ADVANCED NUMERICAL ANALYSIS

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⇒ Iterative Methods for Solving Non-Linear Equations $f(x)=0$.

There are two types of methods
(i) $f(x)=0$, on $[a, b]$
(ii) $f(x)=0$, no interval given

1:- $f(x)=0$, on $[a, b]$

There are mainly two types of methods

(a) Bisection method (Interval Halving)
(b) Regula Falsi Method (Secant Method, Inverse Interpolation Method)

* Bisection Method:-

Consider $f(x)=0$ on $[a, b]$

Technique:- (i) $f(x)$ must be continuous on the given interval $[a, b]$
(ii) Calculate $f(a)$ and $f(b)$. Find $f(a)\,f(b)$
(iii) If $f(a)\,f(b) < 0$, then $f(x)=0$ has a solution in the interval.
(iv) Find x using the arithmatic

mean, that is $x = \frac{a+b}{2}$

(v) Calculate $f(x)$, Find $f(a)f(x)$ and $f(x)f(b)$

(vi) Choice of the interval halving the solution, if $f(a)f(x) < 0$, then $f(x) = 0$ has a solution in $[a, x]$. Consequently $f(x) = 0$ has no solution in $[x, b]$

(vii) Find $x_1 = \frac{a+x}{2}$, find $f(x_1)$ and repeat the whole process

(viii) If $|x_{n+1} - x_n| \le \varepsilon$ for given $\varepsilon > 0$ then stop the process

Example: Using Bisection method calculate $\sqrt{2}$, take $\varepsilon = 0.001$

Solution

Let $x = \sqrt{2}$

This implies $x^2 = 2$

so $x^2 - 2 = 0$

Take $f(x) = x^2 - 2$

As $x^2 - 2 = 0$ so $f(x) = 0$

Let us take the interval $[a, b] = [1, 2]$

Now

$f(a) = f(1) = (1)^2 - 2 = -1$

$f(b) = f(2) = 2^2 - 2 = 2$

and $f(a)f(b) = (-1)(2) = -2 < 0$

So this implies that $f(x)$ has the solution in $[1, 2]$

Iteration 1:- Let $x_1 = \frac{a+b}{2} = \frac{1+2}{2} = \frac{3}{2}$

So $x_1 = 1.5$

Now we have two intervals
$[1, 1.5]$ and $[1.5, 2]$
Now to check whether the solution exist in the 1st interval or in 2nd.
Take $[a, x_1] = [1, 1.5]$

$f(a) = f(1) = -1$

$f(x_1) = f(1.5) = (1.5)^2 - 2 = 0.25$

Now $f(a)f(x_1) = -1(0.25) = -0.25 < 0$
As $f(a)f(x_1) < 0$ so the solution exist in $[1, 1.5]$, consequently solution does not exist in $[1.5, 2]$

Iteration 2:- Now to divide the interval $[a, x_1] = [1, 1.5]$ into two parts

Let $x_2 = \frac{a+x_1}{2} = \frac{1+1.5}{2}$

So $x_2 = 1.25$

So we have two intervals
$[a, x_2]$ and $[x_2, x_1]$
$[1, 1.25]$ $[1.25, 1.5]$

Now $f(a) = -1$

$f(x_2) = (1.25)^2 - 2 = -0.4375$

and $f(a)f(x_2) = (-1)(-0.4375)$
$= 0.4375 > 0$

So solution does not exist in

this interval. So solution will exist in $[x_2, x_1] = [1.25, 1.5]$

Iteration 3:-

Let $x_3 = \frac{1.25 + 1.5}{2}$

This implies $x_3 = 1.375$

Intervals $[x_2, x_3] = [1.25, 1.375]$
$[x_3, x_1] = [1.375, 1.5]$

Now
$f(x_2) = f(1.25) = -0.4375$
$f(x_3) = f(1.375) = 1.375^2 - 2$
$= -0.1094$

$f(x_2)\, f(x_3) = (-0.4375)(-0.1094)$
$= 0.04786 > 0$

So solution does not exist in this interval. consequently solution will exist in 2nd interval $[x_3, x_1] = [1.375, 1.5]$

Iteration 4:-

Let $x_4 = \frac{x_3 + x_1}{2} = \frac{1.375 + 1.5}{2}$

this implies $x_4 = 1.4375$

Now we have two intervals
$[x_3, x_4] = [1.375, 1.4375]$ and
$[x_4, x_1] = [1.4375, 1.5]$

for first interval.

$f(x_3) = f(1.375) = -0.1094$

$f(x_4) = f(1.4375) = (1.4375)^2 - 2$

$= 0.0664$

Now

$f(x_3)f(x_4) = (-0.1094)(0.0664)$

$= -0.0073 < 0$

So solution exist in this interval.

Itration 5:- Let $x_5 = \frac{x_3 + x_4}{2} = \frac{1.375 + 1.4375}{2}$

So $x_5 = 1.40625$

And we have two intervals

$[x_3, x_5] = [1.375, 1.40625]$ &

$[x_5, x_4] = [1.40625, 1.43,75]$

Now $f(x_3) = f(1.375) = -0.1094$

$f(x_5) = f(1.40625) = (1.40625)^2 - 2$

$= -0.02246$

And $f(x_3)f(x_5) = (-0.1094)(-0.02246)$

$= 0.00246 > 0$

So solution does not exist in this interval. So definitly solution will lie in 2nd interval

$[x_5, x_4] = [1.40625, 1.4375]$

Iteration 6:-

$x_6 = \frac{x_5 + x_4}{2} = \frac{1.40625 + 1.4375}{2}$

$$\Rightarrow \quad x_6 = 1.42187$$

And interval $[x_5, x_6] = [1.40625, 1.42187]$
and $[x_6, x_4] = [1.42187, 1.4375]$
for 1st interval

$f(x_5) = f(1.40625) = -0.02246$

$f(x_6) = f(1.42187) = (1.42187)^2 - 2$

$= 0.02171$

& $f(x_5) f(x_6) = (-0.02246)(0.02171)$

$= 0.00049 < 0$

So solution lies in the interval
$[x_5, x_6] = [1.40625, 1.42187]$

Iteration 7:- $x_7 = \dfrac{x_5 + x_6}{2}$

$$\Rightarrow x_7 = 1.4141$$

Intervals $[x_5, x_7] = [1.40625, 1.4141]$
and $[x_7, x_6] = [1.4141, 1.42187]$

$f(x_5) = f(1.40625) = -0.02246$

$f(x_7) = f(1.4141) = 1.4141^2 - 2 = -0.000$

and $f(x_5) f(x_7) = (-0.02246)(-0.0003)$

$= 0.0000072 > 0$

So solution does not exist in this interval. This implies solution will exist in interval $[x_7, x_6] = [1.4141, 1.42187]$

Iteration 8:-

$$x_8 = \frac{x_7 + x_6}{2} = \frac{1.4141 + 1.42187}{2}$$

$$\Rightarrow \quad x_8 = 1.4179$$

Intervals $[x_7, x_8] = [1.4141, 1.4179]$

and $[x_8, x_6] = [1.4179, 1.42187]$

$$f(x_7) = f(1.4141) = -0.0003$$

$$f(x_8) = f(1.4179) = 0.0104$$

$$f(x_7) f(x_8) = (-0.0003)(0.0104)$$

$$= -3.13 \times 10^{-6} < 0$$

So solution lies in this interval $[x_7, x_8] = [1.4141, 1.4179]$

Iteration 9:-

$$x_9 = \frac{x_7 + x_8}{2} = \frac{1.4141 + 1.4179}{2}$$

$$\Rightarrow \quad x_9 = 1.416$$

Now $|x_9 - x_8| = |1.416 - 1.4179|$

$$= 0.0019 = \varepsilon$$

So the solution is

$$x = 1.416$$

Error Estimate for Bisection Method:-

If x_n is the approximate solution obtained by using the bisection method, then

$$\text{Error Estimate} = |x_n - x| \leq \frac{b-a}{2^n}$$

Note:- The Bisection method in mathematics is a root finding method that repeatedly bisects an interval and then selects a subinterval in which a root must lie for further processing. It is a very simple and robust method, but it is also relatively slow. Because of this it is often used to obtain a rough approximation to a solution which is then used as a starting point for more rapidly converging methods. The method is also called the Interval Halving method, Binary search method or the Dichotomy method.

* Regula Falsi Method:-

Technique:- (i) $f(x)$ is a continuous function on $[a, b]$

(ii) If $f(a) f(b) < 0$ then $f(x) = 0$ has a solution in $[a, b]$

(iii) Calculate $x_1 = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$

(iv) Find $f(x_1)$ and calculate $f(a) f(x_1)$ and $f(x_1) f(b)$, using (ii) select the interval in which $f(x) = 0$ has a solution.

(v) If $|x_{n+1} - x_n| \leq \varepsilon$ for given $\varepsilon > 0$ then x_{n+1} is the approximate solution of $f(x) = 0$

Example:- Find the square root of 2 using regula falsi method with $\varepsilon = 0.001$

Solution

Let $x = \sqrt{2}$

Then $x^2 = 2$

This implies $x^2 - 2 = 0$

Take $f(x) = x^2 - 2$

As $x^2 - 2 = 0$ So $f(x) = 0$

Take the interval $[1, 2]$

Now

$f(1) = 1^2 - 2 = -1$

$f(2) = 2^2 - 2 = 2$

And $f(1)\,f(2) = (-1)(2) = -2 < 0$
So the function $f(x)=0$ has the solution in the interval
$[a,b] = [1,2]$

1st Iteration:- choose

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = \frac{1(2) - 2(-1)}{2-(-1)}$$

This implies $x_1 = \frac{4}{3}$

So we have two intervals
$[a, x_1] = [1, \frac{4}{3}]$ and
$[x_1, b] = [4/3, 2]$

Now to check whether the solution exist in 1st interval or in 2nd.

$f(a) = f(1) = -1$
$f(x_1) = f(\frac{4}{3}) = (\frac{4}{3})^2 - 2 = \frac{-2}{9}$

and $f(a)\,f(x_1) = (-1)(-\frac{2}{9}) = \frac{2}{9} > 0$
So the solution does not exist in this interval, consequently solution will exist in interval
$[x_1, b] = [4/3, 2]$

2nd Iteration:- Let $a = \frac{4}{3}$, $b = 2$

$$x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = \frac{\frac{4}{3}(2) - 2(\frac{-2}{9})}{2-(\frac{-2}{9})}$$

this implies $x_2 = \dfrac{\frac{8}{3}+\frac{4}{9}}{2+\frac{2}{9}} = \dfrac{\frac{24+4}{9}}{\frac{18+2}{9}} = \dfrac{28}{20}$

So $x_2 = 1.4$

and intervals

$[a, x_2] = [1.333, 1.4]$ & $[x_2, b] = [1.4, 2]$

Now $f(a) = f(1.333) = -0.2222$

and $f(x_2) = f(1.4) = (1.4)^2 - 2 = -0.04$

This implies ~~so that~~ $f(a)\,f(x_2) = (-0.222)(-0.04)$
$= 0.0088 > 0$

So the solution does not exist in this interval, consequently solution will lie in $[x_2, b] = [1.4, 2]$

3rd Iteration:- Let $a = 1.4$ and $b = 2$

$$x_3 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{1.4\,f(2) - 2\,f(1.4)}{f(2) - f(1.4)}$$

$$= \frac{(1.4)(2) - 2(-0.04)}{2 - (-0.04)} = \frac{2.88}{2.04}$$

So $x_3 = 1.4118$

Now again we have two intervals

$[a, x_3] = [1.4, 1.4118]$ and

$[x_3, b] = [1.4118, 2]$

for first interval

$f(a) = f(1.41) = -0.04$
$f(x_3) = f(1.4118) = 1.4118^2 - 2 = -0.0068$

So $f(a)f(x_3) = (-0.04)(-0.0068)$
$= 0.00027 > 0$

So the solution does not exist in 1st interval, consequently solution will exist in interval
$[x_3, b] = [1.4118, 2]$

4th Iteration $[a, b] = [1.4118, 2]$

$$x_4 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = \frac{1.4118(2) - 2(-0.0068)}{2 - (-0.0068)}$$

$$= \frac{2.8372}{2.0068} = 1.4137$$

So two intervals

$[a, x_4] = [1.4118, 1.4137]$ and
$[x_4, b] = [1.4137, 2]$

$f(a) = f(1.4118) = -0.0068$
and $f(x_4) = f(1.4137) = (1.4137)^2 - 2$
$= -0.0014$

Now $f(a)f(x_4) = (-0.0068)(-0.0014)$
$= 0.000009 > 0$

As $f(a)f(x_4) > 0$, so solution

does not exist in this interval.
So solution will exist in interval
$[x_4, b] = [1.4137, 2]$

5th Iteration:- $a = 1.4137$, $b = 2$

$$x_5 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{1.4137 f(2) - 2f(1.4137)}{f(2) - f(1.4137)}$$

$$= \frac{1.4137(2) - 2(-0.0014)}{2 - (-0.0014)} = \frac{2.8302}{2.0014}$$

So $x_5 = 1.4141$

So, we have two intervals
$[a, x_5] = [1.4137, 1.4141]$ and
$[x_5, b] = [1.4141, 2]$

Now $f(a) = f(1.4137) = -0.0014$

$f(x_5) = f(1.4141) = (1.4141)^2 - 2 = -0.00032$

And $f(a) f(x_5) = (-0.0014)(-0.00032)$
$= 0.00000045 > 0$

As $f(a) f(x_5) > 0$ so solution does not exist in this interval
This implies solution will be in
$[x_5, b] = [1.4141, 2]$

6th Iteration:- $a = 1.4141$, $b = 2$

$$x_6 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{(1.4141) f(2) - 2f(1.4141)}{f(2) - f(1.4141)}$$

This implies $x_6 = \dfrac{1.41411(2) - 2(-0.00032)}{2 - (-0.00032)}$

$$\Rightarrow x_6 = \frac{2.82884}{2.00032} = 1.41419$$

As $|x_6 - x_5| = |1.41419 - 1.41411|$

$$= 0.00009 < \varepsilon$$

So Approximate solution is

$$x = 1.41419$$

Modified Regula Falsi Method:-

In the regula Falsi method, one end (Point) of the interval remains fixed, while the other end (Point) of the intervals moves to the approximate solution. This is the main drawback of this method.

To overcome this draw back one can modify the Regula Falsi method as follows

(i) Find $x_1 = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$

(ii) Calculate $f(x_1)$ and find

$f(a)\, f(x_1)$ and $f(x_1)\, f(b)$

(iii) If $f(x_1) f(a) < 0$, then the given equation $f(x) = 0$ has solution in the interval $[a, x_1]$. This mean solution does not exist in $[x_1, b]$

(iv) Select the new interval $[a, x_1]$. This means that point 'a' is fixed.

(v) Find $\frac{f(a)}{2}$ and calculate

$$x_2 = \frac{a f(x_1) - x_1 \frac{f(a)}{2}}{f(x_1) - \frac{f(a)}{2}}$$

(vi) Find $f(x_2)$, $f(x_1) f(x_2)$ & $f(a) f(x_2)$ if $f(x_1) f(x_2) < 0$, then the equation $f(x) = 0$ has the solution in $[x_2, x_1]$

(vii) $$x_3 = \frac{x_2 \frac{f(x_1)}{2} - x_1 f(x_2)}{\frac{f(x_1)}{2} - f(x_2)}$$

and so on

Example:- Calculate $\sqrt{2}$ using Modified Regula Falsi Method.

Solution

Let $x = \sqrt{2}$

Then $x^2 = 2$ so $x^2 - 2 = 0$

Take $f(x) = x^2 - 2$

As $x^2 - 2 = 0$ so $f(x) = 0$

Let us take the interval $[a, b] = [1$

Now

$$f(a) = f(1) = (1)^2 - 2 = -1$$

$$f(b) = f(2) = (2)^2 - 2 = 2$$

and

$$f(a)\,f(b) = (-1)(2) = -2 < 0$$

As $f(a)\,f(b) < 0$, so the equation $f(x) = 0$ has the solution in interval $[a, b] = [1, 2]$

Iteration 1:-

Let $x_1 = \dfrac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$

$$= \frac{1\,f(2) - 2\,f(1)}{f(2) - f(1)} = \frac{1(2) - 2(-1)}{2 - (-1)}$$

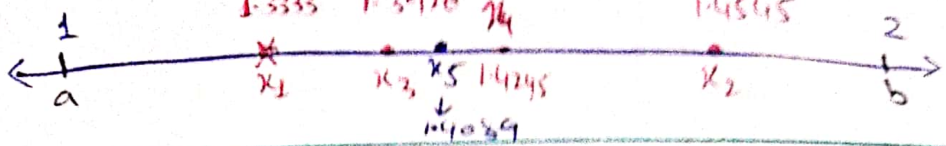
This implies $x_1 = 1.3333$

Now we have two intervals $[a, x_1] = [1, 1.3333]$ & $[x_1, b] = [1.333, 2]$

Now $f(x_1) = f(1.3333) = (1.3333)^2 - 2$

$$= -0.2222$$

$f(a) = f(1) = 1$



and $f(a)\, f(x_1) = (-1)(-0.2222)$

$= 0.2222 > 0$

As $f(a)\, f(x_1) > 0$ so equation $f(x) = 0$ has no solution in $[a, x_1] = [1, 1.3333]$. Consequently the solution will lie in interval $[x_1, b] = [1.3333, 2]$

Here $b = 2$ is fixed.

Now we will calculate $\frac{f(b)}{2}$

$$f(b) = f(2) = 2 \Rightarrow \frac{f(b)}{2} = \frac{2}{2} = 1$$

Iteration 2:-

$$x_2 = \frac{x_1 \frac{f(b)}{2} - b\, f(x_1)}{\frac{f(b)}{2} - f(x_1)}$$

$$= \frac{1.3333(1) - 2(-0.2222)}{1 - (-0.2222)} = \frac{1.777}{1.2222}$$

This implies $x_2 = 1.4545$

Now we have two interval $[x_1, x_2] = [1.3333, 1.4545]$ and $[x_2, b] = [1.4545, 2]$

for first interval

$$f(x_1) = f(1.3333) = -0.2222$$

$$f(x_2) = f(1.4545) = (1.4545)^2 - 2$$

$$= 0.1156$$

And

$$f(x_1)\, f(x_2) = (-0.2222)(0.1156)$$

$$= -0.0257 < 0$$

As $f(x_1)f(x_2) < 0$ so equation $f(x)$ has solution in $[x_1, x_2] = [1.3333, 1.4545]$

The fix point is $x_1 = 1.3333$

Now $\frac{f(x_1)}{2} = \frac{f(1.3333)}{2} = \frac{-0.2222}{2}$

This implies $\frac{f(x_1)}{2} = -0.1111$

Iteration 3:-

Now $x_3 = \dfrac{x_1 f(x_2) - x_2 \frac{f(x_1)}{2}}{f(x_2) - \frac{f(x_1)}{2}}$

$$\Rightarrow x_3 = \frac{1.3333(0.1156) - 1.4545(-0.1111)}{0.1156 - (-0.1111)}$$

$$= \frac{0.3157}{0.2267} = 1.3926$$

So $x_3 = 1.3926$

Now we have to interval

$[x_1, x_3] = [1.3333, 1.3926]$ &

$[x_3, x_2] = [1.3926, 1.4545]$

for first interval

$f(x_1) = f(1.3333) = -0.2222$

$f(x_3) = f(1.3926) = 1.3926^2 - 2 = -0.0607$

$f(x_1)f(x_3) = (-0.2222)(-0.0607)$

$= 0.0135 > 0$

As $f(x_1)f(x_3) > 0$ so equation $f(x)=0$ has no solution in $[x_1, x_3] = [1.333, 1.3926]$ consequently solution will exist in $[x_3, x_2] = [1.3926, 1.4545]$

Here $x_2 = 1.4545$ is fixed point

Now $\frac{f(x_2)}{2} = \frac{f(1.4545)}{2} = \frac{0.1156}{2}$

This implies $\frac{f(x_2)}{2} = 0.0578$

Iteration 4:-

$$x_4 = \frac{x_3 \frac{f(x_2)}{2} - x_2 f(x_3)}{\frac{f(x_2)}{2} - f(x_3)}$$

$$\Rightarrow x_4 = \frac{1.3926(0.0578) - 1.4545(-0.0607)}{0.0578 - (-0.0607)}$$

$$= \frac{0.1688}{0.1185}$$

This implies $x_4 = 1.4245$

Intervals to check

$[x_3, x_4] = [1.3926, 1.4245]$

$[x_4, x_2] = [1.4245, 1.4545]$

for 1st Interval

$f(x_3) = f(1.3926) = -0.0607$

$f(x_4) = f(1.4245) = (1.4245)^2 - 2$

$= 0.0292$

and $f(x_3)\,f(x_4) = (-0.0607)(0.0292)$

$= -0.0017 < 0$

As $f(x_3)\,f(x_4) < 0$ so solution will exist in $[x_3, x_4] = [1.3926, 1.4245]$

fixed point is $x_3 = 1.3926$

So $\dfrac{f(x_3)}{2} = \dfrac{-0.0607}{2} = -0.03035$

Iteration 5:-

$$x_5 = \frac{x_3 f(x_4) - x_4 \frac{f(x_3)}{2}}{f(x_4) - \frac{f(x_3)}{2}}$$

$$\Rightarrow x_5 = \frac{1.3926(0.0292) - 1.4245(-0.03035)}{0.0292 - (-0.03035)}$$

$$= \frac{0.0839}{0.05955}$$

This implies $x_5 = 1.4089$

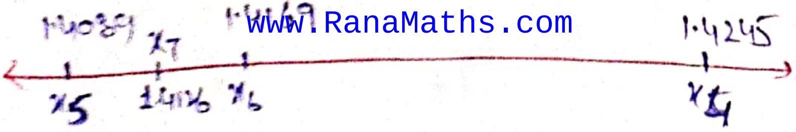
Now we have two intervals
$[x_3, x_5] = [1.3926, 1.4089]$ &
$[x_5, x_4] = [1.4089, 1.4245]$

Now
$f(x_3) = f(1.3926) = -0.0607$

$f(x_5) = f(1.4089) = (1.4089)^2 - 2$

$= -0.015$

and
$f(x_3)f(x_5) = (-0.0607)(-0.015) > 0$



As $f(x_3)f(x_5) > 0$ so solution does not exist in $[x_3, x_5]$ so definitly solution will exist in $[x_5, x_4]$
$= [1.4089, 1.4245]$
Here $x_4 = 1.4245$ is fixed point

