

General form of PDE's.

01-02-2016

$$f(x, y, \dots, u, u_x, u_y, \dots, u_{xx}, u_{yy}, \dots) = 0 \rightarrow (1)$$

$$\sum_{i,j=1}^n A_{ij} u_{x_i} u_{x_j} + \sum_{i=1}^n b_i u_{x_i} + F(x_1, x_2, \dots) = G(x_1, x_2, \dots, x_n) \rightarrow (2)$$

$$A_{ij} = A_{ji}$$

both the forms (1) & (2) are identical.

Where "f" is any general function of independent variables x, y, \dots , "u" is the unknown function. $u_x, u_y, \dots, u_{xx}, u_{yy}$ are the partial derivatives & the subscripts on the dependent variables denote the derivative.

$$u_{xy} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{LHS is performed left to right \& vice-versa.}$$

Laplace Eq,

$$u_{xx} + u_{yy} = 0$$

$$\text{1-D wave Eq } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\text{1-D Heat Eq, } \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Solution should be differentiable in the domain of given D.E.

Where

$$u = u(x, y, z, \dots)$$

which satisfies (i) identically is a suitable domain "D" of n-dimensional space \mathbb{R}^n in the independent variables x, y, \dots , such functions (if they exist) are called the solution of (1).

Example:-

$$u_{xx} - u_{yy} = 0$$

has solutions

$$u = (x+y)^3, \quad u = \sin(x-y)$$

Polynomial Continuous

Again Continuous

Unique solution are obtained when, the D.Eq is a subject to some conditions.

The order of PDE is the highest derivative of PDE, appearing in the D.Eq.

Example:-

$$u_{xxy} + x u_{yy} + 8u = f_y \rightarrow 3^{\text{rd}} \text{ order}$$

$$u_{xx} + 2x u_{xy} + u_{yy} = e^y \rightarrow 2^{\text{nd}} \text{ order.}$$

A PDE is said to be Linear if

- 1):- Linear in unknown function.
- 2):- All of its partial derivatives with coefficients depending only on the independent variables.

If dependent variable becomes exponential then PDE is non-linear.

A PDE is Quasi-linear if it is linear in the highest derivative of the unknown function.

If force terms are non-linear but rest of the PDE is linear is said to be almost linear.

Force term is called input.

Zero force term implies homogeneity.

$$u_{xx} - u_{yy} = 0$$

$$u = C_1(x+y)^3 + C_2 \sin(x-y)$$

$$u_{xx} = 6C_1(x+y) - C_2 \sin(x-y)$$

$$u_{yy} = 6C_1(x+y) - C_2 \sin(x-y)$$

$$\Rightarrow u_{xx} - u_{yy} = 6C_1(x+y) - 6C_1(x+y) - C_2 \sin(x-y) + C_2 \sin(x-y) = 0$$

Superposition principle always satisfies linear homogeneous PDE.

$$\text{Example :- } U_x - U_y = 0$$

$$\xi = x + y, \quad \eta = x - y$$

$$U_x = U_\xi \xi_x + U_\eta \eta_x = U_\xi + U_\eta$$

$$U_y = U_\xi \xi_y + U_\eta \eta_y = U_\xi - U_\eta$$

$$\text{So, } U_x - U_y = 2U_\eta = 0 \Rightarrow 2U = f(\xi)$$

$$\text{OR simply } \Rightarrow U = f(\xi) = f(x + y)$$

thus "U" can have solution

$$(x+y)^n, \cos n(x+y), \sin n(x+y)$$

$$\exp(n(x+y)), \quad n = 1, 2, 3, \dots$$

Adv. PDE

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Operator.

An operator is a mathematical rule which when applied to a function produces another function.

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Linear:-

$$L [c_1 u_1 + c_2 u_2] = c_1 L [u_1] + c_2 L [u_2]$$

$$L \left[\sum_{i=1}^n c_i u_i \right] = \sum_{i=1}^n c_i L [u_i]$$

Associative:-

$$L+M = M+L$$

$$LM \neq ML \quad \leftarrow \text{(not hold)}$$

Example

$$L = \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial y}$$

$$M = \frac{\partial^2}{\partial y^2} - y \frac{\partial}{\partial x}$$

$$(L+M)+N = L+(M+N)$$

also holds

$$L(MN) = (LM)N$$

$$L(M+N) = LM + LN$$

Result: If u_1 & u_2 are solutions of a linear inhomogeneous Eq. then

$u_1 - u_2$ is a solution of corresponding homogeneous solution.

Boundary & Initial conditions.

Conditions are given on discrete points in ODE.

But in PDE on curves.

DNR

Dirichlet-Neumann-Robin's.

Solution of a BVP.

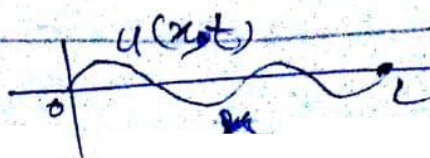
By a solution to a BVP on an open region D we mean a function u that satisfies the Differential Eq. on D and is continuous on $D \cup \partial D$ and satisfies the boundary condition on ∂D .

The B.C.s are linear if they respect a linear relationship b/w u & its partial derivative on ∂D .

For second order PDE linear B.C.s can take one of the following three forms.

(i):- The B.C's specify the values of unknown function u on the boundary this type of the B.C is called Dirichlet B.C's.

Example: If $u(x,t)$ is the displacement of a vibrating string and its ends are fixed at $x=0$ & $x=L$



$$u(0, t) = 0$$

$$u(L, t) = 0$$

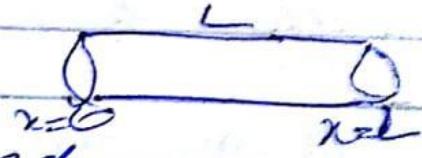
(2) The B.C's specify the value of the derivative of u in the direction normal to boundary written as

$$\frac{\partial u}{\partial n}$$

This type is called Neumann conditions.

Example: Suppose $u(x, t)$ is the temperature in a rod of length 'L' if the rod is perfectly insulated at $x=0$ and

$x=L$ the heat flux at these points is zero.



$$\frac{\partial u(0, t)}{\partial x} = 0 \quad \& \quad \frac{\partial u(L, t)}{\partial x} = 0$$

(3) The B.C's specify a linear relationship b/w u & its normal derivative on the

boundary these are referred as mixed B.C's or Robin B.C's.

General form

$$\left[\alpha u + \beta \frac{\partial u}{\partial n} \right]_{\partial D} = f(x)$$

α, β are constants.

Example: Suppose in Example 2 poor insulated is used at the ends of the rod then B.C's takes the form.

$$u(0,t) + \frac{\partial u(0,t)}{\partial x} = 0$$

$$u(L,t) + \frac{\partial u(L,t)}{\partial x} = 0$$

Note:- Similar to superposition principle for the solution to linear PDE's, we have the superposition principle for linear B.C's.

If u_1 & u_2 are solutions of linear homogeneous PDE with the linear B.C's

$$\left[\alpha u_1(\bar{x}) + \beta \frac{\partial u_1(\bar{x})}{\partial \bar{n}} \right] \Big|_{\partial D} = f(\bar{x}) \Big|_{\partial D}$$

$$\left[\alpha u_2(\bar{x}) + \beta \frac{\partial u_2(\bar{x})}{\partial \bar{n}} \right] \Big|_{\partial D} = g(\bar{x}) \Big|_{\partial D}$$

where α, β are constants.

Then

$w = u_1 + u_2$ is a solution of PDE that satisfy

$$\left[\alpha w(\bar{x}) + \beta \frac{\partial w(\bar{x})}{\partial \bar{n}} \right] \Big|_{\partial D} = \left[f(\bar{x}) + g(\bar{x}) \right] \Big|_{\partial D}$$

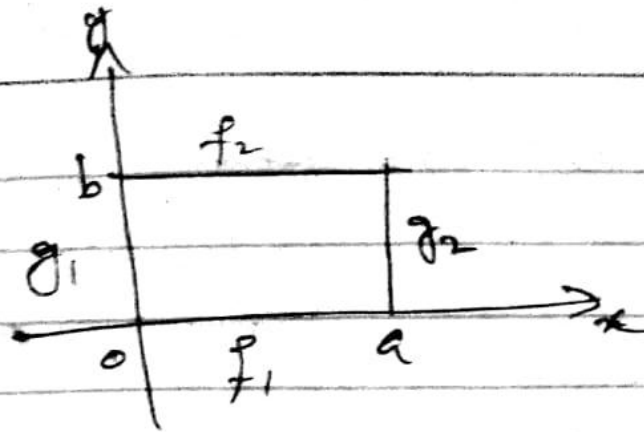
Expt Consider

$$\nabla^2 u = 0 \quad \text{in rectangle}$$

with the following linear B.C.

$$\left. \begin{aligned} u(x, 0) &= f_1(x) \\ u(x, b) &= f_2(x) \end{aligned} \right\} \longrightarrow \textcircled{1}$$

$$\left. \begin{aligned} u(0, y) &= g_1(y) \\ u(a, y) &= g_2(y) \end{aligned} \right\} \longrightarrow \textcircled{2}$$



We strike the problem into two parts along x -axis & along y -axis.

x -axis \Rightarrow

$$\nabla^2 u_1 = 0$$

$$u_1(x, 0) = f_1(x)$$

$$u_1(x, b) = f_2(x)$$

$$u_1(0, y) = 0$$

$$u_1(a, y) = 0$$

y -axis.

$$\nabla^2 u_2 = 0$$

$$u_2(x, 0) = 0$$

$$u_2(x, b) = 0$$

$$u_2(0, y) = g_1(y)$$

$$u_2(a, y) = g_2(y)$$

Obviously if we solve u_1 & u_2 then $u_1 + u_2$ is a solution of Laplace Eq. which satisfies ① & ②.

Note: Neumann B.C's usually do not specify a unique solution of B.V.P.

Because

\Rightarrow derivative of constant is zero.

An Eq. written in the form

$$f(x, y, z, p, q) = 0 \rightarrow \text{①}$$

is called first order PDE. where

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

x, y are independent.

z is dependent variable.

$$x^2 + y^2 + (z-c)^2 = a^2 \rightarrow \text{②}$$

with a & c are arbitrary constants.

$$2x + 2(z-c)p = 0$$

$$2y + 2(z-c)q = 0$$

$$\Rightarrow x + (z-c)p = 0 \Rightarrow z-c = \frac{-x}{p}$$

$$y + (z-c)q = 0 \Rightarrow z-c = \frac{-y}{q}$$

Now compare these Eq's.

$$\Rightarrow \frac{x}{p} = \frac{y}{q}$$

$$y p - x q = 0 \rightarrow \textcircled{3}$$

that is set of all spheres with centres on the z -axis is characterized by Eq. $\textcircled{3}$.

Example.

$$x^2 + y^2 = (z - c)^2 \tan^2 \alpha$$

circular cone.

$$\Rightarrow \frac{x}{p} = \frac{(z - c) \tan^2 \alpha}{p}$$

$$\frac{y}{q} = \frac{(z - c) \tan^2 \alpha}{q}$$

$$\Rightarrow x = (z - c) \tan^2 \alpha \frac{p}{q}$$

$$y = (z - c) \tan^2 \alpha \frac{q}{p}$$

$$\Rightarrow \frac{x}{p} = \frac{y}{q}$$

$$\Rightarrow y p - x q = 0 \rightarrow \textcircled{3}$$

Note: It means that more than one geometrical entities can be described by the same Eq.

All surfaces of revolution with this property (the spheres & the cone ... what have in common that they are surfaces of revolution which have one or an axis of symmetry) are characterized by Eq. of the form

$$(i) \quad z = f(x^2 + y^2) \Rightarrow \text{Some case } \textcircled{B}$$

$$(ii) \quad z = f(x+it) + g(x-it) \Rightarrow \text{Laplace Eq.}$$

Sol: (i) $z = f(x^2 + y^2) \rightarrow \textcircled{A}$

Diff \textcircled{A} w.r.t x

$$\frac{\partial z}{\partial x} = f'(x^2 + y^2) \cdot 2x$$

$$\Rightarrow \frac{p}{x} = 2f'(x^2 + y^2) \rightarrow (i)$$

where $p = \frac{\partial z}{\partial x}$

Diff. (A) w.r.t y .

$$\frac{\partial z}{\partial y} = f'(x^2 + y^2) \cdot 2y$$

$$\Rightarrow \frac{q}{y} = 2f'(x^2 + y^2) \rightarrow (ii)$$

comparing (i) & (ii).

$$\frac{q}{y} = \frac{p}{x}$$

$$\Rightarrow \boxed{py - qx = 0}$$

Solution (ii) $z = f(x+it) + g(x-it) \rightarrow (B)$

Two times Diff. (B) w.r.t x .

$$\frac{\partial z}{\partial x} = f'(x+it) + g'(x-it)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+it) + g''(x-it) \rightarrow (i)$$

Two times Diff. (B) w.r.t " t ".

$$\frac{\partial z}{\partial t} = f'(x+it) \cdot i + g'(x-it) \cdot (-i)$$

$$\frac{\partial^2 z}{\partial t^2} = f''(x+it) \cdot i^2 + g''(x-it) \cdot (-i)^2$$

$$\frac{\partial^2 z}{\partial t^2} = -f''(x+it) - g''(x-it) \rightarrow (ii)$$

By adding (ii) & (i).

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$$

Linear PDE.

08-02-2016

Any Eq. of the form

$$Pp + Qq = R \longrightarrow \textcircled{1}$$

where P, Q & R are given functions of x, y & z which don't involve p, q and derivatives. $\textcircled{1}$ is known as first order PDEs also Lagrange Eq.

Results: The General solution of linear PDE (1) is $F(u, v) = 0$

where F is an arbitrary function & $u(x, y, z) = C_1$ &

$v(x, y, z) = C_2$ forms a solution of the auxiliary Lagrange Eq.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

General PDE

$$AU_{xx} + BU_y + CU = D$$

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU = G$$

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Example:-

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z$$

By comparing:- with $Pp + Qq = R$

$$P = x^2, Q = y^2, R = (x+y)z$$

aux. Eq.

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \longrightarrow \textcircled{1}$$

Now

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$\int x^{-2} dx = \int y^{-2} dy$$

$$-\frac{1}{x} = -\frac{1}{y} + C$$

$$\Rightarrow \frac{1}{y} - \frac{1}{x} = C$$

Let $u = \frac{1}{y} - \frac{1}{x}$

$$\frac{dx - dy}{x^2 - y^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

$$\frac{dx - dy}{(x+y)(x-y)} = \frac{dz}{(x+y)z}$$

$$\frac{d(x-y)}{x-y} = \frac{dz}{z}$$

Integrating.

$$\ln(x-y) = \ln z + \ln A$$

$$x-y = zA$$

$$F(u, v) = 0 \quad \frac{x-y}{z} = A = V$$

$$\text{If } u_i(x_1, x_2, \dots, x_n, z) = C_i$$

$$(i = 1, 2, \dots, n).$$

If u_i are independent solutions of

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \dots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

then the relation

$$\Phi(u_1, u_2, \dots, u_n) = 0,$$

In which Φ is arbitrary.

is a general solution of linear PDE.

$$P_1 \frac{\partial z}{\partial x_1} + P_2 \frac{\partial z}{\partial x_2} + \dots + P_n \frac{\partial z}{\partial x_n} = R$$

If $u = u(x, y, z)$
which satisfy

$$(y-z) \frac{\partial u}{\partial x} + (z-x) \frac{\partial u}{\partial y} + (x-y) \frac{\partial u}{\partial z} = 0$$

Show that u contains x, y and z only in the combination

$$x+y+z \text{ and } x^2+y^2+z^2$$

$$P_1 = y-z, P_2 = z-x, P_3 = x-y, R = 0$$

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{du}{0}$$

By adding.

$$\frac{dx+dy+dz}{y-z+z-x+x-y} = \frac{dx+dy+dz}{0} = \frac{dz}{x-y}$$

$$\Rightarrow x+y+z = C_1$$

$$\frac{x dx + y dy + z dz}{x(y-z) + y(z-x) + z(x-y)} = \frac{dy}{z-x}$$

$$\frac{x dx + y dy + z dz}{\cancel{xy - xz} + \cancel{yz - yx} + \cancel{zx - zy}} = \frac{dy}{z-x}$$

$$\Rightarrow x^2 + y^2 + z^2 = C_2$$

$$\text{Now } \frac{dz}{x-y} = \frac{du}{0}$$

$$0 = du$$

$$\Rightarrow u = C_3$$

$$\Phi(x+y+z, x^2+y^2+z^2, u) = 0$$

$$\Rightarrow u = f(x+y+z, x^2+y^2+z^2)$$

Integral surfaces passing through the given curves

$$u(x(t), y(t), z(t)) = 0, \quad v(x(t), y(t), z(t)) = 0$$

Find the integral surface of the linear PDE

$$x(y^2+z)P - y(z+x^2)Q = (x^2-y^2)Z \quad \rightarrow (**)$$

containing the straight line

$$x+y=0, \quad z=1$$

$$\frac{dx}{x(y^2+z)} = \frac{-dy}{y(z+x^2)} = \frac{dz}{(x^2-y^2)z}$$

$$\frac{x dx + y dy}{\cancel{x}y^2 + \cancel{x}z - y\cancel{z} + -y\cancel{x}^2} = \frac{dz}{x^2z - y^2z}$$

$$x dx + y dy = dz$$

$$\frac{x^2 + y^2}{2} = z + C_1$$

$$x+y=0$$

$$x=-y$$

$$y=t$$

$$x=-t$$

$$z=1$$

$$\frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{\cancel{y^2+z} \cancel{yz} \cancel{x^2+x^2} \cancel{yz}} = \frac{dz}{(x-y^2)z}$$

$$\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0$$

$$\ln x + \ln y + \ln z = \ln C_1$$

$$\ln (xyz) = \ln C_1$$

$$\boxed{xyz = C_1}$$

$$\frac{x^2}{2} + \frac{y^2}{2} - z = C_2$$

$$x+y=0$$

$$\Rightarrow x = -y \quad \text{let } y = t$$

$$x = -t$$

$$z = 1$$

$$f(t)(t)(1) = C_1$$

$$-t^2 = C_1 \rightarrow \textcircled{1}$$

$$\frac{t^2}{2} + \frac{t^2}{2} - 1 = C_2$$

$$t^2 - 1 = C_2$$

$$-9 - 1 = C_2$$

$$C_1 + C_2 + 1 = 0$$

$$xy^2 + \frac{x^2}{2} + \frac{y^2}{2} - z + 1 = 0$$

Put $\boxed{x = -y}$ $\boxed{z = 1}$

$$(-y)(y)(1) + \frac{y^2}{2} + \frac{y^2}{2} - 1 + 1 = 0$$

$$-y^2 + y^2 = 0$$

$$0 = 0$$

Diff \textcircled{A} w.r.t x .

$$xy^2 + yz + x - p = 0$$

$$p(xy - 1) = -yz - x$$

$$p = \frac{yz + x}{1 - xy}$$

Diff (A) wrt y .

$$xyv + xz + y - v = 0$$

$$v(xy - 1) = -y - xz$$

$$v = \frac{(xz + y)}{(1 - xy)}$$

Put p & q in (*)

$$x(y^2 + z) \frac{yz + x}{1 - xy} - y(z + x^2) \frac{xz + y}{1 - xy} = (x^2 - y^2)z$$

$$(y^2 + z)(xyz + x^2) - (z + x^2)(xyz + y^2) = (x^2 - y^2)z(1 - xy)$$

$$y^2(xyz + x^2) + xyz^2 + x^2z - xyz^2 - y^2z - x^2(xyz + y^2) = (x^2 - y^2)z(1 - xy)$$

$$xy^2z + xyz^2 + (x^2 - y^2)z - x^3yz$$

$$= (x^2 - y^2)z(1 - xy)$$

$$xyz(y^2 - x^2) + (x^2 - y^2)z = "$$

$$(y^2 - x^2)(xy - 1)z = (y^2 - x^2)z(xy - 1)$$

The General 2nd order Linear PDE
in \dots one independent variable
 $u = u(x, y)$ can be written in the
form:-

$$A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F u = 0$$

where A, B, C are
functions of x, y independent
variables & don't vanish
simultaneously.

D, E & F are also function
of x & y .

