,y,t)$ of the membrane satisfies the two dimensional wave equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \longrightarrow (1)$$

We suppose that boundary is given such that all four sides are fixed with zero displacement.



Rectangular Membrane

$$u(0,y,t)=0 \qquad u(x,0,t)=0 \longrightarrow (2)$$

$$u(L,y,t)=0 \qquad u(x,H,t)=0 \longrightarrow (2)$$

We ask what is the displacement at time t if the initial position & velocity are given.

$$u(x,y,0) = \alpha(x,y) \longrightarrow ④$$

$$u_t(x,y,0) = \beta(x,y) \longrightarrow ⑤$$

As we indicated earlier, since the partial differential equation and the boundry conditions are linear & homogeneous, we apply method of separation of variables. first we separate only the time variable by seeking product solutions in the form

$$u(x,y,t) = h(t)\phi(x,y) \longrightarrow ⑥$$

According to earlier calculation we are able to introduce a separation constant $-\lambda$, & the following two equations result.

$$\frac{d^2h}{dt^2} = -\lambda c^2 h \Rightarrow h'' + \lambda c^2 h = 0 \longrightarrow ⑦$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\lambda\phi \Rightarrow \phi_{xx} + \phi_{yy} = -\lambda\phi \longrightarrow ⑧$$

$$\Rightarrow \phi_{xx} + \phi_{yy} + \lambda\phi = 0$$ [Still a PDE

Assume $\phi(x,y) = f(x)\, g(y) \longrightarrow ⑨$

$$\Rightarrow \phi_{xx}(x,y) = f''(x)\, g(y)$$

$$\phi_{yy}(x,y) = f(x)\, g''(y)$$

PDE ② becomes

$$f''(x)\, g(y) + f(x)\, g''(y) + \lambda f(x)\, g(y) = 0 \longrightarrow ⑩$$

$$\Rightarrow f''(x)\,g(y) = -f(x)\,g''(y) - \lambda f(x)\,g(y)$$

Dividing both sides by $f(x)\,g(y)$

$$\Rightarrow \frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)} - \lambda$$

$$\Rightarrow \underbrace{\frac{f''}{f}}_{\text{only function of } x} = \underbrace{-\left(\frac{g''}{g} + \lambda\right)}_{\text{only function of } y} = -\mu \quad (\text{a constant}) \longrightarrow (11)$$

$$\Rightarrow \frac{f''}{f} = -\mu \Rightarrow f'' + \mu f = 0 \longrightarrow (12)$$

and $u(0,y,t) = 0 \Rightarrow \phi(0,y)\,h(t) = 0$

$$\Rightarrow \phi(0,y) = 0 \Rightarrow f(0)\,g(y) = 0$$

$$\Rightarrow f(0) = 0$$

Similarly $u(L,y,t) = 0 \Rightarrow f(L) = 0$

$$\Rightarrow f'' + \mu f = 0 \; ; \text{ with}$$

$$f(0) = 0 \quad \& \quad f(L) = 0$$

$$\Rightarrow f_n(x) = \sin\left(\frac{n\pi}{L}x\right);\ n = 1, 2, 3, \ldots \longrightarrow (*)$$

$$\because \mu_n = \left(\frac{n\pi}{L}\right)^2$$

And $-\dfrac{g''}{g} - \lambda = -\mu$

$$\Rightarrow g'' + (\lambda - \mu)\,g = 0 \longrightarrow (13)$$

and boundary conditions

$$u(x,0,t) = \phi(x,0)\,h(t) = 0$$

$$\Rightarrow \phi(x,0) = 0 \Rightarrow f(x)\,g(0) = 0$$

$\Rightarrow g(0) = 0$ & similarly $g(H) = 0$

Eigen values $\lambda - \mu_n = \left(\frac{m\pi}{H}\right)^2$; $m = 1, 2, 3, \ldots \longrightarrow (14)$

& $g_m(y) = \sin\left(\frac{m\pi}{H} y\right)$; $m = 1, 2, 3, \ldots \longrightarrow (**)$

The separation constant λ_{nm} determine from equ (14)

$$\lambda_{nm} = \mu_n + \left(\frac{m\pi}{H}\right)^2$$

$$\Rightarrow \lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 ; \quad n = 1, 2, 3, \ldots \quad m = 1, 2, 3, \ldots \longrightarrow (15)$$

The two dimensional eigen value problem has eigen values λ_{nm} given by (15) and eigen functions given by the product of two one-dimensional eigen functions. Using the notation $\phi_{nm}(x, y)$ for the two dimensional eigenfunction corresponding to the eigen values λ_{nm} we have

$$\phi_{nm}(x, y) = \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{m\pi}{H} y\right) \longrightarrow (16)$$

$$n = 1, 2, 3, \ldots ; \quad m = 1, 2, 3, \ldots$$

And also we have

$$h(t) = A_{nm} \cos c\sqrt{\lambda_{nm}}\, t + B_{nm} \sin c\sqrt{\lambda_{nm}}\, t$$

The product solution is

$$U_{nm}(x, y, t) = \left[A_{nm} \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{m\pi}{H} y\right) \cos c\sqrt{\lambda_{nm}}\, t\right] + \left[B_{nm} \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{m\pi}{H} y\right) \sin c\sqrt{\lambda_{nm}}\, t\right] \longrightarrow (17)$$

$$n, m = 1, 2, 3, \ldots$$

The principle of superposition implies that we should consider a linear combination of all possible product solution. Thus, we must include both families, summing over both n and m

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} \cos c\sqrt{\lambda_{nm}}\, t$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} \sin c\sqrt{\lambda_{nm}}\, t \longrightarrow (18)$$

The two families of coefficients A_{nm} and B_{nm} hopefully will be determined from the two initial conditions. for example

$u(x,y,0) = \alpha(x,y)$ implies that

$$\alpha(x,y) = \sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty} A_{nm} \sin\frac{n\pi x}{L}\right] \sin\frac{m\pi y}{H} \longrightarrow (19)$$

The series in equ (19) is an example of what is called double fourier series. For fixed x, we note that $\sum_{n=1}^{\infty} A_{nm} \sin n\pi x/L$ depends only on m. It must be the coefficient of the fourier sine series in y of $\alpha(x,y)$ over $0<y<H$. From our theory of Fourier sine series, we therefore know that we may easily determine the coefficients

$$\sum_{n=1}^{\infty} A_{nm} \sin\frac{n\pi x}{L} = \frac{2}{H}\int_0^H \alpha(x,y) \frac{m\pi y}{H}\, dy \longrightarrow (20)$$

for each m equation (20) is valid for all x, the right hand side is a function of

x (not y, because y is integrated from 0 to H). For each m, the left hand side is a Fourier sine series of the right hand side, $\frac{2}{H}\int_0^H \alpha(x,y)\sin\frac{m\pi y}{H}\,dy$. The coefficients of this Fourier sine series in x are easily determined.

$$A_{nm} = \frac{2}{L}\int_0^L \left[\frac{2}{H}\int_0^H \alpha(x,y)\sin\frac{m\pi y}{H}\,dy\right]\sin\frac{n\pi x}{L}\,dx \longrightarrow (21)$$

This may be simplified to one double integral over the entire rectangular region, rather than two iterated one dimensional integrals. In this manner we have determined one set of coefficients from one of the initial conditions.

The other coefficients B_{nm} can be determined in a similar way. In particular, from equ (18) $\frac{\partial u}{\partial t}(x,y,0) = \beta(x,y)$ implies that

$$\beta(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\lambda_{mn}}\, B_{nm}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H} \longrightarrow (22)$$

Thus again using a Fourier sine series in y and a Fourier sine series in x, we obtain

$$c\sqrt{\lambda_{mn}}\, B_{nm} = \frac{4}{LH}\int_0^L\int_0^H \beta(x,y)\sin\frac{m\pi y}{H}\sin\frac{n\pi x}{L}\,dy\,dx \longrightarrow (23)$$

The solution of our initial value problem is the doubly finite series given by equ (18), where the coefficients are determined by equ (21) & (23).

We have shown that when all three independent variables separate for a partial differential equation, there results three ordinary differential equations, two of which are

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eigen value problems. In general, for a partial differential equation in N variables that completely separates. There will be N ordinary differential equations, $N-1$ which of are one dimensional eigen value problems (to determine $N-1$ separation constants). We have already shown for $N=3$ & $N=2$.

⇒ Heat Conduction: Any Region:-

We will analyze the flow of thermal energy in any two dimensional region. We begin by seeking product solution of the form

$$u(x,y,t) = h(t)\,\phi(x,y) \longrightarrow ①$$

for two dimensional heat equation, assuming constant thermal properties and no sources, substituting equ ① in heat equation

$$u_t = k(u_{xx} + u_{yy}) \longrightarrow ②$$

implies
$$h'(t)\,\phi(x,y) = k\left[\phi_{xx}\,h(t) + \phi_{yy}\,h(t)\right] \longrightarrow ③$$

Dividing equ ③ by $k\,h(t)\,\phi(x,y)$

$$\Rightarrow \frac{h'(t)}{k\,h(t)} = \frac{\phi_{xx} + \phi_{yy}}{\phi(x,y)} \longrightarrow ④$$

$$\Rightarrow \underbrace{\frac{h'(t)}{k\,h(t)}}_{\text{function } t \text{ only}} = \underbrace{\frac{\phi_{xx} + \phi_{yy}}{\phi(x,y)}}_{\text{function of } x \text{ \& } y} = -\lambda \longrightarrow ⑤$$

From eqn ⑤ we have two equations.

$$h'(t) = -\lambda k\, h(t) \longrightarrow (6)$$

& $$\phi_{xx} + \phi_{yy} = -\lambda \phi \longrightarrow (7)$$

The eigen value λ relates to the decay rate of time dependent part. The eigen value λ determined by the boundry value problem, again consisting of the partial differential equation (6,7) with a corresponding boundry condition on the entire boundry of the region.

For heat flow in any three dimensional region heat equation is

$$u_t = k(u_{xx} + u_{yy} + u_{zz}) \longrightarrow (8)$$ & Product solution is

$$u(x,y,z,t) = h(t)\,\phi(x,y,z) \longrightarrow (9)$$

may still be sought, and after separating variables, we obtain equation similar to eqn (6) and (7)

$$\left.\begin{aligned} &\frac{dh}{dt} = -\lambda k h \\ &\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -\lambda\phi \end{aligned}\right\} \longrightarrow (10)$$

The eigen value λ is determined by finding those values of λ for which non trivial solution of eqn (10) exist, subject to a homogeneous boundry condition on the entire boundry.

⇒ Eigen Values & Eigen Functions of Regular SL Equation w.r.t different Boundry Conditions. $u'' + \lambda u = 0$

Boundry Conditions	Eigen Values	Eigen Functions
(i) $u(0) = 0 = u(\ell)$	$\lambda = \left[\frac{n\pi}{\ell}\right]^2$, $n = 1, 2, 3, \dots$	$u_n(x) = \sin\left(\frac{n\pi}{\ell}x\right)$, $n = 1, 2, 3, \dots$
(ii) $u(0) = 0 = u'(\ell)$	$\lambda = \left[\frac{(2n+1)\pi}{2\ell}\right]^2$, $n = 0, 1, 2, 3, \dots$	$u_n(x) = \sin\left[\frac{(2n+1)\pi}{2\ell}x\right]$
(iii) $u'(0) = 0 = u(\ell)$	$\lambda = \left[\frac{(2n+1)\pi}{2\ell}\right]^2$, $n = 0, 1, 2, \dots$	$u_n(x) = \cos\left[\frac{(2n+1)\pi}{2\ell}x\right]$, $n = 0, 1, 2, \dots$
(iv) $u'(0) = 0 = u'(\ell)$	$\lambda = \left[\frac{n\pi}{\ell}\right]^2$, $n = 0, 1, 2, \dots$	$u_n(x) = \cos\left[\frac{n\pi}{\ell}x\right]$, $n = 0, 1, 2, \dots$

$$u(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$$

&

In case of heat equation

$$h(t) = c\,e^{-k\lambda t}$$

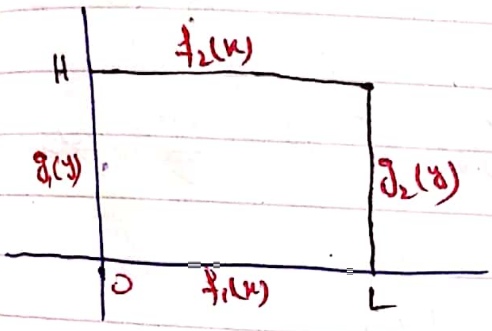
$$= c\,e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \qquad \because \lambda = \left(\frac{n\pi}{\ell}\right)^2$$

Laplace Equation:-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow u = u(x,y)$$

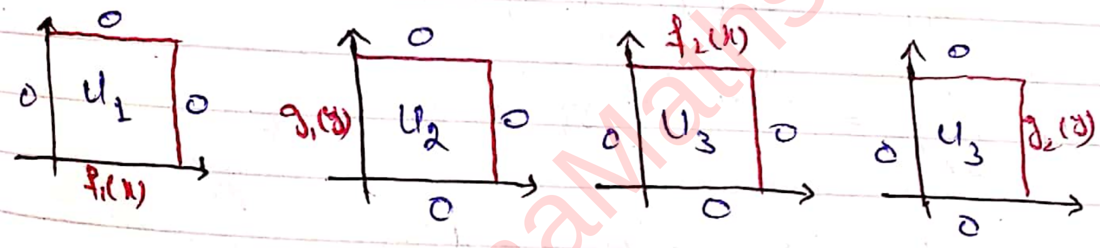
$u(0,y) = g_1(y)$, $u(L,y) = g_2(y)$
$u(x,0) = f_1(x)$, $u(x,H) = f_2(x)$

All conditions are non-homo



We divide sing problem into four problems

$$u = u_1 + u_2 + u_3 + u_3$$



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = \frac{\partial^2(u_1+u_2+u_3+u_4)}{\partial x^2} + \frac{\partial^2(u_1+u_2+u_3+u_4)}{\partial y^2}$$

Satisfied the Laplace equation.

Now for Boundry conditions

$$u(0,y) = u_1(0,y) + u_2(0,y) + u_3(0,y) + u_4(0,y)$$

$$= 0 + g_1 + 0 + 0$$

$$\Rightarrow u(0,y) = g_1$$

$$u(L,y) = u_1(L,y) + u_2(L,y) + u_3(L,y) + u_4(L,y)$$

$$= 0 + 0 + 0 + g_2$$

$$= g_2$$

similarly all satisfied.

Question:- If Possible, solve Laplace's Equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$\frac{\partial u}{\partial x}(0, y, z) = 0 \qquad u(x, 0, z) = 0 \qquad u(x, y, 0) = f(x, y)$

$\frac{\partial u}{\partial x}(L, y, z) = 0 \qquad u(x, \omega, z) = 0 \qquad u(x, y, H) = 0$

Solution

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \longrightarrow (1)$$

Let $u = u(x, y, z)$

$$\Rightarrow u = f(x)\, g(y)\, h(z)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = f''(x)\, g(y)\, h(z)$$

$$\frac{\partial^2 u}{\partial y^2} = f(x)\, g''(y)\, h(z)$$

$$\frac{\partial^2 u}{\partial z^2} = f(x)\, g(y)\, h''(z)$$

Therefore equ (1) becomes

$$f''(x)g(y)h(z) + f(x)g''(y)h(z) + f(x)g(y)h''(z) = 0$$

dividing by $f(x)g(y)h(z)$ implies

$$\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} + \frac{h''(z)}{h(z)} = 0$$

$$\Rightarrow \frac{f''(x)}{f(x)} = -\left[\frac{g''(y)}{g(y)} + \frac{h''(z)}{h(z)}\right] = -\lambda$$

(constant)

$\longrightarrow (*)$

$$\Rightarrow \frac{f''(x)}{f(x)} = -\lambda \quad \Rightarrow f''(x) = -\lambda f(x)$$

$$\Rightarrow f''(x) + \lambda f(x) = 0 \longrightarrow ②$$

And corresponding boundry value conditions

$$\frac{\partial u}{\partial x}(0,y,z) = 0 \Rightarrow f'(0)\,g(y)\,h(z) = 0$$

$$\Rightarrow f'(0) = 0 \quad \because g(y)\,h(z) \neq 0$$

$$\frac{\partial u}{\partial x}(L,y,z) = 0 \Rightarrow f'(L) = 0$$

from ② characteristic equ' n

$$D^2 + \lambda = 0 \Rightarrow D^2 = -\lambda$$

$$\Rightarrow D = \pm i\sqrt{\lambda}$$

$$\Rightarrow f(x) = C_1 \cos\sqrt{\lambda}\,x + C_2 \sin\sqrt{\lambda}\,x$$

And after putting the boundry value conditions we have

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 ; \; n = 0,1,2,\cdots$$

$$\Rightarrow f_n(x) = \cos\left(\frac{n\pi}{L}x\right) ; \; n = 0,1,2,3,\cdots \longrightarrow ③$$

Now again from (*)

$$-\frac{g''(y)}{g(y)} - \frac{h''(z)}{h(z)} = -\lambda$$

$$\Rightarrow \frac{g''(y)}{g(y)} + \frac{h''(z)}{h(z)} - \lambda = 0$$

$$\Rightarrow \frac{g''(y)}{g(y)} = -\left[\frac{h''(z)}{h(z)} - \lambda\right] = -\mu \quad (\text{a constant}) \longrightarrow (**)$$

$$\Rightarrow \frac{g''(y)}{g(y)} = -\mu$$

$$\Rightarrow g'' + \mu g = 0 \longrightarrow (4)$$

This is another eigen value problem in y variable. B.Cs are

$$u(x,0,z) = 0 \Rightarrow f(x)\, g(0)\, h(z) = 0$$

$$\Rightarrow g(0) = 0$$

$$u(x,\omega,z) = 0 \Rightarrow g(\omega) = 0$$

$$\Rightarrow \mu_m = \left[\frac{m\pi}{\omega}\right]^2 \text{ and}$$

$$g_m(y) = \sin\left(\frac{m\pi}{\omega} y\right); \; m = 1,2,3,\ldots \longrightarrow (5)$$

Now from $(**)$

$$-\frac{h''(z)}{h(z)} + \lambda = -\mu$$

$$\Rightarrow \frac{h''(z)}{h(z)} = \lambda + \mu$$

$$\Rightarrow h''(z) = h(z)\,[\lambda + \mu)$$

$$\Rightarrow h''(z) - h(z)\,(\lambda + \mu) = 0$$

Characteristic eqn is

$$D^2 - (\lambda + \mu) = 0$$

$\Rightarrow D^2 = \lambda + \mu \quad \Rightarrow D = \pm\sqrt{\lambda+\mu}$

$\Rightarrow h_{nm}(z) = A_{nm}\, e^{\sqrt{\lambda+\mu}\, z} + B_{nm}\, e^{-\sqrt{\lambda+\mu}\, z}$

Product solution is given by

$$u(x,y,z) = \sum_m \sum_n \cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{\omega}y\right)\left[A_{mn}e^{\sqrt{\lambda+\mu}\, z} + B_{mn}e^{-\sqrt{\lambda+\mu}\, z}\right]$$

$\because u(x,y,0) = f(x,y)$

$$\Rightarrow f(x,y) = \sum_{m,n} \cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{\omega}y\right)\left[A_{mn}+B_{mn}\right]$$

$\&\ u(x,y,H) = 0 \Rightarrow$

$$0 = \sum_{m,n} \cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{\omega}y\right)\left[A_{mn}e^{+\sqrt{\lambda+\mu}\, H} + B_{mn}e^{-\sqrt{\lambda+\mu}\, z}\right.$$

Question:- Solve $\frac{d^2 g}{d\theta^2} + \mu g = 0 \quad -\pi < \theta < \pi$

$$g(-\pi) = g(\pi)$$

$$\frac{dg}{d\theta}(-\pi) = \frac{dg}{d\theta}(\pi)$$

Solution $\frac{d^2 g}{d\theta^2} + \mu g = 0$

Case I $\mu > 0$

Characteristic equation is

$$D^2 + \mu = 0 \Rightarrow D^2 = -\mu$$

$$\Rightarrow D = \pm i\sqrt{\mu}$$

$$\Rightarrow g(\theta) = C_1 \cos\sqrt{\mu}\,\theta + C_2 \sin\sqrt{\mu}\,\theta$$

$$g(\pi) = C_1 \cos\sqrt{\mu}\,\pi + C_2 \sin\sqrt{\mu}\,\pi$$

$$g(-\pi) = C_1 \cos\sqrt{\mu}\,\pi - C_2 \sin\sqrt{\mu}\,\pi$$

$g(\pi) = g(-\pi)$ implies

$$\cancel{C_1 \cos\sqrt{\mu}\,\pi} + C_2 \sin\sqrt{\mu}\,\pi = \cancel{C_1 \cos\sqrt{\mu}\,\pi} - C_2 \sin\sqrt{\mu}\,\pi$$

$$\Rightarrow 2C_2 \sin\sqrt{\mu}\,\pi = 0 \longrightarrow (1)$$

$$g'(\theta) = -C_1\sqrt{\mu}\sin\sqrt{\mu}\,\theta + C_2\sqrt{\mu}\cos\sqrt{\mu}\,\theta$$

$$g'(\pi) = -C_1\sqrt{\mu}\sin\sqrt{\mu}\,\pi + C_2\sqrt{\mu}\cos\sqrt{\mu}\,\pi$$

$$g'(-\pi) = C_1\sqrt{\mu}\sin\sqrt{\mu}\,\pi + C_2\sqrt{\mu}\cos\sqrt{\mu}\,\pi$$

$g'(\pi) = g'(-\pi)$ implies

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$$2C_1\sqrt{\mu}\sin\sqrt{\mu}\,\pi = 0 \longrightarrow ②$$

Suppose $C_2 = 0$

$\Rightarrow 2C_1\sqrt{\mu}\sin\sqrt{\mu}\,\pi = 0$

$\Rightarrow \sin\sqrt{\mu}\,\pi = 0$

$\Rightarrow \sqrt{\mu}\,\pi = n\pi,\ n = 0, 1, 2, 3, \ldots$

$\Rightarrow \sqrt{\mu} = n \Rightarrow \mu = n^2,\ n = 0, 1, 2, 3, \ldots$

$\Rightarrow g(\theta) = C_1\cos\sqrt{\mu}\,\theta$

$\Rightarrow g(\theta) = C_1\cos n\theta,\ n = 0, 1, 2, 3 \ldots$

If $C_1 = 0 \Rightarrow 2C_2\sin\sqrt{\mu}\,\pi = 0$

$\Rightarrow \sin\sqrt{\mu}\,\pi = 0$

$\sqrt{\mu}\,\pi = m\pi \Rightarrow \sqrt{\mu} = m$

$\Rightarrow \mu = m^2$

$g(\theta) = C_2\sin\sqrt{\mu}\,\theta \Rightarrow g(\theta) = C_2\sin m\theta,\ m = 1, 2, 3, \ldots$

$\Rightarrow g(\theta) = C_1\cos n\theta + C\sin m\theta \quad n = 0, 1, 2, \ldots;\ m = 1, 2, \ldots$

$\Rightarrow g(\theta) = C_0 + C_n\cos n\theta + C_m\sin m\theta;\quad n, m = 1, 2, 3, \ldots$

$$\Rightarrow g(\theta) \sim C_0 + \sum_{n=1}^{\infty} C_n\cos n\theta + \sum_{m=1}^{\infty} C_m\sin m\theta$$

This is full fourier series

for $\mu < 0$ solution not hold.

⇒ Vibrating Circular Membrane:-
& Bessel Function:- A

An interesting applicati[on] to both one dimensional (S.L) and multi-dimensional eigen value problems occurs when considering the vibration of a circular membrane. The vertical displacement u satisfies the two dimensional wave equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

The geometry suggests that we use polar co-ordinates, in which case $u = u(r,\theta,t)$ was assume that the membrane has zero displacement at the circular boundry $r = a$

Boundry Condition: $u(a,\theta,t) = 0$

The initial position & velocity are given

$$\text{IC:} \quad u(r,\theta,0) = \alpha(r,\theta)$$
$$\frac{\partial u}{\partial t}(r,\theta,0) = \beta(r,\theta)$$

⇒ Separation of Variable:-

We first separate out the time variable by seeking product solutions

$$u(r,\theta,t) = \phi(r,\theta)h(t)$$

Then as shown earlier →①

$$\frac{d^2 h}{dt^2} = -\lambda c^2 h \quad \to ②$$

where λ is a seperation constant

from ② $c\sqrt{\lambda}$ (if $\lambda > 0$)

$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$\Rightarrow \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] + \frac{1}{r^2}[$

for Three D

$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] + \frac{1}{r^2}\left[\frac{\partial^2}{\partial \theta^2}\right.$

$+ \frac{\partial^2 u}{\partial z^2}$

are the natural frequencies of vibration. In addition $\phi(r,\theta)$ satisfies the two dimensional eigen value problem

$$\nabla^2\phi + \lambda\phi = 0 \longrightarrow ③$$

with $\phi = 0$ on the entire boundry, $r = a$

$$\phi(a,\theta) = 0 \longrightarrow ④$$

we will attempt to obtain solution of equ ③ in product form appropriate for polar coordinates.

$$\phi(r,\theta) = f(r)\, g(\theta) \longrightarrow ⑤$$

since for the circular membrane $0<r<a$, $-\pi<\theta<\pi$. This is equivalent to originally seeking solution to the wave equation in the form of a product of functions of each independent variable $u(r,\theta,t) = f(r)\,g(\theta)\,h(t)$. we substitute equ ⑤ into equ ③; in polar coordinates

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\phi}{\partial r}\right] + \frac{1}{r^2}\left[\frac{\partial^2\phi}{\partial\theta^2}\right] \quad \text{and}$$

Thus $\nabla^2\phi + \lambda\phi = 0$ becomes

$$\frac{g(\theta)}{r}\frac{d}{dr}\left[r\frac{df}{dr}\right] + \frac{f(r)}{r^2}\frac{d^2g}{d\theta^2} + \lambda f(r)\, g(\theta) = 0 \longrightarrow ⑥$$

r and θ may separate by multiplying by r^2 and dividing by $f(r)\,g(\theta)$

$$-\frac{1}{g}\frac{d^2g}{d\theta^2} = \frac{r}{f}\frac{d}{dr}\left[r\frac{df}{dr}\right] + \lambda r^2 = \mu \longrightarrow ⑦$$

We introduce a second separation constant in the form μ because our experience with circular regions (sec 2.4.2 & sec 2.5.2) suggests that $g(\theta)$ must oscillate in order to satisfy the periodic conditions in θ. Our three differential equations with two separation constants are thus

$$\frac{d^2h}{dt^2} = -\lambda c^2 h \longrightarrow (8)$$

$$\frac{d^2g}{d\theta^2} = -\mu g \longrightarrow (9)$$

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu) f = 0 \longrightarrow (10)$$

Two of these equations must be eigen value problems. However ignoring the initial conditions, the only given boundry condition is $f(a) = 0$, which follows from $u(a,\theta,t) = 0$ or $\phi(\alpha,\theta) = 0$.
We must remember that $-\pi < \theta < \pi$ and $0 < r < a$. Thus both θ and r defined over finite intervals. As such there should be boundry conditions at both ends. The periodic nature of solution in θ implies that

$$\left.\begin{aligned} g(-\pi) = g(\pi) \quad , \quad \frac{dg}{d\theta}(-\pi) = \frac{dg}{d\theta}(\pi) \end{aligned}\right\} \rightarrow (11)$$

We already have a condition at $r = a$, polar coordinates are singular at $r = 0$, a singularity condition must be introduced there. Since the displacement must be finite, we conclude that

$$|f(0)| < \infty$$

Explanation:- $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2}\right)\right] \longrightarrow (1)$$

Let $u(r,\theta,t) = \phi(r,\theta)\,h(t)$

$$\frac{\partial^2 u}{\partial t^2} = h''(t)\,\phi(r,\theta) \quad , \quad \frac{\partial u}{\partial r} = \frac{\partial \phi}{\partial r}\,h(t)$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 \phi}{\partial \theta^2}\,h(t)$$; Thus equ (1) becomes

$$\frac{d^2h}{dt^2}\phi(r,\theta) = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}h(t)\right) + \frac{1}{r^2}\left(\frac{\partial^2 \phi}{\partial \theta^2}h(t)\right)\right]$$

$$\Rightarrow \frac{d^2h}{dt^2}\phi(r,\theta) = c^2\left[\frac{h(t)}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{h(t)}{r^2}\left(\frac{\partial^2 \phi}{\partial \theta^2}\right)\right] \quad \because h(t) \text{ constant}$$

Dividing both sides by $c^2 h(t)\,\phi(r,\theta)$ implies

$$\frac{h''(t)}{c^2 h(t)} = \frac{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 \phi}{\partial \theta^2}\right)}{\phi(r,\theta)} = -\lambda \text{ (constant)} \longrightarrow (2)$$

$$\Rightarrow \frac{h''(t)}{c^2 h(t)} = -\lambda \quad \Rightarrow h''(t) = -\lambda c^2 h(t)$$

$$\Rightarrow h''(t) + \lambda c^2 h(t) \longrightarrow (3)$$

This is ODE in 't' variable.

Now from (2)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 \phi}{\partial \theta^2}\right) + \lambda\,\phi(r,\theta) = 0 \longrightarrow (4)$$

Now let $\phi(r,\theta) = f(r)\,g(\theta)$

$$\frac{\partial \phi}{\partial r} = f'(r)\,g(\theta) \quad , \quad \frac{\partial^2 \phi}{\partial \theta^2} = f(r)\,g''(\theta)$$

So equ (4) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r f'(r) g(\theta)\right] + \frac{1}{r^2}\left[f(r) g''(\theta)\right] + \lambda f(r) g(\theta) = 0$$

$$\Rightarrow \frac{g(\theta)}{r}\frac{\partial}{\partial r}\left[r\frac{df}{dr}\right] + \frac{f(r)}{r^2}\left[\frac{d^2 g}{d\theta^2}\right] + \lambda f(r) g(\theta) = 0$$

$\div$ by $f(r)\,g(\theta)$ & $\times$ by r^2 implies that

$$\frac{r}{f}\frac{d}{dr}\left[r\frac{df}{dr}\right] + \frac{1}{g}\left[\frac{d^2 g}{d\theta^2}\right] + \lambda r^2 = 0$$

$$\Rightarrow \frac{r}{f}\frac{d}{dr}\left[r\frac{df}{dr}\right] + \lambda r^2 = -\frac{1}{g}\frac{d^2 g}{d\theta^2} = \mu \text{ (constant)}$$

$$\Rightarrow \boxed{\frac{d^2 g}{d\theta^2} = -\mu g} \longrightarrow (5)$$

$$\boxed{r\frac{d}{dr}\left[r\frac{df}{dr}\right] + (\lambda r^2 - \mu) f = 0} \longrightarrow (6)$$

eqn (5) & (6) are ODE's in θ, r, respectively

Eigen value Problems (One Dimensional)

After separating variables we have obtain two eigen value problems. We are quite familiar with the θ-eigen value problem (9) with boundry conditions (11). Although it is not a regular Sturm-Liouville problem due to the periodic boundry conditions, we know that the eigen values are

$$\mu_m = m^2, \quad m = 0, 1, 2, \ldots \longrightarrow (12)$$

the corresponding eigen functions are both

$$g(\theta) = \sin m\theta \quad \& \quad g(\theta) = \cos m\theta \longrightarrow (13)$$

although for $m=0$ this reduces to one eigen function (not two as for $m \neq 0$). This eigen value problem generates a full fourier series in θ, as already know. m is the number of crests in the θ-direction.

For each integral value of m equ (13) helps to define as eigen value problem for λ.

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right)+(\lambda r^2-m^2)f=0 \longrightarrow (14)$$

$$\left.f(a)=0 \quad , \quad |f(0)|<\infty \right\} \longrightarrow (15)$$

since equ (14) has non-constant coefficients it is not surprising that equ (14) can be put as the S.L form by dividing it by r

$$\frac{d}{dr}\left(r\frac{df}{dr}\right)+\left(\lambda r-\frac{m^2}{r}\right)f=0 \longrightarrow (16)$$

It can be written as

$$\frac{d}{dr}\left\{r\frac{df}{dr}\right\}-\frac{m^2}{r}f(r)+\lambda r f(r)=0$$

Compare with S.L eque

$$\frac{d}{dx}\left\{P(x)\frac{dy}{dx}\right\}+q(x)y(x)+\lambda\sigma(x)f(x)=0$$

or $\mathcal{L}f+\lambda r f=0$ where

$\mathcal{L}=\frac{d}{dr}\left[r\frac{d}{dr}\right]-\frac{m^2}{r}$ by

comparing to general S.L differential equation

$$\frac{d}{dx}\left[P(x)\frac{d\phi}{dx}\right]+q\phi+\lambda\sigma\phi=0$$

with independent variable r, we have that $x=r$, $P(r)=r$, $\sigma(r)=r$ and $q(r)=-m^2/r$. Our problem is not a regular sturm Liouville problem due to the behaviour at the origin $(r=0)$

1) The boundary condition at $r=0$ is not of the correct form.

2) $p(r) = 0$ & $\sigma(r) = 0$ at $r = 0$ (and hence is not +ve every where)

3) $q(r) \to \infty$ as $r \to 0$ [and $q(r)$ is not continuous] for $m \neq 0$

However we claim that all the statements concerning regular S-L problem are still valid for this important singular Sturm-Liouville equation. To begin with there are an infinite number of eigen-values (for each m). We designate the eigen values as λ_{nm}, where $m = 0, 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$ and the eigen functions $f_{nm}(r)$. for each fixed m, these eigen functions are orthogonal with weight r [see sec 7.7.21], since it can be shown that the boundry terms vanish in green's formula [see Exercise 5.5.1]. Thus

$$\int_0^a f_{mn_1} f_{mn_2} \, r\,dr = 0 \quad \text{for } n_1 \neq n_2$$

Shortly, we will state more explicit facts about these eigenfunctions.

⇒ Bessel Differential Equation:-

The r dependent seperation of variables solutions satisfies a "singular" "Sturm Liouville Equation (16). An alternative form is obtained by using the product rule of differentiation and by multiplying by r.

$$r^2\frac{d^2f}{dr^2}+r\frac{df}{dr}+(\lambda r^2-m^2)f=0 \longrightarrow (17)$$

There is some additional analysis of eqn (17) that can be performed. Eqn (17) contains two parameters, m and λ. We already know that m is integer, but the allowable values of λ are as yet unknown. It would be quite tedious to solve numerically eqn (17) for various values of λ (for different values of m) Instead we might notice that the simple scaling transformation

$$z=\sqrt{\lambda}\,r$$

remove the dependence of the differential equation on λ

$$z^2\frac{d^2f}{dz^2}+z\frac{df}{dz}+(z^2-m^2)f=0 \longrightarrow (18)$$

It is known as Bessel Differential equation of order m.

⇒ **Bessel Functions & Their Asymptotic Properties**:- We continue to discuss Bessel differential equation of order m

$$z^2\frac{d^2f}{dz^2}+z\frac{df}{dz}+(z^2-m^2)f=0 \longrightarrow (18)$$

A motivated by the previous discussion, we claim there are two types of solutions, solutions that are well behaved near $z=0$ and solutions that are singular at $z=0$. Different values of m yields a different differential equation. Its corresponding solution will depend on m. We introduce the standard notation for a well-behaved solution of equ (18) $J_m(Z)$, called the Bessel function of the 1st kind of order m. In similar vein, $Y_m(Z)$, called the Bessel function of the 2nd kind of order m.

We can solve a lot of problems using Bessel's Differential equation by just remembering that $Y_m(Z) \to \pm\infty$ as $Z \to 0$.

The general solution of of any linear homogeneous second order differential equation is a linear combination of two independent solutions. Thus the general solution of B.D.E (18) is

$$f = C_1 J_m(Z) + C_2 Y_m(Z) \longrightarrow (19)$$

Precise definition of $J_m(Z)$ and $Y_m(Z)$ are given in sec 7.8. However for our immediate purposes, we simply note that they satisfy the following asymptotic properties for small Z ($Z \to 0$)

$$J_m(Z) \sim \begin{cases} 1 & \text{if } m = 0 \\ \dfrac{1}{2^m m!} Z^m & \text{if } m > 0 \end{cases}$$

$$Y_m(Z) \sim \begin{cases} \dfrac{2}{\pi} \ln Z & m = 0 \\ -\dfrac{2^m (m-1)!}{\pi} Z^{-m} & m > 0 \end{cases}$$

⇒ Eigen Value Problem Involving Bessel Functions :-

In this we determine the eigen values of singular SL problem (m fixed)

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)f = 0 \longrightarrow (1)$$

$$f(a) = 0 \longrightarrow (2) \qquad |f(0)| < \infty \longrightarrow (3)$$

By change of variables $z = \sqrt{\lambda}\, r$ equ (1) becomes

$$z^2 \frac{d^2 f}{dz^2} + z\frac{df}{dz} + (z^2 - m^2) f = 0 \longrightarrow (4)$$

The general solution of equ (4) is a linear combination of Bessel functions, $f = C_1 J_m(z) + C_2 Y_m(z)$.
The scale change implies that in terms of the radial coordinate r

$$f = C_1 J_m(\sqrt{\lambda}\, r) + C_2 Y_m(\sqrt{\lambda}\, r) \longrightarrow (5)$$

Applying the homogeneous boundary conditions will determine eigen values. $f(0)$ must be finite. However $Y_m(0)$ is infinite. Thus $C_2 = 0$, implies that

$$f = C_1 J_m(\sqrt{\lambda}\, r) \longrightarrow (6)$$

Thus the condition $f(a) = 0$ determine the eigen values

$$J_m(\sqrt{\lambda}\, a) = 0 \longrightarrow (7)$$

We see that $\sqrt{\lambda}\, a$ must be zero of the Bessel function $J_m(z)$. (Later in sec 7.8.1) We show that a Bessel function is a decaying oscillation. There is an infinite number of zeros of Bessel function $J_m(z)$. Let z_{mn} designate the nth zero of $J_m(z)$. Then

$$\sqrt{\lambda}\, a = z_{mn}$$

$$\text{or} \quad \lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2 \longrightarrow (8)$$

for each m there is an infinite number of eigenvalues. and equ (8) is analogous to $\lambda = \left(\frac{n\pi}{L}\right)^2$ where $n\pi$ are zeros of $\sin x$.