Now $\dfrac{f(x_4)}{2} = \dfrac{0.0292}{2} = 0.0146$

Iteration 6:-

$$x_6 = \frac{x_5 \frac{f(x_4)}{2} - x_4 f(x_5)}{\frac{f(x_4)}{2} - f(x_5)}$$

$$\Rightarrow x_6 = \frac{1.4089(0.0146) - 1.4245(-0.015)}{0.0146 - (-0.015)}$$

$$= \frac{0.04194}{0.0296}$$

This implies $x_6 = 1.4169$
We have two new intervals
$[x_5, x_6] = [1.4089, 1.4169]$ &
$[x_6, x_4] = [1.4169, 1.4245]$
Now
$f(x_5) = f(1.4089) = -0.015$
$f(x_6) = f(1.4169) = 1.4169^2 - 2 = 0.0076$
and $f(x_5)f(x_6) = (-0.015)(0.0076)$
$= -0.00011 < 0$
as $f(x_5)f(x_6) < 0$ so solution will exist in interval $[x_5, x_6] = [1.4089, 1.4169]$
Here $x_5 = 1.4089$ is fixed

Now $\frac{f(x_5)}{2} = \frac{-0.015}{2} = -0.0075$

Iteration 7:-

$$x_7 = \frac{x_5 f(x_6) - x_6 \frac{f(x_5)}{2}}{f(x_6) - \frac{f(x_5)}{2}}$$

$$\Rightarrow x_7 = \frac{1.4089(0.0076) - (1.4169)(-0.0075)}{0.0076 - (-0.0075)}$$

$$= \frac{0.02133}{0.0151}$$

This implies $x_7 = 1.4126$

Now the two new intervals are

$[x_5, x_7] = [1.4089, 1.4126]$ and

$[x_7, x_6] = [1.4126, 1.4169]$

for first interval

$f(x_5) = f(1.4089) = -0.015$

$f(x_7) = f(1.4126) = (1.4126)^2 - 2 = -0.0046$

and $f(x_5) f(x_7) = 0.000069 > 0$

As $f(x_5) f(x_7) > 0$ so solution of $f(x) = 0$ does not exist in this interval. Consequently solution will lie in $[x_7, x_6] = [1.4126, 1.4169]$

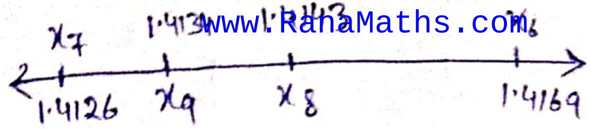
The fixed point is $x_6 = 1.4169$

Now $\frac{f(x_6)}{2} = \frac{f(1.4169)}{2} = \frac{0.0076}{2}$

$= 0.0038$

Iteration 8:-

$$x_8 = \frac{x_7 \frac{f(x_6)}{2} - x_6 f(x_7)}{\frac{f(x_6)}{2} - f(x_7)}$$



$$\Rightarrow x_8 = \frac{1.4126(0.0038) - 1.4169(-0.0046)}{0.0038 - (-0.0046)}$$

$$= \frac{0.011886}{0.0084}$$

This implies $x_8 = 1.4143$

Now we have two new intervals

$[x_7, x_8] = [1.4126, 1.4143]$ and

$[x_8, x_6] = [1.4143, 1.4169]$

Now

$$f(x_7) = f(1.4126) = -0.0046$$

$$f(x_8) = f(1.4143) = (1.4143)^2 - 2 = 0.00024$$

And

$$f(x_7)f(x_8) = (-0.0046)(0.00024) = -0.0000011 < 0$$

As $f(x_7)f(x_8) < 0$ so solution will lie in $[x_7, x_8] = [1.4126, 1.4143]$

Here $x_7 = 1.4126$ is fixed, So

$$\frac{f(x_7)}{2} = \frac{-0.0046}{2} = -0.0023$$

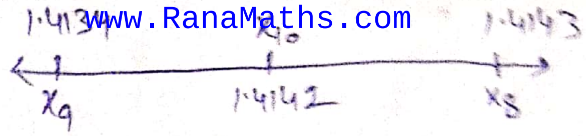
Iteration 9 :-

$$x_9 = \frac{x_7 f(x_8) - x_8 \frac{f(x_7)}{2}}{f(x_8) - \frac{f(x_7)}{2}}$$

$$\Rightarrow x_9 = \frac{1.4126(0.00024) - 1.4143(-0.0023)}{0.00024 - (-0.0023)}$$

$$= \frac{0.00359}{0.0025}$$

This implies $x_9 = 1.4134$



Now we have two intervals
$[x_7, x_9] = [1.4126, 1.4134]$ and
$[x_9, x_8] = [1.4134, 1.4143]$

Now

$$f(x_9) = f(1.4134) = (1.4134)^2 - 2$$
$$= -0.0023$$
$$f(x_8) = f(1.4143) = 0.00024$$

And

$$f(x_9)f(x_8) = (-0.0023)(0.00024)$$
$$= -0.00000055 < 0$$

As $f(x_9)f(x_8) < 0$ so solution will exist in interval $[x_9, x_8] = [1.4134, 1.4143]$

Here $x_8 = 1.4143$ is fixed, so

$$\frac{f(x_8)}{2} = \frac{0.00024}{2} = 0.00012$$

Iteration 10:-

$$x_{10} = \frac{x_9 \frac{f(x_8)}{2} - x_8 f(x_9)}{\frac{f(x_8)}{2} - f(x_9)}$$

$$\Rightarrow x_{10} = \frac{1.4134(0.00012) - 1.4143(-0.0023)}{0.00012 - (-0.0023)}$$

$$= \frac{0.0034}{0.00242}$$

$$\Rightarrow x_{10} = 1.41426$$

So the interval
$[x_9, x_{10}] = [1.4134, 1.41426]$ &
$[x_{10}, x_8] = [1.41426, 1.4134]$

$f(x_9) = f(1.4134) = -0.0023$

$f(x_{10}) = f(1.41426) = (1.41426)^2 - 2 = 0.00013$

$f(x_9)f(x_{10}) = (-0.0023)(0.00013) < 0$

As $f(x_9)f(x_{10}) < 0$ so solution will lie in $[x_9, x_{10}] = [1.4134, 1.41426]$

Here $x_9 = 1.4134$ is fixed, so

$$\frac{f(x_9)}{2} = \frac{-0.0023}{2} = -0.00115$$

Iteration 11:-

$$x_{11} = \frac{x_9 f(x_{10}) - x_{10} \frac{f(x_9)}{2}}{f(x_{10}) - \frac{f(x_9)}{2}}$$

$$\Rightarrow x_{11} = \frac{1.4134(0.00013) - 1.41426(-0.00115)}{0.00013 - (-0.00115)}$$

$$\Rightarrow x_{11} = 1.4142$$

As $|x_{10} - x_{11}| = |1.41426 - 1.4142|$

$= 0.00006 < \varepsilon$

So Approximate solution is

$$\boxed{x_{11} = 1.4142}$$

* Consider a non-linear equation $f(x)$. This equation $f(x)=0$ has a solution for all value of x.

In most of the problems it is very difficult to find the exact solution of non-linear equation $f(x)=0 \longrightarrow (1)$

In these situations we try to find the approximate solution by reformulation the equation as

$$x = g(x) \longrightarrow (2)$$

Where $g(x)$ is an arbitrary function. We study those conditions on the function $g(x)$ under which the equation $x=g(x)$ has a fixed point in the interval.

In general

$$f(x)=0 \iff x=g(x)$$

e.g $f(x) = x^2-2 = 0$

(i) $x^2-2=0$ from this

$$x = \frac{2}{x} \cong g_1(x)$$

(ii) $x+x^2-2=x \Rightarrow x(1+x)=2+x$

$$\Rightarrow x = \frac{2+x}{1+x} = g_2(x)$$

(iii) $x = \left(x - \frac{x^2-2}{2x}\right) = g_3(x)$

Definition (Fix Point) A point p is called the fix point of $g(x)$, if
$$g(p) = p$$

Method of Finding The Approximate Solution Using Fix Point Formulation

Algorithm 1:- For a given x_0 find x_{n+1} by iterative scheme
$$x_{n+1} = g(x_n) \quad ; \quad n = 0, 1, 2, \cdots$$
This method is called explicit method. For x_0 find
$$x_1 = g(x_0)$$
$$x_2 = g(x_1)$$
$$x_3 = g(x_2)$$
$$\vdots \quad \vdots \quad \vdots$$
$$x_{n+1} = g(x_n)$$

Algorithm 2:- For a given x_0 find x_{n+1} by the iterative scheme
$$x_{n+1} = g(x_{n+1})$$
This algorithm is called

implicit method.

To impliment the implicit method we always use the predictor and corrector techniques, Consequently the implicit method is equal to two-step method.

Algorithm 3:- (Two-step Method)

For a given x_0 find x_{n+1} by the iterative scheme as

$$y_n = g(x_n) \quad ; \text{ Predictor}$$

$$x_{n+1} = g(y_n) \quad ; \text{ Corrector}.$$

$$f(x) = 0 \iff x = g(x) \ ;$$

$g(x)$ is an arbitrary function

Example $f(x) = x^2 - 2 \ ; \ [-2, 2]$

$x = -1 \Rightarrow f(-1) = (-1)^2 - 2 = -1 \in [-2, 2]$

$x = 2 \Rightarrow f(2) = (2)^2 - 2 = 2 \in [-2, 2]$

Consequently $x = -1$ and $x = 2$ are two fix points for $g(x) = x^2 - 2$ on $[-2, 2]$

Problem:- How to find fix point of an arbitrary function $g(x)$ ... ?

Theorem- 1) Let $g(x)$ be a continuous function on $[a,b]$. The function $g(x)$ has fixed point in $[a,b]$ if, for all $x \in [a,b]$; $g(a), g(b) \in [a,b]$ and $g(x) \in [a,b]$

~~Proof~~

At $x = a \in [a,b]$
$g(a) = a \in [a,b]$

At $x = b \in [a,b]$
$g(b) = b \in [a,b]$

otherwise we have
$g(a) - a > 0$ & $g(b) - b < 0$

Consider the arbitrary function
$H(x) = g(x) - x$

Find
$H(a) = g(a) - a > 0$
$H(b) = g(b) - b < 0$

using the mean value theorem, there exist a point $p \in [a,b]$ such that
$H(p) = 0$

Thus $H(p) = g(p) - p = 0$
or $g(p) = p$ has a fixed point in $[a,b]$

2) If $g(x)$ has derivative on (a,b) and there exist a constant

say $0 < k < 1$ such that

$|g'(\xi)| \leq k < 1$; for $a \leq \xi \leq b$

Then it has a unique fixed point.

Proof Let $p \neq q$ be two fixed point, then

$p = g(p)$

& $q = g(q)$

Consider $|p-q| = |g(p)-g(q)|$

$\Rightarrow |p-q| = \left|\frac{g(p)-g(q)}{(p-q)}(p-q)\right|$

$= \left|\frac{g(p)-g(q)}{p-q}\right| |p-q|$

$= g'(\xi)\,|p-q|$; $p < \xi < q$

$\Rightarrow |p-q| \leq k\,|p-q|$

This implies $(1-k)|p-q| \leq 0$

$\Rightarrow$ since $1-k > 0$, so $k < 1$

This implies $|p-q| \leq 0$

$\Rightarrow |p-q| = 0$

$\Rightarrow p = q$ uniqueness

★ Consider the equation

$$x = g(x) \longrightarrow ①$$

Find x_{n+1} for given x_0 by the iterative scheme $x_{n+1} = g(x_n) \longrightarrow ②$ $n = 1, 2, \ldots$

↳ We want to know under what conditions $x_{n+1} \xrightarrow{n\to\infty} x_n$, the exact solution $\left(\lim_{n\to\infty} |x_{n+1} - x| = 0\right)$

Consider

$$|x_{n+1} - x| = |g(x_n) - g(x)|$$

$$\Rightarrow |x_{n+1} - x| = \left|\frac{g(x_n) - g(x)}{x_n - x}(x_n - x)\right|$$

$$= \left|\frac{g(x_n) - g(x)}{x_n - x}\right| |x_n - x|$$

$$= g'(\xi)\,|x_n - x| \quad ; \; x_n < \xi < x$$

$$\leq k\,|x_n - x|$$

$$\leq k\left(k\,|x_{n-1} - x|\right)$$

$$= k^2\,|x_{n-1} - x|$$

$$\vdots \qquad \vdots \qquad \vdots$$

$\because k < 1$

$\lim_{n\to\infty} k^n = 0$

$$\leq k^n\,|x_0 - x|$$

$$\Rightarrow \lim_{n\to\infty} |x_{n+1} - x| \leq \lim_{n\to\infty}\left(k^n\,|x_0 - x|\right)$$

$$= \lim_{n\to\infty} k^n\,|x_0 - x| = 0$$

$$\Rightarrow \lim_{n\to\infty} |x_{n+1} - x| = 0$$

$$\Rightarrow \lim_{n\to\infty} |x_{n+1}| = x$$

Theorem 2.3:- Let $g(x)$ be a differentiable continuous function on $[a,b]$, then the approximate solution x_{n+1} obtained from the iteration scheme

$$x_{n+1} = g(x_n) \quad ; n = 0, 1, 2, \dots$$

converge to exact solution x satisfying $x = g(x)$, provided there exist a constant $k<1$ such that

$$g'(\xi) \leq k < 1 \quad ; \quad a \leq \xi \leq b$$

Assignment (Corollary 2.4) If g satisfies the hypothesis of theorem 2.3, The bounds of the error involved in using x_n to approximate x are given by

$$|x_n - x| \leq k^n \max |x_0 - a, b - x_0|$$

and $$|x_n - x| \leq \frac{k^n}{1-k} |x_1 - x_0| \quad \forall\, n \geq 1$$

Proof:

As $x_n \in [a,b]$ for $n \geq 1$

~~Now by~~

$$|x_{n+1} - x| = |g(x_n) - g(x)|$$

$$= \left| \frac{g(x_n) - g(x)}{x_n - x} (x_n - x) \right|$$

$$= \left| \frac{g(x_n) - g(x)}{x_n - x} \right| |x_n - x|$$

$$= g'(\xi) |x_n - x| \quad ; x_n < \xi < x$$

$$\leq k |x_n - x| \quad \because g'(\xi) \leq k$$

$$\leq k (k |x_{n-1} - x|)$$

$$= k^2 |x_{n-1} - x|$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$|x_{n+1} - x| \leq k^n |x_0 - x| \longrightarrow ①$$

$$\leq k^n \max\{x_0 - a, b - x_0\}$$

Now for $n \geq 1$

$$|x_{n+1} - x_n| = |g(x_n) - g(x_{n-1})|$$

$$\leq k |x_n - x_{n-1}| \quad \because \text{using above procedure}$$

$$\leq k (k |x_{n-1} - x_{n-2}|)$$

$$= k^2 |x_{n-1} - x_{n-2}|$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$|x_{n+1} - x_n| \leq k^n |x_1 - x_0| \longrightarrow ②$$

Thus for $m > n \geq 1$

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - \cdots + x_{n+1} - x_n|$$

Since by adding & subtracting $(x_{m-1} + x_{m-2} + \cdots + x_{n+1})$

$$\Rightarrow |x_m - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$

$$\leq k^{m-1}|x_1 - x_0| + k^{m-2}|x_1 - x_0| + \cdots + k^n |x_1 - x_0| \quad \text{using equ②}$$

$$= k^n |x_1 - x_0| + k^{n+1}|x_1 - x_0| + \cdots + k^{m-2}|x_1 - x_0| + k^{m-1}|x_1 - x_0|$$

$$= k^n |x_1 - x_0| \{1 + k + k^2 + \cdots + k^{m-1-n}\}$$

By theorem $\lim_{m \to \infty} x_m = x$

This implies

$$\lim_{m \to \infty} |x_m - x_n| \leq \lim_{m \to \infty} k^n |x_1 - x_0| \sum_{i=0}^{m-n-1} k^i$$

$$\Rightarrow |x - x_n| \leq k^n |x_1 - x_0| \sum_{i=0}^{\infty} k^i$$

But $\sum_{i=0}^{\infty} k^i$ is a geometric

series with ratio k ; $0<k<1$
This series converges to $\frac{1}{1-k}$
This implies

$$|x-x_n| \leq \frac{k^n}{1-k}|x_1-x_0|$$

So $|x_n-x| \leq \frac{k^n}{1-k}|x_1-x_0|$ Proved

---

* $f(x)=0 \iff x=g(x)$; $g(x)$ is an arbitrary function

p is the fixed point of $g(x)$,
that is $g(p)=p$

This alternative fixed point formula $x=g(x)$ is used to suggest the iterative methods.

1) For given x_0, compute x_{n+1} by
$x_{n+1}=g(x_n)$; $n=0,1,2,\dots$
2) $x_{n+1}=g(x_{n+1})$; $n=0,1,2,\dots$

* Taylor Series:-

$$f(x)=f(r)+(x-r)f'(r)+\frac{(x-r)^2}{2}f''(r)+\cdots$$

Note that $(x-r)^2 << (x-r)$,
Assume that $(x-r)^2 \cong 0$ $\because |x-r|<1$
Also $f'(r)\neq 0$, $f''(r)\neq 0$, ...

$\Rightarrow f(x)=f(r)+(x-r)f'(r)+O(r^2)$

$$\Rightarrow \quad f(x) = f(r) + (x-r)f'(r) \quad ; f'(r$$

since $f(x)=0$, so

$$f(r) + (x-r)\,f'(r) = 0$$

Solving for x, we have

$$x = r - \frac{f(r)}{f'(r)} \quad ; f'(r) \neq 0$$

that is

$$x = x - \frac{f(x)}{f'(x)} \quad ; f'(x) \neq 0$$

$$= g(x)$$

where $g(x) = x - \dfrac{f(x)}{f'(x)}$

Algorithm (Newton Method) For a given x_0, find x_{n+1} by iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ; n = 0, 1, 2, \cdots$$

This method is known as Newton Method and the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

is called Newton formula.

Example:- Find $\sqrt{2}$ using Newton Method

Solution Let $x = \sqrt{2}$

Then $x^2 = 2 \Rightarrow x^2 - 2 = 0$

Take $f(x) = x^2 - 2$

As $x^2 - 2 = 0$, So $f(x) = 0$

Now Initial guess is 1

So

$$f(1) = (1)^2 - 2 = -1$$

$$f'(x) = 2x \quad \& \quad f'(1) = 2(1) = 2$$

$$x_0 = 1$$

1st Iteration:-

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(-1)}{2}$$

$$= 1 + \frac{1}{2} = \frac{3}{2}$$

Now

$$f(x_1) = f\left(\tfrac{3}{2}\right) = \left(\tfrac{3}{2}\right)^2 - 2 = 0.25$$

$$f'(x_1) = f'\left(\tfrac{3}{2}\right) = 2\left(\tfrac{3}{2}\right) = 3$$

2nd Iteration:-

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{3}{2} - \frac{0.25}{3}$$

$$\Rightarrow x_2 = 1.4167$$

So

$$f(x_2) = f(1.4167) = (1.4167)^2 - 2 = 0.0069$$

$$f'(x_2) = f'(1.4167) = 2(1.4167) = 2.8333$$

3rd Iteration:-

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.4167 - \frac{0.0069}{2.8333}$$

$$\Rightarrow \quad x_3 = 1.41423$$

So $f(x_3) = f(1.41423) = (1.41423)^2 - 2 = 0.00005$

$f'(x_3) = f'(1.41423) = 2(1.41423) = 2.82846$

4th Iteration:-

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.41423 - \frac{0.00005}{2.82846}$$

This implies

$$x_4 = 1.41421$$

Now

$$|x_4 - x_3| = |1.41421 - 1.41423|$$

$$= 0.00002 < \varepsilon$$

This implies $x_4 = 1.41421$ is the approximate solution.

