ADVANCED ANALYSIS And MEASURE THEORY

S.No. Contents P. No

Advance Analysis

Measure Theory

ADVANCE ANALYSIS

"SET THEORY" CH#1

⇒ Equivalent Set:-

Two sets A and B (May be finite or infinite) are said to be equivallent if there exist a function $f: A \to B$ which is one-one and onto (i.e bijective). We write it as $A \sim B$ and read it as "A is equivalent to B"

Example:- $N = \{1, 2, 3, \ldots\}$, $E = \{2, 4, 6, \ldots\}$

Define $f: N \longrightarrow E$ by $f(n) = 2n$

f is 1-1:- Let $f(n_1) = f(n_2)$

$\Rightarrow 2n_1 = 2n_2 \Rightarrow n_1 = n_2$

$\Rightarrow$ f is one-one

f is onto:- for each $2n \in E$ there exist an element $n \in N$ s.t

$f(n) = 2n$

$\Rightarrow$ f is onto.

$\therefore$ f is bijective $\Rightarrow N \sim E$

Exampl 2:- $A = [1, 2]$ and $B = [3, 5]$

Define $f: A \longrightarrow B$ by $f(x) = 2x + 1$

f is 1-1:- Let $f(x_1) = f(x_2)$

$\Rightarrow 2x_1 + 1 = 2x_2 + 1 \Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2 \Rightarrow$ f is 1-1

f is onto:- for each $2x + 1 \in B$ there exist $x \in A$ s.t $f(x) = 2x + 1$

So f is onto

$\therefore$ f is bijective $\Rightarrow A \sim B$

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⇒ Infinite Set :-

A set is said to be infinite set if it is equivalent to any of its proper subset e.g $N = \{1, 2, 3, \dots\}$
$M = \{2, 4, 6, \dots\}$

Clearly $M \subset N$ And Also $N \sim M$

$\Rightarrow$ N is an infinite set.

⇒ Denumerable Set :-

A set is said to be Denumerable if it is equivalent to the set of natural numbers.

Example 1:- $M = \{3, 6, 9, 12, \dots\}$, $N = \{1, 2, 3, 4, \dots\}$

Define $f : N \longrightarrow M$ by $f(n) = 3n$

Since f is one - one and onto (clearly)

So $N \sim M$

$\therefore$ M is denumerable set.

Secondly we define this function as $f : M \longrightarrow N$ by $f(n) = \frac{n}{3}$

Which is one - one and onto.

یعنی کہ اگر 2 sets کو Equivalent شو کرنا ہے تو یا پہلے سے دوسرے سیٹ تک فنکشن Define کر دیا یا پھر دوسرے سیٹ سے پہلے تک جیسے آسانی سمجھو کوئی فرق نہیں پڑتا۔ یعنی کہ اگر $A \sim B$ تو Definitly $B \sim A$ ہو گا

Example 2:- $N = \{1, 2, 3, 4, \dots\}$
$K = \{-3, -6, -9, \dots\}$

Define $f : N \rightarrow K$ by $f(n) = -3n$

Since f is 1-1 and onto

So $N \sim K$

$\Rightarrow$ K is denumerable.

⇒ Countable Set :-

A set is said to be countable if it is either finite or denumerable.

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Sets

- Finite set
  - Countable set
- Infinite set
  - Denumerable → Countable set
  - Non-denumerable → Un-countable set

⇒ Uncountable Set:-

A set is said to be uncountable if it is neither finite nor denumerable i.e R, $[0,1]$

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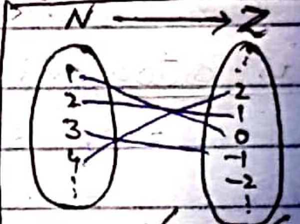
Question Show that set Z of integer is Denumerable.

Solution

Define $f: N \longrightarrow Z$

by $f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{1-x}{2} & \text{if } x \text{ is odd} \end{cases}$



یعنی N کے odd terms کو -ve پر اور Even Terms کو +ve پر Map کرتا (ہے)

f is 1-1:- Let $f(x_1) = f(x_2)$

If x_1 and x_2 are even

then $\frac{x_1}{2} = \frac{x_2}{2} \Rightarrow x_1 = x_2$

If x_1 and x_2 are odd Then

$\frac{1-x_1}{2} = \frac{1-x_2}{2} \Rightarrow x_1 = x_2$

$\Rightarrow f$ is 1-1

f is onto:- Since every $\frac{x}{2}, \frac{1-x}{2} \in Z$ is the image of some $x \in N$ under f.

Therefore f is onto.

$\Rightarrow f$ is bijective $\Rightarrow N \sim Z$

$\Rightarrow Z$ is denumerable.

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Theorem:- Every infinite sequence of distinct elements is denumerable.

Proof:-

Let $S = \{a_1, a_2, a_3, \ldots\}$ be an infinite sequence of distinct elements i.e $a_i \neq a_j \ \forall \ i \neq j$

To prove S is denumerable

For This we have to prove $N \sim S$

Define $f : N \to S$ by $f(n) = a_n$

f is 1-1:- Let $f(n_1) = f(n_2)$

$\Rightarrow a_{n_1} = a_{n_2} \quad \Rightarrow n_1 = n_2 \quad [\because$ elements are distinct.

f is onto:- Since every $a_n \in S$ There exist $n \in N$ s.t $f(n) = a_n$

$\Rightarrow f$ is on-to

Hence f is bijective $\Rightarrow N \sim S$

$\Rightarrow S$ is denumerable

$\Rightarrow$ Every infinite sequence of distinct element is denumerable.

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Theorem:- Prove that $\mathbb{N} \times \mathbb{N}$ is denumerable.

Proof:-

As $N = \{1, 2, 3, 4, \ldots\}$

$N \times N = \{1, 2, 3, 4, \ldots\} \times \{1, 2, 3, 4, \ldots\}$

$$= \begin{cases} (1,1), (1,2), (1,3), \ldots \\ (2,1), (2,2), (2,3), \ldots \\ (3,1), (3,2), (3,3), \ldots \\ \ldots \\ \ldots \end{cases}$$

$\boxed{A \times B = \{(a, b) : a \in A \ \& \ b \in B\}}$

It can be written as

$N \times N = \{(1,1), (1,2), (2,1), (3,1), (2,2), (1,3), \ldots\}$

Rough:- $N \times N = \{ (1,1), (1,2), (1,3), (1,4), \ldots$
$(2,1), (2,2), (2,3), (2,4), \ldots$
$(3,1), (3,2), (3,3), (3,4), \ldots$
$(4,1), (4,2), (4,3), (4,4), \ldots$
$(5,1), (5,2), (5,3), (5,4), \ldots$
$\ldots\ldots\ldots$

***

Theorem:- Let $M = \{0\} \cup N$ show that $M \times M$ is denumerable.

Proof:

Every $n \in N$ can be uniquely written as $n = 2^{r}(2s+1)$, where $r, s \in M$

$f: N \to M \times M$
$f(n) = (?, ?)$
$5 = 2^{0}(2(2)+1)$

5 کے Correspond 2 یونیک نمبرز ہے
$5 \to (0,2)$ So (Whole) میں
$16 = 2^{4}(2(0)+1)$
$16 \to (4,0)$ کو Map کرے گا
اگر 2 Order Pair برابر ہوں تو
ان کی correspon ویلیوز بھی برابر ہیں

Define $f: N \longrightarrow M \times M$ by
$f(n) = (r, s)$

f is 1-1:- Let $f(n_1) = f(n_2)$

$(r_1, s_1) = (r_2, s_2)$

$\Rightarrow r_1 = r_2$ & $s_1 = s_2$

$\Rightarrow 2^{r_1} = 2^{r_2}$ & $2s_1 = 2s_2$

then $2^{r_1}(2s_1+1) = 2^{r_2}(2s_2+1)$

$\Rightarrow n_1 = n_2$ $\Rightarrow f$ is 1-1

f is onto:- Since for every $(r, s) \in M \times M$ there exist a natural number n s.t $f(n) = (r, s)$
So f is onto
Therefore f is bijective
$\Rightarrow N \sim M \times M$
$\Rightarrow M \times M$ is denumerable.

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Theorem: If A and B are denumerable sets then $A\times B$ is denumerable.

Proof:

Let $A=\{a_1, a_2, a_3, \dots\}$

$B=\{b_1, b_2, b_3, \dots\}$

Clearly A and B are denumerable sets.

To prove $A\times B$ is denumerable.

$A\times B=\{a_1, a_2, a_3, \dots\}\times\{b_1, b_2, b_3, \dots\}$

$$=\begin{cases}(a_1,b_1),(a_1,b_2),(a_1,b_3),\dots\\(a_2,b_1),(a_2,b_2),(a_2,b_3),\dots\\(a_3,b_1),(a_3,b_2),(a_3,b_3),\dots\\\dots\dots\dots\dots\end{cases}$$

Which can be written as

$A\times B=\{(a_1,b_1),(a_1,b_2),(a_2,b_1),(a_3,b_1),(a_2,b_2),$
$(a_1,b_3),\dots\}$

Which is an infinite sequence of distinct elements and is denumerable.

Hence $A\times B$ is denumerable.

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Question: If $A=\{1,3,5,7,\dots\}$

$B=\{2,4,6,8,\dots\}$

Show that $A\times B$ is denumerable.