Classifications

(i):

if **DISCRIMINANT**

$$B^2 - AC < 0 \quad \text{elliptic}$$

(ii)

$$B^2 - AC = 0 \quad \text{parabola.}$$

(iii)

$$B^2 - AC > 0 \quad \text{Hyperbola.}$$

$$u_{xx} + u_{yy} = 0$$

$$A=1, B=0, C=1$$

$$B^2 - AC = -1 < 0 \quad \text{Elliptic}$$

$$u_{xx} = d^2 u_t \leftarrow$$

$$A=1, B=0, C=0$$

$$B^2 - AC = 0 \Rightarrow \text{parabolic}$$

correct

$$u_{xx} = d^2 u_{yy}$$

$$u_{xx} - d^2 u_{yy} = 0$$

$$A=1, B=0, C=-d^2$$

$$B^2 - AC = +d^2 > 0 \text{ Hyperbolic.}$$

$$u_{xx} = \frac{1}{c^2} u_{tt}$$

$$u_{xx} - \frac{1}{c^2} u_{tt} = 0$$

$$A=1, B=0, C=-\frac{1}{c^2}$$

$$B^2 - AC = \frac{1}{c^2} > 0 \text{ Hyperbolic.}$$

Reduction of 2nd order PDE
to the canonical form

Consider

$$A u_{xx} + 2B u_{xy} + C u_{yy} + H(x, y, u, u_x, u_y) = 0$$

Where A, B, C are functions
of x, y and u have continuous
first and 2nd order partial derivatives.

let us transform

$$\xi = \xi(x, y)$$

$$\eta = \eta(x, y)$$

$$J(\xi, \eta) = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$$

in the region under consideration.

$$(\xi, \eta)$$

$$\bar{A}u_{\xi\xi} + 2\bar{B}u_{\xi\eta} + \bar{C}u_{\eta\eta} + \bar{H}(\xi, \eta, u, u_{\xi}, u_{\eta}) = 0 \quad \rightarrow \textcircled{2}$$

where

$$\bar{A} = A(\xi_x)^2 + 2B\xi_x\xi_y + C(\xi_y)^2$$

$$\bar{C} = A(\eta_x)^2 + 2B\eta_x\eta_y + C(\eta_y)^2$$

$$\bar{B} = A(\xi_x\eta_x + \eta_x\xi_x) + B(\xi_x\eta_y + \xi_y\eta_x) + C\xi_y\eta_y$$

The resulting Eq. (2) is in the same form to the original Eq. (1).

The nature of the Eq. remains invariant under such a transformation if the jacobian does not vanish.

$$\bar{B}^2 - A\bar{C} = J^2 (B^2 - AC)$$

Consider

$$A u_{xx} + 2B u_{xy} + C u_{yy} + H(u_x, u_y, u) = 0$$

We suppose first that none of A, B, C is zero.

Let (ξ, η) be new variables such that

$$\bar{A} = 0$$

$$\& \bar{C} = 0$$

then

$$\left. \begin{aligned} A(\xi_x)^2 + 2B\xi_x\eta_x + C(\eta_x)^2 &= 0 \\ \& A(\eta_x)^2 + 2B\eta_x\xi_x + C(\xi_x)^2 &= 0 \end{aligned} \right\} \rightarrow (3)$$

These two Eq. are the same type. Therefore we can write

$$A(\Phi_x)^2 + 2B\Phi_{xy} + C(\Phi_y)^2 = 0$$

$\div A$

$$(\Phi_x)^2 + 2\frac{B}{A}\Phi_{xy} + \frac{C}{A}(\Phi_y)^2 = 0 \rightarrow **$$

\rightarrow which is quadratic in (Φ_x/Φ_y) .

$$\left(\frac{\Phi_x}{\Phi_y}\right)^2 + \frac{2B}{A}\left(\frac{\Phi_x}{\Phi_y}\right) + \frac{C}{A} = 0$$

$$\frac{\Phi_x}{\Phi_y} = \frac{-\frac{2B}{A} \pm \sqrt{\frac{4B^2}{A^2} - \frac{4C}{A}}}{2} = \frac{1}{A}[-B \pm \sqrt{B^2 - AC}]$$

$$A\Phi_x = [B \pm \sqrt{B^2 - AC}]\Phi_y$$

$$A\Phi_x + [B \pm \sqrt{B^2 - AC}]\Phi_y = 0$$

$$\left[A\Phi_x + (B + \sqrt{B^2 - AC})\Phi_y \right] \left[A\Phi_x + (B - \sqrt{B^2 - AC})\Phi_y \right] = 0$$

$$\Rightarrow \left\{ \begin{array}{l} A\Phi_x + (B + \sqrt{B^2 - AC})\Phi_y = 0 \rightarrow \textcircled{4} \\ A\Phi_x + (B - \sqrt{B^2 - AC})\Phi_y = 0 \rightarrow \textcircled{5} \end{array} \right.$$

$$\left\{ \begin{array}{l} A\Phi_x + (B + \sqrt{B^2 - AC})\Phi_y = 0 \rightarrow \textcircled{4} \\ A\Phi_x + (B - \sqrt{B^2 - AC})\Phi_y = 0 \rightarrow \textcircled{5} \end{array} \right.$$

\rightarrow Characteristics Eq.s / Lagrange EV
Note that the solution of

$\textcircled{4}$ & $\textcircled{5}$ is the sol. of (**).

To solve (4) & (5) the corresponding auxiliary Eq.s will be

$$\frac{dx}{A} = \frac{dy}{B + \sqrt{B^2 - AC}}$$

$$\& \frac{dx}{A} = \frac{dy}{B - \sqrt{B^2 - AC}}$$

Solutions of (4) & (5) called the characteristics family of curves in $x-y$ -plane along which

$\xi = \text{constant}$ & $\eta = \text{constant}$.

$$\left. \begin{aligned} \Phi_1(x, y) &= C_1 \\ \Phi_2(x, y) &= C_2 \end{aligned} \right\} \text{characteristics curves.}$$

Hence the transformation will be

$$\xi = \xi(x, y) = \Phi_1(x, y)$$

$$\eta = \eta(x, y) = \Phi_2(x, y)$$

Example:

$$y^2 u_{xx} - x^2 u_{yy} = 0 \rightarrow \textcircled{1}$$

$$A = y^2, \quad B = 0, \quad C = -x^2$$

$$B^2 - AC = x^2 y^2 > 0 \text{ Hyperbolic.}$$

$$x \neq 0 \neq y$$

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

$$\frac{dy}{dx} = \frac{x}{y}, \quad \frac{dy}{dx} = -\frac{x}{y}$$

$$y dy = x dx$$

$$\frac{y^2}{2} + \frac{x^2}{2} = C_2$$

$$\frac{y^2}{2} - \frac{x^2}{2} = C_1$$

$$\text{Let } \xi = \frac{y^2}{2} - \frac{x^2}{2}$$

$$\eta = \frac{y^2}{2} + \frac{x^2}{2}$$

$$u_x = u_{\xi} \xi_x + u_{\eta} \eta_x$$

$$= -x u_{\xi} + x u_{\eta}$$

$$u_{xx} = -u_{\xi} - x [u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x]$$

$$+ 1 \cdot u_{\eta} + x [u_{\eta\eta} \eta_x + u_{\eta\xi} \xi_x]$$

$$= -u_{\xi} - x [u_{\xi\xi} (-x) + x u_{\xi\eta}]$$

$$+ u_{\eta} + x [u_{\eta\eta} (x) + (-x) u_{\eta\xi}]$$

$$= u_{\eta} - u_{\xi} + x^2 [u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}]$$

Answer
will be

$$u_{\xi\eta} = \frac{\eta}{2(\xi^2 - \eta^2)} \quad u_{\xi\xi} = \frac{\xi}{2(\xi^2 - \eta^2)} u_{\eta\eta}$$

$$u_{xy} = u_{\xi} \xi_y + u_{\eta} \eta_y = y u_{\xi} + y u_{\eta}$$

$$u_{yy} = u_{\xi} + u_{\eta} + y^2 [u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}]$$

put these values in (1)

$$y^2 [u_{\eta} - u_{\xi} + x^2 [u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}]] - x^2 [u_{\xi} + u_{\eta} + y^2 [u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}]] = 0$$

$$\Rightarrow y^2 u_\eta - y^2 u_\xi + 2xy^2 (2u_{\xi\eta}) - x^2 u_\xi - x^2 u_\eta = 0$$

$$\Rightarrow u_{\xi\eta} = \frac{(y^2 - x^2) u_\eta}{4x^2 y^2} - \frac{(y^2 + x^2) u_\xi}{4x^2 y^2}$$

$$2\xi = y^2 - x^2 \quad \& \quad 2\eta = y^2 + x^2$$

$$4\xi^2 = y^4 + x^4 - 2x^2 y^2 \quad ; \quad -4\eta^2 = y^4 + x^4 + 2x^2 y^2$$

$$\Rightarrow 4(\eta^2 - \xi^2) = 4x^2 y^2$$

$$\Rightarrow u_{\xi\eta} = \frac{2\xi u_\eta}{4(\eta - \xi^2)} - \frac{2\eta}{4(\eta - \xi^2)} u_\xi$$

$$u_{\xi\eta} = \frac{\xi}{2(\eta - \xi^2)} u_\eta - \frac{\eta}{2(\eta - \xi^2)} u_\xi$$

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$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

For parabola

$$\sqrt{B^2 - AC} = 0$$

Example:-

$$\frac{\partial^2 u}{\partial x^2} - 2x \frac{\partial^2 u}{\partial x \partial y} + x^2 \frac{\partial^2 u}{\partial y^2} - 2xy = 0$$

$$A=1, B=-2x, C=x^2$$

$$B^2 - AC = 4x^2 - x^2 = 0$$

 \Rightarrow Parabolic Eq.

Then

$$\frac{dy}{dx} = \frac{-x \pm x}{1}$$

$$y^2 = \frac{x^2}{2} + C$$

$$y \pm \frac{x}{\sqrt{2}} = C = f(x, y)$$

Let $\eta = x$ (condition: Jacobian should not be zero.)

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$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} x & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$= x u_\xi + u_\eta$$

$$u_{xx} = u_\xi + x [u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x]$$

$$+ u_{\eta\xi} \xi_x + u_{\eta\eta} \eta_x$$

$$= u_\xi + x (u_{\xi\xi} (x) + u_{\xi\eta} (1))$$

$$+ u_{\eta\xi} (x) + u_{\eta\eta} (1)$$

$$= u_\xi + x^2 u_{\xi\xi} + 2x u_{\xi\eta} + u_{\eta\eta}$$

$$\frac{\partial u}{\partial y} = u_\xi \xi_y + u_\eta \eta_y$$

$$= u_\xi$$

$$\frac{\partial^2 u}{\partial x \partial y} = u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x$$

$$= x u_{\xi\xi} + u_{\xi\eta}$$

$$u_{yy} = u_{\xi\xi} \xi_y + u_{\xi\eta} \eta_y$$

$$\boxed{u_{yy} = u_{\xi\xi}}$$

$$\begin{aligned} x^2 u_{\xi\xi} + 2xy u_{\xi\eta} + y^2 u_{\eta\eta} + u_{\xi} \\ - 2x^2 u_{\xi\xi} - 2xy u_{\xi\eta} + x^2 u_{\xi\xi} - 2u_{\xi} = 0 \end{aligned}$$

$$\boxed{u_{\eta\eta} - u_{\xi} = 0}$$

Example:

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0$$

result will be $\boxed{u_{\eta\eta} = 0}$ ①

$$A = x^2, B = xy, C = y^2$$

$$B^2 - AC = x^2 y^2 - x^2 y^2 = 0 \quad (\text{Parabolic PDE})$$

$$\frac{dy}{dx} = \frac{+xy}{x^2} = +\frac{y}{x}$$

$$\frac{1}{y} dy = +\frac{1}{x} dx$$

$$\Rightarrow \ln y = + \ln x + \ln c$$

$$\frac{y}{x} = c$$

$$\Rightarrow \xi = \frac{y}{x} \quad \text{let } \eta = x$$

$$u_x = -\frac{y}{x^2} u_{\xi} + u_{\eta}$$

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$$u_{xx} = y[u_{\xi\xi} y + u_{\xi\eta}] + u_{\eta\xi} y + u_{\eta\eta}$$

$$u_{xx} = y^2 u_{\xi\xi} + 2y u_{\xi\eta} + u_{\eta\eta}$$

$$u_{xy} = u_{\xi} + y[u_{\xi\xi} x + u_{\xi\eta} (0)] + u_{\eta\xi} x + 0$$

$$u_{xy} = u_{\xi} + xy u_{\xi\xi} + x u_{\xi\eta}$$

$$u_y = x u_{\xi}$$

$$u_{yy} = x^2 u_{\xi\xi}$$

$$x^2 [y^2 u_{\xi\xi} + 2y u_{\xi\eta} + u_{\eta\eta}] + 2xy [u_{\xi} + xy u_{\xi\xi} + x u_{\xi\eta}] + y^2 x^2 u_{\xi\xi} = 0$$

$$4x^2 y^2 u_{\xi\xi} + 4y(x^2) u_{\xi\eta} + x^2 u_{\eta\eta} + 2xy u_{\xi} = 0$$

$$4\xi^2 u_{\xi\xi} + 4\xi\eta u_{\xi\eta} + \eta^2 u_{\eta\eta} + 2\xi\eta u_{\xi} = 0$$

$$u_{xx} = \frac{2y}{x^3} u_{\xi} + \frac{y^2}{x^4} u_{\xi\xi} - \frac{2y}{x^2} u_{\xi\eta}$$

$$u_{xy} = -\frac{1}{x^2} u_{\xi} - \frac{y}{x^3} u_{\xi\xi} + \frac{1}{x} u_{\eta\xi}$$

$$u_y = \frac{1}{x} u_{\xi}$$

$$u_{yy} = \frac{1}{x^2} u_{\xi\xi}$$

Put values in (1).

$$2 \frac{y}{x^3} u_{\xi} + \frac{y^2}{x^4} u_{\xi\xi} - \frac{2y}{x^2} u_{\xi\eta} + x^2 u_{\eta\eta} - \frac{2y}{x^2} u_{\xi} - \frac{y^2}{x^3} u_{\xi\xi} + \frac{1}{x} u_{\eta\xi} - \frac{2y^2}{x^2} u_{\xi\xi} + 2y u_{\eta\xi} + \frac{y^2}{x^2} u_{\xi\eta} = 0$$

$$\Rightarrow x^2 u_{\eta\eta} = 0$$

$$\Rightarrow u_{\eta\eta} = 0 \quad \because x^2 \neq 0$$

$$\Rightarrow u_{\eta} = f(\xi)$$

$$u = \int f(\xi) d\eta$$

$$= f(\xi) \cdot \eta + g(\xi)$$

If $B^2 - AC < 0$

$$\Rightarrow \phi_1 \pm i\phi_2$$

then take

$$\phi_1 = \xi, \quad \phi_2 = \eta.$$

Example:

$$u_{xx} + x^2 u_{yy} = 0$$

$$A=1, \quad B=0, \quad C=x^2$$

$$B^2 - AC = 0 - x^2 < 0$$

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

$$\Rightarrow \frac{dy}{dx} = \pm ix$$

$$\text{Then } y = \pm i \frac{x^2}{2} + c$$

$$y \pm i \frac{x^2}{2} = c$$

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$$u = \phi_1 \pm i\phi_2$$

$$\phi_1 = \xi = y, \quad \phi_2 = \eta = \frac{x^2}{2}$$

$$\frac{\sigma(\xi, \eta)}{\sigma(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ x & 0 \end{vmatrix} = -x \neq 0$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$= x u_\eta$$

$$u_{xx} = u_\eta + x [u_\eta \xi_x + u_{\eta\eta} \eta_x]$$

$$= u_\eta + x^2 u_{\eta\eta}$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi$$

$$u_{yy} = u_{\xi\xi} \xi_y + u_{\xi\eta} \eta_y$$

$$= u_{\xi\xi}$$

$$u_{xx} + x^2 u_{yy} = 0$$

$$x^2 u_{\eta\eta} + u_\eta + x^2 u_{\xi\xi} = 0$$

$$\frac{x^2}{2} = \eta$$

$$2\eta u_{\eta\eta} + u_\eta + 2\eta u_{\xi\xi} = 0$$

Example:- Reduced into canonical form & find the General solution of wave Eq. with the Cauchy data.

$$u_{xx} = \frac{1}{c^2} u_{tt}$$

$$\text{Cauchy data} \Rightarrow u(x, 0) = f(x)$$

$$u_x(x, 0) = g(x)$$

Conditions on curves not on a point.

Solution:-

$$u_{xx} - \frac{1}{c^2} u_{tt} = 0 \longrightarrow \textcircled{A}$$

$$A=1, B=0, C=-\frac{1}{c^2}$$

$$B^2 - AC = 0 + \frac{1}{c^2} > 0$$

$$\frac{dt}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

$$\frac{dt}{dx} = \pm \sqrt{\frac{1}{c^2}} = \pm \frac{1}{c} \quad \text{then}$$

$$\pm c dt = dx \quad \pm ct + x = C_1$$

$$x = \pm ct + C_1$$

$$\rightarrow x \pm ct = C_1$$

$$\xi = x + ct$$

$$\eta = x - ct$$

$$u_x = u_\xi + u_\eta$$

$$u_{xx} = u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta}$$

$$u_t = cu_\xi - cu_\eta$$

$$u_{tt} = c^2 u_{\xi\xi} - 2c^2 u_{\eta\xi} + c^2 u_{\eta\eta}$$

$$\textcircled{A} \Rightarrow u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta} - u_{\xi\xi} + 2u_{\eta\xi} - u_{\eta\eta} = 0$$

$$4u_{\eta\xi} = 0$$

$$\text{Final Result is } u_{\xi\eta} = 0$$

$$u_\xi = f(\xi)$$

$$u = \int f(\xi) d\xi + \psi(\eta)$$

$$u = \phi(\xi) + \psi(\eta)$$

$$= \phi(x+ct) + \psi(x-ct) \rightarrow \textcircled{B}$$

↙ This wave travel to left.

Using conditions.

$$u(x,0) = f(x)$$

$$\Rightarrow f(x) = \phi(x) + \psi(x) \rightarrow \textcircled{1}$$

$$\& \quad u_t = c\phi'(x+ct) - c\psi'(x-ct)$$

$$\text{using } u_t(x,0) = g(x)$$

$$g(x) = c\phi'(x) - c\psi'(x)$$

→ $\textcircled{2}$

From ① Diff. w.r.t x

$$f'(x) = \phi'(x) + \psi'(x)$$

$$\phi'(x) = f'(x) - \psi'(x)$$

put in ②

$$g(x) = c f'(x) - c \psi'(x) - c \phi'(x)$$

$$\boxed{\psi'(x) = \frac{f'(x)}{2} - \frac{g(x)}{2c}}$$

Now put $\psi'(x) = f'(x) - \phi'(x)$ in

Eq. ①.

$$g(x) = c \phi'(x) - c f'(x) + c \phi'(x)$$

$$\boxed{\frac{g(x)}{2c} + \frac{f'(x)}{2} = \phi'(x)}$$

On integrating.

$$\phi(x) = \frac{1}{2c} \int g(z) dz + \frac{1}{2} f(x) + K_1$$

$$\& \psi(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(z) dz + K_2$$

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By adding.

$$\phi(x) + \psi(x) = f(x) + k_1 + k_2$$

$$\therefore f(x) = \phi(x) + \psi(x)$$

$$\text{So, } f(x) = f(x) + k_1 + k_2$$

$$\Rightarrow k_1 = -k_2$$

$$\text{Let } k_1 = k \Rightarrow k_2 = -k$$

$$\Rightarrow \phi(x+ct) = \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_0^{x+ct} g(z) dz + k$$

$$\& \psi(x-ct) = \frac{1}{2} f(x-ct) - \frac{1}{2c} \int_0^{x-ct} g(z) dz - k$$

Put these values in (B)

$$u(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_0^{x+ct} g(z) dz + k + \frac{1}{2} f(x-ct) - \frac{1}{2c} \int_0^{x-ct} g(z) dz - k$$

$$u(x,t) = \frac{1}{2} [f(x+ct) + f(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z) dz$$

which is known as D'Alembert solution of PDE.