★ Taylor Series:-

$$f(x) = f(r) + (x-r)f'(r) + \frac{1}{2}(x-r)^2 f''(r)$$

assume $(x-r)^3 \simeq 0$, $(x-r)^4 \simeq 0, \ldots$

So as $f(x) \simeq 0$

This implies

$$\frac{1}{2}(x-r)^2 f''(r) + (x-r)f'(r) + f(r) = 0$$

$$\Rightarrow (x-r)^2 f''(r) + 2(x-r)f'(r) + 2f(r) = 0 \longrightarrow (A)$$

This equation (A) is quadratic in $(x-r)$, so

$$x-r = \frac{-2f'(r) \pm \sqrt{[-2f'(r)]^2 - 4f''(r)(2f(r))}}{2f''(r)}$$

$$= \frac{-2f'(r) \pm \sqrt{4(f'(r))^2 - 8f''(r)f(r)}}{2f''(r)}$$

$$= \frac{-2f'(r) \pm \sqrt{4\{(f'(r))^2 - 2f''(r)f(r)\}}}{2f''(r)}$$

$$= \frac{\cancel{2}\left\{-f'(r) \pm \sqrt{(f'(r))^2 - 2f''(r)f(r)}\right\}}{\cancel{2}f''(r)}$$

$$\Rightarrow x = r + \frac{-f'(r) \pm \sqrt{(f'(r))^2 - 2f''(r)f(r)}}{f''(r)}$$

So

$$x = r - \frac{f'(r) \mp \sqrt{(f'(r))^2 - 2f''(r)f(r)}}{f''(r)}$$

Now rationalizing w.r.t negative sign

$$x = r - \frac{f'(r) - \sqrt{(f'(r))^2 - 2f''(r)f(r)}}{f''(r)} \times \left\{\frac{f'(r) + \sqrt{f'(r)^2 - \ldots}}{f'(r) + \sqrt{f'(r)^2 - 2\ldots}}\right.$$

$$= r - \frac{\{f'(r)\}^2 - \left\{\sqrt{(f'(r))^2 - 2f''(r)f(r)}\right\}^2}{f''(r)\left\{f'(r) + \sqrt{(f'(r))^2 - 2f''(r)f(r)}\right\}}$$

$$= r - \frac{\cancel{\{f'(r)\}^2} - \cancel{\{f'(r)\}^2} + 2f''(r)f(r)}{}$$

$$= r - \frac{2\cancel{f''(r)}f(r)}{\cancel{f''(r)}\left\{f'(r) + \sqrt{(f'(r))^2 - 2f''(r)f(r)}\right\}}$$

$$= r - \frac{2f(r)}{f'(r) + f'(r)\sqrt{1 - \frac{2f''(r)f(r)}{(f'(r))^2}}}$$

$$= r - \frac{2f(r)}{f'(r) + f'(r)\left\{1 - \frac{2f(r)f''(r)}{f'(r)^2}\right\}^{1/2}}$$

This implies

$$x = r - \frac{2f(r)}{f'(r) + f'(r)\left\{1 - \frac{1}{2}\frac{2f(r)f''(r)}{f'(r)^2}\right\}}$$

Since using binomial series ang neglecting higher terms.

$$\Rightarrow x = r - \frac{2f(r)}{f'(r) + f'(r) - \frac{f(r)f''(r)}{f'(r)}}$$

$$= r - \frac{2f(r)}{2f'(r) - \frac{f(r)f''(r)}{f'(r)}}$$

$$= r - \frac{2f(r)}{\frac{2(f'(r))^2 - f(r)f''(r)}{f'(r)}}$$

$$= r - \frac{2f(r)f'(r)}{2(f'(r))^2 - f(r)f''(r)}$$

$$\Rightarrow x = x - \frac{2f(x)f'(x)}{2(f'(x))^2 - f'(x)f''(x)}$$

Algorithm (Halley Method)

For a given find x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2(f'(x_n))^2 - f(x_n)f''(x_n)}$$

$n = 0, 1, 2, \ldots$

Example:- Find $\sqrt{2}$ by using Halley method

Solution

Let $x = \sqrt{2}$

This implies $x^2 = 2 \Rightarrow x^2 - 2 = 0$

take $f(x) = x^2 - 2$

As $x^2 - 2 = 0$, so $f(x) = 0$

Let $x_0 = 1$

$f'(x) = 2x$, $f''(x) = 2$

Now

$f(1) = (1)^2 - 2 = -1$

$f'(1) = 2(1) = 2$

$f''(1) = 2$

1st Iteration:-

$$x_1 = x_0 - \frac{2f(x_0)f'(x_0)}{2(f'(x_0))^2 - f(x_0)f''(x_0)}$$

$$= 1 - \frac{2(-1)(2)}{2(2)^2 - (-1)(2)}$$

This implies $x_1 = 1.4$

Now

$$f(x_1) = f(1.4) = (1.4)^2 - 2 = -0.04$$

$$f'(x_1) = f'(1.4) = 2(1.4) = 2.8$$

$$f''(x_1) = f''(1.4) = 2$$

2nd Iteration:-

$$x_2 = x_1 - \frac{2f(x_1)f'(x_1)}{2(f'(x_1))^2 - f(x_1)f''(x_1)}$$

$$= 1.4 - \frac{2(-0.04)(2.8)}{2(2.8)^2 - (-0.04)(2)}$$

This implies

$$x_2 = 1.4142$$

$$f(x_2) = f(1.4142) = (1.4142)^2 - 2 = -0.00000$$

$$f'(x_2) = f'(1.4142) = 2(1.4142) = 2.8284$$

$$f''(x_2) = f''(1.4142) = 2$$

3rd Iteration:-

$$x_3 = x_2 - \frac{2f(x_2)f'(x_2)}{2(f'(x_2))^2 - f(x_2)f''(x_2)}$$

$$= 1.4142 - \frac{2(-0.000001)(2.8284)}{2(2.8284)^2 - (-0.000001)(2)}$$

This implies $x_3 = 1.4142$

Since $|x_3 - x_2| = |1.4142 - 1.4142| = 0$

So $x_3 = 1.4142$ is the approximate solution.

*From Taylor series

$$(x-r)^2 f''(r) + 2(x-r) f'(r) + 2f(r) = 0$$

$$(x-r) f'(r) = f(r) - \frac{(x-r)^2}{2} f''(r)$$

$$x = r - \frac{f(r)}{f'(r)} - \frac{(x-r)^2}{2} \frac{f''(r)}{f'(r)}$$

$$c + N(x) = g(x)$$

$$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(x_{n+1} - x_n)^2}{2} \frac{f''(x_n)}{f'(x_n)},$$

$n = 0, 1, 2, \dots$

Algorithm:- (Implicit) For given x_0 find x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}(x_{n+1} - x_n)^2 \frac{f''(x_n)}{f'(x_n)}$$

$n = 0, 1, 2, \dots$

To impliment this method, we use the predictor corrector technique, We take Newton method as predictor and an Algorithm (implicit) as corrector.

Algorithm:- For a given x_0 Compute x_{n+1} by the iterative scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}(y_n - x_n)^2 \frac{f''(x_n)}{f'(x_n)}$$

$n = 0, 1, 2, \ldots$

Algorithm (Householder Method) For a given x_0 find x_{n+1} by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\left(\frac{f(x_n)}{f'(x_n)}\right)^2 \frac{f''(x_n)}{f'(x_n)};$$

$n = 0, 1, 2, \ldots$

Taylor Series:-

$$f(x) = f(r) + (x-r)f'(r) + \frac{(x-r)^2}{2!} f''(r) + \cdots$$

assume $(x-r)^3 \cong 0$, $(x-r)^4 = 0, \ldots$

In this case

$$(x-r)^2 f''(r) + 2(x-r) f'(r) + 2f(r) = 0 \quad \rightarrow ①$$

Solving for x, we can rewrite equ ① as

$$(x-r)\left[(x-r) f''(r) + 2f'(r)\right] + 2f(r) = 0$$

$$x\left[(x-r) f''(r) + 2f'(r)\right] = -2f(r) + r\left[(x-r) f''(r) + 2f'(r)\right]$$

Therefore

$$x = r - \frac{2f(r)}{2f'(r) + (x-r) f''(r)} = C + N(x)$$

This fixed point formula is used to suggest the following iteration.

Algorithm (Implicit) For a given x_0 compute x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)}{2f'(x_n) + (x_{n+1} - x_n) f''(x_n)}$$

To impliment this method, we use the technique of predictor corrector. In this case we use Newton method as a predictor and the implicit method as corrector, Thus we find the following iteration method

Algorithm (Two Step Method) for a given x_0 find x_{n+1} by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{2f'(x_n) + (y_n - x_n) f''(x_n)}$$

Eliminating y_n, we have

Algorithm (Halley Method) For a given x_0 find x_{n+1} by

$$x_{n+1} = x_n - \frac{2f(x_n)}{2f'(x_n) - \frac{f(x_n)}{f'(x_n)} f''(x_n)}$$

$$= x_n - \frac{2f(x_n)}{\frac{(2f'(x_n))^2 - f(x_n)f''(x_n)}{f'(x_n)}}$$

This implies

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2(f'(x_n))^2 - f(x_n)f''(x_n)}$$

HouseHolder Method $\Leftrightarrow$ Halley Method

$$x_{n+1} = x_n - \frac{2 f(x_n) f'(x_n)}{2 (f'(x_n))^2 - f(x_n) f''(x_n)} \quad \text{(Halley)} \longrightarrow ①$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\left(\frac{f(x_n)}{f'(x_n)}\right)^2 \frac{f''(x_n)}{f'(x_n)} \quad \text{(House Holder)} \longrightarrow ②$$

From equation ①

$$x_{n+1} = x_n - \frac{2 f(x_n) f'(x_n)}{2(f'(x_n))^2 \left[1 - \frac{f(x_n) f''(x_n)}{2 (f'(x_n))^2}\right]}$$

$$= x_n - \frac{f(x_n)}{f'(x_n) \left[1 - \frac{f(x_n) f''(x_n)}{2 (f'(x_n))^2}\right]}$$

$$= x_n - \frac{f(x_n)}{f'(x_n)} \left[1 - \frac{f(x_n) f''(x_n)}{2 (f'(x_n))^2}\right]^{-1}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\left(\frac{f(x_n)}{f'(x_n)}\right)^2 \frac{f''(x_n)}{f'(x_n)}$$

⇒ Variational Iteration Method:

Consider the equation $f(x) = 0$ ⟶ ①

This can be written as $\lambda f(x) = 0$, where λ is a unknown constant. Rewriting this equation

$$x = x + \lambda f(x) = g(x)$$

Consider the auxiliary function

$$H(x) = x + \lambda f(x) \longrightarrow ②$$

where λ is unknown constant. This unknown constant can be found by using optimality condit

Optimality Condition:-

1) Find $H'(x)$
2) Take $H'(x) = 0$ and find λ

Then subtitute the known value of λ into equ ②

$$H(x) = x + \lambda f(x)$$

$$H'(x) = x + \lambda f'(x)$$

Take $H'(x) = 0 \Rightarrow \lambda = -\frac{1}{f'(x)}$

Therefore

$$H(x) = x - \frac{f(x)}{f'(x)}$$

Thefore

$$x = H(x) = x - \frac{f(x)}{f'(x)}$$

Algorithm:- For a given x_0 find x_{n+1} by iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ; \quad n = 0, 1, 2, \dots$$

* Consider the equation

$$f(x) = 0 \longrightarrow (1)$$

Rewrite the equation as

$$\lambda \phi(x) f(x) = 0$$

$$\Rightarrow \quad x = x + \lambda \phi(x) f(x)$$

Take the auxiliary function

$$H(x) = x + \lambda \phi(x) f(x)$$

Here λ is unknown constant

$$H'(x) = 1 + \lambda [\phi'(x) f(x) + \phi(x) f'(x)]$$

Take $H'(x) = 0$ implies

$$1 + \lambda [\phi'(x) f(x) + \phi(x) f'(x)] = 0$$

$$\Rightarrow \quad \lambda = \frac{-1}{f'(x)\phi(x) + \phi'(x) f(x)}$$

Therefore,

$$H(x) = x - \frac{f(x)\, \phi(x)}{f'(x)\phi(x) + \phi'(x) f(x)}$$

Thus
$$x = x - \frac{f(x)\,\phi(x)}{f'(x)\,\phi(x) + \phi'(x)\,f(x)} \longrightarrow (A)$$

Algorithm:- (General) for a given x_0 find x_{n+1} by the iterative scheme
$$x_{n+1} = x_n - \frac{f(x_n)\,\phi(x_n)}{f'(x_n)\,\phi(x_n) + \phi'(x_n)\,f(x_n)} \longrightarrow (2)$$

★ If $\phi(x) = 1 \Rightarrow \phi'(x) = 0$

Put in (2) implies
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

★ If $\phi(x) = x \Rightarrow \phi'(x) = 1$

Put in (2) implies
$$x_{n+1} = x_n - \frac{x_n\,f(x_n)}{x_n f'(x_n) + f(x_n)}$$

★ If $\phi(x) = e^{\alpha x}$
$$\Rightarrow \phi'(x) = \alpha e^{\alpha x} = \alpha\,\phi(x)$$

Therefore (2) implies
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + \alpha f(x_n)}$$

$f(x) = x^2 - 2x$
$f'(x) = 2x - 2$
at $x = 1$
$f'(1) = 0$
Newton Meth does not work

★ If $\phi(x) = e^{1/f(x)}$

$$\Rightarrow \phi'(x) = e^{1/f(x)} \left[\frac{-f'(x)}{(f(x))^2} \right]$$

Put in ② implies

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \frac{f'(x_n) f(x_n)}{(f(x_n))^2}}$$

$$= x_n - \frac{(f(x_n))^2}{f'(x_n) f(x_n) - f'(x_n)}$$

Now consider

$$H(x) = x + \lambda f(\phi(x))$$

In this case

$$H'(x) = 1 + \lambda f'(\phi(x)) \phi'(x)$$

Take $H'(x) = 0$ implies

$$\lambda = \frac{-1}{f'(\phi(x)) \phi'(x)}$$

equation ② implies

$$x_{n+1} = x_n - \frac{f(\phi(x_n))}{f'(\phi(x_n)) \phi'(x_n)}$$

Predictor:- $y_n = \phi(x_n) = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Corrector:-

$$x_{n+1} = x_n - \frac{f(y_n)}{f'(y_n)\, y_n'}$$

★ **Zeros of Multiplicity:-** Let α be a zero of non-linear equation $f(x)=0$, then we say that α is a zero of multiplicity of order m if

$$f(x) = (x-\alpha)^m \phi(x) \; ; \quad \lim_{x\to\alpha} \phi(x) \neq 0$$

There are two cases

★ If m is known:- Then it is called known multiplicity.

★ If m is unknown:- Then it is called unknow zero of multiplicity.

★ If m is a zero of multiplicity, then $f(\alpha)=0$, $f'(\alpha)=0$, $f''(\alpha)=0, \ldots$ $f^{m-1}(\alpha)=0$ and $f^{(m)}(\alpha) \neq 0$

Special Cases:- If $f(\alpha)=0$, $f'(\alpha)\neq 0$ then $x=\alpha$ is the simple root (simple zero)

★ If $f(\alpha)=0$, $f'(\alpha)=0$ & $f''(\alpha)\neq 0$ then $x=\alpha$ is zero of multiplicity 2 and so on

Newton Method:-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \; ; \; n = 0, 1, 2, \ldots$$

★ If m is the known zero of multiplicity.

Algorithm:- (Modified Newton Method)

For a given x_0, find x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{m\, f(x_n)}{f'(x_n)} \; ; \; n = 0, 1, 2, \cdots$$

$$f(\alpha) = 0,\; f'(\alpha) = 0, \ldots, f^{(m-1)}(\alpha) = 0 \;,\; f^{(m)}(\alpha) \neq 0$$

★ If m is unknow zero of multiplicity

Algorithm:- (Modified Newton Method)

For a given x_0 find x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)\, f'(x_n)}{(f'(x_n))^2 - f(x_n)\, f''(x_n)} \; ; \; n = 0, 1, \ldots$$

This method is very much similar to Halley method which is

$$x_{n+1} = x_n - \frac{2 f(x_n)\, f'(x_n)}{2(f'(x_n))^2 - f(x_n)\, f''(x_n)}$$

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Therefore

$$x = H(x) = x - \frac{f(x)}{f'(x)}$$

Algorithm:- For a given x_0 fi... x_{n+1} by iterative sch...

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ; \; n = 0, 1, 2, \ldots$$

* Consider the equation

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$$H(x) = x + \lambda \phi(x) f(x)$$

Here λ is unknown constant

$$H'(x) = 1 + \lambda [\phi'(x) f(x) + \phi(x) f'(x)]$$

Take $H'(x) = 0$ implies

$$1 + \lambda [\phi'(x) f(x) + \phi(x) f'(x)] = 0$$

$$\Rightarrow \lambda = \frac{-1}{f'(x)\phi(x) + \phi'(x) f(x)}$$

Therefore.

$$H(x) = x - \frac{f(x)\,\phi(x)}{f'(x)\phi(x) + \phi'(x) f(x)}$$

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$$x = x - \frac{f(x)\ \phi(x)}{f'(x)\phi(x) + \phi'(x) f(x)} \longrightarrow (A)$$

Algorithm:- (General) for a given x_0 find x_{n+1} by the iterative scheme

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Put in (2) implies

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

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$$\Rightarrow \phi'(x) = e^{1/f(x)}\left[\frac{-f'(x)}{(f(x))^2}\right]$$

Put in ② implies

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \frac{f'(x_n)f(x_n)}{(f(x_n))^2}}$$

$$= x_n - \frac{(f(x_n))^2}{f'(x_n)f(x_n) - f'(x_n)}$$

Now consider

$$H(x) = x + \lambda f(\phi(x))$$

In this case

$$H'(x) = 1 + \lambda f'(\phi(x))\,\phi'(x)$$

Take $H'(x) = 0$ implies

$$\lambda = \frac{-1}{f'(\phi(x))\,\phi'(x)}$$

equation ② implies

$$x_{n+1} = x_n - \frac{f(\phi(x_n))}{f'(\phi(x_n))\,\phi'(x_n)}$$

Predictor:- $y_n = \phi(x_n) = x_n - \dfrac{f(x_n)}{f'(x_n)}$

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There are two cases

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* If m is unknown:- Then it is called unknow zero of multiplicity.

* If m is a zero of multiplicity, then $f(\alpha)=0$, $f'(\alpha)=0$, $f''(\alpha)=0, \dots$ $f^{m-1}(\alpha)=0$ and $f^{(m)}(\alpha) \neq 0$

Special Cases:- If $f(\alpha)=0$, $f'(\alpha)\neq 0$ then $x=\alpha$ is the simple root (simple zero)

* If $f(\alpha)=0$, $f'(\alpha)=0$ & $f''(\alpha)\neq 0$ then $x=\alpha$ is zero of multiplicity 2 and so on

Newton Method:-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \; ; \; n = 0, 1, 2, \ldots$$

★ If m is the known zero of multiplicity.

Algorithm:- (Modified Newton Method)
For a given x_0, find x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{m\, f(x_n)}{f'(x_n)} \; ; \; n = 0, 1, 2, \cdots$$

$$f(\alpha) = 0, \; f'(\alpha) = 0, \ldots, \; f^{(m-1)}(\alpha) = 0 \; , \; f^{(m)}(\alpha) \neq 0$$

★ If m is unknow zero of multiplicity

Algorithm:- (Modified Newton Method)
For a given x_0 find x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)\, f'(x_n)}{(f'(x_n))^2 - f(x_n)\, f''(x_n)} \; ; \; n = 0$$

This method is very much similar to Halley method which

$$x_{n+1} = x_n - \frac{2 f(x_n)\, f'(x_n)}{2(f'(x_n))^2 - f(x_n)\, f''(x_n)}$$

⇒ Variational Iteration Method:-

* If m is known then,

$$H(x) = x + \lambda f(x)^{1/m}$$

where λ is unknow parameter (constant), which is determined by using the optimality condition

$$H'(x) = 1 + \lambda \left[\frac{1}{m} f(x)^{\frac{1}{m}-1} \right] f'(x)$$

$$= 1 + \frac{\lambda}{m} f(x)^{\frac{1}{m}-1} f'(x)$$

Take $H'(x) = 0$ implies

$$1 + \frac{\lambda}{m} f(x)^{\frac{1}{m}-1} f'(x) = 0$$

$$\Rightarrow \lambda = \frac{-m}{f^{\frac{1}{m}-1}(x)\, f'(x)}$$

So

$$H(x) = x - \frac{m}{f^{\frac{1}{m}-1}(x)\, f'(x)} f(x)^{\frac{1}{m}}$$

$$= x - \frac{m}{f'(x)} f(x)^{\cancel{\frac{1}{m}} - \cancel{\frac{1}{m}} + 1}$$

$$= x - \frac{m f(x)}{f'(x)}$$

So

$$x_{n+1} = x_n - \frac{m f(x_n)}{f'(x_n)}$$

* $f(x) = 0$; zero of multiplicity m then $\frac{f(x)}{f'(x)}$ has zeros of multiplicity 1

* m is unknown zero of Multiplicity. In this case we consider the auxiliary function

$$H(x) = x + \lambda \frac{f(x)}{f'(x)}$$

Now

$$H'(x) = 1 + \lambda \left[\frac{f'(x) f'(x) - f(x) f''(x)}{(f'(x))^2} \right]$$

$$= 1 + \lambda \left\{ \frac{(f'(x))^2 - f(x) f''(x)}{(f'(x))^2} \right\}$$

Take $H'(x) = 0$

$$\Rightarrow \quad \lambda = -\frac{(f'(x))^2}{(f'(x))^2 - f(x) f''(x)}$$

So

$$H(x) = x - \left[\frac{(f'(x))^2}{(f'(x))^2 - f(x) f''(x)} \right] \frac{f(x)}{f'(x)}$$

$$\Rightarrow \quad x = x - \frac{f'(x) f(x)}{(f'(x))^2 - f(x) f''(x)}$$

So Iterative scheme is

$$x_{n+1} = x_n - \frac{f'(x_n) f(x_n)}{(f'(x_n))^2 - f(x_n) f''(x_n)}$$

* If $g(x)$ has a unique fix point x then $|x_{n+1} - x| \leq k^{n+1} |x_0 - x|; 0 \leq k < 1$

$$\lim_{n\to\infty} \frac{|x_{n+1} - x|}{|x_0 - x|} \leq k^{n+1} = \lambda$$

$$\Rightarrow |x_{n+1} - x| \leq \lambda |x_0 - x|^{\alpha}$$

* Criteria of Finding Rate of Convergence :-

Let x_{n+1} be the approximate solution obtained by an iterative method. If for a given exact solution x, there exist λ (constant) and α s.t

$$\lim_{n\to\infty} \frac{|x_{n+1} - x|}{|x_0 - x|^{\alpha}} = \lambda$$

Then we say that $x_{n+1} \xrightarrow{\text{as } n\to\infty} x$ at the rate of order α. OR equivalently

$$|x_{n+1} - x| \leq \lambda |x_0 - x|^{\alpha}$$

If $\alpha = 1$ it is called the linear convergence

If $\alpha = 2$ it is called Quadratic convergence

If $\alpha = 3$ it is " cubic "

and so on...

Sometimes the definition of convergence is not easy to verify, to overcome this drawback, we use the following criteria to find the rate of convergence

Criteria:- 1) Decompose $f(x)=0 \Leftrightarrow x=g(x)$

(i) If $g(P)=P$

(ii) find $g'(x)$ and calculate $g'(P)$ at $x=P$

(iii) If $g'(P)\neq 0$ then the convergence is linear

2) If $g(P)=P$, $g'(P)=0$ & $g''(P)\neq 0$ then the convergence is quadratic

3) If $g(P)=P$, $g'(P)=0$, $g''(P)=0$ and $g'''(P)\neq 0$ then cubic convergence.