⇒ **Initial Value Problem for a Vibrating Membrane:-** The vibration $u(r,\theta,t)$ of a circular membrane are describe by the two dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \longrightarrow (1)$$

$$\text{B.C} \quad u(a,\theta,t) = 0 \longrightarrow (2)$$

$$\text{I.C} \quad \left.\begin{aligned} u(r,\theta,0) &= \alpha(r,\theta) \\ \frac{\partial u}{\partial t}(r,\theta,0) &= \beta(r,\theta) \end{aligned}\right\} \longrightarrow (3)$$

When we apply the method of seperation of variables we obtain four families of product solutions

$$u(r,\theta,t) = f(r)\, g(\theta)\, h(t)$$

$$\Rightarrow u(r,\theta,t) = J_m(\sqrt{\lambda_{mn}}\, r) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} \begin{Bmatrix} \cos c\sqrt{\lambda_{mn}}\, t \\ \sin c\sqrt{\lambda_{mn}}\, t \end{Bmatrix} \longrightarrow (4)$$

To simplify the algebra, we will assume that the membrane is initially at rest

$$\frac{\partial u}{\partial t}(r,\theta,0) = \beta(r,\theta) = 0$$

$\because R(t) = c_1 \cos c\sqrt{\lambda_{mn}}\, t + c_2 \sin c\sqrt{\lambda_{mn}}\, t$

$R'(t) = -c_1 c\sqrt{\lambda_{mn}} \sin c\sqrt{\lambda_{mn}}\, t + c_2 c\sqrt{\lambda_{mn}} \cos c\sqrt{\lambda_{mn}}\, t$

$R'(0) = 0$

$\Rightarrow 0 + c_2\sqrt{\lambda_{mn}} \cos c_2\sqrt{\lambda_{mn}}\, t$

$\Rightarrow c_2\sqrt{\lambda_{mn}}\,(1) = 0 \Rightarrow c_2 = 0$

Thus, The $\sin c\sqrt{\lambda_{mn}}\, t$ terms in equ ④ will not be necessary. Then according to principle of superposition we attempt to satisfy the initial values problem by considering the infinite linear combination of the ~~reading~~ remaining product solutions

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}}\, r) \cos m\theta \cos c\sqrt{\lambda_{mn}}\, t$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}}\, r) \sin m\theta \cos c\sqrt{\lambda_{mn}}\, t \longrightarrow 5$$

The initial position $u(r,\theta,0) = \alpha(r,\theta)$ implies that

$$\alpha(r,\theta) = \sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\right] \cos m\theta$$

$$+ \sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}}\, r)\right] \sin m\theta \longrightarrow ⑥$$

By properly arranging the terms in equ ⑥, we see that this is an ordinary Fourier series in θ. Their fourier coefficients are Fourier-Bessel series (note that m is fixed). Thus the coefficients may be determined by the orthogonality of $J_m(\sqrt{\lambda_{mn}}\, r)$ with weight r. As such we can determine the coefficients by repeated application of one dimensional orthogonality.

Two families of coefficients A_{mn} and B_{mn} (including $m=0$) can be determined from the initial condition since the periodicity in θ yielded two eigenfunctions corresponding to each eigenvalue.

However, It is somewhat easier to determine all the coefficients using two dimensional orthogonality. Recall that for the two dimensional eigen value problem,

$$\nabla^2 \phi + \lambda \phi = 0$$

with $\phi = 0$ on the circle of radius a, the two dimensional eigenfunctions are the doubly infinite families.

$$\phi_\lambda(r,\theta) = J_m(\sqrt{\lambda_{mn}}\, r) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix}$$

Thus,

$$\alpha(r,\theta) = \sum_\lambda A_\lambda \phi_\lambda(r,\theta) \longrightarrow (7)$$

where $\sum_\lambda$ stands for a summation over all eigenfunctions [actually two double sums] including both $\sin m\theta$ and $\cos m\theta$ as in (6). These eigenfunctions $\phi_\lambda(r,\theta)$ are orthogonal (in a two dimensional sence) with weight 1. We then immediately calculate A_λ (representing both A_{mn} and B_{mn})

$$A_\lambda = \frac{\iint \alpha(r,\theta)\phi_\lambda(r,\theta)\, dA}{\iint \phi_\lambda^2(r,\theta)\, dA}$$

Here $dA = r\,dr\,d\theta$. In two dimensions the weight function is constant. However, for geometric reasons $dA = r\,dr\,d\theta$. Thus the weight r that appears in the one dimensional orthogonality of Bessel functions is just a geometric factor.

Circularly Symmetric Case:-

In this subsection, as an example, we consider the vibration of a circular membrane, with $u=0$ on the circular boundry, in the case in which the initial conditions are circularly symmetric (meaning independent of θ). we could consider this as a special case of the general problem. The symmetry of the problem, including the initial conditions suggests that the entire solution should be circularly symmetric; there should be no dependence on the angle θ. Thus

$$u = u(r,t) \quad \text{and} \quad \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \quad \because \frac{\partial^2 u}{\partial \theta^2} = 0$$

The mathematical formulation is thus

PDE
$$\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \longrightarrow (1)$$

BC:
$$u(a,t) = 0 \longrightarrow (2)$$

IC
$$\left.\begin{aligned} u(r,0) &= \alpha(r) \\ \frac{\partial u}{\partial t}(r,0) &= \beta(r) \end{aligned}\right\} \longrightarrow (3)$$

We will apply method of separation of variable. Looking for product solution

$$u(r,t) = \phi(r)\,h(t) \longrightarrow (4)$$

Yelds

$$\frac{1}{c^2}\frac{1}{h}\frac{d^2h}{dt^2} = \frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda \quad \longrightarrow (5)$$

where $-\lambda$ is introduced because we suspect that the displacement oscillates in time. The time-dependent equation

$$\frac{d^2h}{dt^2} = -\lambda c^2 h$$

has solutions $\sin c\sqrt{\lambda}t$ & $\cos c\sqrt{\lambda}t$, $\lambda > 0$. The eigen value problem for the separation constant is

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0 \quad \longrightarrow (6)$$

$$\phi(a) = 0 \;, \; |\phi(0)| < \infty \quad \longrightarrow (7)$$

since equ (6) is the form of SL problem, we immidiately know that eigenfunctions corresponding to distinct eigenvalues are orthogonal with weight r.

From the Rayleigh quotient we could show that $\lambda > 0$, Thus we may use the transformation

$$z = \sqrt{\lambda}\, r \quad \longrightarrow (8)$$

In this case (6) becomes

$$\frac{d}{dz}\left(z\frac{d\phi}{dz}\right) + z\phi = 0 \quad \text{or} \quad z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + z^2\phi = 0 \quad \longrightarrow (9)$$

We may recall Bessel's differential equation of order m is

$$z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + (z^2 - m^2)\phi = 0 \quad \longrightarrow (10)$$

with solutions being Bessel functions of order m, $J_m(z)$ and $Y_m(z)$. A comparison with eqn ⑩ shows that eqn ⑨ is Bessel's differential equation of order 0. The general solution of eqn ⑨ is thus a linear combination of zeroth order B. functions

$$\phi = C_1 J_0(z) + C_2 Y_0(z) = C_1 J_0(\sqrt{\lambda}\, r) + C_2 Y_0(\sqrt{\lambda}\, r) \longrightarrow (11)$$

In terms of the radial variable. The singularity condition at the origin ie $|\phi(0)| < \infty$ shows that $C_2 = 0$, since $Y_0(\sqrt{\lambda}\, r)$ has a logarithmic singularity at $r=0$

$$\Rightarrow \phi = C_1 J_0(\sqrt{\lambda}\, r) \longrightarrow (12)$$

Finally the eigen values are determined by the condition at $r=a$, in which

$$J_0(\sqrt{\lambda}\, r) = 0 \longrightarrow (13)$$

Thus, $\sqrt{\lambda}\, a$ must be a zero of the zeroth Bessel function. We thus obtain an infite number of eigen values, which we label $\lambda_1, \lambda_2, \ldots$

We have obtained to infinite families of product solutions

$J_0(\sqrt{\lambda_n}\, r) \sin c\sqrt{\lambda_n}\, t$ & $J_0 \sqrt{\lambda_n} \cos c\sqrt{\lambda_n}\, t$

According to the principle of superposition we have

$$u(r,t) = \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_n}\, r) \cos c\sqrt{\lambda_n}\, t + \sum_{n=1}^{\infty} b_n J_0(\sqrt{\lambda_n}\, r) \sin c\sqrt{\lambda_n}\, t \longrightarrow (14)$$

As before, we determine the coefficients a_n and b_n from the initial conditions $u(r,0) = \alpha(r)$ implies that

$$\alpha(r) = \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_n}\, r) \longrightarrow (15)$$

The coefficients a_n are thus the fourier-Bessel coefficients (of order 0) of $\alpha(r)$. Since $J_0(\sqrt{\lambda_n}\, r)$ forms an orthogonal set with weight r, we can easily determine a_n,

$$a_n = \frac{\int_0^a \alpha(r) J_0(\sqrt{\lambda_n}\, r)\, r\, dr}{\int_0^a J_0^2(\sqrt{\lambda_n}\, r)\, r\, dr} \longrightarrow (16)$$

In a similar manner $\frac{\partial u}{\partial t}(r,0) = \beta(r)$ determines b_n.

Question 7.7.1:- Solve as simply as possible

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

with $u(a,\theta,t) = 0$, $u(r,\theta,0) = 0$
and $\frac{\partial u}{\partial t}(r,\theta,0) = \alpha(r)\sin 3\theta$

Solution Hint $u = f(r)h(t)\sin 3\theta$

P.D.E $$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2}\right)\right]$$

substitution for u

$$f \sin 3\theta \frac{d^2 h}{dt^2} = c^2\left[h\sin 3\theta \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{1}{r^2}\, 9\sin 3\theta\, f h\right]$$

$$\Rightarrow \frac{1}{c^2 h}\frac{d^2 h}{dt^2} = \frac{1}{f}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{9}{r^2}f\right] = -\lambda$$

$$\Rightarrow \frac{d^2 h}{dt^2} + \lambda c^2 h = 0$$

$$\& \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda - \frac{9}{r^2}\right)f = 0$$

B.C $\quad u(a,\theta,t) = 0 \Rightarrow f(a)\sin 3\theta \, h(t) = 0$

$$\Rightarrow \boxed{f(a) = 0}$$

I.C's $\quad u(r,\theta,0) = 0 \Rightarrow f(r)\sin 3\theta \, h(0) = 0$

$$\Rightarrow \boxed{h(0) = 0}$$

and $\frac{\partial u}{\partial t}(r,\theta,0) = \alpha(r)\sin 3\theta$

$$\Rightarrow f(r)\sin 3\theta \frac{dh}{dt}(0) = \alpha(r)\sin 3\theta$$

$$\Rightarrow \boxed{f(r)\frac{dh}{dt}(0) = \alpha(r)}$$

Solution to ODE for $f(r)$ is

$$f(r) = C_1 J_3(\sqrt{\lambda}\, r) + C_2 Y_3(\sqrt{\lambda}\, r)$$

Apply $|f(0)| < \infty \Rightarrow C_2 = 0$

use $f(a) = 0 \Rightarrow J_3(\sqrt{\lambda}\, a) = 0$

$$\Rightarrow \sqrt{\lambda}\, a = Z_{3n}$$

{nth zero of $J_3(Z) = Z_{3n}$

$$\Rightarrow \lambda_n = \left(\frac{Z_{3n}}{a}\right)^2$$

Thus

$$\boxed{f_n(r) = C_n J_3(\sqrt{\lambda_n}\, r)}$$

Now solve ODE for $h(t)$

$$h(t) = A\cos(c\sqrt{\lambda}t) + B\sin(c\sqrt{\lambda}t)$$

using homogeneous boundry condition

$$h(0) = 0 \Rightarrow A = 0$$

$$\Rightarrow h_n(t) = B_n \sin(c\sqrt{\lambda_n}\, t)$$

Product solution

$$u = f(r)\, h(t) \sin 3\theta$$

$$u_n = f_n(r)\, h_n(t) \sin 3\theta$$

using Principle of superposition

$$u = \sum_{n=1}^{\infty} u_n$$

$$\Rightarrow u = \sum_{n=1}^{\infty} a_n J_3(\sqrt{\lambda_n}\, r) \sin(\sqrt{c\,\lambda_n}\, t) \sin 3\theta$$

Now find a_n from ICs $\frac{\partial u}{\partial t}(r,\theta,0) = \alpha(r)\sin 3\theta$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} a_n J_3(\sqrt{\lambda_n}\, r)\, c\sqrt{\lambda_n} \cos(c\sqrt{\lambda_n}\, t) \sin 3\theta$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n J_3(\sqrt{\lambda_n}\, r)\, c\sqrt{\lambda_n}\, \cancel{\sin 3\theta} = \alpha(r)\, \cancel{\sin 3\theta}$$

use orthogonality : apply $\int_0^a (\cdot)\, J_3(\sqrt{\lambda_m}\, r)\, r\,dr$

$$\Rightarrow \sum_{n=1}^{\infty} a_n c\sqrt{\lambda_n} \int_0^a J_3(\sqrt{\lambda_n}\, r)\, J_3(\sqrt{\lambda_m}\, r)\, r\,dr = \int_0^a \alpha(r)\, J_3(\sqrt{\lambda_m}\, r)\, r\,dr$$

$$\Rightarrow a_m c\sqrt{\lambda_m} \int_0^a \left(J_3(\sqrt{\lambda_m}\, r)\right)^2 r\,dr = \int_0^a \alpha(r)\, J_3(\sqrt{\lambda_m}\, r)\, r\,dr$$

Thus
$$a_n = \frac{\frac{1}{c\sqrt{\lambda_n}}\int_0^a \alpha(r)\, J_3(\sqrt{\lambda_n}\, r)\, r\, dr}{\int_0^a \left(J_3(\sqrt{\lambda_n}\, r)\right)^2 r\, dr}$$

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NON HOMOGENEOUS PROBLEMS.

⇒ Steady State or Equilibrium State Temperature Distribution:-

If u depends upon t also then the flow of heat is time dependent or non-steady. And when $t \to \infty$ i.e after a long time the non-steady temprature tends to become steady. If $r(x)$ represents steady flow of temperature then

$$u(x,t) \longrightarrow r(x) \quad \text{when } t \to \infty$$

Now if we put

$$\lim_{t \to \infty} u(x,t) = r(x) \quad \text{then from ①}$$

$$\boxed{\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \to ① \\ u(x,0) &= f(x) \to ② \\ u(0,t) &= T_0 \to ③ \\ u(a,t) &= T_1 \to ④ \end{aligned}}$$

$$\frac{\partial^2 r}{\partial x^2} = 0 \longrightarrow ⑤ \qquad \because \frac{\partial u}{\partial t} = 0 \text{ at } t \to \infty$$

from ③ $\quad r(0) = T_0 \longrightarrow ⑥$

from ④ $\quad r(a) = T_1 \longrightarrow ⑦$

Now from ⑤

$$r(x) = Ax + B$$

$$r(0) = T_0 \quad \Rightarrow \quad B = T_0$$

$$r(a) = T_1 \quad \Rightarrow \quad Aa + B = T_1 \quad \Rightarrow A = \frac{T_1 - T_0}{a}$$

Hence

$$r(x) = \left[\frac{T_1 - T_0}{a}\right]x + T_0 \longrightarrow ⑧$$

which is required solution of steady state problem.

⇒ Trasient Temperature Distribution:

If we define

$$v(x,t) = u(x,t) - r(x) \longrightarrow ⑨$$

Then $\lim\limits_{t\to\infty} v(x,t) = r(x) - r(x)$
$= 0$

i.e $v(x,t)$ is non-zero only for that value of t which are not very large. $v(x,t)$ is called transient temperature distribution.

Now from ⑨

$$u(x,t) = v(x,t) + r(x)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} + \frac{dr}{dx}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + 0 \qquad \because \frac{d^2 r}{dx^2} = 0 \text{ from ⑤}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2}$$

Also

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t}$$

using these values equ ① take the form

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial v}{\partial t} \longrightarrow ⑩$$

Now by ⑨ $v(x,0) = u(x,0) - r(x) = f(x) - r(x) = g(x)$

$\Rightarrow v(x,0) = g(x)$

$v(0,t) = u(0,t) - v(0) = T_0 - T_0 = 0$

$v(a,t) = u(a,t) - v(a) = T_1 - T_1 = 0$

Hence by ⑨ the given problem reduces to

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial v}{\partial t}$$

$$v(x,0) = f(x) - r(x)$$

$$v(0,t) = 0 \quad , \quad v(a,t) = 0$$

Now the boundry conditions are homogeneous, so the system can be solved by standard method of separation of variables (S.O.V)

⇒ Heat Flow With Sources And Non-Homogeneous Boundry Conditions:

★ Time Independent Boundry Conditions:-

Consider the heat flow (without sources) in a uniform rod of length L with the temprature fixed at the left end at $A°$ and right at $B°$. If the initial condition is prescribed, the mathematical problem for the temperature $u(x,t)$ is

$$\text{PDE:} \quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$$

$$\text{BC1:} \quad u(0,t) = A \longrightarrow (2)$$

$$\text{BC2:} \quad u(L,t) = B \longrightarrow (3)$$

$$\text{I.C:} \quad u(x,0) = f(x) \longrightarrow (4)$$

The method of seperation of variable can not be used directly since for even this simple example the boundry conditions are not homogeneous.

⇒ Equilibrium Temperature:-

To analyze

this problem, we first obtain an equilibrium temprature distribution, $u_E(x)$. If such a temperature distribution exists, it must satisfy the steady-state (time independent) heat equation

$$\frac{d^2 u_E}{dx^2} = 0 \longrightarrow (5)$$

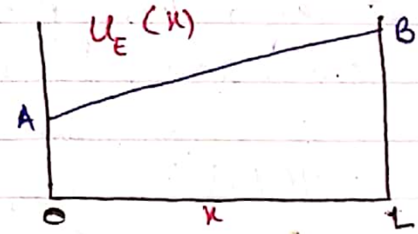
As well as the given boundry conditions

$$u_E(0) = A \longrightarrow (6) \qquad u_E(L) = B \longrightarrow (7)$$

We ignore the initial conditions in defining an equilibrium temprature distribution. (As shown in sec 1.4) Equ (5) implies that the temprature distribution is linear, and the unique one that satisfies equ (2) & (3) can be determined geometrically or algebraically

$$u_E(x) = A + \frac{B-A}{L} x \longrightarrow (8)$$

which is sketched as



*** Displacement From Equilibrium:-** For more general initial conditions, we consider the temperature displacement from the equilibrium temprature

$$v(x,t) = u(x,t) - u_E(x) \longrightarrow (9)$$

Instead of solving for $u(x,t)$, we will determine $v(x,t)$. Since

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} \quad \& \quad \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}$$

note that $u_E(x)$ is linear in x

It follows that $v(x,t)$ also satisfies the heat equation

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} \longrightarrow (10)$$

Furthermore, both $u(x,t)$ and $u_E(x)$ equal A at $x=0$, and equal B at $x=L$, and hence their difference is zero at $x=0$ & at $x=L$

$$v(0,t) = 0 \longrightarrow (11) \quad \& \quad v(a,t) = 0 \longrightarrow (12)$$

Initially $v(x,t)$ equals the difference between the given initial temprature and the equilibrium temprature

$$v(x,0) = f(x) - u_E(x) \longrightarrow (13)$$

Fortunately, the mathematical problem for $v(x,t)$ is a linear homogeneous partial differential equation with linear homogeneous boundry condition. Thus $v(x,t)$ can be determined by the method of separation of variables. In fact, this problem is one we have encountered frequently. Hence, we note that

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} \longrightarrow (14)$$

where the initial condition implies that

$$f(x) - u_E(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \longrightarrow (15)$$

Thus a_n equal the fouries sine coefficients of $f(x) - u_E(x)$

$$\Rightarrow a_n = \frac{2}{L}\int_0^L [f(x) - u_E(x)] \sin\frac{n\pi x}{L}\,dx \longrightarrow (16)$$

from eqn ⑨ we easily obtain the desired temperature $u(x,t) = u_E(x) + v(x,t)$, Thus

$$u(x,t) = u_E(x) + \sum_{n=1}^{\infty} a_n \sin e^{-k\left(\frac{n\pi}{L}\right)^2 t} \longrightarrow (17)$$

where a_n is given in eqn (16) & $u_E(x)$ is given by ⑧. As $t \to \infty$, $u(x,t) \to u_E(x)$ irrespective of the initial condition. The temperature approaches its equilibrium distribution for all initial conditions.

⇒ Steady Non-Homogeneous Terms:-

The previous method also works if there are steady sources of thermal energy:

$$\text{PDE}: \quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x) \longrightarrow (18)$$

$$\text{BC}: \quad \left.\begin{aligned} u(0,t) &= A \\ u(L,t) &= B \end{aligned}\right\} \longrightarrow (19)$$

$$\text{IC}: \quad u(x,0) = f(x) \longrightarrow (20)$$

If an equilibrium solution exists (see exercise 1.4.6 for a somewhat different example where an equilibrium solution does not exist), then we determine it and again consider the displacement from equilibrium,

$$v(x,t) = u(x,t) - u_E(x)$$

we can show that $v(x,t)$ satisfies a linear homogeneous partial differential equation (10) with linear homogeneous B.Cs. Thus again $u(x,t) \to u_E(x)$ as $t \to \infty$.

⇒Time-dependent Non Homogeneous Terms:-

Unfortunately, non homogeneous problems are not always as easy to solve as the previous examples. In order to clarify the situation, we again consider the heat flow in a uniform rod of length L. However we make two substantial changes. First we introduce temperature-dependent heat sources distribution in a prescribed way throughout the rod. Thus, the temprature will solve the following non-homogeneous partial differential equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \longrightarrow (21)$$

$$\text{B.Cs}: \left.\begin{aligned} u(0,t) &= A(t) \\ u(L,t) &= B(t) \end{aligned}\right\} \longrightarrow (22)$$

$$\text{I.C}: \quad u(x,0) = f(x) \longrightarrow (23)$$

The mathematical problem defined by (21) consists of non-homogeneous partial differential equation with non-homogeneous boundry conditions.

Related Homogeneous Boundry Conditions:-

We claim that we can not always reduce this problem to a homogeneous PDE with homogeneous boundry conditions, as we did for the first example of this section. Instead, we will find it quite useful to note that we can always transform our problem into one with homogeneous boundry conditions, although in general the PDE will remain non-homogeneous. We consider any reference temperature distribution $r(x,t)$ (the simpler the better) with only the property that it satisfy the given non-homogeneous boundry conditions. In our example this mean only that

$$r(0,t) = A(t), \quad r(L,t) = B(t)$$

It is usually not difficult to obtain many candidates for $r(x,t)$. Perhapes the simplest choice is

$$r(x,t) = A(t) + \frac{x}{L}(B(t) - A(t)) \longrightarrow (24)$$

Again the difference between the desired solution $u(x,t)$ and the chooçen function $r(x,t)$ [not not necessary an equilibrium solution] is employed

$$v(x,t) = u(x,t) - r(x,t) \longrightarrow (25)$$

Since both $u(x,t)$ and $r(x,t)$ satisfy the same linear (although non homogeneous) boundry condition at both $x=0$ and $x=L$. It follows that $v(x,t)$ satisfies the related homogeneous boundry conditions

$$v(0,t) = 0 \longrightarrow (26) \qquad v(L,t) = 0 \longrightarrow (27)$$

The PDE satisfied by $v(x,t)$ is derived by substituting

$$u(x,t) = v(x,t) + r(x,t)$$

into the heat equation with sources.

Thus,

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \left[Q(x,t) - \frac{\partial r}{\partial t} + k\frac{\partial^2 r}{\partial x^2} \right]$$

$$\Rightarrow \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \bar{Q} \longrightarrow (28)$$

In general the partial differential equation for $v(x,t)$ is of the same type as for $u(x,t)$, but with a different nonhomogeneous terms. since $r(x,t)$ usually does not satisfy the homogeneous heat equation. The initial condition is also usually altered.

$$v(x,0) = f(x) - r(x,0)$$

$$= f(x) - A(0) - \frac{x}{L}[B(0) - A(0)] \equiv g(x) \longrightarrow (29)$$

It can be seen that in general only the boundry conditions have been made homogeneous. In sec 8.3 we will develop a method to analyze nonhomogeneous problem with homogenous boundry conditions.

⇒ Method of Eigenfunction Expansion:

(With Homogeneous Boundry Conditions)

PDE : $\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \bar{Q}(x,t) \longrightarrow (1)$

B.C : $\left.\begin{array}{l} v(0,t) = 0 \\ v(L,t) = 0 \end{array}\right\} \longrightarrow (2)$

IC : $v(x,0) = g(x) \longrightarrow (3)$

We solve this problem by the method of eigenfunction expansion. Consider the eigen functions of the related homogeneous problem. The related homogeneous problem is

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \longrightarrow (4)$$

$$u(0,t) = 0 \quad , \quad u(L,t) = 0$$

The eigen functions of this related homogeneous problem satisfy

$$\frac{d^2\phi}{dx^2} + \lambda \phi = 0 \longrightarrow (5)$$

$$\phi(0) = 0 \quad , \quad \phi(L) = 0$$

We know that the eigenvalues are $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, 3, \ldots$ and the corresponding eigen functions are $\phi_n(x) = \sin\left(\frac{n\pi}{L}x\right)$. However, the eigen functions will be different for other problems. We assume that the eigenfunctions (of related homogeneous problem) are known, and we designate them $\phi_n(x)$. The eigenfunction satisfy a SL eigen value problem and as such they are complete (The piecewise smooth function may be expanded in a series of these eigenfunctions). The method of eigen function expansion, employed to solve the non-homogeneous problem (1) with homogeneous boundary conditions (2), consisting of expanding the unknown solution $v(x,t)$ in a series of the related homogeneous eigenfunctions.

$$V(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x) \longrightarrow (6)$$

For each fixed t, $V(x,t)$ is a function of x, and hence $V(x,t)$ will have a generalized Fourier series. In our example $\phi_n(x) = \frac{\sin n\pi x}{L}$, and this series in an ordinary Fourier sine series. The generalized fourier coefficients are a_n, but the coefficients will vary as t changes. Thus the generalized fourier coefficients are functions of time, $a_n(t)$.

The initial condition satisfied if

$$g(x) = \sum_{n=1}^{\infty} a_n(0)\,\phi_n(x)$$

due to the orthogonality of eigen functions we can determine the initial values of the generalized Fourier coefficients.

$$a_n(0) = \frac{\int_0^L g(x)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx} \longrightarrow (7)$$

Now we proceed to term-by-term differentiative $V(x,t)$

$$\frac{\partial V}{\partial t} = \sum_{n=1}^{\infty} \frac{da_n(t)}{dt}\,\phi_n(x)$$

$$\frac{\partial^2 V}{\partial x^2} = \sum_{n=1}^{\infty} a_n(t)\frac{d^2\phi_n(x)}{dx^2} = -\sum_{n=1}^{\infty} a_n(t)\,\lambda_n\,\phi_n(x)$$

since $\phi_n(x)$ satisfies $d^2\phi_n/dx^2 + \lambda_n\phi_n = 0$. Substituting these results into equ (1) yields

$$\sum_{n=1}^{\infty}\left[\frac{da_n}{dt}+\lambda_n k a_n\right]\phi_n(x) = \bar{Q}(x,t) \longrightarrow (8)$$

The left hand side is generalized fourier series for $\bar{Q}(x,t)$. Due to the orthogonality of $\phi_n(x)$. we obtain a 1st order differential equation for $a_n(t)$

$$\frac{da_n}{dt}+\lambda_n k a_n = \frac{\int_0^L \bar{Q}(x,t)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx} \equiv \bar{q}_n(t) \longrightarrow (9)$$

The right hand side is a known function of time (and n), namely, the fourier coefficient of $\bar{Q}(x,t)$:

$$\bar{Q}(x,t) = \sum_{n=1}^{\infty}\bar{q}_n(t)\,\phi_n(x)$$

Equation (9) requires an initial condition, and sure enough $a_n(0)$ equals the generalized fourier coefficients of the initial condition.

Equ (9) is a non-homogeneous linear first order equation. To solve it is to multiply it by I.F

$$\text{I.F is } e^{\int k\lambda_n dt} = e^{k\lambda_n t}$$

$$\Rightarrow e^{k\lambda_n t}\left[\frac{da_n}{dt}+\lambda_n k a_n\right] = \bar{q}_n e^{\lambda_n k t}$$

$$\Rightarrow \frac{d}{dt}\left[a_n e^{\lambda_n k t}\right] = \bar{q}_n e^{\lambda_n k t}$$

Integrating from 0 to t yields

$$a_n(t)e^{\lambda_n k t} - a_n(0)e^{k\lambda_n(0)} = \int_0^t \bar{q}_n(t)\,e^{k\lambda_n t}$$

$$\Rightarrow a_n(t)e^{\lambda_n kt} - a_n(0) = \int_0^t \bar{q}_n(t)\, e^{\lambda_n kt}\, dt$$

$$\Rightarrow a_n = \left[a_n(0) + \int_0^t \bar{q}_n(t)e^{\lambda_n kt}\, dt\right]e^{-k\lambda_n t} \longrightarrow (10)$$

Note that $a_n(t)$ is in the form of a constant, $a_n(0)$, times the homogeneous solution $e^{-\lambda_n kt}$ plus a particular solution. This completes the method of eigenfunction expansions. The solution of our nonhomogeneous partial differential equation with homogeneous boundry conditions is

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\, \phi_n(x)$$

where $\phi_n(x) = \sin n\pi x/L$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$. $a_n(t)$ is given by equ (10), $\bar{q}_n(t)$ is given by (9) and $a_n(0)$ is given by (7). The solution is rather complicated.

As a check, if the problem was homogeneous, $Q(x,t) = 0$. Then the solution simplifies to

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\, \phi_n(x)$$

where $a_n(t) = a_n(0)\, e^{-\lambda_n kt}$

and $a_n(0)$ is given by equ (7), exactly the solution obtained by seperation o.v.

Example (Sec 8.3):- As an elementary example suppose that for $0<x<\pi$ (i.e $L=\pi$)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin 3x\, e^{-t}$$

subject to $u(0,t)=0$, $u(\pi,t)=1$

$$u(x,0) = f(x).$$

Hence $r(x,t) = 0 + \frac{1-0}{\pi}x = \frac{x}{\pi}$

To make the boundry conditions homogeneous, we introduce the displacement from equilibrium

$v(x,t) = u(x,t) - x/\pi$ in which case

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \sin 3x\, e^{-t} \quad \text{subject to}$$

$$v(0,t) = 0 \quad , \quad v(\pi,t) = 0$$

$$v(x,0) = f(x) - x/\pi$$

The eigenfunctions are $\sin\frac{n\pi x}{L} = \sin nx$ ($\because L=\pi$) and Thus

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin nx \longrightarrow (11)$$

This eigenfunction expansion is substituted into the PDE, yielding

$$\sum_{n=1}^{\infty}\left[\frac{da_n}{dt} + n^2 a_n\right]\sin nx = \sin 3x\, e^{-t}$$

Thus the unknown fourier coefficients

$$\frac{da_n}{dt} + n^2 a_n = \frac{\int_0^{\pi}\sin 3x\, e^{-t}\sin nx\, dx}{\int_0^{\pi}\sin^2 nx\, dx}$$

Thus unknown Fourier sine coefficients satisfy

$$\frac{da_n}{dt} + n^2 a_n = \begin{cases} 0 & n \neq 3 \\ e^{-t} & n = 3 \end{cases}$$

The solution of this does not require (10)

$$a_n(t) = \begin{cases} a_n(0)e^{-n^2 t} & n \neq 3 \\ \frac{1}{8}e^{-t} + \left[a_3(0) - \frac{1}{8}\right]e^{-9t} & n=3 \end{cases}$$

where $\longrightarrow$ (12)

$$a_n(0) = \frac{2}{\pi}\int_0^{\pi}\left(f(x) - \frac{x}{\pi}\right)\sin nx \, dx \longrightarrow (13)$$

The solution to the original nonhomogeneous problem is given by $u(x,t) = v(x,t) + \frac{x}{\pi}$, where v satisfies eqn (11) with $a_n(t)$ determined from eqn (12) & (13)

Question 7.2.1:- For a vibrating membrane of any shape, show that the displacement u satisfies $\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$ after separating time. $\nabla^2\phi = -\lambda\phi$

Already Solved.

Question 7.2.2:- For heat equation (in any two D region) i.e $\partial u/\partial t = k(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2)$, show that
$\nabla^2\phi = -\lambda\phi$

Already Solved.

Question 7.2.3:-(a) obtain product solution $\phi = f(x)g(y)$ of, of $\nabla^2\phi = -\lambda\phi$ that satisfies $\phi = 0$ on the four sides of a rectangle.

Solution Let $\phi(x,y) = f(x)\,g(y)$

$$\Rightarrow f''(x)\,g(y) + g''(y)\,f(x) = -\lambda f(x)\,g(y)$$

$$\Rightarrow \frac{f''}{f} + \frac{g''}{g} = -\lambda \quad \Rightarrow \frac{f''}{f} = -\lambda - \frac{g''}{g} = -\mu$$

$$\Rightarrow f'' + \mu f = 0 \longrightarrow (1) \quad \&$$

$$g'' + g(\lambda - \mu) = 0 \longrightarrow (2)$$

Now we have to solve equ (1) & (2) in order to get product solution $\phi = f\cdot g$

Consider equ (1) $f'' + \mu f = 0$

Given that $\phi = 0$ on all four sides

$\Rightarrow \phi(0,y,t) = 0$, $\phi(L,y,t) = 0$

$\phi(x,0,t) = 0$, $\phi(x,H,t) = 0$

Auxiliary equation for ① is

$m^2 + \mu = 0$

Case 1:- If $\mu = 0$ $\boxed{\Rightarrow m^2 = 0 \quad \Rightarrow m = 0, 0}$

$\Rightarrow f(x) = C_1 + C_2 x$ B.Cs $f(0) = 0 \Rightarrow C_1 = 0$

$f(L) = 0 \Rightarrow C_1 + C_2 L = 0$

$\Rightarrow C_2 L = 0$

$L \neq 0 \Rightarrow C_2 = 0$

So we get a trivial solution for $\mu = 0$.

Case 2:- If $\mu < 0$. Let $\mu = -p^2$

$\Rightarrow m^2 - p^2 = 0 \quad \Rightarrow m = \pm p$

$\Rightarrow f(x) = C_1 e^{px} + C_2 e^{-px}$

B.Cs $f(0) = 0 \Rightarrow C_1 + C_2 = 0$

& $f(L) = 0 \Rightarrow C_1 e^{Lp} + C_2 e^{-Lp} = 0$

$\Rightarrow 2C_1 \sinh Lp = 0$

$\Rightarrow \boxed{C_1 = 0} \quad \because \sinh Lp \neq 0$

$\Rightarrow C_2 + 0 = 0 \Rightarrow \boxed{C_2 = 0}$

So we again get a trivial solution for $\mu < 0$

Case 3:- If $\mu > 0$. Let $\mu = p^2$

$\Rightarrow m^2 + p^2 = 0 \quad \Rightarrow m = \pm ip$

$\Rightarrow f(x) = C_1 \cos px + C_2 \sin px$

B.Cs $f(0) = 0 \Rightarrow \boxed{C_1 = 0}$

$f(L) = 0 \Rightarrow C_2 \sin Lp = 0$

for non trivial solution $C_2 \neq 0$

$\Rightarrow \sin Lp = 0 \Rightarrow Lp = n\pi$, $n = 1, 2, 3, \ldots$

$\Rightarrow p = \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$

$\Rightarrow \mu_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, 3, \ldots$

& $f(x) = C_2 \sin \frac{n\pi}{L} x$ is eigen function. $\longrightarrow$ ⊛

Now for equ ②

$$g'' + (\lambda - \mu) g = 0$$

which is the same equation as before and we have same b.cs. So the non-trivial solution exists for $(\lambda - \mu) > 0$

Let $\lambda - \mu = q^2$ so

$$g(y) = c_1 \cos qy + c_2 \sin qy$$

B.Cs $g(0) = 0 \Rightarrow c_1 = 0$

$g(H) = 0 \Rightarrow c_2 \sin qH = 0$

$\Rightarrow C_2 \neq 0 \quad \therefore \sin qH = 0 \Rightarrow qH = m\pi$

$\Rightarrow q = \frac{m\pi}{H}$, $m = 1, 2, 3, \ldots$

$\Rightarrow \lambda - \mu = \left(\frac{m\pi}{H}\right)^2$, $m = 1, 2, 3, \ldots$

$\Rightarrow \lambda_{mn} = \left(\frac{m\pi}{H}\right)^2 + \mu = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2$; $m, n = 1, 2, 3, \ldots$

& eigen functions are

$$g_{mn} = C_2 \sin\left(\frac{m\pi}{H}\right) y \longrightarrow ⊛⊛$$

from ⊛ & ⊛⊛ we have

$$\phi(x,y) = C_{2_n} \sin\left(\frac{n\pi}{L} x\right) C_{2_m} \sin\left(\frac{m\pi}{H} y\right)$$

$m, n = 1, 2, 3, \ldots$

Question 7.2.3:- (b) using part (a) solve the initial value problem for a vibrating rectangular membrane (fixed on all sides)

Already Solved page 132

Question 7.2.3:- (c) using part (a) solve initial value problem for heat equation with zero temprature on all sides.

See page 135

Question 7.3.1:- Consider the heat equation in a two dimensional rectangular region $0<x<L$, $0<y<H$, $\partial u/\partial t = k[\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2]$ subject to the initial condition $u(x,y,0) = f(x,y)$

Solve the initial value problem and analyze the temprature $t \to \infty$ if the boundry conditions are.

(a) $u(0,y,t)=0$, $u(L,y,t)=0$

$u(x,0,t)=0$, $u(x,H,t)=0$

Solution Given equation is

$$\frac{\partial u}{\partial t} = k\left[\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2\right]$$

using separation of variables $u(x,y,t) = \phi(x,y)\,h(t) \longrightarrow *$

we get

$$h'\phi = kH\nabla^2\phi$$

$$\Rightarrow \frac{h'}{kh} = \frac{1}{\phi}\nabla^2\phi = -\lambda \text{ (a sep constant)}$$

$$\Rightarrow h' + k\lambda h = 0 \longrightarrow (1)$$

$$\& \quad \nabla^2\phi + \lambda\phi = 0 \longrightarrow (2)$$

Now equ ① is ODE and ② is still PDE. So ② can still be simplified to two ODE because of being homogeneous with homogeneous boundry conditions. So

$$\phi(x,y) = f(x)\cdot g(y) \quad \text{in ②}$$

equ $\Rightarrow f_n(x) = C_{2n} \sin \frac{n\pi}{L}x$ with $\mu_n = \left(\frac{n\pi}{L}\right)^2$, $n=1,2,3,\dots$

& $g_m(y) = C_{2m} \sin \frac{m\pi}{H}y$ & $\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$, $n, m = 1,2,3,\dots$

$$\Rightarrow \phi(x,y) = C_{2n} C_{2m} \sin \frac{n\pi}{L}x \sin \frac{m\pi}{H}y$$

Now for equ ①

$$h' + k\lambda h = 0$$

A.E is $m + k\lambda = 0 \Rightarrow m = -k\lambda$

for $\lambda = \lambda_{mn}$, $m = -k\lambda_{mn}$

$$\Rightarrow H(t) = C_1 e^{-k\lambda_{mn}t} \longrightarrow ③$$

$$\Rightarrow u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{2n}C_{2m} \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right) e^{-k\lambda_{mn}t} \longrightarrow ④$$

$$\Rightarrow u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right) e^{-k\lambda_{mn}t}$$

To find value of A_{nm}, we use initial condition which is $u(x,y,0) = f(x,y)$

$$\Rightarrow f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right) (1)$$

$$\Rightarrow A_{nm} = \frac{2}{L}\int_0^L \left[\frac{2}{H}\int_0^H \left(\sin \frac{m\pi y}{H} \cdot f(x,y)\,dy\right)\sin\left(\frac{n\pi}{L}x\right) dx\right.$$

And since we have rectangular region

So two iterated one dimensional integrals can be simplified a single double integral as

$$A_{nm} = \frac{4}{LH}\int_0^L\int_0^H f(x,y)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)dy\,dx \longrightarrow (5)$$

Solution of initial value problem is given by equ (4) & coefficients are determined by (5).