Solution:

Define $f: N\longrightarrow A$ by

$f(n)=2n-1$

f is 1-1:- Let $f(n_1)=f(n_2)$

$\Rightarrow 2n_1-1=2n_2-1$

$\Rightarrow 2n_1=2n_2 \quad \Rightarrow n_1=n_2$

$\Rightarrow f$ is 1-1

f is onto:- Since every $2n-1 \in A$ is the image of some $n \in N$ so f is onto

$\Rightarrow f$ is bijective

$\Rightarrow N \sim A$

$\Rightarrow A$ is denumerable.

Now define

$$g : N \longrightarrow B \text{ by } g(n) = 2n$$

g is 1-1:- Let $g(n_1) = g(n_2)$

$\Rightarrow 2n_1 = 2n_2 \quad \Rightarrow n_1 = n_2$

$\Rightarrow g$ is $1-1$

g is onto:- Since every $2n \in B$ there exist $n \in N$ s.t $g(n) = 2n$. So g is onto

$\Rightarrow g$ is bijective

$\Rightarrow N \sim B \quad \Rightarrow B$ is denumerable.

Now A and B are denumerable. Then $A \times B$ is also denumerable.

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Theorem Let $\{A_i\}$ be a family of denumerable sets which are disjoint pairwise. Then $\bigcup_i A_i$ is also denumerable. OR

The union of pairwise disjoint denumerable sets is also denumerable.

Proof

Let $A_i = \{a_{i1}, a_{i2}, a_{i3}, a_{i4}, \ldots\}$

To prove $\bigcup_i A_i$ is denumerable.

Define $f : \bigcup_i A_i \longrightarrow N \times N$ by

$$f(a_{ij}) = (i, j)$$

f is 1-1:- Let $f(a_{i_1 j_1}) = f(a_{i_2 j_2})$

$\Rightarrow (i_1, j_1) = (i_2, j_2)$

$f : \bigcup_i A_i \longrightarrow N$ Suppose

$f(a_{13}) = 1+3 = 4$

اب 1-1 نہیں رہا کیونکہ

$f(a_{22}) = 2+2 = 4$

$f : \bigcup_i A_i \to N \times N$ لیا

اور a_{ij} کو Define کیا

$\Rightarrow i_1 = i_2$ & $j_1 = j_2$

So $a_{i_1 j_1} = a_{i_2 j_2}$

f is onto:- Since for every $(i,j) \in N \times N$ There exist $a_{ij} \in \bigcup_i A_i$

$\Rightarrow f$ is onto

$\Rightarrow f$ is bijective

So

$\bigcup_i A_i \sim N \times N \sim N$

$\Rightarrow \bigcup_i A_i \sim N$

$\Rightarrow \bigcup_i A_i$ is denumerable.

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Note:- Union of countable sets is countable.

Question Show that set Q is countable.

Solution

First we show that Q^+ is countable. Define

$f: Q^+ \longrightarrow N \times N$ by

$f(p/q) = (p,q)$

f is 1-1:- Let $f(p_1/q_1) = f(p_2/q_2)$

$\Rightarrow (p_1, q_1) = (p_2, q_2)$

$\Rightarrow p_1 = p_2$ & $q_1 = q_2$

$\Rightarrow p_1/q_1 = p_2/q_2$

$\Rightarrow f$ is 1-1

f is onto:- since every $(p,q) \in N \times N$

is image of some $p/q \in Q^+$ under f
$\Rightarrow f$ is onto.
Hence f is bijective
$\Rightarrow Q^+ \sim N \times N$
As $N \times N$ is countable $\Rightarrow Q^+$ is countable.
Now
Define $g: Q^- \to Q^+$ by
$$g(-p/q) = p/q$$
g is 1-1:- Let $g(-p_1/q_1) = g(-p_2/q_2)$
$\Rightarrow p_1/q_1 = p_2/q_2 \Rightarrow -p_1/q_1 = -p_2/q_2$
$\Rightarrow g$ is 1-1
g is onto:- Since every $p/q \in Q^+$ is the image of some $-p/q \in Q^-$ under g
Therefore g is onto
$\Rightarrow g$ is bijective
$\Rightarrow Q^- \sim Q^+$
$\Rightarrow Q^-$ is countable ($\because Q^+$ is countable
$\rightarrow Q$ is countable being the union of countable sets.

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Theorem:- Prove that the set of all points in the plane with rational coordinate is denumerable.

Proof:

$S = \{P \in \mathbb{R} \times \mathbb{R} :$ coordinate of P are rational $\}$

Define $f: S \to Q \times Q$ by

$f: S \to N \times N$ is not because $f(-2,3) = (-2,3) \notin N$

$f(P) = (x, y)$ where $x, y \in Q$

f is 1-1:- $f(P_1) = f(P_2)$

$\Rightarrow (x_1, y_1) = (x_2, y_2)$

$\Rightarrow x_1 = x_2$ & $y_1 = y_2$

$\Rightarrow P_1 = P_2$

f is onto:- Since every $(x, y) \in Q \times Q$ is the image of some $P \in S$ under f. Hence f is bijective.

$\Rightarrow S \sim Q \times Q \sim N$

$\Rightarrow S$ is denumerable.

Question Prove that $A \sim A \times \{1\}$

Proof:

Define $f : A \longrightarrow A \times \{1\}$

by $f(x) = (x, 1) \; \forall \; x \in A$

f is 1-1:- Let $f(x) = f(y)$

$\Rightarrow (x, 1) = (y, 1)$

$\Rightarrow x = y \quad \Rightarrow f$ is 1-1

f is onto:-

Since every $(x, 1)$ is the image of some $x \in A$ under f

$\Rightarrow f$ is onto

Hence f is bijective

So $A \sim A \times \{1\}$

——— ★★★ ———

V. Imp

Theorem.- Prove that the set $\mathbb{R}$ of reals is infinite.

Proof:

To prove $\mathbb{R}$ is infinite we have to prove that $\mathbb{R}$ is equivalent to any of its proper subset.

Consider $A = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \subset \mathbb{R}$

Define $f : A \to \mathbb{R}$ by

$$f(x) = \tan x$$

$\mathbb{N}$ is countable and infinite. So countable sets may be infinite

$f\left(-\frac{\pi}{2}\right) = \tan\left(-\frac{\pi}{2}\right) = -\infty$

$f(\pi/2) = \tan\left(\frac{\pi}{2}\right) = \infty$

[illegible] onto [illegible] 1-1 [illegible] End Point [illegible] Correspond [illegible] Pts [illegible]

f is 1-1:- Let $f(x) = f(y)$

$$\Rightarrow \tan x = \tan y$$

$$\Rightarrow \tan^{-1}(\tan x) = \tan^{-1}(\tan y)$$

$$\Rightarrow x = y$$

$$\Rightarrow f \text{ is } 1-1$$

f is onto:- Since every $\tan x \in \mathbb{R}$ is the image of some $x \in A$ under f

$$\Rightarrow f \text{ is onto}$$

Hence f is bijective

$\Rightarrow A \sim \mathbb{R}$ so $\mathbb{R}$ is infinite

————

Question: Show that the set of resiprocals of natural numbers is denumerable.

Proof:

Let $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$

Define $f : \mathbb{N} \to S$ by $f(n) = \frac{1}{n}$

f is 1-1:- Let $f(n_1) = f(n_2)$

$$\Rightarrow \frac{1}{n_1} = \frac{1}{n_2} \Rightarrow n_1 = n_2$$

$$\Rightarrow f \text{ is } 1-1$$

f is onto:- Since for $\frac{1}{n} \in S$ $\exists$ a $n \in N$

s.t $f(n) = \frac{1}{n}$

$\Rightarrow f$ is onto

Hence f is bijective

$\Rightarrow S \sim N$

So S is denumerable.

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Question Consider the cocentric circles.

$C_1 = \{(x,y) : x^2+y^2 = a^2\}$

$C_2 = \{(x,y) : x^2+y^2 = b^2\}$, $a<b$

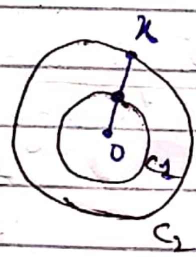
Show that $C_1 \sim C_2$

Proof

Consider the function

$f : C_2 \longrightarrow C_1$ defined by

$f(x)$ = Point of intersection of the line joing x to O and C_1



Clearly f is both 1-1 and onto.

So $C_2 \sim C_1$

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$\Rightarrow$ Charactristic Function:-

A function

$f_A : X \longrightarrow \{0,1\}$ defined by

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \text{ where } A \in 2^X \end{cases}$$

where $A \subseteq X$

Theorem:- Let X be a non-empty set and $C(X)$ be the class of characteristic functions defined on X. Then prove $2^X \sim C(X)$

Proof)

Let A be any subset of X i.e $A \in 2^X$

Define $f: 2^X \longrightarrow C(X)$ by

$$f(A) = \varphi_A$$

> 2^X Mean power set of X it is Also denoted $P(X)$
>
> [illegible] Subset $\subseteq X$ ہیں
>
> [illegible] Charac Funct(s) ہیں

f is 1-1:- Let

$$f(A_1) = f(A_2)$$

$$\Rightarrow \varphi_{A_1} = \varphi_{A_2}$$

$$\Rightarrow \varphi_{A_1}(x) = \varphi_{A_2}(x) \quad \forall x \in X \longrightarrow ①$$

Let $x \in A_1 \Rightarrow \varphi_{A_1}(x) = 1$ ($\because$ by def. of ch. function

Also $\varphi_{A_2}(x) = \varphi_{A_1}(x) = 1$ using ①

$$\Rightarrow \varphi_{A_2}(x) = 1$$

$\Rightarrow x \in A_2$ $\because$ by definition of characteristic function

$$\Rightarrow A_1 \subseteq A_2$$

Similarly $A_2 \subseteq A_1$

$$\Rightarrow A_1 = A_2$$

$$\Rightarrow f \text{ is } 1-1$$

f is onto:- Since every $\varphi_A \in C(X)$ is the image of some $A \in 2^X$ under f

$\Rightarrow f$ is onto. Hence f is bijective

$$\Rightarrow 2^X \sim C(X)$$

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⇒ Algebraic Numbers:-

Solution of polynomials is called an Algebraic Number.

e.g:- $x^2 - 9 = 0$ [This is a polynomial of 2nd degree]

$\Rightarrow x = \pm 3$

Here ± 3 is Algebraic Number.

