ADVANCED ANALYTICAL DYNAMICS

INSTRUCTOR:- Prof. Dr. AFTAB KHAN

⇒ **Dynamics:-** Dynamics is the science of motion. In this branch we shall study different types of equations of motion.

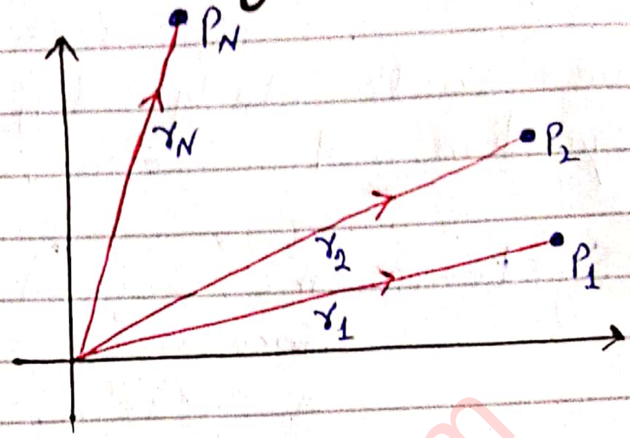
System:- The object on which we focus our attention is called a system.

Degree of Freedom:- The number of independent co-ordinates require to determine the configuration of a dynamical system is called its degree of freedom.

Degree of Freedom = {Total number of coordinates − Number of constraints

Generalized Coordinates:- The smallest possible number of coordinates of a dynamical system such that ① There values determine the configuration of the system ② They may be varied arbitrarily and independently without violating any constraint of the system, are called generalized co-ordinates

Lagrange Equation of Motion:-



Let us consider a dynamical system of N particles, whose configuration at any time 't' is specified by 'n' generalized coordinates $q_1, q_2, \ldots, q_n$

$$\Rightarrow \quad \underline{r}_1 = \underline{r}_1(q_1, q_2, \ldots, q_n, t)$$

$$\underline{r}_2 = \underline{r}_2(q_1, q_2, \ldots, q_n, t)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\underline{r}_n = \underline{r}_n(q_1, q_2, \ldots, q_n, t)$$

$$\Rightarrow \quad \underline{r}_i = \underline{r}_i(q_1, q_2, \ldots, q_n, t) \; ; i = 1, 2, \ldots, n$$

$$\Rightarrow \quad \dot{\underline{r}}_i = \frac{\partial \underline{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \underline{r}_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial \underline{r}_i}{\partial q_n}\dot{q}_n + \frac{\partial \underline{r}_i}{\partial t} \quad (1)$$

By using Einstein summation convension

$$\dot{\underline{r}}_i = \frac{\partial \underline{r}_i}{\partial q_s}\dot{q}_s + \frac{\partial \underline{r}_i}{\partial t} \qquad s = 1, 2, \ldots n \quad \text{(1)}$$

$$\ddot{r}_i = \frac{\partial \dot{r}_i}{\partial q_s}\dot{q}_s + \frac{\partial r_i}{\partial q_s}\ddot{q}_s + \frac{\partial \dot{r}_i}{\partial t} \longrightarrow (2)$$

Differentiating equ (1) w.r.t $\dot{q}_s$ implies

$$\frac{\partial \dot{r}_i}{\partial \dot{q}_s} = \frac{\partial r_i}{\partial q_s} \longrightarrow (3)$$

And differentiating equ (2) w.r.t $\ddot{q}_s$

$$\Rightarrow \frac{\partial \ddot{r}_i}{\partial \ddot{q}_s} = \frac{\partial r_i}{\partial q_s} \longrightarrow (4)$$

and so on implies

$$\frac{\partial r_i}{q_s} = \frac{\partial \dot{r}_i}{\partial \dot{q}_s} = \frac{\partial \ddot{r}_i}{\partial \ddot{q}_s} = \cdots$$

Now by using D-Alembert principal, that is

$$(F_i - m_i\ddot{r}_i)\,\delta r_i = 0 \longrightarrow (5)$$

Since $r_i = r_i(q_s, t)$

$$\Rightarrow \delta r_i = \frac{\partial r_i}{\partial q_s}\delta q_s \qquad ; \delta t = 0$$

Put in (5)

$$(F_i - m_i\ddot{r}_i)\frac{\partial r_i}{\partial q_s}\delta q_s = 0$$

$$\Rightarrow \left(F_i\frac{\partial r_i}{\partial q_s} - m_i\ddot{r}_i\frac{\partial r_i}{\partial q_s}\right)\delta q_s = 0$$

$$\Rightarrow F_i \frac{\partial r_i}{\partial q_s} - m_i \ddot{r}_i \frac{\partial r_i}{\partial q_s} = 0 \quad \because \delta q_s \neq 0$$

$$\Rightarrow m_i \ddot{r}_i \frac{\partial r_i}{\partial q_s} = Q_s \quad ; \text{ where } Q_s = F_i \frac{\partial r_i}{\partial q_s} \longrightarrow (6)$$

Now Consider

$$\frac{d}{dt}\left[\dot{r}_i \frac{\partial r_i}{\partial q_s}\right] = \ddot{r}_i \frac{\partial r_i}{\partial q_s} + \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_s}$$

$$\Rightarrow \ddot{r}_i \frac{\partial r_i}{\partial q_s} = \frac{d}{dt}\left[\dot{r}_i \frac{\partial r_i}{\partial q_s}\right] - \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_s} \longrightarrow (7)$$

$$\Rightarrow m \ddot{r}_i \frac{\partial r_i}{\partial q_s} = \frac{d}{dt}\left[m \dot{r}_i \frac{\partial r_i}{\partial q_s}\right] - m \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_s}$$

As we know that the Kinetic energy of ith particle is

$$\boxed{P.E = V = mg q_s}$$

$$T = \frac{1}{2} m_i \dot{r}_i^2$$

$$\frac{\partial T}{\partial \dot{q}_s} = \frac{1}{2} m_i \left(2 \dot{r}_i \frac{\partial \dot{r}_i}{\partial \dot{q}_s}\right)$$

$$= m_i \dot{r}_i \frac{\partial \dot{r}_i}{\partial \dot{q}_s}$$

$$\Rightarrow \frac{\partial T}{\partial \dot{q}_s} = m_i \dot{r}_i \frac{\partial r_i}{\partial q_s} \longrightarrow (8)$$

$$\text{since } \frac{\partial \dot{r}_i}{\partial \dot{q}_s} = \frac{\partial r_i}{\partial q_s}$$

Also

$$\frac{\partial T}{\partial q_s} = \frac{1}{2} m_i \left(2 \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_s}\right)$$

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$$\Rightarrow \frac{\partial T}{\partial q_s} = m \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_s} \longrightarrow (9)$$

using (8) and (9) equ (7) becomes

$$m_i \ddot{r}_i \frac{\partial r_i}{\partial q_s} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s}$$

Thus equation (6) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} - Q_s = 0$$

In Holonomic system

$$Q_s = -\frac{\partial V}{\partial q_s}$$

$$\boxed{\begin{array}{l} V = mgy \\ \frac{\partial V}{\partial y} = mg \end{array}}$$

So
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} + \frac{\partial V}{\partial q_s} = 0 \longrightarrow (10)$$

Lagrange defined a function

$$L = T - V$$

$$\frac{\partial L}{\partial \dot{q}_s} = \frac{\partial T}{\partial \dot{q}_s} \quad ; \quad \frac{\partial V}{\partial \dot{q}_s} = 0$$

$$\frac{\partial L}{\partial q_s} = \frac{\partial T}{\partial q_s} - \frac{\partial V}{\partial q_s} = \frac{\partial}{\partial q_s}(T - V)$$

where $T = T(q_s, \dot{q}_s, t)$ & $V = V(q_s, t)$

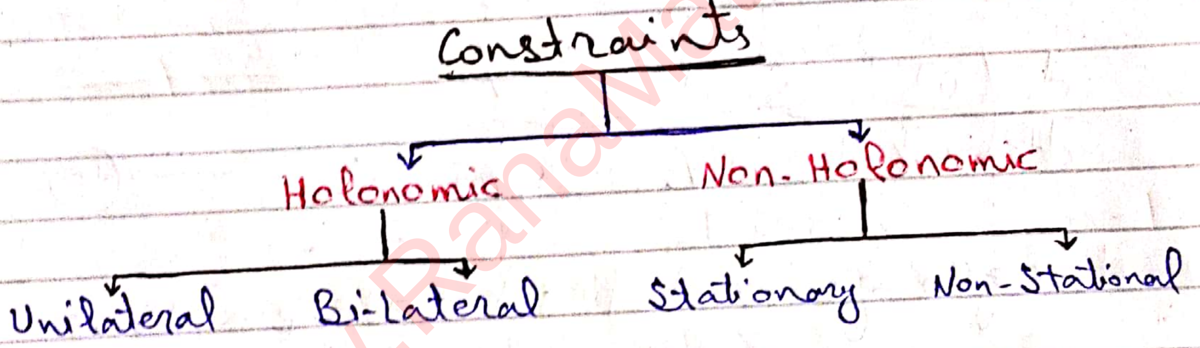
So equ (10) implies

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0$$

which is Lagrange equation of Motion

Constraint:- Any thing which limits the motion of a dynamical system is called constraint. OR constraint is a restriction which is imposed on a dynamical system.

Classification of Constraints:- The constraints that are involved in a dynamical system can be classified as follows



Holonomic Constraints:- If the condition of the constraints are expressed by relation connecting the co-ordinates and possibly time then these constraints are called Holonomic Constraints.

If there are N particles in a dynamical system and "2" constraints are imposed on the system (All constraints are Holonomic) then the relation are of the form

$$f_\ell = f_\ell(x_1, x_2, x_3, \dots x_{3N}, t) = 0 \longrightarrow (1)$$

where $\ell = 1, 2, 3, \dots, r)$, $r < 3N$

The Holonomic constraints may also be expressed as

$$f_\ell = f_\ell(x_1, x_2, x_3, \dots x_{3N}, t) \lessgtr 0 \longrightarrow (2)$$

$\ell = 1, 2, \dots, r$, $r < 3N$

The holonomic constraints are Bi-lateral if the relation representing the constraints are equation, otherwise Uni-lateral.

Non-Holonomic Constraints:-

The constraints are said to be Non-Holonomic is the conditions are connected by the relation involving co-ordinates, velocities of coordinates and possibly time. The relation may be expressed as

$$\phi_\ell = \phi_\ell(x_1, x_2, \dots x_{3N}, \dot{x}_1, \dot{x}_2, \dots \dot{x}_{3N}, t) = 0$$

where $\ell = 1, 2, \dots, r$ & $r < 3N$

The limitation may be expressed as

$$A_{\ell\nu} \dot{x}_\nu + A_\ell = 0 \longrightarrow (*)$$

$\ell = 1, 2, \dots, r$ & $\nu = 1, 2, 3, \dots 3N$

where $A_{\ell\nu}$ and A_ℓ are functions of coordinates and possibly time

If time is involve explicitly in equ ⊛ then the constraints are said to be non-stationary otherwise stationary.

Note:- For stationary non-Holonomic constraints equ ⊛ can be written as

$$\therefore A_{lv}\dot{x}_v = 0$$

Holonomic Dynamical System:- A system subject to Holonomic constraints is said to be Holonomic dynamical system i.e. all constraints are Holonomic.

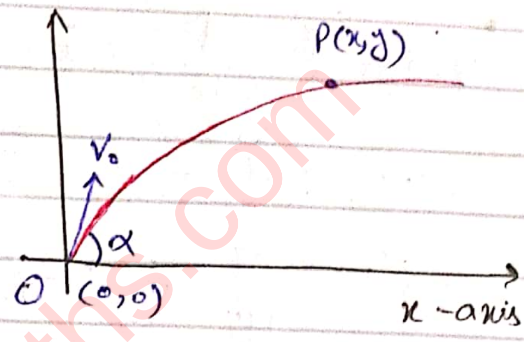
Non-Holonomic Dynamical System:- If constraints imposed on a system are non-Holonomic (non necessarily all) Then the system is said to be Non-Holonomic dynamical system.

Applications of Lagrange Equation of Motion :-

Question:- Derive Lagrang equation of motion for a projectile.

Solution

Let a particle of mass m is projected in space with velocity V_0 making an angle α with x-axis. Let at any time 't' particle is at point $P(x,y)$.



at $t=0$; $x=0$, $y=0$
$\dot{x} = V_0 \cos\alpha$, $\dot{y} = V_0 \sin\alpha$

Here x and y are generalized coordinates therefore corresponding Lagrange equation of motions are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \longrightarrow (1)$$

since $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0$

&

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \longrightarrow (2)$$

As

$$L = T - V$$

for this problem

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \quad \& \quad V = mgy$$

Therefore $L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy$

Now $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$, $\frac{\partial L}{\partial \dot{y}} = m\dot{y}$

$\frac{\partial L}{\partial x} = 0$ & $\frac{\partial L}{\partial y} = -mg$

from ① $\frac{d}{dt}(m\dot{x}) - 0 = 0$

$\Rightarrow m\ddot{x} = 0 \quad \Rightarrow \quad \ddot{x} = 0 \longrightarrow ③$

from ② $\frac{d}{dt}(m\dot{y}) - (-mg) = 0$

$\Rightarrow m\ddot{y} + mg = 0 \quad \Rightarrow m(\ddot{y}+g) = 0$

$\Rightarrow \quad \ddot{y} = -g \longrightarrow ④$

Consider

$\ddot{x} = 0 \quad \Rightarrow \quad \dot{x} = C_1$

at $t=0$; $\dot{x} = V_0 \cos\alpha$

$\Rightarrow \quad C_1 = V_0 \cos\alpha$

$\dot{x} = V_0 \cos\alpha$

$\Rightarrow \quad x = (V_0 \cos\alpha)t + C_2$

at $t=0$; $x=0 \quad \Rightarrow \quad C_2 = 0$

$\Rightarrow \quad x = (V_0 \cos\alpha)t \longrightarrow ⑤$

Now consider

$\ddot{y} = -g$

$$\Rightarrow \dot{y} = -gt + c_3$$

at $t=0$; $\dot{y} = V_0 \sin\alpha$

$$\Rightarrow \boxed{c_3 = V_0 \sin\alpha}$$

So $\dot{y} = -gt + V_0 \sin\alpha$

$$\Rightarrow y = (V_0 \sin\alpha)t - \frac{1}{2}gt^2 + c_4$$

at $t=0$; $y=0$

$$\Rightarrow \boxed{c_4 = 0}$$

So $y = (V_0 \sin\alpha)t - \frac{1}{2}gt^2 \longrightarrow (6)$

from (5) calculate t and put in (6)

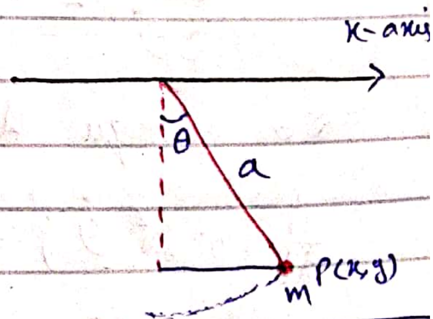
$$y = x\tan\alpha - \frac{gx^2}{2V_0^2}\sec^2\alpha$$

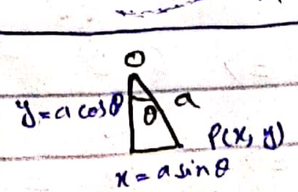
is the Lagrange equation of motion for projectile.

Question:- Derive Lagrange equation of motion for simple pendulam.

Solution

Let a bob of mass m is attatched with a thread of length 'a' and is moving in a vertical plan. Let at any time 't' pendulam is at point $P(x,y)$





$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j}$$

making an angle θ with vertical as shown in figure.

Here θ is generalized coordinate, so Lagrange equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \longrightarrow (1)$$

As $L = T - V$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2), \quad V = -mgy$$

$$x = a\sin\theta, \quad y = a\cos\theta$$

$$\Rightarrow \dot{x} = (a\cos\theta)\dot{\theta} \quad \& \quad \dot{y} = (-a\sin\theta)\dot{\theta}$$

Therefore

$$T = \frac{1}{2}ma^2\dot{\theta}^2 \quad \& \quad V = -mga\cos\theta$$

$$\Rightarrow L = \frac{1}{2}ma^2\dot{\theta}^2 + mga\cos\theta \longrightarrow (2)$$

Now $\dfrac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta}$ & $\dfrac{\partial L}{\partial \theta} = -mga\sin\theta$

So equ (1) becomes

$$\frac{d}{dt}(ma^2\dot{\theta}) - (-mga\sin\theta) = 0$$

$$\Rightarrow a\ddot{\theta} + g\sin\theta = 0$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{a}\sin\theta$$

As θ is small so $\sin\theta \approx \theta$

$$\Rightarrow \ddot{\theta} \approx -\frac{g}{a}\theta \quad \text{or} \quad \ddot{\theta} \propto -\theta$$

$$\boxed{\ddot{x} \propto x}$$

D'Alembert Principle:- Let us consider a dynamical system of N particles whose configuration at any time 't' is specified by 'n' generalized coordinates $q_1, q_2, \ldots q_n$. The position vector of ith particle is given by $r_i = r_i(q_1, q_2, \ldots, q_n, t) \longrightarrow ①$

Let F_i be the total external force on the ith particle. Then by principle of virtual work

$$F_i \delta r_i = 0 \longrightarrow ②$$

Where δr_i is virtual displacement of the ith particle.