Newton Method:-

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

This implies that $x=g(x)$

$$g(x)=x-\frac{f(x)}{f'(x)}$$

at $x=P$

$$g(P)=P-\frac{f(P)}{f'(P)}$$

$f(x)=0 \Leftrightarrow x=g(x)$
at $x=P \Rightarrow f(P)=0$

$$=P-\frac{0}{f'(P)}=P$$

$$g(P) = P$$

Now

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{f'(x)\,f'(x) - f(x)\,f''(x)}{(f'(x))^2}$$

$$= 1 - \frac{(f'(x))^2 - f(x)\,f''(x)}{(f'(x))^2} = \frac{f(x)\,f''(x)}{(f'(x))^2}$$

at $x = p$

$$g'(P) = 1 - \frac{(f'(P))^2 - f(P)\,f''(P)}{(f'(P))^2}$$

$$g'(P) = 1 - \frac{(f'(P))^2 - 0}{(f'(P))^2} = 1 - 1 + 0$$

$$= 0$$

Now

$$g''(x) = \frac{d}{dx}\left[\frac{f(x)\,f''(x)}{(f'(x))^2}\right]$$

$$= \frac{(f'(x))^2\{f'(x)\,f''(x) + f(x)\,f'''(x)\} - f(x)\,f''(x)\{2f'(x)\,f''(x)\}}{(f'(x))^4}$$

$$= \frac{(f'(x))^3 f''(x) + f(x)(f'(x))^2 f'''(x) - 2f(x)\,f'(x)(f''(x))^2}{(f'(x))^4}$$

at $x = P$

$$g''(P) = \frac{(f'(P))^3 f''(P) + 0 - 2(0)}{(f'(P))^4}$$

$$\Rightarrow g''(P) = \frac{f''(P)}{f'(P)} \neq 0$$

Therefore the convergence is Quadra

$$* \quad x_{n+1} = x_n - \frac{f(x)}{f'(x_n)} \; ; \; f(x_n) = 0 \; , \; f'(x_n) = 0, \; f''(x_n) \neq 0$$

Check the convergence ?

$$g(x) = x - \frac{f(x)}{f'(x)}$$

at $x = P$

$$g(P) = P - \frac{f(P)}{f'(P)} \quad \left(\frac{0}{0}\right)$$

So by L, Hospital rule

$$g(P) = P - \lim_{x \to P} \frac{f(x)}{f'(x)}$$

$$= P - \lim_{x \to P} \frac{f'(x)}{f''(x)} = P - \frac{f'(P)}{f''(P)}$$

$$= P - \frac{0}{P} = P - 0$$

$$\Rightarrow g(P) = P$$

Now

$$g'(x) = \frac{f(x) \, f''(x)}{(f'(x))^2}$$

at $x = P$

$$g'(P) = \frac{f(P) \, f''(P)}{(f'(P))^2}$$

$$\Rightarrow g'(p) = \lim_{x \to p} \frac{f(x) f''(x)}{(f'(x))^2}$$

$$= \lim_{x \to p} \frac{f'(x) f''(x) + f(x) f'''(x)}{2 f'(x) f''(x)}$$

$$\cancel{= \frac{f'(p) f''(p) + (f''(p))^2}{2 f'(p) f''(p)} \neq 0}$$

$$\cancel{g(p) \neq 0 \quad (\text{Linear Convergence})}$$

$$\Rightarrow g'(p) = \lim_{x \to p} \left[\frac{1}{2} + \frac{f(x) f'''(x)}{2 f'(x) f'(x)} \right]$$

$$= \frac{1}{2} + \lim_{x \to p} \frac{f'(x) f'''(x) + f(x) f^{iv}(x)}{2 f'(x) + 2 f^{iv}(x)}$$

$$= \frac{1}{2} \neq 0$$

at $x = p$

$g(p) = p$, $g'(p) = \frac{1}{2} \neq 0$

So the rate of convergence of Newton method is linear

H.W.- To check convergence of Modified Newton Method

$$g(x) = x - \frac{2 f(x)}{f'(x)} ; \quad f(x) = 0,\ f'(x) = 0 \ \& \ f''(x) \neq 0$$

* Suppose if 2 is unknown

$$g(x) = x - \frac{f(x) f'(x)}{(f(x))^2 - f(x) f''(x)}$$

{ This is newton method of unknown zero of multiplicity

Trapezoidal Rule for Integration:

If $f(x)$ is integrable, then

★ $$\int_a^b f(x)\,dx = \frac{b-a}{2}\left(f(a)+f(b)\right)$$

or $$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{f(a)+f(b)}{2}$$

★ $f'(x)$ also integrable

★ $$\int_a^b f'(x)\,dx = f(b)-f(a)$$

★ $$\int_a^x f'(x)\,dx = f(x)-f(a)$$

★ $$\int_a^{x^2} f'(x)\,dx = 2x\,f(x^2)-f(a)$$

★ $$\int_a^{g(x)} f'(x)\,dx = f(g(x))\,g'(x)-f(a)$$

★ $$\int_{h(x)}^{g(x)} f'(x)\,dx = f(g(x))\,g'(x)-f(h(x))\,h'(x)$$

Labitiz Theorem

★ $$\int_a^x f'(x)\,dx = \frac{x-a}{2}\left[f'(a)+f'(x)\right]$$

$$= f(x)-f(a)$$

From the discussion we have

$$f(x) - f(a) = \frac{x-a}{2}\left[f'(x) + f'(a)\right]$$

$$f(x) = f(a) + \left(\frac{x-a}{2}\right)\left[f'(a) + f'(x)\right] \longrightarrow (1)$$

Given $f(x) = 0$

$$\Rightarrow f(a) + \left(\frac{x-a}{2}\right)\left[f'(a) + f'(x)\right] = 0$$

or

$$(x-a)\left[f'(a) + f'(x)\right] = -2f(a) \longrightarrow (*)$$

or

$$x\left[f'(a) + f'(x)\right] = -2f(a) + a\left[f'(a) + f'(x)\right]$$

or

$$\boxed{x = a - \frac{2f(a)}{f'(a) + f'(x)}}$$

If $f'(x) = 0$, Then

$$x = a - \frac{2f(a)}{f'(a)} \quad \Big\{ \text{Modified Newton Method}$$

Now

$$x = a - \frac{2f(a)}{f'(a) + f'(x)}$$

$$= a - \frac{2f(a)}{f'(a) + f'(a) + (x-a)f''(a)}$$

$$= a - \frac{2f(a)}{2f'(a) + (x-a)f''(a)} \quad \Big\{ \text{Halley Method}$$

By (*) $(x-a)f'(a) = -2f(a) - (x-a)f'(x)$

or

$$xf'(a) = af'(a) - 2f(a) - (x-a)f'(x)$$

$$x = a - \frac{2f(a)}{f'(a)} - (x-a)\frac{f'(x)}{f'(a)}$$

$$x = a - \frac{2f(a)}{f'(a)} \qquad (x-a) \cong 0$$

$$\star \int_a^b f(x)\,dx = \frac{b-a}{2n}\left[f(a) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(b)\right]$$

$$\star \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \; ; \; a \le c \le b$$

$$\star \quad = \int_a^{\frac{a+b}{2}} f(x)\,dx + \int_{\frac{a+b}{2}}^b f(x)\,dx$$

$$= \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] \quad \because n=1$$

Trapezoidal Rule:-

$$\int_a^b f(x)\,dx$$

subdivide the interval $[a, b]$ s.t $a = x_0 \le x_1 \le x_2 \le \dots \le x_{n-1} \le x_n = b$

We take $h = x_{i+1} - x_i \quad i = 0, 1, \dots, n$

(uniform sub division)

$$x_{i+1} = x_i + h$$

For $i=0$ $\quad x_1 = x_0 + h \quad = a + h$

for $i=1$ $\quad x_2 = x_1 + h \quad = a + h + h$

$$= a + 2h$$

$\vdots \qquad \vdots$

$$x_n = a + nh$$

But

$$x_n = B$$

Therefore $b = a + nh$

$\Rightarrow nh = b - a$

Thus $\boxed{n = \frac{b-a}{h}}$ Number of intervals

or $\boxed{h = \frac{b-a}{n}}$ Width of the interval

$$\star \int_a^b f(x)\,dx = \frac{b-a}{2n}\Big[\underset{1}{f(a)} + \underbrace{2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1})}_{2} + \underset{1}{f(b)}\Big]$$

$$= \frac{b-a}{2n}\Big[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\Big]$$

(Trapezoidal Rule)

Example:- Evaluate $\int_1^3 (e^{2x} + x^2 + 1)\,dx$; $n=3$

Solution

Here $a = 1$, $b = 3$

Since $n = 3$, so

$$h = \frac{b-a}{n} = \frac{3-1}{3} = \frac{2}{3} = 0.67$$

h is the width of the sub intervals

Now

$$x_0 = a = 1$$
$$x_1 = a + h = 1 + 0.67 = 1.67$$
$$x_2 = a + 2h = 1 + 2(0.67) = 2.34$$
$$x_3 = a + 3h = 1 + 3(0.67) = 3$$

So $f(x) = e^{2x} + x^2 + 1$

$$f(x_0) = f(1) = e^{2(1)} + (1)^2 + 1 = 9.389$$

$$f(x_1) = f(1.67) = e^{2(1.67)} + (1.67)^2 + 1 = 32.008$$

$$f(x_2) = f(2.34) = e^{2(2.34)} + (2.34)^2 + 1 = 114.246$$

$$f(x_3) = f(3) = e^{2(3)} + (3)^2 + 1 = 413.429$$

Now by using Trapezoidal rule

$$\int_1^3 (e^{2x} + x^2 + 1)\,dx = \frac{b-a}{2n}\left[f(a) + 2f(x_1) + 2f(x_2) + f(b)\right]$$

$$= \frac{3-1}{2(3)}\left[9.389 + 2(32.008) + 2(114.246) + 413.429\right]$$

$$= \frac{2}{6}\left[715.326\right] = 238.442$$

So

$$\int_1^3 (e^{2x} + x^2 + 1)\,dx = 238.442$$

★ Consider the linear equation

$f(x) = 0$; $[a, b]$

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left[f(a) + f(b)\right] \quad ; [a, b]$$

or

$$\int_a^x f(x)\,dx = \frac{x-a}{2}\left[f(a) + f(x)\right] \quad ; \quad [a, x]$$

If $f'(x)$ is integrable then

$$\int_a^x f'(x)\,dx = \frac{x-a}{2}\left[f'(x) + f'(a)\right] \longrightarrow (1)$$

using Trapezoidal rule (n=1) Also using the fundamental theorem of calculus, we have

$$\int_a^x f'(x)\,dx = f(x) - f(a) \longrightarrow ②$$

from ① and ② we have

$$f(x) - f(a) = \frac{x-a}{2}\left[f'(a) + f'(x)\right]$$

From this we have

$$f(x) - f(a) = \frac{x-a}{2}\left[f'(a) + f'(x)\right]$$

From this we have

$$f(x) = f(a) + \frac{x-a}{2}\left[f'(a) + f'(x)\right]$$

$$= f(a) + (x-a)\left[\frac{f'(a) + f'(x)}{2}\right]$$

Since $f(x) = 0$, so

$$2f(a) + (x-a)\left[f'(a) + f'(x)\right] = 0$$

Solving this equation for x, we have

$$x = a - \frac{2f(a)}{f'(a) + f'(x)} = g(x)$$

Thus

$$x = a - \frac{2f(a)}{f'(x) + f'(a)}$$

This fix point formulation

is used to suggest the following iterative method.

Algorithm 1:- $(x_{n+1} = g(x_n))$

for a given x_0 find x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n)}$$

$$= x_n - \frac{2f(x_n)}{2f'(x_n)}$$

$$= x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{(Newton Method)}$$

**

Algorithm 2:- $(x_{n+1} = g(x_{n+1}))$

for a given x_n find x_{n+1} by the following iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_{n+1})}$$

This is implicit method. To use this method we use predictor corrector technique. One usually use Algorithm 1 (Newton Method) as predictor and Algorithm 2 as corrector Method.

Algorithm 3:- For given x_0 find x_{n+1} by the scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(y_n) + f'(x_n)}$$

★ using Taylor series from Algorithm 3, we have

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n) + (y_n - x_n)f''(x_n)}$$

$$= x_n - \frac{2f(x_n)}{2f'(x_n) + (y_n - x_n)f''(x_n)}$$

★ Eliminating y_n, we have

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2(f'(x_n))^2 - f(x_n)f''(x_n)}$$

(Halley Method)

★★ **Algorithm 4:-** For given x_0 find x_{n+1} by the iterative scheme

$$y_n = x_n - \frac{2f(x_n)}{f'(x_n)}$$ $\left\{ f'(x) = 0,\ f'(x) \neq 0 \right.$ M.N.M

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(y_n) + f'(x_n)}$$

$$= x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n) + (y_n - x_n) f''(x_n)}$$

Eliminating y_n, we have

$$x_{n+1} = x_n - \frac{f(x)\, f'(x_n)}{(f'(x_n))^2 - f(x_n)\, f''(x_n)}$$

(Modified Newton Method)

★ In Trapezoidal rule the order of error estimate is $O(h^4)$

Algorithm 5:- (Two step Method) For a given x_0 find x_{n+1} by the iterative scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(y_n) + f'(x_n)}, \quad f'(x_n) \neq 0$$

Algorithm 6:- for a given x_0 find x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \left(\frac{f(x_n)}{f'(x_n)}\right)^2 \frac{f''(x_n)}{f'(x_n)}$$

House holder Method

Example:- Find $\sqrt{2}$ using Algorith 2 to 6.

Solution

Algorithm 5

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \longrightarrow (1)$$

$$x_{n+1} = x_n - \frac{2 f(x_n)}{f'(y_n) + f'(x_n)} \longrightarrow (2)$$

Let $x = \sqrt{2}$

This implies $x^2 = 2$

$x^2 - 2 = 0$

Take $f(x) = x^2 - 2$ and $x_0 = 1$

As $x^2 - 2 = 0$ So $f(x) = 0$

Now

$$f(x) = x^2 - 2 \longrightarrow (3)$$

$$f'(x) = 2x \longrightarrow (4)$$

$$f''(x) = 2 \longrightarrow (5)$$

$f(x_0) = f(1) = (1)^2 - 2 = -1$

$f'(x_0) = f'(1) = 2(1) = 2$

Iteration 1:- $n = 0$

$$y_0 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-1}{2}$$

$$\boxed{y_0 = 1.5}$$

So $f'(y_0) = f'(1.5) = 2(1.5) = 3$

So

$$x_1 = x_0 - \frac{2f(x_0)}{f'(y_0) + f'(x_0)}$$

$$= 1 - \frac{2(-1)}{3+2}$$

$$\Rightarrow \boxed{x_1 = 1.4}$$

Now

$$f(x_1) = f(1.4) = (1.4)^2 - 2 = -0.04$$

$$f'(x_1) = f'(1.4) = 2(1.4) = 2.8$$

Iteration 2:- $n=1$

$$y_1 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.4 - \frac{-0.04}{2.8}$$

$$\boxed{y_1 = 1.4142}$$

and $f'(y_1) = f'(1.4142) = 2(1.4142) = 2.8284$

So

$$x_2 = x_1 - \frac{2f(x_1)}{f'(y_1) + f'(x_1)}$$

$$= 1.4 - \frac{2(-0.04)}{2.8284 + 2.8}$$

$$\Rightarrow \boxed{x_2 = 1.4142}$$

This is the approximate solution

Algorithm 3:-

$$x_{n+1} = x_n - \frac{2 f(x_n) f'(x_n)}{2 (f'(x_n))^2 - f(x_n) f''(x_n)} \longrightarrow (A)$$

$f(x) = x^2 - 2$ $\quad f(x_0) = f(1) = -1$

$f'(x) = 2x$ $\quad f'(x_0) = 2$

$f''(x) = 2$ $\quad f''(x_0) = 2$

Iteration 1 $\quad n = 0$

$$x_1 = x_0 - \frac{2 f(x_0) f'(x_0)}{2 (f'(x_0))^2 - f(x_0) f''(x_0)}$$

$$= 1 - \frac{2(-1)(2)}{2(2)^2 - (-1)(2)}$$

$$\Rightarrow \boxed{x_1 = 1.4}$$

$f(x_1) = f(1.4) = -0.04$

$f'(x_1) = f'(1.4) = 2.8$

$f''(x_1) = 2$

Iteration 2:- $\quad n = 1$

$$x_2 = x_1 - \frac{2 f(x_1) f'(x_1)}{2 (f'(x_1))^2 - f(x_1) f''(x_1)}$$

$$= 1.4 - \frac{2(-0.04)(2.8)}{2(2.8)^2 - (-0.04)(2)}$$

$$\Rightarrow \boxed{x_2 = 1.4142}$$

$f(x_2) = f(1.4142) = (1.4142)^2 - 2 = -0.0000$

$f'(x_2) = f'(1.4142) = 2(1.4142) = 2.8284$

$f''(x_2) = 2$

Iteration 3:- $n = 2$

$$x_3 = x_2 - \frac{2 f(x_2) f'(x_2)}{2(f'(x_2))^2 - f(x_2) f''(x_2)}$$

$$= 1.4142 - \frac{2(-0.00003)(2.8284)}{2(2.8284)^2 - (-0.00003)(2)}$$

$$\Rightarrow \boxed{x_3 = 1.4142}$$

As $|x_3 - x_2| = |1.4142 - 1.4142|$

$\cong 0$

So $x_3 = 1.4142$ is approximate solution.

Algorithm 6

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \left(\frac{f(x_n)}{f'(x_n)}\right)^2 \frac{f''(x_n)}{f'(x_n)}$$

$f(x) = x^2 - 2$

$f'(x) = 2x$

$f''(x) = 2$

Let $x_0 = 1$

$f(x_0) = f(1) = -1$

$f'(x_0) = f'(1) = 2$

$f''(x_0) = 2$

Iteration 1:- $n = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \left(\frac{f(x_0)}{f'(x_0)}\right)^2 \frac{f''(x_0)}{f'(x_0)}$$

$$= 1 - \frac{-1}{2} - \left(\frac{-1}{2}\right)^2 \frac{\cancel{2}}{\cancel{2}}$$

$$\boxed{x_1 = 1.25}$$

$f(x_1) = f(1.25) = (1.25)^2 - 2 = -0.4375$

$f'(x_1) = f'(1.25) = 2(1.25) = 2.5$

$f''(x_1) = 2$

Iteration 2:- $n = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} - \left(\frac{f(x_1)}{f'(x_1)}\right)^2 \frac{f''(x_1)}{f'(x_1)}$$

$$= 1.25 - \frac{-0.4375}{2.5} - \left(\frac{-0.4375}{2.5}\right)^2 \frac{2}{2.5}$$

$$\Rightarrow \quad x_2 = 1.4005$$

$f(x_2) = f(1.4005) = 1.4005^2 - 2 = -0.0386$

$f'(x_2) = f'(1.4005) = 2(1.4005) = 2.801$

$f''(x_2) = 2$

Iteration 3:- $n=2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} - \left(\frac{f(x_2)}{f'(x_2)}\right)^2 \frac{f''(x_2)}{f'(x_2)}$$

$$= 1.4005 - \frac{-0.0386}{2.801} - \left(\frac{-0.0386}{2.801}\right) \frac{2}{2.801}$$

$$\Rightarrow \boxed{x_3 = 1.4142}$$

$$f(x_3) = f(1.4142) = (1.4142)^2 - 2 = -0.00003$$

$$f'(x_3) = f'(1.4142) = 2(1.4142) = 2.8284$$

$$f''(x_3) = 2$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} - \left(\frac{f(x_3)}{f'(x_3)}\right)^2 \frac{f''(x_3)}{f'(x_3)}$$

$$= 1.4142 - \frac{(-0.00003)}{2.8284} - \left(\frac{-0.00003}{2.8284}\right)^2 \frac{2}{2.8284}$$

$$\Rightarrow \boxed{x_4 = 1.4142}$$

As $|x_4 - x_3| \simeq 0$

So $x_4 = 1.4142$ is the approximate solution.

Algorithm 4

$$x_{n+1} = x_n - \frac{f(x_n)\, f'(x_n)}{(f'(x_n))^2 - f(x_n)\, f''(x_n)}$$

Let $x_0 = 1$

$f(x) = x^2 - 2 \Rightarrow f(x_0) = -1$

$f'(x) = 2x \Rightarrow f'(x_0) = 2$

$f''(x) = 2 \Rightarrow f''(x_0) = 2$

Iteration 1:- $n = 0$

$$x_1 = x_0 - \frac{f(x_0)\, f'(x_0)}{(f'(x_0))^2 - f(x_0)\, f''(x_0)}$$

$$= 1 - \frac{(-1)(2)}{(2)^2 - (-1)(2)}$$

$$\Rightarrow \boxed{x_1 = 1.3333}$$

Now

$f(x_1) = f(1.3333) = (1.3333)^2 - 2 = -0.2222$

$f'(x_1) = f'(1.3333) = 2(1.3333) = 2.6667$

$f''(x_1) = 2$

Iteration 2:- $n = 1$

$$x_2 = x_1 - \frac{f(x_1)\, f'(x_1)}{(f'(x_1))^2 - f(x_1)\, f''(x_1)}$$

$$= 1.3333 - \frac{(-0.2222)(2.6667)}{(2.6667)^2 - (-0.2222)(2)}$$

$$\Rightarrow \boxed{x_2 = 1.4117}$$

Now

$f(x_2) = f(1.4117) = (1.4117)^2 - 2 = -0.0071$

$f'(x_2) = f'(1.4117) = 2(1.4117) = 2.8234$

$f''(x_2) = 2$

Iteration 3:- $n = 2$

$$x_3 = x_2 - \frac{f(x_2)\, f'(x_2)}{(f'(x_2))^2 - f(x_2)\, f''(x_2)}$$

$$= 1.4117 - \frac{(-0.0071)(2.8234)}{(2.8284)^2 - (-0.0071)(2)}$$

$$\Rightarrow \boxed{x_3 = 1.4142}$$

Now

$f(x_3) = f(1.4142) = -0.00003$

$f'(x_3) = f'(1.4142) = 2.8284$

$f''(x_3) = 2$

Iteration 4:- $n = 3$

$$x_4 = x_3 - \frac{f(x_3)\, f'(x_3)}{(f'(x_3))^2 - f(x_3)\, f''(x_3)}$$

$$= 1.4142 - \frac{(-0.00003)(2.8284)}{(2.8284)^2 - (-0.00003)(2)}$$

$$\Rightarrow \boxed{x_4 = 1.4142}$$

This is approximate solution

★ If $f(x)$ has 'α' zero of multiplicity of order 'm' then $f(x)/f'(x)$ has 'α' zero of multiplicity of order 1

$$f(x) = (x-\alpha)^m \phi(x) \quad ; \quad \lim_{x \to \alpha} \phi(x) = 0$$

$$f'(x) = m(x-\alpha)^{m-1} \phi(x) + (x-\alpha)^m \phi'(x)$$

$$\frac{f(x)}{f'(x)} = \frac{(x-\alpha)^m \phi(x)}{m(x-\alpha)^{m-1} \phi(x) + (x-\alpha)^m \phi'(x)}$$

$$= \frac{(x-\alpha)^m \phi(x)}{(x-\alpha)^{m-1}\left[m\phi(x) + (x-\alpha)\phi'(x)\right]}$$

$$= \frac{(x-\alpha)\phi(x)}{m\phi(x) + (x-\alpha)\phi'(x)}$$

$$\frac{f(x)}{f'(x)} = (x-\alpha) R(x) \quad ; \quad (\text{order } 1)$$

where $R(x) = \dfrac{\phi(x)}{m\phi(x) + (x-\alpha)\phi'(x)}$

where $\lim_{x \to \alpha} R(x) = 0$

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★ From Trapezoidal Rule:

$$f(x) = 2f(a) + (x-\alpha)[f'(x) + f'(a)]$$

As $f(x) = 0$

So, $2f(a) + (x-\alpha)[f'(x) + f'(a)]$

$$\Rightarrow (x-\alpha)f'(x) + (x-\alpha)f'(a) = -2f(a)$$

$$(x-\alpha)f'(a) = -2f(a) - (x-\alpha)f'(x)$$

Therefore

$$x = a - \frac{2f(a)}{f'(a)} - (x-a)\frac{f'(x)}{f'(a)}.$$

Algorithm:- For a given x_0 find x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)} - (x_{n+1} - x_n)\frac{f'(x_{n+1})}{f'(x_n)}$$