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Question Prove that the set A of all Algebraic Numbers is denumerable.

Proof)

Let $A_i = \{x : x \text{ is solution of } P_i(x) = 0\}$

$$A = \bigcup_{i \in \mathbb{N}} A_i$$

Since polynomial of degree "n" has at the most 'n' roots. Therefore each A_i is finite and Therefore each A_i is countable.

As each A_i is countable and countable union of countable sets is countable $\Rightarrow A$ is countable

As A is not finite

$\Rightarrow A$ is denumerable

i.e The set of all Algebraic Numbers is denumerable.

———***

Rough

$A_1 = \{a_1\} : a_1$ is the solution of $P_1(x) = 0$ i.e Polyno of 1st deg

$A_2 = \{a_2, a_3\} : a_2, a_3$ are " " " $P_2(x) = 0$ " " " 2nd deg

$A_3 = \{a_4, a_5, a_6\} : a_4, a_5, a_6$ are " " " $P_3(x) = 0$ " " " 3rd deg

* تمام درجوں کی polynomials کے solution کے تمام sets جمع کریں تو A بنے گا۔

Question:- Show that Q' (Set of irrational numbers) is non denumerable.

Solution

Suppose Q' is denumerable

Also Q is denumerable

$\Rightarrow Q \cup Q'$ is denumerable ($\because$ being union of denumerable sets)

$\Rightarrow \mathbb{R}$ is denumerable i.e countable

A contradiction. Because The set of real numbers is uncountable.

So our supposition is wrong

Hence Q' is not denumerable.

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Imp

Theorem:- Every subset of a denumerable set is finite or denumerable.

Proof

Let A be a denumerable set and B be a subset of A.

If $B = \phi$ Then B is finite, Because empty set is finite.

If $B \neq \phi$ Then

Let a_{n_1} be the first element of B

a_{n_2} " " 2nd " " "

a_{n_3} " " 3rd " " "

and so on

Thus B can be written as

$B = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$.

Now if B is bounded Then B is finite.

If B is not Bounded Then

$B = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\} \sim \{n_1, n_2, n_3, \ldots\} \sim N$

$\Rightarrow B \sim N \quad \Rightarrow B$ is denumerable.

————————

Imp

Theorem:- Every Infinite set Contains a Denumerable subset.

Proof:-

Let A be an infinite set
Define a mapping $f: 2^A \longrightarrow A$ by

$$f(A) = a_1 \text{ for some } a_1 \in A$$
$$f(A \setminus \{a_1\}) = a_2 \quad " \quad " \quad a_2 \in A \setminus \{a_1\}$$
$$f(A \setminus \{a_1, a_2\}) = a_3 \quad " \quad " \quad a_3 \in A \setminus \{a_1, a_2\}$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$f(A \setminus \{a_1, a_2, a_3, \dots, a_{n-1}\}) = a_n \text{ for some } a_n \in A \setminus \{a_1 \dots a_{n-1}\}$$

Then the set $\{a_1, a_2, a_3, \dots, a_n, \dots\}$ is a subset of A and is denumerable.

Hence every infinite set contains a denumerable subset.

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V. Imp

Theorem:- Prove that $[0,1]$ is non denumerable.

Proof:-

Suppose that $A = [0,1]$ is denumerable (i.e countable)

Then $A = \{x_1, x_2, x_3, \dots\}$ i.e A can be written in the form of an infinite sequence of distinct elements.

Also each element of A can be written in the form of an infinite decimal as follows.

$$x_1 = 0.a_{11}\, a_{12}\, a_{13} \dots\dots a_{1n} \dots$$
$$x_2 = 0.a_{21}\, a_{22}\, a_{23} \dots\dots a_{2n} \dots$$
$$x_3 = 0.a_{31}\, a_{32}\, a_{33} \dots\dots a_{3n} \dots$$
$$\dots \dots \dots \dots$$

$x_n = 0.a_{n1}\,a_{n2}\,a_{n3}\ldots\ldots a_{nn}\ldots\ldots$

where $a_{ij} \in \{0,1,2,3,\ldots,9\}$ and each decimal contains infinite number of non-zero elements. Here we write 1 as $0.99987543776\ldots\ldots$ and $\frac{1}{2} = 0.5 = 0.49999\ldots\ldots$

Now construct a real number $y \in A$ which implies $y = 0.b_1 b_2 b_3\ldots\ldots b_n\ldots\ldots$ ~~Sad~~ s.t

$b_1 \neq a_{11}$ and $b_1 \neq 0$

$b_2 \neq a_{22}$ and $b_2 \neq 0$

$b_3 \neq a_{33}$ and $b_3 \neq 0$ and so on.

Now

$b_1 \neq a_{11} \Rightarrow y \neq x_1$

$b_2 \neq a_{22} \Rightarrow y \neq x_2$

$b_3 \neq a_{33} \Rightarrow y \neq x_3$ and so on

$\ldots\ldots b_n \neq a_{nn} \Rightarrow y \neq x_n$

$\Rightarrow y \notin A \;\; \forall n$ Which contradicts the fact that $y \in A$. Thus our assumption that A is denumerable is wrong.

Hence $A = [0,1]$ is non denumerable.

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Theorem Prove that $A = [a,b]$ is non-denumerable OR show that $[a,b] \sim [0,1]$

Proof:

First we show that $[a,b] \sim [0,1]$

Define $f : [0,1] \longrightarrow [a,b]$ by $f(x) = a + (b-a)x$

پوری Interval Length $(b-a)$ کو 1 سے چھوٹے [illegible] سے ضرب [illegible] پوائنٹ [illegible]

f is 1-1:- Let $f(x) = f(y)$

$\Rightarrow a + (b-a)x = a + (b-a)y$

$\Rightarrow (b-a)x = (b-a)y$ $\because$ by cancel law

$\Rightarrow \quad x = y$

<u>f is onto:-</u> Since every $a+(b-a)x \in [a,b]$ is the image of some $x \in [0,1]$

$\Rightarrow f$ is onto

So $[0,1] \sim [a,b]$

Since $[0,1]$ is non denumerable, so $[a,b]$ is also non-denumerable.

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Question Show that the set of real numbers $\mathbb{R}$ is non-denumerable.

Solution

First we show that the subset $[0,1]$ of $\mathbb{R}$ is non-denumerable. Then As $[0,1]$ is non denumerable, so $\mathbb{R}$ is non denumerable $\because \mathbb{R} \supset [0,1]$

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Theorem:- Show that $(0,1)$ is non-denumerable.

OR show that $(0,1) \sim [0,1]$

Proof

We can write

$[0,1] = \{0, \cancel{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots\} \cup A$

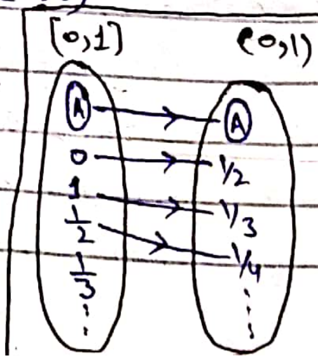
where $A = [0,1] \setminus \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$

Also $(0,1) = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} \cup A$

where $A = (0,1) \setminus \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

Define a mapping $f:[0,1] \to (0,1)$ by

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+2} & \text{if } x = \frac{1}{n} \\ & n = 1, 2, 3, \dots \\ x & \text{if } x \in A \end{cases}$$



Clearly by diagram f is both one-one and onto $\Rightarrow f$ is bijective

So $(0,1) \sim [0,1]$

As $[0,1]$ is non-denumerable

$\Rightarrow (0,1)$ is also non denumerable.

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Question Show that any open interval (a,b) is non-denumerable.

Proof

We have to prove

$$(0,1) \sim (a,b)$$

Define $f:(0,1) \longrightarrow (a,b)$ by

$$f(x) = a+(b-a)x$$

f is 1–1:- Let $f(x) = f(y)$

$$\Rightarrow a+(b-a)x = a+(b-a)y$$

$$\Rightarrow (b-a)x = (b-a)y$$

$$\Rightarrow x = y$$

$$\Rightarrow f \text{ is } 1-1$$

f is onto:- Since every $a+(b-a)x \in (a,b)$ is the image of some $x \in (0,1)$

$$\Rightarrow f \text{ is onto}$$

Hence f is bijective.

So $(0,1) \sim (a,b)$

As $(0,1)$ is non denumerable

So (a,b) is also non denumerable.

——————*

Question:- Show that $(0,1]$ is non-denumerable.