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22-02-2016

$$u(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z) dz$$

$$= \frac{1}{2} \sin(x+ct) + \frac{1}{2} \sin(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} \cos z dz$$

Question 2

$$u_{tt} = c^2 u_{xx} \quad u(x,0) = \sin x, \quad u_t(x,0) = \cos x$$

$$= \frac{1}{2} (2 \sin x \cos ct) + \frac{1}{2c} (\sin z)_{x-ct}^{x+ct}$$

$$= \sin x \cos ct + \frac{1}{2c} (\sin(x+ct) - \sin(x-ct))$$

$$= \sin x \cos ct + \frac{1}{c} \cos x \sin ct$$

Example:-

$$u_x + \cos x u_y + u = xy \rightarrow \text{①}$$

characteristic Eq. $(Pp + Qv = R)$

$$\frac{dx}{1} = \frac{dy}{\cos x} = \frac{du}{xy - u}$$

$$\frac{dx}{1} = \frac{dy}{\cos x}$$

$$\int \cos x dx = \int dy$$

$$\sin x = y + c$$

$$y - \sin x = c$$

$$\text{Let } \eta = x$$

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} \cos x & -1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0$$

Transforms Eqs

$$\xi = \sin x - y$$

$$\eta = x$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$\Rightarrow u_x = \cos x u_\xi + u_\eta$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_y = -u_\xi$$

Put in Eq. ①

$$\cancel{\cos x u_\xi} + u_\eta - \cancel{\cos x u_\xi} + u = xy$$

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$$\therefore y = \sin \eta - \xi, \quad \eta = x$$

$$u \eta + u = \eta (\sin \eta - \xi) \rightarrow (*)$$

$$I.F = e^{\int d\eta} = e^{\eta}$$

multiplying I.F with Eq. (*).

$$e^{\eta} u \eta + e^{\eta} u = e^{\eta} \eta (\sin \eta - \xi)$$

$$\int \frac{\partial}{\partial \eta} (e^{\eta} u) d\eta = \int e^{\eta} \eta (\sin \eta - \xi) d\eta$$

$$e^{\eta} u = \underbrace{\int \eta e^{\eta} \sin \eta d\eta}_{I_1} - \xi \underbrace{\int e^{\eta} \eta d\eta}_{I_2} + f(\xi)$$

$$I_1 = \int \frac{e^{\eta} \eta \sin \eta d\eta}{I \quad II}$$

$$= -e^{\eta} \eta \cos \eta + \int (e^{\eta} + \eta e^{\eta}) \cos \eta d\eta$$

$$= -e^{\eta} \eta \cos \eta + \int \frac{e^{\eta} \cos \eta d\eta}{II \quad I} + \int \frac{e^{\eta} \eta \cos \eta d\eta}{I \quad II}$$

$$= -e^{\eta} \eta \cos \eta + e^{\eta} \cos \eta + \int e^{\eta} \sin \eta d\eta + \eta e^{\eta} \sin \eta - \int (e^{\eta} + \eta e^{\eta}) \sin \eta d\eta$$

$$I_1 = -e^{\eta} \eta \cos \eta + \eta e^{\eta} \sin \eta + e^{\eta} \cos \eta - I_1$$

$$I_1 = -\frac{1}{2} e^{\eta} \eta \cos \eta + \frac{1}{2} \eta e^{\eta} \sin \eta + \frac{1}{2} e^{\eta} \cos \eta$$

$$I_2 = \int \frac{e^{\eta} \eta d\eta}{II \quad I} = e^{\eta} \eta - \int e^{\eta} d\eta = e^{\eta} \eta - e^{\eta}$$

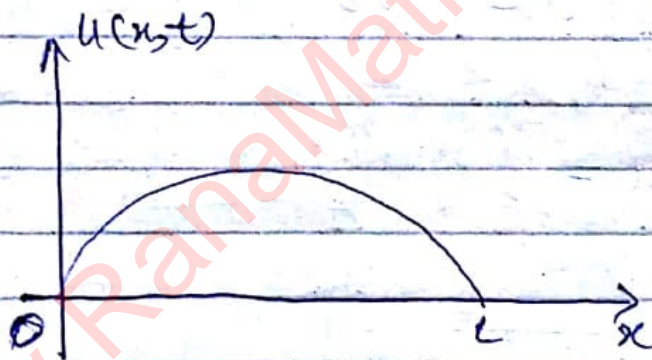
$$e^{\eta} u = \frac{1}{2} e^{\eta} \eta \cos \eta + \frac{1}{2} \eta e^{\eta} \sin \eta + \frac{1}{2} e^{\eta} \cos \eta - \xi \eta e^{\eta} + \xi e^{\eta} + f(\xi)$$

$$u = -\xi(\eta - 1) + \frac{1}{2} \eta (\sin \eta - \cos \eta) + \frac{1}{2} \cos \eta + e^{-\eta} f(\xi)$$

Wind is not a wave.

Derivation of the Wave

Eq. of the vibrating string.

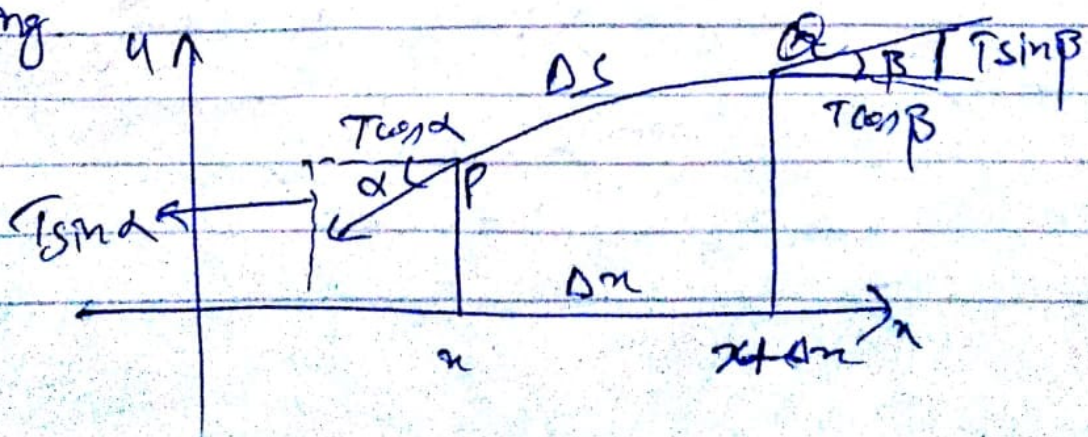


Let us consider a stretched string of length L fixed at both ends. We want to determine the Eq. of motion which characterises the position $u(x,t)$ of the string in time t after initial disturbance. In order to obtain a simple Eq. we make the following assumptions.

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- 1):- The string is flexible and elastic \rightarrow (tension will be tangential)
- 2):- The tension in the string is uniform throughout its length.
- 3):- The weight of the string is small compare with the tension in the string.
- 4) The wave disturbance is small, i.e. the displacement in the string is very small compare with length of the string.
- 5) The slope of the displaced string at any point is small compare with unity, (α is very small).
- 6) There is only (pure) transverse vibrations.
- 7) The Gravity affect negligible.

Consider a small portion of string



Consider an element Δs in a displaced string

The forces acting on the string are Tension T acting at point P & Q in the direction of tangent to the element Δs . The resulting force acting vertically on the mass ($m = \mu \Delta s$) of element Δs .

$$\text{Resulting force} = T \sin \beta - T \sin \alpha$$

$$= m a$$

$$= \mu \Delta s u_{tt}$$

where μ is the line density.

$$\Rightarrow \frac{T(\sin \beta - \sin \alpha)}{\Delta s} = \mu u_{tt}$$

If angles (α, β) are very small then

$$\sin \alpha \approx \tan \alpha$$

$$\sin \beta \approx \tan \beta$$

$$S_{2j} \quad \frac{T (\tan \beta - \tan \alpha)}{\Delta s} = \int \mu_{tt}$$

$$\frac{T \left[\left. \frac{dy}{dx} \right|_{x+\Delta x} - \left. \frac{dy}{dx} \right|_x \right]}{\Delta s} = \int \mu_{tt}$$

Since slope of string is small

then $\Delta s \approx \Delta x$

$$\frac{T \left[\left. \frac{dy}{dx} \right|_{x+\Delta x} - \left. \frac{dy}{dx} \right|_x \right]}{\Delta x} = \int \mu_{tt}$$

when $\Delta x \rightarrow 0$

$$T \mu_{xx} = \int \mu_{tt}$$

$$\mu_{tt} = \left(\frac{T}{\rho} \right) \mu_{xx} = c^2 \mu_{xx}$$

which is called one dimensional wave Eq.

Note: If there is any external force F per unit length acting on the string then new Eq. will be

$$c^2 \mu_{tt} = c^2 \mu_{xx} + \frac{F}{\rho}$$

24-02-2016

Derive the wave Eqn for vibrating membrane.

① The membrane is flexible & elastic.

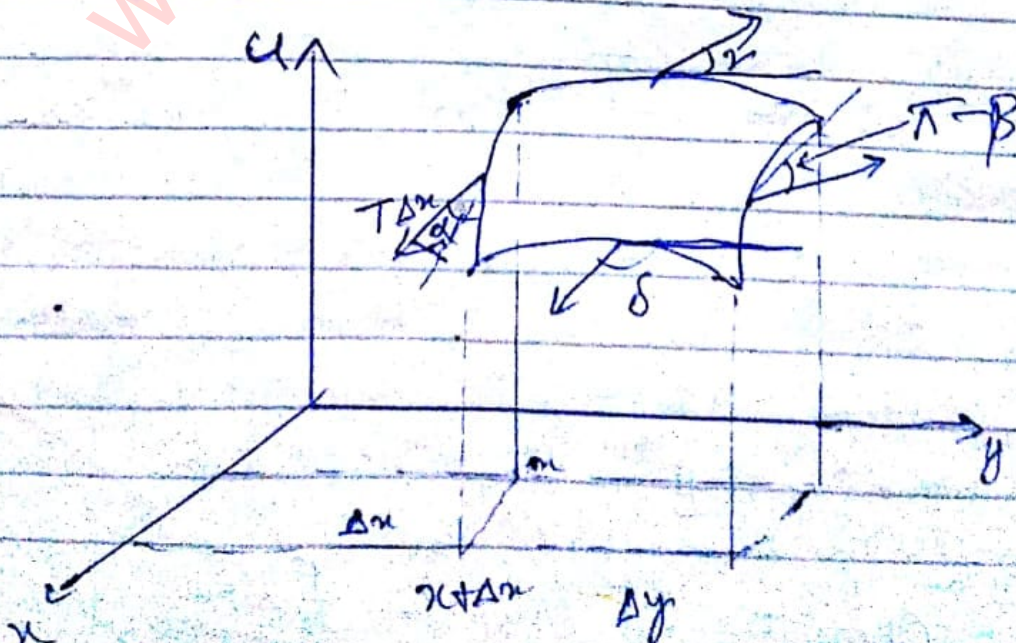
② The tension is constant throughout its length.

③ The weight is small as compare to its tension.

④ The deflection is small as compare to the

⑤ The slope of the displaced membrane at any point is small & compare with unity.

⑥ Only transverse vibrations.



$$\Delta A = \Delta x \cdot \Delta y$$

Let T be the tension in the membrane per unit length.

The forces acting on membrane vertically are

$$-T \Delta x \sin \alpha + T \Delta x \sin \beta + T \Delta y \sin \gamma + T \Delta y \sin \delta = \underbrace{\rho \Delta A}_{\text{mass}} U_{tt}$$

Angles are very small.

$$\text{So, } \sin \alpha \approx \tan \alpha$$

$$+T \Delta x [-\tan \beta - \tan \alpha]$$

$$+T \Delta y [\tan \delta - \tan \gamma] = \rho \Delta x \Delta y U_{tt}$$

$$\tan \alpha = U_y(x, y)$$

$$\tan \beta = U_y(x_1, y_1)$$

$$\tan \delta = U_x(x, y_1)$$

$$\tan \gamma = U_x(x, y)$$

$$\Rightarrow T \Delta x [u_y(x, y + \Delta y) - u_y(x, y)] \\ + T \Delta y [u_x(x + \Delta x, y) - u_x(x, y)] = \int \Delta x \Delta y u_{tt}$$

$$T \left[\frac{u_y(x, y + \Delta y) - u_y(x, y)}{\Delta y} \right]$$

$$+ T \left[\frac{u_x(x + \Delta x, y) - u_x(x, y)}{\Delta x} \right] = \int u_{tt}$$

$$T [u_{yy} + u_{xx}] = \int u_{tt}$$

$$u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt}$$

$$\text{where } \frac{1}{c^2} = \frac{f}{T}$$

$$\text{Eq. of Continuity} \Rightarrow c^2 = \frac{T}{f}$$

Objective:-

Derive a model. Eq. for traffic flow on a highway without exits and with one entrance and one leave.

Assumptions:-

(1). we assume that the highway is infinite.

8.5

2) we define the car density $f(x, t)$

$$f(x, t) = \frac{\text{No. of the cars on } [x, x+\Delta x]}{\Delta x}$$

where

Δx must be large compared to the car length.

3) - we assume that there are no accident on the highway. (if any then negligible).

④ They hold car conservation law.

The rate at which the no. of cars on the segment $[a, b]$ is changing equals the rate at which they are entering, the rate at which they are leaving.

Car flux is

$$q(x, t) = f(x, t) u(x, t)$$

where

u is car speed at x .

The number N of cars in segment at any time t .

$$n(t, a, b) = \int_a^b f(x, t) dx$$

The rate of change in the quantity is

$$\frac{dn}{dt} = \int_a^b \frac{\partial f}{\partial t} dx$$

This rate must be equal to

$$= q(a, t) - q(b, t)$$

$$= - \int_a^b \frac{\partial q}{\partial x} dx$$

$$\Rightarrow \int_a^b \left[\frac{\partial f}{\partial t} + \frac{\partial q}{\partial x} \right] dx = 0$$

$$\int_a^b \left[\frac{\partial f}{\partial t} + \frac{\partial (fu)}{\partial x} \right] dx = 0$$

$$\Rightarrow \frac{\partial f}{\partial t} + \frac{\partial (fu)}{\partial x} = 0$$

$$\frac{\partial f}{\partial t} + \text{div } \vec{J} = 0 \quad \text{f is charge density}$$

$$\vec{J} \rightarrow \text{current density}$$

Hibier man.

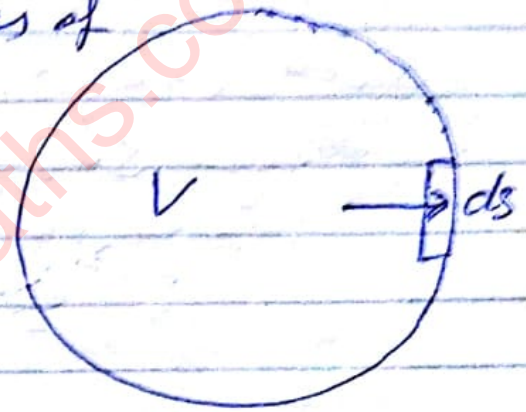
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Laplace & Poisson Eq.

Derive the Laplace & Poisson Eq. at

According to Gauss flux theorem in electrostatics "the total flux of the electrostatic intensity \vec{E} across the closed surface S is equal to four times of the total charge enclosed within the surface S ."



$$\int_S \vec{E} \cdot d\vec{s} = 4\pi \int_V \rho dv$$

where ρ is the volume density of the charge, S is surface that bounds volume V .

$$\int_V \text{div } \vec{E} dv = 4\pi \int_V \rho dv$$

$$\Rightarrow \int_V [\text{div } \vec{E} - 4\pi \rho] dv = 0$$

$$\Rightarrow \text{div } \vec{E} = 4\pi \rho$$

which is Poisson Eq

and

$$\vec{E} = -\nabla\phi$$

where ϕ is potential.

then

$$-\nabla^2\phi = 4\pi\rho \rightarrow \text{Poisson Eqn}$$

when there is no charge.

(free surface)

$$\nabla^2\phi = 0$$

03-03-2016

Method of Separation of variables:-

$$A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F u = 0$$

$$u(x, y) = X(x) Y(y)$$

$$u_{xx} + u_{yy} + u_{zz} = 0 \quad \text{--- (1)}$$

$$\text{Let } u(x, y, z) = X(x) Y(y) Z(z)$$

$$u_x = X' Y Z$$

$$u_{xx} = X'' Y Z$$

$$u_{yy} = X Y'' Z$$

$$u_{zz} = X Y Z''$$

Then

$$\text{(1)} \Rightarrow X'' Y Z + X Y'' Z + X Y Z'' = 0$$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0 \quad \text{assuming that } X \neq 0, Y \neq 0, Z \neq 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = -\frac{Z''}{Z}$$

How it is possible R.H.S is independent of x & y but L.H.S is dependent only when R.H.S is constant.

So, let

$$\frac{X''}{X} + \frac{Y''}{Y} = -\frac{Z''}{Z} = -\lambda^2$$

Separation constant.

$$\Rightarrow \frac{Z''}{Z} = \lambda^2$$

$$\& \frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2$$

$$Z'' - \lambda^2 Z = 0$$

$$D^2 - \lambda^2 = 0$$

$$D = \pm \lambda$$

$$Z = Ae^{\lambda z} + Be^{-\lambda z} = \begin{cases} e^{-\lambda z} \\ e^{\lambda z} \end{cases}$$

Now

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2$$

$$\Rightarrow \frac{X''}{X} = -\lambda^2 - \frac{Y''}{Y}$$

Now this is possible only

$$\frac{X''}{X} = -\lambda^2 - \frac{Y''}{Y} = -\mu^2$$

$$\Rightarrow \frac{X''}{X} = -\mu^2$$

$$\Rightarrow D^2 = \pm i\mu$$

$$X = \begin{cases} \sin \mu x \\ \cos \mu x \end{cases}$$

Now

$$-\frac{Y''}{Y} - \lambda^2 = -\mu^2$$

$$\frac{Y''}{Y} = \mu^2 - \lambda^2$$

Case I $\mu^2 > \lambda^2$, Case II $\mu^2 < \lambda^2$

$$D^2 = \mu^2 - \lambda^2$$

$$D = \pm \sqrt{\mu^2 - \lambda^2}$$

$$\Rightarrow Y = \begin{cases} e^{\sqrt{\mu^2 - \lambda^2} x} \\ e^{-\sqrt{\mu^2 - \lambda^2} x} \end{cases}$$

Result will be after putting values of X, Y & Z in $u(x, y, z) = X(x) Y(y) Z(z)$

Example:-

$$x^2 u_{xx} + x u_x + 9uy = 0$$

like Cauchy Ev.

$$\left\{ \begin{array}{l} x^2 \frac{d^2 y}{dx^2} + x \frac{d^2 y}{dx^2} + \dots + xy' + y = 0 \end{array} \right.$$

Let $u = xy$

then

$$x^2 x''y + x x'y + 9xy' = 0$$

Dividing by xy

$$x^2 \frac{x''}{x} + x \frac{x'}{x} + 9 \frac{y'}{y} = 0$$

$$x^2 \frac{x''}{x} + x \frac{x'}{x} = -9 \frac{y'}{y} = -\mu^2$$

$$9 \frac{y'}{y} = \mu^2$$

$$\Rightarrow D = \frac{\mu^2}{9}$$

Solution will be μ^2

$$y = e^{\frac{\mu^2}{9} x}$$

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$$x^2 \frac{x''}{x} + x \frac{x'}{x} = -\mu^2 \rightarrow \textcircled{A}$$

Put $x = e^t$

Now $x(x) = x(t)$

$$x' = \frac{dx}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{dx}{dt} \cdot \frac{1}{e^t} = \frac{1}{x} \frac{dx}{dt}$$

$$xx' = \dot{x} \Rightarrow \frac{d}{dx}(xx') = \frac{d}{dt}(\dot{x}) \cdot \frac{dt}{dx}$$

$$x' + xx'' = \ddot{x} \cdot \frac{1}{x}$$

$$xx' + x^2 x'' = \ddot{x}$$

$$\rightarrow x^2 x'' = \ddot{x} - x'$$

$$\text{Eq. } \textcircled{A} \Rightarrow \frac{\ddot{x}}{x} - \frac{x'}{x} + \frac{x'}{x} = -\mu^2$$

$$\frac{\ddot{x}}{x} = -\mu^2$$

$$D^2 x = -\mu^2$$

$$D = \pm i\mu$$

$$x = \begin{cases} \cos \mu t \\ \sin \mu t \end{cases} \Rightarrow x(x) = \begin{cases} \cos \mu \ln x \\ \sin \mu \ln x \end{cases}$$

Orthogonal & orthonormal set of functions."