This is an implicit scheme

If $x_{n+1} - x_n \simeq 0$ (x_n is the nbhd of x_{n+1})

Then

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}$$

★ From Trapezoidal Rule:

$$(x-a)f'(x) + (x-a)f'(a) = -2f(a)$$

Solving for x, we have

$$(x-a)f'(x) = -2f(a) - (x-a)f'(a)$$

$$x = a - \frac{2f(a)}{f'(x)} - (x-a)\frac{f'(a)}{f'(x)}$$

Algorithm:- For a given x_0 find x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)} - (x_{n+1} - x_n)\frac{f'(x_n)}{f'(x_n)}$$

$$= 2x_n - x_{n+1} - \frac{2f(x_n)}{f'(x_n)}$$

Algorithm:- For a given x_0 find x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_{n+1})} - (x_{n+1} - x_n)\frac{f'(x_n)}{f'(x_{n+1})}$$

Simpson Rule for Integration:-

$f(x) = 0 \iff x = g(x)$; $h = x_{i+1} - x_i$

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f(c) + f(b)\right]$$

$$= \frac{h}{3}\left[f(a) + 4f(c) + f(b)\right]$$

In general:-

$$\int_a^b f(x)\,dx = \frac{h}{3n}\left[\overset{1}{f(a)} + \overset{4}{4f(x_1)} + \overset{2}{2f(x_2)} + 4f(x_3) + \dots + 2f(x_{n-2}) + \underset{4}{4f(x_{n-1})} + \underset{1}{f(b)}\right]$$

$$n = \frac{b-a}{h}$$

Example:- Evaluate $\int_1^2 (x^2+x+1)\,dx$ if $n=4$

Solution:-

$$n = \frac{b-a}{h} \Rightarrow h = \frac{b-a}{n}$$

$$\Rightarrow h = \frac{2-1}{4} = \frac{1}{4} = 0.25$$

Now $f(x_0) = f(a) = 1$, $f(x_1) = 1.25$

$f(x_2) = 1.5$, $f(x_3) = 1.75$

$f(x_4) = 2 = f(b)$

As $f(x) = x^2 + x + 1$

$f(a) = f(1) = (1)^2 + 1 + 1 = 3$

$f(x_1) = f(1.25) = (1.25)^2 + 1.25 + 1 = 3.8125$

$f(x_2) = f(1.5) = (1.5)^2 + 1.5 + 1 = 4.75$

$f(x_3) = f(1.75) = (1.75)^2 + 1.75 + 1 = 5.8125$

$f(b) = f(2) = (2)^2 + 2 + 1 = 7$

$$\int_1^2 f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \frac{0.25}{3}\left[3 + 4(3.8125) + 2(4.75) + 4(5.8125) + 7\right]$$

$$\Rightarrow \int_1^2 (x^2+x+1)\,dx = \frac{0.25}{3}[58]$$

$$=(1.2083)\times 4$$

Now for Exact Solution:-

$$\int_1^2 (x^2+x+1)\,dx = \frac{x^3}{3}+\frac{x^2}{2}+x\Big|_1^2$$

$$=\left\{\frac{(2)^3}{3}+\frac{(2)^2}{2}+2\right\}-\left\{\frac{(1)^3}{3}+\frac{(1)^2}{2}+1\right\}$$

$$=(6.6667)-(1.8333)$$

$$* \int_a^b f'(x)\,dx = \frac{b-a}{6}\left[f'(a)+4f'\left(\frac{a+b}{2}\right)+f'(b)\right]$$

for $n=2$

$$\int_a^x f'(x)\,dx = \frac{x-a}{6}\left[f'(a)+4f'\left(\frac{a+x}{2}\right)+f'(x)\right] \rightarrow (1)$$

$$\int_a^x f'(x)\,dx = f(x)-f(a) \longrightarrow (2)$$

from (1) and (2)

$$f(x) = f(a)+\frac{x-a}{6}\left[f'(a)+4f'\left(\frac{a+x}{2}\right)+f'(x)\right]$$

Since $f(x)=0$

So

$$(x-a)\left[f'(a)+4f'\left(\frac{x+a}{2}\right)+f'(x)\right] = -6f(a)$$

$$\therefore \quad x = a - \frac{6f(a)}{f'(x)+4f'\left(\frac{x+a}{2}\right)+f'(a)}$$

$$x = g(x) \quad ; \quad x_{n+1} = g(x_n)$$

$$\therefore \quad x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n)+4f'\left(\frac{x_n+x_n}{2}\right)+f'(x_n)}$$

$$= x_n - \frac{6f(x_n)}{6f'(x_n)}$$

$$= x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{(Newton Method)}$$

* Similarly

$$\int_a^b f'(x)\,dx = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

Algorithm:- For a given x_0, find x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_{n+1})+4f'\left(\frac{x_{n+1}+x_n}{2}\right)+f'(x_n)}$$

This is implicit method, This implicit Algorithm is equivalent to

Algorithm:- (Two step Method)

For given x_0, find x_{n+1} by the iterative scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n) + 4f'\left(\frac{x_n+y_n}{2}\right) + f'(y_n)}$$

Example:- Calculate $\sqrt{2}$; $x_0 = 1$

Solution

Let $x = \sqrt{2}$

$\Rightarrow x^2 = 2 \quad \Rightarrow x^2 - 2 = 0$

Take $f(x) = x^2 - 2$

As $x^2 - 2 = 0$, So $f(x) = 0$

Now

$f(x) = x^2 - 2 \Rightarrow f(x_0) = -1$

$f'(x) = 2x \Rightarrow f'(x_0) = 2$

Iteration 1:-

$$y_0 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-1}{2}$$

$$\Rightarrow \boxed{y_0 = 1.5}$$

Now $f'(y_0) = 2(1.5) = 3$

$$f'\left(\frac{x_0+y_0}{2}\right) = f'\left(\frac{1+1.5}{2}\right) = f'(1.25) = 2.5$$

So

$$x_1 = x_0 - \frac{6f(x_0)}{f'(x_0) + 4f'\left(\frac{x_0+y_0}{2}\right) + f'(y_0)}$$

$$x_1 = 1 - \frac{6(-1)}{2+4(2.5)+3}$$

$$= 1 + \frac{6}{15}$$

$$\Rightarrow \boxed{x_1 = 1.4}$$

Now $f(x_1) = f(1.4) = 1.4^2 - 2 = -0.04$

$f'(x_1) = 2(1.4) = 2.8$

Iteration 2:-

$$y_1 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.4 - \frac{-0.04}{2.8}$$

$$\Rightarrow \boxed{y_1 = 1.4142}$$

Now $f'(y_1) = 2(1.4142) = 2.8284$

$$f'\left(\frac{x_1+y_1}{2}\right) = f'\left(\frac{1.4+1.4142}{2}\right)$$

$$= f'(1.4071) = 2(1.4071) = 2.8142$$

So

$$x_2 = x_1 - \frac{6f(x_1)}{f'(x_1)+4f'\left(\frac{x_1+y_1}{2}\right)+f'(y_n)}$$

$$= 1.4 - \frac{6(-0.04)}{2.8+4(2.8142)+2.8284}$$

$$\Rightarrow \boxed{x_2 = 1.4142}$$

Is the solution.

* $$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Also
$$(x-a)\left[f'(a) + 4f'\left(\frac{x+a}{2}\right) + f'(x)\right] = -6f(a)$$

$$x = a - \frac{6f(a)}{f'(a)} - (x-a)\frac{f'(x) + 4f'(x+a)}{f'(a)}$$

If $(x-a) \cong 0$

$$x = a - \frac{6f(a)}{f'(a)}$$

Here $f(x) = (x-a)^6\,\phi(x)$; $\lim_{x\to a} \phi(x) = 0$

Algorithm:- For a given x_0 find x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n)} \; ; \; n = 0, 1, 2, \ldots$$

$(f(x) = 0, f'(x) = 0, \ldots f^{(5)}(x) = 0, f^{(6)}(x) \neq 0)$

$$g(x) = x - \frac{6f(x)}{f'(x)}$$

$$g(P) = p - \lim_{x\to p} \frac{6f'(x)}{f''(x)} \quad \because \text{By L-Hospital Rule}$$

$$= p - \lim_{x\to p} \frac{6f''(x)}{f'''(x)}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$g(P) = p - \lim_{x\to p} \frac{6f^{(5)}(p)}{f^{(6)}(p)}$$

$$= p - \frac{6(0)}{f^{(6)}(p)} = p - 0$$

$$\Rightarrow \quad g(P) = P$$

Algorithm:-

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n)} - (x_{n+1} - x_n)$$
$$\left[f'(x_{n+1}) + 4f'\left(\frac{x_n + x_{n+1}}{2}\right)\right]$$

★

$$(x-a)[f'(a) + f'(x_n)] = -6f(x) - (x-a)4f'\left(\frac{x+a}{2}\right)$$

$$\Rightarrow x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n) + f'(x_{n+1})} - (x-a)\frac{4f'\left(\frac{x_n + x_{n+1}}{2}\right)}{f'(x_n) + f'(x_{n+1})}$$

★————————★

★ Find Taylor Series of about a

$$f\left(\frac{x+a}{2}\right) = f(a) + \cdots \qquad ?$$

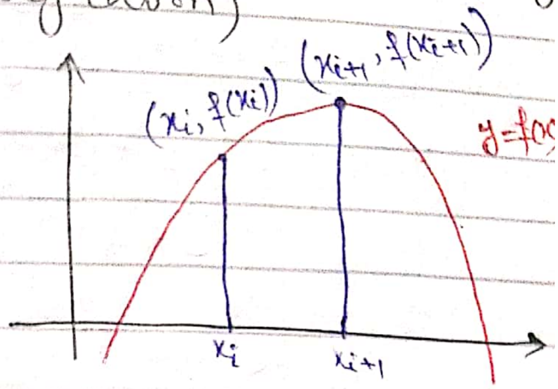
$$= f(a) + \left(\frac{x+a}{2} - a\right)f'(a) + \frac{1}{2!}\left(\frac{x+a}{2} - a\right)^2 f''(a)$$
$$+ \cdots$$

$$= f(a) + \left(\frac{x+a-2a}{2}\right)f'(a) + \frac{1}{2!}\left(\frac{x+a-2a}{2}\right)f''(a)$$
$$+ \cdots$$

$$= f(a) + \left(\frac{x-a}{2}\right)f'(a) + \frac{1}{2!}\left(\frac{x-a}{2}\right)^2 f''(a) + \cdots$$

Interpolation:- (Data Network, Forecasting Integration)

Consider
$y = f(x)$
consider the interval $[a,b]$
subdivide it as



$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

distinct points $x_0 \neq x_1 \neq x_2 \neq \cdots \neq x_n$

$x_i = x_0 + ih$

* Interpolation mean that we find a polynomial $P(x)$ such that

$$P(x_i) = f(x_i) \qquad i = 0,1,2,\cdots,n$$
$$P(x_1) = f(x_1)$$
$$P(x_2) = f(x_2)$$
$$\vdots \qquad \vdots$$
$$P(x_n) = f(x_n)$$

* $f(x) = f(r) + (x-r) f'(r)$
$= c + x \underline{f'(r)} - \underline{r f'(r)}$ constant
$= c + mx$

$$0 = \begin{vmatrix} 1 & 1 & 1 \\ a & x & b \\ f(a) & P(x) & f(b) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & x-a & b-a \\ f(a) & P(x)-f(a) & f(b)-f(a) \end{vmatrix}$$

$= (x-a)\{f(b) - f(a)\} - (b-a)\{P(x) - f(a)\}$

This implies that

$$P(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

$$= \frac{f(b)-f(a)}{b-a}x - \frac{f(b)-f(a)}{b-a}a + f(a)$$

$$= \frac{(b-a)f(a) - a f(b) + a f(a)}{b-a} + \frac{f(b)-f(a)}{b-a}x$$

$$= \frac{b f(a) - a f(b)}{b-a} + \frac{f(b)-f(a)}{b-a}x$$

$$\because P(x) = f(x) = 0 \quad \text{(given)}$$

So $$\frac{b f(a) - a f(b)}{b-a} + \frac{f(b)-f(a)}{b-a}x = 0$$

$$x\,\frac{f(b)-f(a)}{b-a} = \frac{a f(b) - b f(a)}{b-a}$$

$$\Rightarrow x = \frac{a f(b) - b f(a)}{f(b)-f(a)}$$

{ Inverse Interpolati[on] Polynomial Method

$$= a - \frac{f(a)}{\frac{f(b)-f(a)}{b-a}}$$

$$= a - \frac{f(a)}{f'(a)}$$

Lagrange Interpolating Polynomial

i) $f(x)$ is continuous function on $[a,b]$

ii) Subdivide the interval $[a,b]$ as

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

$$x_i - x_{i+1} \neq 0 \quad \text{or} \quad x_i \neq x_{i+1}$$

Find the polynomial, denoted by $P(x)$, using the condition that

$x_0 \neq x_1 \neq x_2 \neq \cdots \neq x_n = b$

such that $f(x_i) = P(x_i)$; $i = 0, 1, 2, \ldots, n$

Find polynomial for $x_0 \neq x_1$, such that

$$P(x) = a_0 + a_1 x \longrightarrow (1)$$

where a_0 and a_1 are unknowns which can be found

$$P(x_0) = f(x_0) \quad \text{and} \quad P(x_1) = f(x_1)$$

Now

$$P(x_0) = a_0 + a_1 x_0 = f(x_0)$$
$$P(x_1) = a_0 + a_1 x_1 = f(x_1)$$

$$\Rightarrow a_0 + a_1 x_0 = f(x_0) \longrightarrow (2)$$
$$a_0 + a_1 x_1 = f(x_1) \longrightarrow (3)$$

Solving equation (2) & (3) simultaneously

$$a_0 + a_1 x_0 = f(x_0)$$
$$\underline{-a_0 - a_1 x_1 = -f(x_1)}$$
$$a_1(x_0 - x_1) = f(x_0) - f(x_1)$$

$$\Rightarrow \boxed{a_1 = \frac{f(x_0) - f(x_1)}{x_0 - x_1}}$$

Again $x_1 a_0 + \cancel{a_1 x_0 x_1} = x_1 f(x_0)$ $\quad$ X ing by x_1

$x_0 a_0 + \cancel{a_1 x_0 x_1} = x_0 f(x_1)$ $\quad$ " " x_0

$$a_0(x_1 - x_0) = x_1 f(x_0) - x_0 f(x_1)$$

$$\Rightarrow \boxed{a_0 = \frac{x_1 f(x_0) - x_0 f(x_1)}{x_1 - x_0}}$$

$\star$ $P(x) = a_0 + a_1 x$

$$P(x) = \frac{x_1 f(x_0) - x_0 f(x_1)}{x_1 - x_0} + \frac{f(x_0) - f(x_1)}{x_0 - x_1} x$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{x_0 - x_1} + \frac{f(x_0) - f(x_1)}{x_0 - x_1} x$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0) + x f(x_0) - x f(x_1)}{x_0 - x_1}$$

$$= \frac{(x - x_1)}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

(Linear)

$\star$ $P(x) = a_0 + a_1 x + a_2 x^2$

$$\Rightarrow P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Quadratic Polynomial

Example:- Find the interpolating polynomial for the function $f(x) = e^x + 1$ at $x = 0, 1$ and calculate $[f(0.5) - P(0.5)]$

Solution

$$f(x) = e^x + 1$$

So

$$f(x_0) = f(0) = e^0 + 1 = 2$$

$$f(x_1) = f(1) = e^1 + 1 = 3.7183$$

Now

$$P(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$= \frac{x-1}{0-1}(2) + \frac{x-0}{1-0}(3.7183)$$

$$= \frac{2x-2}{-1} + \frac{3.7183x}{1}$$

$$= -2x + 2 + 3.7183x$$

$$P(x) = 2 + 1.7183x$$

Now $f(0.5) = e^{0.5} + 1 = 2.6487$

$$P(0.5) = 2 + 1.7183(0.5) = 2.8592$$

Now

$$|f(0.5) - P(0.5)| = |2.6487 - 2.8592|$$

$$= 0.2105$$

Example:- Find Interpolating Polynomial for the function

$$f(x) = x^2+x+1 \quad ; \quad x_0=0,\ x_1=1,\ x_2=2$$

Solution Now

$$f(x_0) = f(0) = (0)^2+0+1 = 1$$

$$f(x_1) = f(1) = (1)^2+1+1 = 3$$

$$f(x_2) = f(2) = (2)^2+2+1 = 7$$

$$P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)}(1) + \frac{(x-0)(x-2)}{(1-0)(1-2)}(3) + \frac{(x-0)(x-1)}{(2-0)(2-1)}(7)$$

$$= \frac{x^2-3x+2}{2} + \frac{x^2-2x}{-1}(3) + \frac{x^2-x}{2}(7)$$

$$= \frac{x^2-3x+2}{2} - 3x^2+6x + \frac{7x^2-7x}{2}$$

$$= x^2-3x+2-6x^2+12x+7x^2-7x$$

$$\boxed{P(x) = 2x^2+2x+2}$$

* Using the Linear Interpolation Evaluate

$$\int_a^b f(x)\,dx \cong \int_a^b P(x)\,dx$$

$$\int_a^b f(x)\,dx \cong \int_a^b \left[\frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b)\right] dx$$

$$= \int_a^b \left(\frac{x-b}{a-b}\right) f(a)\,dx + \int_a^b \left(\frac{x-a}{b-a}\right) f(b)\,dx$$

$$= \frac{f(a)}{a-b} \left.\frac{(x-b)^2}{2}\right|_a^b + \frac{f(b)}{b-a} \left.\frac{(x-a)^2}{2}\right|_a^b$$

$$= \frac{f(a)}{a-b}\left\{-\frac{(a-b)^2}{2}\right\} + \frac{f(b)}{b-a}\left\{\frac{(b-a)^2}{2}\right\}$$

$$= \frac{f(a)}{b-a} \cdot \frac{(a-b)^2}{2} + \frac{f(b)}{b-a} \frac{(b-a)^2}{2}$$

$$= \frac{(b-a)^2}{2(b-a)} \left\{ f(a) + f(b) \right\}$$

$$= \frac{b-a}{2} \left(f(a) + f(b) \right)$$

MUHAMMAD TAHIR

SP18 - PMT - 005

Interpolation: $y = f(x)$

Polynomial which approximate the given function

$[a,b] = a = x_0 < x_1 < x_2 < \cdots < x_n = b$

Notation: $f(x) = f[x]$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$ First Difference $\approx f'(x)$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

2nd Difference $\approx f''(x_0)$

In general

$$f[x_0, x_1, x_2, \ldots x_{n-1}, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots x_{n-1}]}{x_n - x_0}$$

x	$f[x]$	First difference	2nd difference
x_0	$f[x_0]$		
x_1	$f[x_1]$	$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$	
x_2	$f[x_2]$	$f[x_1, x_2] = ?$	$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$
$\vdots$	$\vdots$		
x_n	$f[x_n]$	$f[x_{n-1}, x_n] = \frac{f[x_n] - f[x_{n-1}]}{x_n - x_{n-1}}$	

Example:- $f(x) = x^2 + 1$; $x = 0$, $x = 1$

Find $f[x_0, x_1]$

Solution

$f[x_0] = f[0] = (0)^2 + 1 = 1$

$$f[x_1] = f[1] = (1)^2 + 1 = 2$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{2-1}{1-0}$$

$$= 2$$

★ Given x_0 & x_1 ; Find $P(x)$

Assuming that the equation of straight line passing through the point x_0 & x_1 is of the form

$$P(x) = a_0 + a_1 x$$

where a_0 and a_1 are unknown constants. Find a_0 and a_1 using interpolating polynomial that is

$$P(x_0) = f(x_0) \text{ and } P(x_1) = f(x_1)$$

$$P(x) = a_0 + a_1(x - x_0)$$

at $x = x_0$

$$P(x_0) = a_0 + a_1(x_0 - x_0)$$

$$= \boxed{a_0 = f[x_0]}$$

at $x = x_1$

$$P(x_1) = a_0 + a_1(x_1 - x_0)$$

$$= f[x_0] + a_1(x_1 - x_0)$$

since $f[x_1] = f[x_0] + a_1(x_1 - x_0)$

$$\therefore \boxed{a_1 = \frac{f[x_1] - f[x_0]}{x_1 - x_0}}$$

Hence $P(x) = f[x_0] + \frac{f[x_1] - f[x_0]}{x_1 - x_0}(x - x_0)$

$\Rightarrow P(x) = f[x_0] + (f[x_1] - f[x_0]) \frac{x - x_0}{x_1 - x_0}$

$= f[x_0] - \frac{x - x_0}{x_1 - x_0} f[x_0] + \frac{x - x_0}{x_1 - x_0} f[x_1]$

$= \frac{(x_1 - x_0) f[x_0] - (x - x_0) f[x_0]}{x_1 - x_0} + \frac{x - x_0}{x_1 - x_0} f[x_1]$

$= \frac{x_1 f[x_0] - \cancel{x_0 f[x_0]} - x f[x_0] + \cancel{x_0 f[x_0]}}{x_1 - x_0}$

$+ \frac{x - x_0}{x_1 - x_0} f[x_1]$

$= \frac{x_1 - x}{x_1 - x_0} f[x_0] + \frac{x - x_0}{x_1 - x_0} f[x_1]$

$\Rightarrow \boxed{P(x) = \frac{x - x_1}{x_0 - x_1} f[x_0] + \frac{x - x_0}{x_1 - x_0} f[x_1]}$

★ $f(x) = 0 = P(x)$

find x ?

Since $P(x) = 0$

$\Rightarrow \frac{x - x_1}{x_0 - x_1} f[x_0] + \frac{x - x_0}{x_1 - x_0} f[x_1] = 0$

$\Rightarrow - \frac{x - x_1}{x_0 - x_1} f[x_0] + \frac{x - x_0}{x_1 - x_0} f[x_1] = 0$

$$\Rightarrow \frac{1}{x_1 - x_0}\left[(x-x_0) f[x_1] - (x-x_1) f[x_0]\right] = 0$$

$$\Rightarrow x f[x_1] - x_0 f[x_1] - x f[x_0] + x_1 f[x_0] = 0$$

$\because x_1 \neq x_0$

$$\Rightarrow x\left(f[x_1] - f[x_0]\right) = x_0 f[x_1] - x_1 f[x_0]$$

$$\Rightarrow x = \frac{x_0 f[x_1] - x_1 f[x_0]}{f[x_1] - f[x_0]}$$

Again

$$x = x_0 - \frac{(x_1 - x_0) f[x_0]}{f[x_1] - f[x_0]}$$

$$= x_0 - \frac{f[x_0]}{\frac{f[x_1] - f[x_0]}{x_1 - x_0}} = x_0 - \frac{f[x_0]}{f[x_0, x_1]}$$

★ using the finite difference scheme $f(x) = 0$; $x_0 \neq x_1$

as

$$x = x_0 - \frac{f[x_0]}{f[x_1, x_0]}$$

This fix point formulation enables us to suggest the following method.