(b) Boundary conditions are given by

$$\frac{\partial u}{\partial x}(0,y,t)=0 \quad , \quad \frac{\partial u}{\partial x}(L,y,t)=0$$

$$\frac{\partial u}{\partial y}(x,0,t)=0 \quad , \quad \frac{\partial u}{\partial y}(x,H,t)=0$$

Solution Given heat equation is

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

By simplifying we get

$$h' + k\lambda h = 0 \longrightarrow (1) \quad \& \quad \nabla^2\phi + \lambda\phi = 0 \longrightarrow (2)$$

from (2) Let $\phi(x,y) = f(x)\,g(y)$

$$\Rightarrow \quad f''/f + g''/g = -\lambda$$

$$\text{or} \quad f''/f = -\lambda - g''/g = -\mu \text{ (a sep constant)}$$

$$\Rightarrow f''/f = -\mu \Rightarrow f'' + \mu f = 0 \longrightarrow (3)$$

$$\& \quad g'' + (\lambda-\mu)g = 0 \longrightarrow (4)$$

first we solve equ (3)

$f'' + \mu f = 0$

Auxiliary equation is $m^2 + \mu = 0$

following three cases arise.

Case 1:- If $\mu = 0 \Rightarrow m^2 = 0 \Rightarrow m = 0, 0$

$f(x) = C_1 + C_2 x$

& boundry conditions are

$f_x(0) = 0$ & $f_x(L) = 0$

Now $f_x = C_2$

Applying B.C $f_x(0) = 0 \Rightarrow C_2 = 0$

$f_x(L) = 0 \Rightarrow C_2 = 0$

Case 2 If $\mu < 0$, let $\mu = -p^2$.

A.E is $m^2 - p^2 = 0 \Rightarrow m = \pm p$

$\Rightarrow f(x) = C_1 e^{px} + C_2 e^{-px}$

$\Rightarrow f'(x) = p[C_1 e^{px} - C_2 e^{-px}]$

$f'(0) = 0 \Rightarrow C_1 - C_2 = 0 \Rightarrow \boxed{C_1 = C_2}$

$f'(L) = 0 \Rightarrow C_1 e^{Lx} - C_2 e^{-Lx} = 0 \Rightarrow 2C_1 \sinh Lx = 0$

$\Rightarrow \boxed{C_1 = 0}$ Trivial sol.

Case 3 If $\mu > 0$, Let $\mu = p^2$

A.E is $m^2 + p^2 = 0 \Rightarrow m = \pm ip$

$\Rightarrow f(x) = C_1 \cos px + C_2 \sin px$

$\Rightarrow f'(x) = p[-C_1 \sin px + C_2 \cos px]$

$f'(0) = 0 \Rightarrow 0 = p(C_2) \Rightarrow C_2 = 0$

$f'(L) = 0 \Rightarrow p[-C_1 \sin Lp] = 0$

$\Rightarrow \sin Lp = 0 \Rightarrow Lp = n\pi$

$\Rightarrow p = \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$

$$\Rightarrow \quad \mu_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

& corresponding eigen functions are

$$f_n(x) = C_{1n} \cos\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, 3, \ldots$$

Now for equ ②

$$g'' + g(\lambda - \mu) = 0$$

A.E is $m^2 + (\lambda - \mu) = 0$

Case 1:- $\lambda - \mu = 0 \Rightarrow$ trivial sol.

Case 2:- $\lambda - \mu < 0 \Rightarrow$ trivial sol.

Case 3:- $\lambda - \mu > 0$, Put $\lambda - \mu = p^2$

$$\Rightarrow g(y) = C_1 e^{py} + C_2 e^{-py}$$

$$\Rightarrow g'(y) = p[-C_1 e^{py} + C_2 e^{-py}]$$

After applying Boundry Conditions

$$\lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2$$

$$\Rightarrow \lambda_{nm} = \left(\frac{m\pi}{H}\right)^2 + \mu_n = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2, \quad n, m = 1, 2, 3, \ldots$$

are eigen values and corresponding eigen functions are

$$g_{nm}(y) = C_{1_{nm}} \cos\left(\frac{m\pi}{H}y\right), \quad m = 1, 2, 3, \ldots \longrightarrow ②$$

$$\Rightarrow \phi(x, y) = f(x)\, g(y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{H}y\right)$$

$$\& \quad H(t) = G e^{-k\lambda_{mn} t} \longrightarrow ②$$

So $u(x, y, t) = \phi(x, y)\, h(t) \Rightarrow$

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{H}y\right)e^{-k\lambda_{mn}t} \longrightarrow (4)$$

Now to find A_{nm} we can use I.C as

$$u(x,y,0) = f(x,y)$$

$$\Rightarrow A_{nm} = \frac{4}{LH}\int_0^L\int_0^H f(x,y)\cos\frac{n\pi x}{L}\cos\frac{n\pi y}{H}\,dx\,dy \longrightarrow (5)$$

(C) Boundry Conditions are given by

$$\frac{\partial u}{\partial x}(0,y,t) = 0 \quad , \quad \frac{\partial u}{\partial x}(L,y,t) = 0$$

$$u(x,0,t) = 0 \quad , \quad u(x,H,t) = 0$$

Solution

Given heat equation have following simplifications

$$h' + k\lambda h = 0 \longrightarrow (1) \qquad f'' + \mu f = 0 \longrightarrow (2)$$

$$g'' + g(\lambda - \mu) = 0 \longrightarrow (3)$$

for equ (2) $f_n(x) = C_{1n}\cos\left(\frac{n\pi}{L}x\right)$,

$$\mu_n = \left(\frac{n\pi}{L}\right)^2 \quad , n = 0,1,2,\ldots$$

for equ (3) we get

$$g_{nm}(y) = C_{2m}\sin\frac{m\pi}{H}y,$$

$$\lambda_{mn} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2, \quad m,n = 1,2,3,\ldots$$

So $\phi(x,y) = g(y)f(x) = \sum\sum A_{nm}\cos\frac{n\pi x}{L}\sin\frac{m\pi y}{H}$

for (1) $h(t) = C_{1mn}e^{-k\lambda_{mn}t}$

So $u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cos\frac{n\pi}{L}x\sin\frac{m\pi}{H}y\,e^{-k\lambda_{mn}t}$

& $A_{nm} = \frac{4}{LH}\int_0^L\int_0^H f(x,y)\cos\frac{n\pi}{L}x\sin\frac{m\pi}{H}y\,dx\,dy$

(d) Boundry Conditions are given by

$u(0,y,t) = 0 \quad , \quad \frac{\partial u}{\partial x}(L,y,t) = 0$

$\frac{\partial u}{\partial y}(x,0,t) = 0 \quad , \quad \frac{\partial u}{\partial y}(x,H,t) = 0$

Solution By separation of variables we have

$h' + k\lambda h = 0 \longrightarrow ① \quad f'' + \mu f = 0 \longrightarrow ②$

$g'' + (\lambda - \mu) g = 0 \longrightarrow ③$

For equ ② $f'' + \mu f = 0$

A.E is $m^2 + \mu = 0$

Case 1:- If $\mu = 0 \Rightarrow m^2 = 0 \Rightarrow m = 0, 0$

$\Rightarrow f(x) = c_1 + c_2 x$

B.Cs $f(0) = 0 \Rightarrow \boxed{c_1 = 0}$, $f'(L) = 0 \Rightarrow \boxed{c_2 = 0}$

Trivial Solution.

Case 2:- If $\mu < 0$. let $\mu = -p^2$

A.E is $m^2 = p^2 \Rightarrow m = \pm p$

$\Rightarrow f(x) = c_1 e^{px} + c_2 e^{-px}$

$f(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$f'(L) = 0 \Rightarrow 2pc_1 \cosh Lx = 0$

$\Rightarrow c_1 = 0 \quad \because \cosh Lx \neq 0$

$\Rightarrow$ Sol. is trivial.

Case 3:- If $\mu > 0$, Let $\mu = p^2$

A.E is $m^2 = -p^2 \Rightarrow m = \pm ip$

$\Rightarrow f(x) = c_1 \cos px + c_2 \sin px$

$f'(x) = p[-c_1 \sin px + c_2 \cos px)$

B.Cs $f(0)=0 \Rightarrow \boxed{C_2=0}$

$f'(L)=0 \Rightarrow -C_1 \sin Lp = 0$

$\Rightarrow \sin Lp = 0 \Rightarrow p = \frac{n\pi}{L}$

$\Rightarrow \mu_n = \left(\frac{n\pi}{L}\right)^2 , n=1,2,3,\ldots$

$\Rightarrow f_n(x) = C_1 \cos\left(\frac{n\pi}{L}\right)x$

Solve on same lines

?

There is doubt

?

Question 7.3.2:- Consider the heat equation in a three dimensional heat-shaped region $0<x<L,\ 0<y<H,\ 0<z<W$

$$\frac{\partial u}{\partial t} = k\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

subject to the initial condition $u(x,y,z,0)=f(x,y,z)$. Solve the initial value problem and analyze the temperature as $t\to\infty$ if the boundary conditions are

(a) $u(0,y,z,t)=0 ,\ \frac{\partial u}{\partial y}(x,0,z,t)=0 ,\ \frac{\partial u}{\partial z}(x,y,0,t)=0$

$u(L,y,z,t)=0 ,\ \frac{\partial u}{\partial y}(x,H,z,t)=0,\ u(x,y,W,t)=0$

Solution Let we use separation of variable technique to solve this problem.

Let $u(x,y,z,t)=\phi(x,y,z)\,h(t)$

$$\Rightarrow \phi(x,y,z)\,h'(t) = k\,h(t)\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right)$$

Hence
$$\frac{h'(t)}{k\,h(t)} = \frac{1}{\phi}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right) = -\lambda$$

We get $h'(t) + k\lambda h = 0 \longrightarrow (1)$

& $\nabla^2 \phi = \lambda \phi = 0 \longrightarrow (2)$

Now Let $\phi(x,y,z) = f(x)\, g(y,z)$

$\therefore (2) \Rightarrow \dfrac{\partial^2 f}{\partial x^2} g(y,z) + \dfrac{\partial^2 g}{\partial y^2} f(x) + \dfrac{\partial^2 g}{\partial z^2} f(x) = -\lambda f g$

$\div$ by fg we get

$$f''/f + \left(\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right)\cdot\frac{1}{g} = -\lambda$$

or $\left(\dfrac{\partial^2 g}{\partial y^2} + \dfrac{\partial^2 g}{\partial z^2}\right)\dfrac{1}{g} = -\lambda - f''/f = -u$

we get two differential equations from above expression

$$\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} = -ug \longrightarrow (3)$$

& $f'' + (\lambda - u) f = 0 \longrightarrow (4)$

with $f(0) = 0 = f(L)$

Since equ (3) is homogeneous with homo- boundry conditions. we can apply separation of variables to get two ODE.

Let $g(y,z) = a(y)\, b(z)$

$\therefore (3) \Rightarrow a'' b + b'' a = -u a b$

$\div$ by $ab \Rightarrow \dfrac{a''}{a} + \dfrac{b''}{b} = -u$

$\Rightarrow -\dfrac{a''}{a} - u = \dfrac{b''}{b} = -\eta$ (Constant)

Hence two ODE's are

$$a''(y) + (\mu - \lambda)\, a(y) \longrightarrow (5)$$

with $a'(0) = a'(H) = 0$

& $b''(z) = -\lambda\, b(z) \longrightarrow (6)$

with $b'(0) = 0 = b(W)$

Thus we have four ODE's (1), (4), (5) & (6)

To Solve Eqn 6:-

$$b''(z) + \lambda\, b(z) = 0$$

A.E is $m^2 + \lambda = 0$

Case 1 If $\lambda = 0$, $\Rightarrow m = 0, 0 \rightarrow b(z) = c_1 + c_2 z$

$b'(0) = 0 \Rightarrow \boxed{c_2 = 0}$ $b(W) = 0 \Rightarrow \boxed{c_1 = 0}$ (T.S)

Case 2 If $\lambda < 0$ Let $\lambda = -p^2 \Rightarrow m^2 = p^2 \Rightarrow m = \pm p$

$\Rightarrow b(z) = c_1 e^{pz} + c_2 e^{-pz}$

$b'(0) = 0 \Rightarrow p(c_1 - c_2) = 0 \Rightarrow c_1 = c_2$

$b(W) = 0 \Rightarrow c_1 e^{Wz} + c_2 e^{-Wz} = 0$

$\Rightarrow 2c_1 \cosh Wz = 0 \Rightarrow c_1 = 0$ (T.S)

Case 3 If $\lambda > 0$ Let $\lambda = p^2 \Rightarrow m = \pm ip$

$\Rightarrow b(z) = c_1 \cos pz + c_2 \sin pz$

$b'(z) = 0 \Rightarrow p(-c_1(0) + c_2) = 0 \Rightarrow c_2 = 0$

$b(W) = 0 \Rightarrow c_1 \cos Wp = 0 \Rightarrow \cos Wp = 0$

$\Rightarrow Wp = (2n+1)\frac{\pi}{2} \Rightarrow \lambda_n = \left((2n+1)\frac{\pi}{2W}\right)^2$, $n = 1, 2, 3, \ldots$

are eigen values and corresponding eigen functions are

$$b_n(z) = c_n \cos\left(\frac{2n+1}{2W}\pi\right)z \longrightarrow (i)$$

To Solve Eqn (5)

$$\alpha''(y) + a(\mu - \lambda) = 0$$

after solving $\mu - \lambda = \left(\frac{m\pi}{H}\right)^2$

$$\Rightarrow \mu_{mn} = \left(\frac{m\pi}{H}\right)^2 + \lambda_n = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{(2n+1)\pi}{2}\right)^2,$$

$m, n = 1, 2, 3, \cdots$

are eigen values & corresponding eigen functions are

$$a_m(y) = C_m \cos \frac{m\pi}{H} y \longrightarrow \text{(ii)}$$

To solve eqn (4)

$$f'' + f(\lambda - \mu) = 0$$

with $f'(0) = 0$ & $f(L) = 0$

$$\Rightarrow \lambda - \mu = \left(\frac{j\pi}{L}\right)^2, \quad j = 1, 2, 3, \cdots$$

$$\Rightarrow \lambda_{jmn} = \left(\frac{j\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{(2n+1)\pi}{2W}\right)^2; \quad j, m, n = 1, 2, 3, \cdots$$

are eigen values & eigen functions are

$$f_{jmn}(x) = C_j \sin\left(\frac{j\pi}{L} x\right) \quad j = 1, 2, 3, \cdots \longrightarrow \text{(iii)}$$

$$\Rightarrow \phi(x, y, z) = \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{jmn} \sin\left(\frac{j\pi}{L} x\right) \cos\left(\frac{m\pi}{H} y\right) \cos\left(\frac{(2n+1)\pi}{2W} z\right)$$

& from (1)

$$h(t) = C_1 e^{-k \lambda_{jmn} t}$$

$$\Rightarrow u(x, y, z, t) = \phi(x, y, z)\, h(t)$$

$$= ?$$

Question 7.3.2 (b) boundry conditions are

$$\frac{\partial u}{\partial x}(0,y,z,t)=0, \quad \frac{\partial u}{\partial y}(x,0,z,t)=0, \quad \frac{\partial u}{\partial z}(x,y,0,t)=0$$

$$\frac{\partial u}{\partial x}(L,y,z,t)=0, \quad \frac{\partial u}{\partial y}(x,H,z,t)=0, \quad \frac{\partial u}{\partial z}(x,y,W,t)=0$$

Solution To solve equ ④ of previous

$$b''(z)+\eta b=0$$

with $b'(0)=0=b'(W)$.

$\Rightarrow \eta_k=\left(\frac{k\pi}{W}\right)^2$ are eigen values &

$$b_k(z)=c_k\cos\frac{k\pi}{W}z \longrightarrow \text{(i)} \quad \text{are eigen functions}$$

To solve equ ③ of previous part.

$$a''(y)+a(y)(\mu-\eta)=0$$

with $a'(0)=0=a'(H)$

$\Rightarrow \mu-\eta_k=\left(\frac{m\pi}{H}\right)^2 \Rightarrow \mu_{mk}=\left(\frac{m\pi}{H}\right)^2+\left(\frac{k\pi}{W}\right)^2;$

$k, m=1,2,3,---$

& $a_m(y)=C_m\cos\left(\frac{m\pi}{H}y\right) \longrightarrow \text{(ii)}$

To solve equ ② of previous part

$$f''+(\lambda-\mu)f=0$$

with $f'(0)=0=f'(L)$

$\Rightarrow \lambda-\mu_{mk}=\left(\frac{n\pi}{L}\right)^2 \Rightarrow \lambda_{mnk}=\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2+\left(\frac{k\pi}{W}\right)^2;$

$m, \cancel{k}, n=1,2,3,---$

& $f_n(x)=C_n\cos\left(\frac{n\pi}{L}x\right) \longrightarrow \text{(iii)}$

To solve equ ① of previous part.

$$h'(t) + k\lambda h(t) = 0$$

$$\Rightarrow h(t) = C_1 e^{-k\lambda_{mnk} t} \longrightarrow \text{(iv)}$$

Product solution is

$$u(x,y,z,t) = \sum_{m,n,k=0}^{\infty} A_{mnk} \cos\frac{n\pi}{L}x \cos\frac{m\pi}{H}y \cos\frac{k\pi}{\omega}z \longrightarrow ⑤$$

&

$$A_{mnk} = \frac{\int_0^L\int_0^H\int_0^\omega f(x,y,z)\cos\frac{n\pi}{L}x \cos\frac{m\pi}{H}y \cos\frac{k\pi}{\omega}z}{\int_0^L \cos^2\frac{n\pi}{L}x\,dx \int_0^H \cos^2\frac{m\pi}{H}y\,dy \int_0^\omega \cos\frac{k\pi}{H}z\,dz} \longrightarrow ⑥$$

equ ⑤ gives solution & equ ⑥ undetermined coefficients A_{mnk}.

Question 7.3.3:- Solve

$$\frac{\partial u}{\partial t} = k_1\frac{\partial^2 u}{\partial x^2} + k_2\frac{\partial^2 u}{\partial y^2}$$ on a

rectangle $(0<x<L,\ 0<y<H)$ subject to

$u(x,y,0) = f(x,y)$ $\quad u(0,y,t) = 0$ $\quad \partial u/\partial y(x,0,t) = 0$

$\quad u(L,y,t) = 0$ $\quad \partial u/\partial y(x,H,t) = 0$

Solution Let $u(x,y,t) = f(x)g(y)h(t)$

$$\Rightarrow f(x)g(y)h'(t) = k_1 f''(x)g(y)h(t) + k_2 f(x)g''(y)h(t)$$

$$\Rightarrow \frac{h'(t)}{h(t)} = \frac{k_1 f''(x)}{f(x)} + \frac{k_2 g''(y)}{g(y)} = -\lambda$$

$$\Rightarrow h'(t) + \lambda h(t) = 0$$

A.E is $D+\lambda=0 \Rightarrow D=-\lambda$

$$\Rightarrow h(t) = C_1 e^{-\lambda t} \longrightarrow (*)$$

Also we have

$$\frac{k_1 f''(x)}{f(x)} = -\lambda - \frac{k_2 g''(y)}{g(y)} = -\mu.$$

$$\Rightarrow k_1 \frac{f''(x)}{f(x)} + \mu f(x) = 0$$

$$\Rightarrow f''(x) + \frac{\mu}{k_1} f(x) = 0 \longrightarrow (1)$$

with $f(0) = 0 = f(L)$

& $k_2 \frac{g''(y)}{g(y)} + (\lambda - \mu) = 0$

$$\Rightarrow g'' + \frac{\lambda-\mu}{k_2} g(y) = 0 \longrightarrow (2)$$

with $g'(0) = 0 = g'(H)$

Solve eqn (1) Eigen values $\frac{\mu_n}{k_1} = \left(\frac{n\pi}{L}\right)^2$

or $\sqrt{\mu_n} = \sqrt{k_1}\left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, \ldots$

& eigen functions are

$$f_n(x) = C_n \sin\left(\frac{n\pi}{L}x\right); \ n = 1, 2, 3, \ldots \longrightarrow (i)$$

Solve eqn (2) $g''(y) + \frac{\lambda-\mu}{k_2} g(y) = 0$

with $g'(0) = 0 = g'(H)$

$$\Rightarrow \frac{\lambda - \mu_n}{k_2} = \left(\frac{m\pi}{H}\right)^2 \Rightarrow \lambda_{mn} - \mu_n = k_2 \left(\frac{m\pi}{H}\right)^2$$

$\Rightarrow \lambda_{mn} = k_2\left(\frac{m\pi}{H}\right)^2 + k_1\left(\frac{n\pi}{L}\right)^2$; $m,n = 1,2,3,\ldots$

& $g_m(y) = C_m \cos\left(\frac{m\pi}{H}y\right) \longrightarrow$ (ii)

from ⊛ $h_{mn}(t) = c_{mn} e^{-\lambda_{mn} t} \longrightarrow$ (iii)

Now by equ (i), (ii) & (iii)

$\Rightarrow u_{mn}(x,y,t) = A_{mn} \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{H}y\right) e^{-\lambda_{mn}t}$

Now by Principle of superposition

$$u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{H}y\right) e^{-\lambda_{mn}t}$$

Now by I.C $u(x,y,0) = f(x,y)$

$$\Rightarrow f(x,y) = \sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{n\pi}{L}x\right)\right]\cos\frac{m\pi}{H}y$$

which is fourier series

$$\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{n\pi}{L}x\right) = \frac{2}{H}\int_0^H f(x,y)\cos\left(\frac{m\pi}{H}y\right)dy$$

where

$$A_{nm} = \frac{2}{L}\int_0^L \frac{2}{H}\int_0^H f(x,y)\cos\frac{m\pi}{H}y \sin\frac{n\pi}{L}x \, dx\, dy$$

Question 7.3.4:- Consider the wave equation for a vibrating rectangular membrane $(0<x<L,\ 0<y<H)$ $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

subject to the initial conditions

$u(x,y,0) = 0$ & $\frac{\partial u}{\partial t}(x,y,0) = f(x,y)$

& boundry Conditions.

(a) $u(0,y,t) = 0$ $\quad$ $u(L,y,t) = 0$

$\frac{\partial u}{\partial y}(x,0,t) = 0$ $\quad$ $\frac{\partial u}{\partial y}(x,H,t) = 0$

Solution Let $u(x,y,t) = f(x)g(y)h(t)$

$\Rightarrow h''(t)f(x)g(y) = c^2[f''gh + g''fh]$

$\div$ by $f(x)g(y)h(t)$

$$\Rightarrow \frac{h''(t)}{c^2 h(t)} = \frac{f''}{f} + \frac{g''}{g} = -\lambda \text{ (constant)}$$

$$\Rightarrow h''(t) + \lambda c^2 h = 0 \longrightarrow (1)$$

& $f''/f + g''/g = -\lambda$

or $f''/f = -\lambda - g''/g = -\mu$ (a constant)

$$\Rightarrow f'' + \mu f = 0 \longrightarrow (2)$$

with $f(0) = 0 = f(L)$

& $g'' + (\lambda - \mu)g = 0 \longrightarrow (3)$

with $g'(0) = g'(H) = 0$

To Solve equ (2):-

$$f'' + \mu f = 0$$

$\Rightarrow f(x) = C_1 \cos px + C_2 \sin px$

After applying Boundry conditions

$\mu = \left(\frac{n\pi}{L}\right)^2$ are eigen values &

$f_n(x) = C_n \sin\left(\frac{n\pi}{L}x\right)$ are eigen functions

where $n = 1, 2, 3, \ldots$

To Solve equ (3):-

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$$g'' + g(\lambda - \mu) = 0$$

after applying Boundry Conditions

$$\lambda - \mu_n = \left(\frac{m\pi}{H}\right)^2 \Rightarrow \lambda_{mn} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2$$

& $g_m(y) = C_m \cos\left(\frac{m\pi}{H} y\right)$, $m = 1, 2, 3, \ldots$

to Solve equ ① $h'' + \lambda c^2 h = 0$

A.E is $m^2 + \lambda c^2 = 0 \Rightarrow m = \pm i\sqrt{\lambda_{mn}}\, c$

$$\Rightarrow h(t) = C_1 \cos\sqrt{\lambda_{mn}}\, ct + C_2 \sin\sqrt{\lambda_{mn}}\, ct$$

& $h'(t) = c\sqrt{\lambda_{mn}}\left[-C_1 \sin\sqrt{\lambda_{mn}}\, ct + C_2 \cos\sqrt{\lambda_{mn}}\, ct\right]$

$h(0) = 0 \Rightarrow C_1 = 0$

$\Rightarrow h(t) = C_2 \sin\sqrt{\lambda_{mn}}\, ct$

Product Solution is

$$u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\cos\frac{m\pi}{H}y\, B_{mn}\sin\sqrt{\lambda_{mn}}\, ct \longrightarrow ③$$

$u_t(x,y,0) = f(x,y) \Rightarrow$

$$f(x,y) = c\sqrt{\lambda_{mn}}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\cos\frac{m\pi}{H}y\,[B_{mn}]$$

$$\Rightarrow B_{mn} = \frac{1}{c\sqrt{\lambda_{mn}}}\,\frac{4}{LH}\int_0^L\int_0^H f(x,y)\sin\frac{n\pi}{L}x\cos\frac{m\pi}{H}y\,dy\,dx \longrightarrow ④$$

equs ③ & ④ gives solutions & values for coefficients respectively.

(b) Boundry Conditions are given by.

$$\frac{\partial u}{\partial x}(0,y,t) = 0, \quad \frac{\partial u}{\partial x}(L,y,t) = 0, \quad \frac{\partial u}{\partial y}(x,0,t) = 0$$

$$\frac{\partial u}{\partial y}(x,H,t) = 0$$

Solution We know on separation of variables we get

$f'' + \mu f = 0 \longrightarrow (1) \quad g'' + (\lambda - \mu) g = 0 \longrightarrow (2)$

$h'' + \lambda c^2 h = 0 \longrightarrow (3)$

$(1) \Rightarrow \quad f(x) = c_1 \cos px + c_2 \sin px \quad ; \mu = p^2$

$\Rightarrow f'(x) = p[-c_1 \sin px + c_2 \cos px]$

B.Cs $\Rightarrow \quad \mu = \left(\frac{n\pi}{L}\right)^2 \quad , n = 0, 1, 2, 3, \ldots$

& $f_n(x) = C_n \cos\left(\frac{n\pi}{L} x\right) \quad ; n = 0, 1, 2, \ldots$

$(2) \Rightarrow \quad g(x) = c_1 \cos qy + c_2 \sin qy \, ; \quad q^2 = (\lambda - \mu)$

$\Rightarrow g'(x) = q[-c_1 \sin qy + c_2 \cos qy]$

$\Rightarrow g_m(y) = C_m \cos\left(\frac{m\pi}{H} y\right),$

$\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 \quad , \begin{matrix} n = 0, 1, 2, \ldots \\ m = 1, 2, 3, \ldots \end{matrix}$

$(3) \Rightarrow \quad h(t) = C_1 \cos \sqrt{\lambda_{mn}}\, ct + C_2 \sin \sqrt{\lambda_{mn}}\, ct$

$h(0) = 0 \quad \Rightarrow \quad \boxed{C_1 = 0}$

$\Rightarrow h(t) = C_{mn} \sin \sqrt{\lambda_{mn}}\, ct$

$\Rightarrow u(x,y,t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \cos \frac{n\pi}{L} x \sin \frac{m\pi}{H} y \, C_{mn} \sin \sqrt{\lambda_{mn}}\, ct \longrightarrow (4)$

$u_t(x,y,0) = f(x,y) \Rightarrow$

$f(x,y) = c\sqrt{\lambda_{mn}} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{mn} \cos \frac{n\pi}{L} x \sin \frac{m\pi}{H} y$

$\Rightarrow C_{mn} = \frac{1}{c\sqrt{\lambda_{mn}}} \frac{4}{LH} \int_0^L \int_0^H \left[\cos \frac{n\pi}{L} x \cos \frac{m\pi}{H} y\right] f(x,y)\, dx\, dy$

Question 7.3.5:- Consider $\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] - k\frac{\partial u}{\partial t}$ with $k > 0$

(a) Give a brief physical interpretation of this equation

(b) Suppose that $u(x,y,t) = f(x)\,g(y)\,h(t)$. What ordinary differential equations are satisfied by f, g and h.

Solution (b) $u(x,y,t) = f(x)\,g(y)\,h(t)$

$$\Rightarrow f(x)\,g(y)\,h''(t) = c^2\left[f''(x)\,g(y)\,h(t) + f(x)\,g''(y)\,h(t)\right] - k\,f(x)\,g(y)\,h'(t)$$

$\div\ f(x)\,g(y)\,h(t) \Rightarrow$

$$\frac{h''(t)}{h(t)} = c^2\left[\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)}\right] - k\,\frac{h'(t)}{h(t)}$$

$$\Rightarrow \frac{h''(t)}{h(t)} + k\,\frac{h'(t)}{h(t)} = c^2\left[\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)}\right]$$

$$\Rightarrow \frac{1}{c^2}\,\frac{h''(t)}{h(t)} + \frac{k}{c^2}\,\frac{h'(t)}{h(t)} = \left[\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)}\right] = -\lambda$$

we have

$$\frac{1}{c^2}\,\frac{h''(t)}{h(t)} + \frac{k}{c^2}\,\frac{h'(t)}{h(t)} + \lambda = 0$$

or $\boxed{h''(t) + k\,h'(t) + c^2\lambda\,h(t) = 0}$ ①

Also we have

$$\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} = -\lambda$$

$$\Rightarrow \frac{f''(x)}{f(x)} = -\lambda - \frac{g''(y)}{g(y)} = -\mu \quad (\text{constant})$$

$$\Rightarrow \boxed{f''(x) + \mu f} = 0 \longrightarrow ②$$

& $\boxed{g''(y) + (\lambda-\mu) g(y) = 0} \longrightarrow ③$

Equ ①, ② and ③ are ODE's in t, x and y respectively.

Question 7.3.6:- Consider the Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

in a right cylinder whose base is arbitrarly shaped. The top is $z=H$ and the bottom is $z=0$. Assume that

$$\frac{\partial u}{\partial z}(x,y,0) = 0 \quad , \quad u(x,y,H) = f(x,y)$$

(a) Seprate the z-variable in general

Question 7.3.7:- If possible, solve Laplace's equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

in a rectangular shaped region, $0<x<L$, $0<y<W$, $0<z<H$, subject to the boundry conditions.

(a) $\frac{\partial u}{\partial x}(0,y,z)=0$, $u(x,0,z)=0$, $u(x,y,0)=f(x,y)$

$\frac{\partial u}{\partial x}(L,y,z)=0$, $u(x,W,z)=0$, $u(x,y,H)=0$

Solution Let $u(x,y,z) = f(x)\, g(y)\, h(z)$

$\Rightarrow f''(x) g(y) h(z) + f(x) g''(y) h(z) + f(x) g(y) h''(z) = 0$

$\div$ by $f(x) g(y) h(z)$

$$\Rightarrow \frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} + \frac{h''(z)}{h(z)} = 0.$$

$$\text{or} \quad \frac{h''(z)}{h(z)} = -\left[\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)}\right] = \lambda$$

$\Rightarrow h''(z) - \lambda h(z) = 0 \longrightarrow (1)$

$\Rightarrow$ A.E is $D^2 - \lambda = 0 \Rightarrow D^2 = \lambda$

$\Rightarrow D = \pm\sqrt{\lambda}$

$\Rightarrow h(z) = c_1 \quad + c_2$

$\longrightarrow (1)$

Also $-\left[\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)}\right] = \lambda$

$$\Rightarrow \frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} = -\lambda$$

or $\dfrac{f''(x)}{f(x)} = -\lambda - \dfrac{g''(y)}{g(y)} = -\mu$ (constant)

$\Rightarrow f''(x) + \mu f(x) = 0 \longrightarrow (2)$

with $f'(0) = 0 = f'(L)$

& $g''(y) + (\lambda - \mu) g(y) = 0 \longrightarrow (3)$

with $g(0) = 0 = g(w)$

By equ (2) $f''(x) + \mu f(x) = 0$

$\Rightarrow \mu_n = \left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, 2, 3, \ldots$

& $f_n(x) = C_n \cos\left(\frac{n\pi}{L} x\right) \longrightarrow (4)$

By equ (3) $g''(y) + (\lambda - \mu) g(y) = 0$

$\Rightarrow \lambda - \mu_n = \left(\frac{m\pi}{w}\right)^2$, $m = 1, 2, 3, \ldots$

$\Rightarrow \lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{w}\right)^2$ $\quad n = 0, 1, 2, \ldots$
$m = 1, 2, 3, \ldots$

& $g_m(y) = C_m \sin\left(\frac{m\pi}{w} y\right) \longrightarrow (5)$

FOURIER TRANSFORM*

*Definition:- Fourier Transform of a function $f(x)$, if it exists, is denoted and defined by

$$\mathcal{F}[f(x)] = F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx$$

In this case $f(x)$ is called the inverse fourier transform of $F(\omega)$ and it is defined as

$$\mathcal{F}^{-1}[F(\omega)] = f(x) = \int_{-\infty}^{\infty} e^{-i\omega x} F(\omega)\, d\omega$$

★ If $F(\omega)$ is the F.T of $f(x)$ and $f(x)$ is the inverse F.T of $F(\omega)$ with

$$F(\omega) = C_1\int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx$$

$$f(x) = C_2\int_{-\infty}^{\infty} e^{-i\omega x} f(\omega)\, d\omega$$

Then C_1 and C_2 be related as follows for each case

(1) $C_1 = \frac{1}{\sqrt{2\pi}}$; $C_2 = \frac{1}{\sqrt{2\pi}}$

(2) $C_1 = \frac{1}{2\pi}$; $C_2 = 1$

(3) $C_1 = 1$; $C_2 = \frac{1}{2\pi}$

Question 10.3.1:- Show that the Fourier Transform is a linear operator, i.e show that

(a) $\mathcal{F}[c_1 f(x) + c_2 g(x)] = c_1 F(\omega) + c_2 G(\omega)$

(b) $\mathcal{F}[f(x)\, g(x)] \neq F(\omega)\, G(\omega)$

Solution

(a) $\mathcal{F}[c_1 f(x) + c_2 g(x)]$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}[c_1 f(x) + c_2 g(x)]\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} c_1 f(x)\, dx + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} c_2 g(x)\, dx$$

$$= c_1\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx\right] + c_2\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} g(x)\, dx\right]$$

$$= c_1 \mathcal{F}[f(x)] + c_2 \mathcal{F}[g(x)]$$

$$= c_1 F(\omega) + c_2 G(\omega)$$

(b) $\mathcal{F}[f(x) g(x)]$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x) g(x)\, dx$$

$$\neq \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} g(x)\, dx$$

$$= F(\omega)\, G(\omega)$$

$$\Rightarrow \mathcal{F}[f(x) g(x)] \neq F(\omega)\, G(\omega)$$

Because in integration for product of two functions we use integration By Part method.

Question 10.3.2:- Show that the inverse Fourier Transform is a linear operator, i.e show that (a) $\mathcal{F}^{-1}[c_1 F(\omega)+c_2 G(\omega)] = c_1 f(x) + c_2 g(x)$

(b) $\mathcal{F}^{-1}[F(\omega)G(\omega)] \neq f(x)\, g(x)$

Solution

(a) $\mathcal{F}^{-1}[c_1 F(\omega)+c_2 G(\omega)]$

$$= \int_{-\infty}^{\infty} e^{-i\omega x}[c_1 F(\omega)+c_2 G(\omega)]\, d\omega$$

$$= \int_{-\infty}^{\infty} e^{-i\omega x}(c_1 F(\omega))\, d\omega + \int_{-\infty}^{\infty} e^{-i\omega x}(c_2 G(\omega))\, d\omega$$

$$= c_1 \int_{-\infty}^{\infty} e^{-i\omega x} F(\omega)\, d\omega + c_2 \int_{-\infty}^{\infty} e^{-i\omega x} G(\omega)\, d\omega$$

$$= c_1 \mathcal{F}^{-1}[F(\omega)] + c_2 \mathcal{F}^{-1}[G(\omega)]$$

$$= c_1 f(x) + c_2 g(x)$$

(b) $\mathcal{F}^{-1}[F(\omega)G(\omega)] = f * g$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, g(x-\bar{x})\, d\bar{x}$$

$$\neq f(x)\, g(x)$$

$$\Rightarrow \mathcal{F}^{-1}[F(\omega)G(\omega)] \neq f(x)\, g(x)$$

Question 10.3.3:- Let $F(\omega)$ be the Fourier Transform of $f(x)$. Show that if $f(x)$ is real, then $F^*(\omega) = F(-\omega)$, where $*$ denotes the complex conjugate.

Solution

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx$$

Taking complex conjugate on both sides

$$\Rightarrow F^*(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx \qquad \because f(x) \text{ is real so } f^*(x) = f(x)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(-\omega)x} f(x)\, dx$$

$$\Rightarrow F^*(\omega) = F(-\omega) \quad \text{Proved.}$$

Question 10.3.4:- Show that

$$\mathcal{F}\left[\int f(x;\alpha)\,d\alpha\right] = \int F(\omega;\alpha)\,d\alpha$$

Solution

$$\mathcal{F}\left[\int f(x;\alpha)\,d\alpha\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\left[\int f(x;\alpha)\,d\alpha\right] dx$$

$$= \frac{1}{2\pi}\int\int_{-\infty}^{\infty} e^{i\omega x} f(x;\alpha)\, dx\, d\alpha$$

$$= \int \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x;\alpha)\, dx\, d\alpha$$

$$= \int \mathcal{F}\left[f(x;\alpha)\right] d\alpha$$

$$= \int F(\omega;\alpha)\, d\alpha \quad \text{Proved.}$$

Question 10.3.5:- If $F(\omega)$ is the Fourier transform of $f(x)$. Show that the inverse fourier transform of $e^{i\omega\beta}F(\omega)$ is $f(x-\beta)$. this result is known as the Shift theorem for the Fourier Transforms.