A finite or infinite seq. of set of real valued functions

$U_n(x)$ ($n=1, 2, \dots$) Defined in $[a, b]$ is said to form an orthogonal set on $[a, b]$ if

$$\text{inner product} \Rightarrow \int_a^b U_n(x) U_m(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

where $L > 0$

If $L=1$ then set of function $U_n(x)$ is said to be orthonormal set.

$$\int_a^b U_n(x) U_m(x) dx = \delta_{mn}$$

also we can make an orthogonal set... an orthonormal set if we take

$$\Rightarrow \frac{U_n(x)}{\sqrt{L}} \Rightarrow \frac{1}{L} \int_a^b U_n(x) U_m(x) dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Some Results.

$$\textcircled{1} \int_{-\pi}^{\pi} \sin mx \cdot \sin nx dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

same for $\cos mx \cos nx$.

$$\textcircled{2} \int_{-\pi}^{\pi} \sin mx \cos nx dx = 0$$

for any choice of m & n .

$$\textcircled{3} \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \begin{cases} 0 & \text{if } m \neq n \\ L/2 & \text{if } m = n \end{cases}$$

Same for $\cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}$

$$\text{But } \int_0^L \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = 0$$

Remark on Eigenvalues & Eigenfunc.

Suppose we are given a set of Eov. involving a parameter λ .

And we required to find the value of λ for which the given set of Eq. have a non-trivial solution. Such type of solutions are called Eigenvalues solution.

For which the eigenvalue the function is called Eigen function.

Spectrum :-

The set of all eigenvalues of a problem is called a spectrum.

Example: Consider

$$\begin{array}{l} 5x + (1-\lambda)y = 0 \\ (1-\lambda)x + 5y = 0 \end{array} \quad \left| \begin{array}{l} Ax = 0 \\ A = \begin{bmatrix} 5 & 1-\lambda \\ 1-\lambda & 5 \end{bmatrix} \end{array} \right.$$

non-trivial solution

$$\text{If } \text{Det}[A - \lambda I] = 0 \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{So, } \begin{vmatrix} 5 & 1-\lambda \\ 1-\lambda & 5 \end{vmatrix} = 0$$

$$25 - (1-\lambda)^2 = 0$$

$$(1-\lambda)^2 = 25$$

$$1-\lambda = \pm 5$$

$$1-\lambda = 5$$

$$\boxed{-4 = \lambda}$$

$$1-\lambda = -5$$

$$\boxed{6 = \lambda}$$

These ^{are} Eigen values of the problem

$$\text{Spectrum} = \{-4, 6\}$$

When $\lambda = -4$

$$5x + 5y = 0$$

$$5x + 5y = 0$$

$$\Rightarrow \boxed{x = -y} \leftarrow \text{infinite solutions}$$

when $\lambda = 6$

$$5x - 5y = 0$$

$$-5x + 5y = 0$$

$$\Rightarrow \boxed{x = y}$$

Questions:

$$U_{xx} = \frac{1}{c^2} U_{tt}$$

$$U_{xx} - \frac{1}{c^2} U_{tt} = 0$$

Let

$$U(x, t) = X(x)T(t)$$

then

$$X''T - \frac{1}{c^2} XT'' = 0$$

$$\frac{X''}{X} - \frac{1}{c^2} \frac{T''}{T} = 0$$

$$\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = d^2$$

$$\Rightarrow \frac{X''}{X} = d^2$$

$$D^2 = d^2$$

$$D = \pm d$$

$$\Rightarrow X(x) = \begin{cases} e^{dx} \\ e^{-dx} \end{cases}$$

$$\frac{1}{c^2} \frac{T''}{T} = d^2$$

$$D^2 = c^2 d^2$$

$$D = \pm cd$$

$$T(t) = \begin{cases} e^{cdt} \\ e^{-cdt} \end{cases}$$

Put in $U(x, t)$.

Question:-

$$U_{xx} + U_{yy} + U_{xy} = 0$$

Let $U(x, y) = X(x) Y(y)$

Then $U_x = X' Y$ $U_{xy} = X' Y'$

$$X'' Y + X Y'' + X' Y' = 0$$

Dividing by $X Y$.

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{X' Y'}{X Y} = 0$$

Diff. w.r.t x .

$$\left(\frac{X''}{X}\right)' + 0 + \frac{Y'}{Y} \left(\frac{X'}{X}\right)' = 0$$

$$\left(\frac{X''}{X}\right)' = -\frac{Y'}{Y} \left(\frac{X'}{X}\right)'$$

$$\Rightarrow \left(\frac{X''}{X}\right)' / \left(\frac{X'}{X}\right)' = -\frac{Y'}{Y} = -\lambda^2$$

$$\Rightarrow \frac{\left(\frac{X''}{X}\right)'}{\left(\frac{X'}{X}\right)'} = -\lambda^2, \quad -\frac{Y'}{Y} = -\lambda^2$$

Solution will be.

$$y' = -\lambda^2 y \Rightarrow y(x) = c_1 e^{-\lambda^2 x}$$

Now

$$\frac{x''}{x} + \frac{y''}{y} + \frac{x'}{x} \cdot \frac{y'}{y} = 0 \quad (*)$$

Put $\frac{y'}{y} = -\lambda^2$

& $y' - \lambda^2 y = 0$

$$\Rightarrow y'' - \lambda^2 y' = 0 \Rightarrow y'' - \lambda^4 y = 0$$

$$\Rightarrow \frac{y''}{y} = \lambda^4$$

$$(*) \Rightarrow \frac{x''}{x} + \lambda^4 + \frac{x'}{x} \lambda^2 = 0$$

$$m^2 + m\lambda^2 + \lambda^4 = 0$$

$$m = \frac{-\lambda^2 \pm \sqrt{\lambda^4 - 4\lambda^4}}{2} = \frac{-\lambda^2 \pm \sqrt{3}\lambda^2}{2}$$

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07-03-2016

show that the eigen functions of the problem form an orthogonal set and then make it orthonormal.

$$\frac{d^2 u}{dx^2} + \lambda u = 0$$

$$u(0) = 0, \quad u(l) = 0$$

$$D^2 = -\lambda$$

$$D = \pm \sqrt{\lambda}$$

$$u = A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x$$

since $u(0) = 0$

$$\Rightarrow B = 0 \quad \text{then } u = A \sin \sqrt{\lambda} x$$

when $u(l) = 0$

$$\Rightarrow A \sin \sqrt{\lambda} l = 0$$

we take $A \neq 0$

then

$$\sin \sqrt{\lambda} l = 0$$

$$\Rightarrow \lambda_n = \left(\frac{n\pi}{l} \right)^2, \quad n = 1, 2, \dots$$

$$u_n = A_n \sin \frac{n\pi}{l} x$$

$$\int_0^l u_n u_m dx = \int_0^l A_n A_m \sin \frac{n\pi}{l} x \sin \frac{m\pi}{l} x dx$$

$$= \begin{cases} 0 & \text{if } m \neq n \\ \frac{l}{2} A_n^2 & \text{if } m = n \end{cases}$$

So, u_n is orthogonal set-

Now

$$u_n = \frac{\sin \frac{n\pi}{l} x}{\sqrt{l/2}} \text{ will be orthonormal.}$$

Example: Solve String Problem.

$$u_{xx} = \frac{1}{c^2} u_{tt} \quad 0 < x < l$$

$$\left. \begin{aligned} u(0, t) &= 0 \\ u(l, t) &= 0 \end{aligned} \right\} t \geq 0$$

$$u(x, 0) = f(x)$$

$$\dot{u}(x, 0) = 0 \quad 0 \leq x \leq l$$

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Sol: Let $u = XT$

$$X''T = \frac{1}{c^2} X \ddot{T}$$

$\div XT$

$$\frac{X''}{X} = \frac{1}{c^2} \frac{\ddot{T}}{T} = \lambda \quad \swarrow \text{Let}$$

$$\frac{X''}{X} = \lambda$$

$$D^2 = \lambda$$

$$D = \pm \sqrt{\lambda}$$

Case I if $\lambda = 0$

$$\frac{X''}{X} = 0$$

$$D^2 = 0$$

$$u(0,t) = 0 \Rightarrow X = A + Bx$$

$$X(0) = 0$$

$$\Rightarrow A = 0$$

$$\text{then } X = Bx$$

$$\text{using } X(l) = 0$$

$$\Rightarrow lB = 0$$

$$\Rightarrow B = 0$$

Trivial solutions

other case \Rightarrow If $\lambda = \mu^2$

$$\frac{X''}{X} = \mu^2$$

$$D^2 = \mu^2$$

$$\Rightarrow D = \pm \mu$$

$$X = A e^{\mu x} + B e^{-\mu x}$$

using $X(0) = 0$

$$\Rightarrow \boxed{0 = A + B} \Rightarrow \boxed{A = -B}$$

then using

$$X'(0) = 0$$

$$\Rightarrow A e^{\mu l} + B e^{-\mu l} = 0$$

$$-B e^{\mu l} + B e^{-\mu l} = 0$$

$$-B \left[\frac{e^{\mu l} - e^{-\mu l}}{2} \right] = 0$$

always

$$\Rightarrow \frac{e^{\mu l} - e^{-\mu l}}{2} \neq 0 \text{ thus } B = 0$$

$$\Rightarrow A = 0$$

Again Trivial solution.

now

$$\text{take, } \lambda = -\mu^2$$

$$\Rightarrow X = A \sin \mu x + B \cos \mu x$$

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Using $X(0) = 0$

$$\Rightarrow \boxed{B=0}$$

Now using $X(l) = 0$ we get

$$X_n = A_n \sin\left(\frac{n\pi}{l}x\right)$$

$$\mu = \frac{n\pi}{l}$$

Now

$$\frac{\ddot{T}}{T} = -c^2 \mu^2$$

$$D^2 = -c^2 \mu^2$$

$$D = \pm i c \mu$$

$$T = F \sin c \mu t + D \cos c \mu t$$

$$\dot{T} = c \mu F \cos c \mu t - \mu c D \sin c \mu t$$

Using $\dot{T}(0) = 0$

$$\Rightarrow \boxed{F=0}$$

$$\text{Then } T = D \cos c \mu t$$

$$u = A_n \sin \frac{n\pi}{l} x \cos(c \mu t)$$

$$u = \sum_1^{\infty} A_n \sin \frac{n\pi}{l} x \cos(c \mu t) \longrightarrow \textcircled{A}$$

where $\boxed{n \neq 0}$.

Note Using

$$u(x, 0) = \phi(x) \quad \text{in (A)}$$

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right)$$

Using $\sin \frac{m\pi}{l}x$

$$\phi(x) \sin \frac{m\pi}{l}x = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{l}x \sin \frac{m\pi}{l}x$$

$$\int_0^l \phi(x) \sin \frac{m\pi}{l}x dx = \int_0^l \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{l}x \sin \frac{m\pi}{l}x dx$$

$$= \sum_{n=1}^{\infty} A_n \delta_{mn} \left(\frac{l}{2}\right)$$

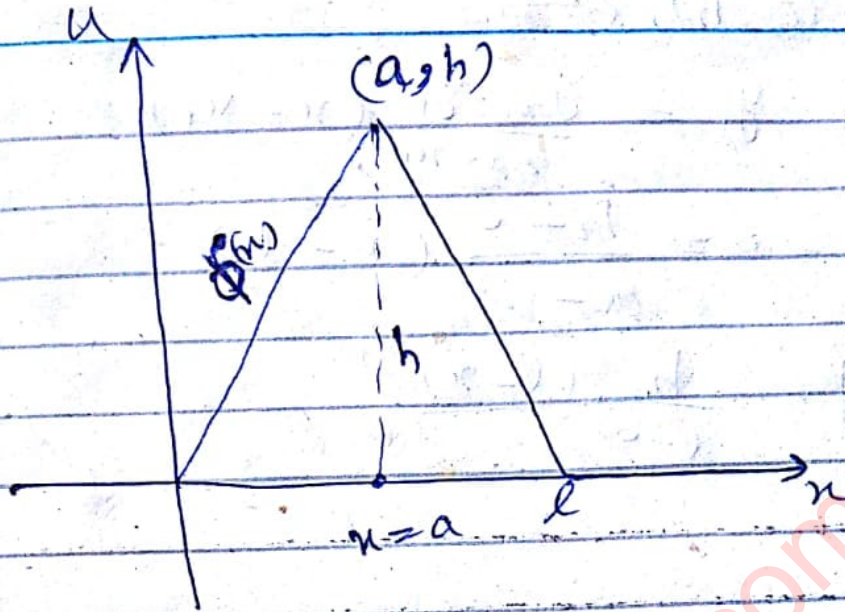
$$= \frac{l}{2} A_{mn}$$

put A_{mn} in (A)

$$u = \sum_{n=1}^{\infty} \left\{ \int_0^l \phi(x) \sin \frac{n\pi}{l}x dx \right\} \sin \frac{n\pi}{l}x \cos(\mu t)$$

$$u(x, 0) = 0$$

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Suppose the string is raised to a height h at $x=a$ and then released. The string will oscillate.

$$u(x,0) = \begin{cases} \frac{hx}{a} & 0 \leq x \leq a \\ \frac{h(l-x)}{l-a} & a \leq x \leq l \end{cases}$$

$$A_n = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi}{l} x dx$$

For $0 \leq x \leq a$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$y - 0 = \frac{h - 0}{a - 0} (x - 0)$$

$$y = \frac{h}{a} x$$

$(l, 0)$, (a, h) for $a \leq x \leq l$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$y - 0 = \frac{h - 0}{a - l} (x - l)$$

$$y = \frac{h \cdot (l - x)}{l - a}$$

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Theorem:-

10-03-2015

Let $f(x)$ be piecewise smooth in $[-\pi, \pi]$. $f(x)$ is periodic upto period 2π , then the Fourier series converges uniformly to 'f' in every closed interval containing no discontinuity.

Theorem:- (Uniqueness Theorem)

There exist at most one solution of the wave eq,

$$u_{xx} = \frac{1}{c^2} u_{tt} \quad 0 < x < l$$

$$t > 0$$

I.C.s

$$\left. \begin{aligned} u(x, 0) &= f(x) \\ u_t(x, 0) &= g(x) \end{aligned} \right\} 0 \leq x \leq l$$

B.C.s

$$\begin{aligned} u(0, t) &= 0 \\ u(l, t) &= 0 \end{aligned}$$

where u is twice continuously differentiable function both x & t .

Proof:- Suppose there ^{are} 2 solutions u_1 & u_2 .

$$V = u_1 - u_2 \longrightarrow \textcircled{1}$$

then

$$u_{1tt} = c^2 u_{1xx}$$

subtract and $u_{2tt} = c^2 u_{2xx}$

use ① $\Rightarrow v_{tt} = c^2 v_{xx} \longrightarrow$ ②

then $v(x,0) = 0$, $t \geq 0$
 $v_x(x,0) = 0$
 $v(x,l) = 0$
 $v_t(x,0) = 0$

energy $\rightarrow I(t) = \frac{1}{2} \int_0^l [c^2 v_x^2 + v_t^2] dx \rightarrow$ ③

which physically represents the total energy of the vibrating string at any time "t".

$$\Rightarrow \frac{dI}{dt} = \frac{1}{2} \int_0^l [c^2 2 v_x v_{xt} + 2 v_t v_{tt}] dx$$

$$= c^2 \int_0^l v_x v_{xt} dx + \int_0^l v_t v_{tt} dx$$

Integrating by parts.

$$= c^2 [v_x(l,t) v_t(l,t) - v_x(0,t) v_t(0,t)] + c^2 \int_0^l v_{xx} v_t dx + (11)$$

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Using the given conditions, we have

$$= -c^2 \int_0^l v_{xx} v_t dx + \int_0^l v_t v_{tt} dx$$

$$= \int_0^l (c^2 v_{xx} + v_{tt}) v_t dx$$

from (2)

$$= 0 \Rightarrow \frac{dI}{dt} = 0$$

$$\Rightarrow I(t) = \text{constant} = K$$

$$I(0) = \frac{1}{2} \int_0^l [c^2 v_x^2(x, 0) + v_t^2(x, 0)] dx$$

$$= 0 \rightarrow \textcircled{5}$$

$$\begin{aligned} I(t) &= K \\ I(0) &= 0 \\ \Rightarrow K &= 0 \end{aligned}$$

from (3), (4) & (5)

$I(t) = 0$ So "v" is zero

hence $u_1 = u_2$

$$\begin{aligned} v_t^2 &= 0 \\ \Rightarrow v &= k \\ \therefore c \Rightarrow (v=0) \end{aligned}$$

Example:- Heat-conduction Problem

$$u_{xx} = \alpha^2 u_t$$

$$u(0, t) = 0, \quad u(l, t) = \bar{T}_0, \quad u(x, 0) = \phi(x)$$

This problem is non-homogeneous.

$$\begin{aligned} \text{Let } w_{xx} &= r \\ w_x &= c_1 \\ w &= c_1 x + c_2 \\ w(0) = 0 &\Rightarrow c_2 = 0 \\ w(l) = \frac{T_0}{l} &\Rightarrow c_1 = \frac{T_0}{l} \\ \text{So, } w(x) &= \frac{T_0}{l} x \end{aligned}$$

$$\text{Let } V(x,t) = U(x,t) - w(x)$$

$$U(x,t) = V(x,t) + \frac{T_0 x}{l}$$

$$U_{xx} = V_{xx} + 0$$

$$U_t = V_t$$

$$V_{xx} = d^2 V_t$$

$$V(0,t) = 0$$

$$V(l,t) = 0$$

$$V(x,0) = \phi(x) - \frac{T_0 \cdot x}{l}$$

The result will be for $V(x,t)$

$$V(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{l} \left(\phi\left(\frac{n\pi x}{l}\right) - \frac{T_0 \cdot \frac{n\pi x}{l}}{l} \right) \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2}{l^2} t} \right] \sin \frac{n\pi x}{l}$$

$$V(x,t) = T(t) X(x)$$

$$V_{xx} = T X''$$

$$V_t = \dot{T} X$$

$$V_{xx} = d^2 V_t \Rightarrow T X'' = d^2 \dot{T} X$$

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$$\Rightarrow \frac{X''}{X} = \frac{T''}{T} = \alpha^2 = -\mu^2$$

$$\alpha^2 \frac{T''}{T} = -\mu^2$$

$$\frac{T''}{T} = -\frac{\mu^2}{\alpha^2}$$

$$D = \pm i \frac{\mu}{\alpha}$$

$$T(t) = A e^{-\frac{\mu^2 t}{\alpha^2}}$$

$$\frac{X''}{X} = -\mu^2$$

$$D^2 = -\mu^2$$

$$D = \pm i \mu$$

$$X(x,t) = E \sin \mu x + F \cos \mu x$$

$$X(0,t) = 0 = 0 + F(1)$$

$$\Rightarrow \boxed{F=0}$$

$$X(x,t) = E \sin \mu x$$

$$X(x,t) = 0 = E \sin(\mu x)$$

$$\sin(\mu x) = 0 \therefore E \neq 0$$

$$\Rightarrow \mu = \frac{n\pi}{l}, \quad n=1, 2, 3, \dots$$

$$V(x,t) = A e^{-\frac{\mu^2 t}{a^2}} \sin\left(\frac{n\pi x}{l}\right)$$

$$V(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} e^{-\frac{\mu^2 t}{a^2}}$$

Inner product

$$\int_0^l \sin \frac{n\pi x}{l} V(x,0) dx = \int_0^l \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

$$\sin \frac{n\pi x}{l} e^{-\frac{\mu^2 t}{a^2}} dx$$

$$\int_0^l \sin \frac{n\pi x}{l} \left(\phi_0 x - \frac{T_0 x}{l}\right) dx = \int_0^l \sum_{n=1}^{\infty} A_n$$

$$\sin \frac{n\pi x}{l} \sin \frac{n\pi x}{l} dx$$

$$= \frac{l}{2} A_n \quad m=n$$

$$\Rightarrow A_n = \frac{2}{l} \int_0^l \sin \frac{n\pi x}{l} \left(\phi_0 x - \frac{T_0 x}{l}\right) dx$$

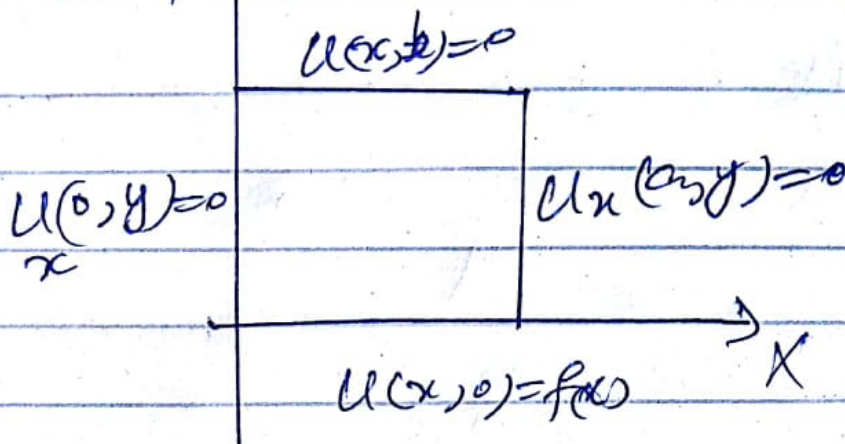
$$\text{So, } V(x,t) = \frac{2}{l} \sum_{n=1}^{\infty} \int_0^l \sin \frac{n\pi x}{l} \left(\phi_0 x - \frac{T_0 x}{l}\right) dx e^{-\frac{\mu^2 t}{a^2}}$$

$$\text{So, } u(x,t) = V(x,t) + \frac{T_0 x}{l} \sin \frac{n\pi x}{l}$$

$$\Rightarrow u(x,t) = \frac{T_0 x}{l} + \frac{2}{l} \sum_{n=1}^{\infty} \int_0^l \sin \frac{n\pi x}{l} \left(\phi_0 x - \frac{T_0 x}{l}\right) dx \sin \frac{n\pi x}{l} e^{-\frac{\mu^2 t}{a^2}}$$

symmetric $\Rightarrow f(x) = f(-x)$ (87)
 Anti-symmetric $\Rightarrow f(x) = -f(-x)$