Algorithm:- (Steffanson Method) For a given x_0 find x_{n+1} by the iterative scheme

$$x_{n+2} = x_n - \frac{f[x_n]}{f[x_{n+1}, x_n]} \quad ; n = 0, 1, \ldots$$

Example. Find $\sqrt{2}$ using Steffanson method ; [1,2]

Solution

Let $x = \sqrt{2}$

then $x^2 - 2 = 0$

Take $f(x) = x^2 - 2$

As $x^2 - 2 = 0$ So $f(x) = 0$

take $x_0 = 1$, $x_1 = 2$

So

$f[x_0] = f[1] = (1)^2 - 2 = -1$

$f[x_1] = f[2] = (2)^2 - 2 = 2$

Iteration 1, $n = 0$

$$x_2 = x_0 - \frac{f[x_0]}{f[x_1, x_0]}$$

$$= x_0 - \frac{f[x_0]}{\frac{f[x_1] - f[x_0]}{x_1 - x_0}} = 1 - \frac{-1}{\frac{2-(-1)}{2-1}}$$

$$= 1 + \frac{1}{3/1} = 1 + 1 \times \frac{1}{3}$$

$$\Rightarrow \boxed{x_2 = 1.3333}$$

Now

$f[x_2] = (1.3333)^2 - 2$

$= -0.2222$

Iteration 2 $\quad n=1$

$$x_3 = x_1 - \frac{f[x_1]}{f[x_2, x_1]}$$

$$= x_1 - \frac{f[x_1]}{\frac{f[x_2]-f[x_1]}{x_2-x_1}} = 2 - \frac{2}{\frac{-0.2222-2}{1.3333-2}}$$

$$= 2 - \frac{2}{3.3332}$$

$$\Rightarrow \boxed{x_3 = 1.3999}$$

$$f[x_3] = (1.3999)^2 - 2 = -0.0403$$

Iteration 3 $\quad n=2$

$$x_4 = x_2 - \frac{f[x_2]}{f[x_3, x_2]}$$

$$= x_2 - \frac{f[x_2]}{\frac{f[x_3]-f[x_2]}{x_3-x_2}}$$

$$= 1.3333 - \frac{-0.2222}{\frac{-0.0403-(-0.2222)}{1.3999-1.3333}}$$

$$= 1.3333 + \frac{0.2222}{2.7312}$$

$$\rightarrow \boxed{x_4 = 1.4142}$$

System of Linear Equations:-

$$AX = b$$

System of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\Leftrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Leftrightarrow \quad AX = b$$

Argument matrix $[A:b]$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

Row wise $\begin{bmatrix} c_1 & 0 & 0 & d_1 \\ 0 & c_2 & 0 & d_2 \\ 0 & 0 & c_3 & d_3 \end{bmatrix}$

$$\Leftrightarrow \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

If $|A| = 0 \Rightarrow A^{-1}$ does not exist

$\Rightarrow A$ is not invertible

If $\begin{bmatrix} c_1 & 0 & 0 & d_1 \\ 0 & c_2 & 0 & d_2 \\ 0 & 0 & 0 & d_3 \end{bmatrix}$ Then system has infinite many solutions

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

Row Echelon

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & c_{11} & c_{12} & c_{13} \\ 0 & 1 & 0 & c_{21} & c_{22} & c_{23} \\ 0 & 0 & 1 & c_{31} & c_{32} & c_{33} \end{array}\right]$$

↳ Singular matrix. We take its inverse

$f(x) = 0 \Leftrightarrow x = g(x)$

You can decompose a matrix as follows

$$A = LU$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\det(A) = \det(LU) = \det(U)$$

* In Gaussian Elimination if row and column interchange is possible (complete Pivot) you can not apply LU decomposition method.

as when you apply complete pivot $u_{11}, u_{12}, \ldots, u_{nn}$ becomes zero

★ If $A = A^t$ then the matrix A is symmetric.

★ If the matrix A is not symmetric we can make it symmetric

Example:-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \\ 1 & 4 & 7 \end{bmatrix}$$ is not symmetric

Find the mean values of off diagonal elements

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 6 & 6 \\ 2 & 6 & 7 \end{bmatrix}$$

$$\frac{2+4}{2} = \frac{6}{2} = 3$$

$$\frac{3+1}{2} = \frac{4}{2} = 2$$

$$\frac{8+4}{2} = \frac{12}{2} = 6$$

It is symmetric

★ If $A = [a_{ij}]$ $i,j = 1,2,\ldots,n$ (not symmetric)

$A^t = [a_{ji}]$ $= j,i = 1,2,\ldots n$

Then $A_s = \left(\frac{a_{ij} + a_{ji}}{2}\right)$ $i,j = 1,2,\ldots n$

★ Consider
$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A^t X = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 & a_{21}x_2 & a_{31}x_3 \\ a_{12}x_1 & a_{22}x_2 & a_{32}x_3 \\ a_{13}x_1 & a_{23}x_2 & a_{33}x_3 \end{bmatrix}$$

XA is not possible

How will you make it possible

$$X^T A = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$X^T A X = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1^2 a_{11} + (a_{12} + a_{21}) x_1 x_2 + x_2^2 a_{22} +$$
$$(a_{31} + a_{13}) x_1 x_3 + x_3^2 a_{33} + (a_{32} + a_{23}) x_2 x_3$$

Consider $Q(x) = a_{11} x_1^2 + a_{22} x_2^2 + a_{33} x_3^2 + C_{21} x_1 x_2$
$$+ C_{32} x_2 x_3 + C_{13} x_1 x_3$$
$$= X^t A X$$

* Discuss The convergence analysis of the Regula falsi method for solving $f(x) = 0$

As Regula Falsi method is

$$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

Replace a by x

$$x = \frac{x f(b) - b f(x)}{f(b) - f(x)}$$

$$g(x) \overset{x = g(x)}{=} \frac{x f(b) - b f(x)}{f(b) - f(x)}$$

$$g(P) = \frac{P f(b) - b f(P)}{f(b) - f(P)}$$

$$= \frac{P \cancel{f(b)}}{\cancel{f(b)}} \qquad \because f(P) = 0$$

$$\Rightarrow \boxed{g(P) = P}$$

$$g(x) = \frac{x f(b) - b f(x)}{f(b) - f(x)}$$

$$g'(x) = \frac{[f(b) - b f'(x)][f(b) - f(x)] - [x f(b) - b f(x)][- f'(x)]}{[f(b) - f(x)]^2}$$

$$g(x) = \frac{[f(b) - b f'(x)][f(b) - f(x)] + f'(x)[x f(b) - b f(x)]}{[f(b) - f(x)]^2}$$

$$g'(x) = \frac{[f(b)]^2 - f(b)f(x) - b f'(x) f(b) + b f(x) f'(x) + x f'(x) f(b) - b f(x) f'(x)}{[f(b) - f(x)]^2}$$

Take $x = p$

$$\Rightarrow g'(p) = \frac{[f(b)]^2 - f(b)\cancel{f(p)}^{0} - b\cancel{f'(p)}^{0} f(b) + b\cancel{f(p)}^{0} f'(p) + p\cancel{f'(p)}^{0} f(b) - b\cancel{f(p)}^{0} f'(p)}{[f(b) - f(p)]^2}$$

$$g'(p) = \frac{[f(b)]^2}{[f(b)]^2} = 1$$

So it is linearly convergent

Question:- If $f(x) = 0$ has zero of multiplicity 2, Then discuss the convergence of the iterative method as

$$x_{n+1} = x_n - \frac{2 f(x_n)}{f'(x_n) + \alpha f(x_n)} ; \quad n = 0, 1, 2, \ldots \quad \alpha \neq 0$$

Solution

$$x_{n+1} = x_n - \frac{2 f(x_n)}{f'(x_n) + \alpha f(x_n)} ; \quad f''(x) \neq 0$$

① first checking fix point

$$g(x) = x - \frac{2f(x)}{f'(x) + \alpha f(x)}$$

at $x = p$

$$g(p) = p - \frac{2f(p)}{f'(p) + \alpha f(p)}$$

$$\Rightarrow g(p) = p - 2\lim_{x \to p} \frac{f(x)}{f'(x) + \alpha f(x)}$$

$$= p - 2\lim_{x \to p} \frac{f'(x)}{\cancelto{0}{f''(x)} + \alpha f'(x)}$$

$$\Rightarrow \boxed{g(p) = p}$$

$$g'(x) = 1 - \frac{2f'(x)[f'(x) + \alpha f(x)] - 2f(x)[f''(x) + \alpha f'(x)]}{[f'(x) + \alpha f(x)]^2}$$

$$\Rightarrow g'(x) = 1 - \frac{2f'(x)}{f'(x) + \alpha f(x)} - \frac{2f(x)[f''(x) + \alpha f'(x)]}{[f'(x) + \alpha f(x)]^2}$$

$$g'(x) = 1 - 2\lim_{x \to p} \frac{f''(x)}{f''(x) + \alpha f'(x)} - 2\lim_{x \to p} \frac{f'(x)[f''(x) + \alpha f'(x)] + f(x)[f'''(x) + \alpha f''(x)]}{2[f'(x) + \alpha f(x)][f''(x) + \alpha f'(x)]}$$

$$g'(x) = 1-2 + \lim_{x\to p} \frac{f'(x)[f''(x)+\alpha f'(x)] + f(x)[f'''(x)+\alpha f''(x)]}{[f'(x)+\alpha f(x)][f''(x)+\alpha f'(x)]}$$

$$= -1 + \lim_{x\to p} \frac{f'(x)}{f'(x)+\alpha f(x)} + \lim_{x\to p} \frac{f(x)[f'''(x)+\alpha f''(x)]}{[f'(x)+\alpha f(x)][f''(x)+\alpha f'(x)]}$$

$$= -1 + \lim_{x\to p} \frac{\cancel{f''(x)}}{\cancel{f''(x)+\alpha f'(x)}}^{1} + \lim_{x\to p} \frac{f'(x)[f'''(x)+\alpha f''(x)] + f(x)[f^{(4)}(x)+\alpha f'''(x)]}{[f''(x)+\alpha f'(x)][f''(x)+\alpha f'(x)] + [f'(x)+\alpha f(x)][f'''(x)+\alpha f''(x)]}$$

$$g'(x) = -1 + 1 + \lim_{x\to p} \frac{0+0}{f''(x)(f''(x))+0}$$

$$\Rightarrow \boxed{g'(x) = 0}$$

$$g''(x) = \frac{d}{dx}\left[\frac{-2f'(x)}{f'(x)+\alpha f(x)} + \frac{2f(x)[f''(x)+\alpha f'(x)]}{[f'(x)+\alpha f(x)]^2}\right]$$

$$= \frac{-2f''(x)[f'(x)+\alpha f(x)] + 2f'(x)[f''(x)+\alpha f'(x)]}{[f'(x)+\alpha f(x)]^2}$$

$$+ \frac{[f'(x)+\alpha f(x)]^2\{2f'(x)[f''(x)+\alpha f'(x)] + 2f(x)[f'''(x)+\alpha f''(x)]\} - 2f(x)[f''(x)+\alpha f'(x)]\cdot 2[f'(x)+\alpha f(x)][f''(x)+\alpha f'(x)]}{[f'(x)+\alpha f(x)]^4}$$

$$\Rightarrow g''(x) = \underbrace{\frac{-2f''(x)}{f'(x)+\alpha f(x)} + \frac{2f(x)[f''(x)+\alpha f'(x)]}{[f'(x)+\alpha f(x)]^2}}_{ⓐ}$$

$$+ \underbrace{\frac{2f'(x)[f''(x)+\alpha f'(x)]}{[f'(x)+\alpha f(x)]^2}}_{ⓑ} + \underbrace{\frac{2f(x)[f'''(x)+\alpha f''(x)]}{[f'(x)+\alpha f(x)]^2}}_{ⓒ}$$

$$- \underbrace{\frac{4f(x)[f''(x)+\alpha f'(x)]^2}{[f'(x)+\alpha f(x)]^3}}_{ⓓ} \longrightarrow (1)$$

Solving ⓐ we have

$$-\frac{2f''(x)}{f'(x)+\alpha f(x)} + \frac{2f'(x)[f''(x)+\alpha f'(x)]}{[f'(x)+\alpha f(x)]^2}$$

$$= \frac{-2f'''(x)}{f''(x)+\alpha f'(x)} + \frac{f''(x)[f''(x)+\alpha f'(x)]+f'(x)[f'''(x)+\alpha f''(x)]}{[f'(x)+\alpha f(x)][f''(x)+\alpha f'(x)]}$$

$$= \frac{-2f'''(x)}{f''(x)} + \frac{f''(x)}{f'(x)+\alpha f(x)} + \frac{f'(x)[f''(x)+\alpha f'(x)]}{[f'(x)+\alpha f(x)][f''(x)+\alpha f'(x)]}$$

$$= \frac{-2f'''(x)}{f''(x)} + \frac{f'''(x)}{f''(x)+\alpha f'(x)} + \text{———}\,''\,\text{———}$$

$$= \frac{-2f'''(x)}{f''(x)} + \frac{f'''(x)}{f''(x)}$$

$$+ \frac{f''(x)[f'''(x)+\alpha f''(x)]+f'(x)[f^{(iv)}(x)+\alpha f'''(x)]}{[f''(x)+\alpha f'(x)]^2+[f'(x)+\alpha f(x)][f'''(x)+\alpha f''(x)]}$$

$$= -\frac{f'''(x)}{f''(x)} + \frac{f''(x)\left[f'''(x) + \alpha f''(x)\right] + 0}{\left[f''(x)\right]^2 + 0}$$

$$= -\frac{f'''(x)}{f''(x)} + \frac{f'''(x)}{f''(x)} + \alpha$$

$$= \alpha \longrightarrow (a)$$

Solving (b) we have

$$\frac{2f'(x)\left[f''(x)+\alpha f'(x)\right]}{\left[f'(x)+\alpha f(x)\right]^2} = \frac{2f''(x)\left[f''(x)+\alpha f'(x)\right] + 2f'(x)\left[f'''(x)+\alpha f''(x)\right]}{2\left[f'(x)+\alpha f(x)\right]\cdot\left[f''(x)+\alpha f'(x)\right]}$$

$$= \frac{f''(x)}{f'(x)+\alpha f(x)} + \frac{f'(x)\left[f'''(x)+\alpha f''(x)\right]}{\left[f'(x)+\alpha f(x)\right]\left[f''(x)+\alpha f'(x)\right]}$$

$$= \frac{f'''(x)}{f''(x)+\alpha f'(x)} + \frac{f''(x)\left[f'''(x)+\alpha f''(x)\right] + f'(x)\left[f^{(4)}(x)+\alpha f'''(x)\right]}{\left[f''(x)+\alpha f'(x)\right]^2 + \left[f'(x)+\alpha f(x)\right]\left[f'''(x)+\alpha f''(x)\right]}$$

$$= \frac{f'''(x)}{f''(x)} + \frac{f''(x)\left[f'''(x)+\alpha f''(x)\right] + 0}{\left[f''(x)\right]^2 + 0}$$

$$= \frac{f'''(x)}{f''(x)} + \frac{f'''(x) + \alpha f''(x)}{f''(x)}$$

$$= \frac{f'''(x)}{f''(x)} + \frac{f'''(x)}{f''(x)} + \alpha$$

$$= \alpha + 2\frac{f'''(x)}{f''(x)} \longrightarrow (b)$$

Solving for c we have

$$\frac{2f(x)\left[f'''(x)+\alpha f''(x)\right]}{\left[f'(x)+\alpha f(x)\right]^2}$$

$$=\frac{2f'(x)\left[f'''(x)+\alpha f''(x)\right]+2f(x)\left[f^{(4)}(x)+\alpha f'''(x)\right]}{2\left[f'(x)+\alpha f(x)\right]\left[f''(x)+\alpha f'(x)\right]}$$

$$=\frac{f'(x)\left[f'''(x)+\alpha f''(x)\right]+f(x)\left[f^{(4)}(x)+\alpha f'''(x)\right]}{\left[f'(x)+\alpha f(x)\right]\left[f''(x)+\alpha f'(x)\right]}$$

$$=\frac{f''(x)\left[f'''(x)+\alpha f''(x)\right]+f'(x)\left[f^{(4)}(x)+\alpha f'''(x)\right]+f'(x)\left[f^{(4)}(x)+\alpha f'''(x)\right]+f(x)\left[f^{(5)}(x)+\alpha f^{(4)}(x)\right]}{\left[f''(x)+\alpha f'(x)\right]^2+\left[f'(x)+\alpha f(x)\right]\left[f''(x)+\alpha f'(x)\right]}$$

$$=\frac{f''(x)\left[f'''(x)+\alpha f''(x)\right]+0+0+0}{\left[f''(x)\right]^2}$$

$$=\frac{f'''(x)+\alpha f''(x)}{f''(x)}$$

$$=\alpha+\frac{f'''(x)}{f''(x)}\longrightarrow (c)$$

Solving for (d) we have.

$$\frac{f(x)\left[f''(x)+\alpha f'(x)\right]^2}{\left[f'(x)+\alpha f(x)\right]^3}$$

$$= \frac{f'(x)[f''(x)+\alpha f'(x)]^2 + 2f(x)[f''(x)+\alpha f'(x)][f'''(x)+\alpha f''(x)]}{3[f'(x)+\alpha f(x)]^2[f''(x)+\alpha f'(x)]}$$

$$= \frac{1}{3}\frac{f'(x)[f''(x)+\alpha f'(x)]}{[f'(x)+\alpha f(x)]^2} + \frac{2}{3}\frac{f(x)[f'''(x)+\alpha f''(x)]}{[f'(x)+\alpha f(x)]^2}$$

$$= \frac{1}{2\cdot 3}\frac{f''(x)[f''(x)+\alpha f'(x)] + f'(x)[f'''(x)+\alpha f''(x)]}{[f'(x)+\alpha f(x)][f''(x)+\alpha f'(x)]}$$

$$+ \frac{1}{3}\frac{f''(x)[f'''(x)+\alpha f''(x)] + f(x)[f^{(iv)}(x)+\alpha f'''(x)]}{[f'(x)+\alpha f(x)][f''(x)+\alpha f'(x)]}$$

$$= \frac{1}{6}\left\{\frac{f''(x)}{f'(x)+\alpha f(x)} + \frac{f'(x)[f'''(x)+\alpha f''(x)]}{[f'(x)+\alpha f(x)][f''(x)+\alpha f'(x)]}\right\}$$

$$+ \frac{1}{3}\frac{f'(x)[f'''(x)+\alpha f''(x)]}{[f'(x)+\alpha f(x)][f''(x)+\alpha f'(x)]} + \frac{1}{3}\frac{f(x)[f^{(4)}(x)+\alpha f'''(x)]}{[f'(x)+\alpha f(x)][f''(x)+\alpha f'(x)]}$$

$$= \frac{1}{6}\left\{\frac{f'''(x)}{f''(x)+\alpha f'(x)}\right\} + \frac{1}{2}\left[\frac{f''(x)[f'''(x)+\alpha f''(x)] + f'(x)[f^{(4)}(x)+\alpha f'''(x)]}{[f''(x)+\alpha f'(x)]^2 + [f'(x)+\alpha f(x)][f'''(x)+\alpha f''(x)]}\right]$$

$$+ \frac{1}{3}\frac{f'(x)[f^{(4)}(x)+\alpha f'''(x)] + f(x)[f^{(5)}(x)+\alpha f^{(4)}(x)]}{[f''(x)+\alpha f'(x)]^2 + [f'(x)+\alpha f(x)][f'''(x)+\alpha f''(x)]}$$

$$= \frac{1}{6}\frac{f'''(x)}{f''(x)} + \frac{1}{2}\left[\frac{f''(x)[f'''(x)+\alpha f''(x)]}{[f''(x)]^2}\right]$$

$$+ \frac{1}{3}\frac{0+0}{[f''(x)]^2}$$

$$= \frac{1}{6}\frac{f'''(x)}{f''(x)} + \frac{1}{2}\frac{f'''(x)}{f''(x)} + \alpha \longrightarrow (d)$$

Putting (a), (b), (c), (d) in (1) we have

$$g''(x) = \alpha + \alpha + 2\frac{f'''(x)}{f''(x)} + \alpha + \frac{f'''(x)}{f''(x)}$$

$$- 4\alpha - \frac{8}{3}\frac{f'''(x)}{f''(x)}$$

$$= -\alpha + \frac{1}{3}\frac{f'''(x)}{f''(x)}$$

$$= \frac{1}{3}\frac{f'''(x)}{f''(x)} - \alpha$$

$$\Rightarrow g''(x) \neq 0$$

So convergence is Quadratic

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System of Linear Equations:-

$$AX = b$$

$\det(A) \neq 0$ (A is non-singular)

Definition:- A matrix 'A' is said to be positive definite if the matrix A is linear and symmetric such that

$$\langle AX, X \rangle > 0 \text{ for all } X \in \mathbb{R}^n$$

Example:-

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

As $A = A^t \Rightarrow A$ is symmetric

$X \in \mathbb{R}^2$

$$\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$X^T A X = \langle AX, X \rangle$$

Now

$$AX = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix}$$

Now

$$X^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

$$X^T A X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix}$$

$$\Rightarrow X^T A X = [x_1(x_1 - x_2) + x_2(x_2 - x_1)]$$

$$= x_1^2 - x_1 x_2 + x_2^2 - x_1 x_2$$

$$= x_1^2 + x_2^2 - 2x_1 x_2$$

$$= (x_1 - x_2)^2 \geq 0$$

$\Rightarrow A$ is +ve definite.

Example

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = A$$

$\Rightarrow A$ is symmetric

$$X \in R^3 \Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Now

$$X^T A X = \langle AX, X \rangle$$

$$AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{bmatrix}$$

and $X^T A X = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{bmatrix}$

$$= x_1(2x_1 - x_2) + x_2(-x_1 + 2x_2 - x_3) + x_3(-x_2 + 2x_3)$$

$$= 2x_1^2 - x_1x_2 - x_1x_2 + 2x_2^2 - x_2x_3 - x_2x_3 + 2x_3^2$$

$$= 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$$

$$= x_1^2 + x_2^2 - 2x_1x_2 + x_2^2 + x_3^2 - 2x_2x_3 + x_1^2 + x_3^2$$

$$= (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_1^2 + x_3^2 \geq 0$$

This implies A is +ve definite.