Solution

$$\mathcal{F}^{-1}[e^{i\omega\beta}F(\omega)] = \int_{-\infty}^{\infty} e^{-i\omega x} e^{i\omega\beta} F(\omega)\, d\omega$$

$$= \int_{-\infty}^{\infty} e^{-i\omega x + i\omega\beta} F(\omega)\, d\omega$$

$$= \int_{-\infty}^{\infty} e^{-i\omega(x-\beta)} F(\omega)\, d\omega$$

In the def. of F.T i will +ve and while applying I.F.T i will be -ve

$$= f(x-\beta)$$

Now:-

$$\mathcal{F}[f(x-\beta)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x-\beta)\, dx$$

Put $x-\beta = t \Rightarrow x = t+\beta$ & $dx = dt$

$x \to \pm\infty \Rightarrow t \Rightarrow \pm\infty$

$$\Rightarrow \mathcal{F}[f(x-\beta)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t+\beta)} f(t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\cdot e^{i\omega\beta} f(t)\, dt$$

$$= e^{i\omega\beta}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt$$

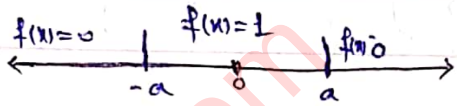
$$= e^{i\omega\beta}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx$$

$$\Rightarrow \mathcal{F}[f(x-\beta)] = e^{i\omega\beta}\,\mathcal{F}[f(x)]$$

$$= e^{i\omega\beta}\,F(\omega)$$

Q 10.3.6:- If $f(x) = \begin{cases} 0 & |x| > a \\ 1 & |x| < a \end{cases}$. determine the Fourier Transform of $f(x)$.

Solution



Given

$$f(x) = \begin{cases} 1 & ; -a < x < a \\ 0 & ; \text{otherwise} \end{cases}$$

$$\mathcal{F}[f(x)] = F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{-a} e^{i\omega x} f(x)\,dx + \int_{-a}^{a} e^{i\omega x} f(x)\,dx + \int_{a}^{\infty} e^{i\omega x} f(x)\,dx\right]$$

$$= 0 + \frac{1}{2\pi}\int_{-a}^{a} e^{i\omega x}\cdot 1\,dx + 0$$

$$= \frac{1}{2\pi}\cdot \left.\frac{e^{i\omega x}}{i\omega}\right|_{-a}^{a} = \frac{1}{2\pi}\cdot\frac{1}{i\omega}\left[e^{i\omega a} - e^{-i\omega a}\right]$$

$$= \frac{1}{\omega\pi}\left[\frac{e^{i\omega a} - e^{-i\omega a}}{2i}\right] = \frac{1}{\pi\omega}\cdot \sin\omega a$$

$$= \frac{1}{\pi}\cdot\frac{\sin\omega a}{\omega}$$

Question 10.3.7:- If $F(\omega) = e^{-|\omega|\alpha}$ $(\alpha > 0)$ determine the inverse Fourier Transform of $F(\omega)$

Solution

$$\mathcal{F}^{-1}\left[e^{-|\omega|\alpha}\right] = \int_{-\infty}^{\infty} e^{-|\omega|\alpha}\, e^{-i\omega x}\, d\omega$$

$$\Rightarrow \mathcal{F}^{-1}\left[e^{-|\omega|\alpha}\right] = \int_{-\infty}^{0} e^{+\omega\alpha}\, e^{-i\omega x}\, d\omega + \int_{0}^{\infty} e^{-\omega\alpha}\, e^{-i\omega x}\, d\omega$$

$$= \int_{-\infty}^{0} e^{(\alpha - ix)\omega}\, d\omega + \int_{0}^{\infty} e^{-(\alpha + ix)\omega}\, d\omega$$

$$= \frac{1}{\alpha - ix}\cdot e^{(\alpha - ix)\omega}\Big|_{-\infty}^{0} + \frac{1}{-(\alpha + ix)}\, e^{-(\alpha + ix)\omega}\Big|_{0}^{\infty}$$

$$= \frac{1}{\alpha - ix}(1 - 0) + \frac{1}{-(\alpha + ix)}(0 - 1)$$

$$= \frac{1}{\alpha - ix} + \frac{1}{\alpha + ix}$$

$$= \frac{\alpha + \cancel{ix} + \alpha - \cancel{ix}}{(\alpha - ix)(\alpha + ix)} = \frac{2\alpha}{(\alpha)^2 - (ix)^2}$$

$$\Rightarrow \mathcal{F}^{-1}\left[e^{-|\omega|\alpha}\right] = \frac{2\alpha}{\alpha^2 + x^2}$$

Q 10.3.8:- If $F(\omega)$ is the Fourier Transform of $f(x)$. show that $-i\, dF/d\omega$ is the Fourier Transform of $x f(x)$.

Solution

$$\mathcal{F}[f(x)] = F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx$$

$$\frac{dF}{d\omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty} (ix)\, e^{i\omega x} f(x)\, dx$$

$$= \frac{i}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} x f(x)\, dx$$

$$= i\, \mathcal{F}[x f(x)]$$

$$\Rightarrow \frac{1}{i}\frac{dF}{d\omega} = \mathcal{F}[x f(x)] \longrightarrow (1)$$

$$\Rightarrow \mathcal{F}[x f(x)] = -i\frac{dF}{d\omega} \quad \text{Proved}$$

Similarly From (1)

$$(-i)^1 \frac{d}{d\omega}[F(\omega)] = \mathcal{F}[x f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} x f(x)\, dx$$

$$\Rightarrow (-i)^1 \frac{d^2}{d\omega^2}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} ix\, e^{i\omega x} x f(x)\, dx$$

$$= \frac{i}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} x^2 f(x)\, dx$$

$$\Rightarrow (-i)^2 \frac{d^2}{d\omega^2}[F(\omega)] = \mathcal{F}[x^2 f(x)]$$

Continuing this process and differentiating upto 'n' times we get

$$\mathcal{F}\{x^n f(x)\} = (-i)^n \frac{d^n}{d\omega^n}(F(\omega))$$

Prove that $\mathcal{F}^{-1}\left[-i\frac{dF}{d\omega}\right] = x f(x)$

$$\mathcal{F}^{-1}\left[-i\frac{dF}{d\omega}\right] = -i\frac{d}{d\omega}\mathcal{F}^{-1}[F(\omega)]$$

$$= -i\frac{d}{d\omega}\int_{-\infty}^{\infty} e^{-i\omega x} F(\omega)\,d\omega$$

$$= -i\int_{-\infty}^{\infty} -ix\, e^{-i\omega x} F(\omega)\,d\omega$$

$$= (-i)^2 x\int_{-\infty}^{\infty} e^{-i\omega x} F(\omega)\,d\omega$$

$$= x\,\mathcal{F}^{-1}[F(\omega)]$$

$$= x f(x) \quad \text{Proved.}$$

Question 10.3.11:- (b) If $F(\omega)$ is the Fourier Transform of $f(x)$, show that $F(\alpha\omega)$ is the Fourier Transform of $\frac{1}{\alpha} f(x/\alpha)$

Solution

$$\mathcal{F}\left[\frac{1}{\alpha} f(x/\alpha)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\frac{1}{\alpha} f\left(\frac{x}{\alpha}\right) dx$$

Put $x/\alpha = x'$ $\Rightarrow$ $x = \alpha x'$ & $dx = \alpha\, dx'$

$$\Rightarrow \mathcal{F}\left[\frac{1}{\alpha} f(x/\alpha)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\alpha x'}\frac{1}{\alpha} f(x')\,\alpha\, dx'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega\alpha)x'} f(x')\,dx'$$

$$\Rightarrow \mathcal{F}\left[\frac{1}{\alpha} f\left(\frac{x}{\alpha}\right)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\alpha\omega)x} f(x)\,dx$$

$$= F(\alpha\omega)$$

$$\frac{1}{\alpha} f\left(\frac{x}{\alpha}\right) = F(\alpha\omega) \quad \text{Proved}$$

* Attenuation Property:-

$$\mathcal{F}\left[e^{\alpha x} f(x)\right] = F(\omega - i\alpha)$$

$$\mathcal{F}\left[e^{\alpha x} f(x)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} e^{\alpha x} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x + \alpha x} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega - i\alpha)x} f(x)\,dx$$

$$\Rightarrow \mathcal{F}\left[e^{\alpha x} f(x)\right] = F(\omega - i\alpha)$$

⇒ Real & Complex Values of F.T

(a) If $f(x)$ is real and even then $F(\omega)$ is real.

(b) If $f(x)$ is real and odd then $F(\omega)$ is pure imaginary.

(c) If $f(x)$ is complex then

$$\mathcal{F}\{\bar{f}(-x)\} = \bar{F}(K)$$

Proof)

(a) $$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx$$

$$\bar{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx \quad \because f \text{ is real}$$

Put $x = -x' \Rightarrow dx = -dx'$, $x \to \pm\infty \Rightarrow x' \to \mp\infty$

$$\Rightarrow \bar{F}(\omega) = \frac{1}{2\pi}\int_{+\infty}^{-\infty} e^{-i\omega(-x')} f(-x')(-dx')$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x'} f(x')\,dx' \quad \because f(x) \text{ is even}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx$$

$\bar{F}(\omega) = F(\omega) \Rightarrow F(\omega)$ is real.

(b) $$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx$$

$$\bar{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx \quad \because f(x) \text{ is real}$$

Put $x = -x' \Rightarrow dx = -dx'$, $x \to \pm\infty \Rightarrow x' \to \mp\infty$

$$\Rightarrow \bar{F}(\omega) = \frac{1}{2\pi}\int_{\infty}^{-\infty} e^{-i\omega(-x')} f(-x')(-dx')$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x'} [-f(x')]\,dx' \quad \because f(x) \text{ is odd}$$

$$= \frac{-1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x'} f(x')\,dx'$$

$$= \frac{-1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx$$

$\Rightarrow \bar{F}(\omega) = -F(\omega)$

$\Rightarrow F(\omega)$ is pure imaginary.

(c) $F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx$

$\Rightarrow \bar{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\bar{f}(x)\, dx \quad \because f(x)$ is complex

Put $x = -x' \Rightarrow dx = -dx'$, $x \to \pm\infty \Rightarrow x' \to \mp\infty$

$\Rightarrow \bar{F}(\omega) = \frac{1}{2\pi}\int_{\infty}^{-\infty} e^{-i\omega(-x')}\bar{f}(-x')(-dx')$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x'}\bar{f}(-x')\, dx'$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\bar{f}(-x)\, dx$

$\Rightarrow \bar{F}(\omega) = \mathcal{F}[\bar{f}(-x)]$

* **Theorem:** If $f(x), f'(x), f''(x), \ldots, f^{(n-1)}(x) \to 0$ as $x \to \pm\infty$ then

$\mathcal{F}\{f^{(n)}(x)\} = (-i\omega)^n F(\omega)$

Proof: We prove the theorem by principle of mathematical induction.

For $n=1$:-

$F\{f'(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f'(x)\, dx$

$= \frac{1}{2\pi}\left[e^{i\omega x} f(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} i\omega\, e^{i\omega x} f(x)\, dx\right]$

$= \frac{1}{2\pi}\left[0 - 0 - \int_{-\infty}^{\infty} (i\omega)\, e^{i\omega x} f(x)\, dx\right]$

$$\Rightarrow \mathcal{F}[f'(x)] = (-i\omega)^1 \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx$$

$$= (-i\omega)^1 F(\omega)$$

So Case-I is satisfied.

Case II:- Now we assume that theorem is true for $n=m$ $\therefore$ i.e

$$\mathcal{F}[f^{(m)}(x)] = (-i\omega)^m F(\omega)$$

Case III:- Now we prove the theorem for $n = m+1$

$$\mathcal{F}[f^{(m+1)}(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f^{(m+1)}(x)\,dx$$

$$= \frac{1}{2\pi}\left[e^{i\omega x} f^{(m)}(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (i\omega)\, e^{i\omega x} f^{(m)}(x)\,dx \right]$$

$$= \frac{1}{2\pi}\left[0 - 0 - \int_{-\infty}^{\infty} (i\omega)\, e^{i\omega x} f^{(m)}(x)\,dx \right]$$

$$= (-i\omega)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f^{(m)}(x)\,dx$$

$$= (-i\omega)\,\mathcal{F}[f^{(m)}(x)] = (-i\omega)(-i\omega)^m F(\omega) \quad \text{by c-2}$$

$$\Rightarrow \mathcal{F}[f^{(m+1)}(x)] = (-i\omega)^{m+1} F(\omega)$$

$\Rightarrow$ Case III is satisfied

So theorem is true for all n

$$\Rightarrow \mathcal{F}[f^{(n)}(x)] = (-i\omega)^n F(\omega)$$

* **Convolution:-** The convolution of two functions $f(x)$ and $g(x)$, defined over $]-\infty, \infty[$ is denoted and defined as

$$f*g = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, g(x-\bar{x})\, d\bar{x}$$

Convolution theorem:- If $F(\omega)$ and $G(\omega)$ are fourier transforms of $f(x)$ and $g(x)$ then $\mathcal{F}[f*g] = F(\omega)\,G(\omega)$

or $\mathcal{F}^{-1}[F(\omega)\,G(\omega)] = f*g$

Proof) We will prove

$$\mathcal{F}^{-1}[F(\omega)G(\omega)] = f*g$$

$$\mathcal{F}^{-1}[F(\omega)G(\omega)] = \int_{-\infty}^{\infty} e^{-i\omega x} F(\omega)\, G(\omega)\, d\omega$$

$$= \int_{-\infty}^{\infty} e^{-i\omega x} F(\omega)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x'} g(x')\, dx'\right] d\omega$$

$$= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(x-x')} F(\omega)\, d\omega\right) g(x')\, dx'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-x')\, g(x')\, dx'$$

$$= g*f$$

$$\Rightarrow \mathcal{F}^{-1}[F(\omega)G(\omega)] = f*g \quad \because \quad f*g = g*f$$

* Parseval's Identities:-

(a) $$\int_{-\infty}^{\infty} F(\omega)G(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,g(-x)\,dx$$

(b) $$\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

Proof (a) By definition of convolution theorem

$$\mathcal{F}^{-1}[F(\omega)G(\omega)] = f(x)*g(x)$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-i\omega x} F(\omega)G(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\,g(x-\bar{x})\,d\bar{x}$$

Put $x = 0$

$$\Rightarrow \int_{-\infty}^{\infty} e^{0} F(\omega)G(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\,g(-\bar{x})\,d\bar{x}$$

$$\Rightarrow \int_{-\infty}^{\infty} F(\omega)G(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,g(-x)\,dx$$

(b) Now Take $g(x) = \bar{f}(-x)$

$$\Rightarrow g(-x) = \bar{f}(x) \longrightarrow (i)$$

$$\Rightarrow \mathcal{F}[g(x)] = \mathcal{F}[\bar{f}(-x)]$$

$$\Rightarrow G(\omega) = \bar{F}(\omega) \quad \text{(already proved)} \longrightarrow (ii)$$

Now $\displaystyle\int_{-\infty}^{\infty} F(\omega)\, G(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, g(-x)\, dx$

$\Rightarrow \displaystyle\int_{-\infty}^{\infty} F(\omega)\, \bar{F}(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, \bar{f}(x)\, dx$ by (i) & (ii)

$\Rightarrow \displaystyle\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)|^2\, dx$ $\quad \because z\bar{z} = |z|^2$

---

* $\mathcal{F}\left[e^{-\alpha x^2}\right] = \dfrac{1}{\sqrt{4\pi\alpha}}\, e^{-\omega^2/4\alpha}$

* $\mathcal{F}\left[\sqrt{\dfrac{\pi}{\alpha}}\, e^{-x^2/4\alpha}\right] = e^{-\alpha\omega^2}$

* $\mathcal{F}\left[\delta(x - x_0)\right] = \dfrac{1}{2\pi}\, e^{i\omega x_0}$

* $\mathcal{F}\left[f(x-\beta)\right] = e^{i\omega\beta}\, F(\omega)$

* $\mathcal{F}\left[\dfrac{2\alpha}{x^2+\alpha^2}\right] = e^{-|\omega|\alpha}$

* $\mathcal{F}\left[\dfrac{\partial f}{\partial t}\right] = \dfrac{\partial}{\partial t}\mathcal{F}\left[f(x)\right] = \dfrac{\partial}{\partial t}F(\omega)$

* $\mathcal{F}\left[\dfrac{\partial f}{\partial x}\right] = (-i\omega)\, F(\omega)$

* $\mathcal{F}\left[\dfrac{\partial^2 f}{\partial x^2}\right] = (-i\omega)^2 F(\omega)$

* $\mathcal{F}\left[x\, f(x)\right] = -i\, \dfrac{dF}{d\omega}$

Solve $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$; $t > 0$, $-\infty < x < \infty$

$$u(x,0) = f(x)$$

Solution Taking Fourier Transform

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = k\,\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\mathcal{F}[u] = k(-i\omega)^2 \mathcal{F}[u]$$

$$\Rightarrow \frac{\partial U}{\partial t} = -k\omega^2 U$$

$$\Rightarrow U(\omega,t) = c(\omega)\, e^{-k\omega^2 t} \longrightarrow (*)$$

Now from initial condition

$$\mathcal{F}[u(x,0)] = \mathcal{F}[f(x)]$$

$$\Rightarrow U(\omega,0) = F(\omega)$$

from $(*)$ $U(\omega,0) = c(\omega)\, e^{-k\omega^2(0)}$

$$\Rightarrow F(\omega) = c(\omega)$$

$$\Rightarrow U(\omega,t) = F(\omega)\, e^{-k\omega^2 t} \qquad (A)$$

Now $\mathcal{F}^{-1}[U(\omega,t)] = \mathcal{F}^{-1}\left[F(\omega)\, e^{-k\omega^2 t}\right]$

Since $\mathcal{F}^{-1}[F(\omega)] = f(x)$

& $\mathcal{F}^{-1}\left[e^{-k\omega^2 t}\right] = \sqrt{\frac{\pi}{kt}}\, e^{-x^2/4kt}$

Since $\mathcal{F}^{-1}\left[F(\omega)\, e^{-k\omega^2 t}\right] = \mathcal{F}^{-1}[F(\omega)] * \mathcal{F}^{-1}\left[e^{-k\omega^2 t}\right]$

So from (A) $u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x}) \sqrt{\frac{\pi}{kt}}\, e^{-(x-\bar{x})^2/4kt}\, d\bar{x}$

$$\Rightarrow \quad u(x,t) = \int_{-\infty}^{\infty} f(\bar{x}) \frac{1}{\sqrt{4\pi kt}} e^{-(x-\bar{x})^2/4kt} \, d\bar{x}$$

** ————————— **

Question 10.4.3 (a):- Solve the diffusion equation with convection

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + c \frac{\partial u}{\partial x} \qquad -\infty < x < \infty$$

$$u(x,0) = f(x)$$

Solution

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + c \frac{\partial u}{\partial x} \longrightarrow (*)$$

Apply F.T on both sides

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = k\,\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] + c\,\mathcal{F}\left[\frac{\partial u}{\partial x}\right]$$

$$= \frac{\partial}{\partial t} U(\omega,t) = k(-i\omega)^2 U(\omega,t) + c(-i\omega) U(\omega,t)$$

$$\Rightarrow \frac{\partial U}{\partial t} = -(k\omega^2 + ic\omega) U$$

$$\Rightarrow U(\omega,t) = C(\omega)\, e^{-(k\omega^2 + ic\omega)t} \longrightarrow (1)$$

Now from I.C

$$u(x,0) = f(x) \Rightarrow U(\omega,0) = F(\omega)$$

So from (1) $\quad C(\omega) = F(\omega)$

$$\Rightarrow U(\omega,t) = F(\omega)\, e^{-(k\omega^2 + ic\omega)t}$$

$$\Rightarrow U(\omega,t) = F(\omega)\, G(\omega) \longrightarrow (A)$$

where $G(\omega) = e^{-(k\omega^2 + ic\omega)t}$

Taking F.I.T of (A) implies

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, g(x-\bar{x})\, d\bar{x} \longrightarrow (B)$$

Now

$$G(\omega) = e^{-(k\omega^2 + ic\omega)t}$$

$$\Rightarrow g(x) = \mathcal{F}^{-1}\left[e^{-(k\omega^2+ic\omega)t}\right]$$

$$= \int_{-\infty}^{\infty} e^{-(k\omega^2+ic\omega)t} \cdot e^{-i\omega x}\, d\omega$$

$$= \int_{-\infty}^{\infty} e^{-k\omega^2 t}\, e^{-i\omega(x+ct)}\, d\omega$$

Let $x+ct = s$

$$\Rightarrow g(x) = \int_{-\infty}^{\infty} e^{-k\omega^2 t}\, e^{-i\omega s}\, d\omega$$

$$= \mathcal{F}^{-1}\left[e^{-k\omega^2 t}\right] = \sqrt{\frac{\pi}{kt}}\, e^{\frac{-s^2}{4kt}}$$

$$\Rightarrow g(x) = \sqrt{\frac{\pi}{kt}}\, e^{\frac{-(x+ct)^2}{4kt}}$$

So from (B)

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x}) \sqrt{\frac{\pi}{kt}}\, e^{\frac{-(x-\bar{x}+ct)^2}{4kt}}\, d\bar{x}$$

MUHAMMAD TAHIR

FA15-RMT-007

Q 10.4.5*:- Consider $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$ $\quad -\infty < x < \infty$

$u(x,0) = f(x)$

(a) Show that a particular solution for the fourier Transform $\bar{U}$ is

$$\bar{U} = e^{-k\omega^2 t}\int_0^t \bar{Q}(\omega,\tau)\, e^{k\omega^2\tau}\, d\tau$$

(b) Determine $\bar{U}$

*(c) Solve for $u(x,t)$ (in the simplest form)

Solution

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

Applying Fourier Transform

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = k\,\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] + \mathcal{F}\left[Q(x,t)\right]$$

$$\Rightarrow \frac{\partial \bar{U}}{\partial t} = k(-i\omega)^2 \bar{U} + \bar{Q}(\omega,t)$$

$$\Rightarrow \frac{\partial \bar{U}}{\partial t} = -k\omega^2 \bar{U} + \bar{Q}(\omega,t)$$

$\because \frac{\partial \bar{U}}{\partial t} + k\omega^2 \bar{U} = \bar{Q}(\omega,t)$

I.F $= k\omega^2$

$\Rightarrow \frac{\partial}{\partial t}\left[e^{k\omega^2 t}\bar{U}\right] = \bar{Q}\, e^{k\omega^2 t}$

$\Rightarrow e^{k\omega^2 t}\bar{U} = \int_0^t \bar{Q}(\omega,t)\, e^{k\omega^2 t}\, dt$

$\Rightarrow \bar{U} = e^{-k\omega^2 t}\int_0^t \bar{Q}(\omega,\tau)\, e^{k\omega^2\tau}\, d\tau$

C.F $= c(\omega)\, e^{-k\omega^2 t}$

& P.I $= e^{-k\omega^2 t}\int_0^t \bar{Q}(\omega,\tau)\, e^{k\omega^2\tau}\, d\tau$

$$\bar{U} = \bar{U}_c + \bar{U}_p$$

$$\Rightarrow \bar{U}(\omega,t) = c(\omega)\, e^{-k\omega^2 t} + \int_0^t \bar{Q}(\omega,\tau)\, e^{-k\omega^2(t-\tau)}\, d\tau \longrightarrow (*)$$

Now from I.C $\quad u(x,0) = f(x)$

$$\Rightarrow \bar{U}(\omega,0) = F(\omega)$$

So from ⊛ $c(\omega) = F(\omega)$

$$\Rightarrow \bar{U}(\omega,t) = F(\omega)e^{-k\omega^2 t} + \int_0^t \bar{Q}(\omega,\tau)e^{-k\omega^2(t-\tau)}\,d\tau$$

Now apply Inverse Fourier Transform

$$\Rightarrow u(x,t) = \mathcal{F}^{-1}\left[F(\omega)e^{-k\omega^2 t}\right] + \mathcal{F}^{-1}\underbrace{\left[\int_0^t \bar{Q}e^{-k\omega^2(t-\tau)}\,d\tau\right]}_{①}$$

$$\Rightarrow u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\sqrt{\frac{\pi}{kt}}\,e^{-\frac{(x-\bar{x})^2}{4kt}}\,d\bar{x} + ①$$

Now ① $\Rightarrow$

$$\int_{-\infty}^{\infty}\left(\int_0^t \bar{Q}\,e^{-k\omega^2(t-\tau)}\,d\tau\right)e^{-i\omega x}\,d\omega$$

$$= \int_0^t\left(\int_{-\infty}^{\infty} \bar{Q}(\omega,\tau)\,e^{-k\omega^2(t-\tau)}\,e^{-i\omega x}\,d\omega\right)d\tau$$

$$= \int_0^t \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\bar{x})\,\tilde{g}(x-\bar{x})\,d\bar{x}\,d\tau \quad \left\{ \because \text{By Convolution theorem} \right.$$

$$= \int_0^t \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(\bar{x},\tau)\sqrt{\frac{\pi}{k(t-\tau)}}\,e^{\frac{-(x-\bar{x})^2}{4k(t-\tau)}}\,d\bar{x}\,d\tau$$

Define $G(x,t) = \frac{1}{2\pi}\sqrt{\frac{\pi}{kt}}\,e^{-\frac{x^2}{4kt}}$

$$\Rightarrow u(x,t) = \int_{-\infty}^{\infty} f(\bar{x})\,G(x-\bar{x})\,d\bar{x} + \int_0^t\int_{-\infty}^{\infty} Q(\bar{x},\tau)\,G(x-\bar{x},t-\tau)\,d\bar{x}\,d\tau$$

Question 10.4.4 (a) Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \gamma u \qquad -\infty < x < \infty$$

$$u(x,0) = f(x)$$

Solution

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \gamma u$$

Applying Fourier Transform

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = k\,\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] - \gamma\,\mathcal{F}[u]$$

$$\Rightarrow \frac{\partial U}{\partial t} = k(-i\omega)^2 U - \gamma U$$

$$\Rightarrow \frac{\partial U}{\partial t} = -k\omega^2 U - \gamma U$$

$$C.F = c(\omega)e^{-k\omega^2 t}$$

$$P.I = -e^{-k\omega^2 t}\gamma\int_0^t U(\omega,t)\,e^{k\omega^2 t}\,dt$$

$$\Rightarrow U(\omega,t) = C.F + P.I$$

$$= c(\omega)e^{-k\omega^2 t} - \gamma\int_0^t U(\omega,t)e^{-k\omega^2(t-\tau)}\,d\tau$$

From I.C $c(\omega) = F(\omega)$

$$\Rightarrow U(\omega,t) = F(\omega)e^{-k\omega^2 t} - \gamma\int_0^t U(\omega,t)e^{-k\omega^2(t-\tau)}\,d\tau$$

Now applying Inverse Fourier Transform

$$\Rightarrow u(x,t) = \mathcal{F}^{-1}\left[F(\omega)e^{-k\omega^2 t}\right] - \gamma \mathcal{F}^{-1}\left[\int_0^t U(\omega,\tau)e^{-k\omega^2(t-\tau)}\,d\tau\right] \quad \text{(A)}$$

Now $\mathcal{F}^{-1}\left[F(\omega)\,e^{-k\omega^2 t}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\sqrt{\frac{\pi}{kt}}\, e^{-\frac{(x-\bar{x})^2}{4kt}}\,d\bar{x}$

& $\mathcal{F}^{-1}\left[\int_0^t U(\omega,\tau)e^{-k\omega^2(t-\tau)}\,d\tau\right] = \int_{-\infty}^{\infty}\int_0^t U(\omega,\tau)e^{-k\omega^2(t-\tau)} e^{-i\omega x}\,d\tau\,d\omega$

$$= \int_0^t \int_{-\infty}^{\infty} U(\omega,\tau)e^{-k\omega^2(t-\tau)}\, e^{-i\omega x}\,d\omega\,d\tau$$

$$= \int_0^t \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\bar{x},\tau)\sqrt{\frac{\pi}{k(t-\tau)}}\, e^{-\frac{(x-\bar{x})^2}{4k(t-\tau)}}\,d\bar{x}\,d\tau.$$

So (A) $\Rightarrow$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\sqrt{\frac{\pi}{kt}}\,e^{-\frac{(x-\bar{x})^2}{4kt}}\,d\bar{x} - \gamma\int_0^t \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\bar{x},\tau)\sqrt{\frac{\pi}{k(t-\tau)}}\,e^{-\frac{(x-\bar{x})^2}{4k(t-\tau)}}\,d\bar{x}\,d\tau$$

★ Fourier Sine & Cosine Transforms

If a function $f(x)$ is defined over the interval $[0,\infty]$, then we can define Fourier sine and Fourier cosine Transforms

$$C[f(x)] = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx \quad \rightarrow \text{cosine}$$

$$S[f(x)] = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx \quad \rightarrow \text{sine}$$

The corresponding Inverse Fourier sine and Inverse Fourier cosine Transforms are

$$f(x) = \int_0^\infty F(\omega)\cos\omega x\,d\omega \quad \rightarrow \text{For Cosine}$$

$$f(x) = \int_0^\infty F(\omega)\sin\omega x\,d\omega \quad \rightarrow \text{For sine}$$

★ Fourier Sine & Cosine Transforms of Derivatives**

If $f(x)$ is real values and $|f(x)| \rightarrow 0$ as $x \rightarrow \infty$, Then

① $$C\left[\frac{df}{dx}\right] = \frac{2}{\pi}\int_0^\infty f'(x)\cos\omega x\,dx$$

$$= \frac{2}{\pi}\left[\cos\omega x\, f(x)\Big|_0^\infty - \int_0^\infty f(x)(-\omega\sin\omega x)\,dx\right]$$

$$= \frac{2}{\pi}\left[-f(0) + \omega\int_0^\infty f(x)\sin\omega x\,dx\right]$$

$$= -\frac{2}{\pi}f(0) + \omega\,S[f(x)]$$

② $S\left[\frac{df}{dx}\right] = \frac{2}{\pi}\int_0^\infty f'(x)\sin\omega x\,dx$

$$= \frac{2}{\pi}\left[\sin\omega x\, f(x)\Big|_0^\infty - \int_0^\infty f(x)\,\omega\cos\omega x\,dx\right]$$

$$= \frac{2}{\pi}\left[0 - 0 - \omega\int_0^\infty f(x)\cos\omega x\,dx\right]$$

$$= -\omega\frac{2}{\pi}C[f(x)]$$

③ $C\left[\frac{d^2f}{dx^2}\right] = \frac{2}{\pi}\int_0^\infty f''(x)\cos\omega x\,dx$

$$= \frac{2}{\pi}\left[\cos\omega x\, f'(x)\Big|_0^\infty - \int_0^\infty (-\omega\sin\omega x)\, f'(x)\,dx\right]$$

$$= -\frac{2}{\pi}f'(0) + \frac{2}{\pi}\omega\left[\sin\omega x\, f(x)\Big|_0^\infty - \int_0^\infty \omega\sin\omega x\, f(x)\,dx\right]$$

$$= -\frac{2}{\pi}f'(0) - \frac{2}{\pi}\omega^2\int_0^\infty f(x)\sin\omega x\,dx$$

$$= -\frac{2}{\pi}f'(0) - \omega^2 C[f(x)]$$

③ $S\left[\frac{d^2f}{dx^2}\right] = \frac{2}{\pi}\int_0^\infty f''(x)\sin\omega x\,dx$

$$= \frac{2}{\pi}\left[\sin\omega x\, f'(x)\Big|_0^\infty - \int_0^\infty f'(x)\,\omega\cos\omega x\,dx\right]$$

$$= \frac{2}{\pi}\left[0 - \omega\left\{\cos\omega x f(x)\Big|_0^\infty - \int_0^\infty f(x)(-\omega\sin\omega x)\,dx\right\}\right]$$

$$= \omega\frac{2}{\pi}f(0) - \omega^2\frac{2}{\pi}\int_0^\infty \sin\omega x\, f(x)\,dx$$

$$= \omega\frac{2}{\pi}f(0) - \omega^2 S[f(x)]$$

⇒ SECTION 10.4:-

Fourier Transform & The Heat Equation:-

Basic Properties:-

1:- The solution of $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ will be

$$u(x,t) = \int_{-\infty}^{\infty} c(\omega)\, e^{-i\omega x} e^{-k\omega^2 t}\, d\omega$$

The initial condition satisfies if

$$f(x) = \int_{-\infty}^{\infty} c(\omega)\, e^{-i\omega x}\, d\omega$$

and $c(\omega)$ the fourier Transform of the initial condition of temprature distribution

$$c(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$$

2:- Influence Function:- The solution becomes by using $c(\omega)$

$$u(x,t) = \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, e^{i\omega\bar{x}}\, d\bar{x}\right) e^{-i\omega x} e^{-k\omega^2 t}\, d\omega \longrightarrow (1)$$

It is complicated so we consider

$$g(x) = \int_{-\infty}^{\infty} e^{-i\omega x} e^{-k\omega^2 t} \, d\omega$$

Thus ① can be written as

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) \left[\int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega(x-\bar{x})} d\omega \right] d\bar{x}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) \, g(x-\bar{x}) \, d\bar{x}$$

where $g(x-\bar{x})$ is influence function.

★ The Guassian function $e^{-k\omega^2 t}$; where $\alpha = kt$ can written as

$$g(x) = \sqrt{\frac{\pi}{kt}} \, e^{-x^2/4kt}$$

Thus solution will be

$$u(x,t) = \int_{-\infty}^{\infty} f(\bar{x}) \frac{1}{\sqrt{4\pi kt}} e^{-(x-\bar{x})^2/4kt} \, d\bar{x}$$

Thus,

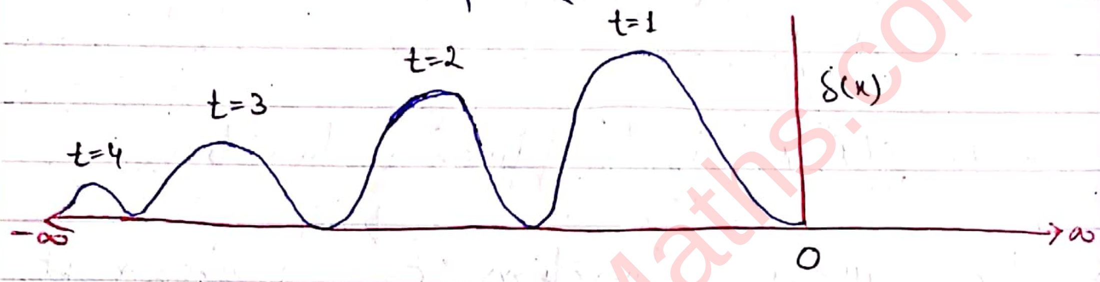
$$G(x,t;\bar{x},0) = \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-\bar{x})^2}{4kt}} = g(x-\bar{x})$$

is called influence function.

Question 10.4.3 (b):- Sketch the graph for $u(x,t)$ for various $t>0$. Comment on the significant of the convection term $c\,\partial u/\partial x$.

Solution

The guassian function deform continuously with time and term ct shift the graph from origin left ←



It depends upon t due to convected term and deform due to guassiant

Question 10.4.4:- (a) Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \gamma u \qquad -\infty<x<\infty$$

$$u(x,0) = f(x)$$

(b) Does your solution suggest a simplifying transformation.

Solution

(a) $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \gamma u$$

Applying fourier Transform

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \mathcal{F}\left[k\frac{\partial^2 u}{\partial x^2}\right] - \mathcal{F}[\gamma u]$$

$$\Rightarrow \frac{\partial}{\partial t}\bar{U}(\omega,t) = k(-i\omega)^2\bar{U}(\omega,t) - \gamma\,\bar{U}(\omega,t)$$

$$\Rightarrow \frac{\partial}{\partial t}\bar{U}(\omega,t) = -k\omega^2\bar{U}(\omega,t) - \gamma\,\bar{U}(\omega,t)$$

$$\Rightarrow \frac{\partial}{\partial t}\bar{U}(\omega,t) = -(k\omega^2+\gamma)\,\bar{U}(\omega,t)$$

$$\Rightarrow \bar{U}(\omega,t) = c(\omega)\,e^{-(k\omega^2+\gamma)t} \longrightarrow ①$$

Now from initial condition

$$u(x,0) = f(x)$$

$$\mathcal{F}[u(x,0)] = \mathcal{F}[f(x)]$$

$$\Rightarrow \bar{U}(\omega,0) = F(\omega) \qquad \text{Put in } ①$$

$$\bar{U}(\omega,0) = c(\omega)$$

$$\Rightarrow c(\omega) = F(\omega)$$

Thus equ ① becomes

$$\bar{U}(\omega,t) = F(\omega)\,G(\omega);$$

where $G(\omega) = e^{-(k\omega^2+\gamma)t}$

Now Take inverse fourier Transform

$$\mathcal{F}^{-1}[\bar{U}(\omega,t)] = \mathcal{F}^{-1}[F(\omega)\,G(\omega)] \longrightarrow ②$$

By Convolution therom

$$\mathcal{F}^{-1}[F(\omega)G(\omega)] = f * g \longrightarrow ⊛$$

Now $\mathcal{F}^{-1}[F(\omega)] = f(x)$

& $\mathcal{F}^{-1}[G(\omega)] = \mathcal{F}^{-1}\left[e^{-(k\omega^2+\gamma)t}\right]$

$$= \int_{-\infty}^{\infty} e^{-(k\omega^2+\gamma)t}\, e^{-i\omega x}\, d\omega$$

$$\Rightarrow \mathcal{F}^{-1}[G(\omega)] = e^{-\gamma t}\int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega x}\, d\omega$$

$$= e^{-\gamma t}\sqrt{\frac{\pi}{kt}}\, e^{\frac{-x^2}{4kt}}$$

Now ② becomes by using convolution theorem

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, g(x-\bar{x})\, d\bar{x}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\, e^{-\gamma t}\sqrt{\frac{\pi}{kt}}\, e^{\frac{-(x-\bar{x})^2}{4kt}}\, d\bar{x}$$

$$= e^{-\gamma t}\frac{1}{2\pi}\sqrt{\frac{\pi}{kt}}\int_{-\infty}^{\infty} f(\bar{x})\, e^{\frac{-(x-\bar{x})^2}{4kt}}\, d\bar{x}$$

$$\Rightarrow u(x,t) = e^{-\gamma t}\frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} f(\bar{x})\, e^{\frac{-(x-\bar{x})^2}{4kt}}\, d\bar{x}$$

(b) Multiplying the solution by $e^{\gamma t}$ we have

$$e^{\gamma t}u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} f(\bar{x})\, e^{\frac{-(x-\bar{x})^2}{4kt}}\, d\bar{x} = f*g$$

where $G(x,t) = \sqrt{\frac{\pi}{kt}}\, e^{-x^2/4kt}$

is influence function of initial condition. Thus

$$\frac{\partial}{\partial t}\left[e^{\eta t}u(x,t)\right] = k\frac{\partial^2 u}{\partial x^2}\left(e^{\eta t}u(x,t)\right)$$

which transform our solution $u(x,t)$ by $e^{\eta t}$.