Question:- $Y \uparrow$



The governing Eq. will be

$$\nabla^2 u = 0$$

$$u(x, 0) = f(x)$$

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Example: (A complete non-homog.) 14. 03-2016

$$u_{tt} = c^2 u_{xx} + F(x)$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x)$$

$$u(0, t) = A, \quad u(l, t) = B$$

Assume that

$$v(x, t) = u(x, t) - W(x)$$

$$u_{xx} = v_{xx} + W_{xx}$$

$$\& \quad u_{tt} = v_{tt}$$

then

$$v_{tt} = c^2 (v_{xx} + W_{xx}) + F(x)$$

$$= c^2 v_{xx} + \boxed{c^2 W_{xx} + F(x)}$$

$$\text{let } c^2 W_{xx} + F(x) = 0$$

then

$$u(x, 0) = f(x)$$

$$\Rightarrow u(x, 0) = f(x) = v(x, 0) + W(x)$$

$$v(x, 0) = f(x) - W(x)$$

$$u_t(x, 0) = g(x) = v_t(x, 0)$$

$$u(0, t) = A = v(0, t) + W(0)$$

$$\Rightarrow v(0, t) = A - W(0)$$

Hyberman \Rightarrow chap # 8

Similarly

$$B = V(l, t) + w(l)$$

$$\Rightarrow V(l, t) = B - w(l)$$

Assume that
 $C \cdot w_{xx} + f(x) = 0$

$$w(0) = A$$

$$w(l) = B$$

then

$$V_{tt} = C^2 V_{xxx}$$

$$V(x, 0) = f(x) - w(x)$$

$$V_t(x, 0) = g(x)$$

$$V(0, t) = 0$$

$$V(l, t) = 0$$

DYS Solution will be of Eq. $w_{xx} = -\frac{1}{C^2} f(x)$

$$w(x) = A + \frac{(B-A)x}{l} + \frac{x^2}{2} \int_0^{\eta} \left[\frac{1}{C^2} \int_0^{\xi} f(\xi) d\xi \right] d\xi$$

$$- \int_0^x \left[\frac{1}{C^2} \int_0^{\eta} f(\xi) d\xi \right] d\eta$$

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Example:-

$$u_{tt} = c^2 u_{xx} + x$$

$$u(x, 0) = 0, \quad u_t(x, 0) = 0$$

$$u(0, t) = 0 = u(l, t)$$

Let $v(x, t) = u(x, t) - w(x)$
 $u = v + w$

$$c^2 w_{xx} + x = 0$$

Integrating $w_{xx} = -\frac{x}{c^2}$
 $w_x = -\frac{x^2}{2c^2} + A_1$

$$w = -\frac{x^3}{6c^2} + A_1 x + B_1$$

Using $w(0) = 0$

$$\Rightarrow B_1 = 0$$

Using $w(l) = 0$

$$\Rightarrow 0 = -\frac{l^3}{6c^2} + A_1 l$$

$$\boxed{A_1 = \frac{l^2}{6c^2}}$$

So, $w = -\frac{x^3}{6c^2} + \frac{l^2}{6c^2} x = \frac{x}{6c^2} (l^2 - x^2)$

$$v(x, t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi c t}{l} \sin \frac{n\pi x}{l}$$

$$v(x, 0) = -w(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

$$V(x,t) = -\frac{2}{l} \sum_{n=1}^{\infty} \int_0^l \frac{x}{6c^2} (l-x^2) \sin \frac{n\pi x}{l} dx \cos \frac{n\pi}{l} ct \sin \frac{n\pi x}{l}$$

$$A_n = -\frac{2}{l} \int_0^l w(x) \sin \frac{n\pi x}{l} dx$$

$$\text{So, } u(x,t) = \frac{x}{6c^2} (l-x^2) - \frac{2}{l} \sum_{n=1}^{\infty} \int_0^l \frac{x}{6c^2} (l-x^2) \sin \frac{n\pi x}{l} dx \cos \frac{n\pi}{l} ct \sin \frac{n\pi x}{l}$$

Related to Question in previous lecture.

A Laplace Eq.

$$\frac{\partial u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(0, y) = g_1(y)$$

$$u(L, y) = g_2(y)$$

$$u(x, 0) = f_1(x)$$

$$u(x, H) = f_2(x)$$

$$\nabla^2 u = 0$$

$$u = f_1$$

$$u = g_2$$

$$u = f_1$$

$$u = g_2$$

$$0 < x < L$$

$$0 < y < H$$

$$\frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0$$

$$u_4(0, y) = g_1(y)$$

$$u_4(L, y) = 0$$

$$u_4(x, 0) = 0$$

$$u_4(x, H) = 0$$

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$$u_4 = X(x) Y(y)$$

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda \text{ or } -\lambda$$

If the separation constant is " $-\lambda$ " then $X(x)$ oscillates & $Y(y)$ is composed of exponential function.

This seems doubtful since $Y(0) = 0$ & $Y(H) = 0$ exponentials in Y are not expected to work.

If we take " $+\lambda$ " then $X(x)$ is exponential & $Y(y)$ oscillates.

Example:

24-03-2016

Let us consider the problem of a finite string within external force acting on it, if ends are fixed then we have

$$u_{tt} - c^2 u_{xx} = f(x,t) \quad \text{--- (1)}$$

$$0 < x < l \\ t > 0$$

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

$$u(0,t) = 0$$

$$u(l,t) = 0$$

We assume the solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{l} \quad \text{--- (2)}$$

$$u_{tt} - c^2 u_{xx} = 0$$

where $u_n(t)$ are to be determined.

$$c^2 \frac{d^2 X}{dx^2} + \lambda^2 X = 0$$

$$\Rightarrow D^2 = -\frac{\lambda^2}{c^2} \Rightarrow D = \pm \frac{\lambda}{c} i$$

We get $u_n(x) = A \sin \frac{\lambda}{c} x + B \cos \frac{\lambda}{c} x$

$$u_n(x) = \sin \frac{n\pi x}{l} \quad \boxed{B=0} \\ \text{where } \frac{\lambda}{c} = \frac{n\pi}{l}$$

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Also assuming,

$$h(x, t) = \sum_{n=1}^{\infty} h_n(t) \sin \frac{n\pi x}{l}$$

Since (2) is solution of (1).

$$\text{from (1)} \Rightarrow u_{tt} = \sum_{n=1}^{\infty} U_n(t) \frac{\sin \frac{n\pi x}{l}}{tt}$$

$$\& \quad u_{xx} = - \sum_{n=1}^{\infty} U_n(t) \left(\frac{n\pi}{l}\right)^2 \sin \frac{n\pi x}{l}$$

$$\begin{aligned} \sum_{n=1}^{\infty} U_n(t) \left(\frac{n\pi}{l}\right)^2 \sin \frac{n\pi x}{l} + \sum_{n=1}^{\infty} U_{ntt} \sin \frac{n\pi x}{l} \\ = \sum_{n=1}^{\infty} h_n(t) \sin \frac{n\pi x}{l} \end{aligned}$$

$$\text{using } \lambda_n = \frac{n\pi}{l} c$$

$$\sum_{n=1}^{\infty} \left[U_n''(t) + \lambda_n^2 U_n(t) \right] \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} h_n(t) \sin \frac{n\pi x}{l}$$

$$\begin{aligned} \text{xing } \left(\sin \frac{n\pi x}{l} \right) \sum_{n=1}^{\infty} \left[U_n''(t) + \lambda_n^2 U_n(t) \right] \int_0^l \sin^2 \frac{n\pi x}{l} dx &= \sum_{n=1}^{\infty} h_n(t) \int_0^l \sin^2 \frac{n\pi x}{l} dx \\ &= h_n(t) \frac{l}{2} \end{aligned}$$

$$\Rightarrow h_n(t) = U_n''(t) + \lambda_n^2 U_n(t)$$

$$U_n(t) = U_{nc}(t) + U_{np}(t)$$

$$D^2 = -\lambda_n^2$$

$$D = \pm i\lambda_n$$

$$(U_n(t))_c = A \cos \lambda_n t + B \sin \lambda_n t$$

$$= A y_1 + B y_2$$

or

$$(U_n(t))_p = V_1(t) y_1 + V_2(t) y_2$$

$$(U_n(t))_p = V_1(t) \cos \lambda_n t + V_2(t) \sin \lambda_n t$$

$$V_{1n}'(t) = \frac{y_2 R}{\omega} = \frac{\sin \lambda_n t h_n(t)}{\lambda_n}$$

R.H.S of previous Eq.

where

$$\omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos \lambda_n t & \sin \lambda_n t \\ -\lambda_n \sin \lambda_n t & \lambda_n \cos \lambda_n t \end{vmatrix} = \lambda_n$$

($\because \lambda_n \neq 0$)

ω Wronskian determinant

$$\Rightarrow V_{1n}(t) = -\frac{1}{\lambda_n} \int h_n(t) \sin \lambda_n t dt$$

$$V_{2n}'(t) = \frac{y_1 R}{\omega} = \frac{\cos \lambda_n t h_n(t)}{\lambda_n}$$

$$\Rightarrow V_{2n}(t) = \frac{1}{\lambda_n} \int \cos \lambda_n t h_n(t) dt$$

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$$\begin{aligned}
 \Rightarrow (U_n(t)) &= -\frac{1}{\Delta_n} \left[\int_0^t h_n(t) \sin \Delta_n t dt \right] \cos(\Delta_n t) \\
 &+ \frac{1}{\Delta_n} \left[\int_0^t h_n(t) \cos \Delta_n t dt \right] \sin(\Delta_n t) \\
 &= \frac{1}{\Delta_n} \left[\int_0^t h_n(\xi) \cos \Delta_n \xi d\xi \right] \sin \Delta_n t \\
 &- \left[\int_0^t h_n(\xi) \sin \Delta_n \xi d\xi \right] \cos \Delta_n t \\
 &= \frac{1}{\Delta_n} \int_0^t h_n(\xi) [\cos \Delta_n \xi \sin \Delta_n t - \sin \Delta_n \xi \cos \Delta_n t] d\xi \\
 &= \frac{1}{\Delta_n} \int_0^t h_n(\xi) \sin \Delta_n (t - \xi) d\xi
 \end{aligned}$$

So,

$$\begin{aligned}
 \textcircled{2} \Rightarrow \therefore U(x, t) &= \sum_{n=1}^{\infty} \left[A_n \cos \Delta_n t + B_n \sin \Delta_n t \right] \\
 &+ \frac{1}{\Delta_n} \int_0^t h_n(\xi) \sin \Delta_n (t - \xi) d\xi \sin \frac{n\pi x}{l}
 \end{aligned}$$

Using $U(x, 0) = f(x)$ -

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

③

By inner product.

$$A_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$B_n = \frac{2}{l} \int_0^l g(x) \sin \frac{n\pi x}{l} dx$$

Example

Now we will consider the I-BVP with the time dependent boundary conditions,

$$u_t - C^2 u_{xx} = h(x,t)$$

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

$$u(0,t) = p(t)$$

$$u(l,t) = q(t)$$

Solution.

We assume the solution in the form,

$$v(x,t) = u(x,t) - s(x,t)$$

$$u(x,t) = v(x,t) + s(x,t)$$

$$v_t - C^2 v_{xx} = h + C^2 s_{xx} - s_{tt}$$

$$u(x,0) = f(x)$$

$$\Rightarrow v(x,0) = f(x) - s(x,0)$$

$$u_t(x,0) = g(x)$$

$$\Rightarrow v_t(x,0) = g(x) - s_t(x,0)$$

$$u(0, t) = p(t)$$

$$\Rightarrow v(0, t) = p(t) - S(0, t)$$

$$u(l, t) = q(t)$$

$$\Rightarrow v(l, t) = q(t) - S(l, t)$$

For homogeneous: -

$$\text{let } h + c^2 S_{xx} - S_{tt} = 0$$

take $S(0, t) = p(t)$ ← using $v(0, t) = 0$
 & $S(l, t) = q(t)$ ← $v(l, t) = 0$

Thus

$$S(x, t) = p(t) + \frac{x}{l} [q(t) - p(t)]$$

↳ this satisfy the upper conditions.

let $S_{xx} = 0$ implies

Now to find $v(x, t)$.

$$v_{tt} - c^2 v_{xx} = h - S_{tt} = H(x, t) \quad \text{assume}$$

$$v(x, 0) = f(x) - S(x, 0) = F(x)$$

$$v_t(x, 0) = g(x) - S_t(x, 0) = G(x)$$

$$v(0, t) = 0, \quad v(l, t) = 0$$

Now this problem can be treated same as the previous example.

DYS
Example:-

$$u_{tt} - u_{xx} = h \uparrow \text{constant}$$

$$u(x, 0) = x(1-x)$$

$$u_t(x, 0) = 0$$

$$u(0, t) = t$$

$$u(1, t) = \sin t$$

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Bar Problem:-

28-03-2016

$$u_t = k u_{xx}$$

$$u(0, t) = A$$

$$u(l, t) = B$$

$$u(x, 0) = f(x)$$

at Equilibrium "Temp." $u_E(x) = 0$

$$u_E(0) = A$$

$$u_E(l) = B$$

$$u_E(x) = A + \frac{(B-A)x}{l}$$

Let $v(x, t) = u(x, t) - u_E(x) \rightarrow (*)$

$$v_t(x, t) = k v_{xx}$$

$$v(0, t) = 0$$

$$v(l, t) = 0$$

$$v(x, 0) = f(x) - u_E(x)$$

Solution will be $v(x, t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x e^{-k \left(\frac{n\pi}{l}\right)^2 t}$

Where $a_n = \frac{2}{l} \int_0^l [f(x) - u_E(x)] \sin \frac{n\pi}{l} x dx$

as $t \rightarrow \infty$ then $u \rightarrow u_E$
 (using $(*)$)

Example:-

$$u_t = k u_{xx} + Q(x, t) \rightarrow \textcircled{1}$$

$$u(0, t) = A(t), \quad u(l, t) = B(t), \quad u(x, 0) = f(x)$$

Let

$$v(x, t) = u(x, t) - r(x, t)$$

$$r(x, t) = A(t) + \frac{x}{l} (B(t) - A(t))$$

$$r(0, t) = A(t)$$

$$r(l, t) = B(t)$$

$$v_t = k v_{xx} + \bar{Q}(x, t) \rightarrow \textcircled{2}$$

$$\bar{Q}(x, t) = Q(x, t) - r_t + k r_{xx}$$

$$v(0, t) = 0$$

$$v(l, t) = 0$$

$$v(x, 0) = f(x) - r(x, 0)$$

$$= f(x) - \left[A(0) + \frac{x}{l} (B(0) - A(0)) \right] \stackrel{\text{Let}}{=} g(x)$$

The related homogeneous problem is

$$u_t = k u_{xx}$$

$$u(0, t) = 0$$

$$u(l, t) = 0$$

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$$\frac{d^2 \phi}{dx^2} + k \phi = 0$$

$$\phi(0) = 0$$

$$\phi(l) = 0$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, n=1, 2, \dots$$

$$\phi_n(x) = \sin \frac{n\pi x}{l}$$

$$V(x,t) = \sum_{n=1}^{\infty} a_n(t) \phi_n(x)$$

$$V(x,0) = g(x)$$

$$g(x) = \sum_{n=1}^{\infty} a_n(0) \phi_n(x)$$

$$a_n(0) = \frac{\int_0^l g(x) \phi_n(x) dx}{\int_0^l \phi_n^2(x) dx}$$

$$V_t = \sum_{n=1}^{\infty} \frac{d}{dt} a_n(t) \phi_n(x)$$

$$V_{xx} = \sum_{n=1}^{\infty} a_n(t) \frac{d^2 \phi_n}{dx^2}$$

$$\textcircled{2} \Rightarrow \sum_{n=1}^{\infty} \left[\frac{d}{dt} a_n(t) + K a_n(t) \right] \phi_n$$

$$= \dots \bar{Q}$$

$$\begin{aligned} \frac{d}{dt} a_n(t) + K \lambda_n a_n & \\ &= \frac{\int_0^l \phi_n \bar{Q} dx}{\int_0^l \phi_n^2 dx} = \bar{V}_n(t) \end{aligned}$$

$$\bar{Q}(x,t) = \sum_{n=1}^{\infty} \bar{V}_n(t) \phi_n(x)$$

$$\frac{d}{dt} a_n(t) + K \lambda_n a_n = \bar{V}_n(t)$$

A linear Eq.

Solution will be;

$$e^{\lambda_n K t} \left(\frac{d a_n}{dt} + \lambda_n K a_n \right) = \bar{V}_n e^{\lambda_n K t}$$

$$a_n(t) e^{\lambda_n K t} - a_n(0) = \int_0^t \bar{V}_n(\tau) e^{\lambda_n K \tau} d\tau$$

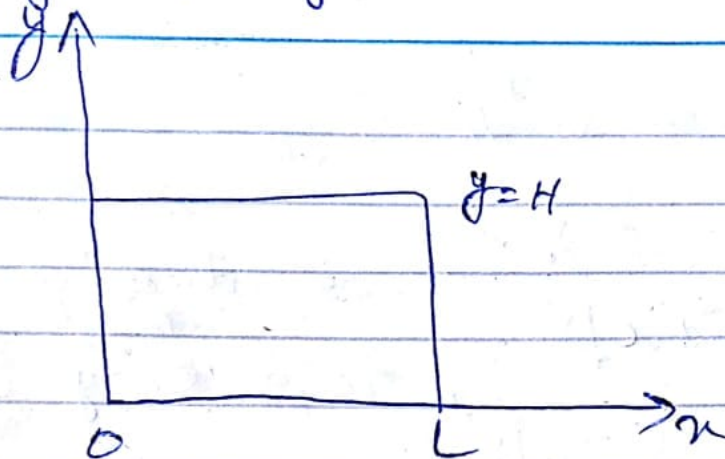
$$a_n(t) = e^{-\lambda_n K t} \left[\frac{\int_0^l g(x) \phi_n(x) dx}{\int_0^l \phi_n^2(x) dx} + \int_0^t \bar{V}_n(\tau) e^{\lambda_n K \tau} d\tau \right]$$

So,

$$V(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n K t} \left[\frac{\int_0^l g(x) \phi_n(x) dx}{\int_0^l \phi_n^2(x) dx} + \int_0^t \bar{V}_n(\tau) e^{\lambda_n K \tau} d\tau \right] \phi_n(x)$$

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Vibrating Rectangular Membrane

31-03-2016

$$u(x, y, t)$$

$$u_{tt} = c^2 [u_{xx} + u_{yy}]$$

→ All the four sides are fixed

Subject to

$$u(0, y, t) = 0$$

$$u(x, 0, t) = 0$$

$$u(L, y, t) = 0$$

$$u(x, H, t) = 0$$

B.C.s

$$u(x, y, 0) = \alpha(x, y)$$

$$u_t(x, y, 0) = \beta(x, y)$$

I.C.s

Let $u(x, y, t) = H(t) \cdot \phi(x, y)$

$$h_{tt} \phi(x, y) = c^2 [h \phi_{xx} + h \phi_{yy}]$$

\div by $c^2 h \phi$

$$\frac{h_{tt}}{c^2 h} = \frac{\phi_{xx} + \phi_{yy}}{\phi} = -\lambda$$

$$\Rightarrow h_{tt} = -\lambda c^2 h$$

$$D^2 = -c^2 \lambda$$

$$D = \pm ic\sqrt{\lambda}$$

oscillations with

frequency

$$h(t) = \begin{cases} \sin \sqrt{\lambda} ct \\ \cos \sqrt{\lambda} ct \end{cases}$$

" λ " is natural frequency of the vibrating membrane & " c " is frequency of oscillation.