From equation ①

$$\delta r_i = \frac{\partial r_i}{\partial q_s} \delta q_s \longrightarrow ③ \quad \because t = 0$$

So equation ② can be written as

$$F_i \frac{\partial r_i}{\partial q_s} \delta q_s = 0 \Rightarrow Q_s \delta q_s = 0$$

where $Q_s = F_i \frac{\partial r_i}{\partial q_s}$ is called generalized force

According to Newton

$$\left\{\begin{aligned} F = ma &= m\frac{dv}{dt} = \frac{d}{dt}(mv) = \frac{d}{dt}(P) \\ &= \frac{dP}{dt} = \dot{P} \end{aligned}\right.$$

2nd law of motion $F_i = \dot{P}_i$
where P_i is the momentum of ith particle.

$$\Rightarrow \quad F_i - \dot{P}_i = 0$$

$$\Rightarrow \quad F_i - m\ddot{r}_i = 0 \; ; \quad \left\{\begin{array}{l}\text{where } m \text{ is the} \\ \text{mass of ith particle}\end{array}\right.$$

$$\Rightarrow \quad (F_i - m\ddot{r}_i)\,\delta r_i = 0$$

$$\Rightarrow \quad (F_i - m\ddot{r}_i)\frac{\partial r_i}{\partial q_s}\,\delta q_s = 0$$

$$\Rightarrow \quad \left[F_i\frac{\partial r_i}{\partial q_s} - m\ddot{r}_i\frac{\partial r_i}{\partial q_s}\right]\delta q_s = 0$$

$$\Rightarrow \quad \left[Q_s - m\ddot{r}_i\frac{\partial r_i}{\partial q_s}\right]\delta q_s = 0$$

This expression is known as D' Alembert Principle.

Question :- Find Lagrange equation of of motion for double Pendu

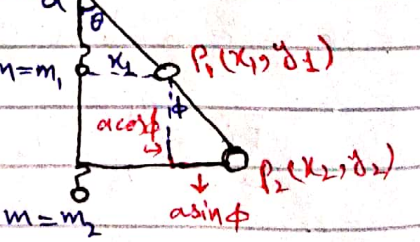
Solution

$x_1 = a\sin\theta$

$y_1 = a\cos\theta$

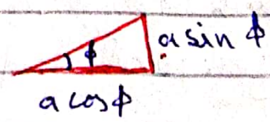
$x_2 = a\sin\theta + a\sin\phi$

$y_2 = a\cos\theta + a\cos\phi$



Now $\dot{x}_1 = a\cos\theta\,\dot{\theta}$

$\dot{y}_1 = -a\sin\theta\,\dot{\theta}$



a = length of thread

$$\dot{x}_2 = a\cos\theta\,\dot{\theta} + a(\cos\phi)\,\dot{\phi}$$

$$\dot{y}_2 = -a\sin\theta\,\dot{\theta} - a\sin\phi\,\dot{\phi}$$

$$T = T_1 + T_2$$

$$= \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2}m\left[(a(\cos\theta)\dot{\theta})^2 + (a(\sin\theta)\,\dot{\theta})^2\right]$$

$$+ \frac{1}{2}m\left[(a(\cos\theta)\dot{\theta} + a(\cos\phi)\dot{\phi})^2 + (a(\sin\theta)\dot{\theta} + a(\sin\phi)\,\dot{\phi})^2\right]$$

$$\Rightarrow T = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}m\{a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}\cos(\theta-\phi)$$

$$V = mgy_1 + mgy_2$$

$$= mga\cos\theta + mg(a\cos\phi + a\cos\theta)$$

As $L = T - V$

$$\therefore L = \frac{1}{2}ma^2\{2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi)\}$$

$$- mga(2\cos\theta + \cos\phi)$$

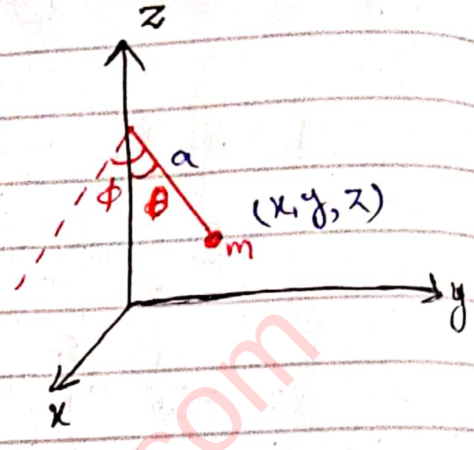
θ and ϕ are generalized coordinates.

So corresponding Langrange equation of motions are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \quad \& \quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0$$

(And so on ... MSc Notes)

Question:- Find Lagrange equation of motion for spherical pendulum



Solution

Let us consider a particle of mass "m" attached with a rod of length "a" and oscilate under the action of gravity. Since the particle is restricted to move in a sphere with fix radius, Therefore this pendulum is called spherical pendulum or spherical equivalator.
Let at any time 't' the particle is at point $P(x,y,z)$. Therefore

$$x = a\sin\theta\cos\phi \quad , \quad y = a\sin\theta\sin\phi$$
$$\text{and} \quad z = a\cos\theta$$

Here ϕ and θ are generalized coordinates. So corresponding Lagrange equation of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \longrightarrow (1)$$

$$\& \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \longrightarrow (2)$$

where $L = T - V$

As $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

& $V = -mgz$

Now $\dot{x} = a\cos\theta\,\dot{\theta}\cos\phi - a\,\sin\theta\sin\phi\,\dot{\phi}$

$\dot{y} = a\cos\theta\,\dot{\theta}\,\sin\phi + a\sin\theta\,\cos\phi\,\dot{\phi}$

$\dot{z} = -a\sin\theta\,\dot{\theta}$

Now

$(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = [(a\cos\theta\,\dot{\theta}\cos\phi - a\sin\theta\sin\phi\dot{\phi})^2$
$+ (a\cos\theta\dot{\theta}\sin\phi + a\sin\theta\cos\phi\,\dot{\phi})^2$
$+ (-a\sin\theta\dot{\theta})^2]$

Therefore $T = \frac{1}{2}m[a^2\dot{\theta}^2 + a^2\dot{\phi}^2\sin^2\theta]$

& $V = -mga\cos\theta$

$\therefore L = T - V$

$= \frac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mga\cos\theta$

Now $\frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta}$; $\frac{\partial L}{\partial \theta} = \frac{1}{2}ma^2(\dot{\phi}^2 2\sin\theta\cos\theta) - mga\sin\theta$

$\frac{\partial L}{\partial \dot{\phi}} = ma^2\dot{\phi}\sin^2\theta$, $\frac{\partial L}{\partial \phi} = 0$

Therefore (1) implies

$\frac{d}{dt}[ma^2\dot{\theta}] - ma^2\dot{\phi}^2\sin\theta\cos\theta + mga\sin\theta = 0$

$\Rightarrow ma^2\ddot{\theta} - ma^2\dot{\phi}^2\sin\theta\cos\theta + mga\sin\theta = 0$

$\Rightarrow \ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta + \cancel{mga}\ g/a\ \sin\theta = 0$

or $\ddot{\theta} + \frac{g}{a}\sin\theta = \dot{\phi}^2 \sin\theta\cos\theta \longrightarrow (3)$

$(2) \Rightarrow \frac{d}{dt}[ma\dot{\phi}\sin^2\theta] - 0 = 0$

$\Rightarrow \dot{\phi}\sin^2\theta = h \longrightarrow (4)$

Equation (3) & (4) are Lagrange equation of motions.

Question:- Find Lagrange equation of motion in spherical coordinates.

Solution

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$\dot{x} = \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi$

$\dot{y} = \dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi$

$\dot{z} = \dot{r}\cos\theta + \dot{r}\sin\theta$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta)$

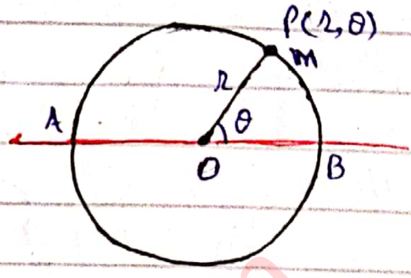
$V = mgr\cos\theta$

L. Equation of Motions are

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} &= 0 \end{aligned}\right\}$$

Question:- Find Lagrange Equation of motion for a particle moving in a circular orbit.

Solution



Let a particle of mass 'm' is moving in a circular orbit with orbitrarly radius "r". Let particle makes angle "θ" with diameter x-axis as shown in figure.

Here r and θ are the generalized co-ordinates. Therefore corresponding Lagrange equation of motions are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \longrightarrow (1)$$

& $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \longrightarrow (2)$$

where $L = T - V$

& $$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \quad \& \quad V = mgy$$

As $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$$

$$\Rightarrow \dot{x}^2 + \dot{y}^2 = \dot{r}^2\cos^2\theta + r^2\sin^2\theta\,\dot{\theta}^2 - \cancel{2r\dot{r}\sin\theta\cos\theta\,\dot{\theta}}$$
$$+ \dot{r}^2\sin^2\theta + r^2\cos^2\theta\,\dot{\theta}^2 + \cancel{2r\dot{r}\sin\theta\cos\theta\,\dot{\theta}}$$

$$\Rightarrow \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

Therefore $T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$

& $V = mg r \sin\theta$

$\Rightarrow$ $L = T - V$

$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - mg r \sin\theta$

Now

$\frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$, $\frac{\partial L}{\partial \theta} = - mg r \cos\theta$

$\frac{\partial L}{\partial \dot{r}} = m \dot{r}$, $\frac{\partial L}{\partial r} = m r \dot{\theta}^2 - mg \sin\theta$

Therefore

① $\Rightarrow$ $\frac{d}{dt}(m\dot{r}) - m r \dot{\theta}^2 + mg \sin\theta = 0$

$\Rightarrow m\ddot{r} - m r \dot{\theta}^2 + mg \sin\theta = 0 \longrightarrow$ ③

② $\Rightarrow$ $\frac{d}{dt}(m r^2 \dot{\theta}) + mg r \cos\theta = 0$

$\Rightarrow r^2 \ddot{\theta} + 2 r \dot{r} \dot{\theta} + g r \cos\theta = 0 \longrightarrow$ ④

Equation ③ and ④ are required equation of motions.

Ignorable Coordinates:- Sometimes it is so happen that some of the generalized coordinates $q_1, q_2, \ldots, q_k$ where $k < n$ do not appear in Lagrangian function although the velocities corresponding to these coordinates are present in "L" (Lagrangian) such coordinates are called cyclic or ignorable coordinates.

For example Lagrangian function in generalized coordinates r and θ is given by

$$L = \frac{1}{2}m(2\dot{r}^2 + r^2\dot{\theta}^2) - mg(l - r)$$

Clearly $\dot{\theta}$ is involved in L but not θ, so θ is ingnorable coordinate and r is non-ignorable.

⇒ Roudh's Equation of Motion:-

Let us consider a dynamical system of N particles whose configuration at any time "t" is spesified by n-generalized coordinates $q_1, q_2, \ldots, q_n$. Then Lagrange equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \longrightarrow (1)$$

Now suppose that we have K ignorable coordinates i.e $q_1, q_2, \dots, q_K$ (K present in Lagranian function. Now Now after K coordinates all

$$\frac{\partial L}{\partial q_i} = 0 \quad ; \quad i = 1, 2, 3, \dots, K \longrightarrow (2)$$

So equation (1) for ignorable coordinate is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, 2, \dots, K$$

using (2) we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$\boxed{\begin{aligned} L &= \tfrac{1}{2} m \dot{q}_i^2 + f(q) \\ \frac{\partial L}{\partial \dot{q}_i} &= m\dot{q}_i = P_i \end{aligned}}$$

Integrating

$$\frac{\partial L}{\partial \dot{q}_i} = P_i \longrightarrow (3)$$

where P_i is the constant of integration and is called generalized momentum. The Lagrange ~~equation~~ function in this case is given by

$$L = L(q_{K+1}, q_{K+2}, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t)$$

$$= L(q_j, \dot{q}_s, t) ; \; s = 1, 2, \dots, n \; \& \; j = K+1, \dots, n$$

Roudh's defines a function

$$R = L - \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \longrightarrow (4)$$

where $R = R(q_j, \dot{q}_j, P_i, t) \longrightarrow (5)$

consider $\delta R = \delta L - \delta\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right)$

$$\delta R = \frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s - \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i - \dot{q}_i\,\delta\left(\frac{\partial L}{\partial \dot{q}_i}\right)$$

$$= \frac{\partial L}{\partial q_j}\delta q_j + \cancel{\frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i} + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j - \cancel{\frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i} - \dot{q}_i\,\delta P_i$$

$\because s = i + j$

$$\Rightarrow \delta R = \frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j - \dot{q}_i\,\delta P_i \longrightarrow (6)$$

from (5) we have

$$\delta R = \frac{\partial R}{\partial q_j}\delta q_j + \frac{\partial R}{\partial \dot{q}_j}\delta \dot{q}_j + \frac{\partial R}{\partial P_i}\delta P_i \longrightarrow (7)$$

Comparing (6) and (7) we have

$$\frac{\partial L}{\partial q_j} = \frac{\partial R}{\partial q_j} \longrightarrow (8a)$$

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial R}{\partial \dot{q}_j} \longrightarrow (8b)$$

$$-\dot{q}_i = \frac{\partial R}{\partial P_i} \longrightarrow (8c)$$

Equation (1) for non ignorable coordinates is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

using (8a) and (8b) we have

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_j}\right) - \frac{\partial R}{\partial q_j} = 0 \;;\quad j = k+1, k+2, \cdots, n \longrightarrow (9)$$

Equation (9) is known as Routh's equation of motion (for non ignorable coordinates)

From (8c) we have

$$\dot{q}_i = -\frac{\partial R}{\partial p_i}$$

Integrating w.r.t "t"

$$q_i = -\int \frac{\partial R}{\partial p_i}\, dt \longrightarrow (10)$$

$i = 1, 2, \ldots, k$

equation (10) gives us ignorable coordinates

Question:- Find Routh's equation of motion for projectile motion.

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2), \quad V = mgy$$

Solution

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy$$

Then

$$\frac{\partial L}{\partial \dot{q}_i} = p_i \Rightarrow \frac{\partial L}{\partial \dot{x}} = p_1$$

(x is ignorable)

$$\Rightarrow m\dot{x} = p_1$$

$$\Rightarrow \dot{x} = \frac{1}{m} p_1 \longrightarrow (1)$$

Define Routh's function as

$$R = L - p_i \dot{q}_i$$

$$R = L - p_1 \dot{q}_1 = L - p_1 \dot{x}$$

$$\Rightarrow R = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy - p_1 \dot{x}$$

By using (1)

$$R = \frac{1}{2}m\left(\frac{1}{m^2}P_1^2 + \dot{y}^2\right) - mgy - P_1\left(\frac{1}{m}P_1\right)$$

$$R = \frac{1}{2}m\dot{y}^2 - mgy - \frac{1}{2m}P_1^2$$

$\therefore$ Routh equation of motion is

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_j}\right) - \frac{\partial R}{\partial q_j} = 0$$

$\boxed{y \text{ is non ignorable}}$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial R}{\partial \dot{y}}\right) - \frac{\partial R}{\partial y} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot{y}) + mg = 0$$

$$\Rightarrow \ddot{y} = -g$$

Question:- Find for a dynamical system with two degree of freedom The

$$T = \frac{1}{2}\left[\frac{\dot{q}_1^2}{a+bq_2^2}\right] + \frac{1}{2}\dot{q}_2^2 \quad \& \quad V = c + dq_2^2$$

where a, b, c and d are constants. Find Routh's equation of motion

Solution

$$L = T - V$$

Here q_1 is ignorable coordinate. Define Routh's function as

$$R = L - P_i\dot{q}_i$$

$$\Rightarrow R = \frac{1}{2}\left[\frac{\dot{q}_1^2}{a+bq_2^2}\right] + \frac{1}{2}\dot{q}_2^2 - c - dq_2^2 - P_1\dot{q}_1$$

where $\frac{\partial L}{\partial \dot{q}_i} = P_i$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_1} = p_1$$

So $\quad \dfrac{\dot{q}_1^2}{a+bq_2^2} = p_1 \Rightarrow \dot{q}_1 = (a+bq_2^2)p_1$

$$\therefore R = \frac{1}{2}\left[\frac{(a+b\cancel{q_2^2})^{\cancel{2}}p_1^2}{(a+\cancel{bq_2^2})}\right] + \frac{1}{2}\dot{q}_2^2 - c - dq_2^2$$
$$- p_1^2(a+bq_2^2)$$

$$\Rightarrow R = \frac{1}{2}\dot{q}_2^2 - c - dq_2^2 - \frac{1}{2}(a+bq_2^2)p_1^2$$

Now Routh equation of motion is

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_2}\right) - \frac{\partial R}{\partial q_2} = 0$$

$$\Rightarrow \frac{d}{dt}(\dot{q}_2) - \{-2dq_2 - bq_2 p_1^2\} = 0$$

$$\Rightarrow \ddot{q}_2 + (2d + bp_1^2)q_2 = 0$$

$$q_2 = A\cos\left(\sqrt{2d+bp_1^2}\right)t + B$$

Ignorable coordinates

$$\dot{q}_i = -\int \frac{\partial R}{\partial p_i}\,dt$$

$$q_1 = -\int \frac{\partial R}{\partial p_1}\,dt$$

$$= +\int (a+bq_2^2)p_1\,dt$$

$\ddot{y} + cy = 0$

$D^2 + c = 0$

$\Rightarrow D^2 = -c$

$D = \pm i\sqrt{c}$

$y = A\cos\sqrt{c}t + B\sin\sqrt{c}t$

$y = A\cos(\sqrt{c}t + B)$

$y = A\cos\sqrt{c}t$

$$\Rightarrow q_1 = p_1 a \int dt + b p_1 \int q_2^2 \, dt$$

$$q_1 = p_1 a t + b p_1 \int A \cos^2 (2d + b p_1^2 + B) t \, dt$$

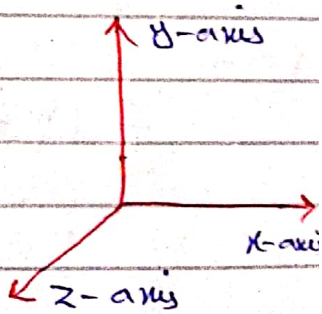
$$\because \int \cos^2 at \, dt = \int \frac{1 + \cos 2at}{2} dt$$

Homework

Find Routh's equation of motion for

* Simple Pendulum
* Spherical polar coordinates (r, θ, ϕ)
* Cylindrical polar coordinates (r, θ, z)

* Routh equation of motion for cylindrical coordinates.