Proof

We can write

$$[0,1] = \{0,1,\tfrac{1}{2},\tfrac{1}{3},\ldots\} \cup A$$

where $A = [0,1] \setminus \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$

Also $(0,1] = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} \cup A$

where $A = (0,1] \setminus \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

Define a mapping $f: [0,1] \to (0,1]$

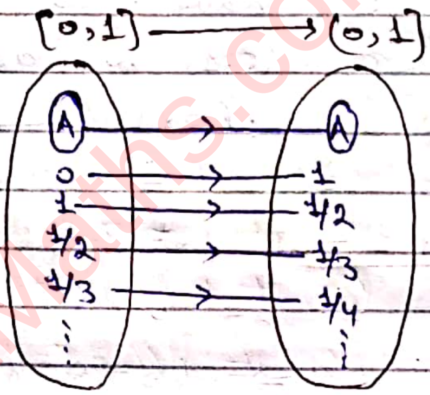
by

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n}, \; n = 1, 2, 3, \dots \\ x & \text{if } x \in A \end{cases}$$

From figure it is clear that f is both 1-1 and onto.

$\therefore f$ is bijective

$\Rightarrow (0,1] \sim [0,1]$



As $[0,1]$ is non denumerable

$\Rightarrow (0,1]$ is also non-denumerable.

*——— ** ———*

Question Show that $[0,1)$ is non-denumerable

OR Show that $[0,1) \sim [0,1]$

Proof

We can write

$[0,1] = \{0, 1, 1/2, 1/3, \dots\} \cup A$

where $A = [0,1] \setminus \{0, 1, 1/2, 1/3, \dots\}$

Also $[0,1) = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} \cup A$

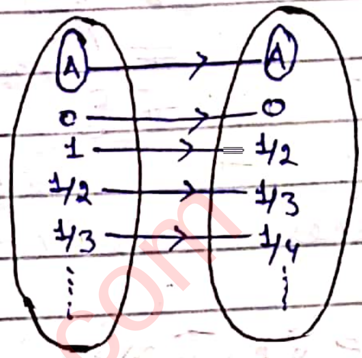
where $A = [0,1) \setminus \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

Define a mapping $f: [0,1] \to [0,1]$

by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n}, \quad n = 1, 2, 3, \ldots \\ x & \text{if } x \in A \end{cases}$$

$[0,1] \longrightarrow [0,1)$



Then obviously f is both one-one and onto

$\Rightarrow f$ is bijective

$\Rightarrow [0,1) \sim [0,1]$

As $[0,1]$ is non-denu-- so $[0,1]$ is also non-denumerable.

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⇒ Cordinal Numbers:-

Cordinal Numbers or cordinals are a generalized kind of numbers used to denote the size of set.

Cantor was the Mathematician who introduced cordinal numbers in 1974.

*** Finite Cordinal Numbers:-**

If A is a finite set then cordinality of A is number of elements in A. i.e $A = \{a, b, c\}$

$\#(A) = 3$ [read as Cordinality of A]

*** Infinite Cordinals:-**

The cordinal numbers of infinite sets are called infinite cordinals.

Note:- (i) Cordinality of denumerable set is denoted by "a" or $\aleph_0$ (alpha-Null) and the cordinality of non-denumerable (uncountable) sets is denoted by "c" or $\mathfrak{c} = \aleph$

Examples:

$$\#(\mathbb{N}) = a = \aleph_0$$
$$\#(\mathbb{Z}) = a = \aleph_0$$
$$\#(\mathbb{Q}) = a = \aleph_0$$
$$\#[0,1] = c = \aleph$$

ii) If two sets are equivallent then their cordinality is same.

⇒ **Definition:-** If α and β be cordinal numbers and let A, B be two disjoint sets such that $\alpha = \#(A)$ and $\beta = \#(B)$ Then

(i) $\alpha + \beta = \#(A \cup B) = \#(A) + \#(B)$

(ii) $\alpha \beta = \#(A \times B) = [\#(A)][\#(B)]$

Examples: (i) Let $A = \{1, 2, 3, 4\}$ & $B = \{a, b, c\}$

$\#(A) = 4$, $\#(B) = 3$

$A \cup B = \{1, 2, 3, 4, a, b, c\}$

$\#(A \cup B) = 7$
$= 4 + 3 = \#(A) + \#(B)$

$\Rightarrow \#(A) + \#(B) = 4 + 3 = \#(A \cup B)$

(ii)

$A \times B = \{(1,a), (2,a), (3,a), (4,a), (1,b), (2,b), (3,b), (4,b), (1,c), (2,c), (3,c), (4,c)\}$

$\#(A \times B) = 12$
$= 4 \times 3 = [\#(A)][\#(B)]$

★★

Theorem:- For any cordinals α, β and γ of disjoint sets. Show that

(i) $\alpha + \beta = \beta + \alpha$ (ii) $\alpha\beta = \beta\alpha$

(iii) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ (iv) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$

(v) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

Proof Let A, B, C be three disjoint sets. such that

$\#(A) = \alpha$, $\#(B) = \beta$ and $\#(C) = \gamma$

(i) $\alpha + \beta = \#(A \cup B)$

$= \#(B \cup A)$ $\because$ union of sets is commutative

$= \#(B) + \#(A)$

$\Rightarrow \alpha + \beta = \beta + \alpha$

(ii) $\alpha\beta = \#(A \times B)$

& $\beta\alpha = \#(B \times A)$

For this we have to prove

$A \times B \sim B \times A$

Define a function $f: A \times B \longrightarrow B \times A$

by $f((a,b)) = (b,a)$, where $a \in A$ & $b \in B$

<u>f is 1-1:</u> $f((a_1, b_1)) = f((a_2, b_2))$

$\Rightarrow (b_1, a_1) = (b_2, a_2)$

$\Rightarrow b_1 = b_2$, $a_1 = a_2$

$\Rightarrow (a_1, b_1) = (a_2, b_2)$

$\Rightarrow f$ is 1-1

Order Pair اس وقت برابر ہو تا ہے جب پہلا element پہلے کے اور دوسرا دوسرے کے برابر ہو

<u>f is onto:-</u> Since every $(b, a) \in B \times A$

is the image of some $(a,b) \in A \times B$ under f $\Rightarrow f$ is onto

Hence f is bijective

So $A \times B \sim B \times A$

$\Rightarrow \#(A \times B) = \#(A \times B)$

$\Rightarrow \alpha\beta = \beta\alpha$

(iii) $\alpha + (\beta + \gamma) = \#(A) + \#(B \cup C)$

$= \#[A \cup (B \cup C)]$

$= \#[(A \cup B) \cup C]$ $\because$ union of sets is associative

$= \#(A \cup B) + \#(C)$

$\Rightarrow \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

(iv) $\alpha(\beta\gamma) = \#[A \times (B \times C)]$

& $(\alpha\beta)\gamma = \#[(A \times B) \times C]$

Define $f: A \times (B \times C) \longrightarrow A \times B \times C$

by $f((a,(b,c))) = (a,b,c)$

Clearly f is both one-one and onto.

$\Rightarrow A \times (B \times C) \sim A \times B \times C$

f is 1-1

$f((a_1,(b_1,c_1))) = f((a_2,(b_2,c_2)))$

$\Rightarrow (a_1,b_1,c_1) = (a_2,b_2,c_2)$

$\Rightarrow a_1 = a_2$, $b_1 = b_2$

& $c_1 = c_2$

$\Rightarrow (a_1,(b_1,c_1)) = (a_2,(b_2,c_2))$

Now define $g: (A \times B) \times C \to A \times B \times C$

by $g(((a,b),c)) = (a,b,c)$

clearly g is both one-one and onto. So $(A \times B) \times C \sim A \times B \times C$

So by Transitive property

$$A\times(B\times C) \sim (A\times B)\times C$$

$$\Rightarrow \#[A\times(B\times C)] = \#[(A\times B)\times C]$$

(V) $$\alpha(\beta+\gamma) = \#[A\times(B\cup C)]$$

$$= \#[(A\times B)\cup(A\times C)]$$

$$= \#(A\times B) + \#(A\times C)$$

$$= \alpha\beta + \alpha\gamma$$

★ ★★★

Theorem: Show that $a+n = a$, where $a = \#(\mathbb{N})$ and n is any finite cardinal. OR Show that

$$\aleph_0 + n = \aleph_0$$

Proof: Let $A = \{a_1, a_2, a_3, \ldots\ldots, a_n\}$

then $\#(A) = n$

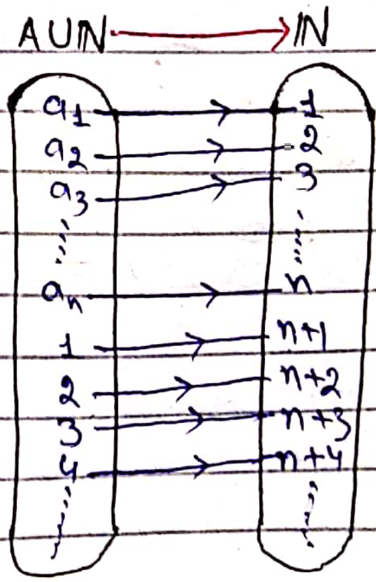
$A\cup\mathbb{N} = \{a_1, a_2, a_3, \ldots, a_n, 1, 2, 3, 4, \ldots\}$

$\mathbb{N} = \{1, 2, 3, 4, \ldots, n, n+1, n+2, n+3, \ldots\}$

Consider a mapping

$f: A\cup\mathbb{N} \longrightarrow \mathbb{N}$ by

$$f(x) = \begin{cases} n & \text{if } x = a_n \\ n+i & \text{if } x = i, \\ & i = 1, 2, \ldots \end{cases}$$



Clearly f is both 1-1 & onto

$\Rightarrow$ f is bijective

So $A \cup N \sim N$

$\Rightarrow$ $\#(A \cup N) = \#(N)$

$\Rightarrow$ $n + a = a$

$\Rightarrow$ $a + n = a$ proved.