---

* Diagonal elements of +ve definite matrix must be positive.

$AX = b$

$\langle AX, X \rangle = \langle b, X \rangle$

$\Rightarrow I[X] = \langle AX, X \rangle - 2\langle b, X \rangle$

* $\langle AX, X \rangle$ = quadratic function

$= \alpha x^2$

* In general, $f(x) = x^2$ is a convex function
* Every convex function has a minimum always

★ With the system of linear equations we can consider the functional denoted by $I[X]$, as

$$I[Y] = \langle AY, Y\rangle - 2\langle b, Y\rangle \quad , Y \in \mathbb{R}^n$$

$$\Leftrightarrow \quad AX = b$$

Definition:- An operator T is said to be linear if

$$\begin{cases} \text{(i)} & T(x+y) = T(x) + T(y) \; ; \forall \; x, y \in \mathbb{R}^n \\ \text{(ii)} & T(\alpha x) = \alpha T(x) \; ; \forall \; x \in \mathbb{R}^n, \alpha \in \mathbb{R} \end{cases}$$

OR

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \; ; \forall \; x, y \in \mathbb{R}^n$$
$$\& \; \alpha, \beta \in \mathbb{R}$$

★ $\alpha x + \beta y$ is called linear combination of the vector x and y.

$\Rightarrow$ If $\alpha + \beta = 1$, then

$$\alpha x + (1-\alpha) y = \alpha x + \beta y$$

and $\alpha + \beta = 1 \quad \Rightarrow \quad \alpha \in [0,1]$
$\qquad\qquad\qquad\qquad\& \; \beta \in [0,1]$

★ S is called subspace if for all $x, y \in S$; $\alpha x + \beta y \in S$, $\alpha, \beta \in \mathbb{R}$

(i) $0 \in S$

$\Rightarrow S = \{ x \in \mathbb{R}^2 : x_1 = x_2 + x_1 x_2 \}$

$$\Rightarrow 0 \in S$$

* Some books use ... , this mean starting from end point to initial point

$\Rightarrow \alpha + \beta = 1 \Rightarrow \alpha x + (1-\alpha) y = \alpha x + \beta y$

In this case $0 \notin S$

$\Rightarrow S = \{x \in \mathbb{R}^n : |x| \leq 1\}$

$y, x \in S \quad : |y| \leq 1 \quad , |x| \leq 1$

Consider $|\alpha x + \beta y| \leq \alpha |x| + \beta |y|$

$\leq \alpha + \beta$

$\therefore \alpha x + \beta y \notin S$

Therefore S is not a subspace

$\Rightarrow$ If $\alpha + \beta = 1$, then $\alpha \in [0,1]$

$|\alpha x + (1-\alpha) y| \leq \alpha |x| + |1-\alpha| |y|$

$\leq \alpha(1) + (1-\alpha) 1 \leq 1$

Definition:- (Convex Set) A set say $C \in \mathbb{R}^n$ is said to be a convex set if for all $x, y \in C$; $t \in [0,1]$

$(1-t) x + t y \in C$

OR

$(1-t) x + t y \in C \quad , \forall \; x, y \in C \; , t \in [0,1]$

* We say that the line segment joining the point $x, y \in C$ remains in C

Note that (i) if $t = 0$

$(1-t) x + t y = x \in C$ (beginning point)

(ii) if $t = 1$

$(1-t) x + t y = y \in C$ (end point)

So $x < y$

* $T(\alpha x+\beta y) = \alpha Tx + \beta Ty$

if $\alpha+\beta=1$

then $\underbrace{T(\alpha x+(1-\alpha)y) = \alpha Tx + (1-\alpha)Ty}_{\text{Convex function}}$,

* Every subspace is convex set but converse is not true

Definition:- A function f on the convex set C is said to be a convex function, if

$$f((1-t)x+ty) \leq (1-t)f(x)+tf(y) \quad \forall\, x,y \in C,\ t\in[0,1]$$

* Not $a=b$

$\Leftrightarrow a\leq b,\ b\leq a \Leftrightarrow b\leq a\leq b$

$a\leq b\leq a$

Examples:- 1) $f(x) = x$

2) $f(x) = |x|$

3) $f(x) = x^2$

Proof 1) $f(x) = x$

Consider

$f((1-t)x+ty) = (1-t)x+ty$

$\leq (1-t)x+ty$

$= (1-t)f(x)+tf(y)$

$\Rightarrow f((1-t)x+ty) \leq (1-t)f(x)+tf(y)$

$\Rightarrow f(x)$ is convex function.

$0 \le (a-b)^2 = a^2 + b^2 + 2ab$
$\Rightarrow 2ab \le a^2 + b^2$

2) $f(x) = |x|$

Consider

$$f((1-t)x + ty) = |(1-t)x + ty|$$
$$\le |(1-t)x| + |ty|$$
$$= (1-t)|x| + t|y|$$
$$= (1-t)f(x) + t f(y)$$

$\Rightarrow f((1-t)x + ty) \le (1-t)f(x) + t f(y)$

$\Rightarrow$ $f(x) = |x|$ is convex function.

3) $f(x) = x^2$

Consider

$$f((1-t)x+ty) = ((1-t)x + ty)^2$$
$$= ((1-t)x)^2 + (ty)^2 + 2(1-t)x \cdot ty$$
$$= (1-t)^2 x^2 + t^2 y^2 + 2xy(1-t)t$$

$\because 2xy \le x^2+y^2$

$$\le (1-t)^2 x^2 + t^2 y^2 + (1-t)t(x^2+y^2)$$
$$= (1+t^2-2t)x^2 + t^2y^2 + (t-t^2)(x^2+y^2)$$
$$= x^2 + x^2t^2 - 2tx^2 + t^2y^2 + tx^2 + ty^2 - t^2x^2 - t^2y^2$$
$$= x^2 - 2tx^2 + tx^2 + ty^2$$
$$= (1-2t+t)x^2 + ty^2$$
$$= (1-t)x^2 + ty^2$$
$$= (1-t)f(x) + t f(y)$$

$\Rightarrow f((1-t)x + ty) \le (1-t)f(x) + t f(y)$

$\Rightarrow f(x) = x^2$ is convex function.

★ Find $u \in \mathbb{R}^n$ such that $\langle u, v \rangle = 0 \ \forall \ v \in \mathbb{R}^n$
then $u = 0$

Find $u \in H$ s.t $\langle u, v \rangle = \langle f, v \rangle, \ \forall \ v \in H$
then $u = f$
Reiz-Frechet Representation Theorem

★ $$\left.\begin{aligned} I[v] &= \langle v, v \rangle - 2 \langle f, v \rangle \\ &= v^2 - 2 f(v) \end{aligned}\right\} \text{Convex}$$

★ Consider the system of linear equations

$$AX = b \longrightarrow (1)$$

which can be written as

$$\langle AX, X \rangle = \langle b, X \rangle \longrightarrow (2)$$

We associate a energy function $I[Y]$ with (2) as

$$I[Y] = \langle AY, Y \rangle - 2 \langle b, Y \rangle \longrightarrow (3)$$

$\boxed{\text{Minimum of (3) is solution of (2)}}$

Theorem:- If the operator A is linear, symmetric, and positive definite, then the minimum X of the functional $I[Y]$, where

$$I[Y] = \langle AY, Y \rangle - 2\langle b, Y \rangle \quad , \forall Y \in \mathbb{R}^n \longrightarrow (4)$$

satisfies

$$\langle AY, Y \rangle = \langle b, Y \rangle \longrightarrow (5)$$

converse is also true

Proof) Let X be the minimum of $I[Y]$, defined by (4). Then

$$I[X] \leq I[Y] \quad \forall \; Y \in \mathbb{R}^n \longrightarrow (6)$$

Since $X, Y \in \mathbb{R}^n$ so for $\varepsilon > 0$

$$Y_\varepsilon = (1-\varepsilon)X + \varepsilon Y$$

$$= X + \varepsilon(Y - X) \in \mathbb{R}^n$$

Replace Y by Y_ε into (6) to have

$$I[X] \leq I[X + \varepsilon(Y-X)]$$

using (4) we have

$$\langle AX, X \rangle - 2\langle b, X \rangle \leq \langle A(X+\varepsilon(Y-X)), X+\varepsilon(Y-X) \rangle - 2\langle b, X+\varepsilon(Y-X) \rangle$$

$$= \langle AX + \varepsilon(AY - AX), X+\varepsilon(Y-X) \rangle - 2\langle b, X \rangle - 2\varepsilon\langle b, Y - X \rangle$$

$$= \langle AX, X \rangle + \varepsilon \langle AX, Y-X \rangle + \varepsilon \langle AY - AX, X \rangle + \varepsilon^2 \langle AY - AX, Y-X \rangle - 2\langle b, X \rangle - 2\varepsilon \langle b, Y-X \rangle$$

Then

$$0 \leq \langle AX, Y-X \rangle + \langle AY - AX, X \rangle + \varepsilon \langle AY - AX, Y-X \rangle - 2\langle b, Y-X \rangle$$

$$\leq \langle AX, Y-X \rangle - 2\langle b, Y-X \rangle + \varepsilon \langle AY - AX, Y-X \rangle$$

After some simplifications we have

$$\langle AX, Y-X \rangle \geq \langle b, Y-X \rangle \longrightarrow (7)$$

Also from (7) we have

$$\langle AX, Y-X \rangle \leq \langle b, Y-X \rangle \longrightarrow (8)$$

from (7) and (8) we have

$$\langle AX, Y-X \rangle = \langle b, Y-X \rangle \longrightarrow (9)$$

take $Y = Y + X \in \mathbb{R}^n$ in (9) to have

$$\langle AX, Y \rangle = \langle b, Y \rangle$$

Thus $X \in \mathbb{R}^n$ is the solution of the system of linear equation $AX = b$.

Conversely, Let X be solution of (1) we have to show that $X \in \mathbb{R}^n$ is the minimum of the function (4)

$I[Y] = \langle AY, Y \rangle - 2\langle b, Y \rangle$

Consider $I[X] - I[Y]$

$= \langle AX, X \rangle - 2\langle b, X \rangle - \langle AY, Y \rangle - 2\langle b, Y \rangle$

$\leq -\langle AX - AY, X - Y \rangle$

≤ 0

$\Rightarrow I[X] \leq I[Y] \quad \forall X, Y \in \mathbb{R}^n$

$\Rightarrow X$ is minimum

$$AX = b$$

$\langle \cdot,\cdot \rangle$
Inner Product
Dot —"—
Scaler —"—
Pairing
Duality

(i) $\langle u,u \rangle \geq 0 \quad \forall\ u \in \mathbb{R}^n$ and
$\langle u,u \rangle = 0$ iff $u = 0$
(ii) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle \ ; \forall\ \alpha \in \mathbb{R}$
(iii) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

H (Hilbert space) $\qquad H^*$ (Dual space)

★ $f \in H^*$, f is linear and continuous
H is Hilbert space
$H \cong H^*$ H is isomorphic to its dual.

$$\underbrace{\langle \underset{H^*}{f}, \underset{H}{u} \rangle}_{\text{Pairing, Duality}} = f(u)$$

★ Define Norm associated with the inner product.

$\|\cdot\|^2 = \langle u, u \rangle$ $\quad \|\cdot\| = \sqrt{\langle u, u \rangle}$
Positive Definite value

★ All the properties of inner product are also enjoyed by norm

Definition:- The norm on the space $\mathbb{R}^n$ is a +ve definite value having the following properties

(i) $\|u\| > 0 \quad \forall\, u \in \mathbb{R}^n$ and

$\|u\| = 0$ iff $u = 0$

(ii) $\|\alpha u\| = |\alpha|\, \|u\| \quad ; \alpha \in \mathbb{R},\ u \in \mathbb{R}^n$

(iii) $\|u+v\| \leq \|u\| + \|v\| \quad \forall\ u, v \in \mathbb{R}^n$

Euclidian Norm:- $\sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$

* On the vector space we have the following norms

(i) $\|u\|_e = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$

Example:- Find $\|u\|_e$ if $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\Rightarrow \|u\|_e = \sqrt{(1)^2 + (2)^2 + (3)^2}$

$= \sqrt{1+4+9} = \sqrt{14}$

(ii) $\|u\|_\infty = \max\limits_{1 \leq i \leq n} \{|x_i|\}$

$= \max\limits_{1 \leq i \leq 3} \{|1|, |2|, |3|\}$

$= \max\limits_{1 \leq i \leq 3} \{1, 2, 3\}$

$= 3$

Result:- In any finite dimensional space all the norms are equivalent; i.e $\alpha \|u\|_\infty \leq \|u\|_e \leq \beta \|u\|_\infty$

* $uv = u^T v$

e.g $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Rightarrow uv = u^T v$

for vector multiplication

Definition:- The matrix norm of a $n \times n$ matrix on $R^n \times R^n$ is a +ve definite value which has the following properties

(i) $\|A\| \geq 0 \quad \forall A \in R^n \times R^n$ and
$\|A\| = 0$ iff A is null matrix

(ii) $\|\alpha A\| = |\alpha| \, \|A\| \quad \forall \alpha \in R, A \in R^n \times R^n$

(iii) $\|A+B\| \leq \|A\| + \|B\| \quad \forall A, B \in R^n \times R^n$

(iv) $\|AB\| \leq \|A\| \, \|B\| \quad \forall A, B \in R^n \times R^n$

In general

$$\|A\|_\infty = \max_{\|X\|=1} \|AX\|_\infty$$

★ If $\|X\| \neq 1$ then we have to normalize it
e.g $\|X\| = 3$, divide every entry of X by 3.

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| = \max_{1 \leq i \leq n} \begin{cases} |a_{11}| + |a_{12}| + \cdots \\ |a_{21}| + |a_{22}| + \cdots \\ \vdots \\ |a_{n1}| + |a_{n2}| + \cdots \end{cases}$$

Examples

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 1 & -1 \\ 1 & 0 & 5 \end{bmatrix}$$

$$\|A\|_\infty = \max_{1 \leq i \leq 3} \begin{cases} |-1| + |2| + |3| \\ |4| + |1| + |-1| \\ |1| + |0| + |5| \end{cases}$$

$$= \max_{1 \le i \le n} \begin{cases} 1+2+3 \\ 4+1+2 \\ 1+0+5 \end{cases}$$

$$= \max_{1 \le i \le 3} \begin{cases} 6 \\ 7 \\ 6 \end{cases}$$

$$\Rightarrow \quad \|A\|_\infty = 7$$

* $\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$

* $\|A\|_2 = \max \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$

---

* Consider the system of linear equations

$$AX = b \longrightarrow ①$$

This can be written as

$$AX - b = 0 \longrightarrow ②$$

Associate the function $F(X)$ with ① as

$$F(X) = AX - b$$

or $F(X) = 0 \iff X = H(X)$

Here $H(X)$ is any arbitrary function.

Algorithm:- For given $X^{(0)}$ find $X^{(k)}$ by the iterative method

$$X^{(k+1)} = H(X^{(k)}), \quad k = 0, 1, 2, \ldots$$

* Consider

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

If $a_{11} \neq 0$, $a_{22} \neq 0$, $a_{33} \neq 0$
Then system of linear equations can be written as

$$x_1 = \frac{b_1}{a_{11}} - 0x_1 - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3$$

$$x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 - 0x_2 - \frac{a_{23}}{a_{22}}x_3$$

$$x_3 = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}}x_1 - \frac{a_{32}}{a_{33}}x_2 - 0x_3$$

$$X = C + TX \quad ; \text{ where}$$

$$C = \begin{bmatrix} -\frac{b_1}{a_{11}} \\ -\frac{b_2}{a_{22}} \\ -\frac{b_3}{a_{33}} \end{bmatrix}, \quad T = \begin{bmatrix} -0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} \\ -\frac{a_{31}}{a_{33}} & -\frac{a_{32}}{a_{33}} & 0 \end{bmatrix}$$

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$$AX = b$$

* Consider $F(X) = AX - b$

$\Rightarrow$ $F(X) = 0 \xleftrightarrow{\text{Decompose}} X = H(X)$

where $H(X)$ is an arbitrary function

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$a_{11} \neq 0 \;,\; a_{22} \neq 0 \;,\; a_{33} \neq 0$$

$$X = C + TX \longrightarrow (1)$$

where T is a suitable matrix

Algorithm:-

For a given $X^{(0)}$ find $X^{(k+1)}$ such that

$$X^{(k+1)} = TX^{(k)} + C \longrightarrow (2)$$

Convergence:-

$$\|X^{(k+1)} - X\| = \|TX^{(k)} + \cancel{C} - TX - \cancel{C}\|$$

$$= \|TX^{(k)} - TX\|$$

$$= \|T(X^{(k)} - X)\|$$

$$\leq \|T\| \, \|X^{(k)} - X\|$$

$$\leq \|T\| \left\{ \|T\| \, \|X^{(k-1)} - X\| \right\}$$

$$= \|T\|^2 \, \|X^{(k-1)} - X\|$$

$$\vdots$$

$$\leq \|T\|^{(k+1)} \, \|X^{(0)} - X\|$$

$$\Rightarrow \lim_{k\to\infty}\left(\|X^{(k+1)}-X\|\right) \le \lim_{k\to\infty}\left\{\|T\|^{(k+1)}\|X^{(0)}-X\|\right\}$$

$$= \lim_{k\to\infty}\left(\|T\|^{(k+1)}\right)\lim_{k\to\infty}\left(\|X^{(0)}-X\|\right)$$

$$= \lim_{k\to\infty}\left(\|T\|^{(k+1)}\right)\|X^{(0)}-X\|$$

.... continuous

★ Convergence if $\|X^{(k-1)}-X\| < 1$

★ Let $A = (a_{ij})$; $i,j = 1,2,\dots,n$
we say that the matrix A is converges if $\lim_{i,j}(a_{ij}) = a_{0,0}$

Example:-

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{4} & 1 & 1 \end{bmatrix}$$

$\|A\|_\infty = \frac{11}{4} > 1$ (So not convergent

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \|A\|_\infty = \frac{3}{4} < 1 \quad \{\text{convergent}\}$$

$$\star \quad \|T^{(k+1)}\| = \|T\|^{(k+1)}$$

* Consider the System of Linear Equations. Then

$$AX = b$$

Let λ be the eigen value of A

Then $AX = \lambda X$

From this we have

$$0 = AX - \lambda X = (A - \lambda I)X$$

$$= \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$$

$AX = \lambda X$
$\not\Rightarrow X(A-\lambda)$
$\not\Rightarrow (A-\lambda)X$
Also $AB \neq BA$

* **Result:-** A matrix 'A' is convergent matrix if $\underbrace{\|A\|}_{\text{(Any norm)}} \leq k < 1$

$$\star \quad f(x) = 0 \Leftrightarrow x = g(x)$$

$$\|g'(x)\| \leq k < 1$$

★ $$AX = b \longrightarrow (1)$$

$$\Leftrightarrow X = TX + C \longrightarrow (2)$$

We decompose matrix A as

$$A = D - L - U \longrightarrow (3)$$

where D is a diagonal matrix. That is

$$D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ -l_{21} & 0 & 0 \\ -l_{31} & -l_{32} & 0 \end{bmatrix} \text{ and } U = \begin{bmatrix} 0 & -u_{12} & -u_{13} \\ 0 & 0 & -u_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

From (1) and (3) we have

$$(D - L - U)X = b$$

OR $$(D - U)X - LX = b$$

OR $$DX - (L+U)X = b$$

$$\Rightarrow DX = b + (L+U)X$$

$$\Rightarrow X = D^{-1}(U+L)X + D^{-1}b$$

$$= T_j X + C$$

where $T_j = D^{-1}(U+L)$

& $C = D^{-1}b$

$$\Rightarrow X = T_j X + C$$

$$\Rightarrow X^{(k+1)} = T_j X^{(k)} + C \quad \text{(Jacobi Method)}$$

Given $X^{(0)}$ find $X^{(k+1)}$ by

$$\begin{bmatrix} x_1^{(0)} \\ x_2^{(1)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix} + \begin{bmatrix} c_1 \\ c_{22} \\ c_{33} \end{bmatrix}$$

$$AX = b \longrightarrow (1)$$

Let $A = D - L - U$

Then $AX = b$ can be written as

$$(D - L - U)X = b \longrightarrow (2)$$

Consider

$$\left.\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 &= b_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 &= b_2 \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 &= b_3 \end{aligned}\right\} \longrightarrow (A)$$

Assume that $a_{11} \neq 0$, $a_{22} \neq 0$, $a_{33} \neq 0$

Then (A) implies

$$x_1 = \frac{b_1}{a_{11}} - 0 \cdot x_1 - \frac{a_{12}}{a_{11}} x_2 - \frac{a_{13}}{a_{11}} x_3$$

$$x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1 - 0 \cdot x_2 - \frac{a_{23}}{a_{22}} x_3$$

$$x_3 = \frac{b_3}{a_{22}} - \frac{a_{31}}{a_{33}} x_1 - \frac{a_{32}}{a_{33}} x_2 - 0 \cdot x_3$$

⇒ **Jacobi Method:-** Given $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$
find $x_1^{(k+1)}, x_2^{(k+1)}$ & $x_3^{(k+1)}$

$$x_1^{(k+1)} = -0\cdot x_1^{(k)} - \frac{a_{12}}{a_{11}} x_2^{(k)} - \frac{a_{13}}{a_{11}} x_3^{(k)} - C_1$$

$$x_2^{(k+1)} = -\frac{a_{21}}{a_{22}} x_1^{(k)} - 0\cdot x_2^{(k)} - \frac{a_{23}}{a_{22}} x_3^{(k)} - C_2$$

$$x_3^{(k+1)} = -\frac{a_{31}}{a_{33}} x_1^{(k)} - \frac{a_{32}}{a_{33}} x_2^{(k)} - 0\cdot x_3^{(k)} - C_3$$

OR

$$X^{(k+1)} = \underbrace{\begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} \\ -\frac{a_{31}}{a_{33}} & -\frac{a_{32}}{a_{33}} & 0 \end{bmatrix}}_{T} X^{(k)} - C$$

$$= T X^{(k)} + C$$

OR

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -a_{21} & 0 & 0 \\ -a_{31} & -a_{32} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -a_{12} & -a_{13} \\ 0 & 0 & -a_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Therefore

$$DX = b + LX + LU$$

$$\Rightarrow DX = b - (-LX) + (-UX)$$
$$= b - (-L - U)X$$

or

$(D - L - U)X = b$, where

$$L = \begin{bmatrix} 0 & 0 & 0 \\ -l_{21} & 0 & 0 \\ -l_{31} & -l_{32} & 0 \end{bmatrix} \text{ and } U = \begin{bmatrix} 0 & -u_{12} & -u_{13} \\ 0 & 0 & -u_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