Solution

$$x = r \sin\theta$$
$$y = r \cos\theta$$
$$z = z$$
$$\dot{x} = \dot{r} \sin\theta + r \cos\theta \, \dot{\theta}$$
$$\dot{y} = \dot{r} \cos\theta - r \sin\theta \, \dot{\theta}$$
$$\dot{z} = \dot{z}$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2)$$

$$V = mgz$$

Therefore

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) - mgz$$

Lagrange Equations of Motion are

For r

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 = 0$$

$$\Rightarrow \ddot{r} - r\dot{\theta}^2 = 0 \longrightarrow (1)$$

For θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) - 0 = 0$$

$$\Rightarrow mr^2\ddot{\theta} = c \longrightarrow (2)$$

For z

$$\frac{d}{dt}(m\dot{z}) - mg = 0$$

$$\Rightarrow \ddot{z} = g \longrightarrow (3)$$

For Routh's function

$$R = L - p_i \dot{q}_i$$

Here θ is ignorable

$$R = L - p_\theta \dot{\theta}$$

Here $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

$$\Rightarrow R = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz - p_\theta\dot{\theta}$$

$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\left(\frac{p_\theta}{mr^2}\right)^2 + \dot{z}^2\right) - mgz - p_\theta\left(\frac{p_\theta}{mr^2}\right)$$

$$\Rightarrow R = \frac{1}{2}m\left(\dot{r}^2 - \frac{P_\theta^2}{m r^2} + \dot{z}^2\right) - mgz$$

Routhian Equation of Motion

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) - \frac{\partial R}{\partial r} = 0 \longrightarrow (A)$$

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{z}}\right) - \frac{\partial R}{\partial z} = 0 \longrightarrow (B)$$

From A:- $\frac{d}{dt}(m\dot{r}) - \frac{P_\theta^2}{2m} \cdot \frac{-2}{r^3} = 0$

$$\Rightarrow m\ddot{r} + \frac{P_\theta^2}{m r^3} = 0$$

$$\Rightarrow \ddot{r} + \frac{P_\theta^2}{m^2 r^3} = 0 \longrightarrow (4)$$

From B:- $\frac{d}{dt}(m\dot{z}) - (-mgz) = 0$

$$\Rightarrow m\ddot{z} + mgz = 0$$

$$\Rightarrow \ddot{z} + gz = 0 \longrightarrow (5)$$

(4) and (5) are required Routheian equation of Motion.

* Routhean Equation of Motion for Simple Pendulum

$x = a\sin\theta$, $\dot{x} = a\cos\theta\,\dot{\theta}$

$y = a\cos\theta$, $\dot{y} = -a\sin\theta\,\dot{\theta}$

$L = T - V$ $\quad \because V = -mgy$

$$\Rightarrow L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$$

$$= \frac{1}{2}m(a^2\dot{\theta}^2\cos^2\theta + a^2\dot{\theta}^2\sin^2\theta) + mga\cos\theta$$

$$\Rightarrow L = \frac{1}{2}m a^2\dot{\theta}^2 + mga\cos\theta$$

There is no ignorable coordinate, so

$$R = L = \frac{1}{2} m a^2 \dot{\theta}^2 + mga\cos\theta$$

$$R = \frac{1}{2} m a^2 \dot{\theta}^2 + mga\cos\theta$$

The Routhcan Equation becomes

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\theta}}\right) - \frac{\partial R}{\partial \theta} = 0$$

$$\frac{d}{dt}(ma^2\dot{\theta}) - (-mga\sin\theta) = 0$$

$$\Rightarrow ma^2\ddot{\theta} + mga\sin\theta = 0$$

$$\Rightarrow a\ddot{\theta} + g\sin\theta = 0 \Rightarrow \ddot{\theta} + \frac{g}{a}\sin\theta = 0$$

If θ is small

$$\ddot{\theta} + \frac{g}{a}\theta = 0 \longrightarrow (A)$$

A is required Routhean Equation of Motion.

★ Routh Equation of Motion for Spherical polar Coordinate.

$x = r\sin\theta\sin\phi$, $y = r\sin\theta\cos\phi$

$$z = r\cos\theta$$

$$\dot{x} = \dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi}$$

$$\dot{y} = \dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi}$$

$$\dot{z} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

$$L = T - V \longrightarrow (1)$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

Now $\dot{x}^2+\dot{y}^2+\dot{z}^2 = [\dot{r}^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta\sin^2\phi\dot{\theta}^2 + r^2\sin^2\theta\cos^2\phi\dot{\phi}^2 + 2\dot{r}r\sin\theta\cos\theta\sin^2\phi\dot{\theta} + 2\dot{r}r\dot{\phi}\sin^2\theta\sin\phi\cos\phi + 2r^2\sin\theta\sin\phi\,\cancel{\cos\theta}\cos\phi\dot{\theta}\dot{\phi} + \dot{r}^2\sin^2\theta\cos^2\phi + r^2\cos^2\theta\cos^2\phi\dot{\theta}^2 + r^2\sin^2\theta\sin^2\phi\dot{\phi}^2 + 2\dot{r}r\sin\theta\cos\phi\cos\theta\cos\phi\dot{\theta} - 2\dot{r}r\sin\theta\cos\phi\sin\theta\sin\phi\dot{\phi} - 2r^2\cos\theta\cos\phi\,\cancel{\sin\theta}\sin\phi\dot{\theta}\dot{\phi} + \dot{r}^2\cos^2\theta + r^2\sin^2\theta\dot{\theta}^2 - 2\dot{r}r\sin\theta\cos\theta\dot{\theta}]$

$$T = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta]$$

$$V = -mgz = -mgr\cos\theta$$

$$\Rightarrow L = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta] + mgr\cos\theta$$

Here ϕ is ignorable coordinate

So $R = L - p_i\dot{q}_i = L - p_\phi\dot{\phi}$

$$R = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta] + mgr\cos\theta - p_\phi\dot{\phi} \longrightarrow (A)$$

Here $p_\phi = \frac{\partial L}{\partial\dot{\phi}} = mr^2\dot{\phi}\sin^2\theta$

$$\Rightarrow \dot{\phi} = \frac{p}{mr^2\sin^2\theta} \longrightarrow (B)$$

Put (B) in (A)

$$R = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta] + mgr\cos\theta$$

$$- \left[\frac{p_\phi^2}{mr^2\sin^2\theta}\right]$$

$$\Rightarrow R = \frac{1}{2} m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta \frac{P_\phi^2}{m^2 r^4 \sin^4\theta}\right]$$

$$+ mgr\cos\theta - \frac{P_\phi^2}{mr^2\sin^2\theta}$$

$$R = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}\frac{P_\phi^2}{mr^2\sin^2\theta} + mgr\cos\theta$$

$$- \frac{P_\phi^2}{mr^2\sin^2\theta}$$

$$\Rightarrow R = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta - \frac{1}{2}\frac{P_\phi^2}{mr^2\sin^2\theta}$$

Routh's Equation of Motion is

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\theta}}\right) - \frac{\partial R}{\partial \theta} = 0$$

$$\Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) - \left(-mgr\sin\theta - \frac{P_\phi^2}{2mr^2} \cdot \frac{-2\cos\theta}{\sin^3\theta}\right) = 0$$

$$\Rightarrow 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + mgr\sin\theta - \frac{P_\phi^2\cos\theta}{mr^2\sin^3\theta} = 0$$

$$\Rightarrow r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + gr\sin\theta - \frac{P_\phi^2\cos\theta}{m^2r^2\sin^3\theta} = 0$$

$$\Rightarrow r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + gr\sin\theta - \frac{P_\phi^2\cos\theta}{m^2r^2\sin^3\theta} = 0$$

If θ is small

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + gr\theta - \frac{P_\phi^2}{m^2r^2\theta^3} = 0 \quad \rightarrow (C)$$

(C) is required Equation.

Hamilton Equation of Motion:-

Let us consider a dynamical system of "N" particles whose configuration at any time 't' is specified by 'n' generalized coordinates (Lagrangian coordinates) $q_1, q_2, \dots, q_n$. Then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \longrightarrow (1)$$

$s = 1, 2, \dots, n$

Hamilton define a function

$$H = p_s \dot{q}_s - L(q_s, \dot{q}_s, t) \longrightarrow (2)$$

where $p_s = \dfrac{\partial L}{\partial \dot{q}_s}$ known as generalized momentum.

Also from (1)

$$\frac{\partial L}{\partial q_s} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)$$

$$\Rightarrow \frac{\partial L}{\partial q_s} = \frac{d}{dt}(p_s) \qquad \because \frac{\partial L}{\partial \dot{q}_s} = p_s$$

$$\Rightarrow \frac{\partial L}{\partial q_s} = \dot{p}_s \longrightarrow (3)$$

Also it gives that

$$H = H(q_s, p_s, t) \longrightarrow (4)$$

From equation (2)

$$\delta H = \delta(p_s \dot{q}_s) - \delta L(q_s, \dot{q}_s, t)$$

$$= (\dot{q}_s \delta p_s + p_s \delta \dot{q}_s) - \left(\frac{\partial L}{\partial q_s}\delta q_s + \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s\right)$$

$$\Rightarrow \delta H = \dot{q}_s \delta p_s + \cancel{p_s \delta \dot{q}_s} - \dot{p}_s \delta q_s - \cancel{p_s \delta \dot{q}_s}$$

$$\Rightarrow \delta H = \dot{q}_s \delta p_s - \dot{p}_s \delta q_s \longrightarrow (5)$$

From equation (4)

$$\delta H = \frac{\partial H}{\partial q_s}\delta q_s + \frac{\partial H}{\partial p_s}\delta p_s \longrightarrow (6)$$

Equation equations (5) and (6) we have

$$\dot{q}_s = \frac{\partial H}{\partial p_s} \longrightarrow (7)$$

$$\& \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} \longrightarrow (8)$$

Equation (7) and (8) are required Hamilton's equation of motion.

Question:- Find Hamilton Equation of Motion for Projectile motion.

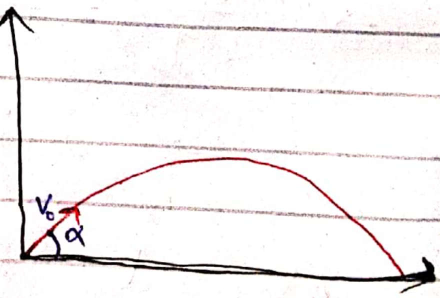
Solution

at $t = 0$

$x = 0$, $y = 0$

$\dot{x} = V_0 \cos\alpha$

$\dot{y} = V_0 \sin\alpha$



$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$V = mgy$$

$$L = T - V$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

We define Hamilton's function

$$H = p_s \dot{q}_s - L$$

$$H = p_1 \dot{q}_1 + p_2 \dot{q}_2 - \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + mgy$$

where $p_1 = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$$\Rightarrow \dot{x} = \frac{1}{m} p_1$$

& $p_2 = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$

$$\Rightarrow \dot{y} = \frac{1}{m} p_2$$

Therefore

$$H = p_1\left(\frac{1}{m} p_1\right) + p_2\left(\frac{1}{m} p_2\right) - \frac{1}{2} m \left\{\frac{p_1^2}{m^2} + \frac{p_2^2}{m^2}\right\} + mgy$$

$$= \frac{1}{2m}(p_1^2 + p_2^2) + mgy$$

$\therefore \quad \dot{q}_1 = \frac{\partial H}{\partial p_1}, \quad \dot{q}_s = \frac{\partial H}{\partial p_2}$

$\dot{p}_1 = -\frac{\partial H}{\partial q_1}$ & $\dot{p}_2 = -\frac{\partial H}{\partial q_2}$

$$\Rightarrow \left.\begin{aligned} \dot{x} &= \frac{1}{m} p_1, \quad \dot{y} = \frac{1}{m} p_2 \\ \dot{p}_1 &= 0, \quad \dot{p}_2 = -mg \end{aligned}\right\}$$

Are required Hamilton Equation of Motions.

* If we Eliminate p from Hamilton equation of motion, we get Lagrange equation of Motion.

Quiz 1

Question:- Find Hamilton Equation of Motion for

$$T = \frac{1}{2}\left(\frac{\dot{q}_1^2}{a+bq_2^2}\right) + \frac{1}{2}\dot{q}_2^2$$

$$V = c + dq_2^2$$

Solution

As $L = T - V$

$$\Rightarrow L = \frac{1}{2}\left(\frac{\dot{q}_1^2}{a+bq_2^2}\right) + \frac{1}{2}\dot{q}_2^2 - c - dq_2^2$$

Hamilton defined a function as

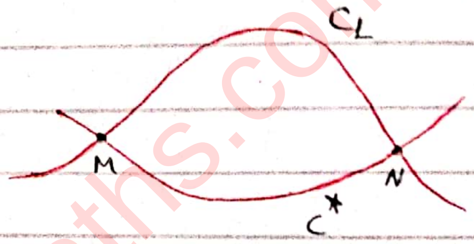
$$H = p_s \dot{q}_s - L$$

⇒ Hamilton Principle :- (Principle of Least Action) The integral

$$I = \int_{t_1}^{t_2} L(q_s, \dot{q}_s, t)\, dt$$

has stationary values along the actual curve C_1 as compare with the neighboring trajectory having the same values at the terminal points

Proof:- Let us consider a dynamical system of N particles whose configuration at any time "t" is specified by n generalized coordinates $q_1, q_2, \dots, q_n$



Since $L(q_s, \dot{q}_s, t)$ is the Lagrangian function. Therefore it must satisfy the Lagrange equation of motion i.e.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \longrightarrow (1)$$

Now consider

$$I = \int_{t_1}^{t_2} L(q_s, \dot{q}_s, t)\, dt$$

$$\delta I = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_s}\delta q_s + \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s\right) dt$$

Consider

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\delta q_s\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\delta q_s + \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\delta q_s\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\delta q_s = \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s$$

$\therefore$

$$\delta I = \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q_s}\delta q_s + \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\delta q_s\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\delta q_s\right\}\right]dt$$

$$= \int_{t_1}^{t_2}\left\{\frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\right\}\delta q_s\, dt + \int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\delta q_s\right)dt$$

$$= \text{———} \; '' \; \text{———} + \frac{\partial L}{\partial \dot{q}_s}\overset{0}{\delta q_s}\Big|_{t_1}^{t_2}$$

$$\Rightarrow \delta I = \int_{t_1}^{t_2}\left\{\frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\right\}\delta q_s\, dt$$

$$\Rightarrow \delta I = 0 \qquad \text{Proved.}$$

Question:- Derive Hamilton Equation of Motion using Hamilton principle.

if $I = \int_{t_1}^{t_2} L(q_s, \dot{q}_s, t)\,dt$

$$\delta I = 0$$

Solution

Hamilton defines a function as

$$H = p_s \dot{q}_s - L$$

$\therefore$

$$I = \int_{t_1}^{t_2}\left[p_s \dot{q}_s - H(p_s, q_s, t)\right]dt$$

Now $\delta I = \int_{t_1}^{t_2} \left\{ (p_s \delta \dot{q}_s + \dot{q}_s \delta p_s) - \left(\frac{\partial H}{\partial q_s} \delta q_s + \frac{\partial H}{\partial p_s} \delta p_s \right) \right\} dt$

Consider

$$\frac{d}{dt}(p_s \delta q_s) = \dot{p}_s \delta q_s + p_s \delta \dot{q}_s$$

or $$p_s \delta \dot{q}_s = \frac{d}{dt}(p_s \delta q_s) - \dot{p}_s \delta q_s$$

$$\delta I = \cancel{p_s \delta q_s}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} (\dot{p}_s \delta q_s - \dot{q}_s \delta p_s) dt - \int \left(\frac{\partial H}{\partial q_s} \delta q_s + \frac{\partial H}{\partial p_s} \delta p_s \right) dt$$

$$\delta I = \int_{t_1}^{t_2} \left[(\dot{q}_s \delta p_s - \dot{p}_s \delta q_s) - \left(\frac{\partial H}{\partial q_s} \delta q_s + \frac{\partial H}{\partial p_s} \delta p_s \right) \right] dt$$

$$= \int_{t_1}^{t_2} \left(\dot{q}_s - \frac{\partial H}{\partial p_s} \right) \delta p_s \, dt - \int_{t_1}^{t_2} \left(\dot{p}_s + \frac{\partial H}{\partial q_s} \right) \delta q_s \, dt$$

$$\left.\begin{aligned} \Rightarrow \quad \dot{q}_s - \frac{\partial H}{\partial p_s} &= c \\ \& \quad \dot{p}_s + \frac{\partial H}{\partial q_s} &= c \end{aligned}\right\}$$

Are Hamilton's equation of motion.