Now

$$\phi_{xx} + \phi_{yy} = -\lambda \phi \rightarrow \textcircled{1}$$

$$\left. \begin{aligned} \phi(0, y) = 0 = \phi(x, 0) \\ \phi(L, y) = 0 = \phi(x, H) \end{aligned} \right\} \rightarrow \textcircled{2}$$

We call $\textcircled{1}$ & $\textcircled{2}$ two dimension eigenvalue problem. It is always linear.

$$\text{Let } \phi(x, y) = f(x) g(y)$$

$$\textcircled{1} \Rightarrow f_{xx} g(y) + f(x) g_{yy} = -\lambda f g$$

$$\Rightarrow \frac{f_{xx}}{f} = -\lambda - \frac{g_{yy}}{g} = -\mu$$

$$\Rightarrow f_{xx} = -\mu f \rightarrow (*)$$

$$f(0) = 0$$

$$f(L) = 0$$

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and

$$f'' + \lambda f = -(\lambda - \mu) f$$

$$f(0) = 0$$

$$f(L) = 0$$

(*) \Rightarrow which is Sturm-Liouville problem in x

$$f(x) = a \sin \sqrt{\mu} x + b \cos \sqrt{\mu} x$$

$$0 = b \Rightarrow b = 0$$

$$f(0) = 0 \Rightarrow b = 0$$

$$f(x) = a \sin \sqrt{\mu} x$$

$$f(L) = 0 \Rightarrow a \sin \sqrt{\mu} L = 0, a \neq 0$$

$$\Rightarrow \sqrt{\mu} L = n\pi$$

since $a \neq 0$
for trivial
sol.

$$\text{Eigenvalues } \Rightarrow \mu_n = \frac{n^2 \pi^2}{L^2} \quad n = 1, 2, \dots$$

$$\Rightarrow f_n(x) = a_n \sin \frac{n\pi}{L} x$$

Eigenfunction

Now,

$$g_{yy} = -(\lambda - \mu_n)g$$

$$D^2 = -(\lambda - \mu_n)$$

assume

$$\lambda > \mu_n$$

$$D = \pm \sqrt{\lambda - \mu_n} i$$

$$g(y) = c \sin(\sqrt{\lambda - \mu_n} y) + d \cos(\sqrt{\lambda - \mu_n} y)$$

$$g(0) = 0 \Rightarrow d = 0$$

$$g(H) = 0 \Rightarrow c \sin \sqrt{\lambda - \mu_n} H = 0$$

$$\Rightarrow \sqrt{\lambda - \mu_n} H = m\pi \quad c \neq 0$$

$$\lambda - \mu_n = \frac{m^2 \pi^2}{H^2} \quad m = 1, 2, \dots$$

$$\lambda = \mu_n + \frac{m^2 \pi^2}{H^2}$$

$$\lambda_{mn} = \mu_n + \left(\frac{m\pi}{H}\right)^2 \quad \text{or} \quad \lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

$n = 1, 2, \dots$

$m = 1, 2, \dots$

there are infinite number of eigenvalues for any "n".

$$\text{So, } g_{mn}(y) = c_m \sin\left(\frac{m\pi}{H}y\right)$$

Therefore

$$\Phi_{mn}(x, y) = (\text{constant}) \sin\frac{n\pi}{L}x \sin\frac{m\pi}{H}y$$

are two dimensional eigen functions corresponding to eigen value λ_{mn} .

$$u(x, y, t) = \begin{cases} \sin\frac{n\pi}{L}x \sin\frac{m\pi}{H}y \sin\sqrt{\lambda_{mn}}t \\ \sin\frac{n\pi}{L}x \sin\frac{m\pi}{H}y \cos\sqrt{\lambda_{mn}}t \end{cases}$$

are two double infinite values

As with the vibrating string each of this

The principle of superposition implies that we should consider a linear combination of all possible

$$U(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y \sin \sqrt{\lambda_{mn}} t$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y \cos \sqrt{\lambda_{mn}} t$$

The two family of coefficient A_{mn} & B_{mn} will be determine from initial conditions.

$$U(x, y, 0) = d(x, y)$$

$$\Rightarrow d(x, y) = \sum_{m=1}^{\infty} \left\{ \sum_{n=1}^{\infty} B_{mn} \sin \frac{n\pi}{L} x \right\} \sin \frac{m\pi}{H} y$$

s.t.

$$\int_0^H d(x, y) \sin \frac{l\pi}{H} y dy$$

$$= \sum_{m=1}^{\infty} \left\{ \sum_{n=1}^{\infty} B_{mn} \sin \frac{n\pi}{L} x \right\} \int_0^H \sin \frac{m\pi}{H} y \sin \frac{l\pi}{H} y dy$$

$$= \sum_{m=1}^{\infty} \left\{ \sum_{n=1}^{\infty} B_{mn} \sin \frac{n\pi}{L} x \right\} \frac{H}{2} \delta_{ml}$$

$$= \sum_{n=1}^{\infty} B_{ln} \sin \frac{n\pi}{L} x \frac{H}{2}$$

$$\Rightarrow B_{mn} = \frac{2}{L} \int_0^L \left[\frac{2}{H} \int_0^H d(x, y) \sin \frac{m\pi}{H} y dy \right] \sin \frac{n\pi}{L} x dx$$

$$= \frac{4}{LH} \int_0^L \int_0^H d(x, y) \sin \frac{m\pi}{H} y \sin \frac{n\pi}{L} x dy dx$$

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Similarly,

$$A_{mn} = \frac{4}{L \cdot H \cdot c \cdot \sqrt{\lambda_{mn}}} \int_0^L \int_0^H \beta(x, y) \sin \frac{m\pi}{H} y dy dx$$

$$A_{mn} = \frac{4}{L \cdot H \cdot c \cdot \sqrt{\lambda_{mn}}} \int_0^L \int_0^H \beta(x, y) \sin \frac{m\pi}{H} y \sin \frac{n\pi}{L} x dx dy \cdot \frac{\sin \frac{n\pi}{L} x}{L}$$

We have shown that all 3 independent variables separate form a PDE. The results are then 3 ODEs to which are eigenvalue problem.

In general, for a PDE in 'n' variables that completely separate there will be n-1 ODE's, n-1 will be one dimensional eigenvalue problem.

Ex. 7.3 Haberman

PDE $u_{tt} = c^2 (u_{xx} + u_{yy})$

$$u(x, 0, t) = u(x, b, t) = 0$$

$$u(0, y, t) = u(a, y, t) = 0$$

$$u(x, y, 0) = \phi(x, y)$$

$$u_t(x, y, 0) = 0$$

Solution will be

$$u(x, y, t) = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\int_0^a \int_0^b \beta(x, y) \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y dy dx \right] \sin \frac{n\pi}{b} y \sin \frac{m\pi}{a} x \cos \sqrt{\lambda_{mn}} t$$

7.4

Question \Rightarrow Answer will be

$$u(x, y, t) = \frac{288}{\pi^3} \sum_{n=1}^{\infty} \sum_{m=1,3,5,\dots}^{\infty} \left(\frac{1}{m}\right)^3 \sin\left(\frac{n\pi x}{3}\right) \sin\left(\frac{m\pi y}{6}\right) e^{-9\lambda_{nm}t}$$

$$\sum_{m=1,3,5,\dots}^{\infty} \left(\frac{1}{m}\right)^3 \sin\left(\frac{n\pi x}{3}\right) \sin\left(\frac{m\pi y}{6}\right) e^{-9\lambda_{nm}t}$$

$$u_t = 9(u_{xx} + u_{yy})$$

$$u(x, 0, t) = 0 = u(x, 6, t)$$

$$u(0, y, t) = 0 = u(3, y, t)$$

$$u(x, y, 0) = -\sin\left(\frac{n\pi x}{3}\right) \sin\left(\frac{m\pi y}{6}\right)$$

$$u_t(x, y, 0) = 0$$

Question: $u_{tt} = c^2(u_{xx} + u_{yy})$

$$u(x, 0, t) = 0 = u(x, \pi, t)$$

$$u(0, y, t) = 0 = u(\pi, y, t)$$

$$u(x, y, 0) = xy(\pi - x)(\pi - y)$$

$$u_t(x, y, 0) = 0$$

The solution will be

$$u(x, y, t) = \frac{64}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^2 n^2} \sin nx \sin my \cos c \sqrt{\lambda_{nm}} t$$

where $m \neq 0, 2, 4, \dots$ & $n \neq 0, 2, 4, \dots$

Bessel's Eq. of order n .

$$r^2 F''(r) + r F'(r) + (r^2 - n^2) F(r) = 0 \quad (111)$$

07-04-2016

Vibrating circular membrane
and the Bessel function.

Introduction:-

The vertical displacement u satisfies the 2-Dimensional wave Eq.

$$u_{tt} = c^2 \nabla^2 u \rightarrow \text{D}$$

The geometry suggest that $u = u(r, \theta, t)$ (polar coordinates).

B.C. $u(a, \theta, t) = 0$ (on circular boundary)

I.C. $u(r, \theta, 0) = \alpha(r, \theta)$

$$u_t(r, \theta, 0) = \beta(r, \theta)$$

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

Let

$$u = \phi(r, \theta) H(t) \quad (\text{separation of variables})$$

$$H(t) = \begin{cases} \sin \sqrt{\lambda} c t \\ \cos \sqrt{\lambda} c t \end{cases}$$

where λ is the separation const.

The natural frequencies of the

Vibrations are $\sqrt{\lambda} C$. $\phi_{ttt} = HC^T \nabla^2 \phi$
 {if $\lambda > 0$ and $\phi(x, 0)$ satisfies the 2-D eigenvalue problem.
 $\frac{H_{tt}}{HC^T} = \frac{\nabla^2 \phi}{\phi} = \lambda$

$$\nabla^2 \phi + \lambda \phi = 0$$

with $\phi(a, 0) = 0$

Let $\phi = f(x)g(y)$, $0 < x < a$
 $-\pi < y < \pi$

$$\left(\frac{1}{f}\right) \frac{\partial}{\partial x} \left(x \frac{\partial f}{\partial x} \right) + \frac{1}{g} \left(\frac{\partial^2 g}{\partial y^2} \right) + \lambda f g = 0$$

$$\frac{1}{f} \frac{\partial}{\partial x} \left(x \frac{\partial f}{\partial x} \right) + \frac{1}{g} \left(\frac{\partial^2 g}{\partial y^2} \right) + \lambda f g = 0$$

$$\frac{1}{f} \frac{\partial}{\partial x} \left(x \frac{\partial f}{\partial x} \right) + \frac{1}{g} \frac{\partial^2 g}{\partial y^2} + \lambda f g = 0$$

$$\frac{1}{f} \frac{\partial^2 f}{\partial x^2} = \frac{x}{f} \left(\frac{\partial f}{\partial x} \right) + \lambda f g = \mu$$

$$\frac{d^2 g}{dy^2} = -\mu g$$

$$g(y) = A \sin \mu y + B \cos \mu y$$

$$x \frac{d}{dx} \left(x \frac{df}{dx} \right) + (x^2 \lambda - \mu) f = 0$$

Two of these Eq.s must be eigenvalues problem the only B.V.C is

$$f(a) = 0$$

$$-\pi < \theta < \pi$$

$$0 < \theta < a$$

So, There should be B.C on both ends.

The periodic nature of solution in θ implies that

$$g(-\pi) = g(\pi)$$

$$\& \frac{dg(-\pi)}{d\theta} = \frac{dg(\pi)}{d\theta}$$

Polar coordinates are singular at $r=0$. a singularity condition must be introduced there. Since the displacement must be finite

$$|f(0)| < \infty$$

Eigen value Problem:-

$$\frac{d^2g}{d\theta^2} = -\mu g$$

$$\Rightarrow g(\theta) = A \sin \sqrt{\mu} \theta + B \cos \sqrt{\mu} \theta$$

Using $g(\pi) = g(0)$

$$A \sin(\sqrt{\mu}(\pi)) + B \cos(\sqrt{\mu}(\pi)) = A \sin \sqrt{\mu} \pi + B \cos \sqrt{\mu} \pi$$

$$A \sin \sqrt{\mu} \pi = 0$$

if $B=0$ then $A \neq 0$

$$\sin \sqrt{\mu} \pi = 0$$

$$\Rightarrow \mu_m = m^2, \quad m=1, 2, \dots$$

$$\frac{dg}{d\theta} = A \sqrt{\mu} \cos \sqrt{\mu} \theta - B \sqrt{\mu} \sin \sqrt{\mu} \theta$$

Using $\frac{dg(\pi)}{d\theta} = \frac{dg(0)}{d\theta}$

$$\Rightarrow A \sqrt{\mu} \cos \sqrt{\mu}(\pi) - B \sqrt{\mu} \sin \sqrt{\mu}(\pi)$$

$$= A \sqrt{\mu} \cos \sqrt{\mu} \pi - B \sqrt{\mu} \sin \sqrt{\mu} \pi$$

$$B \sqrt{\mu} \sin \sqrt{\mu} \pi = 0$$

If $A=0$ & $B \neq 0$

$$\text{the } \mu_m = m^2, \quad m=1, 2, \dots$$

Eigen Value Problem:-

11-04-2016

$$g(\theta) = A \sin m\theta + B \cos m\theta$$

m is the no. of crests in θ direction. for each integral value of ' m ' we have

$$r \frac{d}{dr} \left(r \frac{df}{dr} \right) + (\lambda r^2 - m^2) f = 0 \rightarrow (i)$$

$$f(a) = 0 \quad \& \quad |f(0)| < \infty$$

$$r^2 \frac{d^2 f}{dr^2} + r \frac{df}{dr} + (\lambda r^2 - m^2) f = 0 \rightarrow (ii)$$

Eq. (ii) contains two parameters ' λ ' & ' m '. ' λ ' is yet to be determined. It would be quite traditional to solve (i) numerically for various values of λ instead we might notice that simple scaling transformation

$z = \sqrt{\lambda} r$ in Eq. (ii) we have

$$\left(\frac{df}{dr} = \frac{df}{dz} \cdot \frac{dz}{dr} \right)$$

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - m^2) f = 0 \rightarrow (iii)$$

(iii) is known as Bessel Eq. of order m .

Standard form of Bessel's differential Eq.

$$\text{is } \frac{d^2 f}{dz^2} + a(z) \frac{df}{dz} + b(z) f = 0$$

If $a(z)$ & $b(z)$ & all their derivatives are finite at $z = z_0$ then z_0 is called ordinary point otherwise z_0 is singular point.

For Bessel diff. Eq. (iii) $z=0$ is a singular point & all other points are ordinary points. All solutions of Bessel diff. Eq. are well behaved at every finite point except possibly at $z=0$.

The only difficulty can occur in the nbhd of $z=0$. We will investigate the expected behaviour of solutions of Bessel diff. Eq. at $z=0$.

If z is very close to zero then $z^2 f \ll m^2 f$ so the Eq. (ii) becomes

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} - m^2 f \approx 0 \Rightarrow \text{Cauchy PDE}$$

$$\text{put } z = e^t \Rightarrow \ddot{z} - \frac{\dot{z}}{z} + \frac{z}{z^2} - m^2 = 0$$

$$\Rightarrow z = A e^{mt} + B e^{-mt}$$

$$\begin{aligned}
 z &= A e^{m \ln z} + B e^{-m \ln z} \\
 &= A e^{(\ln z)^m} + B e^{(\ln z)^{-m}} \\
 &= A z^m + B z^{-m}, \quad m > 0
 \end{aligned}$$

If $m = 0$

$$D = 0$$

$$z = a + bt = a + b \ln z \quad m = 0$$

So, $f \approx \begin{cases} a + b \ln z & m = 0 \\ A z^m + B z^{-m} & m > 0 \end{cases}$

We see that the independent

solutions can be chosen that one solution is well behaved and other is ill-behaved at $z = 0$.

$$f = C_1 J_m(z) + C_2 Y_m(z)$$

$J_m(z)$ is called first kind of Bessel function & $Y_m(z)$ is called second kind of Bessel function.

We simply note that this satisfy asymptotic property for small z .

$$J_m(z) \sim \begin{cases} 1 \text{ (constant)} & m=0 \\ \frac{1}{2^m m!} z^m & , m > 0 \end{cases}$$

$$Y_m(z) \sim \begin{cases} \frac{2}{\pi} \ln z & m=0 \\ -\frac{2^m (m-1)!}{\pi} z^{-m} & m > 0 \end{cases}$$

We see that J_m is bounded where as Y_m is not bounded at $z \rightarrow 0$.

Eigen Value Problem involving Bessel functions.

We determine the Eigen value of the singular SL problem

$$\frac{d}{dr} \left(r \frac{df}{dr} \right) + \left(dr - \frac{m^2}{r} \right) f = 0$$

$$f(a) = 0$$

$$|f(a)| < \infty$$

$$r f'' + f' + \left(dr - \frac{m^2}{r} \right) f = 0$$

$$r^2 f'' + r f' + (\lambda r^2 - m^2) f = 0 \quad (19)$$

Similar to previous

\Rightarrow

$$f = C_1 J_m(\sqrt{\lambda} r) + C_2 Y_m(\sqrt{\lambda} r)$$

$$f(a) = 0$$

$$\Rightarrow 0 = C_1 J_m(\sqrt{\lambda} a) + C_2 Y_m(\sqrt{\lambda} a)$$

also to justify

$$|f(0)| < \infty$$

we take $C_2 = 0$

$$f(r) = C_1 J_m(\sqrt{\lambda} r)$$

$$J_m(\sqrt{\lambda} a) = 0$$

$$\Rightarrow \sqrt{\lambda} a = 0$$

There is an infinite no. of zeros of each Bessel function J_m .

Let

z_{mn} designate the n th zero of $J_m(z)$.

$$\sqrt{\lambda} a = z_{mn}$$

$$\Rightarrow \lambda_{mn} = \left(\frac{z_{mn}}{a} \right)^2$$

For each n there is infinite no. of eigenvalues. Thus solution

$$u(r, \theta, t) = \left\{ \begin{array}{l} J_m(\sqrt{\lambda_{mn}} r) \cos m\theta \\ J_m(\sqrt{\lambda_{mn}} r) \sin m\theta \end{array} \right\} \left\{ \begin{array}{l} \cos \sqrt{\lambda_{mn}} t \\ \sin \sqrt{\lambda_{mn}} t \end{array} \right\}$$

Assume that

$$\frac{\partial u(r, \theta, t)}{\partial t} = f(r, \theta) = p$$

Thus the $\sin \sqrt{\lambda_{mn}} t$ terms in (*) is not necessary to include in this solution.

Thus we can write

$$= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}} r) \cos m\theta \cos \sqrt{\lambda_{mn}} t$$

$$+ \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}} r) \sin m\theta \cos \sqrt{\lambda_{mn}} t$$

Use the I.C.

$$u(r, \theta, 0) = d(r, \theta)$$

$$\Rightarrow \alpha(r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} J_n(\sqrt{\lambda_{mn}} r) \cos m\theta$$

$$+ \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} B_{mn} J_n(\sqrt{\lambda_{mn}} r) \sin m\theta$$

It is somewhat easier to determine all coefficients using the 2-D orthogonality property.

Let

$$\phi_{\lambda}(r, \theta) = J_n(\sqrt{\lambda_{mn}} r) \begin{cases} \cos(m\theta) \\ \sin(m\theta) \end{cases}$$

Thus

$$\alpha(r, \theta) = \sum_{\lambda} A_{\lambda} \phi_{\lambda}(r, \theta)$$

where the \sum_{λ} stands for

a summation over all eigenvalues (actually two double sums including both $\cos m\theta$ & $\sin m\theta$)

which implies

$$A_{\lambda} = \frac{\iint \alpha(r, \theta) \phi_{\lambda}(r, \theta) dA}{\iint \phi_{\lambda}^2(r, \theta) dA}$$

$$\text{where } dA = r dr d\theta$$

where r is not a weight
function its a geometric
function.