V.V. Imp

Theorem:- Prove that $a + \beta = \beta$, where β is any infinite cordinal and "a" is the cordinality of a denumerable set.

Proof:- Let A be an infinite set and $B = \{b_1, b_2, b_3, \ldots\}$ be a denumerable set such that

$$A \cap B = \phi \quad \& \quad \#(B) = a$$

$$\text{and } \#(A) = \beta$$

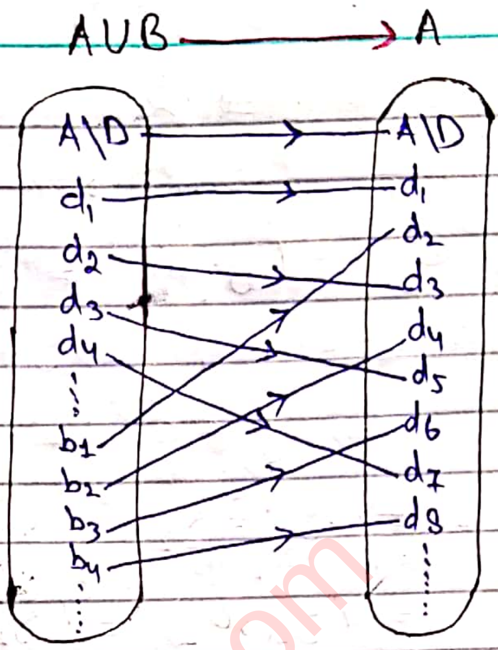
We show that $A \cup B \sim A$

Let $D = \{d_1, d_2, d_3, d_4, \ldots\}$ be a denumerable subset of A.

Because every infinite set contains a denumerable subset.

Define a mapping $f : A \cup B \longrightarrow A$ by

$$f(x) = \begin{cases} x & \text{if } x \in A \setminus D \\ d_{2n-1} & \text{if } x = d_n \in D, \; n = 1, 2, 3, \ldots \\ d_{2n} & \text{if } x = b_n \in B, \; n = 1, 2, 3, \ldots \end{cases}$$



obviously f is both 1 - 1 and onto

$\Rightarrow A \cup B \sim A$

$\Rightarrow \#(A \cup B) = \#(A)$

$\Rightarrow \beta + a = \beta$

$\Rightarrow a + \beta = \beta$

***

Question:- Give examples of two sets A and B such that A and B have cordinality "c" but

(i) $A \setminus B$ is finite (ii) $A \setminus B$ is denumerable

Solution (i) Let $A = [0, 1]$, $B =]0, 1[$

$\Rightarrow \#(A) = c$ and $\#(B) = c$

& $A \setminus B = \{0, 1\}$ is finite.

(ii) Let $A = \mathbb{R}$ and $B = \mathbb{Q}'$

$\#(A) = c$, $\#(B) = c$

$A \setminus B = \mathbb{R} \setminus \mathbb{Q}' = \mathbb{Q}$

which is denumerable.

***

Question:- Show that $ac = c$

Proof: $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$

and $A =]0, 1[$

Clearly $\#(Z) = a$ and $\#(A) = c$

Define a function $f : Z \times A \to \mathbb{R}$

by $f(z, a') = z + a'$

<u>f is 1-1</u>:- Let $f(z_1, a_1) = f(z_2, a_2)$

$\Rightarrow z_1 + a_1 = z_2 + a_2$

$\Rightarrow z_1 = z_2 \;,\; a_1 = a_2$

$\Rightarrow (z_1, a_1) = (z_2, a_2)$

$\Rightarrow f$ is 1-1

Problem with mapping

$f : Z \times A \to A$ is

$f(z, a') = z + a'$

Let $z = 5, a' = 0.5$

$f(5, 0.5) = 5 + 0.5$

$= 5.5 \notin]0,1[$

$\because z$ is integer and a are decimals & integer + decimal = integer + decimal iff integer = integer & decimal = decimal

<u>f is onto</u>:- since every $z + a' \in \mathbb{R}$ is the image of some $(z, a') \in Z \times A$ under f. Therefore f is onto

Hence f is bijective

$\Rightarrow Z \times A \sim \mathbb{R} \Rightarrow \#(Z \times A) = \#(\mathbb{R})$

$\Rightarrow ac = c$

* * *

* $\Rightarrow$ **Definition**:- Let A and B be two sets such that A is equivallent to a subset of B. Then A is said to preceds B and is denoted by

$A \lesssim B$

Moreover if A is not equivallent to B $(A \nsim B)$ Then A strictly proceeds B and is written as

$A < B$

Example: $\mathbb{N}$ the set of natural numbers is equivallent to a proper subset of $\mathbb{R}$ so $\mathbb{N} \lesssim \mathbb{R}$. Further $\mathbb{N} < \mathbb{R}$ as $\mathbb{N} \nsim \mathbb{R}$

⇒ Definition:-

If A and B are two sets such that $\#(A) = \alpha$, $\#(B) = \beta$ and if $A \lesssim B$ then $\alpha \leq \beta$. Moreover if A is strictly preceed B i.e $A < B$ then $\alpha < \beta$. If $A \sim B$ then $A \lesssim B$ and $B \lesssim A$ or

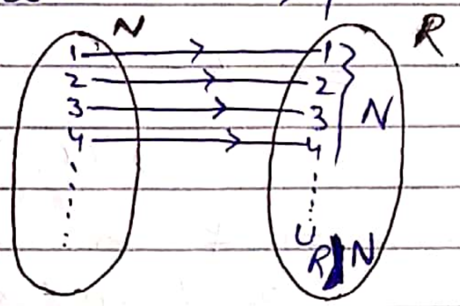
$A \lesssim B$ and $B \lesssim A \Rightarrow A \sim B$

*** Remark:-** (i) $X \lesssim Y$ if there exist an injection (1-1) from $X \longrightarrow Y$

(ii) $X \subseteq Y \Rightarrow X \lesssim Y$

(iii) $X < Y$ if $X \lesssim Y$ but $X \nsim Y$

$\boxed{A \lesssim B \quad f: A \to B \text{ is } 1\text{-}1}$



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Question:- Show that $c^2 = c$ or $c \cdot c = c$ where $c = \#[0,1]$

Proof: Let $A = [0,1]$ and $x, y \in [0,1]$ Then x and y can be written uniquelly in the form of an infinite decimals

$x = 0 \cdot x_1 x_2 x_3 x_4 \ldots\ldots$

$y = 0 \cdot y_1 y_2 y_3 y_4 \ldots\ldots$

where $x_i, y_i \in \{0, 1, 2, \ldots, 9\}$

Define a mapping

$f: A \times A \longrightarrow A$ by

$f(x, y) = 0 \cdot x_1 y_1 x_2 y_2 x_3 y_3 \cdots\cdots$

$\forall \ (x, y) \in A \times A$

f is an injection:-

Let $f((x, y)) = f((x', y'))$

$\Rightarrow 0 \cdot x_1 y_1 x_2 y_2 x_3 y_3 \cdots\cdots = 0 \cdot x_1' y_1' x_2' y_2' x_3' y_3' \cdots\cdots$

$\Rightarrow x_1 = x_1', \ y_1 = y_1', \ x_2 = x_2', \ y_2 = y_2',$

$x_3 = x_3', \ y_3 = y_3' \cdots\cdots$

$\Rightarrow 0 \cdot x_1 x_2 x_3 \cdots\cdots = 0 \cdot x_1' x_2' x_3' \cdots\cdots$ and

$0 \cdot y_1 y_2 y_3 \cdots\cdots = 0 \cdot y_1' y_2' y_3' \cdots\cdots$

$\Rightarrow x = x'$ & $y = y'$

$\Rightarrow (x, y) = (x', y')$

$\Rightarrow f$ is $1-1$

So $A \times A \lesssim A$

$\Rightarrow \#(A \times A) \leq \#(A)$

$\Rightarrow c \cdot c \leq c \longrightarrow (1)$

Moreover $A \sim \{(0, x) : x \in A\} \subseteq A \times A$

$\Rightarrow A \lesssim A \times A$

$\Rightarrow \#(A) \leq \#(A \times A)$

$\Rightarrow c \leq c \cdot c \longrightarrow (2)$

From (1) and (2) $c \cdot c = c$

or $c^2 = c$

———————

Question:- Family of all pairwise disjoint intervals of real numbers is countable.

Solution

Let $\{T_i : i \in I\}$ be any family of interval of real numbers. Now any interval of real numbers must contain atleast one rational number.
Further $T_i \neq T_j \Rightarrow q_i \neq q_j$
where q_i, q_j are rational numbers in interval T_i, T_j respectively.
$\therefore$ $\{T_i : i \in I\}$ is equivalent to a subset of rational numbers.
As set of rational numbers is countable so its every subset is countable.
$\Rightarrow$ $\{T_i : i \in I\}$ is countable.

Question:- Prove that the set of Transedential numbers has cordinality c. OR Set T is Transedential numbers has power of continuum.

Solution

As we know that

$$\mathbb{R} = A \cup T \longrightarrow ①$$

where A is the set of algebraic numbers and T is the set of Transedential numbers.
As we know that the set of algebraic numbers is denumerable

$$\therefore \quad \#(A) = a$$

$$\text{\&} \quad \#(\mathbb{R}) = c$$

So from ①

$$\#(\mathbb{R}) = \#(A \cup T)$$

$$\Rightarrow \quad c = \#(A) + \#(T)$$

$$\Rightarrow \quad c = a + \#(T) \longrightarrow ②$$

Notations
if $A = \{a, b, c\}$
$\#(A) = 3$ or
$|A| = 3$ or
$n(A) = 3$ or
$\text{card}(A) = 3$

As we know that if β is any infinite cordinal number then

$$\alpha + \beta = \beta$$

As c is an infinit cordinal so

$$\alpha + c = c \longrightarrow (3)$$

from equ (2) & (3)

$$\boxed{\begin{array}{l} a + \#(T) = c \\ a + c = c \end{array}}$$

$$\#(T) = c$$

So T has power of continuam.