Question:- Derive Lagrange Equation of motion using Hamilton's Principle

Solution

In Hamilton principle

$$I = \int_{t_1}^{t_2} L(q_s, \dot{q}_s, t)\, dt$$

$$\delta I = \int_{t_1}^{t_2} \delta L(q_s, \dot{q}_s, t)\, dt$$

$$= \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s \right\} dt \longrightarrow (1)$$

Consider

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \delta q_s\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) \delta q_s + \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s$$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \delta q_s\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) \delta q_s$$

Put in (1)

$$\delta I = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\right] \delta q_s\, dt + \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \delta q_s\right) dt$$

For Hamilton principle $\delta I = 0$

$$\Rightarrow \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\right] \delta q_s\, dt + \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \delta q_s\right) dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\right] \delta q_s\, dt + \frac{\partial L}{\partial \dot{q}_s} \delta q_s \Big|_{t_1}^{t_2} = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \right) \right] \delta q_s \, dt = 0$$

$$\Rightarrow \frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \right) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \right) - \frac{\partial L}{\partial q_s} = 0$$

QUIZ 2

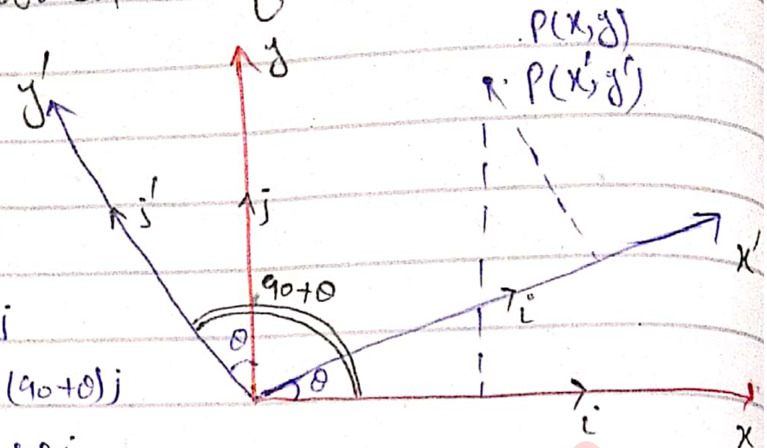
Question:- A bead of mass m is threaded on the loop and moves without friction. The position of the bead at any time 't' is as follows

$$x = a \sin\phi \cos\omega t$$
$$y = a \sin\phi \sin\omega t$$
$$z = a - a\cos\phi$$

where a and ϕ are constants and ϕ is a function of t. Find Lagrange and Hamilton equation of motion.

Solution

Transformation of Axis



$i' = \cos\theta\, i + \sin\theta\, j$

$j' = \cos(90+\theta) i + \sin(90+\theta) j$

$\Rightarrow j' = -\sin\theta\, i + \cos\theta\, j$

$$\overline{OP} = \overline{OP}'$$

$$x_i + y_j = x'i' + y'j'$$

$$x_i + y_i = x'(\cos\theta i + \sin\theta j) + y'(-\sin\theta i + \cos\theta j)$$

$$= (\cos\theta x' - \sin\theta y') i + (\sin\theta x' - \cos\theta y') i$$

$$\Rightarrow x = \cos\theta\, x' - \sin\theta\, y' , \quad y = \sin\theta\, x' + \cos\theta\, y'$$

$$x_1 = \cos\theta\, x_1' - \sin\theta\, x_2' , \quad x_2 = \sin\theta\, x_1' + \cos\theta\, x_2'$$

$$\Rightarrow x_1 = a_{11} x_1' + a_{12} x_2' , \quad x_2 = a_{21} x_1' + a_{22} x_2' \quad \text{]} (1)$$

where $a_{ij} = \cos(\text{angle between } x_i \text{ and } x_j')$

$\therefore a_{11} = \cos\theta$, $a_{12} = \cos(90+\theta) = -\sin\theta$

$a_{21} = \cos(90-\theta) = \sin\theta$, $a_{22} = \cos\theta$

from (1) $x_i = a_{i1} x_1' + a_{i2} x_2'$

$$\Rightarrow \boxed{x_i = a_{ij} x_j'}$$

This is the equation of transformation for translation

$\boxed{x_i = a_{ij} x_j' + a_i}$ This is the equation of transformation for rotation & translation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \Rightarrow \text{Matrix form}$$

Question:- Prove that Lagrange's Equation of Motion is invarient

Proof) Let us consider the system of N-particles, whose configuration is specified by 'n' generalized (Lagrangian) coordinates i.e. $q_1, q_2, q_3, \ldots, q_n$

We know that Lagrange equation of motion is written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \; ; \; s = 1, 2, \ldots, n$$

Now Q_k, $k = 1, 2, \ldots, n$ is another system of coordinates. We transform system of coordinates from $q_s \to Q_k$ s.t

$$Q_k = a_{sk} q_s + a_s \qquad s = k = 1, 2, \ldots, n \longrightarrow (1)$$

$$q_s = a_{sk} Q_k + a_s \longrightarrow (2)$$

Where a_{sk} and a_s are constants

From (1) $\qquad \dot{Q}_k = a_{sk} \dot{q}_s$

Then $\quad L(q_s, \dot{q}_s, t) = K(Q_k, \dot{Q}_k, t)$

$$\frac{\partial L}{\partial \dot{q}_s} = \frac{\partial K}{\partial \dot{Q}_k} \frac{\partial \dot{Q}_k}{\partial \dot{q}_s} = \frac{\partial K}{\partial \dot{Q}_k}(a_{sk})$$

Also $\quad \dfrac{\partial L}{\partial q_s} = \dfrac{\partial K}{\partial Q_k} \dfrac{\partial Q_k}{\partial q_s} = \dfrac{\partial K}{\partial Q_k}(a_{sk})$

Therefore Lagrange Equation of motion becomes

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{Q}_k}\, q/_{sk}\right) - \left(\frac{\partial K}{\partial Q_k}\, q/_{sk}\right) = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{Q}_k}\right) - \frac{\partial K}{\partial Q_k} = 0$$

$\Rightarrow$ Lagrange equation of Motion is invariant.

Assignment 1

Question:- Prove that Hamilton equation of motion are invariant.

Proof) Let us consider the system of N particles whose configuration is specified by 'n' Lagrangian coordinates i.e $q_1, q_2, \dots, q_n$

Then the corresponding Lagrange equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0$$

Hamilton define a function

$$H = \dot{q}_s p_s - L \longrightarrow (1)$$

$$H = H(q_s, p_s, t) \longrightarrow (2)$$

And corresponding Hamilton equations of motion are

$$\dot{p}_s = -\frac{\partial H}{\partial q_s} \longrightarrow (3)$$

$$\dot{q}_s = \frac{\partial H}{\partial p_s} \longrightarrow (4)$$

where $p_s = \frac{\partial L}{\partial \dot{q}_s}$

Now Q_k, P_k ; $k = 1, 2, \ldots, n$ is another system of coordinates. We transform system of coordinates from $q_s \to Q_k$ and $p_s \to P_k$ such that

$$Q_k = a_{sk} q_s + a_s \longrightarrow (5)$$

$$q_s = a_{sk} Q_k + a_s \longrightarrow (6)$$

$$P_k = a_{sk} p_s + a_s \longrightarrow (7)$$

$$p_s = a_{sk} P_k + a_s \longrightarrow (8)$$

From 6

$$\dot{q}_s = a_{sk} \dot{Q}_k$$

From 8

$$\dot{p}_s = a_{sk} \dot{P}_k$$

Now

$$\frac{\partial H}{\partial q_s} = \frac{\partial K}{\partial Q_k} \frac{\partial Q_k}{\partial q_s}$$

$$= \frac{\partial K}{\partial Q_k} (a_{sk})$$

$$\frac{\partial H}{\partial p_s} = \frac{\partial K}{\partial P_k} \frac{\partial P_k}{\partial q_s}$$

$$= \frac{\partial K}{\partial P_k} . (a_{sk})$$

So Hamilton equations of motion becomes

$$\cancel{a_{sk}} \dot{Q}_k = \frac{\partial K}{\partial P_k} \cancel{a_{sk}}$$

$$\Rightarrow \dot{Q}_k = \frac{\partial K}{\partial P_k} \longrightarrow (9)$$

And

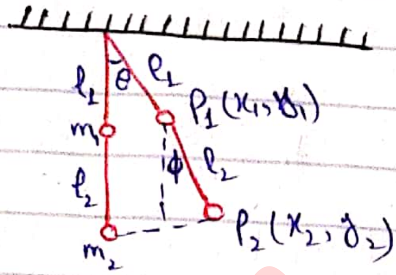
$$\cancel{a_{sk}} \dot{P}_k = -\frac{\partial K}{\partial q_k} \cancel{a_{sk}}$$

$$\Rightarrow \dot{P}_k = -\frac{\partial K}{\partial q_k} \longrightarrow (10).$$

Equation (9) and (10) are required Hamilton equations of motion in Q_k coordinate system.
Hence, Hamilton equation of motion are invariant.

Routh Equation of Motion for Double Pendulum:-

For simplicity take
$m_1 = m_2 = m$
$l_1 = l_2 = l$



$x_1 = l\sin\theta \quad , \quad \dot{x}_1 = l\cos\theta\,\dot{\theta}$
$y_1 = l\cos\theta \quad , \quad \dot{y}_1 = -l\sin\theta\,\dot{\theta}$

$x_2 = l\sin\theta + l\sin\phi \quad , \quad \dot{x}_2 = l\cos\theta\,\dot{\theta} + l\cos\phi\,\dot{\phi}$
$y_2 = l\cos\theta + l\cos\phi \quad , \quad \dot{y}_2 = -l\sin\theta\,\dot{\theta} - l\sin\phi\,\dot{\phi}$

$$T = T_1 + T_2$$

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2}m(l^2\cos^2\theta\,\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\theta}^2) + \frac{1}{2}m(l^2\cos^2\theta\,\dot{\theta}^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l^2\cos\theta\cos\phi\,\dot{\theta}\dot{\phi} + l^2\sin^2\theta\,\dot{\theta}^2 + l^2\sin^2\phi\,\dot{\phi}^2 + 2l^2\sin\theta\sin\phi\,\dot{\theta}\dot{\phi})$$

$$= \frac{1}{2}m(l^2\dot{\theta}^2) + \frac{1}{2}m(l^2\dot{\theta}^2 + l^2\dot{\phi}^2 + 2l^2\dot{\theta}\dot{\phi}[\cos(\theta-\phi)])$$

$$T = \frac{1}{2}ml^2[2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\phi}\dot{\theta}\cos(\theta-\phi)]$$

$$V = -mg(y_1 + y_2) = -mg(l\cos\theta + l\cos\phi)$$

$$L = \frac{1}{2}ml^2[2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\phi}\dot{\theta}\cos(\theta-\phi)] + mgl(\cancel{2}\cos\theta + \cos\phi)$$

There is no ignorable coordinate,
So
$$L = T - V = R$$

By Routh equation of motion

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot\theta}\right) - \frac{\partial R}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(2ml^2\dot\theta + \frac{2}{2}\dot\phi ml^2 \cos(\theta-\phi)\right) + ml^2\frac{2}{2}\dot\phi\dot\theta\sin(\theta-\phi)$$
$$- 2mgl\sin\theta = 0$$

$$\Rightarrow \frac{d}{dt}\left(2ml^2\dot\theta + ml^2\dot\phi\cos(\theta-\phi)\right) + ml^2\dot\theta\dot\phi\sin(\theta-\phi)$$
$$+ 2mgl\sin\theta = 0$$

$$\Rightarrow 2ml^2\ddot\theta + ml^2\ddot\phi\cos(\theta-\phi) - ml^2\dot\phi\dot\theta\sin(\theta-\phi)$$
$$- ml^2\dot\phi^2\sin(\theta-\phi) + ml^2\dot\phi\dot\theta\sin(\theta-\phi)$$
$$+ 2mgl\sin\theta = 0$$

Taking ml^2 common

$$2\ddot\theta + \ddot\phi\cos(\theta-\phi) + \dot\phi^2\sin(\theta-\phi) + \frac{2g}{l}\sin\theta = 0$$

Now
$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot\phi}\right) - \frac{\partial R}{\partial \phi} = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}ml^2(2\dot\phi) + \frac{1}{2}ml^2(2\dot\theta\cos(\theta-\phi))\right)$$
$$- ml^2\dot\phi\dot\theta\sin(\theta-\phi) - mgl\sin\phi = 0$$

$$\ddot\phi + \dot\theta^2\sin(\theta-\phi) + \ddot\theta\cos(\theta-\phi) - \frac{g}{l}\sin\phi = 0$$

Question- Prove that Hamilton's equation of motion are invarient

Solution

We know that the Hamilton is given by

$$H = p_i \dot{q}_i - L(q_i, \dot{q}_i, t) \longrightarrow ①$$

where $$\frac{\partial L}{\partial \dot{q}_i} = p_i$$

Now consider the transformation from

$$(q_s, p_s) \longrightarrow H(Q_k, P_k)$$

From Hamilton's equation

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \text{and} \quad \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}\right\} \longrightarrow ②$$

as $$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = \frac{-\partial H}{\partial q_s}$$

$$Q_K = Q_K(p_s, q_s, t)$$

$$P_K = P_K(p_s, q_s, t)$$

Now consider

$$\dot{Q}_K = \frac{\partial Q_K}{\partial q_s}\dot{q}_s + \frac{\partial Q_K}{\partial p_s}\dot{p}_s \longrightarrow ③$$

putting values $\dot{q}_s$ and $\dot{p}_s$. From ② in ③

$$\dot{Q}_K = \frac{\partial Q_K}{\partial q_s}\frac{\partial H}{\partial p_s} + \left(\frac{-\partial H}{\partial q_s}\right)\frac{\partial Q_K}{\partial p_s} \longrightarrow ④$$

Consider the transformation of Hamilton's

$$H(q_s, p_s, t) \longrightarrow H(Q_k, P_k, t)$$

$$\left.\begin{aligned} \frac{\partial H}{\partial p_s} &= \frac{\partial H}{\partial Q_k}\frac{\partial Q_k}{\partial p_s} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial p_s} \\ \frac{\partial H}{\partial q_s} &= \frac{\partial H}{\partial Q_k}\frac{\partial Q_k}{\partial q_s} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial q_s} \end{aligned}\right\} \longrightarrow (a)$$

Put (a) in (4)

$$\dot{Q}_k = \frac{\partial Q_k}{\partial q_s}\left[\frac{\partial H}{\partial Q_k}\frac{\partial Q_k}{\partial p_s} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial p_s}\right]$$

$$- \frac{\partial P_k}{\partial p_s}\left[\frac{\partial H}{\partial Q_k}\frac{\partial Q_k}{\partial q_s} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial q_s}\right]$$

$$= \cancel{\frac{\partial Q_k}{\partial q_s}\frac{\partial H}{\partial Q_k}\frac{\partial Q_k}{\partial p_s}} + \frac{\partial Q_k}{\partial q_s}\frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial p_s}$$

$$- \cancel{\frac{\partial P_k}{\partial p_s}\frac{\partial H}{\partial Q_k}\frac{\partial Q_k}{\partial q_s}} - \frac{\partial P_k}{\partial p_s}\frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial q_s}$$

$$= \frac{\partial H}{\partial P_k}\left[\frac{\partial Q_k}{\partial q_s}\frac{\partial P_k}{\partial p_s} - \frac{\partial Q_k}{\partial p_s}\frac{\partial P_k}{\partial q_s}\right]$$

$$\Rightarrow \dot{Q}_k = \frac{\partial H}{\partial P_k}[Q_k, P_k]$$

where $[Q_k, P_k] = \frac{\partial Q_k}{\partial q_s}\frac{\partial P_k}{\partial p_s} - \frac{\partial Q_k}{\partial p_s}\frac{\partial P_k}{\partial q_s}$

Since $[Q_i, P_j] = \delta_{ij}$

and $\delta_{ij} = 1$ if $i = j$

$$\Rightarrow [Q_k, P_k] = 1$$

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Hence $\dot{Q}_k = \frac{\partial H}{\partial P_k} \longrightarrow (A)$