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14-04-2016

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = \frac{1}{c^2} u_{tt}$$

$$\Rightarrow \frac{1}{r} \times \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{1}{c^2} u_{tt} \rightarrow \textcircled{1}$$

we are looking for ω periodic functions in t .

$$u = F(r) e^{i\omega t}$$

$$\text{Then } u' = F' e^{i\omega t}$$

$$u'' = F'' e^{i\omega t}$$

$$u_{tt} = \omega^2 F e^{i\omega t}$$

$$\textcircled{1} \Rightarrow \left[F''(r) + \frac{1}{r} F'(r) \right] e^{i\omega t} = -\frac{\omega^2}{c^2} F e^{i\omega t}$$

$$F''(r) + \frac{1}{r} F'(r) = -\frac{\omega^2}{c^2} F$$

$$r^2 F''(r) + r F'(r) + \frac{\omega^2}{c^2} r^2 F = 0$$

Bessel Eq. of zero order.

$$F(r) = A J_0\left(\frac{\omega r}{c}\right) + B Y_0\left(\frac{\omega r}{c}\right)$$

$$\text{If } A = c_1 + c_2$$

$$B = c(c_1 - c_2)$$

$$\Rightarrow F(r) = c_1 \left[J_0\left(\frac{\omega r}{c}\right) + i Y_0\left(\frac{\omega r}{c}\right) \right] + c_2 \left[J_0\left(\frac{\omega r}{c}\right) - i Y_0\left(\frac{\omega r}{c}\right) \right]$$

رانا شکور احمد

MS (Mathematics)

$$= C_1 H_0^{(1)}\left(\frac{\omega r}{c}\right) + C_2 H_0^{(2)}\left(\frac{\omega r}{c}\right)$$

$$H_0^{(1)} = \sqrt{\frac{2}{\pi x}} e^{i(x - \frac{\pi}{4})}$$

$$H_0^{(2)} = \sqrt{\frac{2}{\pi x}} e^{-i(x - \frac{\pi}{4})}$$

} Hankel function.

D.Y.S.
Question

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

Solution will be

$$u = \left(\frac{\omega r}{c}\right)^{\frac{1}{2}} \left[A J_{\frac{1}{2}}\left(\frac{\omega r}{c}\right) + B Y_{\frac{1}{2}}\left(\frac{\omega r}{c}\right) \right]$$

Fourier Transform; 25-04-2016

Let $f(x)$ be a funcⁿ defined on $(-\infty, \infty)$ & it is piecewise continuous, differentiable at each finite interval and it is absolutely integrable on $(-\infty, \infty)$

i.e. $\int_{-\infty}^{\infty} f(x) dx$ exists.

Thus Fourier transform of $f(x)$ is

$$F[f(x)] = F(\alpha)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

$$f(x) = \mathcal{L}^{-1}[F(\alpha)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha$$

Fourier Sine Trans.

$$f_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \alpha x dx$$

$$= f_s[F(\alpha)] \text{ or } S[f(x)]$$

$$\mathcal{L}^{-1}[f_s(\alpha)] = f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(\alpha) \sin \alpha x d\alpha$$

Similarly for Fourier Cosine Transformation

$$f_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x dx$$

$$= F_c [f(x)]$$

&

$$L^{-1}[F_c(\omega)] = f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\omega) \cos \omega x d\omega$$

Question:

$f(x) = e^{-x^2/2}$ find Fourier Trans. of this Gaussian function.

$$F[f(x), \omega] = F(\omega)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2i\omega x)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-i\omega)^2}{2} - \frac{\omega^2}{2}} e^{\frac{\omega^2}{2}} dx$$

$$= \frac{e^{\omega^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} dt$$

$$\text{put } x - ia = t$$

$$dx = dt$$

$$= \frac{e^{-x^2/2}}{\sqrt{\pi}} \cdot \sqrt{\frac{\pi}{2}}$$

$$= e^{-x^2/2}$$

$$\int_{-\infty}^{\infty} e^{-xt^2} dt$$

$$= \sqrt{\frac{\pi}{\alpha}}$$

The Gaussian itself is a Fourier function.

Question: Find the Fourier trans. of function.

$$f(x) = \begin{cases} |x| & |x| < a \\ 0 & |x| > a \end{cases}$$

& evaluate.

$$\int_{-\infty}^{\infty} \frac{\sin \alpha x \cos \alpha x}{\alpha} d\alpha$$

$$\int_0^{\infty} \frac{\sin \alpha a}{\alpha} d\alpha$$

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a f(x) e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{i\alpha x}}{i\alpha} \Big|_{-a}^a$$

$$= \frac{1}{\sqrt{2\pi}} i\alpha \left[\frac{e^{i\alpha a} - e^{-i\alpha a}}{2i} \right]$$

$$= \frac{1}{\sqrt{\pi}} \frac{1}{\alpha} \left[\frac{e^{i\alpha a} - e^{-i\alpha a}}{2i} \right]$$

$$= \begin{cases} \frac{1}{\alpha} \sqrt{\frac{2}{\pi}} \sin(\alpha a) & \text{if } \alpha > 0 \end{cases}$$

$$\begin{cases} \sqrt{\frac{2}{\pi}} a & \text{if } \alpha = 0 \end{cases}$$

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Now

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha x} F(\alpha) d\alpha \quad \alpha > 0$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{\alpha} \sin \alpha a d\alpha$$

$$= \begin{cases} 1 & |x| \leq a \\ 0 & |x| > a \end{cases}$$

$$\Rightarrow \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \alpha a}{\alpha} [\cos \alpha x - i \sin \alpha x] d\alpha$$

$$= \begin{cases} 1 & \text{if } |x| \leq a \\ 0 & \text{if } |x| > a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha} d\alpha = \begin{cases} \pi & |x| \leq a \\ 0 & |x| > a \end{cases}$$

if $x=0$

$$\int_{-\infty}^{\infty} \frac{\sin \alpha a}{\alpha} d\alpha = \pi$$

Even function. then we can write

$$2 \int_0^{\infty} \frac{\sin \alpha x}{\alpha} d\alpha = \pi \quad \text{if } x > 0$$

$$\int_0^{\infty} \frac{\sin \alpha x}{\alpha} d\alpha = \frac{\pi}{2} \quad \text{if } x = 0$$

Question:- Find Fourier cosine & Fourier sine

$f(x) = e^{-bx}$ and evaluate

$$(i) \int_0^{\infty} \frac{\cos \alpha x}{\alpha^2 + b^2} d\alpha$$

$$(ii) \int_0^{\infty} \frac{\alpha \sin \alpha x}{\alpha^2 + b^2} d\alpha$$

Sol:-

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \alpha x d\alpha$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-bx} \cos \alpha x dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\cos \alpha x \frac{e^{-bx}}{-b} - \frac{\alpha}{b} \int_0^{\infty} e^{-bx} \sin \alpha x dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[0 + \frac{1}{b} \right] - \frac{\alpha}{b} I_2 \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{b} - \frac{\alpha}{b} I_2 \right]$$

$$\Rightarrow I_1 = \frac{1}{b} - \frac{\alpha}{b} I_2 \quad \text{--- (*)}$$

and let

$$I_2 = \int_0^{\infty} e^{-bx} \sin \alpha x \, dx$$

$$I_2 = \frac{-e^{-bx}}{b} \sin \alpha x + \frac{\alpha}{b} \int_0^{\infty} e^{-bx} \cos \alpha x \, dx \quad \rightarrow I_1$$

$$I_2 = \frac{\alpha}{b} I_1$$

$$(*) \Rightarrow I_1 = \frac{1}{b} - \frac{\alpha^2}{b^2} I_1$$

$$I_1 \left[1 + \frac{\alpha^2}{b^2} \right] = \frac{1}{b}$$

$$I_1 \left[\frac{b^2 + \alpha^2}{b^2} \right] = \frac{1}{b}$$

$$I_1 = \frac{b}{b^2 + \alpha^2}$$

$$\Rightarrow I_2 = \frac{\alpha}{b} \left(\frac{b}{b^2 + \alpha^2} \right) = \frac{\alpha}{b^2 + \alpha^2}$$

$$\text{So, } f_c(\alpha) = \sqrt{\frac{2}{\pi}} \cdot \frac{b}{\alpha^2 + b^2}$$

$$\& f_s(\alpha) = \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\alpha^2 + b^2}$$

then

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(\alpha) \cos \alpha x \, d\alpha$$

$$e^{-bx} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{b}{\alpha^2 + b^2} \cos \alpha x \, d\alpha$$

$$\Rightarrow \int_0^{\infty} \frac{\cos \alpha x}{\alpha^2 + b^2} \, d\alpha = \frac{\pi}{2b} e^{-bx}$$

Similarly,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(\alpha) \sin \alpha x \, d\alpha$$

$$e^{-bx} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\alpha}{\alpha^2 + b^2} \right) \sin \alpha x \, d\alpha$$

$$\Rightarrow \int_0^{\infty} \frac{\alpha \sin \alpha x}{\alpha^2 + b^2} \, d\alpha = \frac{\pi}{2} e^{-bx}$$

Transform of Derivatives:- 28-04-2016

$$C\left[\frac{df}{dx}\right] = ? \quad \text{"C" is for cosine.}$$

$$C\left[\frac{df}{dx}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{df}{dx} \cos \omega x dx \quad \text{Taking } \frac{2}{\pi} \text{ here}$$

$$= \sqrt{\frac{2}{\pi}} \left[f(x) \cos \omega x \Big|_0^{\infty} + \int_0^{\infty} f(x) \omega \sin \omega x dx \right] \quad \text{f(x) is bounded.}$$

We assume that $f(x)$ is bounded ($f(x) \rightarrow 0$ when $x \rightarrow \infty$)

$$= \sqrt{\frac{2}{\pi}} \left[-f(\infty) + \omega \int_0^{\infty} f(x) \sin \omega x dx \right]$$

$$C\left[\frac{df}{dx}\right] = \sqrt{\frac{2}{\pi}} \left[-f(\infty) + \omega S[f(x)] \right] \quad \text{--- (1)}$$

Now

$$S\left[\frac{df}{dx}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{df}{dx} \sin \omega x dx$$

$$= \sqrt{\frac{2}{\pi}} \left[f(x) \sin \omega x \Big|_0^{\infty} - \omega \int_0^{\infty} f(x) \cos \omega x dx \right]$$

$$= -\omega C[f(x)] \quad \text{--- (2)}$$

Note:- If a PDE contains first derivative w.r.t. x ,

then \rightarrow Potential ~~Partial~~ variable to be transform^{ed} the Fourier sine or cosine transform will never work. Since sine and

cosine transform of first derivatives always involve the other type of semi infinite transform.

$$(ii) \quad C \left[\frac{d^2 f}{dx^2} \right]$$

$$C \left[\frac{d^2 f}{dx^2} \right] = \frac{\sqrt{2}}{\sqrt{\pi}} \left[\left. \frac{df(x)}{dx} \cos wx \right|_0^{\infty} + \int_0^{\infty} \frac{df(x)}{dx} w \sin wx dx \right]$$

Assumption on $f(x)$ & $\frac{df(x)}{dx}$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \left[-\frac{df(0)}{dx} \right] + \frac{\sqrt{2}}{\sqrt{\pi}} w \int_0^{\infty} \frac{df}{dx} \sin wx dx$$

From ①

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \left[\frac{d f(x)}{dx} \right] - w^2 C [f(x)]$$

$$S \left[\frac{d^2 f}{dx^2} \right] = \frac{\sqrt{2}}{\sqrt{\pi}} \left[\left. \frac{df(x)}{dx} \sin wx \right|_0^{\infty} - \int_0^{\infty} \frac{df(x)}{dx} w \cos wx dx \right]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} [0] - \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\infty} \frac{df(x)}{dx} w \cos wx dx$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} w f(0) - w^2 S [f(x)]$$

Note: If $f(x)$ is given use Fourier sine transform and if $\frac{df(x)}{dx}$ given

use fourier cosine transform.

$$(ii) F_c [e^{-at^2}] = ?$$

$$\Rightarrow I(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-at^2} \cos \alpha t \, dt$$

$$I = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-at^2} \cos \alpha t \, dt$$

$$\frac{dI}{d\alpha} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-at^2} t \sin \alpha t \, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{d(e^{-at^2})}{2a} \sin \alpha t \, dt$$

$$= \frac{1}{\sqrt{2\pi a}} \int_0^{\infty} d(e^{-at^2}) \sin \alpha t \, dt$$

Integration by parts gives

$$= \frac{-\alpha}{\sqrt{2\pi a}} \int_0^{\infty} e^{-at^2} \cos \alpha t \, dt$$

$$= \frac{-\alpha}{2a} I$$

$$\frac{dI}{d\alpha} + \frac{\alpha}{2a} I = 0$$

$$\int \frac{dI}{I} = -\int \frac{\alpha}{2a} d\alpha$$

$$\ln I = -\frac{\alpha^2}{4a} + \ln A$$

$$I = A e^{-\alpha^2/4a}$$

at $\alpha = 0$

$$I(0) = A$$

$$I = I(0) e^{-\alpha^2/4a}$$

$$I(0) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-at^2} dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sqrt{\pi}}{\sqrt{a}} \right] \frac{1}{2}$$

$$\int_{-\infty}^{\infty} e^{-at^2} dt = \sqrt{\frac{\pi}{a}}$$

↳ Even func.

$$= 2 \int_0^{\infty} e^{-at^2} dt$$

$$I(0) = \frac{1}{\sqrt{2a}}$$

$$I = \frac{1}{\sqrt{2a}} e^{-\alpha^2/4a} \rightarrow \text{Answer.}$$

Heat Eq. on a semi infinite interval:-

$$u_t = K u_{xx} \longrightarrow \textcircled{1}$$

$$\text{B.C's } u(0, t) = f(t) \longrightarrow \textcircled{2}$$

$$\text{I.C. } u(x, 0) = f(x) \longrightarrow \textcircled{3}$$

The B.C. at $x=0$ is non-homo.
So, we can't apply separation of variables condition is on u
So using

Sine Fourier transform $\textcircled{1}$

$$\text{Let } \bar{U}_s(\omega, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} u(x, t) \sin \omega x dx$$

Taking fourier sine transform of (1), we have

$$\begin{aligned} \frac{d(\bar{U}_s)}{dt} &= K \left[\sqrt{\frac{2}{\pi}} \omega u(0, t) - \omega^2 \bar{U}_s(x, t) \right] \\ &= K \left[\sqrt{\frac{2}{\pi}} \omega g(t) - \omega^2 \bar{U}_s \right] \end{aligned}$$

Taking fourier sine transform of I.C. gives,

$$\bar{U}_s(\omega, 0) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x dx$$

$$\frac{d(\bar{U}_s)}{dt} + \omega^2 K \bar{U}_s = \sqrt{\frac{2}{\pi}} K \omega g(t)$$

$$\frac{d}{dt} (e^{\omega^2 K t} \bar{U}_s) = \sqrt{\frac{2}{\pi}} K \omega g(t) e^{\omega^2 K t}$$

$$e^{\omega^2 K t} \bar{U}_s = \sqrt{\frac{2}{\pi}} K \omega \int g(t) e^{\omega^2 K t} dt$$

$$e^{\omega^2 K t} \bar{U}_s(x, t) = \sqrt{\frac{2}{\pi}} K \omega \int g(t) e^{\omega^2 K t} dt + A(\omega)$$

$$\bar{U}_s = \frac{\sqrt{2} K \omega e^{-\omega^2 K t}}{\sqrt{\pi}} \int e^{\omega^2 K t} g(t) dt + A(\omega) e^{-\omega^2 K t}$$

Special case

$$\text{If } g(t) = u_0$$

$$f(x) = 0$$

05-05-2016

$$\Rightarrow \bar{u}_{s,t} = k \left[\frac{\sqrt{2}}{\pi} \omega g(t) - \omega^2 \bar{u}_s \right]$$

$$= k \left[\frac{\sqrt{2}}{\pi} \omega u_0 - \omega^2 \bar{u}_s \right]$$

$$\frac{d\bar{u}_s}{dt} + k\omega^2 \bar{u}_s = \frac{\sqrt{2}}{\pi} k\omega u_0$$

$$I F = e^{\int k\omega^2 dt} = e^{\omega^2 t}$$

$$\bar{u}_s e^{\omega^2 t} = \int \frac{\sqrt{2}}{\pi} k\omega u_0 e^{\omega^2 t} dt + A(\omega)$$

$$\bar{u}_s e^{-\omega^2 t} = \frac{\sqrt{2}}{\pi} k\omega u_0 \frac{e^{\omega^2 t}}{\omega^2} + A(\omega)$$

Applying $\bar{u}_s(\omega, 0) = \frac{\sqrt{2}}{\pi} \int_0^\infty f(x) \sin \omega x dx$

$$\Rightarrow \bar{u}_s(\omega, 0) = 0 \quad \because f(x) = 0$$

$$0 = \frac{\sqrt{2}}{\pi} \frac{u_0}{\omega} + A(\omega)$$

$$\Rightarrow A(\omega) = -\frac{\sqrt{2}}{\pi} \frac{u_0}{\omega}$$

So,

$$\bar{u}_s e^{\omega^2 t} = \frac{\sqrt{2}}{\pi} \frac{u_0}{\omega} e^{\omega^2 t} - \frac{\sqrt{2}}{\pi} \frac{u_0}{\omega}$$

$$\bar{u}_s(\omega, t) = \frac{\sqrt{2}}{\pi} \frac{u_0}{\omega} - \frac{\sqrt{2}}{\pi} \frac{u_0}{\omega} e^{-\omega^2 t}$$

By Inverse Fourier ^{sine} Transform.

$$F_s^{-1} [U_s(\omega, t)] = F_s^{-1} \left[\frac{\sqrt{2}}{\sqrt{\pi}} \frac{u_0}{\omega} (1 - e^{-\omega^2 kt}) \right]$$

$$u(x, t) = \frac{2}{\pi} \frac{u_0}{\omega} \int_0^{\infty} \frac{(1 - e^{-\omega^2 kt})}{\omega} \sin \omega x \, d\omega$$

To solve $\int_0^{\infty} e^{-\omega^2 kt} \frac{\sin \omega x}{\omega} d\omega$

also $\operatorname{erf}(x) = \frac{2}{\pi} \int_0^x e^{-u^2} du$

Put $\omega^2 kt = \alpha^2$

$$2\omega kt \, d\omega = \alpha \, d\alpha$$

$$d\omega = \frac{\alpha \, d\alpha}{\omega kt}$$

$$\int_0^{\infty} e^{-\alpha^2} \frac{\sin(\alpha x)}{\alpha} d\alpha = \frac{\pi}{2} \operatorname{erf}(x)$$

$$\int_0^{\infty} \frac{\sin \alpha x}{\alpha} d\alpha = \frac{\pi}{2}$$

Put $\omega = \frac{\alpha}{\sqrt{kt}}$

$$\Rightarrow d\omega = \frac{\alpha \, d\alpha}{\alpha kt} \sqrt{kt}$$

$$d\omega = \frac{d\alpha}{\sqrt{kt}}$$

$$\Rightarrow \int_0^{\infty} e^{-\omega^2 kt} \frac{\sin \omega x}{\omega} d\omega = \int_0^{\infty} \frac{\sin\left(\frac{\alpha x}{\sqrt{kt}}\right)}{\frac{\alpha}{\sqrt{kt}}} e^{-\alpha^2} \frac{d\alpha}{\sqrt{kt}}$$

$$= \int_0^{\infty} \frac{\sin\left(2\alpha \left(\frac{x}{2\sqrt{kt}}\right)\right)}{\alpha} e^{-\alpha^2} d\alpha$$

$$= \frac{\pi}{2} \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right)$$

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examples

Sankar

Thus

$$u(x,t) = \frac{2}{\pi} u_0 \left[\frac{\pi}{2} - \frac{\pi}{2} \operatorname{erf} \left(\frac{x}{2\sqrt{kt}} \right) \right]$$

$$= u_0 \left[1 - \operatorname{erf} \left(\frac{x}{2\sqrt{kt}} \right) \right]$$

$$= u_0 \operatorname{erf}_{\text{comp}} \left(\frac{x}{2\sqrt{kt}} \right)$$

if $g(t) = 0$ $u_x = k u_{xx}$; $u(0,t) = 0$

$$u(a,t) = 0, \quad u(a,0) = f(x)$$

Statement 2: Illustration of Theorem
for Eigenvalue problem:-

$$\nabla^2 \phi + \lambda \phi = 0 \quad \text{Helmholtz Eq.} \quad \textcircled{1}$$

For separation of variables Heat & wave Eq
i.e. $u(x,y,z,t) = \phi(x,y,z) h(t)$ we
get eigenvalue problem for spatial coordinates.

$$\text{With } a\phi + b\nabla\phi \cdot \hat{n} = 0 \quad \textcircled{2}$$

where \hat{n} is normal to the surface

$$\nabla \cdot (p \nabla \phi) + q\phi + \lambda \delta \phi = 0 \quad \text{SL-Prob.}$$

$$\textcircled{1} \ \& \ \textcircled{2} \Rightarrow p=1, \ q=0, \ \delta=1$$

Properties:-

(1):- All eigenvalues are real.

e.g $\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$

$$m, n = 1, 2, 3, \dots$$

(2):- There exist infinite no. of eigen values. There is a smallest eigen value but no larger value.