Question 10.4.7:- (a) Solve the linearized Korteweg-deVries equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^3 u}{\partial x^3} \qquad -\infty < x < \infty$$

$$u(x,0) = f(x)$$

(b) Use convolution theorem to simplify.

Solution (a) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^3 u}{\partial x^3}$

Applying fourier transform

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \mathcal{F}\left[k\frac{\partial^3 u}{\partial x^3}\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\bar{U}(\omega,t) = (-i\omega)^3 k\,\bar{U}(\omega,t)$$

$$\Rightarrow \frac{\partial}{\partial t}\bar{U}(\omega,t) = -ik\omega^3\,\bar{U}(\omega,t)$$

$$\Rightarrow \frac{\partial}{\partial t}\bar{U}(\omega,t) + ik\omega^3\,\bar{U}(\omega,t) = 0$$

$$\Rightarrow \bar{U}(\omega,t) = c(\omega)e^{-ik\omega^3 t} \longrightarrow (1)$$

Now from initial condion

$$\mathcal{F}[u(x,0)] = \mathcal{F}[f(x)]$$

$$\Rightarrow U(\omega,0) = F(\omega)$$

So from (1) $\quad c(\omega) = F(\omega)$

$\Rightarrow \bar{U}(\omega,t) = F(\omega)\, e^{-ik\omega^3 t}$

Take fourier inverse Transform

$\mathcal{F}^{-1}[\bar{U}(\omega,t)] = \mathcal{F}^{-1}[F(\omega)\, e^{-ik\omega^3 t}]$

$= \mathcal{F}^{-1}[F(\omega)\, G(\omega)], \longrightarrow (2)$

where $G(\omega) = e^{-ik\omega^3 t}$

(b) Now By using convolution theorem

$\mathcal{F}^{-1}[F(\omega) G(\omega)] = f * g$

Now $\mathcal{F}^{-1}[F(\omega)] = f(x)$

& $\mathcal{F}^{-1}[G(\omega)] = \int_{-\infty}^{\infty} e^{-ik\omega^3 t}\, e^{-i\omega x}\, d\omega$

Question 10.4.8:- Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \begin{array}{l} 0 < x < L \\ -\infty < y < \infty \end{array}$$

S/to $\quad u(0, y) = g_1(y)$

$\quad u(L, y) = g_2(y)$

Solution

We apply fourier transform w.r.t 'y'; since y has infinite domain

Thus,

We take fourier Transform of u w.r.t y

$$\Rightarrow \mathcal{F}[u(x,y)] = \bar{U}(x,\omega)$$

& $\quad u(x,y) = \int_{-\infty}^{\infty} \bar{U}(x,\omega)\, e^{-i\omega y}\, d\omega$

Now, Take fourier transform of given PDE

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 u}{\partial y^2}\right] = 0$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\bar{U}(x,\omega) + (-i\omega)^2\bar{U}(x,\omega) = 0$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\bar{U}(x,\omega) - \omega^2\bar{U}(x,\omega) = 0$$

characteristic equation is

$$D^2 - \omega^2 = 0 \Rightarrow D^2 = \omega^2$$

$$\Rightarrow D = \pm\omega$$

$$\Rightarrow \bar{U}(x,\omega) = A(\omega)\sinh \omega x + B(\omega)\cosh \omega x \quad \text{--- (1)}$$

Now By initial conditions

$$\bar{U}(0,\omega) = G_1(\omega) \quad \& \quad \bar{U}(L,\omega) = G_2(\omega)$$

Thus from ①

$$\bar{U}(0,\omega) = B(\omega) \quad \Rightarrow G_1(\omega) = B(\omega)$$

and $\bar{U}(L,\omega) = A(\omega)\sinh\omega L + G_1(\omega)\cosh\omega L = G_2(\omega)$

$$\Rightarrow A(\omega) = \frac{G_2(\omega) - G_1(\omega)\cosh\omega L}{\sinh\omega L}$$

Thus

$$\bar{U}(x,\omega) = \left[\frac{G_2(\omega) - G_1(\omega)\cosh\omega L}{\sinh\omega L}\right]\sinh\omega x + G_1(\omega)\cosh\omega x$$

$$= G_2(\omega)\frac{\sinh\omega x}{\sinh\omega L} + G_1(\omega)\left[\cosh\omega x - \frac{\cosh\omega L\sinh\omega x}{\sinh\omega L}\right]$$

$$= G_2(\omega)\frac{\sinh\omega x}{\sinh\omega L} + G_1(\omega)\left[\frac{\cosh\omega x\sinh\omega L - \cosh\omega L\sinh\omega x}{\sinh\omega L}\right]$$

As we know that

$$\cosh x\sinh y - \cosh y\sinh x = \sinh(y-x)$$

$$\Rightarrow \bar{U}(x,\omega) = G_2(\omega)\frac{\sinh\omega x}{\sinh\omega L} + G_1(\omega)\frac{\sinh\omega(L-x)}{\sinh\omega L}$$

$$\Rightarrow u(x,y) = \int_{-\infty}^{\infty}\left[\bar{U}(x,\omega)\right]e^{-i\omega y}\,d\omega$$

$$\& \quad G_i(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g_i(y)\,e^{i\omega y}\,dy$$

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OR:- We can solve it by taking solution

$$\bar{U}(x,\omega) = C_1(\omega)e^{-\omega x} + C_2(\omega)e^{\omega x}$$

In this case

$$C_1(\omega) = \frac{G_2(\omega) + \omega e^{-\omega L} G_1(\omega)}{\omega(e^{\omega L} + e^{-\omega L})}$$

&

$$C_2(\omega) = \frac{G_1(\omega)e^{\omega L} - G_2(\omega)}{\omega(e^{\omega L} + e^{-\omega L})}$$

★ Derivation of the Inverse Fourier Transform of a Gaussian ($e^{-\alpha\omega^2}$)

The inverse Fourier Transform of a Gaussian $e^{-\alpha\omega^2}$ is given by

$$g(x) = \int_{-\infty}^{\infty} e^{-\alpha\omega^2} e^{-i\omega x}\, d\omega$$

$$g'(x) = \int_{-\infty}^{\infty} -i\omega\, e^{-\alpha\omega^2} e^{-i\omega x}\, d\omega$$

can be simplified using an integration by part

$$g'(x) = (0-0) - \frac{-i}{2\alpha}\int_{-\infty}^{\infty} e^{-\alpha\omega^2}(2\alpha\omega)\, e^{-i\omega x}\, d\omega$$

$$= \frac{i}{2\alpha}\int_{-\infty}^{\infty} \frac{d}{d\omega}\left(e^{-\alpha\omega^2}\right) e^{-i\omega x}\, d\omega$$

Integrate by Parts

$$g'(x) = \frac{i}{2\alpha}\left[e^{-i\omega x} e^{-\alpha\omega^2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-i\omega x} (-ix) e^{-\alpha\omega^2} d\omega \right]$$

$$= \frac{i^2 x}{2\alpha} \int_{-\infty}^{\infty} e^{-i\omega x} e^{-\alpha\omega^2} d\omega$$

$$\Rightarrow g'(x) = \frac{-x}{2\alpha} g(x)$$

$$\Rightarrow g'(x) + \frac{x}{2\alpha} g(x) = 0$$

$$\Rightarrow D + \frac{x}{2\alpha} = 0$$

is ODE & its solution will be

$$g(x) = g(0) e^{\left(\frac{-x}{2\alpha}\right) x} = g(0) e^{\frac{-x^2}{2\alpha}}$$

where

$$g(0) = \int_{-\infty}^{\infty} e^{-\alpha\omega^2} d\omega$$

Put $z = \sqrt{\alpha}\,\omega \Rightarrow dz = \sqrt{\alpha}\, d\omega$

$$\Rightarrow g(0) = \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-z^2} dz = \frac{\sqrt{\pi}}{\sqrt{\alpha}} \quad \because \int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$$

$$\Rightarrow g(0) = \sqrt{\frac{\pi}{\alpha}}$$

Thus $g(x) = \sqrt{\frac{\pi}{\alpha}}\, e^{-x^2/4\alpha}$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\alpha\omega^2} e^{-i\omega x}\, d\omega = \sqrt{\frac{\pi}{\alpha}}\, e^{-x^2/4\alpha}$$

$$\Rightarrow \mathcal{F}^{-1}\left[e^{-\alpha\omega^2}\right] = \sqrt{\frac{\pi}{\alpha}}\, e^{-x^2/4\alpha}$$

Example:- Evaluate Fourier Transform of Gaussian function defined by

$g(x) = Ne^{-\alpha x^2}$, where N and α are constant and $\alpha > 0$

Solution

$$\mathcal{F}\{g(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} g(x)\, dx$$

$$\Rightarrow \mathcal{F}[g(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} Ne^{-\alpha x^2} dx$$

$$= \frac{N}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x - \alpha x^2} dx$$

$$= \frac{N}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha\left[x^2 - \frac{i\omega}{\alpha}x\right]} dx$$

$$= \frac{N}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha\left[x^2 - 2\left(\frac{i\omega}{2\alpha}\right)x + \left(\frac{i\omega}{2\alpha}\right)^2 - \left(\frac{i\omega}{2\alpha}\right)^2\right]} dx$$

$$= \frac{N}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha\left\{\left[x - \frac{i\omega}{2\alpha}\right]^2 + \frac{\omega^2}{4\alpha^2}\right\}} dx$$

$$= \frac{N}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha\left[x - \frac{i\omega}{2\alpha}\right]^2} \cdot e^{-\frac{\omega^2}{4\alpha}} dx$$

$$\Rightarrow \mathcal{F}[g(x)] = \frac{N}{2\pi}\cdot e^{\frac{-\omega^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-\left[\sqrt{\alpha}\left(x-\frac{i\omega}{2\alpha}\right)\right]^2} dx \longrightarrow (1)$$

Put $\sqrt{\alpha}\left[x-\frac{i\omega}{2\alpha}\right] = y$

$$\Rightarrow \sqrt{\alpha}\, dx = dy \Rightarrow dx = \frac{1}{\sqrt{\alpha}} dy$$

So equ (1) becomes

$$\mathcal{F}[g(x)] = \frac{N}{2\pi} e^{\frac{-\omega^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-y^2} \cdot \frac{1}{\sqrt{\alpha}} dy$$

$$= \frac{N}{\sqrt{4\pi^2\alpha}} e^{\frac{-\omega^2}{4\alpha}} \sqrt{\pi} \quad \because \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$$

$$\Rightarrow \mathcal{F}[g(x)] = G(\omega) = \frac{N}{\sqrt{4\pi\alpha}} e^{-\omega^2/4\alpha}$$

Example:- Inverse Fourier Transform of Gaussian $e^{-\alpha\omega^2}$ (Alternate)

Solution

Let $G(\omega) = e^{-\alpha\omega^2}$

$$\mathcal{F}^{-1}[G(\omega)] = \int_{-\infty}^{\infty} e^{-\alpha\omega^2} e^{-i\omega x} d\omega$$

$$= \int_{-\infty}^{\infty} e^{-\alpha\left(\omega^2+\frac{i\omega}{\alpha}x\right)} d\omega$$

$$= \int_{-\infty}^{\infty} e^{-\alpha\left[\omega^2+2\left(\frac{ix}{2\alpha}\right)\omega+\left(\frac{ix}{2\alpha}\right)^2-\left(\frac{ix}{2\alpha}\right)^2\right]} d\omega$$

$$\Rightarrow \mathcal{F}^{-1}[G(\omega)] = \int_{-\infty}^{\infty} e^{-\alpha\left[\left(\omega+\frac{ix}{2\alpha}\right)^2+\frac{x^2}{4\alpha^2}\right]} d\omega$$

$$= \int_{-\infty}^{\infty} e^{-\alpha\left[\omega+\frac{ix}{2\alpha}\right]^2} \cdot e^{-\frac{x^2}{4\alpha}} d\omega$$

$$= e^{-\frac{x^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-\left[\sqrt{\alpha}\left(\omega+\frac{ix}{2\alpha}\right)\right]^2} d\omega$$

Put $\sqrt{\alpha}\left[\omega+\frac{ix}{2\alpha}\right] = z$

$$\Rightarrow \sqrt{\alpha}\, d\omega = dz \Rightarrow d\omega = \frac{1}{\sqrt{\alpha}} dz$$

$$\Rightarrow \mathcal{F}^{-1}[G(\omega)] = g(x) = e^{-\frac{x^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-z^2} \cdot \frac{1}{\sqrt{\alpha}} dz$$

$$\Rightarrow g(x) = \frac{1}{\sqrt{\alpha}} e^{-\frac{x^2}{4\alpha}} \cdot \sqrt{\pi} \quad \because \int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$$

$$\Rightarrow g(x) = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{x^2}{4\alpha}}$$

SECTION 10.5:-

* Fourier Sine & Cosine Transforms.

Fourier Sine Transform Table

$$S[f(x)] = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx$$

$$f(x) = \int_0^\infty F(\omega)\sin\omega x\,d\omega$$

$$S\left[\frac{df}{dx}\right] = -\omega C[f(x)]$$

$$S\left[\frac{d^2f}{dx^2}\right] = \frac{2}{\pi}\omega f(0) - \omega^2 S[f(x)]$$

$$\frac{x}{x^2+\beta^2} \longrightarrow e^{-\omega\beta}$$

$$e^{-\varepsilon x} \longrightarrow \frac{2}{\pi}\frac{\omega}{\varepsilon^2+\omega^2}$$

$$\frac{1}{\pi}\int_0^\infty f(\bar{x})[g(x-\bar{x})-g(x+\bar{x})]d\bar{x}$$

$$\frac{1}{\pi}\int_0^\infty g(\bar{x})[f(x-\bar{x})-f(x+\bar{x})]d\bar{x}$$

$\longrightarrow S[f(x)]C[g(x)]$

Convolution

$$1 \rightarrow \frac{2}{\pi}\frac{1}{\omega}$$

Fourier Cosine Transform Table

$$C[f(x)] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx$$

$$f(x) = \int_0^\infty f(\omega)\cos\omega x\,d\omega$$

$$C\left[\frac{df}{dx}\right] = -\frac{2}{\pi}f(0) + \omega S[f(x)]$$

$$C\left[\frac{d^2f}{dx^2}\right] = -\frac{2}{\pi}f'(0) - \omega^2 C[f(x)]$$

$$\frac{\beta}{x^2+\beta^2} \longrightarrow e^{-\omega\beta}$$

$$e^{-\varepsilon x} \longrightarrow \frac{2}{\pi}\frac{\varepsilon}{\omega^2+\varepsilon^2}$$

$$e^{-\alpha x^2} \longrightarrow \frac{1}{\sqrt{4\pi\alpha}} e^{-\omega^2/4\alpha}$$

$$\int_0^\infty g(\bar{x})[f(x-\bar{x}) + f(x+\bar{x})]d\bar{x} \longrightarrow F(\omega)G(\omega)$$

★ Heat Equation on a Semi-Infinite Interval:-

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = g(t) \qquad \text{(Non Homo B.C)}$$

$$u(x,0) = f(x) \qquad 0<x<\infty$$

Boundry condition suggest that we use Fourier sine Transform.

$$S\left[\frac{\partial u}{\partial t}\right] = S\left[k\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\bar{U}(\omega,t) = k\left[\frac{2}{\pi}\omega\, u(0,t) - \omega^2\bar{U}(\omega,t)\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\bar{U}(\omega,t) = k\left[\frac{2}{\pi}\omega\, g(t) - \omega^2\bar{U}(\omega,t)\right]$$

$$\Rightarrow \frac{\partial \bar{U}}{\partial t} + k\omega^2 \bar{U} = k\frac{2}{\pi}\omega g(t) \longrightarrow (1)$$

with initial condition

$$\bar{U}(\omega,0) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx$$

Thus for solution of equ (1)

$$I.F = e^{\int k\omega^2 dt} = e^{k\omega^2 t}$$

$$\Rightarrow \frac{\partial}{\partial t}\left[e^{k\omega^2 t}\bar{U}\right] = \frac{k2}{\pi}\omega g(t)e^{k\omega^2 t}$$

$$\Rightarrow e^{k\omega^2 t}\bar{U} = \frac{2k}{\pi}\omega\int g(t)e^{k\omega^2 t}dt + c$$

$$\Rightarrow \bar{U} = \frac{2k\omega}{\pi}e^{-k\omega^2 t}\int g(t)e^{k\omega^2 t}dt + ce^{-k\omega^2 t}$$

$$\Rightarrow \bar{U} = \frac{2k\omega}{\pi}\int g(\bar{t})\,e^{-k\omega^2(t-\bar{t})}d\bar{t} + ce^{-k\omega^2 t}$$

$$\Rightarrow \bar{U} = \frac{2}{\pi}k\int g(\bar{t})\,e^{-k\omega^2(t-\bar{t})}\cdot\omega\,d\bar{t} + ce^{-k\omega^2 t}$$

$$\Rightarrow u(x,t) = \frac{2}{\pi}k\int g(\bar{t})\left(\int_{-\infty}^{\infty} e^{-k\omega(t-\bar{t})}\cdot\omega\sin\omega x\,d\omega\right)d\bar{t}$$
$$+ c\int_{-\infty}^{\infty} e^{-k\omega^2 t}\sin\omega x\,d\omega$$

$$\Rightarrow u(x,t) = \frac{2}{\pi}k\int_{-\infty}^{\infty}\left(\int g(\bar{t})\,e^{-k\omega(t-\bar{t})}d\bar{t}\right)\cdot\omega\sin\omega x\,d\omega$$
$$+ c\int_{-\infty}^{\infty} e^{-k\omega^2 t}\sin\omega x\,d\omega.$$

Example 10.5.6:- $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

$u(0,t) = 0 \; ; \; u(x,0) = f(x) \qquad 0 < x < \infty$

Solution

Applying Fourier sine Transform

$$\frac{\partial}{\partial t}\bar{U} = k\left[\frac{2}{\pi}\cancel{u(0,t)} - \omega^2 \bar{U}\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\bar{U} = -k\omega^2\bar{U} \qquad \because u(0,t)=0$$

$$\Rightarrow \bar{U}(\omega,t) = c(\omega)\, e^{-k\omega^2 t} \longrightarrow ①$$

Now from initial condition

$$\mathcal{F}_s[\bar{U}(x,0)] = \mathcal{F}_s[f(x)]$$

$$\Rightarrow \bar{U}(\omega,0) = F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx$$

from ① $\Rightarrow F(\omega) = c(\omega)$

$$\Rightarrow \bar{U}(\omega,t) = F(\omega)e^{-k\omega^2 t}$$

From this step we can use Convolution theorem

$$\Rightarrow u(x,t) = \int_0^\infty F(\omega)\, e^{-k\omega^2 t}\sin\omega x\, d\omega$$

$$u(x,t) = \int_0^\infty F(\omega)\, e^{-k\omega^2 t}\,\frac{e^{i\omega x}}{2i}\, d\omega \longrightarrow (*)$$

Continue

Integrating Property for odd & even functions

$$\int_{-\infty}^{0} + \int_0^\infty = 0 \quad \text{for odd function}$$

$$\int_{-\infty}^{0} + \int_0^\infty = 2\int_0^\infty \quad \text{for even function}$$

If $f(x)$ is odd, the $\sin \omega x$ is also odd our over all function will become even

Now consider

$$\frac{F(\omega)}{2i} = \frac{2}{\pi}\int_0^\infty f(x)\,\frac{\sin \omega x}{2i}\,dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\,\frac{\sin \omega x}{2i}\,dx$$

$$\Rightarrow \frac{F(\omega)}{2i} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{-i\omega x}\,dx$$

use in (A) implies

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-k\omega^2 t}\int_{-\infty}^{\infty} f(\bar{x})\,\cancel{e^{-i\omega x}}\,\cancel{e^{i\omega x}}\,d\bar{x}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\bar{x})\,e^{-k\omega^2 t}\,d\bar{x}\,d\omega$$

From this step we can use Convolution theorem directly

$$= \int_{-\infty}^{\infty} f(\bar{x})\cdot\frac{1}{2\pi}\sqrt{\frac{\pi}{kt}}\,e^{-\frac{(x-\bar{x})^2}{4kt}}\,d\bar{x}$$

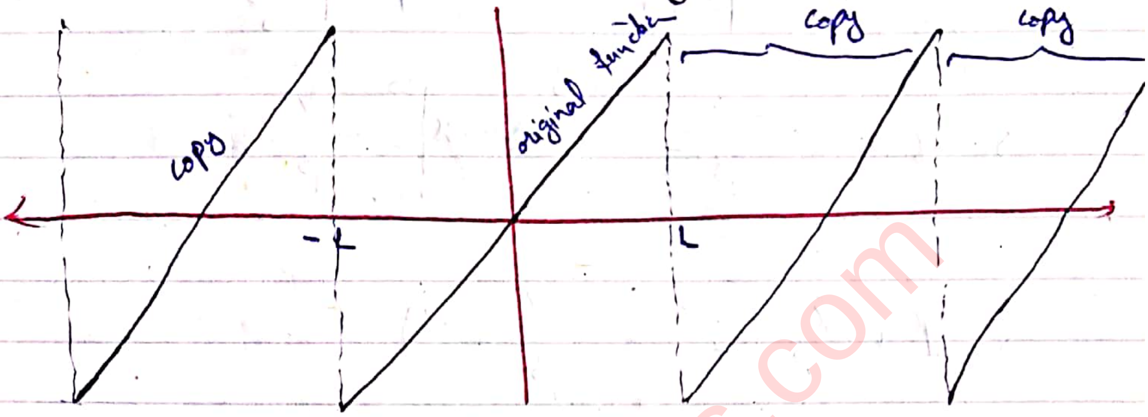
$$= \sqrt{\frac{1}{4\pi kt}}\int_{-\infty}^{\infty} f(\bar{x})\,e^{-\frac{(x-\bar{x})^2}{4kt}}\,d\bar{x}$$

$$= \frac{1}{\sqrt{4\pi kt}}\left[\int_{-\infty}^{0} -f(-\bar{x})\,e^{-\frac{(x-\bar{x})^2}{4kt}}\,d\bar{x} + \int_0^\infty f(\bar{x})\,e^{-\frac{(x-\bar{x})^2}{4kt}}\,d\bar{x}\right]$$

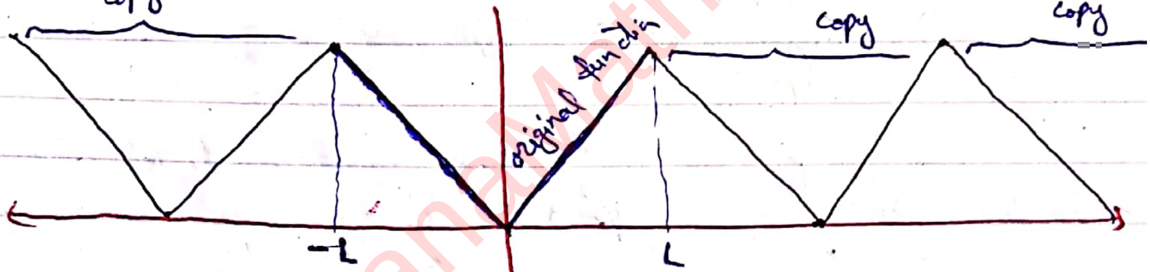
$$\bar{x} \longrightarrow -\bar{x}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} f(\bar{x}) \left[e^{-(x-\bar{x})^2/4kt} - e^{-(x+\bar{x})^2/4kt} \, d\bar{x} \right]$$

* Fourier Series (Graphycal Representation)



Fourier cosine (Even)



Fourier sine (odd)

Exercise 10.5

Question 10.5.1:- Consider $F(\omega) = e^{-\omega\beta}$, $\beta > 0$ $(\omega \geq 0)$

(a) Derive the Inverse Fourier sine transform of $F(\omega)$

(b) Derive the Inverse Fourier cosine transform of $F(\omega)$

Solution

$$F(\omega) = e^{-\omega\beta}$$

$$\mathcal{F}^{-1}[F(\omega)] = \int_0^{\infty} e^{-\omega\beta} \sin\omega x \, d\omega = I$$

$$\Rightarrow \mathcal{F}^{-1}[F(\omega)] = \cancel{\frac{-1}{\beta} e^{-\omega\beta} \sin\omega x}\Big|_0^\infty + \frac{x}{\beta}\int_0^\infty e^{-\omega\beta}\cos\omega x\, d\omega$$

$$= \frac{x}{\beta}\left[\frac{-1}{\beta} e^{-\omega\beta}\cos\omega x\Big|_0^\infty - \frac{x}{\beta}\int_0^\infty e^{-\omega\beta}\sin\omega x\, d\omega\right.$$

$$\Rightarrow I = \frac{-x}{\beta^2} e^{-\omega\beta}\cos\omega x\Big|_0^\infty - \frac{x^2}{\beta^2} I$$

$$\Rightarrow \left(1 + \frac{x^2}{\beta^2}\right) I = \frac{-x}{\beta^2}(-1)$$

$$\Rightarrow \frac{\beta^2 + x^2}{\beta^2} I = \frac{x}{\beta^2}$$

$$\Rightarrow I = \frac{x}{\beta^2 + x^2}$$

$$\Rightarrow \mathcal{F}^{-1}_S[F(\omega)] = \frac{x^2}{\beta^2 + x^2}$$

(b) $$\mathcal{F}^{-1}[F(\omega)] = \int_0^\infty e^{-\omega\beta}\cos\omega x\, d\omega = I$$

$$\Rightarrow I = \frac{-e^{-\omega\beta}}{\beta}\cos\omega x\Big|_0^\infty - \frac{x}{\beta}\int_0^\infty e^{-\omega\beta}\sin\omega x\, d\omega$$

$$= \frac{-1}{\beta}(-1) - \frac{x}{\beta}\left[\cancel{-\frac{1}{\beta} e^{-\omega\beta}\sin\omega x}\Big|_0^\infty + \frac{x}{\beta}\int_0^\infty e^{-\omega\beta}\cos\omega x\, d\omega\right]$$

$$= \frac{1}{\beta} - \frac{x^2}{\beta^2}\int_0^\infty e^{-\omega\beta}\cos\omega x\, d\omega$$

$$\Rightarrow \left(1+\frac{x^2}{\beta^2}\right) I = \frac{1}{\beta}$$

$$\Rightarrow \frac{x^2+\beta^2}{\beta^2} I = \frac{1}{\beta}$$

$$\Rightarrow (x^2+\beta^2) I = \beta$$

$$\Rightarrow I = \mathcal{F}^{-1}_C[F(\omega)] = \frac{\beta}{x^2+\beta^2}$$

Question 10.5.12:- Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad (x>0)$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad , \quad u(x,0) = f(x)$$

Solution Boundry condition suggests that we should use fourier cosine transform.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$C[u(x,t)] = \bar{U}(\omega,t) = \frac{2}{\pi}\int_0^\infty u(x,t)\cos\omega x\, dx$$

Thus
$$\frac{\partial \bar{U}}{\partial t} = k\left[-\frac{2}{\pi}\frac{\partial u}{\partial x}(0,t) - \omega^2\bar{U}\right]$$

$$\Rightarrow \frac{\partial \bar{U}}{\partial t} = k\left[-\frac{2}{\pi}(0) - \omega^2\bar{U}(\omega,t)\right] \quad \because \frac{\partial u}{\partial x}(0,t)=0$$

$$\Rightarrow \frac{\partial \bar{U}}{\partial t} + \omega^2\bar{U} = 0$$

$$\Rightarrow \bar{U}(\omega,t) = A(\omega)e^{-k\omega^2 t}$$

Now by initial condition

$$u(x,0) = f(x) \Rightarrow \bar{U}(x,0) = F(\omega)$$

Thus we have

$$\bar{U}(\omega,0) = A(\omega) = F(\omega)$$

Thus solution will be

$$\bar{U}(\omega,t) = F(\omega)\, e^{-k\omega^2 t} \longrightarrow (1)$$

Now using inverse fourier transform

$$u(x,t) = \int_0^\infty F(\omega)\, e^{-k\omega^2 t} \cos\omega x \, d\omega$$

take $g(\omega) = e^{-k\omega^2 t}$

$$\Rightarrow u(x,t) = \int_0^\infty F(\omega)\, g(\omega) \cos\omega x \, d\omega$$

$$= C^{-1}[f(\omega)\, g(\omega)] \quad \because F(\omega) = f(\omega)$$

By convolution theorem.

$$u(x,t) = \frac{1}{\pi}\int_0^\infty f(\bar{x})[g(x-\bar{x}) + g(x+\bar{x})]\, d\bar{x}$$

where $g(x) = C^{-1} g(\omega) = \sqrt{\frac{\pi}{4kt}}\, e^{\left(\frac{-x^2}{4kt}\right)}$

$$\Rightarrow u(x,t) = \frac{1}{\pi}\sqrt{\frac{\pi}{4kt}}\int_0^\infty f(\bar{x})\left[e^{\frac{-(x-\bar{x})^2}{4kt}} + e^{\frac{-(x+\bar{x})^2}{4kt}}\right] d\bar{x}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_0^\infty f(\bar{x})\left[e^{\frac{-(x-\bar{x})^2}{4kt}} + e^{\frac{-(x+\bar{x})^2}{4kt}}\right] d\bar{x}$$

Question 10.5.9 :- Let $S[f(x)]$ designate the fourier sine Transform

(a) Show that
$$S[e^{-\varepsilon x}] = \frac{2}{\pi}\frac{\omega}{\varepsilon^2+\omega^2} \quad \text{for } \varepsilon>0$$
show that $\lim_{\varepsilon\to 0+} S[e^{-\varepsilon x}] = \frac{2}{\pi\omega}$.
We will let $S[1] = 2/\pi\omega$. Why is not $S[1]$ technically defined.

(b) Show that $S^{-1}\left[\frac{2/\pi}{\omega}\right] = \frac{2}{\pi}\int_0^\infty \frac{\sin z}{z}dz$ which is know to equal 1.

Solution

$$S[e^{-\varepsilon x}] = \frac{2}{\pi}\int_0^\infty e^{-\varepsilon x}\sin\omega x\,dx$$

$$\Rightarrow I = S[e^{-\varepsilon x}] = \frac{2}{\pi}\left[\cancel{\frac{-1}{\varepsilon}e^{-\varepsilon x}\sin\omega x\Big|_0^\infty}^{\,0} - \int_0^\infty \frac{e^{-\varepsilon x}}{-\varepsilon}(\omega\cos\omega x)dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\omega}{\varepsilon}\int_0^\infty e^{-\varepsilon x}\cos\omega x\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\omega}{\varepsilon}\left\{\frac{e^{-\varepsilon x}}{-\varepsilon}\cos\omega x\Big|_0^\infty + \int_0^\infty \frac{e^{-\varepsilon x}}{-\varepsilon}\omega\sin\omega x)dx\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{\omega}{\varepsilon}\left[\left(0+\frac{1}{\varepsilon}\right)\right]\right] + \frac{2}{\pi}\cdot\frac{-\omega^2}{\varepsilon^2}\int_0^\infty e^{-\varepsilon x}\sin\omega x\,d$$

$$= \frac{2\omega}{\pi\varepsilon^2} - \frac{\omega^2}{\varepsilon^2}I$$

$$\Rightarrow I + \frac{\omega^2}{\varepsilon^2}I = \frac{2\omega}{\pi\varepsilon^2}$$

$$\Rightarrow \left(\frac{\varepsilon^2+\omega^2}{\cancel{\varepsilon^2}}\right)I = \frac{2}{\pi}\frac{\omega}{\cancel{\varepsilon^2}}$$

$$\Rightarrow I = \frac{2}{\pi}\,\frac{\omega}{\varepsilon^2+\omega^2}$$

Thus $$S\left[e^{-\varepsilon x}\right] = \frac{2}{\pi}\,\frac{\omega}{\varepsilon^2+\omega^2} \qquad \text{Proved}$$

Let $$\lim_{\varepsilon\to 0^+} S\left[e^{-\varepsilon x}\right] = \lim_{\varepsilon\to 0^+}\frac{2}{\pi}\,\frac{\omega}{\varepsilon^2+\omega^2}$$

$$= \frac{2}{\pi}\left(\frac{\omega}{\omega^2}\right)$$

$$\Rightarrow \lim_{\varepsilon\to 0^+} S\left[e^{-\varepsilon x}\right] = \frac{2}{\pi\omega}$$

$$\Rightarrow S[1] = \frac{2/\pi}{\omega}$$

If we take direct

$$S[1] = \frac{2}{\pi}\int_0^\infty 1\cdot\sin\omega x\,dx$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\,dx$$

$$= \frac{2}{\pi}\left.\frac{-\cos\omega x}{\omega}\right|_0^\infty$$

$$= \frac{2}{\pi}\left[-\left(0-\frac{1}{\omega}\right)\right] = \frac{2}{\pi}\left[\frac{1}{\omega}\right]$$

$$\Rightarrow S[1] = \frac{2/\pi}{\omega} \qquad \text{Proved.}$$

Question 10.5.15:- Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \begin{matrix} 0<x<L \\ 0<y<\infty \end{matrix}$$

$u(x,0) = 0$

$u(0,y) = g_1(y)$; $u(L,y) = g_2(y)$

Solution

We apply fourier transform on y and x consider as initial condition. & initial condition suggest that will will apply fourier sine transform

$$\Rightarrow S\left[\frac{\partial^2 u}{\partial x^2}\right] + S\left[\frac{\partial^2 u}{\partial y^2}\right] = 0$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\overline{U} + \left[+\frac{2}{\pi}\omega\, u(x,0) - \omega^2\,\overline{U}(x,\omega)\right] = 0$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\overline{U} + 0 - \omega^2\,\overline{U} = 0 \qquad \because u(x,0) = 0$$

$$\Rightarrow \frac{\partial^2\overline{U}}{\partial x^2} - \omega^2\,\overline{U} = 0$$

Ch equ is $D^2 - \omega^2 = 0 \Rightarrow D^2 = \omega^2$

$$\Rightarrow D = \pm\omega$$

$$\Rightarrow \overline{U}(x,\omega) = A(\omega)\sinh\omega x + B(\omega)\cosh\omega x \longrightarrow ①$$

Now by initial condition

$\overline{U}(0,\omega) = G_1(\omega)$ & $\overline{U}(L,\omega) = G_2(\omega)$

Thus from ①

$$\overline{U}(0,\omega) = B(\omega) \Rightarrow G_1(\omega) = B(\omega)$$

and $U(L,\omega) = A(\omega)\sinh\omega L + B(\omega)\cosh(\omega L)$

$$\Rightarrow G_2(\omega) = A(\omega)\sinh\omega L + G_1(\omega)\cosh\omega L$$

$$\Rightarrow A(\omega) = \frac{G_2(\omega) - G_1(\omega)\cosh\omega L}{\sinh\omega L}$$

$$\Rightarrow \bar{U}(x,\omega) = \left[\frac{G_2(\omega) - G_1(\omega)\cosh\omega L}{\sinh\omega L}\right]\sinh\omega x + G_1(\omega)\cosh\omega x$$

$$= G_2(\omega)\frac{\sinh\omega x}{\sinh\omega L} + G_1(\omega)\left[\cosh\omega x - \frac{\cosh\omega L\sinh\omega x}{\sinh\omega L}\right]$$

$$= G_2(\omega)\frac{\sinh\omega x}{\sinh\omega L} + G_1(\omega)\left[\frac{\cosh\omega x\sinh\omega L - \cosh\omega L\sinh\omega x}{\sinh\omega L}\right]$$

$$\Rightarrow U(x,\omega) = G_2(\omega)\frac{\sinh\omega x}{\sinh\omega L} + G_1(\omega)\frac{\sinh\omega(L-x)}{\sinh\omega L}$$

$$\Rightarrow u(x,y) = \int_{-\infty}^{\infty} U(x,\omega)\sin\omega y\, d\omega$$

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GREEN'S FUNCTIONS FOR TIME-INDEPENDENT PROBLEMS ***

Question 17 :- Consider $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$

$$u(x,0) = g(x)$$

(a) Use Green's formula instead of term-by-term spatial differentiation if

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

Solution

Solution by separation of variable is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x \, e^{-k\left(\frac{n\pi}{L}\right)^2 t} \longrightarrow (1)$$

where $g(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}$

Thus $a_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$

So (1) will be

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L g(x_0)\frac{\sin n\pi x_0}{L}dx_0\right]\sin\frac{n\pi x}{L}e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

So

$$u(x,t) = \int_0^L g(x_0)\left[\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right]dx_0$$

Then $G(x,t;x_0,0) = \sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

So $u(x,t) = \int_0^L g(x_0)\,G(x,t;x_0,0)\,dx_0$

where $G(x,t;x_0,0)$ is the influence of the initial condition.