Consider $\dot{P}_k = \frac{\partial P_k}{\partial q_s}\dot{q}_s + \frac{\partial P_k}{\partial p_s}\dot{p}_s \longrightarrow (5)$

Putting values of $\dot{q}_s$ and $\dot{p}_s$ from (2) in (5)

$$\dot{P}_k = \frac{\partial P_k}{\partial q_s}\frac{\partial H}{\partial p_s} - \frac{\partial P_k}{\partial p_s}\frac{\partial H}{\partial q_s}$$

$$= \frac{\partial P_k}{\partial q_s}\left[\frac{\partial H}{\partial Q_k}\frac{\partial Q_k}{\partial p_s} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial p_s}\right]$$

$$- \frac{\partial P_k}{\partial p_s}\left[\frac{\partial H}{\partial Q_k}\frac{\partial Q_k}{\partial q_s} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial q_s}\right]$$

$$= \frac{\partial P_k}{\partial q_s}\frac{\partial H}{\partial Q_k}\frac{\partial Q_k}{\partial p_s} + \frac{\partial P_k}{\partial q_s}\cancel{\frac{\partial H}{\partial P_k}}\frac{\partial P_k}{\partial p_s}$$

$$- \frac{\partial P_k}{\partial p_s}\frac{\partial H}{\partial Q_k}\frac{\partial Q_k}{\partial q_s} - \frac{\partial P_k}{\partial p_s}\frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial q_s}$$

$$= \frac{\partial H}{\partial Q_k}\left[\frac{\partial P_k}{\partial q_s}\frac{\partial Q_k}{\partial p_s} - \frac{\partial P_k}{\partial p_s}\frac{\partial Q_k}{\partial q_s}\right]$$

$$= \frac{\partial H}{\partial Q_k}[P_k, Q_k]$$

$$\Rightarrow \dot{P}_k = \frac{\partial H}{\partial Q_k} \longrightarrow (B)$$

From equ (2), (A) and (B) The Hamilton equations are invariant under the given transformations

Non-Holonomic System:- A system is said to be non Holonomic if equation of constraints are function of velocity
e.g:- Equation of constraints may be

$$a\dot{x} + \dot{y} = 0$$
$$ax\dot{y} + c = 0$$
$$\dot{x} + \dot{y} < c$$

Lagrange Equation of Motion in a Non-Holonomic System:-

Since the system is non-holonomic, constraints can be written as

$$a_{ks}\dot{q}_s + a_k = 0 \qquad \begin{matrix} s = 1,2,\cdots,n \\ k = 1,2,\cdots,m \end{matrix} \longrightarrow (1)$$

$\Rightarrow$ degree of freedom of system will be $(n-m)$

For system we can write

$$(F_i - m_i\ddot{r}_i)\delta r_i = 0$$

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s}\right]\delta q_s = 0 \longrightarrow (2)$$

Here degree of freedom is $(n-m)$

$\therefore$ Equ (1)

$$a_{ks}\delta q_s = 0 \longrightarrow (3) \qquad \begin{matrix} k = 1,2,\cdots,m \\ s = 1,2,\cdots,n \end{matrix}$$

Multiply equ (1) of equation (3) by λ_1 and equation (2) of equation (3) by λ_2 and so on multiplying mth equation of equation ③ by λ_m and then we get

$$a_{ks}\,\delta q_s = 0$$

$$\Rightarrow (\lambda_1 a_{11} + \lambda_2 a_{21} + \lambda_3 a_{31} + \cdots + \lambda_m a_{m1})\,\delta q_1 + (\lambda_1 a_{12} + \lambda_2 a_{22} + \lambda_3 a_{32} + \cdots + \lambda_m a_{m2})\,\delta q_2 + \cdots + (\lambda_1 a_{1n} + \lambda_2 a_{2n} + \lambda_3 a_{3n} + \cdots + \lambda_m a_{mn})\,\delta q_n = 0$$

$$\Rightarrow (\lambda_k a_{k1})\,\delta q_1 + (\lambda_k a_{k2})\,\delta q_2 + (\lambda_k a_{k3})\,\delta q_3 + \cdots + (\lambda_k a_{kn})\,\delta q_n = 0$$

$$\Rightarrow \lambda_k a_{ks}\,\delta q_s = 0 \longrightarrow ④ \quad \begin{matrix} k = 1, 2, \ldots, m \\ s = 1, 2, \ldots, n \end{matrix}$$

Subtract equ ④ from ②

$$\left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} - \lambda_k a_{ks} \right\} \delta q_s = 0$$

Since degree of freedom is $n-m$

$\therefore \delta q_s \neq 0 \quad \forall s$

Now we choose λ_k ; $k = 1, 2, \ldots, m$

s.t remaining $\delta q_s \neq 0$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} - \lambda_k a_k = 0$$

Which is known as Lagrange equation of motion for non-Holonomic system

Lagrange Brackets:- If q_s and p_s are functions of u and v Then Lagrange brackets are defined as

$$\{u,v\} = \frac{\partial q_s}{\partial u}\frac{\partial p_s}{\partial v} - \frac{\partial p_s}{\partial u}\frac{\partial q_s}{\partial v}$$

Prove that (i) $\{u,v\} = 0$

(ii) $\{u,v\} = -\{v,u\}$

(iii) $\{q_i, q_j\} = 0$ (iv) $\{p_i, p_j\} = 0$

(v) $\{p_i, q_j\} = -\delta_{ij}$ (vi) $\{q_i, p_j\} = \delta_{ij}$

Solution ii) $\{u,v\} = \frac{\partial q_s}{\partial u}\frac{\partial p_s}{\partial v} - \frac{\partial p_s}{\partial u}\frac{\partial q_s}{\partial v}$

$$= -\left\{\frac{\partial p_s}{\partial u}\frac{\partial q_s}{\partial v} - \frac{\partial q_s}{\partial u}\frac{\partial p_s}{\partial v}\right\}$$

$$= -\left\{\frac{\partial q_s}{\partial v}\frac{\partial p_s}{\partial u} - \frac{\partial p_s}{\partial v}\frac{\partial q_s}{\partial u}\right\}$$

$$= -\{v,u\}$$

(iii) $\{q_i, q_j\} = \frac{\partial q_s}{\partial q_i}\frac{\partial p_s}{\partial q_j} - \frac{\partial p_s}{\partial q_i}\frac{\partial q_s}{\partial q_j}$

$$= \delta_{si}(0) \mp 0\,\delta_{sj}$$

$$= 0$$

$\Rightarrow \{q_i, q_j\} = 0$

(iv) $\{p_i, p_j\} = \frac{\partial q_s}{\cancel{\partial p_i}}^{0}\frac{\partial p_s}{\partial p_j} - \frac{\partial p_s}{\partial p_i}\frac{\partial q_s}{\cancel{\partial p_i}}^{0}$

$$\Rightarrow \{P_i, P_j\} = (0)\delta_{sj} - \delta_{si}(0)$$
$$= 0$$

(v) $$\{P_i, q_j\} = \frac{\partial q_s}{\partial P_i}\frac{\partial P_s}{\partial q_j} - \frac{\partial P_s}{\partial P_i}\frac{\partial q_s}{\partial q_j}$$

$$= (0)(0) - \delta_{si}\delta_{sj}$$
$$= -\delta_{ij}$$
$$= 0 \text{ if } i \neq j \quad , \quad -1 \text{ if } i = j$$

(vi) $$\{q_i, P_j\} = \frac{\partial q_s}{\partial q_i}\frac{\partial P_s}{\partial P_j} - \frac{\partial P_s}{\partial q_i}\frac{\partial q_s}{\partial P_j}$$

$$= \delta_{si}\delta_{sj} - (0)(0)$$

$$= \delta_{ij} = \begin{cases} 0 \, ; & i \neq j \\ 1 \, ; & i = j \end{cases}$$

Homework:- Prove that Lagrange Brackets are invarient.

Proof

Let
$$Q_m = Q_m(P_k, q_k)$$
$$P_m = P_m(q_k, P_k)$$

be a canonical transformation

We have to prove that

$$\{u, v\}_{q,p} = \{u, v\}_{Q,P}$$

Consider

$$\{u, v\}_{q,P} = \frac{\partial q_k}{\partial u}\frac{\partial P_k}{\partial v} - \frac{\partial P_k}{\partial u}\frac{\partial q_k}{\partial v}$$

$$\{u,v\}_{q,p} = \left[\frac{\partial q_k}{\partial Q_m}\frac{\partial Q_m}{\partial u} + \frac{\partial q_k}{\partial P_m}\frac{\partial P_m}{\partial u}\right]\left[\frac{\partial p_k}{\partial Q_n}\frac{\partial Q_n}{\partial v} + \frac{\partial p_k}{\partial P_n}\frac{\partial P_n}{\partial v}\right] - \left[\frac{\partial p_k}{\partial Q_m}\frac{\partial Q_m}{\partial u} + \frac{\partial p_k}{\partial P_m}\frac{\partial P_m}{\partial u}\right]\left[\frac{\partial q_k}{\partial Q_n}\frac{\partial Q_n}{\partial v} + \frac{\partial q_k}{\partial P_n}\frac{\partial P_n}{\partial v}\right]$$

$$= \left[\frac{\partial q_k}{\partial Q_m}\frac{\partial Q_m}{\partial u}\frac{\partial p_k}{\partial Q_n}\frac{\partial Q_n}{\partial v} + \frac{\partial q_k}{\partial Q_m}\frac{\partial Q_m}{\partial u}\frac{\partial p_k}{\partial P_n}\frac{\partial P_n}{\partial v}\right]$$

$$+ \left[\frac{\partial q_k}{\partial P_m}\frac{\partial P_m}{\partial u}\frac{\partial p_k}{\partial Q_n}\frac{\partial Q_n}{\partial v} + \frac{\partial q_k}{\partial P_m}\frac{\partial P_m}{\partial u}\frac{\partial p_k}{\partial Q_n}\frac{\partial Q_n}{\partial v}\right]$$

$$- \left[\frac{\partial p_k}{\partial Q_m}\frac{\partial Q_m}{\partial u}\frac{\partial q_k}{\partial Q_n}\frac{\partial Q_n}{\partial v} + \frac{\partial p_k}{\partial Q_m}\frac{\partial Q_m}{\partial u}\frac{\partial q_k}{\partial P_n}\frac{\partial P_n}{\partial v}\right]$$

$$- \left[\frac{\partial p_k}{\partial P_m}\frac{\partial P_m}{\partial u}\frac{\partial q_k}{\partial Q_n}\frac{\partial Q_n}{\partial v} + \frac{\partial p_k}{\partial P_m}\frac{\partial P_m}{\partial u}\frac{\partial q_k}{\partial P_n}\frac{\partial P_n}{\partial v}\right]$$

$$= \frac{\partial Q_m}{\partial u}\frac{\partial Q_n}{\partial v}\left[\frac{\partial q_k}{\partial Q_m}\frac{\partial p_k}{\partial Q_n} - \frac{\partial q_k}{\partial Q_n}\frac{\partial p_k}{\partial Q_m}\right]$$

$$+ \frac{\partial Q_m}{\partial u}\frac{\partial P_n}{\partial v}\left[\frac{\partial q_k}{\partial Q_m}\frac{\partial p_k}{\partial P_n} - \frac{\partial q_k}{\partial P_n}\frac{\partial p_k}{\partial Q_m}\right]$$

$$+ \frac{\partial P_m}{\partial u}\frac{\partial Q_n}{v}\left[\frac{\partial q_k}{\partial P_m}\frac{\partial p_k}{\partial Q_n} - \frac{\partial q_k}{\partial Q_n}\frac{\partial p_k}{\partial P_m}\right]$$

$$+ \frac{\partial P_m}{\partial u}\frac{\partial P_n}{\partial v}\left[\frac{\partial q_k}{\partial P_m}\frac{\partial p_k}{\partial P_n} - \frac{\partial q_k}{\partial P_n}\frac{\partial p_k}{\partial P_m}\right]$$

$$= \frac{\partial Q_m}{\partial u}\frac{\partial Q_n}{\partial v}\{Q_m, Q_n\} + \frac{\partial Q_m}{\partial u}\frac{\partial P_n}{\partial v}\{Q_m, P_n\}$$

$$+ \frac{\partial P_m}{\partial u}\frac{\partial Q_n}{\partial v}\{P_m, Q_n\} + \frac{\partial P_m}{\partial u}\frac{\partial P_n}{\partial v}\{P_m, P_n\}$$

$$= \frac{\partial Q_m}{\partial u}\frac{\partial Q_n}{\partial v}(0) + \frac{\partial Q_m}{\partial u}\frac{\partial P_n}{\partial v}\delta_{mn}$$

$$+ \frac{\partial P_m}{\partial u}\frac{\partial Q_n}{\partial v}(-\delta_{mn}) + 0$$

$$= \frac{\partial Q_m}{\partial u}\frac{\partial P_m}{\partial v} - \frac{\partial P_m}{\partial u}\frac{\partial Q_m}{\partial v}$$

$$\Rightarrow \{u,v\}_{q,p} = \{u,v\}_{Q,P}$$

Quiz:- Lagrange function is given by
$$L = \frac{1}{2}m[\dot{x}^2+\dot{y}^2+\dot{z}^2] + A\dot{x} + B\dot{y} + C\dot{z} - V$$

where A, B, C and V are functions of x, y, z. Find

i) Langrange Equation of motion.

Solution

Lagrange's equation of motion is as follows

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0$$

For x:- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$$\Rightarrow \frac{d}{dt}(m\dot{x}+A) - \left(\frac{\partial A}{\partial x}\dot{x} + \frac{\partial B}{\partial x}\dot{y} + \frac{\partial C}{\partial x}\dot{z} - \frac{\partial V}{\partial x}\right) = 0$$

$$\Rightarrow m\ddot{x} + \cancel{\frac{\partial A}{\partial x}\dot{x}} + \frac{\partial A}{\partial y}\dot{y} + \frac{\partial A}{\partial z}\dot{z} - \cancel{\frac{\partial A}{\partial x}\dot{x}} - \frac{\partial B}{\partial x}\dot{y}$$

$$- \frac{\partial C}{\partial x}\dot{z} + \frac{\partial V}{\partial x} = 0$$

$$\Rightarrow m\ddot{x} + \left(-\frac{\partial B}{\partial x} + \frac{\partial A}{\partial y}\right)\dot{y} + \left(-\frac{\partial C}{\partial x} + \frac{\partial A}{\partial x}\right)\dot{z} + \frac{\partial V}{\partial x} = 0 \longrightarrow (A)$$

For y:- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$

$$\Rightarrow \frac{d}{dt}(m\dot{y} + B) - \left(\frac{\partial A}{\partial y}\dot{x} + \frac{\partial B}{\partial y}\dot{y} + \frac{\partial C}{\partial y}\dot{z} - \frac{\partial V}{\partial y}\right) = 0$$

$$\Rightarrow m\ddot{y} + \cancel{\frac{\partial B}{\partial y}\dot{y}} + \frac{\partial B}{\partial y}\dot{x} + \frac{\partial B}{\partial y}\dot{z} - \frac{\partial A}{\partial y}\dot{x} - \cancel{\frac{\partial B}{\partial y}\dot{y}}$$

$$- \frac{\partial C}{\partial y}\dot{z} + \frac{\partial V}{\partial y} = 0$$

$$\Rightarrow m\ddot{y} + \left(\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\right)\dot{x} + \left(\frac{\partial B}{\partial y} - \frac{\partial C}{\partial y}\right)\dot{z} + \frac{\partial V}{\partial y} = 0 \longrightarrow (B)$$

Similarly For z.