Notation:- If $\alpha = \#(A)$, then we let

$$2^{\alpha} = \#(2^{A})$$

$\boxed{\begin{array}{l} 2^{A} = \text{family of all subsets of } A \\ \quad = \text{Power set of } A \end{array}}$

Example Let $A = \{a, b, c\}$

$\#(A) = 3$ & let $\#(A) = \alpha = 3$

$$2^{A} = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

$$\Rightarrow \#(2^{A}) = 8 = 2^{3} = 2^{\#(A)}$$

$$\Rightarrow \#(2^{A}) = 2^{\#(A)}$$

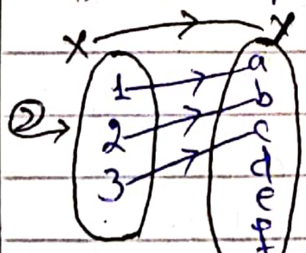
$$\Rightarrow \#(2^{A}) = 2^{\alpha}$$

Remark:- 1) If $X \lesssim Y$ then $|X| \leq |Y|$

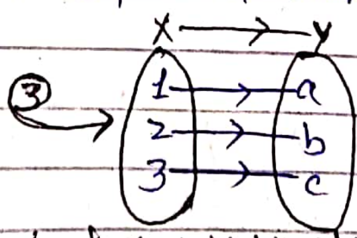
2) $X \lesssim Y$ if there exist an injection (one-one) $f : X \longrightarrow Y$

3) $X \lesssim Y$ and $Y \lesssim X$ if there exist bijection from $X \rightarrow Y$

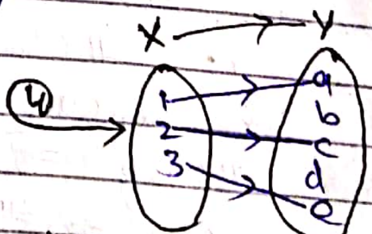
4) $X \prec Y$ if $X \lesssim Y$ & $X \not\sim Y$

(2) 

$f : X \rightarrow Y$ is one-one so $X \lesssim Y$

(3) 

$f : X \rightarrow Y$ bijective so $X \lesssim Y$ & $Y \lesssim X$

(4) 

$X \lesssim Y$ & $X \not\sim Y$ so $X \prec Y$

Theorem:- Prove that $c = 2^a$, where $c = \#(\mathbb{R})$, $a = \#(\mathbb{N})$, OR

Show that $\#(\mathbb{R}) = 2^{\#(\mathbb{N})}$

Proof:-

Define a function $f : \mathbb{R} \longrightarrow 2^{\mathbb{Q}}$ by $f(a) = \{x \in \mathbb{Q} : x < a\}$ $\forall\, a \in \mathbb{R}$

f is injection:-

Let $a, b \in \mathbb{R}$ s.t $a \neq b$

Suppose $a < b$ Then by rational density theorem there exist a rational number "r" between a, b i.e $a < r < b$

$$\Rightarrow r \in f(b), \quad r \notin f(a)$$

$$\Rightarrow f(a) \neq f(b)$$

which shows that f is 1-1

$$\Rightarrow \mathbb{R} \lesssim 2^{\mathbb{Q}}$$

$$\Rightarrow \#(\mathbb{R}) \leq \#(2^{\mathbb{Q}})$$

$$\Rightarrow c \leq 2^a \longrightarrow ①$$

Consider

$C(\mathbb{N})$, the collection of all characteristic functions defined on $\mathbb{N}$, then

$$C(\mathbb{N}) \sim 2^{\mathbb{N}}$$

$$\Rightarrow \#[C(\mathbb{N})] = \# 2^{\mathbb{N}}$$

$$\Rightarrow \#(C(\mathbb{N})) = 2^a$$

Define $F : C(\mathbb{N}) \longrightarrow [0,1]$ as

$$F(f) = 0.f(1)f(2)f(3)f(4)\ldots\ldots \quad \forall\, f \in C(\mathbb{N})$$

we show that F is 1-1

Let $f, g \in C(\mathbb{N})$ s.t $f \neq g$

$$F(f) = 0.f(1)f(2)f(3)\ldots\ldots$$

$$F(g) = 0.g(1)g(2)g(3)\ldots\ldots$$

$$\Rightarrow F(f) \neq F(g) \quad \because f \neq g$$

$\Rightarrow$ F is an injection

$\Rightarrow C(N) \lesssim [0,1]$

$\Rightarrow \#(C(N)) \leq \#[0,1]$

$\Rightarrow 2^{a} \leq c \longrightarrow ②$

from ① and ②

$c = 2^{a}$

**

⇒ Cantor's Theorem:-

For any set A, $A < 2^{A}$ and hence if $\alpha = \#(A)$ then $\alpha < 2^{\alpha}$, $2^{\alpha} = \#(2^{A})$.

If α is the cordinality of A and 2^{α} is the cordinality of power set of A. Determine whether $\alpha < 2^{\alpha}$ OR

For any set A, $A < P(A)$ and hence

$|A| < |P(A)|$

Proof)

To prove $|A| < |2^{A}|$ $(A < 2^{A})$ we have to prove

(i) $A \lesssim 2^{A}$

(ii) $A \not\sim 2^{A}$

Define a mapping $f : A \longrightarrow 2^{A}$ by

$f(a) = \{a\}$, $a \in A$

$\therefore$ This function is an injection but not onto.

$\Rightarrow A \lesssim 2^{A}$ ...... ①

Now we show $A \not\sim 2^{A}$

Suppose there exist a mapping $g : A \longrightarrow 2^{A}$ which is both one-

one and onto.

Let B be the set of all those elements of A which do not belong to their images. i.e

$$B = \{x \in A : x \notin g(x)\}$$

where $B \subseteq A$ also $B \in 2^A$

Since g is suppose to be onto so $\exists$ an element $b \in A$ s.t $g(b) = B$

Now there are two possibilities for "b"

1) $b \in B \Rightarrow b \notin g(b) = B \Rightarrow b \notin B$ which is impossible

2) $b \notin B \Rightarrow b \in g(b) = B \Rightarrow b \in B$ which is also impossible.

Which shows that there does not exist any bijection from $A \longrightarrow 2^A$

$$\boxed{\text{Hence } A < 2^A \Rightarrow \#(A) < \#(2^A) \Rightarrow \alpha < 2^\alpha}$$

Hence $A \not\sim 2^A$ ...... ②

from ① & ② $A < 2^A \Rightarrow \#(A) < \#(2^A)$

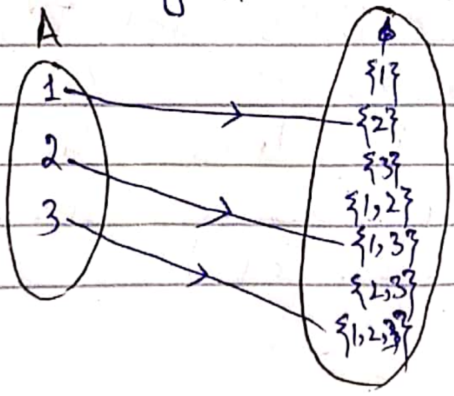
$\Rightarrow \alpha < 2^\alpha$

***

Example $A = \{1, 2, 3\}$

$2^A = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$g : A \longrightarrow 2^A$



$g(1) = \{2\}$ means $1 \notin g(1) = \{2\}$

$g(2) = \{1,3\}$ means $2 \notin g(2) = \{1,3\}$

$g(3) = \{1,2,3\}$ means $3 \in g(3) = \{1,2,3\}$

So $B = \{1, 2\}$ both elements does not belong to their images.

⇒ Exponent of Cordinal Numbers:-

If A and B are two non-empty set and B^A denote the collection of all functions from set A into set B. then if $\alpha = \#(A)$ & $\beta = \#(B)$ then

$$\beta^\alpha = \#(B^A)$$

Example

$A = \{a, b, c\}$, $B = \{0, 1\}$

$\#(A) = 3$, $\#(B) = 2$

functions from $A \to B$

$f_1 = \{(a,0),(b,0),(c,0)\}$, $f_2 = \{(a,0),(b,0),(c,1)\}$

$f_3 = \{(a,0),(b,1),(c,0)\}$, $f_4 = \{(a,1),(b,0),(c,0)\}$

$f_5 = \{(a,0),(b,1),(c,1)\}$, $f_6 = \{(a,1),(b,1),(c,0)\}$

$f_7 = \{(a,1),(b,0),(c,1)\}$, $f_8 = \{(a,1),(b,1),(c,1)\}$

$B^A = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$

$\#(B^A) = 8 = 2^3 = \beta^\alpha$

$\Rightarrow \beta^\alpha = \#(B^A)$

where $\alpha = 3 = \#(A)$ & $\beta = 2 = \#(B)$

Rough

$A \times B = \{(a,0),(b,0),(c,0),(a,1),(b,1),(c,1)\}$

Any subset of $A \times B$ is called binary relation from $A \to B$

* Some binary relation are functions which satisfy some conditions & other or not.