From equation (2)

$$DX - LX - UX = b$$
$$DX = b + LX + UX$$
$$= b + (L + U)X$$

Therefore

$$X = D^{-1}(L+U)X + D^{-1}b$$
$$= T_j X + C$$

Example :-
$$2x_1 - x_2 + x_3 = -1$$
$$2x_1 + 2x_2 + 2x_3 = 4$$
$$-x_1 - x_2 + 2x_3 = -5$$

Find
$$T_j = D^{-1}(L+U)$$

Solution

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 4 \\ -5 \end{bmatrix}$$

Therefore

$$U = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$L+U = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

Now

$$D^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$T_j = D^{-1}(L+U)$$

$$= \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.5 & -0.5 \\ -1 & 0 & -1 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

$\|T_j\|_\infty = 2 > 1$ (not convergent)

Gauss Seidal:- Given $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$ find $x_1^{(k+1)}, x_2^{(k+1)}$ & $x_3^{(k+1)}$ by

$$x_1^{(k+1)} = -0 \cdot x_1^{(k)} - \frac{a_{12}}{a_{11}} x_2^{(k)} - \frac{a_{13}}{a_{11}} x_3^{(k)} - C_1$$

$$x_2^{(k+1)} = \frac{-a_{21}}{a_{22}} x_1^{(k+1)} - 0 \cdot x_2 - \frac{a_{23}}{a_{22}} x_3^{(k)} - C_2$$

$$x_3^{(k+1)} = \frac{-a_{31}}{a_{33}} x_1^{(k+1)} - \frac{a_{32}}{a_{33}} x_2^{(k+1)} - 0 \cdot x_3 - C_3$$

$$\star \quad X = (D-L)^{-1} UX + (D-L)^{-1} b$$

$$= T_{gs} X + (D-L)^{-1} b$$

Now from previous example

$$D-L = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

To find $(D-L)^{-1}$

Augmented matrix is

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 1 & 0 & 0 & 1/2 & 0 \\ 1 & 1 & -2 & 0 & 0 & -1 \end{array}\right] \begin{array}{l} \frac{1}{2} R_1 \\ 1/2\, R_2 \\ -R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 1 & -2 & -1/2 & 0 & -1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & -2 & 0 & -1/2 & -1 \end{array}\right] R_3 - R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 1/4 & 1/2 \end{array}\right] -\frac{1}{2} R_3$$

$$\Rightarrow (D-L)^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \end{bmatrix}$$

$$T_{gs} = (D-L)^{-1} U$$

$$= \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T_{gs} = \begin{bmatrix} 0 & 0.5 & -0.5 \\ 0 & -0.5 & -0.5 \\ 0 & 0 & -0.5 \end{bmatrix}$$

$$\|T_{gs}\|_\infty = 1 \quad \& \quad \|T_{gs}\|_1 = 1.5 > 1$$

not convergent

* If λ is the eigen value of the matrix A, Then $(AX = \lambda X)$
If A^{-1} exist then

$$X = A^{-1}\lambda X = \lambda A^{-1}X$$

$$\Rightarrow \frac{1}{\lambda}X = A^{-1}X$$

$\Rightarrow$ eigen value of $A^{-1} = 1/\lambda$

* If λ is the eigen value of A Then find the eigen value of A^n?

Definition:- (Spectral Radius of Matrix A) The spectral radius of matrix A is denoted by $\rho(A)$ and is defined as

$$\rho(A) = \max|\lambda|$$

where λ is the eigen value of matrix A.

* The norm $\|\cdot\|_2$ of matrix A is defined as

$$\|A\|_2 = \sqrt{\rho(A^tA)}$$

where A^t is the transpose of A.

$\Rightarrow$ For any norm $\|\cdot\|$ we always have

$$\rho(A) \leq \|A\|$$

i.e $\|A\|_2 \leq \|A\|$

Proof) Let λ be the eigen value

of A. Then

$$|\lambda| = |\lambda| \|X\| = \|\lambda X\| = \|AX\| \quad \because AX = \lambda X$$

$$\leq \|A\| \|X\|$$

$$\leq \|A\| \quad \because \|X\| = 1$$

Therefore

$$\rho(A) \leq \max |\lambda| \leq \|A\|$$

Convergence of Regula Falsi

$$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x + x = x + \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$2x = x + \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x = \frac{x}{2} + \frac{1}{2} \frac{a f(b) - b f(a)}{f(b) - f(a)} = g(x) \longrightarrow (1)$$

By iterative scheme

$$x_{n+1} = g(x_n)$$

$$\Rightarrow g(x_n) = \frac{x_n}{2} + \frac{1}{2} \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{2} \frac{a f(b) - b f(a)}{f(b) - f(a)} \longrightarrow (2)$$

From (1) & (2)

$$|x_{n+1} - x| = |g(x_n) - g(x)|$$

$$= \left| \frac{x_n}{2} + \frac{1}{2}\frac{af(b) - bf(a)}{f(b) - f(a)} - \frac{x}{2} - \frac{1}{2}\frac{af(b) - bf(a)}{f(b) - f(a)} \right|$$

$$= \frac{1}{2}|x_n - x|$$

$$\Rightarrow \frac{|x_{n+1} - x|}{|x_n - x|} = \frac{1}{2}$$

$\alpha = 1$ linear.

Convergence of Bisection

Example:- Discuss the convergence of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$ by the spectral radius.

Solution

As $\|A\|_2 = \sqrt{\rho(A^t A)}$

Now

$$A^t = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

$$A^t A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow A^t A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 6 \end{bmatrix}$$

Let λ be the eigen value of $A^t A$

$$\Rightarrow |A^t A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & 0 \\ 1 & 0 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 0 & 6-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & 0 \\ 1 & 6-\lambda \end{vmatrix} + 1\begin{vmatrix} -1 & 2-\lambda \\ 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(2-\lambda)(6-\lambda) + 1(-1)(6-\lambda) + 1(-1)(2-\lambda) = 0$$

$$\Rightarrow (4-2\lambda-2\lambda+\lambda^2)(6-\lambda) - 1(6-\lambda) - 1(2-\lambda) = 0$$

$$\Rightarrow (4-4\lambda+\lambda^2)(6-\lambda) - 6 + \lambda - 2 + \lambda = 0$$

$$\Rightarrow 24 - 24\lambda + 6\lambda^2 - 4\lambda + 4\lambda^2 - \lambda^3 - 6 + \lambda - 2 + \lambda = 0$$

$$\Rightarrow 16 - 26\lambda + 10\lambda^2 - \lambda^3 = 0$$

$$\Rightarrow \lambda^3 - 10\lambda^2 + 26\lambda - 16 = 0$$

$$\Rightarrow \lambda_1 = 6.25 \;,\quad \lambda_2 = 2.85 \quad \& \quad \lambda_3 = 0.9$$

So $\max|\lambda| = |6.25|$

$$\Rightarrow \rho(A^t A) = 6.25$$

$$\|A\|_2 = \sqrt{6.25}$$

$$= 2.5 > 1$$ (So not convergent

Also $\|A\|_2 \leq \|A\|_\infty = 3$

$$\|A\|_2 \leq \|A\|_1 = 4$$

———————

* A matrix A is convergent if

$$\lim_{n\to\infty} A^n = 0$$

i.e $$\lim_{n\to} A^n = [0]_{i,j}$$

Example:- Discuss the convergence of the matrix

$$A = \begin{bmatrix} 1/2 & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2^2} & 0 \\ \frac{1}{8}+\frac{1}{8} & \frac{1}{2^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2^2} & 0 \\ \frac{2}{2^3} & \frac{1}{2^2} \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} \frac{1}{2^2} & 0 \\ \frac{2}{2^{2+1}} & \frac{1}{2^2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2^3} & 0 \\ \frac{2}{2^4}+\frac{1}{2^4} & \frac{1}{2^3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2^3} & 0 \\ \frac{3}{2^{3+1}} & \frac{1}{2^3} \end{bmatrix}$$

and so on, we have

$$A^n = \begin{bmatrix} \frac{1}{2^n} & 0 \\ \frac{2}{2^{n+1}} & \frac{1}{2^n} \end{bmatrix}$$

so

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} \begin{bmatrix} \frac{1}{2^n} & 0 \\ \frac{n}{2^{n+1}} & \frac{1}{2^n} \end{bmatrix}$$

$$= \lim_{n\to\infty} \begin{bmatrix} \frac{1}{2^n} & 0 \\ \frac{1}{2^{n+1}\ln 2} & \frac{1}{2^n} \end{bmatrix}$$ using L,Hospital rule on $[a]_{2,2}$

$$\Rightarrow \lim_{n\to\infty} A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow A$ is convergent matrix

Example:- Find the eigen values of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

Solution

Let λ be the eigen value of A, so

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & -1 \\ -1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} - 0 + 2\begin{vmatrix} 0 & 1-\lambda \\ -1 & 1 \end{vmatrix} = 0$$

$$(1-\lambda)\left((1-\lambda)^2+1\right) + 2\left(0 + 1(1-\lambda)\right) = 0$$

$$(1-\lambda)(1+\lambda^2-2\lambda+1) + 2(1-\lambda) = 0$$

Taking $(1-\lambda)$ as common

$$\Rightarrow (1-\lambda)\{1+\lambda^2-2\lambda+1+2\} = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2-2\lambda+4) = 0$$

$$\Rightarrow 1-\lambda = 0 \qquad \lambda^2-2\lambda+4 = 0$$

$$\Rightarrow \boxed{\lambda = 1} \qquad \lambda = \frac{-(-2) \pm \sqrt{(-2)^2-4(1)(4)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-16}}{2}$$

$$= \frac{2(1 \pm \sqrt{1-4})}{2}$$

$$= 1 \pm \sqrt{-3}$$

$$\Rightarrow \lambda = 1 \pm \sqrt{3}\,i$$

Example:- Find the eigen values of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Solution Let λ be the eigen value of A

so $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ -1 & 2-\lambda \end{vmatrix} + 0 = 0$$

$$\Rightarrow (1-\lambda)\{(2-\lambda)^2 - 1\} - 1(2-\lambda+1) = 0$$

$$\Rightarrow (1-\lambda)(4+\lambda^2-4\lambda-1) - 1(3-\lambda) = 0$$

$$4+\lambda^2-4\lambda-1-4\lambda-\lambda^3+4\lambda^2+\lambda-3+\lambda = 0$$

$$-\lambda^3+5\lambda^2-6\lambda = 0$$

$$\Rightarrow -\lambda(\lambda^2-5\lambda+6) = 0$$

$-\lambda = 0$

$\Rightarrow \boxed{\lambda = 0}$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\lambda^2 - 3\lambda - 2\lambda + 6 = 0$$

$$\lambda(\lambda-3) - 2(\lambda-3) = 0$$

$$(\lambda-3)(\lambda-2) = 0$$

$\boxed{\lambda = 3}$, $\boxed{\lambda = 2}$

$\Rightarrow \lambda = 0, 2, 3$ are eigen values.

$$A = \begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix} \quad \text{find inverse}$$

$$\text{Adj } A = \begin{bmatrix} 2 & -2 \\ -1.0001 & 1 \end{bmatrix}$$

and $|A| = \begin{vmatrix} 1 & 2 \\ 1.0001 & 2 \end{vmatrix}$

$$= 2 - 2.0002$$

$$= -0.0002$$

So $A^{-1} = \dfrac{1}{|A|} \text{Adj } A$

$$= \frac{1}{-0.0002} \begin{bmatrix} 2 & -2 \\ -1.0001 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -10000 & 10000 \\ 5000.5 & -5000 \end{bmatrix}$$

$$\|A^{-1}\|_\infty = \max \begin{cases} |-10000| + |10000| \\ |5000.5| + |-5000| \end{cases}$$

$$= \max \begin{cases} 20000 \\ 10000.5 \end{cases}$$

$$\Rightarrow \|A^{-1}\|_\infty = 20000$$

& $\|A\|_\infty = 3.0001$

$$\Rightarrow \|A^{-1}\|_\infty \|A\|_\infty = (20000)(3.0001)$$
$$= 60002$$

$$A^{-1}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\|I\|_\infty = \|AA^{-1}\|_\infty = 1$$

$$\Rightarrow 1 = \|AA^{-1}\|_\infty \leq \|A\| \, \|A^{-1}\|$$

$$\Rightarrow \|A\| \, \|A^{-1}\| \geq 1$$

$\Rightarrow$ Matrix is Ill Condition

Differential Equations:-

1) Initial value Problems.

$$\frac{dy}{dt} = f(t,y); \quad a \leq t \leq b \quad y(a) = y(t_0) = \alpha \quad \boxed{y = y(t)}$$

2) Boundary value Problems

$$\frac{d^2y}{dx^2} = f(x,y); \quad a \leq x \leq b \quad y(0) = \alpha, \; y(b) = \beta$$

★ $Tu = u^2$

$$\|Tu - Tv\| = \|u^2 - v^2\|$$
$$= \|(u+v)(u-v)\|$$
$$\leq \|u+v\| \, \|u-v\|$$
$$= \beta \|u-v\|$$

$\Rightarrow \|Tu - Tv\| \leq \beta \|u-v\|$; where $\beta = \|u+v\|$

β is variable
So Tu is not Lipchitz Continuous

★ **Initial Value Problem**:- Consider the initial value problem

$$\frac{dy}{dt} = f(t,y) ; \quad y(t_0) = \alpha \quad (t_0 \leq t \leq t_n)$$

$\{$ To solve the problem first check that it is Lipchitz continuous

Taylor Series Method:-

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i) \, y'(t_i) + R_n$$
$$= y(t_i) + (t_{i+1} - t_i) \, f(t_i, y_i)$$

$$\Rightarrow y(t_{i+1}) = y(t_i) + h f(t_i, y(t_i))$$

$y_{i+1} = y_i + h\,f(t_i, y_i)$ First order Taylor series Method (Euler Method)

Algorithm:- Given $y_0 = y(t_0)$ find $y_1, y_2, \ldots y_n$ by the iterative scheme

$$y_{i+1} = y_i + h\,f(t_i, y_i); \quad i = 0, 1, 2, \ldots$$

* $\frac{dy}{dx} - f(t, x) = 0 \iff y = c + h\,N(y)$

Example:- Find the appropriate solution of

$$\frac{dy}{dt} = y - t^2 + 1; \quad y(0) = 0.5, \quad N = 10$$
$$0 \le t \le 2, \quad h = 0.2$$

Solution

$$y_{i+1} = y_i + h\,f(t_i, y_i) \longrightarrow (1)$$

Here $f(t_i, y_i) = y_i - (t_i)^2 + 1$

for $t = 0$

$$y_1 = y_0 + h\,(y_0 - (t_0)^2 + 1)$$
$$= 0.5 + 0.2(0.5 - (0)^2 + 1)$$
$$= 0.5 + 0.2(1.5)$$
$$= 0.8$$

$\Rightarrow \boxed{y_1 = 0.8}$

Now $t_1 = t_0 + h = 0 + 0.2$

$= 0.2$

for $i=1$

$$y_2 = y_1 + h(y_1 - (t_1)^2 + 1)$$
$$= 0.8 + 0.2(0.8 - (0.2)^2 + 1)$$
$$= 0.8 + 0.2(0.8 - 0.04 + 1)$$
$$= 0.8 + 0.2(1.76)$$
$$\Rightarrow \boxed{y_2 = 1.152}$$

Taylor Series Method:-

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2!} y''(t_i)$$
$$= y_i + h f(t_i, y_i) + \frac{h^2}{2} f'(t_i, y_i)$$

for $i=0$

$$y_1 = y_0 + h(y_0 - (t_0)^2 + 1) + \frac{h^2}{2} f'(t_0, y_0)$$

Now $f(t_i, y_i) = y - t^2 + 1$

$$f'(t_i, y_i) = y' - 2t$$
$$= y - t^2 - 2t + 1 \quad \because y' = y - 2t + 1$$

So $y_1 = y_0 + h(y_0 - (t_0)^2 + 1) + \frac{h^2}{2}(y_0 - (t_0)^2 - 2t_0 + 1)$

$$= 0.5 + 0.2\{0.5 - (0)^2 + 1\} + \frac{(0.2)^2}{2}(0.5 - (0)^2 - 2(0) + 1)$$
$$= 0.5 + 0.2(1.5) + \frac{0.04}{2}(0.5 + 1)$$
$$\rightarrow \boxed{y_1 = 0.83}$$

Dual Space:-(H^*) The set of linear and continuous functions (mappings, operators). In general $H^* \neq H$ (However assume $H^* \subseteq H$)

Definition. (Convex Set) A set K in H is said to be convex if
$(1-t)u + tv \in K \; ; \forall \; u, v \in K, t \in [0,1]$

At $t=0$ $\quad (1-t)u + tv = u \in K$

At $t=1$ $\quad (1-t)u + tv = v \in K$

At $t = \frac{1}{2}$ $\quad (1-t)u + tv = \frac{u+v}{2}$ arithmetic mean

Note that $0 \notin K$

* The convex set K does not contain zero in general.

* Also $u+v \notin K$

Definition:- (Convex function) A function f on the convex set K is said to be convex function if
$f((1-t)u + tv) \leq (1-t) f(u) + t f(v)$
$\forall \; u, v \in K \; \& \; t \in [0,1]$

for $t = \frac{1}{2}$

$$f\left(\frac{u+v}{2}\right) \leq \frac{f(u) + f(v)}{2} \quad \forall \; u, v \in K$$

This is known as "Jensen mid-convex function" (1905)

* If the inequality in the definition of convex function is reserved then it is called concave function.

Examples- 1) $f(x) = |x|$

$\forall \; u, v \in K, \; t \in [0,1]$

Consider

$$f((1-t)u+tv) = |(1-t)u+tv|$$
$$\leq |(1-t)u| + |tv|$$
$$= (1-t)|u| + t|v|$$
$$= (1-t)f(u) + t f(v)$$

$$\Rightarrow f((1-t)u+tv) \leq (1-t)f(u) + t f(v)$$

2) $f(x) = x$

$\forall \; u, v \in K, \; t \in [0,1]$

$$f((1-t)u+tv) = (1-t)u+tv$$
$$\leq (1-t)u + tv$$
$$= (1-t)f(u) + t f(v)$$

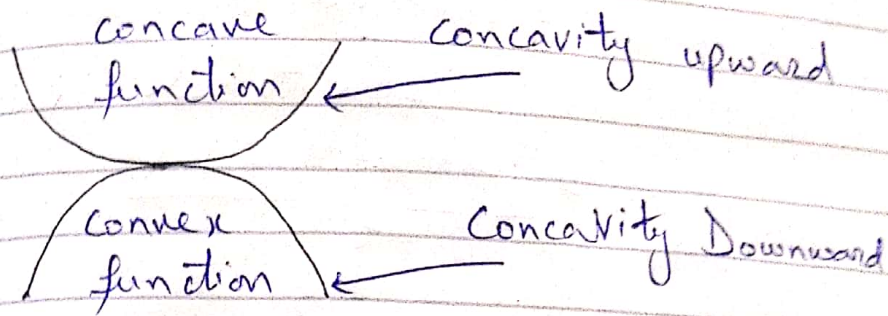
Also

$$(1-t)f(u) + t f(v) \leq f((1-t)u+tv)$$

Therefore $f(x) = x$ is also a concave function.

* If a function both convex

and concave then $f(x)$ is linear.



The Derivative of a Convex Function:-

we say that, if

$$\lim_{t \to 0}\left(\frac{f(u+tv)-f(u)}{t}\right) = \langle f'(u), v \rangle$$

Frechet Derivative

Theorem:- Let f be a differentiable convex function in the convex set K. Then $u \in K$ is the minimum of f if and only if $u \in K$ satisfies

$$\langle f'(u), v-u \rangle \geq 0 \; ; \; \forall \, v \in K$$

Proof)

Let $u \in K$ be a minimum of f over K. Then

$$f(u) \leq f(v) \; ; \; \forall \; v \in K \longrightarrow ①$$

Since K is a convex set, so for all $u, v \in K$, $t \in [0,1]$

$$v_t = (1-t)u + tv = u + t(u-v) \in K \longrightarrow ②$$

from ① and ② we have

$f(u) \leq f(v) = f(u+t(v-u))$

Since f is differentiable so

$$0 \leq \lim_{t \to 0} \frac{f(u+t(v-u)) - f(v)}{t}$$

$$= \langle f'(u), v-u \rangle$$

that is $\langle f'(u), v-u \rangle \geq 0 \; ; \forall \; v \in K$

Conversely let $u \in K$ satisfy

$\langle f'(u), v-u \rangle \geq 0 \; ; \; \forall \; v \in K$

we have to show that $u \in K$ is the minimum of f.

Since f is a convex function so

$f(u+t(v-u)) \leq f(u) + t(f(v) - f(u))$

From this we have

$$f(v) - f(u) \geq \lim_{t \to 0} \left(\frac{f(u+t(v-u)) - f(v)}{t} \right)$$

$$= \langle f'(u), u-v \rangle \geq 0$$

Thus $f(u) \geq f(v) \quad \forall \; v \in K$

From the definition of minimum, we see that $u \in K$ is the minimum of f.

Special Case:- If $K = H$ in the whole space Then $f'(u) = 0$

Consider $u \in H$

$\langle f'(u), v - u \rangle \geq 0 \; ; \; \forall \; v \in K \longrightarrow ③$

Replace u by $v + u$ in ③ to have

$\langle f'(u), v \rangle \geq 0 \quad \forall \; v \in H \longrightarrow ④$

Replace v by $-v + u$ in ③ to have

$\langle f'(u), -v \rangle \geq 0 ; \quad \forall \; v \in H \longrightarrow ⑤$

or $\langle f'(u), v \rangle \leq 0 \longrightarrow ⑥$

From ④ and ⑥ we have

$\langle f'(u), v \rangle = 0 \; , \; \forall \; v \in H$

Thus $f'(u) = 0$

Boundary value Problem:- Consider the Boundary value problem such that

$$\left. \begin{aligned} -\frac{d^2y}{dx^2} + y = f(x) \; ; \quad a \leq x \leq b \\ y(a) = 0 \; , \quad y(b) = 0 \end{aligned} \right\} \longrightarrow ①$$

Where $f(x)$ is linear continuous function on $[a,b]$. Boundary conditions are homogeneous.