(ii) Hamilton's Equation of Motion:-

$$H = (P_x - A)\dot{x} + (P_y - B)\dot{y} + (P_z - C)\dot{z}$$
$$- \frac{1}{2}m(\dot{x} + \dot{y} + \dot{z}) + V$$

where $P_x = \frac{\partial L}{\partial \dot{x}} \Rightarrow P_x = m\dot{x} + A \Rightarrow \dot{x} = \frac{1}{m}(P_x - A)$

$P_y = \frac{\partial L}{\partial \dot{y}} \Rightarrow P_y = m\dot{y} + B \Rightarrow \dot{y} = \frac{1}{m}(P_y - B)$

$P_z = \frac{\partial L}{\partial \dot{z}} \Rightarrow P_z = m\dot{z} + C \Rightarrow \dot{z} = \frac{1}{m}(P_z - C)$

$$\Rightarrow H = \frac{1}{m}(P_x - A)^2 + \frac{1}{m}(P_y - B)^2 + \frac{1}{m}(P_z - C)^2 + V$$
$$- \frac{1}{2}m\left[\frac{1}{m^2}(P_x - A)^2 + \frac{1}{m^2}(P_y - B)^2 + \frac{1}{m^2}(P_z - C)^2\right]$$

$$\Rightarrow H = \frac{1}{2m}\left[(P_x - A)^2 + (P_y - B)^2 + (P_z - C)^2\right] + V$$

Now Hamilton's equations of motion

$$\dot{q}_j = \frac{\partial H}{\partial P_j} \quad , \quad \dot{P}_j = -\frac{\partial H}{\partial q_j}$$

$$\Rightarrow \dot{x} = \frac{\partial H}{\partial P_x} \quad , \quad \dot{P}_x = -\frac{\partial H}{\partial x}$$

$$\dot{x} = \frac{1}{m}(P_x - A)$$

$$\dot{y} = \frac{\partial H}{\partial P_y}$$

$$\Rightarrow \dot{y} = \frac{1}{m}(P_y - B)$$

$$\dot{z} = \frac{\partial H}{\partial P_z}$$

$$\Rightarrow \dot{z} = \frac{1}{m}(P_z - C)$$

$$\dot{P}_x = \frac{1}{2m}\left[2(P_x - A)\left(-\frac{\partial A}{\partial x}\right) + 2(P_y - B)\left(-\frac{\partial B}{\partial x}\right) + 2(P_z - C)\left(-\frac{\partial C}{\partial x}\right)\right] - \frac{\partial V}{\partial x}$$

$$\Rightarrow \dot{P}_x = \frac{1}{m}\left[(P_x - A)\frac{\partial A}{\partial x} + (P_y - B)\frac{\partial B}{\partial x} + (P_z - C)\frac{\partial C}{\partial x}\right] - \frac{\partial V}{\partial x}$$

similar

$$\dot{P}_y = \frac{1}{m}\left[(P_x - A)\frac{\partial A}{\partial y} + (P_y - B)\frac{\partial B}{\partial y} + (P_z - C)\frac{\partial C}{\partial y}\right] - \frac{\partial V}{\partial y}$$

$$\dot{P}_z = \frac{1}{m}\left[(P_x - A)\frac{\partial A}{\partial z} + (P_y - B)\frac{\partial B}{\partial z} + (P_z - C)\frac{\partial C}{\partial z}\right] - \frac{\partial V}{\partial z}$$

Question:- Find Hamilton's equation of motion for a given Hamilton fn

$$H = P_1 q_1 - P_2 q_2 - a q_1^2 + b q_2^2$$

where a, b are constants. Hence Prove that

(i) $\dfrac{P_2 - b q_2}{q_1} = \text{constant}$ (ii) $q_1 q_2 = \text{constant}$

(iii) $\log q_1 = t$

Solution Given that

$$H = q_1 p_1 - p_2 q_2 - a q_1^2 + b q_2^2 \longrightarrow (*)$$

Hamilton's equation of motion are as follows

$$\dot{q}_s = \frac{\partial H}{\partial p_s} \quad \text{and} \quad \dot{p}_s = -\frac{\partial H}{\partial q_s}$$

$$\Rightarrow \dot{q}_1 = \frac{\partial H}{\partial p_1} \qquad \dot{p}_1 = -\frac{\partial H}{\partial q_1}$$

&

$$\dot{q}_2 = \frac{\partial H}{\partial p_2} \qquad \dot{p}_2 = -\frac{\partial H}{\partial q_2}$$

By using (*)

$$\dot{q}_1 = q_1 \longrightarrow (1) \qquad \dot{p}_1 = 2a q_1 - p_1 \longrightarrow (3)$$

$$\dot{q}_2 = -q_2 \longrightarrow (2) \qquad \dot{p}_2 = p_2 - 2b q_2 \longrightarrow (4)$$

Equation (1) $\rightarrow$ (4) are required Hamilton's equations of motion.

i) To prove $\dfrac{p_2 - b q_2}{q_1} = \text{constant}$

Consider $\dfrac{d}{dt}\left(\dfrac{p_2 - b q_2}{q_1}\right)$

$$= \frac{q_1(\dot{p}_2 - b\dot{q}_2) - (p_2 - b q_2)\dot{q}_1}{q_1^2}$$

using ① to ④ in this eqn

$$\frac{d}{dt}\left(\frac{p_2 - bq_2}{q_1}\right) = \frac{q_1\{(p_2 - 2bq_2) + b(q_2)\} - [p_2 - bq_2](\dot{q}_1)}{q_1^2}$$

$$= \frac{\cancel{q_1 p_2} - \cancel{2bq_1q_2} + \cancel{bq_1q_2} - \cancel{p_2q_1} + \cancel{bq_1q_2}}{q_1^2}$$

$$= 0$$

So $\frac{p_2 - bq_2}{q_1}$ is constant.

(ii) To prove $q_1 q_2$ is constant

$$\frac{d}{dt}(q_1q_2) = q_1\dot{q}_2 + q_2\dot{q}_1$$

$$= -\cancel{q_1q_2} + \cancel{q_2q_1} = 0$$

So $q_1 q_2$ is constant.

(iii) To prove $\log q_1 = t$

Consider $\frac{d}{dt}(\log q_1) = \frac{1}{q_1}\cdot\dot{q}_1 = \frac{q_1}{q_1} = 1$

Integrate w.r.t t

$\Rightarrow \log q_1 = t + c$

$\dot{q}_1 = q_1$

$\frac{dq_1}{dt} = q_1 \Rightarrow \frac{dq_1}{q_1} = dt$

Integrating

$\log q_1 = t$

Question:- Define Bilinear covarient and show that the expression $\delta p_i\, dq_i - \delta q_i\, dp_i$ is invarient under a canonical transformation.

Solution

Given expression is

$$\delta p_i\, dq_i - \delta q_i\, dp_i \longrightarrow (1)$$

Since this expression is linear in $dp, \delta p$ and also linear in $\delta q, dq$ so it is called bilinear covarient.

Next we show that this expression is invarient under a canonical transformation

$$(q_i, p_i) \longrightarrow (Q_k, P_k) \; ; \; i, k = 1, 2, \cdots, n$$

i.e $q_i = q_i(Q_k, P_k)$ & $p_i = p_i(Q_k, P_k)$

Then
$$\delta p_i = \frac{\partial p_i}{\partial Q_k}\delta Q_k + \frac{\partial p_i}{\partial P_k}\delta P_k$$

$$dq_i = \frac{\partial q_i}{\partial Q_k} dQ_k + \frac{\partial q_i}{\partial P_k} dP_k$$

Therefore

$$\delta p_i dq_i - \delta q_i dp_i = \left(\frac{\partial p_i}{\partial Q_k}\delta Q_k + \frac{\partial p_i}{\partial P_k}\delta P_k\right)\left(\frac{\partial q_i}{\partial Q_k} dQ_k + \frac{\partial q_i}{\partial P_k} dP_k\right) - \left(\frac{\partial q_i}{\partial Q_k}\delta Q_k + \frac{\partial q_i}{\partial P_k}\delta P_k\right)$$

$$\left(\frac{\partial p_i}{\partial Q_k} dQ_k + \frac{\partial p_i}{\partial P_k} dP_k\right)$$

$$=\left(\frac{\partial P_i}{\partial Q_k}\frac{\partial q_i}{\partial Q_k}-\frac{\partial q_i}{\partial Q_k}\frac{\partial}{\partial Q_k}\right)\delta Q_k\, dQ_k$$
$$+\left(\frac{\partial P_i}{\partial Q_k}\frac{\partial q_i}{\partial P_k}-\frac{\partial q_i}{\partial Q_k}\frac{\partial P_i}{\partial P_k}\right)\delta Q_k\, dP_k$$
$$+\left(\frac{\partial P_i}{\partial P_k}\frac{\partial q_i}{\partial Q_k}-\frac{\partial q_i}{\partial P_k}\frac{\partial P_i}{\partial Q_k}\right)\delta P_k\, dQ_k$$
$$+\left(\frac{\partial P_i}{\partial P_k}\frac{\partial q_i}{\partial P_k}-\frac{\partial q_i}{\partial P_k}\frac{\partial P_i}{\partial Q_k}\right)\delta P_k\, dP_k$$

$$=\{Q_k,Q_k\}^{0}\delta Q_k\, dQ_k+\{Q_k,P_k\}\delta Q_k\, dP_k$$
$$+\{P_k,Q_k\}\delta P_k\, dQ_k+\{P_k,P_k\}^{0}\delta P_k\, dP_k$$

$$=\delta P_k\, dQ_k-\delta Q_k\, dP_k$$

Question:- Using the method of bi-linear co-varient, show that the transformation

$$Q=\cot^{-1}\left[\frac{p}{m\omega q}\right]\longrightarrow (1)\qquad P=\frac{p^2+m^2\omega^2q^2}{2m\omega}\longrightarrow (2)$$

is canonical.

Solution

From given transformation (1) and (2) we have

$$dQ=\frac{-1}{1+\frac{p^2}{m^2\omega^2q^2}}\cdot\frac{1}{m\omega}\left\{\frac{q\,dp-p\,dq}{q^2}\right\}$$

Question:- Using the method of bilinear covariant show that transformation

$$Q = \frac{\sqrt{2q}}{\sqrt{k}} \cos p \quad \& \quad P = \sqrt{2q}\sqrt{k}\, \sin p$$

is canonical.

Solution

As we know that a transformation is canonical if

$$\delta P dQ - \delta Q dP = \delta p dq - \delta q dp$$

Consider

$$dQ = \frac{1}{\sqrt{k}} \sqrt{2}\, \frac{1}{2} q^{-\frac{1}{2}} dq \cos p + \frac{\sqrt{2q}}{\sqrt{k}} (-\sin p)\, dp$$

$$= \frac{1}{\sqrt{2k}\sqrt{q}} \cos p\, dq - \frac{\sqrt{2q}}{\sqrt{k}} \sin p\, dp$$

Now

$$dP = \sqrt{2k}\left\{\frac{1}{2} q^{-\frac{1}{2}} dq \sin p\right\} + \sqrt{2q}\sqrt{k} \cos p\, dp$$

$$= \frac{\sqrt{k}}{\sqrt{2}\sqrt{q}} \sin p\, dq + \sqrt{2q}\sqrt{k} \cos p\, dp$$

Consider

$$\delta P dQ - \delta Q dP = \left[\frac{\sqrt{k}}{\sqrt{2}\sqrt{q}} \sin p\, \delta q + \sqrt{2q}\sqrt{k} \cos p\, \delta p\right]$$

$$\left[\frac{1}{\sqrt{2k}\sqrt{q}} \cos p\, dq - \frac{\sqrt{2q}}{\sqrt{k}} \sin p\, dp\right] -$$

$$\left[\frac{\cos p}{\sqrt{2k}\sqrt{q}} \delta q - \frac{\sqrt{2q}}{\sqrt{k}} \sin p\, \delta p\right]\left[\frac{\sqrt{k}}{\sqrt{2q}}\right.$$

$$\left. \sin p\, dq + \sqrt{2q}\sqrt{k} \cos p\, dp\right]$$

$$= \frac{1}{2q} \cancel{\sin p} \cos p\, \delta q\, dq - \sin^2 p\, dp\, \delta q$$

$$+ \cos^2 p\, \delta p\, dq - 2q\, \cancel{\sin p} \cos p\, \delta p\, dp$$

$$-\frac{1}{2q}\cancel{\sin p\cos p\, dq\,\delta q} - \cos^2 p\, dP\,\delta q$$

$$+\sin^2 p\,\delta P\,dq + \cancel{2q\sin p\cos p\,\delta P\,dp}$$

$$= (\cos^2 p + \sin^2 p)\,\delta P\,dq - (\cos^2 p + \sin^2 p)\,\delta q\,dP$$

$$= \delta P\,dq - \delta q\,dP$$

$$\Rightarrow \delta P\,dQ - dP\,\delta Q = \delta P\,dq - \delta q\,dP$$

Therefore transformations are canonical

⇒ **Poisson Bracket:-** If u, v are function of q_i and p_i then Poisson bracket is defined as

$$\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

and it is defined as $[u, v]$

$$\Rightarrow [u, v] = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

Properties:- i) $[u, u] = 0 = [v, v]$

ii) $[u, v] = -[v, u]$

iii) $[u, v+w] = [u, v] + [u, w]$

iv) $\frac{\partial}{\partial t}[u, v] = \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right]$

v) $[q_i, q_j] = 0$

vi) $[P_i, q_j] = -\delta_{ij}$

vii) $[P_i, P_j] = 0$

Proof:

i) $[u,u] = \dfrac{\partial u}{\partial q_i}\dfrac{\partial u}{\partial P_i} - \dfrac{\partial u}{\partial P_i}\dfrac{\partial u}{\partial q_i}$

$= \dfrac{\partial u}{\partial q_i}\cancel{\dfrac{\partial u}{\partial P_i}} - \dfrac{\partial u}{\partial q_i}\cancel{\dfrac{\partial u}{\partial P_i}}$

$= 0$

similarly $[v,v] = 0$

ii) $[u,v] = \dfrac{\partial u}{\partial q_i}\dfrac{\partial v}{\partial P_i} - \dfrac{\partial u}{\partial P_i}\dfrac{\partial v}{\partial q_i}$

$= -\left\{\dfrac{\partial v}{\partial q_i}\dfrac{\partial u}{\partial P_i} - \dfrac{\partial v}{\partial P_i}\dfrac{\partial u}{\partial q_i}\right\}$

$[u,v] = -[v,u]$

iii) $[u, v+w] = \dfrac{\partial u}{\partial q_i}\dfrac{\partial}{\partial P_i}[v+w] - \dfrac{\partial u}{\partial P_i}\dfrac{\partial}{\partial q_i}[v+w]$

$= \dfrac{\partial u}{\partial q_i}\dfrac{\partial v}{\partial P_i} + \dfrac{\partial u}{\partial q_i}\dfrac{\partial w}{\partial P_i} - \dfrac{\partial u}{\partial P_i}\dfrac{\partial v}{\partial q_i}$

$- \dfrac{\partial u}{\partial P_i}\dfrac{\partial w}{\partial q_i}$

$= \dfrac{\partial u}{\partial q_i}\dfrac{\partial v}{\partial P_i} - \dfrac{\partial u}{\partial P_i}\dfrac{\partial v}{\partial q_i}$

$+ \dfrac{\partial u}{\partial q_i}\dfrac{\partial w}{\partial P_i} - \dfrac{\partial u}{\partial P_i}\dfrac{\partial w}{\partial q_i}$

$= [u,v] + [u,w]$

iv) $\frac{\partial}{\partial t}[u,v] = \frac{\partial}{\partial t}\left\{\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right\}$

$$= \frac{\partial}{\partial t}\left\{\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i}\right\} - \frac{\partial}{\partial t}\left\{\frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right\}$$

$$= \left(\frac{\partial}{\partial t}\frac{\partial u}{\partial q_i}\right)\frac{\partial v}{\partial p_i} + \frac{\partial u}{\partial q_i}\left(\frac{\partial}{\partial t}\frac{\partial v}{\partial p_i}\right) - \left(\frac{\partial}{\partial t}\frac{\partial u}{\partial p_i}\right)\frac{\partial v}{\partial q_i} - \frac{\partial u}{\partial p_i}\left(\frac{\partial}{\partial t}\frac{\partial v}{\partial q_i}\right)$$

$$= \frac{\partial}{\partial t}\frac{\partial u}{\partial q_i}\cdot\frac{\partial v}{\partial p_i} - \frac{\partial}{\partial t}\frac{\partial u}{\partial p_i}\cdot\frac{\partial v}{\partial q_i} + \frac{\partial u}{\partial q_i}\cdot\frac{\partial}{\partial t}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\cdot\frac{\partial}{\partial t}\frac{\partial v}{\partial q_i}$$

$$= \frac{\partial}{\partial q_i}\frac{\partial u}{\partial t}\cdot\frac{\partial v}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial u}{\partial t}\cdot\frac{\partial v}{\partial q_i} + \frac{\partial u}{\partial q_i}\frac{\partial}{\partial p_i}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial p_i}\frac{\partial}{\partial q_i}\frac{\partial v}{\partial t}$$

$$= \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right]$$

v) $[q_i, q_j] = \frac{\partial q_i}{\partial q_i}\frac{\partial q_j}{\partial p_i} - \frac{\partial q_i}{\partial p_i}\frac{\partial q_j}{\partial q_i}$

$$0 - 0(\delta_{ji})$$

$$= 0$$

similarly $[p_i, p_j] = 0$

vii) $[q_i, p_j] = \frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k}$.