⇒ Heat Equation With Source Term:-

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \longrightarrow ①$$

with $\quad u(0,t) = 0 \quad , \quad u(L,t) = 0$

$$u(x,0) = g(x)$$

Solution

Consider homogeneous problem and we have solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{L}$$

By using eigen function expansion method ① becomes

$$\frac{da_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 a_n = \frac{2}{L}\int_0^L Q(x,t)\sin\frac{n\pi x}{L}\,dx = q_n(t)$$

$$\Rightarrow \quad \frac{da_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 a_n = q_n(t) \quad \text{(say)}$$

Then the fourier series of $Q(x,t)$ will be

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\,\frac{\sin n\pi x}{L}$$

Consider $\quad \dfrac{da_n}{dt} + k\lambda_n a_n = q_n(t)$

I.F is $\quad e^{\int k\lambda_n dt} = e^{k\lambda_n t}$

Thus we have

$$\frac{d}{dt}\left[a_n e^{k\lambda_n t}\right] = q_n(t)\,e^{k\lambda_n t}$$

$$\Rightarrow \quad a_n e^{k\lambda_n t}\Big|_0^t = \int_0^t q_n(t_0)\,e^{\lambda_n k t_0}\,dt_0$$

$$\Rightarrow a_n(t)\, e^{k\lambda_n t} - a_n(0)\, e^{k\lambda_n(0)} = \int_0^t q_n(t_0)\, e^{\lambda_n k t_0}\, dt_0$$

$$\Rightarrow a_n(t)\, e^{k\lambda_n t} = a_n(0) + \int_0^t q_n(t_0)\, e^{\lambda_n k t_0}\, dt_0$$

$$\Rightarrow a_n(t) = a_n(0)\, e^{-k\lambda_n t} + e^{-k\lambda_n t}\int_0^t q_n(t_0)\, e^{\lambda_n k t_0}\, dt_0$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n(0)\, e^{-k\lambda_n t} + e^{-k\lambda_n t}\int_0^t q_n(t_0)\, e^{\lambda_n k t_0}\, dt_0\right]\sin\frac{n\pi x}{L}$$

with $\quad g(x) = \sum_{n=1}^{\infty} a_n(0)\sin\frac{n\pi x}{L}$

$$\Rightarrow a_n(0) = \frac{2}{L}\int_0^L g(x_0)\, \sin\frac{n\pi}{L}x_0\, dx_0$$

So

$$u(x,t) = \sum_{n=1}^{\infty}\left[\left(\frac{2}{L}\int_0^L g(x_0)\sin\frac{n\pi x_0}{L}\, dx_0\right) e^{-k\lambda_n t}\right.$$

$$\left. + e^{-k\lambda_n t}\int_0^t\left(\frac{2}{L}\int_0^L Q(x_0,t_0)\sin\frac{n\pi x_0}{L}\,dx_0\right) e^{k\lambda_n t_0}\, dt_0\right]\sin\frac{n\pi x}{L}$$

$$\Rightarrow u(x,t) = \int_0^L g(x_0)\left[\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}\, e^{-k\lambda_n t}\right] dx_0$$

$$+ \int_0^L\int_0^t Q(x_0,t_0)\left[\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2(t-t_0)}\right] dt_0\, dx_0$$

$$\Rightarrow u(x,t) = \int_0^L g(x_0)\, G(x,t;x_0,0)\, dx_0 + \int_0^L\int_0^t Q(x_0,t_0)\, G(x,t;x_0,t_0)\, dt_0\, dx_0$$

where

$$G(x,t-t_0;x_0,t_0) = \sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2(t-t_0)}$$

Green function only depend upon time.

Exercise 9.2

Question 9.2.1:- Consider $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$

$$u(x,0) = g(x)$$

in all cases obtain formula instead to (9-2-20) introducing a Green's function.

a) Use Green's function instead of term-by term spatial differentiation if

$$u(0,t) = 0, \quad u(L,t) = 0$$

Solution

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

$$\text{with } u(x,0) = g(x)$$

The solution of homogeneous problem with homogeneous conditions is

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\frac{n\pi x}{L}$$

Plug into PDE, and use term by term differentiation w.r.t x

$$\sum_{n=1}^{\infty}\frac{da_n(t)}{dt}\sin\frac{n\pi x}{L} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

$$\Rightarrow \frac{da_n(t)}{dt} = \frac{\int_0^L\left[k\frac{\partial^2 u}{\partial x^2} + Q(x,t)\right]\sin\frac{n\pi x}{L}\,dx}{\int_0^L\left(\sin\frac{n\pi x}{L}\right)^2 dx} \longrightarrow (*)$$

By using green's formula, we get

$$\Rightarrow \int\left(u\frac{\partial^2 v}{\partial x^2} - v\frac{\partial^2 u}{\partial x^2}\right)dx = u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\Big|_0$$

$$\int_0^L\left(u\frac{\partial^2}{\partial x^2}\left(\sin\frac{n\pi x}{L}\right) - \sin\frac{n\pi x}{L}\frac{\partial^2 u}{\partial x^2}\right)dx = \left[u\frac{d}{dx}\sin\frac{n\pi x}{L} - \sin\frac{n\pi x}{L}\frac{\partial u}{\partial x}\right)\Big|_0^L$$

By using boundry conditions
$u(0,t) = u(L,t) = 0$, we get

$$\int_0^L \left[-\left(\frac{n\pi}{L}\right)^2 u \sin\frac{n\pi x}{L} - \sin\frac{n\pi x}{L}\frac{\partial^2 u}{\partial x^2}\right] dx =$$

$$\cancel{u(L)}^{\,0}\frac{d}{dx}\cancel{\sin\frac{n\pi L}{L}}^{\,0} - \sin\frac{n\pi L}{L}\frac{\partial u}{\partial x} - u(0)\frac{d}{dx}\sin\frac{n\pi x}{L} + \cancel{\sin(0)\frac{\partial u}{\partial x}}^{\,0}$$

$$\Rightarrow -\int_0^L \left(\frac{n\pi}{L}\right)^2 u \sin\frac{n\pi x}{L}\,dx - \int_0^L \sin\frac{n\pi x}{L}\frac{\partial^2 u}{\partial x^2}\,dx = 0$$

$$\Rightarrow -\int_0^L \left(\frac{n\pi}{L}\right)^2 u \sin\frac{n\pi x}{L}\,dx = +\int_0^L \sin\frac{n\pi x}{L}\frac{\partial^2 u}{\partial x^2}\,dx$$

Thus equ (*) becomes

$$\frac{d a_n(t)}{dt} = \frac{k\int_0^L \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi x}{L}\,dx}{\int_0^L \left(\sin\left(\frac{n\pi x}{L}\right)\right)^2 dx} + \frac{\int_0^L Q(x,t)\sin\left(\frac{n\pi x}{L}\right)dx}{\int_0^L \left(\sin\left(\frac{n\pi x}{L}\right)\right)^2 dx}$$

$$= \frac{-k\int_0^L \left(\frac{n\pi}{L}\right)^2 u \sin\frac{n\pi x}{L}\,dx}{\int_0^L \left(\sin\frac{n\pi x}{L}\right)^2 dx} + \frac{\int_0^L Q(x,t)\sin\frac{n\pi x}{L}\,dx}{\int_0^L \left(\sin\frac{n\pi x}{L}\right)^2 dx}$$

$$= -k\left(\frac{n\pi}{L}\right)^2 a_n(t) + q_n(t)$$

where $a_n(t) = \dfrac{\int_0^L u \sin\frac{n\pi x}{L}\,dx}{\int_0^L \left(\sin\frac{n\pi x}{L}\right)^2 dx}$, $q_n(t) = \dfrac{\int_0^L Q(x,t)\sin\frac{n\pi x}{L}\,dx}{\int_0^L \sin^2\frac{n\pi x}{L}\,dx}$

Thus we left with

$$\frac{d a_n(t)}{dt} + k\left(\frac{n\pi}{L}\right)^2 a_n(t) = q_n(t)$$

$$I.F = e^{k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\Rightarrow \frac{d}{dt}\left[e^{k\left(\frac{n\pi}{L}\right)^2 t}\cdot a_n\right] = q_n(t)\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\Rightarrow a_n e^{k\left(\frac{n\pi}{L}\right)^2 t}\Big|_0^t = \int_0^t q_n(\bar{t})\, e^{k\left(\frac{n\pi}{L}\right)^2 \bar{t}}\, d\bar{t}$$

$$\Rightarrow a_n(t) e^{k\left(\frac{n\pi}{L}\right)^2 t} - a_n(0) = \int_0^t q_n(\bar{t})\, e^{k\left(\frac{n\pi}{L}\right)^2 \bar{t}}\, d\bar{t}$$

$$\Rightarrow a_n(t) e^{k\left(\frac{n\pi}{L}\right)^2 t} = a_n(0) + \int_0^t q_n(\bar{t})\, e^{k\left(\frac{n\pi}{L}\right)^2 \bar{t}}\, d\bar{t}$$

$$\Rightarrow a_n(t) = a_n(0)\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} + e^{-k\left(\frac{n\pi}{L}\right)^2 t}\int_0^t q_n(\bar{t})\, e^{k\left(\frac{n\pi}{L}\right)^2 \bar{t}}\, d\bar{t}$$

Now

$$u(x,0) = \sum_{n=1}^{\infty} a_n(0) \sin\frac{n\pi x}{L}$$

$$\Rightarrow g(x) = \sum_{n=1}^{\infty} a_n(0) \sin\frac{n\pi x}{L}$$

$$\Rightarrow a_n(0) = \frac{\int_0^L g(\bar{x}) \sin\frac{n\pi\bar{x}}{L}\, d\bar{x}}{\int_0^L \sin^2\frac{n\pi\bar{x}}{L}\, d\bar{x}} = \frac{2}{L}\int_0^L g(\bar{x}) \sin\frac{n\pi\bar{x}}{L}\, d\bar{x}$$

Put in value of $u(x,t)$

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{\int_0^L g(\bar{x}) \sin\frac{n\pi\bar{x}}{L}\, d\bar{x}}{\int_0^L \sin^2\frac{n\pi x}{L}\, dx}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right] \sin\frac{n\pi x}{L} +$$

$$\sum_{n=1}^{\infty}\left[e^{-k\left(\frac{n\pi}{L}\right)^2 t}\int_0^t q_n(\bar{t})\, e^{k\left(\frac{n\pi}{L}\right)^2 \bar{t}}\, d\bar{t}\right] \sin\frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} \frac{2}{L}\int_0^L g(\bar{x}) \sin\frac{n\pi\bar{x}}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}\, d\bar{x} +$$

$$\sum_{n=1}^{\infty}\left[e^{-k\left(\frac{n\pi}{L}\right)^2 t}\, \frac{2}{L}\int_0^L\int_0^t Q(\bar{x},\bar{t}) \sin\frac{n\pi\bar{x}}{L}\, dx\, e^{-k\left(\frac{n\pi}{L}\right)^2 \bar{t}}\, d\bar{t}\right] \sin\frac{n\pi x}{L}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi \bar{x}}{L} dx\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L}\, d\bar{x}$$

$$+ \sum_{n=1}^{\infty} \frac{2}{L} \int_0^L \int_0^t Q(\bar{x},\bar{t}) \sin \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 (t-\bar{t}_0)}$$

Thus

$$u(x,t) = \int_0^L g(\bar{x}_0)\, G(x,t;\bar{x}_0,0)\, dx_0 + \int_0^L \int_0^t Q(\bar{x},\bar{t})\, G(x,t;\bar{x},\bar{t})\, d\bar{t}\, d\bar{x}$$

where

$$G(x,t;\bar{x},\bar{t}) = \sum_{n=1}^{\infty} \frac{2}{L} \sin \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 (t-t_0)}$$

(b) Modify part (a) if

$$u(0,t) = A(t) \quad \text{and} \quad u(L,t) = B(t)$$

Do not reduce to a problem with homogeneous boundary conditions.

Solution

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

Hence solution of homogeneous eqn is

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L}$$

Put in PDE and get the result

$$\sum_{n=1}^{\infty} \frac{da_n(t)}{dt} \sin \frac{n\pi x}{L} = k \frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

$$\Rightarrow \frac{da_n(t)}{dt} = \frac{\int_0^L \left[k \frac{\partial^2 u}{\partial x^2} + Q(x,t)\right] \sin \frac{n\pi x}{L}\, dx}{\int_0^L \sin^2 \frac{n\pi x}{L}\, dx} \longrightarrow (A)$$

Now take the Green's formula for

$$\int_0^L \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi x}{L}$$

$$\int_0^L \left[u \frac{d^2}{dx^2} \sin\frac{n\pi x}{L} - \sin\frac{n\pi x}{L}\frac{\partial^2 u}{\partial x^2}\right] dx = u\frac{d}{dx}\sin\frac{n\pi x}{L} - \sin\frac{n\pi x}{L}\frac{\partial u}{\partial x}\Big|_0^L$$

$$\Rightarrow -\int_0^L \left(\frac{n\pi}{L}\right)^2 u \sin\frac{n\pi x}{L}\, dx - \int_0^L \sin\frac{n\pi x}{L}\frac{\partial^2 u}{\partial x^2}\, dx$$

$$= \frac{n\pi}{L} u(L) \cos\frac{n\pi L}{L} - \frac{n\pi}{L} u(0) \cos\frac{n\pi 0}{L} - 0$$

$$= \frac{n\pi}{L}\left[B(t)(-1)^n - A(t)\right]$$

$$\Rightarrow \int_0^L \frac{\partial^2 u}{\partial x^2} \cdot \sin\frac{n\pi x}{L}\, dx = -\int_0^L \left(\frac{n\pi}{L}\right)^2 u \sin\frac{n\pi x}{L}\, dx - \frac{n\pi}{L}\left[B(t)(-1)^n - A(t)\right]$$

So ⊛ becomes

$$\frac{d\,a_n(t)}{dt} = \frac{\int_0^L k \frac{\partial^2 u}{\partial x^2} \sin\frac{n\pi x}{L}\, dx}{\int_0^L \sin^2\frac{n\pi x}{L}\, dx} + \frac{\int_0^L Q(x,t) \sin\frac{n\pi x}{L}\, dx}{\int_0^L \sin^2\frac{n\pi x}{L}}$$

$$= \frac{-k\left(\frac{n\pi}{L}\right)^2 \int_0^L u \sin\frac{n\pi x}{L}\, dx}{\int_0^L \sin^2\frac{n\pi x}{L}\, dx} - \frac{\frac{n\pi}{L}\left[B(t)(-1)^n - A(t)\right]}{\int_0^L \sin^2\frac{n\pi x}{L}\, dx} + q_n(t)$$

$$= -k\left(\frac{n\pi}{L}\right)^2 a_n(t) - \frac{2nk\pi}{L^2}\left[B(t)(-1)^n - A(t)\right] + q_n(t)$$

$$\Rightarrow \frac{da_n(t)}{dt} + k\left(\frac{n\pi}{L}\right)^2 a_n(t) = -\frac{2nk\pi}{L^2}\left[B(t)(-1)^n - A(t)\right] + q_n(t)$$

$$I.F = e^{k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\Rightarrow \frac{d}{dt}\left[e^{k\left(\frac{n\pi}{L}\right)^2 t} a_n(t)\right] = \left[\frac{-2n\pi k}{L^2}\left(B(t)(-1)^n - A(t)\right) + q_n(t)\right] e^{k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\Rightarrow a_n(t)\,e^{k\left(\frac{n\pi}{L}\right)^2 t} - a_n(0) = \int_0^t \left[-\frac{2nk\pi}{L}\left(B(t)(-1)^n - A(t)\right) + q_n\right] e^{k\left(\frac{n\pi}{L}\right)^2 t}\,dt$$

$$\Rightarrow a_n(t) = a_n(0)\,e^{-k\left(\frac{n\pi}{L}\right)^2 t} + e^{-k\left(\frac{n\pi}{L}\right)^2 t}\int_0^t \left[-\frac{2nk\pi}{L}\left(B(t)(-1)^n - A(t)\right) + q_n\right] e^{k\left(\frac{n\pi}{L}\right)^2 t}\,dt$$

$$\Rightarrow a_n(t) = \frac{2}{L}\int_0^L g(\bar{x})\sin\frac{n\pi\bar{x}}{L}\,e^{-k\left(\frac{n\pi}{L}\right)^2 t}\,d\bar{x}$$

$$+ e^{-k\left(\frac{n\pi}{L}\right)^2 t}\int_0^t \left[\left[-\frac{2nk\pi}{L}\left(B(\bar{t})(-1)^n - A(\bar{t})\right)\right]d\bar{t} + \frac{2}{L}\int_0^L \left(Q(x,\bar{t})\sin\frac{n\pi x}{L}\,dx\right)\right] e^{+k\left(\frac{n\pi}{L}\right)^2 \bar{t}}\,d\bar{t}$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\frac{n\pi x}{L}$$

Thus

$$u(x,t) = \int_0^L \sum_{n=1}^{\infty} g(\bar{x})\sin\frac{n\pi\bar{x}}{L}\sin\frac{n\pi x}{L}\,e^{-k\left(\frac{n\pi}{L}\right)t}\,d\bar{x}$$

$$+ \int_0^t\int_0^L \sum_{n=1}^{\infty} Q(\bar{x},\bar{t})\sin\frac{n\pi\bar{x}}{L}\sin\frac{n\pi x}{L}\,e^{-k\left(\frac{n\pi}{L}\right)^2(t-\bar{t})}\,d\bar{x}\,d\bar{t}$$

$$- \int_0^t \left[\sum_{n=1}^{\infty}\frac{n\pi k}{L}\left(B(\bar{t})(-1)^n - A(\bar{t})\right)\sin\frac{n\pi x}{L}\,e^{-k\left(\frac{n\pi}{L}\right)^2(t-\bar{t})}\right]d\bar{t}$$

$$\Rightarrow u(x,t) = \int_0^L g(\bar{x})\,G(x,t;\bar{x},0)\,dx + \int_0^t\int_0^L Q(\bar{x},\bar{t})\,G(x,t;\bar{x},\bar{t})\,d\bar{x}\,d\bar{t}$$

$$- \int_0^L \sum_{n=1}^{\infty}\left(\frac{n\pi k}{L}\left(B(t)(-1)^n - A(t)\right)\right)\frac{2}{L}\sin\frac{n\pi x}{L}\,e^{-k\left(\frac{n\pi}{L}\right)^2(t-\bar{t})}\,d\bar{t}$$

where $G(x,t;\bar{x},\bar{t}) = \sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi\bar{x}}{L}\sin\frac{n\pi x}{L}\,e^{-k(t-\bar{t})\left(\frac{n\pi}{L}\right)^2}$

(C) Solve using any method if

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0$$

Solution

The solution of $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

with given boundry conditions is

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos \frac{n\pi}{L} x$$

$$\Rightarrow \frac{d a_n(t)}{dt} = \frac{\int_0^L \left[k \frac{\partial^2 u}{\partial x^2} + Q(x,t)\right] \cos \frac{n\pi x}{L} \, dx}{\int_0^L \cos^2 \frac{n\pi x}{L} \, dx} \longrightarrow (1)$$

using green's formula

$$\int_0^L \left[u \frac{d^2}{dx^2}\left(\cos \frac{n\pi x}{L}\right) - \cos \frac{n\pi x}{L} \frac{\partial^2 u}{\partial x^2}\right] dx = \left[u \frac{d}{dx} \cos\left(\frac{n\pi x}{L}\right) - \cos \frac{n\pi x}{L} \frac{\partial u}{\partial x}\right]_0^L$$

$$\Rightarrow \int_0^L \left[-u \left(\frac{n\pi}{L}\right)^2 \cos \frac{n\pi x}{L} - \cos \frac{n\pi x}{L} \frac{\partial^2 u}{\partial x^2}\right] dx = \frac{-n\pi}{L} u(L) \cancel{\sin n\pi}^{0} - \cos n\pi \cancel{\frac{\partial u}{\partial x}}^{0}(L)$$
$$+ \frac{n\pi}{L} \cancel{\sin(0)}^{0} + \cos(0) \cancel{\frac{\partial u}{\partial x}}^{0}(0)$$

$$\Rightarrow -\left(\frac{n\pi}{L}\right)^2 \int_0^L u \cos \frac{n\pi x}{L} dx - \int_0^L \cos \frac{n\pi x}{L} \frac{\partial^2 u}{\partial x^2} dx = 0$$

$$\Rightarrow \int_0^L \cos \frac{n\pi x}{L} \frac{\partial^2 u}{\partial x^2} = -\left(\frac{n\pi}{L}\right)^2 \int_0^L u \cos \frac{n\pi x}{L} dx$$

Thus $$\frac{d a_n(t)}{dt} = \frac{2}{L} \int_0^L k \frac{\partial^2 u}{\partial x^2} \cos \frac{n\pi x}{L} dx + \frac{2}{L} \int_0^L Q(x,t) \cos \frac{n\pi x}{L} dx$$

$$\Rightarrow \frac{d a_n(t)}{dt} + k \left(\frac{n\pi}{L}\right)^2 \frac{2}{L} \int_0^L u \cos \frac{n\pi x}{L} dx = q_n$$

$$\Rightarrow \frac{d a_n(t)}{dt} + k \left(\frac{n\pi}{L}\right)^2 a_n(t) = q_n$$

$$I.F = e^{k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\Rightarrow \frac{d}{dt}\left[e^{k\left(\frac{n\pi}{L}\right)^2 t} a_n(t)\right] = q_n e^{k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\Rightarrow a_n(t)\, e^{k\left(\frac{n\pi}{L}\right)^2\bar{t}}\Big|_0^t = \int_0^t q_n\, e^{k\left(\frac{n\pi}{L}\right)^2\bar{t}}\, d\bar{t}$$

$$\Rightarrow a_n(t)\, e^{k\left(\frac{n\pi}{L}\right)^2 t} - a_n(0) = \int_0^t q_n(\bar{t})\, e^{k\left(\frac{n\pi}{L}\right)^2\bar{t}}\, d\bar{t}$$

$$\Rightarrow a_n(t) = a_n(0)\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \int_0^t q_n(\bar{t})\, e^{k\left(\frac{n\pi}{L}\right)^2 (t-\bar{t})}\, d\bar{t}$$

$$\Rightarrow a_n(t) = \int_0^L g(\bar{x}) \cos\frac{n\pi\bar{x}}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}\, d\bar{x} + \frac{2}{L}\int_0^t\int_0^L Q(\bar{x},\bar{t}) \cos\frac{n\pi\bar{x}}{L}\, e^{k\left(\frac{n\pi}{L}\right)^2 (t-\bar{t})}\, d\bar{x}\, d\bar{t}$$

And $u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos\frac{n\pi x}{L}$

$$\Rightarrow u(x,t) = \int_0^L g(\bar{x}) \frac{2}{L}\sum_{n=0}^{\infty}\left(\cos\frac{n\pi\bar{x}}{L}\cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right) d\bar{x}$$
$$+ \int_0^t\int_0^L Q(\bar{x},\bar{t}) \frac{2}{L}\sum_{n=0}^{\infty}\cos\frac{n\pi\bar{x}}{L}\cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 (t-\bar{t})}\, d\bar{x}\, d\bar{t}$$

$$\Rightarrow u(x,t) = \int_0^L g(x_0)\, G(x,t;x_0,0)\, dx_0 +$$
$$\int_0^t\int_0^L Q(\bar{x},\bar{t})\, G(x,t;\bar{x},\bar{t})\, d\bar{x}\, d\bar{t}$$

where $G(x,t;\bar{x},\bar{t}) = \frac{2}{L}\cos\frac{n\pi\bar{x}}{L}\cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 (t-\bar{t})}$

(d) Use Green's formula Instead of term-by term differentiation if

$$\frac{\partial u}{\partial t}(0,t) = A(t), \quad \frac{\partial u}{\partial t}(L,t) = B(t)$$

Solution: The solution of PDE $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ will be

$$\frac{da_n(t)}{dt} = \frac{\int_0^L \left[k \frac{\partial^2 u}{\partial x^2} + Q(x,t)\right]}{\int_0^L \cos^2 \frac{n\pi x}{L}\, dx} \cos \frac{n\pi x}{L}\, dx$$

$$\Rightarrow \frac{da_n(t)}{dt} = \frac{2}{L}\int_0^L \left[k \frac{\partial^2 u}{\partial x^2} + Q(x,t)\right] \cos \frac{n\pi x}{L}\, dx \longrightarrow (*)$$

using green's function formula

$$\int_0^L \left(u \frac{d^2}{dx^2} \cos \frac{n\pi}{L}x - \cos \frac{n\pi x}{L} \frac{\partial^2 u}{\partial x^2}\right) dx = u \frac{d}{dx} \cos \frac{n\pi x}{L} - \cos \frac{n\pi x}{L} \frac{du}{dx}\Big|_0^L$$

$$\Rightarrow \int_0^L -u\left(\frac{n\pi}{L}\right)^2 \cos \frac{n\pi x}{L}\, dx - \int_0^L \cos \frac{n\pi x}{L} \frac{\partial^2 u}{\partial x^2} = \left[-u \frac{n\pi}{L} \sin \frac{n\pi x}{L} - \cos \frac{n\pi x}{L} \frac{du}{dx}\right]\Big|_0^L$$

$$= -\frac{n\pi}{L} u(L) \cancel{\sin n\pi}^{\,0} - \cos \frac{n\pi x}{L} \frac{\partial u}{\partial x}(L)$$

$$+ \frac{n\pi}{L} u(0) \cancel{\sin(0)}^{\,0} + \cos(0) \frac{\partial u}{\partial x}(0)$$

$$\Rightarrow -\int_0^L u\left(\frac{n\pi}{L}\right)^2 \cos \frac{n\pi x}{L}\, dx - \int_0^L \cos \frac{n\pi x}{L} \frac{\partial^2 u}{\partial x^2}\, dx = (-1)^{n+1} B(t) + A(t)$$

$$\Rightarrow \int_0^L \cos \frac{n\pi x}{L} \frac{\partial^2 u}{\partial x^2} = -\left(\frac{n\pi}{L}\right)^2 \int_0^L u \cos \frac{n\pi x}{L}\, dx + (-1)^n B(t) + A(t)$$

So (*) will be

$$\frac{da_n(t)}{dt} = \frac{2}{L}\int_0^L k \frac{\partial^2 u}{\partial x^2} \cos \frac{n\pi x}{L}\, dx + \frac{2}{L}\int_0^L Q(x,t) \cos \frac{n\pi x}{L}\, dx$$

$$= -\left(\frac{n\pi}{L}\right)^2 \frac{2k}{L} \int_0^L u \cos \frac{n\pi x}{L}\, dx + \frac{2k}{L}\left((-1)^{n+1} B(t) + A(t)\right) + q_n$$

$$\Rightarrow \frac{da_n(t)}{dt} + k\left(\frac{n\pi}{L}\right)^2 a_n(t) = \frac{2k}{L}\left[(-1)^{n+1}B(t) + A(t)\right] + q_n$$

$$\text{I.F} = e^{+k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\Rightarrow \frac{d}{dt}\left[a_n(t)\, e^{k\left(\frac{n\pi}{L}\right)^2 t}\right] = \int_0^t \left[\frac{2k}{L}\left((-1)^{n+1}B(t) + A(t)\right) + q_n\right] e^{k\left(\frac{n\pi}{L}\right)^2 t} dt$$

$$\Rightarrow a_n(t) e^{k\left(\frac{n\pi}{L}\right)^2 t} - a_n(0) = \int_0^t \frac{2k}{L}\left((-1)^{n+1}B(\bar{t}) + A(\bar{t}) + q_n(\bar{t})\right) e^{k\left(\frac{n\pi}{L}\right)^2 \bar{t}}\, d\bar{t}$$

$$\Rightarrow a_n(t) = a_n(0)e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \int_0^t \frac{2k}{L}\left((-1)^{n+1}B(\bar{t}) + A(\bar{t}) + q_n(\bar{t})\right) e^{-k\left(\frac{n\pi}{L}\right)^2 (t-\bar{t})}\, d\bar{t}$$

As $u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos\frac{n\pi x}{L}$

$$\Rightarrow u(x,t) = \sum_{n=0}^{\infty}\left[\frac{2}{L}\int_0^L g(\bar{x})\cos\frac{n\pi\bar{x}}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\frac{n\pi x}{L}\, d\bar{x}\right]$$

$$+ \sum_{n=0}^{\infty}\left[\frac{2}{L}\int_0^t\int_0^L Q(\bar{x},\bar{t})\cos\frac{n\pi\bar{x}}{L}\cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 (t-\bar{t})}\, d\bar{x}\, d\bar{t}\right.$$

$$+ \int_0^t k\, B(\bar{t})\frac{2}{L}\sum_{n=0}^{\infty}(-1)^{n+1}\cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 (t-\bar{t})}\, d\bar{t}$$

$$+ \int_0^t k\, A(\bar{t})\frac{2}{L}\sum_{n=0}^{\infty}\cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 (t-\bar{t})}\, d\bar{t}$$

$$\Rightarrow u(x,t) = \frac{2}{L}\int_0^L g(\bar{x})\, G(x,t;\bar{x},0) + \int_0^t k\, B(\bar{t})\, G(x,t;0,\bar{t})\, d\bar{t}$$

$$+ \int_0^t k\, A(\bar{t})\, G(x,t;0,\bar{t})\, d\bar{t} + \int_0^t\int_0^L Q(\bar{x},\bar{t})\, G(x,t;\bar{x},\bar{t})\, d\bar{x}\, d\bar{t}$$

Question 9.2.3:- Solve by method of eigenfunction expansion

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

$$u(0,t) = 0 \quad , \quad u(x,0) = f(x)$$

$$u(L,t) = 0 \quad , \quad \frac{\partial u}{\partial t}(x,0) = g(x)$$

Define functions (in the simplest possible way) such that a relationship similar to 9.2.20 exists.

Solution

Related homogeneous problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Hence solution $u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L}$

for homogeneous boundry conditions we have

$$\sum_{n=1}^{\infty} \frac{d^2 a_n(t)}{dt} \sin \frac{n\pi x}{L} = -c^2 \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 a_n(t) \sin \frac{n\pi x}{L} + Q(x,t)$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[\frac{d^2 a_n(t)}{dt} + c^2 a_n(t) \left(\frac{n\pi}{L}\right)^2 \right] \sin \frac{n\pi x}{L} = Q(x,t)$$

$$\Rightarrow \frac{d^2 a_n(t)}{dt} + c^2 a_n(t) \left(\frac{n\pi}{L}\right)^2 = \frac{\int_0^L Q(x,t) \sin \frac{n\pi x}{L} \, dx}{\int_0^L \sin^2 \frac{n\pi x}{L} \, dx}$$

$$\Rightarrow \frac{d^2 a_n(t)}{dt} + c^2 \left(\frac{n\pi}{L}\right)^2 a_n(t) = \frac{2}{L} \int_0^L Q(x,t) \sin \frac{n\pi x}{L} \, dx$$

$$\Rightarrow \frac{d^2 a_n(t)}{dt} + c^2\left(\frac{n\pi}{L}\right)^2 a_n(t) = q_n(t)$$

For Complimentary Function

$$\frac{d^2 a_n(t)}{dt} + c^2\left(\frac{n\pi}{L}\right)^2 a_n(t) = 0$$

$$\Rightarrow D^2 = -c^2\left(\frac{n\pi}{L}\right)^2 \quad \Rightarrow D = \pm i c\left(\frac{n\pi}{L}\right)$$

So

$$a_{n_c}(t) = C_1 \cos c\left(\frac{n\pi}{L}\right)t + C_2 \sin c\left(\frac{n\pi}{L}\right)t$$

$$a_{n_p}(t) = U_1 A_{1np} + U_2 U_{2np}$$

where $A_{1np} = \cos c\left(\frac{n\pi}{L}\right)t$, $A_{2np} = \sin c\left(\frac{n\pi}{L}\right)t$

$$A'_{1np} = -\frac{cn\pi}{L}\sin c\left(\frac{n\pi}{L}\right)t, \quad A'_{2np} = \frac{cn\pi}{L}\cos c\left(\frac{n\pi}{L}\right)t$$

$$W = W(A_{1np}, A_{2np}) = \begin{vmatrix} \cos c\left(\frac{n\pi}{L}\right)t & \sin c\left(\frac{n\pi}{L}\right)t \\ -\frac{cn\pi}{L}\sin c\left(\frac{n\pi}{L}\right)t & \frac{cn\pi}{L}\cos c\left(\frac{n\pi}{L}\right)t \end{vmatrix}$$

$$= \frac{cn\pi}{L}\left(\cos^2 \frac{cn\pi}{L} + \sin^2 \frac{cn\pi}{L}\right)$$

$$\Rightarrow W = \frac{cn\pi}{L}$$

$$W_1 = \begin{vmatrix} 0 & \sin c\left(\frac{n\pi}{L}\right)t \\ q_i(t) & \frac{cn\pi}{L}\cos c\left(\frac{n\pi}{L}\right)t \end{vmatrix}$$

$$\Rightarrow W_1 = -q_i(t)\sin c\left(\frac{n\pi}{L}\right)t$$

& $W_2 = \begin{vmatrix} \cos c\left(\frac{n\pi}{L}\right)t & 0 \\ -\frac{cn\pi}{L}\sin c\left(\frac{n\pi}{L}\right)t & q_i(t) \end{vmatrix}$

$$\Rightarrow W_2 = q_i(t)\cos c\left(\frac{n\pi}{L}\right)t$$

$$u_1' = \frac{W_1}{W} = \frac{-q_i(t)\sin c\left(\frac{n\pi}{L}\right)t}{c\frac{n\pi}{L}}$$

$$\Rightarrow u_1 = \frac{L}{cn\pi}\int_0^t -q_i(\bar{t})\sin c\left(\frac{n\pi}{L}\right)\bar{t}\,d\bar{t}$$

$$u_2' = \frac{W_2}{W} = \frac{q_i(t)\cos c\left(\frac{n\pi}{L}\right)t}{cn\pi/L}$$

$$\Rightarrow u_2 = \frac{L}{cn\pi}\int_0^t q_i(\bar{t})\cos c\left(\frac{n\pi}{L}\right)\bar{t}\,d\bar{t}$$

$$\Rightarrow u_{np}(t) = \frac{-L}{cn\pi}\int_0^t q_i(\bar{t})\sin c\left(\frac{n\pi}{L}\right)\bar{t}\cos c\frac{n\pi}{L}t\,d\bar{t}$$

$$+\frac{L}{cn\pi}\int_0^t q_i(\bar{t})\cos\frac{cn\pi}{L}\bar{t}\sin c\frac{n\pi}{L}t\,d\bar{t}$$

$$\Rightarrow a_{np} = \frac{L}{cn\pi}\int_0^t q_i(\bar{t})\sin c\frac{n\pi}{L}(t-\bar{t})\,d\bar{t}$$

$\because \cos\alpha\sin\beta - \sin\alpha\cos\beta = \sin(\beta-\alpha)$

$$a_n(t) = a_{nc} + a_{np}$$

$$\Rightarrow a_n(t) = C_1\cos\frac{cn\pi}{L}t + C_2\sin c\frac{n\pi}{L}t + \frac{L}{cn\pi}\int_0^t q_i(\bar{t})\sin c\frac{n\pi}{L}(t-\bar{t})\,d\bar{t}$$

$$\Rightarrow \frac{\partial}{\partial t} a_n(t) = -C_1 c\frac{n\pi}{L}\sin c\frac{n\pi}{L}t + C_2\frac{cn\pi}{L}\cos c\frac{n\pi}{L}t + \frac{\partial}{\partial t}\frac{L}{cn\pi}\int_0^t q_i(\bar{t})\sin c\frac{n\pi}{L}(t-\bar{t})\,d\bar{t}$$

$$\frac{\partial}{\partial t} a_n(0) = C_2\frac{cn\pi}{L} \Rightarrow g(x) = C_2\frac{cn\pi}{L}$$

$$\Rightarrow C_2 = \frac{L}{cn\pi} g(x)$$

$$\boxed{a(0) = f(x) \Rightarrow C_1 = f(x)}$$

Thus we left with

$$a_n(t) = f(x)\cos c\frac{n\pi}{L}t + \frac{L}{cn\pi} g(x)\sin c\frac{n\pi}{L}t + \frac{L}{cn\pi}\int_0^t Q(\bar{x},\bar{t})\sin\frac{n\pi x}{L}\,d\bar{x}\,\sin c\frac{n\pi}{L}(t-\bar{t})\,d\bar{t}$$

when

$$C_1 = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$\frac{cn\pi}{L}C_2 = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

$$\Rightarrow C_2 = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

?

Now

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\frac{n\pi}{L}x$$

$$= \sum_{n=1}^{\infty}\left[\left(C_1\cos\frac{n\pi ct}{L} + C_2\sin\frac{cn\pi t}{L}\right) + \frac{L}{n\pi c}\int_0^t q_n(\bar{t})\sin c\frac{n\pi}{L}(t-\bar{t})\,d\bar{t}\right]\sin\frac{n\pi}{L}x$$

$$= \sum_{n=1}^{\infty}\frac{2}{L}\int_0^L f(\bar{x})\sin\frac{n\pi\bar{x}}{L}\,d\bar{x}\,\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} +$$

$$\sum_{n=1}^{\infty} \frac{2}{L}\int_0^L g(\bar{x}) \sin\frac{n\pi\bar{x}}{L} \sin\frac{n\pi x}{L} \frac{\cos\frac{n\pi ct}{L}}{\frac{n\pi c}{L}}\, d\bar{x}$$

$$+\sum_{n=1}^{\infty}\int_0^L\int_0^t Q(\bar{x},\bar{t})\frac{2}{L}\sin\frac{n\pi\bar{x}}{L}\sin\frac{n\pi x}{L}\frac{\sin\frac{n\pi ct}{L}}{\frac{n\pi c}{L}}\, d\bar{x}\, d\bar{t}$$

Introducing green's function

$$G(x,t;\bar{x},\bar{t}) = \sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi\bar{x}}{L}\sin\frac{n\pi x}{L}\, d\bar{x}\, d\bar{t}$$

$$u(x,t) = \int_0^t\int_0^L Q(\bar{x},\bar{t})\, G(x,t;\bar{x},\bar{t})\, d\bar{x}\, d\bar{t}$$

$$+\int_0^t g(\bar{x})\, G(x,t;\bar{x},0)\, d\bar{x} + \int_0^L f(\bar{x})\frac{\partial G}{\partial t}(x,t;\bar{x},0)\, d\bar{x}$$

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MUHAMMAD TAHIR WATTOO ***

M.S. MATHEMATICS

COMSATS UNIVERSITY ISLAMABAD *

FA15-RMT-007

Question 9.3.5:- Consider $\frac{d^2u}{dx^2} = f(x)$, with $u(0) = 0$, $\frac{du}{dx}(L) = 0$

(a) Solve by direct integration.

Solution

By integration

$$\frac{du}{dx} = \int_L^{x} f(\bar{x})\,d\bar{x}$$ using B.C at $\frac{du}{dx}$ i.e $\frac{du}{dx}(L) = 0$

Again integrating

$$u(x) = \int_0^x \Big(\underbrace{\int_L^{x_0} f(\bar{x})\,d\bar{x}}_{I} \Big) \underbrace{dx_0}_{II}$$ using $u(0)=0$

Do integration by parts

Derivative of $I = f(\bar{x})\,d\bar{x}\Big|_L^{x_0} = f(x_0)\,dx_0$

$$\Rightarrow u(x) = \int_L^{x_0} f(\bar{x})\,d\bar{x}\cdot x_0 \Big|_0^x - \int_0^x f(x_0)\,dx_0 \cdot x_0$$

$$= x \cdot \int_L^x f(\bar{x})\,d\bar{x} - \int_0^x x_0 f(x_0)\,dx_0$$

MODELING***

⇒ The Heat (OR Diffusion) Equation:-

Objective:- Build a model that describes the temperature distribution in a metal as function of position and time.

Discussion:- Consider the heat conduction Problem in a rod of length L, made of homogeneous metal with constant cross sectional area A and rod is completely insulated along its lateral edges.

Flux:- Consider a flow of certain physical quantity (such as a mass, energy, heat etc). The flux $q(x,t)$ of this flow is defined as a vector in the direction of flow at (x,t) whose magnitude is given by the amount of quantity crossing a unit area normal to the flow in unit time; i.e

$$|q(x,t)| = \lim_{\Delta S\to 0,\Delta t\to 0} \frac{\text{Quantity Passing through } \Delta S \text{ in } [t,\Delta t]}{\Delta S \Delta t}$$

where ΔS is small surface are at x that is normal to the flow, Δt is time.