$f_9 = \{(a,0),(a,1),(b,0)\}$ is binary relation but not function

Question:- Let $\alpha = \#(A)$ then $2^{\alpha} = \#(2^A)$. Show that $\#(2^A) = \#(B^A)$, where $\#(B) = 2$

Solution

Let $B = \{0, 1\} \Rightarrow \#(B) = 2$

and $B^A = C(A) \sim$ set of all characteristic functions from $A \to B$

Define $f: 2^A \longrightarrow B^A = C(A)$ by

$$f(A_1) = \chi_{A_1}$$

f is 1-1:- Let $f(A_1) = f(A_2)$

$$\Rightarrow \chi_{A_1} = \chi_{A_2} \quad \Rightarrow \chi_{A_1}(x) = \chi_{A_2}(x) \quad \forall x \in A$$

If $x \in A_1$ then $\chi_{A_1}(x) = 1$

As $\chi_{A_1}(x) = \chi_{A_2}(x)$

$$\Rightarrow \chi_{A_2}(x) = 1 \quad \Rightarrow x \in A_2$$

So $A_1 \subseteq A_2$

Similarly $A_2 \subseteq A_1$

So $A_1 = A_2$

$$\Rightarrow f \text{ is } 1-1$$

f is onto:- Since every $\chi_A \in C(A)$ is the image of some $A \in 2^A$ under f.

So f is onto.

$$\Rightarrow f \text{ is bijective}$$

$$\Rightarrow 2^A \sim B^A \quad \Rightarrow \#(2^A) = \#(B^A)$$

***

2^A = Power set of A

B^A = All functions from $A \to B$

اگر B میں صرف 2 elements ہوں تو $A \to B$ تک تمام فنکشنز کی تعداد برابر ہو گی $A \to B = \{0, 1\}$ تک characteristic فنکشنز کی تعداد = P(A) میں subsets کی تعداد

⇒ Restriction:-

Consider a function $f: A \to S$. Let B be a subset of A. Then f induces a function f' on B defined by $f'(b) = f(b)\ \forall\ b \in B$. This function f' is called restriction of f to B. It is some times denoted by

$$f|B$$

Example 1

Consider a function

$$g = \{(1,3),(2,6),(3,11),(4,18),(5,27)\}$$

and

$$g' = \{(1,3),(3,11),(5,27)\}$$

observe that g' is a subset of g. Thus g' is the restriction of g to $B = \{1,3,5\}$ The set of first elements of g'. Now note that B is the subset of $A = \{1,2,3,4,5\}$ the set of first elements of g.

Example 2

Let $f: \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = x^2$$

Recall that f is not 1-1. e.g

$$f(2) = f(-2) = 4$$

Consider the restriction f to non-negative real numbers $D = [0, \infty)$. Then f/D is 1-1.

★——★★——★

Theorem:- Prove that $\alpha^{\beta}\cdot\alpha^{\gamma} = \alpha^{\beta+\gamma}$, where $\alpha = \#(A)$, $\beta = \#(B)$ and $\gamma = \#(C)$. where A, B and C are mutually disjoint.

Proof

$$\alpha^{\beta} = \#(A^{B}) \quad , \quad \alpha^{\gamma} = \#(A^{C})$$

$$\alpha^{\beta}\cdot\alpha^{\gamma} = \text{cordinality of } (A^{B}\times A^{C})$$

$$\beta+\gamma = \#(B\cup C)$$

$$\alpha^{\beta+\gamma} = \#(A^{B\cup C})$$

To prove $\alpha^{\beta+\gamma} = \alpha^{\beta}\cdot\alpha^{\gamma}$, we prove that $A^{B\cup C} \sim A^{B}\times A^{C}$. Define a mapping

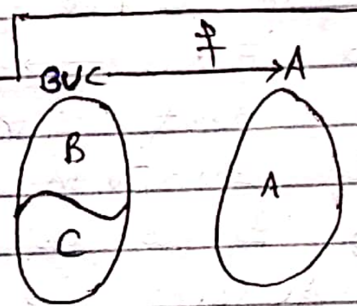
$F: A^{B\cup C} \longrightarrow A^{B}\times A^{C}$ by

$$F(f) = (f/_B, f/_C)$$

Here $f: B\cup C \longrightarrow A$, where $f/_B$ and $f/_C$ are restriction of f to B and C respectively.

i.e $f/_B : B \longrightarrow A$

$f/_C : C \longrightarrow A$



$f \in A^{B\cup C} \Rightarrow f: B\cup C \longrightarrow A$

$f/_B : B \to A \Rightarrow f/_B \in A^{B}$

$f/_C : C \to A \Rightarrow f/_C \in A^{C}$

$\Rightarrow f/_B, f/_C \in A^{B}\times A^{C}$

we prove f is a bijective function.

F is 1-1:- Let $F(f) = F(g)$

$$\Rightarrow (f/_B, f/_C) = (g/_B, g/_C)$$

$$\Rightarrow f/_B = g/_B \quad , \quad f/_C = g/_C$$

$\Rightarrow f = g$

$\Rightarrow F$ is 1-1

F is onto:- since for every $(f|_B, f|_C) \in A^B \times A^C$. There exist $f \in A^{B\cup C}$ under F

$\Rightarrow F$ is onto

So F is bijective

$\Rightarrow A^{B\cup C} \sim A^B \times A^C$

$\Rightarrow \#(A^{B\cup C}) = \#(A^B \times A^C)$

$\Rightarrow \alpha^{\beta+\gamma} = \alpha^\beta . \alpha^\gamma$

⇒ Extension:-

Consider a function $f: A \to S$. Suppose B be a superset of A, ie $A \subseteq B$. Let $F: B \longrightarrow S$ be a function on B such that for every $a \in A$ $F(a) = f(a)$. This function F is called an extension of f to B. We note that such an extension is rarely unique.

Example

Let f be a function on the non-negative real numbers $D = [0, \infty)$ defined by $f(x) = x$. Then the absolute value function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

is an extension of f to the set $\mathbb{R}$ of all real numbers. Clearly the identity function $I_R: R \to R$ is also an extension of f to R

Question:- If α, β, γ are cordinals and $\alpha \leq \beta$, then

(i) $\alpha + \gamma \leq \beta + \gamma$ (ii) $\alpha\gamma \leq \beta\gamma$

(iii) $\alpha^{\gamma} \leq \beta^{\gamma}$ (iv) $\gamma^{\alpha} \leq \gamma^{\beta}$

Solution Let A, B, C be the three sets such that $\#(A) = \alpha$, $\#(B) = \beta$ and $\#(C) = \gamma$

Let $A \subseteq B$ & $B \cap C = \phi$

i) Now $\alpha + \gamma = \#(A \cup C)$

and $\beta + \gamma = \#(B \cup C)$

As $A \subseteq B \Rightarrow A \cup C \subseteq B \cup C$

$\Rightarrow \#(A \cup C) \leq \#(B \cup C)$

$\Rightarrow \alpha + \gamma \leq \beta + \gamma$

ii) Now $A \subseteq B$

$\Rightarrow A \times C \subseteq B \times C \Rightarrow \#(A \times C) \leq \#(B \times C)$

$\Rightarrow \alpha\gamma \leq \beta\gamma$

iii) Now $\#(A^C) = \alpha^{\gamma}$ & $\#(B^C) = \beta^{\gamma}$

Let $f \in A^C \Rightarrow f: C \longrightarrow A$ is function

$\Rightarrow f: C \longrightarrow B$ is function ($\because A \subseteq B$)

$\Rightarrow f \in B^C \Rightarrow A^C \subseteq B^C$

$\Rightarrow \#(A^C) \leq \#(B^C)$

$\Rightarrow \alpha^{\gamma} \leq \beta^{\gamma}$

iv) $\gamma^{\alpha} = \#(C^{A})$ and $\gamma^{\beta} = \#(C^{B})$

Let $f \in C^{A} \Rightarrow f : A \to C$

Now as $A \subseteq B$. so Let $f^{*} : B \longrightarrow C$ be its extension on B.

$$\Rightarrow f^{*} \in C^{B}$$

Now define $F : C^{A} \longrightarrow C^{B}$ by

$$F(f) = f^{*}$$

Thus if $f \neq g$ then $f^{*} \neq g^{*}$

$\Rightarrow F(f) \neq F(g)$ $\boxed{\because \text{By contra positive definition}}$

$\Rightarrow F$ is 1-1

$$\Rightarrow \#(C^{A}) \leq \#(C^{B})$$

*———***———*

Theorem:- If A is countable, B is denumerable and $A \cap B = \phi$. Then $A \cup B$ is denumerable.

Proof Here arises two cases of A

Case I:- If A is denumerable, then

$$A = \{a_1, a_2, a_3, \ldots\ldots\}$$

Let $B = \{b_1, b_2, b_3, \ldots\ldots\}$

If $A \cap B = \phi$. so

$$A \cup B = \{a_1, a_2, a_3, \ldots\ldots, b_1, b_2, b_3, \ldots\ldots\}$$

Define $f : \mathbb{N} \longrightarrow A \cup B$ by

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \ldots\ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & \ldots\ldots \end{array}$$

Then $\mathbb{N} \sim A \cup B \Rightarrow A \cup B$ is denumerable.