Smallest $\lambda_{1,1}$

(3):- Corresponding to an eigenvalue there may be many eigenfunction (unlike Regular SL problems).

Example $\lambda_{nm} = \frac{\pi^2}{4H} [n^2 + m^2]$

$$\phi_{nm} = \frac{\sin n\pi x}{2H} \sin \frac{m\pi}{H} y$$

$$\lambda_{41} = \lambda_{22}$$

but

$$\phi_{41} = \sin \frac{2\pi}{H} x \sin \frac{\pi}{H} y$$

$$\phi_{22} = \sin \frac{\pi}{H} x \sin \frac{2\pi}{H} y$$

$$\Rightarrow \phi_{41} \neq \phi_{22}$$

$$m=1, n=2 \Rightarrow d_{41} = d_{22}$$

$$d_{(2n)m} = d_{(2m)n}$$

Put $n=3, m=5$

$$d_{65} = \frac{\pi^2}{4H} [36 + 4(25)]$$

$$d_{103} = \frac{\pi^2}{4H} [100 + 36]$$

$$\Rightarrow d_{65} = d_{103}$$

A:- The eigenfunction form a complete set meaning that any piecewise smooth func. $f(x,y) / \phi(x,y)$ can be represented as generalized fourier series of the eigen-function.

$$f(x,y) \sim \sum_{\lambda} a_{\lambda} \phi_{\lambda}(x,y)$$

~~or~~

$$f(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \frac{\sin \frac{n\pi x}{L}}{L} \frac{\sin \frac{m\pi y}{H}}{H}$$

(5) The eigenfunctions belonging to different eigen values are orthogonal relative to the weight function over the entire region.

$$\iint_R \phi_{\lambda_1} \phi_{\lambda_2} dx dy = 0$$

if $\lambda_1 \neq \lambda_2$

We may assume that (5) is valid even $\lambda_1 = \lambda_2$ as long as ϕ_{λ_1} and ϕ_{λ_2} are different (or independent).

(6):- An eigenvalue λ can be related to eigenfunction by the Rayleigh quotient.

$$\lambda = \frac{-\int \phi \nabla \phi \cdot \hat{n} ds + \iint_R |\nabla \phi|^2 dx dy}{\iint_R \phi^2 dx dy}$$

Consider

$$\nabla^2 \phi + \lambda \phi = 0$$

multiply by ϕ from left.

$$\Rightarrow \phi \nabla^2 \phi + \lambda \phi^2 = 0$$

$$\Rightarrow \iint_R \phi \nabla^2 \phi \, dx \, dy + \lambda \iint_R \phi^2 \, dx \, dy = 0$$

$$\Rightarrow \lambda = - \frac{\iint_R \phi \nabla^2 \phi \, dx \, dy}{\iint_R \phi^2 \, dx \, dy}$$

$$\nabla \cdot (fg) = f \nabla \cdot g + g \nabla f$$

~~$$\phi \nabla^2 \phi + \lambda \phi^2$$~~

~~$$\iint_R \phi [\phi \nabla^2 \phi + \lambda \phi^2] \, dx \, dy$$~~

~~$$\iint_R \phi^2 \, dx \, dy$$~~

$$\text{if } f = \phi$$

$$g = \nabla \phi$$

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$$\nabla \cdot (\phi \nabla \phi) = \phi (\nabla \cdot \nabla \phi) + \nabla \phi \cdot (\nabla \phi)$$

$$\nabla \cdot (\phi \nabla \phi) = \phi \nabla^2 \phi + |\nabla \phi|^2$$

$$\Rightarrow \phi \nabla^2 \phi = \nabla \cdot (\phi \nabla \phi) - |\nabla \phi|^2$$

Put in λ .

$$\lambda = \frac{-\iint_R [\nabla \cdot (\phi \nabla \phi) - |\nabla \phi|^2] dx dy}{\iint_R \phi^2 dx dy}$$

$$= \frac{-\iint_R \nabla \cdot (\phi \nabla \phi) dx dy + \iint_R |\nabla \phi|^2 dx dy}{\iint_R \phi^2 dx dy}$$

$$\lambda = \frac{-\oint \phi \nabla \phi \cdot \hat{n} ds + \iint_R |\nabla \phi|^2 dx dy}{\iint_R \phi^2 dx dy} \quad \left| \begin{array}{l} \text{Using Gauss Diverg.} \\ \text{theorem.} \\ \iint_R \nabla \cdot \vec{F} dx dy = \oint \vec{F} \cdot \hat{n} ds \end{array} \right.$$

Example:- we consider any region in which the B.C is $\phi = 0$ on the entire boundary.

then

$$\oint \phi \nabla \phi \cdot \hat{n} ds \quad \leftarrow \text{on boundary.}$$

$$\iint_R \phi^2 dx dy \neq 0 \quad \leftarrow \text{on region.}$$

If $\lambda = 0$ then

$$\iint_R |\phi|^2 dx dy = 0$$

$$\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j}$$

$$\rightarrow \frac{\partial \phi}{\partial x} = 0 \quad \& \quad \frac{\partial \phi}{\partial y} = 0$$

$\Rightarrow \phi$ is equal to zero everywhere.

Therefore ϕ is zero.

but ϕ not an eigenfunction

hence $\lambda > 0$ is not an eigenfunction.

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Self adjoint Green's Formula

09-05-2016

Q5

$$(i) \nabla^2 u = \nabla \cdot (\nabla u)$$

$$(ii) \nabla \cdot (a \vec{B}) = a \nabla \cdot \vec{B} + \nabla a \cdot \vec{B}$$

$$\nabla \cdot (\underline{u} \cdot \nabla \underline{v}) = u \cdot \nabla^2 \underline{v} + \nabla u \cdot \nabla \underline{v}$$

$$\nabla \cdot (\underline{v} \nabla u) = v \cdot \nabla^2 u + \nabla v \cdot \nabla u \quad \rightarrow \textcircled{1}$$

$$\iint_R (u \nabla^2 v - v \nabla^2 u) \, dx \, dy = \iint_R (\nabla \cdot (u \nabla v - v \nabla u)) \, dx \, dy$$

$$= \oint (u \nabla v - v \nabla u) \cdot \vec{n} \, ds$$

If u & v are functions such that

$$u \nabla v - v \nabla u = 0$$

then these are self adjoint operators.

Then

$$\iint_R (u \nabla^2 v - v \nabla^2 u) \, dx \, dy = 0$$

Green Identity

Dirac Delta function.

$$\delta_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon} & \text{if } |t| < \epsilon \\ 0 & \text{if } |t| > \epsilon \end{cases}$$

$$\int_{-\infty}^{\infty} \delta_\epsilon(t) dt = \int_{-\epsilon}^{\epsilon} \delta_\epsilon(t) dt = 1$$

$$\int_{-\infty}^{\infty} f(t) \delta_\epsilon(t) dt = \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(t) dt = f\left(\frac{\epsilon}{2}\right)$$

Def.

$$\delta(t) = \lim_{\epsilon \rightarrow 0} \delta_\epsilon(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

This limiting $\delta(t)$ define by ① & ②

Properties:-

$$(2) \int_{-\infty}^{\infty} f(t) \delta(t) dt = f(0)$$

Condition: $f(t)$ should be const

$$(3) \int_{-\infty}^{\infty} \delta(t-a) f(t) dt = f(a)$$

$$(4) \delta(-t) = \delta(t)$$

$$(5) \delta(at) = \frac{1}{|a|} \delta(t), \quad a > 0$$

$$(6) \int_{-\infty}^{\infty} f(t) \delta'(t) dt = -f'(0)$$

$$(7) \int_{-\infty}^{\infty} \delta'(t-a) f(t) dt = -f'(a)$$

Consider Differential Eq.

$$L u(x) = f(x) \longrightarrow (1)$$

where L is ordinary linear operator. $f(x)$ is known function while $u(x)$ is unknown function.

$$L^{-1}(f(x)) = u(x) = L^{-1}(L u) = \int G(x, \xi) f(\xi) d\xi \longrightarrow (2)$$

\swarrow Kernel
Green's function.

apply L-operator.

$$\Rightarrow f(x) = \int L G(x, \xi) f(\xi) d\xi \rightarrow (3)$$

$$L G(x, \xi) = \delta(x - \xi) \rightarrow (4)$$

where $\delta(x - \xi)$ is called Dirac delta

The solution of the Eq. (4) is called singularity sol of (1).

Consider

$$\int_{-\infty}^{\infty} \delta'(x) \phi(x) dx = -\phi'(0)$$

for every test func. $\phi(x)$
we define a heaviside
unit step function.

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$\phi(x)$ is a function of a real variable x , which possess derivative of all orders & vanishes outside of a finite interval such functions are called test function.

test func. 151

$$\int_{-\infty}^{\infty} H'(x) \phi(x) dx = H(x) \phi(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H(x) \phi'(x) dx$$

$$= - \int_0^{\infty} \phi'(x) dx = -\phi(x) \Big|_0^{\infty}$$

$$= \phi(0)$$

$$= \int \delta(x-0) \phi(x) dx$$

$$\Rightarrow H'(x) = \delta(x-0)$$

$$\int_{-\infty}^{\infty} H'(x-\xi) \phi(x) dx = \int_{-\infty}^{\infty} \delta(x-\xi) \phi(x) dx = \phi(\xi)$$

$$\frac{d}{dx} [(x-\xi) H(x-\xi)] = (x-\xi) H'(x-\xi) + H(x-\xi)$$

integration

$$\int H'(x-\xi) (x-\xi) dx = 0$$

$$(x-\xi) H(x-\xi) = \int_{-\infty}^{\infty} H(x-\xi) d\xi$$

Example: $\frac{d^2 u}{dx^2} = f(x)$

Green func
by Heaviside.

$$u(x=0) = 0 = u(x)$$

$$\angle G(x, \xi) = \delta(x-\xi)$$

$$\Rightarrow \frac{d^2 G(x, \xi)}{dx^2} = \delta(x - \xi)$$

$$\frac{dG}{dx} = \int \delta(x - \xi) dx = \int H'(x - \xi) dx$$

$$\frac{dG}{dx} = H(x - \xi) + A(\xi)$$

$$\Rightarrow G = \int H(x - \xi) dx + xA(\xi) + B(\xi)$$

$$G(x, \xi) = (x - \xi) H(x - \xi) + xA(\xi) + B(\xi)$$

where A & B can be determined by boundary conditions.

$$u(x) = \int_0^x (x - \xi) H(x - \xi) f(\xi) d\xi + x \int_{-\infty}^{\infty} A(\xi) f(\xi) d\xi + \int_{-\infty}^{\infty} B(\xi) f(\xi) d\xi$$

$$u(0) = 0 \Rightarrow 0 = \int_{-\infty}^{\infty} B(\xi) f(\xi) d\xi$$

$$\Rightarrow \boxed{B(\xi) = 0}$$

Now

$$u(1) = 0$$

$$0 = \int_0^1 (1 - \xi) H(1 - \xi) f(\xi) d\xi + \int_{-\infty}^{\infty} A(\xi) f(\xi) d\xi$$

$$A(\xi) = (1 - \xi) H(1 - \xi) \quad \text{if } 0 < \xi < 1$$

$$= 0 \quad \text{if } \xi > 1$$

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$$\Rightarrow \boxed{A(\xi) = \xi - 1}$$

$$G(x, \xi) = (x - \xi) + (x - \xi) - x(1 - \xi)$$

$$0 \leq \xi \leq 1$$

$$\Rightarrow G(0, \xi) = 0 = G(1, \xi)$$

12-05-2016

Establish G.F using variation of parameters.

$$\textcircled{1} \quad L y = [p(x)y']' + q(x)y = -f(x) \quad a < x < b$$

$$\textcircled{2} \quad \alpha y(a) + \beta y'(a) = 0$$

$$\textcircled{3} \quad \alpha_1 y(b) + \beta_1 y'(b) = 0$$

If $\lambda = 0$ is not an eigenvalue of the associated S.L system

$$y(x) = u_1 y_1 + u_2 y_2$$

Here y_1 & y_2 are solutions of the homogeneous Eq.

To be specific we determine y_1 & y_2 upto a constant multiply by requiring that y_1 satisfied condition $\textcircled{1}$ & y_2 satisfied condition $\textcircled{2}$.

$$\textcircled{1} \Rightarrow p y'' + p' y' + q y = -f(x)$$

$$y'' + \frac{p'}{p} y' + \frac{q}{p} y = -\frac{f(x)}{p}$$

$$c_1' = -\frac{y_2 R}{w}$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$c_1' = \frac{+y_2 f(x)}{p(x)w}$$

$$\& c_2' = -\frac{y_1 f(x)}{p(x)w}$$

We determine c_1 & c_2 uniquely and obtained

$$y(x) = y_2(x) \int_a^x \frac{y_1(\xi) f(\xi)}{p(\xi)w(\xi)} d\xi - y_1(x) \int_x^b \frac{y_2(\xi) f(\xi)}{p(\xi)w(\xi)} d\xi$$

$$\text{Let } G(x, \xi) = \begin{cases} -\frac{y_2(x)y_1(\xi)}{p(\xi)w(\xi)} & a \leq \xi \leq x \\ -\frac{y_1(x)y_2(\xi)}{p(\xi)w(\xi)} & x \leq \xi \leq b \end{cases}$$

$$\Rightarrow y = \int_a^b G(x, \xi) f(\xi) d\xi$$

$$\text{Property } G(x, \xi) = G(\xi, x)$$

Example:-

$$y'' = -f$$

$$y(0) = 0 = y'(l)$$

$$D^2 = 0$$

$$\Rightarrow y = A + Bx$$

$$y_1 = A$$

$$y_2 = Bx$$

$$y_1(0) = 0$$

$$\Rightarrow A = 0$$

$$y'(l) = 0 \Rightarrow B = 0$$

$\Rightarrow y = 0$ ^{$\lambda = 0$} there is not an eigenvalue.

Assume that

$$y_1 = 1$$

$$y_2 = x$$

$$p(x) = 1$$

$$w(x) = 1$$

$$G(x, \xi) = \begin{cases} -\frac{y_2(x) y_1(\xi)}{p(\xi) w(\xi)} = -x \\ -\frac{y_1(x) y_2(\xi)}{p(\xi) w(\xi)} = -\xi \end{cases}$$

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$$w = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = 1$$

$$(py')' + (v + dR)y = 0$$

To obtain Fourier representation of Green's function:

let $\{\lambda_n\}_{n \geq 1}$ be the eigenvalues and $\{\phi_n\}_{n \geq 1}$ a set of normalized eigenfunctions for the Eq.

$$(py')' + v y + \delta(x - \xi) y = 0 \quad \text{where } \delta = 1 \text{ weight function}$$

then the Green's func. is written as

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{\phi_n(x) \phi_n(\xi)}{\lambda_n \delta}$$

This series is absolutely uniform & convergent b/w interval $[a, b]$.

Example:-

$\lambda = 0$ not an eigenvalue.

$$\phi_n'' + \lambda \phi_n = 0$$

$$u'' = -f(x)$$

$$D^2 = \lambda$$

$$u(x) = 0 = u'(l)$$

$$D = \pm \sqrt{\lambda} i$$

$$\phi_n(0) = 0 = \phi_n'(l)$$

$$\phi_n = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x$$

$$\phi_n(0) = 0 \Rightarrow A = 0$$

$$\phi_n'(l) = 0 \Rightarrow 0 = AB \cos \sqrt{\lambda} l$$

$$B \neq 0 \text{ but } \cos \sqrt{\lambda} l = 0$$

$$\sqrt{\lambda} = \frac{(2n-1)\pi}{2l}$$

$$\therefore \phi_n(x) = B_n \sin \sqrt{\lambda} x$$

To normalize

$$\int_0^l \phi_n^2 dx = \int_0^l B_n^2 \sin^2 \sqrt{\lambda} x dx$$

$$= \frac{B_n^2}{2} \left(x - \frac{\sin 2\sqrt{\lambda} x}{2\sqrt{\lambda}} \Big|_0^l \right)$$

$$= \frac{B_n^2}{2} l$$

$$\text{Let } B_n = \sqrt{\frac{2}{l}}$$

then

$$\int_0^l \phi_n^2 dx = 1$$

$$\text{So, } \phi_n = \sqrt{\frac{2}{l}} \sin \sqrt{\lambda} x$$

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{\sqrt{\frac{2}{l}} \sin \sqrt{\lambda} x \sqrt{\frac{2}{l}} \sin \sqrt{\lambda} \xi}{\left(\frac{(2n-1)\pi}{2l}\right)^2}$$

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{\left(\sqrt{\frac{2}{l}}\right)^2 \sin \sqrt{\lambda} x \sin \sqrt{\lambda} \xi}{\frac{((2n-1)\pi)^2}{(2l)^2}}$$

$$= \sum_{n=1}^{\infty} \frac{2l \sin \sqrt{\lambda} x \sin \sqrt{\lambda} \xi}{((2n-1)\pi)^2}$$

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Properties:-

$$(p(x)y') + q(x)y + r(x)y = +f(x)$$

$$R_0(y) = \alpha y(a) + \beta y'(a) = 0$$

$$R_1(y) = \alpha y(b) + \beta y'(b) = 0$$

$$y = \int_a^b G(x, \xi) f(\xi) d\xi$$

①:- The func. $G(x, \xi)$ will satisfies

$$L(G) = 0 \quad \text{if } x \neq \xi$$

$$[a, \xi) \cup (\xi, b]$$

②. G satisfies the same B.C.s.

$$R_0(G) = 0$$

$$R_1(G) = 0$$

③. The func. G is continuous for $a \leq \xi$ & $a \leq b$

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④. The partial derivatives are continuous ~~at~~ ^{except} $x = \xi$

$$\left(\frac{\partial G}{\partial x}\right)(\xi+0, \xi) - \left(\frac{\partial G}{\partial x}\right)(\xi-0, \xi) = -\frac{1}{P(\xi)}$$

⑤. The func. G is symmetric

$$G(x, \xi) = G(\xi, x) \text{ for } a \leq \xi, x \leq b$$

Question:- $u'' + \lambda u = -f(x)$
 $u'' + \lambda u = 0$

$$u(0) = 0 = u(1)$$

1st step

no eigenvalue at $\lambda = 0$

$$\lambda = 0 \Rightarrow u'' = 0$$

$$u = A + Bx$$

Use

Property ① \Rightarrow

$$G(0) = 0 = G(1)$$

$$G'' + \lambda G = 0$$

$$G = A + Bx \quad x \neq \xi$$

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$$u'' + \lambda u = 0$$

$$u(0) = 0, \quad u(1) = 0$$

For $\lambda = 0$

$$u'' = 0$$

$$m = 0$$

$$u = A + Bx$$

$$u(0) = 0 \Rightarrow A = 0$$

$$u(1) = 0 \Rightarrow B = 0$$

$\lambda = 0$ is not an eigenvalue.

Step 2.

$$LG = 0, \quad \text{at } x \neq \xi$$

$$\Rightarrow G''(x, \xi) = 0$$

$$G(x, \xi) = \begin{cases} A_1 + B_1 x & 0 \leq x < \xi \\ A_2 + B_2 x & \xi < x \leq 1 \end{cases}$$

Step 3.

$$G(0, \xi) = 0$$

$$G(1, \xi) = 0$$

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Applying $G(0, \xi) = 0 \Rightarrow A_1 = 0$

$$G(1, \xi) = 0 \Rightarrow a_1 + b_1 = 0$$

$$\Rightarrow a_1 = -b_1$$

$$G(x, \xi) = \begin{cases} \beta_1 x & 0 \leq x < \xi \\ -b_1 + b_1 x & \xi < x \leq 1 \end{cases}$$

Step # 4. Since G is continuous on $[0, 1]$.

Therefore G is continuous at $x = \xi$

$$B_1(\xi) = b_1(\xi - 1)$$

$$B_1 = \frac{b_1(\xi - 1)}{\xi}$$

$$\therefore G(x, \xi) = \begin{cases} \frac{b_1(\xi - 1)}{\xi} x & 0 \leq x < \xi \\ b_1(x - 1) & \xi < x \leq 1 \end{cases}$$

Step 5. $G'(\xi + 0, \xi) = G'(\xi - 0, \xi) = -\frac{1}{\rho(\xi)}$

$$b_1 \xi - b_1(\xi - 1) = -\xi$$

$$\boxed{b_1 = -\xi}$$

$$G(x, \xi) = \begin{cases} -(\xi - 1)x & 0 \leq x < \xi \\ -\xi(x - 1) & \xi < x \leq 1 \end{cases}$$

$$y'' = -f$$

$$y(0) = 0$$

$$y'(1) = 0$$