Thus the approximate amount of

surface quantity passing through a surface ΔS in time Δt is given by

$$Q(x,t,\Delta S,\Delta t) \approx |q(x,t)|\,\Delta S\,\Delta t$$

Basic Law of Thermodynamics:- A change ΔQ in the amount of heat in a body of mass m is accompanied by a change Δu in its equilibrium temprature. The relationship between these changes is given by

$$\Delta Q = c\,m\,\Delta u$$

Here $c(x)$ = Specific heat of the material at which the body is made. i.e, The amount of heat required to raise the temperature 1°C of a body of unit mass.

Fourier Law of Heat Conduction:- Heat is tranported by diffusion in the direction opposite to the temperature gradiant and at a rate proportional to it. Thus, the heat flux $q(x,t)$ is related to the temperature gradiant by

$$q(x,y) = -k\,\text{grad}\,u(x,t) = -k\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

where $u(x,y,z,t)$ is the temperature at (x,y,z), at time t and k is the thermal conductivity of the material.

Note:- Remember that the gradiant of a function increases more rapidly while in the direction opposite to it the function decreases more rapidly. Thus, a restatement of the fourier law is that " Heat flow in the direction in which the temperature decreases most rapidly". This is the reason of minus sign in the above equation.

Principle of Enegy Conservation:- The total amount of energy in an isolated system remains constant over time.

* Approximation & Idealization:-

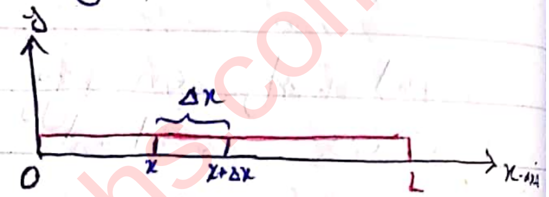
* We assume that rod is homogeneous, It follows that c, k, ρ are independent of the position x. Also we further assume for a prototype model that c, k, ρ are independent of the temperature u.
* the length of the rod remains constant in spite of the changes in its temperature.
* We also assume that the rod in perfectly insulated along its lateral surface (Idealization). Hence, heat can flow only in the horizontal direction, since a vertical flow will lead to heat accumulation along the edges, which is forbidden by the Fourier law of conduction. Therefore we

infer that the temperature on a vertical cross section of the rod must be the same. Thus the temperature u depends only on x and t; that is

$$u = u(x,t)$$

★ We assume that the heat flows in the rod from left to right, which requires the left side to be warmer than the right side.

⇒ Modeling:-



We consider an infinitesimal element of the rod between x and $x+\Delta x$ and write the equation for energy conservation in it

$$\Delta V = \text{Volume of the element} = A\,\Delta x$$

where 'A' is the cross sectional area of the element.

$$\Delta m = \text{mass of the element} = \rho A \Delta x = \rho \Delta V$$

Also the amount of heat at time t is

$$Q(x,t,\Delta x) = c\,\Delta m\, u(x,t)$$

From above two equations, we obtain

$$Q(x,t,\Delta x) = c\rho A\Delta x\, u(x,t)$$

The rate of change in heat is

$$\frac{dQ}{dt} = c\rho A\Delta x \frac{\partial u(x,t)}{\partial t}$$

Heat flowing in $= q(x,t)\,A$

Heat flowing out $= q(x+\Delta x,t)\,A$

By principle of heat

conservation "The rate of change must equal the rate at which heat is flowing in less than the rate at which it is flowing out". Hence

$$\frac{dQ}{dt} = q(x,t)A - q(x+\Delta x,t)A$$
$$= A[q(x,t) - q(x+\Delta x,t)]$$

Substituting the value of $\frac{dQ}{dt} = c\rho A\Delta x \frac{\partial u(x,t)}{\partial t}$ in above equation, we obtain

$$c\rho A\Delta x \frac{\partial u(x,t)}{\partial t} = A[q(x,t) - q(x+\Delta x,t)]$$

$$\Rightarrow c\rho \frac{\partial u(x,t)}{\partial t} = \frac{q(x,t) - q(x+\Delta x,t)}{\Delta x}$$

taking limit $\Delta x \to 0$, we get

$$c\rho \frac{\partial u(x,t)}{\partial t} = \lim_{\Delta x \to 0} \frac{q(x,t) - q(x+\Delta x,t)}{\Delta x}$$

$$= -\frac{\partial q}{\partial x}$$

By Fourier law of heat conduction in one dimension gives

$$q(x,t) = -k\left(\frac{\partial u}{\partial x}\right)$$

From above two equations we have

$$c\rho \frac{\partial u(x,t)}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{K} \frac{\partial u(x,t)}{\partial t} \qquad K = \frac{k}{c\rho}$$

Thus the heat equation in one dimension is

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{K} \frac{\partial u(x,t)}{\partial t}$$

where $K = \frac{k}{c\rho}$ is called the thermal diffusivity.

***Initial Conditions:-** Since one dimension heat equation is first order in t, it needs only one initial condition, which is normally taken to be

$$u(x,0) = f(x) \qquad 0 < x < L$$

This means prescribing the initial distribution of temperature in the rod.

***Boundry Conditions:-** The equation is of 2nd order in with respect to the space variable, so we need two boundry conditions. There are three main types of such conditions prescribed at the end points $x=0$ and $x=L$ which has physical significance.

* The temperature may be at one end point; for example

$$u(0,t) = \alpha(t), \qquad t>0$$

* If the rod is insulated at an end point then the heat flux at that end must be zero. This is equivalent to the derivative $U_x = \partial u/\partial x$ being equal to zero; for example

$$\frac{\partial u(L,t)}{\partial x} = 0, \qquad t>0$$

More generally, if the flow of heat at each end point to be

$\beta(t)$ then the above condition becomes

$$\frac{\partial u(L,t)}{\partial x} = \beta(t), \quad t > 0$$

When one end point is in contact with another medium, we use Newton's law of cooling, which states that "heat flux at the end point is proportional to the difference of the rod and the temperature of the external medium". For example

$$u_x(0,t) = H[u(0,t) - U(t)], \quad t > 0$$

where $U(t)$ is the known temperature of the external medium and $H > 0$ is the heat transfer coefficient. Owing to the convention concerning the direction of heat flux, at the other end point this type of condition becomes

$$u_x(0,t) = -H[u(0,t) - U(t)], \quad t > 0$$

Remark:- * Only one boundry condition is prescribed at each end point.

* the boundry condition at $x = 0$ may differ from that at $x = L$.
* It is easily verified that the heat equation is linear.
* The one dimensional heat equation is the simplest example of a so called parabolic equation.
* If a uniform rod is bent into a ring at the ends at $x = 0$ and $x = L$ are joind. Then appropriate boundry

conditions would be

$$u(0,t) = u(L,t) \quad , t>0$$
$$u_x(0,t) = u_x(L,t) \quad , t>0$$

* Initial Boundry Value Problem:-

Definition:- A partial differential equation associated with the initial and boundry conditions is known as the initial boundry value problem.

If only initial conditions or boundry conditions are present, then we have an initial or boundry value problem respectively.

* Examples:-

Example 1:- The initial boundry value problem modeling heat conduction in a one dimensional uniform rod with sources, insulated lateral surface, and temperature prescribed at both end points is of the form.

$$u_t(x,t) = k\,u_{xx}(x,t) + \underbrace{Q(x,t)}_{\text{source term}}, \quad 0<x<L$$

with boundry conditions

$$u(0,t) = \alpha(t) \quad , \; t>0$$
$$u(L,t) = \beta(t) \quad , \; t>0$$

and initial condition is

$$u(x,0) = f(x) \quad , \; 0<x<L$$

Example 2:- If the near end point is insulated and for one is kept in a medium of constant zero temperature, and if the rod contains no sources, then the corresponding initial value problem is

$$u_t(x,t) = k\,u_{xx}(x,t), \quad 0<x<L$$

with boundry conditions

$$u_x(0,t) = 0 \quad , \; t>0$$

$$u_x(L,t) + h\,u(L,t) = \beta(t) \quad , \; t>0$$

and initial condition is

$$u(x,0) = f(x) \; ; \quad 0<x<L$$

where Q, α, β and f are given functions.

* **Solution:-** By a classical solution of an initial boundry value problem we understand a function $u(x,t)$ that satisfies pointwise the given partial differential equation, boundry conditions and initial conditions every where in the region where the problem is formulated.

Remark:- If the functions α, β and f are sufficiently smooth to ensure that u, u_t, u_x and u_{xx} are continuous in G and upto the boundry of G including the two corner points, then the initial boundry value problem has at most one solution.

* **Assignment:-** Generalize a prototype model for the case when the heat

is generated in the rod at a rate of $r(x,t)$ per unit volume.

⇒ The Wave Equation:-

Objective:- Construct a prototype model for the transverse vibration of a string with fixed ends.

Background:- Generally, the wave phenomena requires the elastic properties of matter and leads to a complicated set of equations. To overcome this difficulty we mak the following simplifying approximations and idealizations so that a prototype model can be constructed by applying Newton's second law of motion $F = ma$, (i.e., The force equals the mass multiplied by the acceleration) to the system under study.

* Approximations & Idealization :-

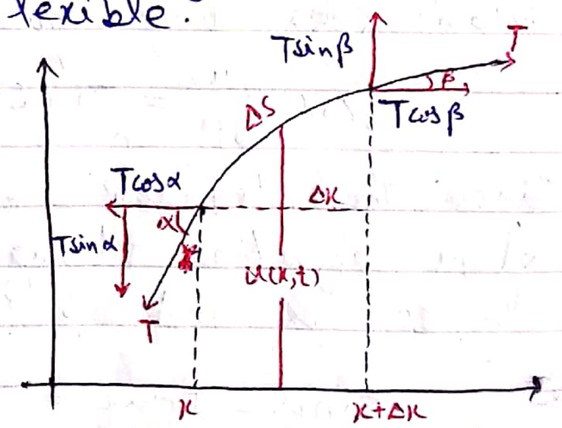
* The string is rigidly attached at its end points.
* The string vibrates in one plane.
* No external force act on the string.
* The string does not suffer from damping forces.
* The string is homogeneous. In particular

this implies that the density ρ and the mass per unit length m of the string are constant.

* The deflection of the string from its equilibrium and its slope are always small. Consequently we are able to make the following two approximations
  (a) The magnitude of the tension force $T(x,t)$ in the string is constant; i.e
  $T(x,t) = T$
  (b) The string is longitudinally i.e., a point on the string moves only in the vertical direction.
* The tension force in the string is always tangential to it. this is usually expressed by saying that the string is assumed to be perfectly flexible.

⇒ Modeling:-

Consider a small segment of the string between x and Δx.



(small segment of the string)

Before we can apply Newton's 2nd law of motion of this segment, we must make the following observations.

1. By approximation 6b the segment is not moving in the horizontal direction.
2. The mass of the segment is $\rho \Delta s$. since we are considering small

deflection, $|u|<<1$. It follows that $\Delta s \approx \Delta x$

3. As $u(x,t)$ is the displacement in the vertical direction therefore acceleration of the segment in vertical direction given by $\partial^2 u/\partial t^2$.

4. The sum of the vertical forces acting on the segment is $T\sin\beta - T\sin\alpha$
$= T(\sin\beta - \sin\alpha)$

Newton's 2nd law of motion is

$$F = ma = m\,\partial^2 u/\partial t^2$$

Since $m = \rho\Delta s \approx \rho\Delta x$, Thus we have

$$F = \rho\Delta x\,\partial^2 u/\partial t^2$$

From observation 4 we know that

$$F = T(\sin\beta - \sin\alpha)$$

From last two equations we get

$$\rho\Delta x\,\frac{\partial^2 u}{\partial t^2} = T(\sin\beta - \sin\alpha)$$

Since the deflection of the string and slope of the string is small and hence α and β are small & so.

$$\sin\alpha \cong \tan\alpha = \frac{\partial u}{\partial x}(x,t)$$

$$\sin\beta \cong \tan\beta = \frac{\partial u}{\partial x}(x+\Delta x,t)$$

This implies

$$\rho\Delta x\,\frac{\partial^2 u}{\partial t^2} = T\left[\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right]$$

Dividing by Δx, we obtain

$$\rho\,\frac{\partial^2 u}{\partial t^2} = \frac{T}{\Delta x}\left[\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right]$$

taking $\Delta x \to 0$, we get

$$\rho \frac{\partial^2 u}{\partial t^2} = T \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x,t)\right]$$

$$= T \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

Thus the wave equation in one dimension is $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ where $c^2 = \frac{T}{\rho}$

Remark:- From figure we note that the sum of the horizontal forces acting on the string segment is $T(\cos\beta - \cos\alpha) \neq 0$. Hence the segment must have an acceleration in the horizontal direction, which contradicts approximation 6b. However, since α, β are small, we can take $T(\cos\beta - \cos\alpha)$ is negligible.

*Examples:-

Example 1:- Derive a model equation for the vibration of the string if a vertical external force $F(x,t)$ per unit length is acting on it.

Approximations & Idealizations:-

* The string is rigidly attached at its end points.

- ★ The string vibrates in one plane.
- ★ No external force act on the string.
- ★ The string does not suffer from damping forces.
- ★ The string is homogeneous. In Particular this implies that the density ρ and the mass per unit length m of the string are constant.
- ★ The deflection of the string from its equilibrium and its slope are always small. Consequently we are able to make the following two approximations.
  - (a) The magnitude of the tension force $T(x,t)$ in the string is constant, $T(x,t) = T$
  - (b) The string is rigid longitudinally i.e a point on the string moves only in the vertical direction
- ★ The tension force in the string is always tangential to it. This is usually expressed by saying that the string is assumed to be perfectly flexible.

⇒ **Modeling:-** By 2nd Law of motion

$$F = ma$$

$$\approx \rho \Delta x \frac{\partial^2 u(x,t)}{\partial t^2} \longrightarrow ①$$

Since,

$$\text{Vertical force} = T(\sin\beta - \sin\alpha) + F(x,t)\Delta x \longrightarrow ②$$

for very small angle α, β we have the following relations

$$\sin\beta \approx \tan\beta = \frac{\partial u}{\partial x}(x+\Delta x, t) \longrightarrow ③$$

$$\sin\alpha \approx \tan\alpha = \frac{\partial u}{\partial x}(x, t) \longrightarrow ④$$

Substituting the values of ③ and ④ in equ ② yields

Vertical force = Total force acting on the segment

$$= T\left[\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x,t)\right] + F(x,t)\Delta x \longrightarrow ⑤$$

From equ ① and ⑤ we obtain

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2}(x,t) = T\left[\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x,t)\right] + F(x,t)\Delta x$$

Dividing by Δx we get

$$\rho \frac{\partial^2 u}{\partial t^2}(x,t) = \frac{T}{\Delta x}\left[\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x,t)\right] + F(x,t)$$

Taking lim $\Delta x \to 0$ in above equation, we have

$$\rho \frac{\partial^2 u}{\partial t^2}(x,t) = T \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x,t)\right] + F(x,t)$$

$$\Rightarrow \rho \frac{\partial^2 u}{\partial t^2}(x,t) = T \frac{\partial^2 u}{\partial x^2} + F(x,t)$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} + \frac{1}{\rho} F(x,t)$$

Finally we have the following wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{\rho} F(x,t)$$

where $c^2 = T/\rho$ is the wave speed.

Example 2:- Derive a model equation for very small vibrations of a vertically suspended chain, whose length is L and whose mass density per unit length ρ is constant.

* Approximations & Idealizations:-

1:- Since the amplitude u of the vibration is small, we assume that a point on the chain does not change its x co-ordinate (See Fig)

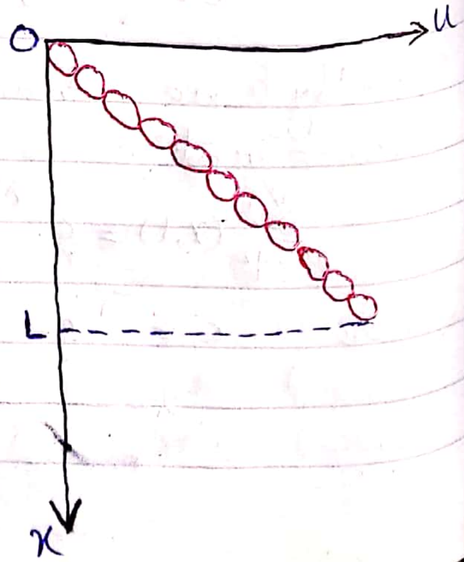
2:- The tension $T(x,t)$ in the chain can not be assumed to be constant in the present situation. In fact, in the equilibrium (vertical) position of the chain: $T(x) = \rho g(L-x) \longrightarrow (a)$

The above equation gives us an acceptable approximation for the tension in the vibrating chain when $|u| \ll 1$ and $\partial u/\partial t \ll 1$

3:- Other approximations and idealization of the prototype model remain intact.

⇒ Modeling:-

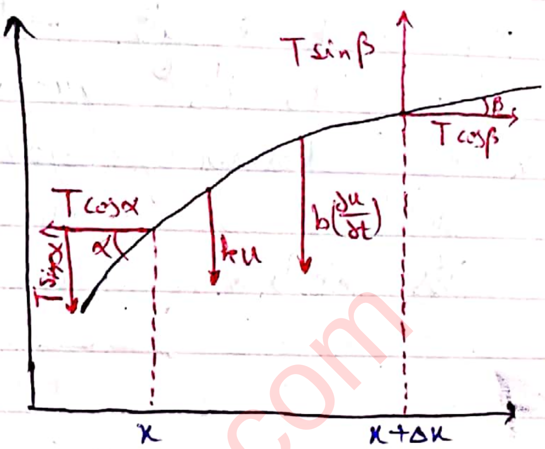
For the construction of mathematical model we once again consider a small section of chain between $[x, x+\Delta x]$



Applying Newton 2nd law of motion in the horizontal direction to such a section we have

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = T(x+\Delta x)\sin\beta - T(x)\sin\alpha \rightarrow (7)$$

For small α and β we have the relation describes in equation (3), (4) using them in eqn (7) we arrive at



$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = T(x+\Delta x)\frac{\partial u}{\partial x}(x+\Delta x,t) - T(x)\frac{\partial u}{\partial x}(x,t)$$

or

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = \frac{1}{\Delta x}\left[T(x+\Delta x)\frac{\partial u}{\partial x}(x+\Delta x,t) - T(x)\frac{\partial u}{\partial x}(x,t)\right]$$

Taking limit $\Delta x \to 0$ we have

$$\rho \frac{\partial^2 u}{\partial t^2} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[T(x+\Delta x)\frac{\partial u}{\partial x}(x+\Delta x,t) - T(x)\frac{\partial u}{\partial x}(x,t)\right]$$

$$= \frac{\partial}{\partial x}\left[T(x)\frac{\partial u}{\partial x}\right]$$

substituting equation (6) in the above equation we obtain

$$\frac{\partial^2 u}{\partial t^2} = g\frac{\partial}{\partial x}\left[T(x)\frac{\partial u}{\partial x}\right]$$

———————————

Assignment:- Derive a model equation for the vibration of the string when its motion is subject to an elastic restraint and a damping force.

Note:- ★ The restraint force can be considered as a force of tru per unit length acting to return the string to its equilibrium position.

★ The damping force is given by $b(\partial u/\partial t)$ per unit length and to oppose its motion.

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Non-Homogeneous Problems

Example $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

B.Cs:- $u(0,t) = A$, $u(L,t) = B$

I.C:- $u(x,0) = f(x)$

Solution

To analyze this problem we first obtain an equilibrium temperature distribution $u_E(x)$. If such temperature distribution exist, it must satisfy the steady state time independent heat equation.

$$\frac{d^2u_E}{dx^2} = 0 \Rightarrow \frac{du_E(x)}{dx} = A_1 \quad \begin{cases} u_E(0) = A \\ u_E(L) = B \end{cases}$$

$$\Rightarrow u_E(x) = A_1 x + B_1$$

$$u_E(0) = A \Rightarrow 0 + B_1 = A \Rightarrow \boxed{B_1 = A}$$

$$u_E(L) = B \Rightarrow A_1 L + A = B$$

$$\Rightarrow A_1 = \frac{B-A}{L}$$

So $u_E(x) = A + \left(\frac{B-A}{L}\right)x$

Let us consider another function

$$V(x,t) = u(x,t) - u_E(x)$$

$$\Rightarrow u(x,t) = V(x,t) + u_E(x)$$

$$\frac{\partial u}{\partial t} = \frac{\partial V}{\partial t} \quad ; \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 V}{\partial x^2} + 0 = \frac{\partial^2 V}{\partial x^2}$$

So original equation becomes

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2}$$

with $V(0,t) = 0 = V(L,t)$

& $V(x,0) = f(x) - u_E(x)$

Now By seperation of variable we get

$$V(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

By Initial condition

$$V(x,0) = f(x) - u_E(x)$$

$$\Rightarrow f(x) - u_E(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}$$

& By orthogonality condition

$$a_n = \frac{2}{L}\int_0^L [f(x) - u_E(x)] \sin\frac{n\pi x}{L} dx$$

& at the end

$$u(x,t) = V(x,t) + u_E(x)$$

$$\Rightarrow u(x,t) = u_E(x) + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

As $t \to \infty$ $u(x,t) \to u_E(x)$

Example:- Steady Non-Homogeneous Terms

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x)$$

$$u(0,t) = A \quad , \quad u(L,t) = B$$

$$u(x,0) = f(x)$$

If an equilibrium solution exists, then we determine it and again consider the displacement from equilibrium

$$V(x,t) = u(x,t) - u_E(x)$$

We can show that $V(x,t)$ satisfies a

linear homogeneous partial differential equation with linear homogenous boundry conditions. Thus again $u(x,t) \rightarrow u_E(x)$ as $t \rightarrow \infty$

Example:- Time Dependent Non-Homo Terms

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \longrightarrow (1)$$

B.Cs:- $\quad u(0,t) = A(t)\ , \quad u(L,t) = B(t)$

I.C $\quad u(x,0) = f(x)$

Solution Consider any reference temprature distribution $R(x,t)$ with the property that it satisfy the given non-homogen boundry conditions

$$R(0,t) = A(t) \quad ; \quad R(L,t) = B(t)$$

We say that solution is of the form

$$R(x,t) = A(t) + \frac{x}{L}[B(t) - A(t)]$$

Introduce $V(x,t) = u(x,t) - R(x,t)$

$$\Rightarrow \frac{\partial V}{\partial t} = \frac{\partial u}{\partial t} - \frac{\partial R}{\partial t}$$

$$\& \quad \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 R}{\partial x^2}$$

So equ (1) $\Rightarrow$ $$\frac{\partial V}{\partial t} + \frac{\partial R}{\partial t} = k\frac{\partial^2 V}{\partial x^2} + k\frac{\partial^2 R}{\partial x^2} + Q(x,t)$$

$$\Rightarrow \frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} + \left[Q(x,t) + k\frac{\partial^2 R}{\partial x^2} - \frac{\partial R}{\partial t}\right]$$

$$\Rightarrow \frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} + \overline{Q}(x,t)$$

with

$v(0,t)=0 \qquad , v(L,t)=0$

$$v(x,0) = f(x) - R(x,0) \equiv g(x)$$

Exercise 8.2

Question 8.2.1:- Solve the heat equation with time independent sources and Boundry conditions.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x)$$

$$u(x,0) = f(x)$$

If an equilibrium solution exists. Analyze the limit as $t \to \infty$. If no equilibrium exists, explain why and reduce the problem to one with homogeneous boundry conditions (But do not solve). Assume

(a) $Q(x) = 0 \quad , u(0,t) = A \quad , \frac{\partial u}{\partial x}(L,t) = B$

Solution An equilibrium satisfy

$$\frac{d^2 u_E}{dx^2} = 0 \qquad \text{with } u_E(0) = A \quad \& \quad \frac{du_E}{dx}(L) = B$$

$$\Rightarrow u_E(x) = C_1 x + C_2$$

$$u_E(0) = A \quad \Rightarrow \quad \boxed{C_2 = A}$$

$$\frac{du_E}{dx} = B \quad \Rightarrow \quad \boxed{C_1 = B}$$

So $\quad u_E(x) = A + Bx$

Introduce the displacement from equilibr

$$v(x,t) = u(x,t) - u_E(x)$$

Satisfying $\frac{\partial V}{\partial t} = k \frac{\partial^2 V}{\partial x^2}$ with

$$V(0,t) = 0 \quad , \quad \frac{\partial V}{\partial x}(L,t) = 0$$

$$V(x,0) = f(x) - U_E(x)$$

Now

By separation of variable

$$V(x,t) = \phi(x)\, h(t)$$

$$\Rightarrow \phi\, h'(t) = k h\, \phi''(x)$$

$$\Rightarrow \frac{1}{kh}\frac{dh}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$

$$\Rightarrow \frac{dh}{dt} = -\lambda k h \quad \Rightarrow h(t) = c\, e^{-k\lambda t}$$

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad ; \phi(0) = 0 , \quad \phi'(L) = 0$$

$$\Rightarrow \phi(x) = A\cos\sqrt{\lambda}\, x + B\sin\sqrt{\lambda}\, x$$

$$\phi(0) = 0 \quad \Rightarrow \boxed{A = 0}$$

$$\Rightarrow \phi(x) = B\sin\sqrt{\lambda}\, x \quad \Rightarrow \phi'(x) = B\sqrt{\lambda}\cos\sqrt{\lambda}\, x$$

$$\phi'(L) = 0 \Rightarrow \cos\sqrt{\lambda}\, L = 0$$

$$\Rightarrow \sqrt{\lambda}\, L = (n + \tfrac{1}{2})\pi \quad \Rightarrow \lambda = \left[\frac{(n+\frac{1}{2})\pi}{L}\right]^2$$

$$\Rightarrow \phi(x) = \sin\frac{(2n+1)\pi}{2L}x$$

So

$$V(x,t) = \sum_{n=0}^{\infty} a_n \sin\frac{(2n+1)\pi}{2L}x\; e^{-\lambda k t}$$

Now By I.C

$$V(x,0) = f(x) - U_E(x) = \sum_{n=0}^{\infty} a_n \sin\frac{(2n+1)\pi}{2L}x$$

$$\Rightarrow g(x) = \sum_{n=0}^{\infty} a_n \sin\frac{(2n+1)\pi}{2L}x$$

$$\Rightarrow a_n = \frac{2}{L}\int_0^L g(x)\sin\frac{(2n+1)\pi}{2L}x\,dx$$

So $\quad u(x,t) = v(x,t) + u_E(x)$

As $t \to \infty \quad u(x,t) \to u_E(x)$

(b) $Q(x) = 0 \quad , \frac{\partial u}{\partial x}(0,t) = 0 \quad , \frac{\partial u}{\partial x}(L,t) = B \neq 0$

Solution

An Equilibrium satisfies

$$\frac{d^2u_E(x)}{dx^2} = 0 \quad \text{with} \quad \frac{du_E}{dx}(0) = 0 \ \& \ \frac{du_E}{dx}(L) = B$$

$$u_E(x) = C_1 x + C_2$$

$$\frac{du_E}{dx} = C_1 \quad ; \quad \frac{du_E}{dx}(0) = 0 \quad \Rightarrow \boxed{C_1 = 0}$$

$$\frac{du_E}{dx}(L) = B \quad \Rightarrow \boxed{C_1 = B}$$

$$\Rightarrow u_E(x) = C_2$$

Introduce displacement from equilibrium

$$v(x,t) = u(x,t) - u_E(x)$$

satisfying $\quad \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} \quad$ with

$$\frac{\partial v}{\partial x}(0,t) = 0 = \frac{\partial v}{\partial x}(L,t)$$

(i) $\quad \frac{dh}{dt} = -k\lambda h \quad \Rightarrow h(t) = e^{-k\lambda t}$

(ii) $\quad \frac{d^2\phi}{dx^2} = -\lambda\phi \quad$ with $\phi'(0) = 0 \ , \ \phi'(L) = 0$

$$\Rightarrow \phi(x) = \cos\frac{n\pi}{L}x, \quad n = 0, 1, 2, 3, \ldots$$

where $\quad \lambda = \left(\frac{n\pi}{L}\right)^2$

So $V(x,t) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi}{L}x \, e^{-\lambda kt}$

I.C $\Rightarrow$ $V(x,0) = f(x) - U_E(x) = g(x)$ (say)

$$\Rightarrow g(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi}{L}x$$

$$\Rightarrow a_n = \frac{2}{L}\int_0^L g(x)\cos\frac{n\pi x}{L}\,dx$$

And
$$u(x,t) = U_E(x) + \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\frac{n\pi}{L}x$$

(C) $Q(x) = 0$, $\frac{\partial u(0,t)}{\partial x} = A \neq 0$, $\frac{\partial u}{\partial x}(L,t) = A$

Solution

An equilibrium exist

$$\frac{d^2 U_E(x)}{dx} = 0; \quad \frac{dU_E(0)}{dx} = A, \quad \frac{dU_E(L)}{dx} = A$$

$$U_E(x) = C_1 x + C_2$$

$$\frac{dU_E}{dx} = C_1; \quad \frac{dU_E(0)}{dx} = A \Rightarrow C_1 = A$$

$$\frac{dU_E(L)}{dx} = A \Rightarrow C_1 = A$$

$$\Rightarrow U_E(x) = Ax + C_2$$

Introduce
$$V(x,t) = u(x,t) - U_E(x)$$

$$\Rightarrow u(x,t) = V(x,t) + U_E(x)$$

$$\frac{\partial u}{\partial t} = \frac{\partial V}{\partial t} \quad \& \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 V}{\partial x^2}$$

Original equation becomes

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} \quad \text{with}$$

$$\frac{\partial V}{\partial x}(0,t) = 0 = \frac{\partial V}{\partial x}(L,t)$$

$$v(x,t) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi}{L}x\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$a_n = \frac{2}{L}\int_0^L g(x)\cos\frac{n\pi}{L}x\,dx$$

and
$$u(x,t) = Ax + C_2 + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi}{L}x\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

where C_2 is still unknown and can be determined by conservation of energy

$$\int_0^L f(x)\,dx = \frac{AL^2}{2} + C_2 L + 0$$

$$\Rightarrow C_2 = -\frac{AL^2}{2} + \frac{1}{L}\int_0^L f(x)\,dx$$

(d) $Q(x) = k$; $u(0,t) = A$, $u(L,t) = B$

Solution

Heat Equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + k$$

For steady state we have $u_E(x)$

$$k\frac{d^2 u_E(x)}{dx^2} + k = 0 ; \quad u_E(0) = A, \quad u_E(L) = B$$

$$\Rightarrow \frac{d^2 u_E(x)}{dx^2} = -1$$

$$\frac{du_E}{dx} = -x + C_1 \Rightarrow u_E(x) = \frac{-x^2}{2} + C_1 x + C_2$$

$$u_E(0) = A \Rightarrow \boxed{C_2 = A}$$

$$\Rightarrow u_E(x) = \frac{-x^2}{2} + C_1 x + A$$

$$u_E(L) = B \Rightarrow -\frac{L^2}{2} + C_1 L + A = B$$

$$\Rightarrow C_1 L = B - A + \frac{L^2}{2} \Rightarrow C_1 = \frac{B-A}{L} + \frac{L}{2}$$

$$\Rightarrow U_E(x) = \frac{-x^2}{2} + \left(\frac{B-A}{L} + \frac{L}{2}\right)x + A$$

Now consider $V(x,t) = U(x,t) - U_E(x)$

$V(0,t) = 0$, $V(L,t) = 0$

$$\Rightarrow \frac{\partial}{\partial t}\left(V + U_E(x)\right) = k\frac{\partial^2}{\partial x^2}\left(V + U_E(x)\right) + k$$

$$\Rightarrow \frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} + k(-1) + k$$

$$\Rightarrow \frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} \;;\; V(0,t) = 0 = V(L,t)$$

I.C $\quad V(x,0) = f(x) - u_E(x) = g(x)$

So

$$V(x,t) = \phi(x)\,h(t)$$

$h(t) = c\,e^{-\lambda k t}$, $\phi(x) = B\sin\frac{n\pi}{L}x$

$$\Rightarrow V(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x\; e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Now

$$V(x,0) = g(x) \Rightarrow g(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x$$

$$\Rightarrow a_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

So

$$U(x,t) = U_E(x) + V(x,t)$$

$$\Rightarrow U(x,t) = -\frac{x^2}{L} + \left(\frac{B-A}{L} + \frac{L}{2}\right)x + A + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\lim_{t\to\infty} U(x,t) = \frac{-x^2}{L} + \left(\frac{B-A}{L} + \frac{L}{2}\right)x + A$$

(e) $Q(x) = k$;

$$\frac{\partial u}{\partial x}(0,t) = 0 \;,\; \frac{\partial u}{\partial x}(L,t) = 0$$

Solution Heat equation is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + k$$

Steady state $k\frac{d^2 u_E}{dx^2} + k = 0$

$$\Rightarrow \frac{d^2 u_E}{dx^2} = -1 \quad \Rightarrow \frac{du_E}{dx} = -x + A_1$$

$$\Rightarrow u_E(x) = \frac{-x^2}{2} + A_1 x + B_1$$

$$\frac{du_E}{dx}(0) = 0 \quad \Rightarrow \boxed{A_1 = 0}$$

$$\frac{du_E}{dx}(L) = 0 \quad \Rightarrow \quad -L = 0$$

So $\quad u_E(x) = \frac{-x^2}{2} + B_1$

Let $\quad v(x,t) = u(x,t) - u_E(x)$

$$\frac{\partial}{\partial t}(v + u_E) = k\frac{\partial^2 u}{\partial x^2}(v + u_E) + k$$

$$\Rightarrow \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} - k + k$$

$$\Rightarrow \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} \quad \text{with}$$

$$\frac{\partial v}{\partial x}(0,t) = 0 \quad , \quad \frac{\partial v}{\partial x}(L,t) = 0$$

Question 8.2.2.- Consider the heat equation with time-dependent sources & boundary conditions

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

$$u(x,0) = f(x)$$

Reduce the problem with homogeneous boundry conditions if

(a) $\frac{\partial u}{\partial x}(0,t) = A(t)$ and $\frac{\partial u}{\partial x}(L,t) = B(t)$

Solution We introduce the refrence temperature $r(x,t)$ such that

$\frac{\partial r}{\partial x}(0,t) = A(t)$, $\frac{\partial r}{\partial x}(L,t) = B(t)$

$\frac{d^2u}{dx^2} = 0 \Rightarrow u = Ax + B$

$\frac{\partial u}{\partial x} = A$

$\frac{\partial u}{\partial x}(0) = A(t) \Rightarrow A = A(t)$

Let $V(x,t) = u(x,t) - r(x,t)$

with $\frac{\partial V}{\partial x}(0,t) = 0$, $\frac{\partial V}{\partial x}(L,t) = 0$

We consider $r(x,t) = A(t)\,x + \frac{x^2}{2L}[B(t) - A(t)]$

which satisfies

$\frac{\partial r}{\partial x}(0,t) = A(t)$, $\frac{\partial r}{\partial x}(L,t) = B(t)$

Now original equation becomes

$$\frac{\partial}{\partial t}(V+r) = k\frac{\partial^2}{\partial x^2}(V+r) + Q(x,t)$$

$$\Rightarrow \frac{\partial V}{\partial t} + \frac{\partial r}{\partial t} = k\frac{\partial^2 V}{\partial x^2} + k\frac{\partial^2 r}{\partial x^2} + Q(x,t)$$

$$\Rightarrow \frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} + \left[Q(x,t) - \frac{\partial r}{\partial t} + k\frac{\partial^2 r}{\partial x^2}\right]$$

$$\Rightarrow \frac{\partial v}{\partial t} = k\frac{\partial^2 V}{\partial x^2} + \bar{Q}(x,t) \quad \text{with } \frac{\partial V}{\partial x}(0,t) = 0 = \frac{\partial V}{\partial x}(L,t)$$

4. Initial condition becomes

$$V(x,0) = u(x,0) - r(x,0)$$

$$= f(x) - A(0)x + \frac{x^2}{2L}[B(0) - A(0)] = g(x)$$

(b) $u(0,t) = A(t)$ and $\frac{\partial u}{\partial x}(L,t) = B(t)$

Solution Introduce the refrence temperature $r(x,t)$

$r(x,t) = A(t) + x B(t)$

satisfying $r(0,t) = A(t)$

$\frac{\partial r}{\partial x}(L,t) = B(t)$

$\frac{d^2u}{dx^2} = 0 \Rightarrow u = Ax + B$

$u(0) = A(t) \Rightarrow \boxed{B = A(t)}$

$u = Ax + A(t)$

$\frac{\partial u}{\partial x} = A$

$\frac{\partial u}{\partial x}(L) = B(t) \Rightarrow \boxed{A = B(t)}$

Let $v(x,t) = u(x,t) - r(x,t)$

Then $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$

$\Rightarrow \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \left[Q - \frac{\partial r}{\partial t} + k\frac{\partial^2 r}{\partial x^2}\right]$

$\Rightarrow \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \bar{Q}(x,t)$

with $v(0,t) = 0$, $\frac{\partial v}{\partial x}(L,t) = 0$

I.C $v(x,0) = u(x,0) - r(x,0)$

$= f(x) - A(0) - xB(0) = g(x)$

c) $\frac{\partial u}{\partial x}(0,t) = A(t)$ and $u(L,t) = B(t)$

Solution Introduce the refrence temperature

$r(x,t) = A(t)x + B(t) - LA(t)$

$= A(t)[x - L] + B(t)$

satisfying $\frac{\partial r}{\partial x}(0,t) = A(t)$

$r(L,t) = B(t)$

$\frac{d^2u}{dx^2} = 0 \Rightarrow u = Ax + B$

$\frac{\partial u}{\partial x} = A$, $\frac{\partial u}{\partial x}(0) = A(t) \Rightarrow \boxed{A = A(t)}$

$\Rightarrow u = A(t)x + B$

$u(L) = B(t)$

$\Rightarrow A(t)L + B = B(t)$

$\Rightarrow \boxed{B = B(t) - LA(t)}$

Let $V(x,t) = u(x,t) - r(x,t)$

equ ① $\Rightarrow \dfrac{\partial V}{\partial t} = k\dfrac{\partial^2 V}{\partial x^2} + \bar{Q}(x,t)$

B.Cs:- $\dfrac{\partial V}{\partial x}(0,t) = 0 \quad , \quad V(L,t) = 0$

I.C:- $V(x,0) = u(x,0) - r(x,0)$

$= f(x) - [A(0)[x-L] - B(0)] = g(x)$

(d) $u(0,t) = 0 \quad , \quad \dfrac{\partial u}{\partial x}(L,t) + h[u(L,t) - B(t)] = 0$

Solution Introduce the refrence temperature

$r(x,t) = \dfrac{hB(t)}{1+hL}\, x$

satisfying $r(0,t) = 0$

$\dfrac{\partial r}{\partial x}(L,t) + h\, r(L,t) = h\, B(t)$

$\dfrac{d^2u}{dx^2} = 0 \Rightarrow u = Ax + B$

$u(0) = 0 \Rightarrow \boxed{B = 0}$

$u = Ax \ , \ \dfrac{\partial u}{\partial x} = A$

$\dfrac{\partial u}{\partial x}(L) + h\, u(L) = h\, B(t)$

$\Rightarrow A + hAL = h\, B(t)$

$\Rightarrow A(1 + hL) = h\, B(t)$

$\Rightarrow \boxed{A = \dfrac{hB(t)}{1+hL}}$

And

$V(x,t) = u(x,t) - r(x,t)$

$\Rightarrow \dfrac{\partial V}{\partial t} = k\dfrac{\partial^2 V}{\partial x^2} + \bar{Q}(x,t)$ with

$V(0,t) = 0 \quad , \quad \dfrac{\partial V}{\partial x}(L,t) + h\, V(L,t) = 0$

& $V(x,0) = g(x)$

(e) $\dfrac{\partial u}{\partial x}(0,t) = 0 \quad , \quad \dfrac{\partial u}{\partial x}(L,t) + h[u(L,t) - B(t)] = 0$

Solution Consider the refrence temperature distribution

$\dfrac{d^2u}{dx^2} = 0 \Rightarrow u = Ax + B$

$\dfrac{\partial u}{\partial x} = A$

$\dfrac{\partial u}{\partial x}(0) = 0 \Rightarrow \boxed{A = 0}$

$$r(x,t) = \frac{h B(t)}{(2+L)L} x^2$$

satisfying $\frac{\partial r}{\partial x}(0,t) = 0$

$\frac{\partial r}{\partial x}(L,t) + h r(L,t) = h B(t)$

$\Rightarrow u = B$

$\frac{\partial u}{\partial x}(L) + h u(L) = h B(t)$

$\Rightarrow 0 + hB = hB(t)$

$\Rightarrow B = B(t)$

$\Rightarrow u(x) = B(t)$

Consider

$$V(x,t) = u(x,t) - r(x,t)$$

Equ ① $\Rightarrow$ $$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} + \bar{Q}(x,t)$$

B.Cs $\Rightarrow$ $$\frac{\partial V}{\partial x}(0,t) = 0 \quad , \quad \frac{\partial V}{\partial x}(L,t) + h V(L,t) = 0$$

I.C $\Rightarrow$ $$V(x,0) = f(x) - \frac{h B(0) x^2}{(2+L)L} = g(x)$$

Question 8.2.3:- Solve the two dimensional heat equation with circulary symmetric time independent sources, boundry conditions, initial condition (inside a circle)

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + Q(r)$$

with $u(r,0) = f(r)$ & $u(a,t) = T$

Solution Let equilibrium temperature distribution

$$u_E(r) = T \qquad \text{with} \qquad u_E(a) = T$$

Let $V(r,t) = u(r,t) - u_E(r)$

① $\Rightarrow$ $$\frac{\partial}{\partial t}(V + u_E) = \frac{k}{r}\left[r\frac{\partial}{\partial r}(V+u_E)\right] + Q(r)$$

$$\Rightarrow \frac{\partial V}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right]$$

with $V(a,t) = 0$

Solve this by

$$V(r,t) = \phi(r)\, h(t)$$

$$\Rightarrow \phi \frac{dh}{dt} = \frac{k}{r}\frac{\partial}{\partial r}\left[r\, h \frac{d\phi}{dr}\right]$$

$$\Rightarrow \frac{1}{kh}\frac{dh}{dt} = \frac{1}{\phi}\frac{1}{r}\frac{d}{dr}\left[r\frac{d\phi}{dr}\right] = -\lambda$$

i) $\frac{dh}{dt} = -\lambda k h \qquad \Rightarrow h(t) = e^{-\lambda k t}$

ii) $\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda\phi$

$$\Rightarrow \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \lambda\phi = 0$$

$$\Rightarrow r^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + \lambda r^2\phi = 0 \longrightarrow ①$$

with $\phi(a) = 0$

① is a Bessel differential equation.