$= \delta_{ik}\delta_{jk} - 0$

$= \delta_{ij}$

Question:- Prove the Jacobi Identity

i.e $[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$

Solution

Consider

$$[u,[v,w]] = \frac{\partial u}{\partial q_i}\frac{\partial}{\partial p_i}[v,w] - \frac{\partial u}{\partial p_i}\frac{\partial}{\partial q_i}[v,w]$$

$$= \frac{\partial u}{\partial q_i}\frac{\partial}{\partial p_i}\left\{\frac{\partial v}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial v}{\partial p_i}\frac{\partial w}{\partial q_i}\right\}$$

$$- \frac{\partial u}{\partial q_i}\frac{\partial}{\partial q_i}\left\{\frac{\partial v}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial v}{\partial p_i}\frac{\partial w}{\partial q_i}\right\}$$

$$= \frac{\partial u}{\partial q_i}\frac{\partial}{\partial p_i}\left\{\frac{\partial v}{\partial q_i}\frac{\partial w}{\partial p_i}\right\} - \frac{\partial u}{\partial q_i}\frac{\partial}{\partial p_i}\left\{\frac{\partial v}{\partial p_i}\frac{\partial w}{\partial q_i}\right\} - \frac{\partial u}{\partial q_i}\frac{\partial}{\partial q_i}\left\{\frac{\partial v}{\partial q_i}\frac{\partial w}{\partial p_i}\right\}$$

$$+ \frac{\partial u}{\partial q_i}\frac{\partial}{\partial p_i}\left\{\frac{\partial v}{\partial p_i}\frac{\partial w}{\partial q_i}\right\}$$

$$= \frac{\partial u}{\partial q_i}\frac{\partial w}{\partial p_i}\frac{\partial^2 v}{\partial p_i \partial q_i} + \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial q_i}\frac{\partial^2 w}{\partial^2 p_i}$$

$$- \frac{\partial u}{\partial q_i}\frac{\partial w}{\partial q_i}\frac{\partial^2 v}{\partial^2 p_i} - \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i}\frac{\partial^2 w}{\partial p_i \partial q_i}$$

$$- \frac{\partial u}{\partial q_i}\frac{\partial w}{\partial p_i}\frac{\partial^2 v}{\partial q_i^2} - \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial q_i}\frac{\partial^2 w}{\partial p_i \partial q_i}$$

$$+ \frac{\partial u}{\partial q_i}\frac{\partial w}{\partial q_i}\frac{\partial^2 v}{\partial p_i^2} + \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i}\frac{\partial^2 w}{\partial p_i \partial q_i}$$

Question:- Prove that Poisson Brackets are invariant under canonical transfor–

Solution

Let
$$\left.\begin{aligned} Q_m &= Q_m(q_k, p_k) \\ P_m &= P_m(q_k, p_k) \end{aligned}\right\} \longrightarrow (1)$$

be a canonical transformation.

We have to prove that

$$[u, v]_{q,p} = [u, v]_{Q,P}$$

Consider

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_k}\frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k}\frac{\partial v}{\partial q_k}$$

$$= \left[\frac{\partial u}{\partial Q_m}\frac{\partial Q_m}{\partial q_k} + \frac{\partial u}{\partial P_m}\frac{\partial P_m}{\partial q_k}\right]\left[\frac{\partial v}{\partial Q_n}\frac{\partial Q_n}{\partial p_k} + \frac{\partial v}{P_n}\frac{\partial P_n}{\partial p_k}\right]$$

$$- \left[\frac{\partial u}{\partial Q_m}\frac{\partial Q_m}{\partial p_k} + \frac{\partial u}{\partial P_m}\frac{\partial P_m}{\partial p_k}\right]\left[\frac{\partial v}{\partial Q_n}\frac{\partial Q_n}{\partial q_k} + \frac{\partial v}{\partial P_n}\frac{\partial P_n}{\partial q_k}\right]$$

$$= \frac{\partial u}{\partial Q_m}\frac{\partial Q_m}{\partial q_k}\frac{\partial v}{\partial Q_n}\frac{\partial Q_n}{\partial p_k} + \frac{\partial u}{\partial Q_m}\frac{\partial Q_m}{\partial q_k}\frac{\partial v}{\partial P_n}\frac{\partial P_n}{\partial p_k}$$

$$+ \frac{\partial u}{\partial P_m}\frac{\partial P_m}{\partial q_k}\frac{\partial v}{\partial Q_n}\frac{\partial Q_n}{\partial p_k} + \frac{\partial u}{\partial P_m}\frac{\partial P_m}{\partial q_k}\frac{\partial v}{\partial P_n}\frac{\partial P_n}{\partial p_k}$$

$$- \frac{\partial u}{\partial Q_n}\frac{\partial Q_n}{\partial p_k}\frac{\partial v}{\partial Q_n}\frac{\partial Q_n}{\partial q_k} - \frac{\partial u}{\partial Q_m}\frac{\partial Q_m}{\partial p_k}\frac{\partial v}{\partial P_n}\frac{\partial P_n}{\partial q_k}$$

$$- \frac{\partial u}{\partial P_m}\frac{\partial P_m}{\partial p_k}\frac{\partial v}{\partial Q_n}\frac{\partial Q_n}{\partial q_k} - \frac{\partial u}{\partial P_m}\frac{\partial P_m}{\partial p_k}\frac{\partial v}{\partial P_n}\frac{\partial P_n}{\partial q_n}$$

$$= \frac{\partial u}{\partial Q_m}\frac{\partial v}{\partial Q_n}\left\{\frac{\partial Q_m}{\partial q_k}\frac{\partial Q_n}{\partial p_k} - \frac{\partial Q_m}{\partial p_k}\frac{\partial Q_n}{\partial q_k}\right\}$$

$$+\frac{\partial u}{\partial Q_m}\frac{\partial v}{\partial P_n}\left\{\frac{\partial Q_m}{\partial q_k}\frac{\partial P_n}{\partial p_k}-\frac{\partial Q_m}{\partial p_k}\frac{\partial P_n}{\partial q_k}\right\}$$

$$+\frac{\partial u}{\partial P_m}\frac{\partial v}{\partial Q_n}\left\{\frac{\partial P_m}{\partial q_k}\frac{\partial Q_n}{\partial p_k}-\frac{\partial P_m}{\partial q_k}\frac{\partial Q_n}{\partial p_k}\right\}$$

$$+\frac{\partial u}{\partial P_m}\frac{\partial v}{\partial P_n}\left\{\frac{\partial P_m}{\partial q_k}\frac{\partial P_n}{\partial p_k}-\frac{\partial P_n}{\partial p_k}\frac{\partial P_m}{\partial q_k}\right\}$$

$$=\frac{\partial u}{\partial Q_m}\frac{\partial v}{\partial Q_n}[Q_m, Q_n]+\frac{\partial u}{\partial Q_m}\frac{\partial v}{\partial Q_n}[Q_m, P_n]$$

$$+\frac{\partial u}{\partial P_m}\frac{\partial v}{\partial Q_n}[P_m, Q_n]+\frac{\partial u}{\partial P_m}\frac{\partial v}{\partial P_n}[P_m, P_n]$$

$$=\frac{\partial u}{\partial Q_m}\frac{\partial v}{\partial P_n}\delta_{mn}-\frac{\partial u}{\partial P_m}\frac{\partial v}{\partial Q_n}(\delta_{mn})$$

$$=\frac{\partial u}{\partial Q_n}\frac{\partial v}{\partial P_n}-\frac{\partial u}{\partial P_n}\frac{\partial v}{\partial Q_n}$$

$$\Rightarrow [u,v]_{q,p}=[u,v]_{Q,P}$$

Question- Derive Hamiltons equation of motion in terms of Poisson bracket.

OR show that

$$\frac{du}{dt}=[u,H]+\frac{\partial u}{\partial t}, \text{ where}$$

$u=u(p_s, q_s, t)$ & H is Hamilton's function

OR

If $A=A(q,p,t)$ then $\dot{A}=[A,H]+\frac{\partial A}{\partial t}$

Solution As

$$[u, H] = \frac{\partial u}{\partial q_s}\frac{\partial H}{\partial p_s} - \frac{\partial u}{\partial p_s}\frac{\partial H}{\partial q_s} \longrightarrow ①$$

By Hamilton's equation of motion

$$\dot{q}_s = \frac{\partial H}{\partial p_s} \quad \& \quad \dot{p}_s = -\frac{\partial H}{\partial q_s}$$

Put in ① we have

$$[u, H] = \frac{\partial u}{\partial q_s}\dot{q}_s + \frac{\partial u}{\partial p_s}\dot{p}_s$$

$$= \frac{\partial u}{\partial q_s}\frac{dq_s}{dt} + \frac{\partial u}{\partial p_s}\frac{dp_s}{dt} \longrightarrow ②$$

given $u = u(q_s, p_s, t)$

$$\frac{du}{dt} = \frac{\partial u}{\partial q_s}\frac{\partial q_s}{\partial t} + \frac{\partial u}{\partial p_s}\frac{dp_s}{dt} + \frac{\partial u}{\partial t}$$

$$= [u, H] + \frac{\partial u}{\partial t} \quad \text{using ②}$$

This is required result.

⇒ Lie Brackets :- (Lie Algebra)

In Lie Algebra elements are operators i.e.

* $X = a_i(\bar{x})\frac{\partial}{\partial x_i} = a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + \cdots$
* $Y = b_i(\bar{x})\frac{\partial}{\partial x_i} = b_1\frac{\partial}{\partial x_1} + b_2\frac{\partial}{\partial x_2} + \cdots$

* $(X \pm Y) = (a_i \pm b_i)\frac{\partial}{\partial x_i}$

* $XY = \left(a_i(\bar{x})\frac{\partial}{\partial x_i}\right)\left(b_i(\bar{x})\frac{\partial}{\partial x_i}\right)$

$$= a_i \frac{\partial b_j}{\partial x_i}\frac{\partial}{\partial x_j} + a_i b_j \frac{\partial^2}{\partial x_i \partial x_j}$$

* $XY(f) = \left(a_i \frac{\partial}{\partial x_i}\right)\left(b_j \frac{\partial f}{\partial x_j}\right)$

$$= a_i \frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j} + a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

* $YX(f) = \left(b_j \frac{\partial}{\partial x_j}\right)\left(a_i \frac{\partial f}{\partial x_i}\right)$

$$= b_j \frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} + b_j a_i \frac{\partial^2 f}{\partial x_j x_i}$$

$\Rightarrow XY(f) \neq YX(f)$

* In Lie Algebra commutative law does not holds.

Lie Brackets:- Let $X, Y \in$ Lie algebra, Then lie brackets are defined as

$$[X, Y] = XY - YX$$

Question:- Prove Jacobi Identity i.e
$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Solution

Consider L.H.S

$$\Rightarrow \left(X[Y,Z]-[Y,Z]X\right)+\left(Y[Z,X]-[Z,X]Y\right)$$
$$+\left(Z[X,Y]-[X,Y]Z\right)$$
$$=\{X(YZ-ZY)-(YZ-ZY)X\}+\{Y(ZX-XZ)$$
$$-(ZX-XZ)Y\}+\{Z(XY-YX)-(XY-YX)Z\}$$
$$=XY\cancel{Z}-X\cancel{Z}Y-Y\cancel{Z}X+ZYX+Y\cancel{Z}X-Y\cancel{X}Z$$
$$-Z\cancel{X}Y+X\cancel{Z}Y+Z\cancel{X}Y-Z\cancel{Y}X-X\cancel{Y}Z+Y\cancel{X}Z$$
$$=0 \quad = \text{R.H.S}$$

Proved

⇒ **Canonically Contact Transformation:-**

A transformation from $(q_s, p_s) \rightarrow (Q_k, P_k)$; $s,k = 1,2,\ldots,n$ is called canonically contact transformation if $p_s dq_s - P_s dQ_s$ is an exact differential

OR

$p_s dq_s - P_s dQ_s = 0$ is an exact equation

Question:- Show that the following transformation is canonically contact
$$Q = \frac{1}{2}(q^2+p^2) \quad ; \quad P = -\tan^{-1}(q/p)$$

Solution: As we know that the given transformation is canonical if

$$p\,dq - P\,dQ = 0 \longrightarrow ①$$

is an exact differential equation

Now

$$dQ = \frac{1}{2}[2p\,dp + 2q\,dq] = p\,dp + q\,dq$$

Consider

$$p\,dq - P\,dQ = p\,dq + \tan^{-1}(q/p)(p\,dp + q\,dq)$$

$$\Rightarrow p\,dq - P\,dQ = [p + q\tan^{-1}(q/p)]\,dq + [p\tan^{-1}(q/p)]\,dp$$

Let

$$M = p + q\tan^{-1}(q/p) \text{ \& } N = p\tan^{-1}(q/p)$$

$$\frac{\partial M}{\partial p} = 1 + q\cdot\frac{1}{1+q^2/p^2}\left(\frac{-q}{p^2}\right)$$

$$= 1 + \frac{-q^2}{p^2+q^2} = \frac{p^2}{p^2+q^2}$$

&

$$\frac{\partial N}{\partial q} = p\cdot\frac{1}{1+q^2/p^2}\cdot\frac{1}{p} = \frac{p^2}{p^2+q^2}.$$

as

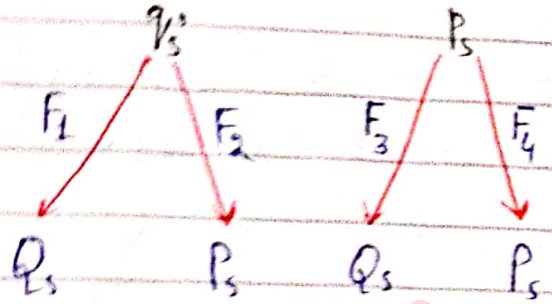
$$\frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$$

$\Rightarrow$ $p\,dq - P\,dQ = 0$ is an exact differential equation. So therefore given transformation is canonical

————————

Generating Function:- There are four types of generating function

(i) $F_1(q_s, Q_s, t)$
(ii) $F_2(q_s, P_s, t)$
(iii) $F_3(p_s, Q_s, t)$
(iv) $F_4(p_s, P_s, t)$



Case 1:- $F = F_1(q_s, Q_s, t)$

$$dF = dF_1 \longrightarrow (1)$$

As we know that

$$dF = p_s\,dq_s - P_s\,dQ_s - (H-K)\,dt \longrightarrow (2)$$

& $$dF_1 = \frac{\partial F_1}{\partial q_s}dq_s + \frac{\partial F_1}{\partial Q_s}dQ_s + \frac{\partial F_1}{\partial t}dt \longrightarrow (3)$$

From (1), (2) and (3) we have

$$p_s\,dq_s - P_s\,dQ_s - (H-K)dt = \frac{\partial F_1}{\partial q_s}dq_s + \frac{\partial F_1}{\partial Q_s}dQ_s + \frac{\partial F_1}{\partial t}dt$$

Equating the differential coefficients

$$\frac{\partial F_1}{\partial q_s} = p_s \longrightarrow (a)$$

$$\frac{\partial F_1}{\partial Q_s} = -P_s \longrightarrow (b)$$

& $$\frac{\partial F}{\partial t} = K - H \longrightarrow (c)$$

If F_1 is indipendent of t explicitly then $K = H$

Case 2:- $F = F_2(q_s, P_s, t)$

$$dF = dF_2 \longrightarrow ①$$

As we know that

$$dF = P_s dq_s - P_s dQ_s - (H-K)dt \longrightarrow ②$$

&

$$dF_2 = \frac{\partial F_2}{\partial q_s} dq_s + \frac{\partial F_2}{\partial P_s} dP_s + \frac{\partial F_2}{\partial t} dt \longrightarrow ③$$

we want to cancel out the terms $P_s dQ_s$. Therefore we put

$$F_2 = F + P_s Q_s$$

$$\Rightarrow dF_2 = dF + P_s dQ_s + Q_s dP_s$$

using ② & ③

$$\frac{\partial F_2}{\partial q_s} dq_s + \frac{\partial F_2}{\partial P_s} dP_s + \frac{\partial F_2}{\partial t} dt = P_s dq_s + Q_s dP_s - (H-$$

Equating differential coefficients

$$\frac{\partial F_2}{\partial q_s} = P_s \longrightarrow (a) \qquad \frac{\partial F_2}{\partial P_s} = Q_s \longrightarrow (b)$$

$$\frac{\partial F_2}{\partial t} = K - H \longrightarrow (c)$$

If F_2 is indipendent of 't' explic
then $K = H$.