Question 8.2.4:- Solve the two dimensional heat equation with time independent boundry conditions.

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Subject to boundry conditions

$u(0,y,t) = 0 \quad , \quad \frac{\partial}{\partial y}u(x,0,t) = 0$

$u(L,y,t) = 0 \quad , \quad u(x,H,t) = g(x)$

and the initial condition

$$u(x,y,0) = f(x,y)$$

Analyze the limit as $t \to \infty$

Solution

For steady state

$$\frac{\partial u}{\partial t} = 0 \quad \Rightarrow$$

$$\frac{\partial^2 U_E}{\partial x^2} + \frac{\partial^2 U_E}{\partial y^2} = 0 \quad \text{with} \longrightarrow ⊛$$

$U_E(0,y) = 0$, $\frac{\partial U_E}{\partial y}(x,0) = 0$

$U_E(L,y) = 0$, $\underline{U_E(x,H) = g(x)}$,
I.C for $U_E(x,y)$

Let
$U_E(x,y) = X(x)Y(y)$

⊛ $\Rightarrow$ $YX'' + XY'' = 0$

$\Rightarrow \frac{X''}{X} = \frac{-Y''}{Y} = -\lambda$

i) $X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$

$\underline{\lambda > 0} \Rightarrow X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$X(0) = 0 \Rightarrow A = 0$

$X(L) = 0 \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$

$\Rightarrow X(x) = B\sin\left(\frac{n\pi}{L}x\right)$

ii) $Y'' - \lambda Y = 0$, $Y'(0) = 0$,

$\underline{\lambda > 0} \Rightarrow Y(y) = A\cosh\sqrt{\lambda}y + B\sinh\sqrt{\lambda}y$

$Y'(y) = \sqrt{\lambda}A\sinh\sqrt{\lambda}y + \sqrt{\lambda}B\cosh\sqrt{\lambda}y$

$Y'(0) = 0 \Rightarrow B = 0$

$\Rightarrow Y(y) = \cosh\frac{n\pi}{L}y$

So $U_E(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x \cos\frac{n\pi}{L}y$

$U_E(x,H) = g(x)$

$\Rightarrow g(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x \cosh\frac{n\pi H}{y}$

$$\Rightarrow a_n = \frac{1}{\cosh \frac{n\pi H}{y}} \left[\frac{2}{L}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx\right]$$

Let $V(x,y,t) = u(x,y,t) - u_E(x,y)$.

$\Rightarrow u(x,y,t) = V(x,y,t) + u_E(x,y)$

$$① \rightarrow \quad \frac{\delta V}{\delta t} = k\left(\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V}{\delta y^2}\right)$$

with $V(0,y,t) = 0$, $\frac{\delta V}{\delta y}(x,0,t) = 0$

$V(L,y,t) = 0$, $V(x,H,t) = 0$

Let $V(x,y,t) = \phi(x,y)\, h(t)$

1) $\frac{dh}{dt} = -\lambda k h \quad \Rightarrow \quad h(t) = e^{-\lambda k t}$

2) $\frac{d^2 f}{dx^2} + \mu f = 0 \quad ; \quad f(0) = 0 = f(L)$

$\Rightarrow \mu = \left(\frac{n\pi}{L}\right)^2 , \quad f(x) = \sin \frac{n\pi}{L} x$

3) $\frac{d^2 g}{dy^2} + (\lambda - \mu) g = 0 \; ; \quad g'(0) = 0 = g(H)$

$\Rightarrow \lambda - \mu = \left[\left(m - \frac{1}{2}\right)\frac{\pi}{H}\right]^2$

$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 + \left[\left(m - \frac{1}{2}\right)\frac{\pi}{H}\right]^2$

& $g(y) = \cos\left(m - \frac{1}{2}\right)\frac{\pi}{H} y$

finally

$$V(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin \frac{n\pi}{L} x \cos\left[\left(m - \frac{1}{2}\right)\frac{\pi}{H} y\right]$$

So $u(x,y,t) = V(x,y,t) + u_E(x,y)$

$$\Rightarrow u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} e^{-k\lambda t} \sin\frac{n\pi x}{L} \cos\left[\left(m-\frac{1}{2}\right)\frac{\pi y}{H}\right]$$

$$+ \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \cos\frac{n\pi y}{H}$$

I.C:- $u(x,y,0) = f(x,y)$ implies

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{n\pi x}{L}\right) \cos\left[\left(m+\frac{1}{2}\right)\frac{\pi y}{H}\right]$$

$$+ \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \cos\frac{n\pi y}{H}$$

& so on.

Question 8.2.5:- Solve the initial value problem for a two dimensional heat equation inside a circle (of radius a) with time independent boundry conditions

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

$$u(a,\theta,t) = g(\theta), \quad u(r,\theta,0) = f(r,\theta)$$

Solution

$$\frac{\partial u}{\partial t} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right] \longrightarrow (1)$$

for steady state $\frac{\partial u}{\partial t} = 0$

$$\Rightarrow \nabla^2 u_E = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \longrightarrow (*)$$

with
$$\left.\begin{aligned} u_E(r,-\pi) &= u_E(r,\pi) \\ \frac{du_E}{d\theta}(r,-\pi) &= \frac{du_E}{d\theta}(r,\pi) \end{aligned}\right\} \text{B.Cs}$$

$$u_E(a,\theta) = g(\theta) \quad \text{I.C}$$

Let $u_E(r,\theta) = \phi(r)\,G(\theta)$

$$(R) \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\phi\frac{dG}{dr}\right) + \frac{1}{r^2}G\frac{d^2\phi}{d\theta^2} = 0$$

$$\Rightarrow -\frac{r}{G}\frac{d}{dr}\left[r\frac{dG}{dr}\right] = \frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = -\lambda$$

1) $\frac{d^2\phi}{d\theta^2} = -\lambda\phi$; $\phi(-\pi) = \phi(\pi)$

$\phi'(-\pi) = \phi'(\pi)$

$\phi(\theta) = A\cos\sqrt{\lambda}\,\phi + B\sin\sqrt{\lambda}\,\phi$

$\phi(-\pi) = \phi(\pi) \Rightarrow \lambda = n^2$

$\phi'(-\pi) = \phi'(\pi) \Rightarrow \lambda = n^2$

$$\Rightarrow \phi(\theta) = A\cos n\theta + B\sin n\theta$$

2) $r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - \lambda G = 0$

$$\Rightarrow r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2 G = 0$$

for $n \neq 0$ $G(r) = C_1 r^n + C_2 r^{-n}$

$G(r) = C_1 r^n$

for $n = 0$ $G(r) = C_1$

$$U_E(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta$$

I.C $\Rightarrow$ $g(\theta) = \sum_{n=0}^{\infty} A_n a^n \cos n\theta + \sum_{n=1}^{\infty} B_n a^n \sin n\theta$

Now Let

$V(r,\theta,t) = u(r,\theta,t) - U_E(r,\theta)$

So (1) $\Rightarrow$

$$\frac{\partial V}{\partial t} = k\nabla^2 V$$

with

$u(a,\theta,t) = 0$

$$v(r,\theta,0) = f(r,\theta) - u_E(r,\theta)$$

which can be solved easily by separation of variable

$$v(r,\theta,t) = \phi(r,\theta)\, h(t)$$

$$\frac{\partial v}{\partial t} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\right]$$

$$\phi\frac{dh}{dt} = \frac{k}{r}\frac{\partial}{\partial r}\left[r h\frac{\partial \phi}{\partial r}\right] + \frac{k}{r^2} h \frac{\partial^2 \phi}{\partial \theta^2}$$

$$\frac{1}{kh}\frac{dh}{dt} = \frac{1}{\phi}\nabla^2\phi = -\lambda$$

1) $\frac{dh}{dt} = -\lambda k h \quad \Rightarrow h(t) = c e^{-\lambda k t}$

2) $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \lambda\phi = 0$

$$\phi(r,\theta) = R(r)\,G(\theta)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r G\frac{dR}{dr}\right] + \frac{1}{r^2} R\frac{d^2 G}{d\theta^2} + \lambda R G = 0$$

$$\Rightarrow \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r^2 = \frac{-1}{G}\frac{d^2 G}{d\theta^2} = \mu$$

★ $\frac{d^2 G}{d\theta^2} = -\mu G$

$\Rightarrow G(\theta) = A\cos\sqrt{\mu}\,\theta + B\sin\sqrt{\mu}\,\theta$ $\qquad G(-\pi) = G(\pi)$

B.C $\Rightarrow \quad \mu = n^2$ $\qquad G'(-\pi) = G'(\pi)$

$\Rightarrow G(\theta) = A\cos n\theta + B\sin n\theta$

★ $\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r^2 = \mu$ $\qquad R(a) = 0$

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda r^2 - \mu) R = 0$$

Solution near $r = 0$

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2 R = 0$$

for $n \neq 0$ $\quad R(r) = C_1 r^n + C_2 r^{-n}$

$$\Rightarrow R(r) = C_1 r^n$$

Question 8.2.6:- Solve the wave equation with time independent sources.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + Q(x)$$

If an "equilibrium" solution exists. Analyze the behaviour for large t. If no equilibrium exists, explain why and reduce the problem to one with homogeneous boundary conditions.

(a) $Q(x) = 0 \qquad u(0,t) = A \qquad u(L,t) = B$

Solution

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow ①$$

for steady state $\frac{d^2 u_E}{dx^2} = 0$; $u_E(0) = A$, $u_E(L) = B$

$$\Rightarrow u_E = A_1 x + B_1$$

$$u_E(0) = A \Rightarrow \boxed{B_1 = A}$$

So $u_E(x) = A_1 x + A$

$$u_E(L) = B \Rightarrow A_1 L + A = B \Rightarrow A_1 L = B - A$$

$$\Rightarrow \boxed{A_1 = \frac{B-A}{L}}$$

$$\Rightarrow u_E(x) = \left(\frac{B-A}{L}\right) x + A$$

Now let $V(x,t) = u(x,t) - u_E(x)$

$\Rightarrow u(x,t) = V(x,t) + u_E(x)$

So eqn ① $\Rightarrow \frac{\partial^2 V}{\partial t^2} = c^2 \frac{\partial^2 V}{\partial x^2}$ with

$V(0,t) = 0 \quad , \quad V(L,t) = 0$

which can be solved by seperation of variables.

$V(x,t) = \phi(x) h(t)$

$\Rightarrow \phi \frac{d^2 h}{dt^2} = c^2 h \frac{d^2 \phi}{dx^2}$

$\Rightarrow \frac{d^2 h}{dt^2} = -\lambda c^2 h \Rightarrow \boxed{h(t) = A \cos c\sqrt{\lambda} t + B \sin c\sqrt{\lambda} t}$

& $\frac{d^2 \phi}{dx^2} = -\lambda \phi \quad ; \quad \phi(0) = 0, \phi(L) = 0$

$\Rightarrow \phi(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$

$\lambda = \left(\frac{n\pi}{L}\right)$ & $\phi(x) = \sin \frac{n\pi}{L} x$

So $V(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos c \frac{n\pi}{L} t + B_n \sin \frac{n\pi}{L} t\right] \sin \frac{n\pi}{L} x$

& $u(x,t) = V(x,t) + u_E(x)$

$= \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi}{L} t + B_n \sin \frac{n\pi}{L} t\right] \sin \frac{n\pi}{L} x + \frac{(B-A)x}{L} + A$

$u(x,0) = f(x)$

$\Rightarrow \sum_{n=1}^{\infty} \left[A_n \sin \frac{n\pi}{L} x\right] + u_E = f(x)$

$\Rightarrow A_n = \frac{2}{L} \int_0^L (f(x) - u_E) \sin \frac{n\pi}{L} x \, dx$

& $B_n = \frac{2}{n\pi L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$

(b) $Q(x) = 1$, $u(0,t) = 0$, $u(L,t) = 0$

Solution

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + 1 \longrightarrow ①$$

For steady state $\dfrac{d^2 u_E}{dx^2} + 1 = 0$

$$\Rightarrow \frac{d^2 u_E}{dx^2} = \frac{-1}{c^2} \qquad \Rightarrow \frac{d u_E}{dx} = \frac{-x}{c^2} + A$$

$$\rightarrow u_E(x) = \frac{-x^2}{2c^2} + Ax + B \qquad ; u_E(0) = 0 = u_E(L)$$

$$u_E(0) = 0 \Rightarrow \boxed{B = 0}$$

$$u_E(L) = 0 \Rightarrow \frac{-x^2}{2c^2} + AL = 0 \Rightarrow \boxed{A = \frac{L}{2c^2}}$$

$$\Rightarrow \boxed{u_E(x) = \frac{-x^2}{2c^2} + \frac{Lx}{2c^2}}$$

Let $v(x,t) = u(x,t) - u_E(x)$

$$\Rightarrow u(x,t) = v(x,t) + u_E(x)$$

$$① \rightarrow \frac{\partial^2}{\partial t^2}(v + u_E) = c^2 \frac{\partial^2}{\partial x^2}(v + u_E) + 1$$

$$\Rightarrow \frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} + \left(-\frac{c^2}{c^2}\right) + 1$$

$$\Rightarrow \frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} \; ; \qquad v(0,t) = 0, \; v(L,t) = 0$$

Thus $v(x,t) = \displaystyle\sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi c t}{L} + B_n \sin \frac{n\pi c t}{L}\right] \sin \frac{n\pi x}{L}$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi c t}{L} + B_n \sin \frac{n\pi c t}{L}\right] \sin \frac{n\pi x}{L} + u_E(x)$$

As $t \longrightarrow \infty$

$$u(x,t) \longrightarrow u_E(x)$$

(C) $Q(x)=1$, $u(0,t)=A$, $u(L,t)=B$

Solution for steady state

$$c^2\frac{d^2u_E}{dx^2}=-1 \Rightarrow u_E(x)=\frac{-x^2}{2c^2}+A_1x+B_1$$

$$u_E(0)=A \rightarrow \boxed{B_1=A}$$

$$u_E(L)=B \Rightarrow \frac{-L}{2c^2}+A_1L+A=B$$

$$\Rightarrow \boxed{A_1=\frac{B-A}{L}+\frac{L}{2c^2}}$$

So $u_E(x)=-\frac{x^2}{2c^2}+\left(\frac{B-A}{L}+\frac{L}{2c^2}\right)x+A$

Let $v(x,t)=u(x,t)-u_E(x)$

gives $\frac{\partial^2 v}{\partial t^2}=c^2\frac{\partial^2 v}{\partial x^2}$; $v(0,t)=0=v(L,t)$

At the end

$$u(x,t)=\sum_{n=1}^{\infty}\left[A_n\cos\frac{n\pi c}{L}t+B_n\sin\frac{n\pi c}{L}t\right]\sin\frac{n\pi x}{L}+u_E(x)$$

$u(x,0)=f(x)$, $\frac{\partial u}{\partial t}(x,0)=g(x)$

$$\Rightarrow A_n=\frac{2}{L}\int_0^L(f(x)-u_E(x))\sin\frac{n\pi}{L}x\,dx$$

$$\& \ B_n=\frac{2}{n\pi L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

(d) $Q(x)=\sin\frac{\pi x}{L}$, $u(0,t)=0$, $u(L,t)=0$

Solution

$$\frac{d^2u_E}{dx^2}=-\frac{1}{c^2}\sin\frac{\pi x}{L}$$

$$\frac{du_E}{dx}=\frac{L}{\pi c^2}\cos\frac{\pi x}{L}+C_1 \Rightarrow u_E=\frac{L^2}{c^2\pi^2}\sin\frac{\pi x}{L}+C_1x+C_2$$

$$u_E(0)=0 \longrightarrow \boxed{C_2=0}$$

$u_E(L) = 0 \quad \Rightarrow \quad \boxed{C_1 = 0}$

$$\Rightarrow \quad u_E(x) = \frac{L^2}{c^2\pi^2} \sin \frac{\pi x}{L}$$

Let $\quad v(x,t) = u(x,t) - u_E(x)$

$$① \Rightarrow \quad \frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2}{\partial x^2}\left(v + u_E(x)\right) + \sin \frac{\pi x}{L}$$

$$= c^2 \frac{\partial^2 v}{\partial x^2} - \sin \frac{\pi x}{L} + \sin \frac{\pi x}{L}$$

$$\Rightarrow \frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} \quad ; \quad v(0,t) = 0 = v(L,t)$$

which is easy to solve & we get $u(x,t)$

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Example:- $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin 3x\, e^{-t}$

$u(0,t) = 0$, $u(\pi,t) = 1$

$u(x,0) = f(x)$

To make the boundry conditions homogen- consider the refrence temperature

$$V(x,t) = u(x,t) - \frac{x}{\pi}$$

where
$$\left.\begin{array}{l} \frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} + \sin 3x\, e^{-t} \;; \\ V(0,t) = 0 \,,\; V(\pi,t) = 0 \,,\; V(x,0) = f(x) - \frac{x}{\pi} \end{array}\right\} (A)$$

Now consider the related homogeneous problem

$$\left.\begin{array}{l} \frac{\partial V}{\partial t} + \frac{\partial^2 V}{\partial x^2} \;; \quad V(0,t) = 0 \\ \qquad\qquad\qquad V(\pi,t) = 0 \end{array}\right\} \longrightarrow (B)$$

and using this problem, by seperation of variable $V(x,t) = \phi(x) h(t)$

$$\Rightarrow \phi \frac{dh}{dt} = h \frac{d^2\phi}{dx^2} \Rightarrow \frac{1}{h}\frac{dh}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$

It gives $\frac{d^2\phi}{dx^2} = -\lambda\phi$; $\phi(0) = 0$, $\phi(\pi) = 0$

$\lambda > 0 \Rightarrow \quad \phi(x) = A\cos\sqrt{\lambda}\, x + B\sin\sqrt{\lambda}\, x$

$\phi(0) = 0 \quad \Rightarrow \boxed{A = 0}$

$\phi(\pi) = 0 \quad \Rightarrow \sin\sqrt{\lambda}\,\pi = 0 \quad \Rightarrow \sqrt{\lambda}\,\pi = n\pi$

$\Rightarrow \lambda = n^2$

$\Rightarrow \phi(x) = \sin nx$

Now, By method of eigen function expansio

Consider $V(x,t) = \sum_{n=1}^{\infty} a_n(t)$:

$\Rightarrow V(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin nx$

$$\frac{\partial V}{\partial t} = \sum_{n=1}^{\infty} \frac{da_n}{dt} \sin nx$$

$\&$ $\frac{\partial V}{\partial x} = \sum_{n=1}^{\infty} a_n(t)\, n \cos nx \Rightarrow \frac{\partial^2 V}{\partial x^2} = -\sum_{n=1}^{\infty} a_n(t) n^2 \sin nx$

$(B) \Rightarrow \sum_{n=1}^{\infty} \frac{da_n}{dt} \sin nx = -\sum_{n=1}^{\infty} a_n(t)\, n^2 \sin nx + \sin 3x\, e^{-t}$

$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{da_n}{dt} + n^2 a_n\right) \sin nx = \sin 3x\, e^{-t}$

Then the unknown fourier sine coefficients satisfy

$$\frac{da_n}{dt} + n^2 a_n = \begin{cases} 0 & \text{if } n \neq 3 \\ e^{-t} & \text{if } n = 3 \end{cases} \longrightarrow (*)$$

Since

$$V(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin nx$$

I.C

$$V(x,0) = f(x) - \frac{x}{\pi} = \sum_{n=1}^{\infty} a_n(0) \sin nx$$

$\Rightarrow \int_0^{\pi} \left(f(x) - \frac{x}{\pi}\right) \sin nx\, dx = a_n(0) \int_0^{\pi} \sin nx \sin nx\, dx$

$\Rightarrow a_n(0) = \frac{2}{\pi} \int_0^{\pi} \left(f(x) - \frac{x}{\pi}\right) \sin nx\, dx$

Consider $(*)$ $\quad I.F = e^{\int n^2 dt} \Rightarrow e^{n^2 t}$

Multipling $(*)$ with I.F

for $n \neq 3$

$$e^{n^2 t}\left(\frac{da_n}{dt} + n^2 a_n\right) = 0$$

$\Rightarrow \frac{d}{dt}\left(a_n e^{n^2 t}\right) = 0 \Rightarrow a_n e^{n^2 t}\Big|_0^t = 0$

$$\Rightarrow a_n e^{n^2t} - a_n(0) = 0$$

$$\Rightarrow a_n(t) = a_n(0)\, e^{-n^2t}$$

*) for $n = 3$

$$e^{n^2t}\left(\frac{da_n}{dt} + n^2 a_n\right) = e^{-t} e^{n^2t}$$

$$\Rightarrow \frac{d}{dt}\left[a_n(t) e^{n^2t}\right] = e^{-t} e^{n^2t} = e^{(n^2-1)t}$$

Integrate from 0 to t

$$\Rightarrow a_n(t) e^{n^2t} - a_n(0) = \frac{1}{n^2-1} e^{(n^2-1)t}\Big|_0^t$$

$$\Rightarrow a_n(t) e^{n^2t} - a_n(0) = \frac{1}{n^2-1} e^{(n^2-1)t} - \frac{1}{n^2-1}$$

$\because n=3$

$$a_3(t) = a_3(0)\, e^{-9t} + \frac{1}{9-1} \cancel{e^{n^2t}} \cdot \cancel{e^{-n^2t}} \cdot e^{-t} - \frac{e^{-9t}}{9-1}$$

$$\Rightarrow a_3(t) = a_3(0) e^{-9t} + \frac{1}{8} e^{-t} - \frac{e^{-9t}}{8}$$

$$= \frac{1}{8} e^{-t} + \left[a_3(0) - \frac{1}{8}\right] e^{-9t}$$

Therefore

$$a_n(t) = \begin{cases} a_n(0)\, e^{-n^2t} & n \neq 3 \\ \frac{1}{8} e^{-t} + \left[a_3(0) - \frac{1}{8}\right] e^{-9} & n = 3 \end{cases}$$

Exercise 8.3

Question 8.3.1:- Solve the initial value problem for the heat equation with time-dependent sources

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \quad ; \quad u(x,0) = f(x)$$

subject to the following boundry conditions

(a) $u(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = 0$

Solution

Associated homogeneous equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \; ; \; u(0,t) = 0 \; , \; \frac{\partial u}{\partial x}(L,t) = 0$$

From this we get

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad ; \quad \phi(0) = 0 \; , \; \phi'(L) = 0$$

$$\Rightarrow \lambda = \left[(n-\tfrac{1}{2})\frac{\pi}{L}\right]^2 \; ; \; \phi(x) = \sin(n-\tfrac{1}{2})\frac{\pi}{L}x$$

Now, By the method of eigenfunction expansion let

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(n-\tfrac{1}{2})\frac{\pi}{L}x$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\frac{da_n}{dt}\phi_n(x) \; ; \; \frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} a_n\frac{d^2\phi_n}{dx^2}$$

$$= \sum_{n=1}^{\infty} a_n(-\lambda\phi_n)$$

So given equ becomes

$$\sum_{n=1}^{\infty}\frac{da_n}{dt}\phi_n(x) = -k\sum_{n=1}^{\infty} a_n\lambda\phi_n + Q(x,t)$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[\frac{da_n}{dt} + \lambda_n k a_n\right]\phi_n(x) = Q(x,t)$$

Let $Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x)$

$$\frac{da_n}{dt} + \lambda_n k a_n = \frac{2}{L}\int_0^L Q(x,t)\phi_n(x)\,dx = q_n(t)$$

$$I.F = e^{\lambda_n k t}$$

So $e^{\lambda_n k t}\left[\frac{da_n}{dt} + \lambda_n k a_n\right] = q_n(t)e^{\lambda_n k t}$

$$\Rightarrow \frac{d}{dt} a_n(t) e^{\lambda_n k t} = q_n(t) e^{\lambda_n k t}$$

$$\Rightarrow a_n(t)e^{\lambda_n k t} - a_n(0) = \int_0^t q_n(t) e^{\lambda_n k t}\,dt$$

$$\Rightarrow a_n(t) = a_n(0)e^{-\lambda_n k t} + e^{-\lambda_n k t}\int_0^t q_n(t)\,e^{\lambda_n k t}\,dt$$

(b) $u(0,t)=0$, $\quad u(L,t)+2\frac{\partial u}{\partial x}(L,t)=0$

Solution

Related homogeneous problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

and $\frac{d^2\phi}{dx^2} = -\lambda\phi$; $\phi(0)=0$, $\phi(L)+2\phi'(L)=0$

$\lambda>0 \Rightarrow \quad \phi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$\phi(0)=0 \Rightarrow \boxed{A=0}$

$\phi(L)+2\phi'(L)=0$

$$\Rightarrow A\cos\sqrt{\lambda}L + B\sin\sqrt{\lambda}L - 2\sqrt{\lambda}A\sin\sqrt{\lambda}L + B\sqrt{\lambda}\cos\sqrt{\lambda}L = 0$$

$$\Rightarrow \sin\sqrt{\lambda}L = -2\sqrt{\lambda}\cos\sqrt{\lambda}L$$

$$\Rightarrow \tan\sqrt{\lambda}L = -2\sqrt{\lambda}/L$$

$$\Rightarrow \sqrt{\lambda}L \approx \left(n-\tfrac{1}{2}\right)\pi$$

$$\Rightarrow \lambda \approx \left[\left(n-\tfrac{1}{2}\right)\frac{\pi}{L}\right]^2$$

$$\Rightarrow \phi(x) = \sin\left(n-\tfrac{1}{2}\right)\frac{\pi x}{L}$$

Now by eigen function expansion

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\,\phi_n(x)$$

So given eqn $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{da_n(t)}{dt}\phi_n = \sum_{n=1}^{\infty} k A_n(t)\frac{d^2\phi}{dx^2} + Q(x,t)$

$$= -\lambda k \sum_{n=1}^{\infty} A_n(t)\phi_n + Q(x,t)$$

$$\Rightarrow \left(\frac{dA_n}{dt} + \lambda k A_n\right)\phi_n = Q(x,t)$$

$$\Rightarrow \frac{dA_n}{dt} + \lambda k A_n = \frac{\int_0^L Q(x,t)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx} = q_n(t)$$

$$\Rightarrow Q(x,t) = \sum q_n(t)\,\phi_n(x)$$

$$I.F = e^{\lambda k t}$$

$$\Rightarrow e^{\lambda k t}\left[\frac{da_n}{dt} + \lambda k A_n\right] = q_n(t)e^{\lambda k t}$$

$$\Rightarrow \frac{d}{dt}\left(A_n e^{\lambda k t}\right) = q_n(t)e^{\lambda k t}$$

$$\Rightarrow A_n(t) = A_n(0)e^{-\lambda k t} + e^{-\lambda k t}\int_0^t q_n(t)e^{\lambda k t}\,dt$$

(c) $u(0,t) = A(t)$, $\dfrac{\partial u}{\partial x}(L,t) = 0$

Solution

Consider the $r(x,t)$

$r(x,t) = A(t)$

$\dfrac{d^2u}{dx^2} = 0 \Rightarrow u = Ax + B$

$u(0) = A(t) \Rightarrow \boxed{B = A(t)}$

$\dfrac{\partial u}{\partial x}(L) = 0 \Rightarrow \boxed{A = 0}$

$\Rightarrow u = A(t)$

with $r(0,t) = A(t)$, $\frac{\partial r}{\partial x}(L,t) = 0$

Now
$$V(x,t) = u(x,t) - A(t)$$

Therefore $\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} + Q(x,t);$ $\longrightarrow$ ①

$V(0,t) = 0$, $\frac{\partial V}{\partial x}(L,t) = 0$

Related homogeneous problem is
$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2}$$

and the eigen value problem of above eq is
$$\frac{d^2\phi}{dx^2} = -\lambda\phi; \quad \phi(0) = 0, \ \phi'(L) = 0$$

$\Rightarrow \phi(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$

$\Rightarrow \lambda = \left((n-\frac{1}{2})\frac{\pi}{L}\right)^2$ & $\phi(x) = \sin(n-\frac{1}{2})\frac{\pi x}{L}$

Then by applying method of eigen function expansion $V(x,t) = \sum_{n=1}^{\infty} A_n(t)\phi_n(x)$

we get $V(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin(n-\frac{1}{2})\frac{\pi x}{L}$

and $u(x,t) = A(t) + \sum_{n=1}^{\infty} A_n(t)\sin(n-\frac{1}{2})\frac{\pi x}{L}$

(d) $u(0,t) = A \neq 0$, $u(L,t) = 0$

Solution

Consider the reference temperature distribution

$r(x,t) = -\frac{A}{L}x + A$

$r(0,t) = A$, $r(L,t) = 0$

So $V(x,t) = u(x,t) - r(x,t)$

$\frac{d^2u}{dx^2} = 0 \Rightarrow u = A_1x + B_1$

$u(0) = A \Rightarrow \boxed{B_1 = A}$

$u(L) = 0 \Rightarrow \boxed{A_1 = -\frac{A}{L}}$

$\Rightarrow u(x) = -\frac{A}{L}x + A$

$$\Rightarrow \frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} + Q(x,t)$$

Related homogeneous problem is

$$\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} ;$$

$$v(0,t) = 0 \quad , \quad v(L,t) = 0$$

& eigen value problem is

$$\frac{d^2\phi}{dx^2} = -\lambda\phi ; \quad \phi(0) = 0 , \phi'(L) = 0$$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 \quad \& \quad \phi(x) = \sin\frac{n\pi}{L}x$$

we get $v(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{L}$

and $u(x,t) = A\left[1 - \frac{x}{L}\right] + \sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{L}$

(e) $\frac{\partial u}{\partial x}(0,t) = A(t) \quad , \quad \frac{\partial u}{\partial x}(L,t) = B(t)$

Solution Consider

$$r(x,t) = A(t)x + \left[\frac{B(t) - A(t)}{L}\right]\frac{x^2}{2}$$

$\frac{d^2u}{dx^2} = 0 \Rightarrow u = Ax + B$

$\Rightarrow \frac{\partial u}{\partial x} = A$

$\frac{\partial u}{\partial x}(0) = A(t) \Rightarrow \boxed{A = A(t)}$

such that

$$\frac{\partial r}{\partial x}(0,t) = A(t) , \quad \frac{\partial r}{\partial x}(L,t) = B(t)$$

So $v(x,t) = u(x,t) - r(x,t)$

Related Homo problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad , \quad \phi'(0) = 0 = \phi(L)$$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 \quad \& \quad \phi(x) = \cos\frac{n\pi}{L}x$$

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t) \cos\frac{n\pi x}{L}$$

$$\Rightarrow u(x,t) = A(t)x + \frac{x^2}{2L}\left[B(t) - A(t)\right] + \sum_{n=1}^{\infty} a_n(t)\cos\frac{n\pi x}{L}$$

(f) $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = 0$

Related homogeneous problem is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \; ; \; \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t)$$

and eigen value problem is

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \; ; \; \phi'(0) = 0 = \phi'(L)$$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 \quad \& \quad \phi(x) = \cos\frac{n\pi}{L}x$$

$$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} a_n(t)\cos\frac{n\pi}{L}x$$

where $a_n(t) = a_n(0)e^{-\lambda kt} + e^{-\lambda kt}\int_0^t q_n(t)\,dt$

Question 8.3.6:- Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin 5x\, e^{-2t}$$

subject to $u(0,t) = 1$, $u(\pi,t) = 0$

and $u(x,0) = 0$

Solution

Consider a refrence temperature distribution

$r(x,t) = 1 - \frac{x}{\pi}$

satisfying

$r(0,t) = 1$, $r(\pi,t) = 0$

Let

$\frac{d^2u}{dx^2} = 0 \Rightarrow u = Ax + B$

$u(0) = 1 \Rightarrow \boxed{B = 1}$

$u(\pi) = 0 \Rightarrow A\pi + 1 = 0$

$\Rightarrow A = -\frac{1}{\pi}$

$\Rightarrow u = -\frac{x}{\pi} + 1$

$V(x,t) = u(x,t) - 2(x,t)$

$$\Rightarrow \frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} + \sin 5x\, e^{-2t} \; ;$$

$V(0,t) = 0 \;,\; V(\pi,t) = 0$ &

$V(x,0) = -2(x,0)$

Consider related eigen value problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad , \phi(0) = 0 \;,\; \phi(\pi) = 0$$

$\Rightarrow \lambda = n^2$ & $\phi(x) = \sin nx$

Now by method of eigen function expansion

$$V(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x)$$

$$= \sum_{n=1}^{\infty} a_n(t) \sin nx$$

So given equ becomes

$$\sum_{n=1}^{\infty} \frac{da_n}{dt} \sin nx = -\sum_{n=1}^{\infty} a_n(t) n^2 \sin nx + \sin 5x\, e^{-2t}$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[\frac{da_n}{dt} + n^2 a_n\right] \sin nx = \sin 5x\, e^{-2t}$$

$$\Rightarrow \frac{da_n}{dt} + n^2 a_n = \begin{cases} 0 & n \neq 5 \\ e^{-2t} & n = 5 \end{cases}$$

I.F $= e^{n^2 t}$

$\Rightarrow \frac{d}{dt}\left[a_n e^{n^2 t}\right] = 0 \Rightarrow a_n(t) = a_n(0)\, e^{-n^2 t}$

$$\frac{d}{dt}\left[a_n e^{n^2 t}\right] = e^{(n^2-2)t}$$

$$\Rightarrow a_n(t) e^{n^2 t} - a_n(0) = \frac{1}{n^2-2} e^{(n^2-2)t} - \frac{1}{n^2-2}$$

$$\Rightarrow a_n(t) = a_n(0)e^{-n^2t} + \frac{1}{n^2-2}e^{-2t} - \frac{e^{-n^2t}}{n^2-2}$$

$\because n=5$ so

$$a_5(t) = \frac{1}{23}e^{-2t} + \left[a_5(0) - \frac{1}{23}\right]e^{-25t}$$

therefore

$$a_n(t) = \begin{cases} a_n(0)\,e^{-n^2t} & n \neq 5 \\ \frac{1}{23}e^{-2t} + \left[a_5(0) - \frac{1}{23}\right]e^{-25t} & n=5 \end{cases}$$

and

$$V(x,t) = \sum_{n=0}^{\infty} a_n(t)\sin nx \quad ; V(x,0) = \frac{x}{\pi} - 1$$

$$\Rightarrow \left(\frac{x}{\pi} - 1\right) = a_n(0)\sin nx$$

$$\Rightarrow a_n(0) = \frac{2}{\pi}\int_0^{\pi}\left(\frac{x}{\pi} - 1\right)\sin nx\,dx$$

Therefore

$$u(x,t) = 1 - \frac{x}{\pi} + \sum_{n=0}^{\infty} a_n(t)\sin nx$$