Case 3:- $F = F_3(P_s, Q_s, t)$

$$\Rightarrow dF = dF_3 \longrightarrow ①$$

As we know that

$$dF = P_s dq_s - P_s dQ_s - (H-K)dt \longrightarrow ②$$

As $dF_3 = \frac{\partial F_3}{\partial P_s} dP_s + \frac{\partial F_3}{\partial Q_s} dQ_s + \frac{\partial F_3}{\partial t} dt \longrightarrow ③$

we want to cancel out the term $P_s dq_s$, Put $F_3 = F - P_s q_s$

$$dF_3 = dF - P_s dq_s - q_s dP_s$$

using ② and ③

$$\frac{\partial F_3}{\partial P_s} dP_s + \frac{\partial F_3}{\partial Q_s} dQ_s + \frac{\partial F_3}{\partial t} dt = \cancel{P_s dq_s} - P_s dQ_s - (H-K)dt - \cancel{P_s dq_s} - q_s dP_s$$

Equating differential coefficients

$$\frac{\partial F_3}{\partial P_s} = -q_s \longrightarrow ⓐ, \quad \frac{\partial F_3}{\partial Q_s} = -P_s \longrightarrow ⓑ$$

$$\frac{\partial F_3}{\partial t} = K-H \longrightarrow ⓒ$$

If F_3 is independent of "t" explicitly then $K = H$

Case 4:- $F = F_4(P_s, P_s, t)$

$$dF = dF_4 \longrightarrow ①$$

As we know that

$$dF = P_s dq_s - P_s dQ_s - (H-K)\,dt \longrightarrow ②$$

As

$$dF_4 = \frac{\partial F_4}{\partial P_s} dP_s + \frac{\partial F_4}{\partial P_s} dP_s + \frac{\partial F_4}{\partial t} dt$$

we want to cancel out the term $P_s dq_s$ & $P_s dQ_s$. Therefore we put

$$F_4 = F - P_s q_s + P_s Q_s$$

$$\Rightarrow dF_4 = dF - P_s dq_s - q_s dP_s + P_s dQ_s + Q_s dP_s$$

using ② and ③

$$\frac{\partial F_4}{\partial P_s} dP_s + \frac{\partial F_4}{\partial P_s} dP_s + \frac{\partial F_4}{\partial t} dt = P_s dq_s - P_s dQ_s - (H-K)dt$$
$$- P_s dq_s - q_s dP_s + P_s dQ_s + Q_s dP_s$$

Equating coefficients of differentials

$$\frac{\partial F_4}{\partial P_s} = -q_s \longrightarrow ⓐ \quad \frac{\partial F_4}{\partial P_s} = Q_s \longrightarrow ⓑ$$

$$\frac{\partial F_4}{\partial t} = K - H \longrightarrow ⓒ$$

If F_4 is independent of t explicitly. Then

$$K = H.$$

Question- Show that the following transf. is canonically contact.

$$Q = \log\left(\tfrac{1}{q}\sin p\right)$$

$$P = q \cot p$$

Solution Consider

$$pdq - PdQ = pdq - q\cot p \, d\left\{\log \tfrac{1}{q}\sin p\right\}$$

$$= pdq - q\cot p \left\{\frac{1}{\frac{1}{q}\sin p} d\left(\frac{\sin p}{q}\right)\right\}$$

$$= pdq - q\cot p \left\{\frac{q}{\sin p} \cdot \frac{q\cos p\,dp - \sin p\,dq}{q^2}\right\}$$

$$= pdq - q\cot p \left\{\frac{q\cos p\,dp - \sin p\,dq}{q\sin p}\right\}$$

$$pdq - PdQ = pdq + \frac{q^2 \cot p \cos p}{q \sin p} dp + \frac{q \cot p \sin p \, dq}{q \sin p}$$

$$= pdq + \cot^2 p \, dp + \cot p \, dq$$

$$= (p + \cot p) dq + \cot^2 p \, dp$$

Let $M = p + \cot p$ & $N = \cot^2 p$

Question :- Show that the transformation

$$Q = \cot^{-1}\left[\frac{p}{m\omega q}\right] \longrightarrow (a), \quad P = \frac{p^2 + m^2\omega^2 q^2}{2m\omega} \longrightarrow (b)$$

is canonical and also find generating function.

Solution

As we know that given transformation is canonical if

$$pdq - PdQ = 0 \longrightarrow (1)$$

is an exact differential equation.

Now

$$dQ = \frac{-1}{1 + \left(\frac{p}{m\omega q}\right)^2} \cdot \frac{1}{m\omega} \left[\frac{q\,dp - p\,dq}{q^2}\right]$$

$$\Rightarrow dQ = \frac{-1}{\frac{m^2\omega^2 q^2 + p^2}{m^2\omega^2 q^2}} \cdot \frac{1}{m\omega q^2}(q\,dp - p\,dq)$$

$$= \frac{-m\omega}{p^2 + m^2\omega^2 q^2}(q\,dp - p\,dq)$$

Consider $p\,dq - P\,dQ = 0$

$$\Rightarrow p\,dq + \frac{p^2 + m^2\omega^2 q^2}{2m\omega} \cdot \frac{m\omega}{p^2 + m^2\omega^2 q^2}(q\,dp - p\,dq) = 0$$

$$\Rightarrow p\,dq + \frac{q}{2}dp - \frac{p}{2}dq = 0$$

$$\Rightarrow \left(p - \frac{p}{2}\right)dq + \frac{q}{2}dp = 0$$

$$\Rightarrow \frac{p}{2}dq + \frac{q}{2}dp = 0$$

$$\Rightarrow p\,dq + q\,dp = 0$$

Let $M = p$ & $N = q$

$$\frac{\partial M}{\partial p} = 1 \quad \& \quad \frac{\partial N}{\partial q} = 1$$

$$\Rightarrow \frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$$

So given transformation is canonical

Let

$$I_1 = \int p\,dq = p\int dq = pq$$

$$\& \quad I_2 = \int q\,dp = q\int dp = pq$$

$$\therefore F = pq \qquad (F = I_1 = I_2)$$

$F_1 = F_1(q, Q, t)$:- As $F_1 = F$

$\Rightarrow F_1 = pq \longrightarrow$ (A)

from (a) $\cot Q = \frac{p}{m\omega q} \Rightarrow p = m\omega q \cot Q$

Put in (A) implies

$F_1 = m\omega q^2 \cot Q$

$F_2 = F_2(q, P, t)$:- $F_2 = F + PQ$

$\Rightarrow F_2 = pq + PQ = pq + P\cot^{-1}\left(\frac{p}{m\omega q}\right)$

As $2m\omega P = p^2 + m^2\omega^2 q^2$

$\Rightarrow p = \sqrt{2m\omega P - m^2\omega^2 q^2}$

$\therefore F_2 = q\sqrt{2m\omega P - m^2\omega^2 q^2} + P\cot^{-1}\left(\frac{\sqrt{2m\omega P - m^2\omega^2 q^2}}{m\omega q}\right)$

$F_3 = F_3(P, Q, t)$:- As $F_3 = F - pq$

$\Rightarrow F_3 = pq - pq = 0$

$F_4 = F_4(p, P, t)$:- As $F_4 = F + PQ - pq$

$= \cancel{pq} + PQ - \cancel{pq}$

$\Rightarrow F_4 = PQ = P\cot^{-1}\left(\frac{p}{m\omega q}\right) \longrightarrow (*)$

As $p^2 + m^2\omega^2 q^2 = 2m\omega P \Rightarrow m\omega^2 q^2 = 2m\omega P - p^2$

$\Rightarrow q = \frac{\pm\sqrt{2m\omega P - p^2}}{m\omega}$

$\Rightarrow F_4 = P\cot^{-1}\left(\frac{p}{m\omega q}\right)$

where $q = \frac{\pm\sqrt{2m\omega P - p^2}}{m\omega}$

Question:- Write Hamilton equation of motion using generating function F_2 of the transformation defined in previous question

Solution

First we show that transformation is canonical and find F_2

$$F_2 = q\sqrt{2m\omega P - m^2\omega^2 q^2} + P\cot^{-1}\left[\frac{\sqrt{2m\omega P - m^2\omega^2 q^2}}{m\omega q}\right]$$

The corresponding Hamilton equation of motion are $\frac{\partial F_2}{\partial q} = p$ & $\frac{\partial F_2}{\partial P} = Q$

Question:- Show that the transformation

$$Q = \log(1+\sqrt{q}\cos p) \longrightarrow ①$$

& $$P = 2\sqrt{q}\sin p(1+\sqrt{q}\cos p) \longrightarrow ②$$

is canonical. Also find F_3

Solution

As we know that transformation is canonical if

$pdq - PdQ = 0$ is an exact differential equation.

Now

$$dQ = \frac{1}{1+\sqrt{q}\cos p}\left\{0 - \sqrt{q}\sin p\,dp + \frac{1}{2\sqrt{q}}\cos p\,dq\right\}$$

$$= \frac{1}{1+\sqrt{q}\cos p}\left[\frac{-2q\sin p\,dp + \cos p\,dq}{2\sqrt{q}}\right]$$

$$= \frac{\cos p\,dq - 2q\sin p\,dp}{2\sqrt{q}(1+\sqrt{q}\cos p)}$$

Consider

$$pdq - PdQ = pdq - 2\sqrt{q}\sin p(\cancel{1+\sqrt{q}\cos p})\frac{\cos p\,dq - 2q\sin p\,dp}{2\sqrt{q}(\cancel{1+\sqrt{q}\cos p})}$$

$$\Rightarrow pdq - \sin p(\cos p\,dq - 2q\sin p\,dp) = 0$$

$$pdq - \sin p\cos p\,dq + 2q\sin^2 p\,dp = 0$$

$$\Rightarrow (p - \sin p\cos p)dq + 2q\sin^2 p\,dp = 0$$

$M = p - \sin p\cos p$ & $N = 2q\sin^2 p$

$$\frac{\partial M}{\partial p} = 1 - (\cos p\cos p - \sin p\sin p)$$

$$= 1 - (\cos^2 p - \sin^2 p) = \sin^2 p + \sin^2 p$$

$$= 2\sin^2 p$$

& $$\frac{\partial N}{\partial q} = 2\sin^2 p$$

$$\Rightarrow \frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$$

So given transformation is canonical

Let

$$I_1 = \int (p - \sin p\cos p)dq$$

$$= (p - \sin p\cos p)\int dq$$

$$= pq - q\sin p\cos p$$

&

$$I_2 = \int 2q\sin^2 p\,dp = 2q\int \frac{1-\cos 2p}{2}dp$$

$$= q\left[p - \frac{\sin 2p}{2}\right] = q\left[p - \frac{2\sin p\cos p}{2}\right]$$

$$= pq - q\sin p\cos p$$

$$\Rightarrow F = I_1 = I_1 = pq - q\sin p\cos p$$

$$\Rightarrow F = pq - q\sin p\cos p$$

$F_3 = F_3(p, Q, t)$, As $F_3 = F - pq$

$$F_3 = pq - q\sin p\cos p - pq$$

$$= -\frac{q}{2}\sin 2p$$

From ① $e^{Q} = 1 + \sqrt{q}\cos p$

$$\Rightarrow \sqrt{q} = (e^{Q} - 1)/\cos p \Rightarrow q = \left(\frac{e^{Q}-1}{\cos p}\right)^2$$

$$\Rightarrow F_3 = -\frac{1}{2}\left(\frac{e^{Q}-1}{\cos p}\right)^2 2\sin p\cos p$$

$$\Rightarrow F_3 = -\tan p\,(e^{Q}-1)^2$$

Question:- Show that the transformation

$$Q = \frac{\sqrt{2q}}{\sqrt{K}}\cos p, \quad P = \sqrt{2K}\sqrt{q}\sin p$$

is contact. Also find generating function (if possible)

Solution

As we know that the given function is canonically contact if

$$p\,dq - P\,dQ = 0 \longrightarrow ①$$

Now

$$dQ = \frac{-\sqrt{2q}}{\sqrt{K}}\sin p\,dp + \cos p\,\frac{1}{\sqrt{K}}\cdot\frac{1}{2}\,\frac{2dq}{\sqrt{2q}}$$

$$= \frac{-\sqrt{2q}}{\sqrt{K}}\sin p\,dp + \frac{dq}{\sqrt{2qK}}\cos p$$

Therefore ① implies

$$pdq - \sqrt{2Kq}\,\sin p\left[\frac{\cos p\,dq}{\sqrt{2qK}} - \frac{\sqrt{2q}}{\sqrt{K}}\sin p\,dp\right] = 0$$

$$\Rightarrow pdq - \sin p(\cos p\,dq - 2q\sin p\,dp) = 0$$

$$\Rightarrow (p - \sin p\cos p)dq + 2q\sin^2 dp = 0$$

$$M = p - \sin p\cos p \quad \& \quad N = 2q\sin^2 p$$

$$\frac{\partial M}{\partial p} = 2\sin^2 p \quad \& \quad \frac{\partial N}{\partial q} = 2\sin^2 p$$

$$\Rightarrow \frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$$

So given equation is canonically contact

$$F = pq - q\sin p\cos p$$

$\underline{F_1 = F_1(q, Q, t)}$:- As $F_1 = F$

$$\Rightarrow F_1 = pq - q\sin p\cos p \longrightarrow ②$$

from given $Q = \frac{\sqrt{2q}}{\sqrt{k}}\cos p \Rightarrow \cos p = \frac{Q\sqrt{K}}{\sqrt{2q}}$

$$\Rightarrow p = \cos^{-1}\left[\frac{Q\sqrt{K}}{\sqrt{2q}}\right]$$

So equation ② becomes

$$F_1 = \cos^{-1}\left[\frac{Q\sqrt{K}}{\sqrt{2q}}\right]q - q\sin\left[\cos^{-1}\left(\frac{Q\sqrt{K}}{\sqrt{2q}}\right)\right]\cos\left[\cos^{-1}\left(\frac{Q\sqrt{K}}{\sqrt{2q}}\right)\right]$$

$\underline{F_2 = F_2(q, P, t)}$:-

As $F_2 = F + PQ$

$$\Rightarrow F_2 = pq - q\sin p\cos p + PQ$$

$$= pq - q\sin p\cos p + P\left[\frac{\sqrt{2q}}{\sqrt{k}}\cos p\right] \longrightarrow (3)$$

Now $P = \sqrt{2K}\sqrt{q}\sin p$

$$\Rightarrow \sin p = \frac{P}{\sqrt{2K}\sqrt{q}} \Rightarrow p = \sin^{-1}\left(\frac{P}{\sqrt{2K}\sqrt{q}}\right)$$

$$\Rightarrow F_2 = pq - q\sin p\cos p + P\left[\frac{\sqrt{2q}}{\sqrt{k}}\cos p\right]$$

where $p = \sin^{-1}\left[\frac{P}{\sqrt{2kq}}\right]$

<u>$F_3 = F_3(P, Q, t)$</u>:- As $F_3 = F - pq$

$$F_3 = \cancel{pq} - q\sin p\cos p - \cancel{pq}$$

$$F_3 = -q\sin p\cos p \longrightarrow (4)$$

As $Q = \frac{\sqrt{2q}}{\sqrt{k}}\cos p \Rightarrow \sqrt{2q} = \frac{\sqrt{K}\,Q}{\cos p}$

$$\Rightarrow 2q = \left(\frac{\sqrt{K}\,Q}{\cos p}\right)^2 \Rightarrow q = \frac{Q^2 K}{2\cos^2 p}$$

So equ (4) becomes

$$F_3 = -\left(\frac{Q^2 K}{2\cos^2 p}\right)\sin p\cos p$$

<u>$F_4 = F_4(P, P, t)$</u>:- As

$$F_4 = F + PQ - pq$$

$$\Rightarrow F_4 = \cancel{pq} - q\sin p\cos p + PQ - \cancel{pq}$$

$$\Rightarrow F_4 = -q\sin p\cos p + PQ \longrightarrow (5)$$

As $P = \sqrt{2Kq}\sin p \Rightarrow q = \frac{P^2}{2K\sin^2 p}$

So eqn (5) becomes

$$F_4 = \frac{-P^2}{2K\sin^2 p}(\sin p\cos p) + P\left[\frac{2\sqrt{q}}{\sqrt{K}}\cos p\right]$$

where $q = \frac{P^2}{2K\sin^2 p}$

⇒ Gibbs Appel Equation of Motion and its Applications:-

Let us consider a system of N particles whose configuration is specified by n generalized coordinates. Let r_i be the position vector of ith particle i.e

$$r_i = r_i(q_s, t) \qquad \begin{matrix} i = 1,2,\cdots,N \\ s = 1,2,\cdots,n \end{matrix}$$

$$\Rightarrow \dot{r}_i = \frac{\partial r_i}{\partial q_s}\dot{q}_s + \frac{\partial r_i}{\partial t}$$

$$\Rightarrow \ddot{r}_i = \frac{\partial r_i}{\partial q_s}\ddot{q}_s + \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_s}\right)\dot{q}_s + \frac{d}{dt}\left(\frac{\partial r_i}{\partial t}\right)$$

$$\frac{\partial \dot{r}_i}{\partial \dot{q}_s} = \frac{\partial r_i}{\partial q_s}$$

$$\& \quad \frac{\partial \ddot{r}_i}{\partial \ddot{q}_s} = \frac{\partial r_i}{\partial q_s} \longrightarrow (1)$$

Now $(m_i \ddot{r}_i - F_i)\delta r_i = 0$

$$\Rightarrow (m_i \ddot{r}_i - F_i)\frac{\partial r_i}{\partial q_s}\delta q_s = 0$$

$$\Rightarrow \left(m_i \ddot{r}_i \frac{\partial r_i}{\partial q_s} - F_i \frac{\partial r_i}{\partial q_s}\right)\delta q_s = 0$$

$$\Rightarrow \left(m_i \ddot{r}_i \frac{\partial r_i}{\partial q_s} - Q_s\right)\delta q_s = 0$$

where $Q_s = F_i \frac{\partial r_i}{\partial q_s}$

and $Q = \frac{\partial r_i}{\partial q_s}$ for conservative Holonomic system

For a holonomic system

$$\delta q_s \neq 0$$

$$\Rightarrow m_i \ddot{r}_i \frac{\partial r_i}{\partial q_s} - Q_s = 0$$

By using equ ①

$$m_i \ddot{r}_i \frac{\partial \dddot{r}_i}{\partial \ddot{q}_s} - Q_s = 0$$

$$m_i \ddot{r}_i \frac{\partial \ddot{r}_i}{\partial \ddot{q}_s} = Q_s \longrightarrow ②$$

Gibbs defined a function G as

$$G = \frac{1}{2} m_i \ddot{r}_i^2$$

$$\frac{\partial G}{\partial \ddot{q}_s} = m_i \ddot{r}_i \frac{\partial \ddot{r}_i}{\partial \ddot{q}_s}$$

Therefore eqn ① becomes

$$\frac{\partial G}{\partial \ddot{q}_s} = \frac{\partial V}{\partial q_s}$$

which is known as Gibbs equation.