Case II:- If A is finite

Then let $A = \{a_1, a_2, a_3, \ldots\ldots, a_n\}$

and $B = \{b_1, b_2, b_3, \ldots\ldots\}$

As $A \cap B = \phi$. so

$A \cup B = \{a_1, a_2, a_3, \ldots\ldots, a_n, b_1, b_2, b_3, \ldots\ldots\}$

Define $f : A \cup B \longrightarrow \mathbb{N}$ by

$f(a_i) = i$ and $f(b_i) = n + i$

Then obviously f is bijective

$$\begin{array}{cccccccc} a_1 & a_2 & a_3 & \cdots & a_n & b_1 & b_2 & b_3 \cdots \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & \cdots & n & n+1 & n+2 & n+3 \cdots \end{array}$$

$\Rightarrow A \cup B \sim \mathbb{N} \Rightarrow A \cup B$ is denumerable.

*———***———*

Question:- Show that set P of all polynomials with integral coefficients is denumerable.

Solution:-

Consider $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots + a_m x^m$

and let $|a_0| + |a_1| + |a_2| + \ldots\ldots + |a_m| = n$

And denote this polynomial with $P(n, m)$

i.e

$P(n, m) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots + a_m x^m$

with $|a_0| + |a_1| + |a_2| + \ldots\ldots + |a_m| = n$.

then

$$P = \bigcup_{(n,m)\in N\times N} P(n,m)$$

Now as for each pair (n,m), $P(n,m)$ are countable and union of countable sets is countable.

So P is countable.

As P is not finite.

So P is denumerable.

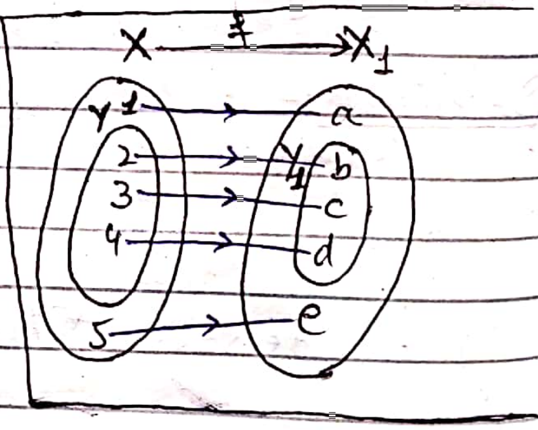
***

⇒ **Schroder Bernstein Theorem:-** states that If $X_1 \subseteq Y \subseteq X$ and $X \sim X_1$ then $X \sim Y$

Proof:- since $X \sim X_1$ so then there exist $f: X \longrightarrow X_1$ which is both 1-1 and onto.

As $Y \subseteq X$ so restriction of f to Y is also 1-1. Here we denote restriction by f. Also ie $f: Y \longrightarrow X_1$ is 1-1

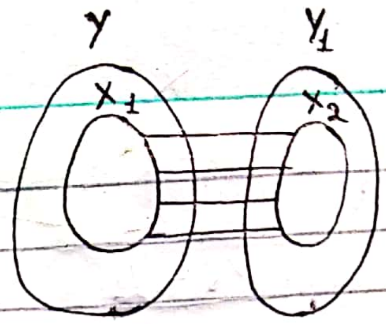
Then there exist some subset Y_1 of X_1 s.t $f: Y \longrightarrow Y_1$ is bijective i.e $Y \sim Y_1$



Note that

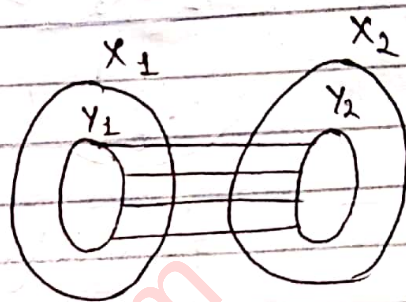
$Y_1 \subseteq X_1 \subseteq Y \subseteq X$

Further as $X_1 \subseteq Y$ and Y is equivalent to Y_1 i.e $Y \sim Y_1$. Then by above argument X_1 is equivalent to some subset of Y_1 say X_2



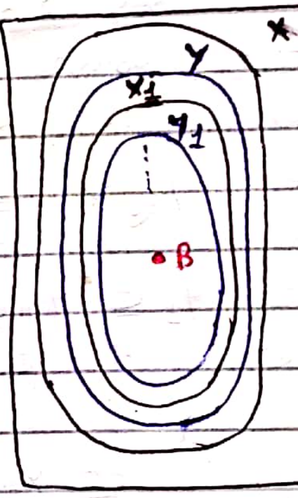
i.e $X_1 \sim X_2$ and
$X_2 \subseteq Y_1 \subseteq X_1 \subseteq Y \subseteq X$

Further as $Y_1 \subseteq X_1$ and $X_1 \sim X_2$, so again by above argument $\exists$ some subset Y_2 of X_2 s.t $Y_1 \sim Y_2$ and
$Y_2 \subseteq X_2 \subseteq Y_1 \subseteq X_1 \subseteq Y \subseteq X$



Continuing this process we obtain the sequence of equivalent sets $X_1, X_2, X_3, \ldots$ and $Y_1, Y_2, Y_3, \ldots$ and
$$\ldots \subseteq Y_2 \subseteq X_2 \subseteq Y_1 \subseteq X_1 \subseteq Y \subseteq X$$

Let $B = X \cap Y \cap X_1 \cap Y_1 \cap X_2 \cap Y_2 \ldots$



Then $X = (X-Y) \cup (Y-X_1) \cup (X_1 - Y_1)$
$\cup (Y_1 - X_2) \cup \ldots \cup B$

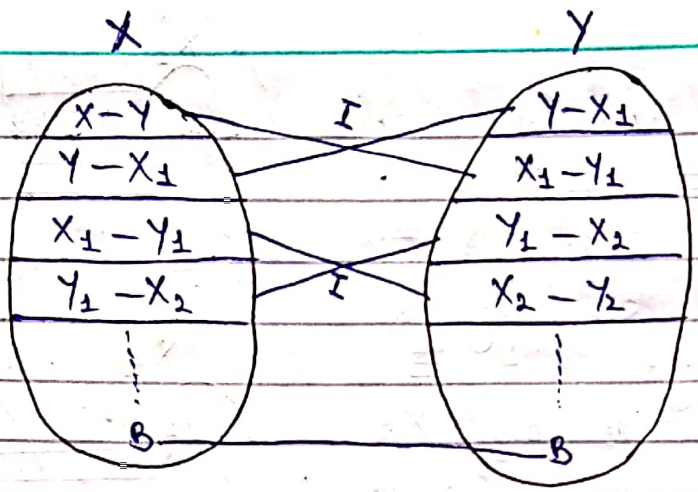
$Y = (Y - X_1) \cup (X_1 - Y_1) \cup (Y_1 - X_2)$
$\cup (\ldots \cup B$

Also note that
$(X - Y) \sim (X_1 - Y_1) \sim (X_2 - Y_2)$
$\sim \ldots$

Specifically $f : X_n - Y_n \longrightarrow X_{n+1} - Y_{n+1}$ is bijective

Here X_0 means X and Y_0 means Y

Define a function $g : X \longrightarrow Y$ by
$$g(x) = \begin{cases} f(x) & \text{if } x \in X_i - Y_i \\ x & \text{if } x \in Y_i - X_{i+1} \text{ or } B \end{cases}$$



Then obviously g is 1-1 and onto
$\Rightarrow X \sim Y$

★——★★★——★

Corollary:- If $X \precsim Y$ and $Y \precsim X$ then
$X \sim Y$

Proof:-
As $X \precsim Y$ so X is equivalent to some subset of Y say Y_1 s.t
$X \sim Y_1 \subseteq Y \longrightarrow ①$

and $Y \precsim X \Rightarrow Y \sim X_1 \subseteq X \longrightarrow ②$

Since $Y_1 \subseteq Y$ and $Y \sim X_1$. So then there exist a subset Y_2 of X_1 s.t
$Y_1 \sim Y_2$ i.e
$Y_1 \sim Y_2 \subseteq X_1$
from ① $X \sim Y_1$ and $Y_1 \sim Y_2$ so
$X \sim Y_2$

Also $Y_2 \subseteq X_1 \subseteq X$ and $X \sim Y_2$ so by schroder Bernstein theorem
$X \sim X_1$

So by ② $Y \sim X_1$ and now $X_1 \sim X$
$\Rightarrow Y \sim X \quad \Rightarrow X \sim Y$

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Corollary:- If $\alpha \leq \beta$ and $\beta \leq \alpha$ then $\alpha = \beta$

Proof:- Let $\alpha = \#(X)$ and $\beta = \#(Y)$

Then $\alpha \leq \beta \Rightarrow X \precsim Y$

$\beta \leq \alpha \Rightarrow Y \precsim X$

Then $X \sim Y \Rightarrow \#(X) = \#(Y)$

$\Rightarrow \alpha = \beta$

*———***———*

Remark:- Show by an example that

1) Cancellation Law do not hold for Cordinal Addition

2) Cancellation Law do not hold for Cordinal multiplication.

Solution

1) Let $\#(N) = a$ & $A = \phi$

& $B = \{x\}$

Then $N \cup A = N$ & $N \cup B = \{x, 1, 2, 3, \ldots\}$

$\Rightarrow \#(N \cup A) = \#(N)$

$\Rightarrow \#(N) + \#(A) = \#(N) \Rightarrow a + 0 = a \longrightarrow ⊛$

Now define $f: N \longrightarrow N \cup B$ by

$f(1) = x$, $f(2) = 1$, $f(3) = 2$, ...

Then obviously f is bijective

$\Rightarrow N \sim N \cup B \Rightarrow \#(N) = \#(N \cup B)$

$\Rightarrow \#(N) + \#(B) = \#(N)$

$\Rightarrow a + 1 = a \longrightarrow ⊛'$

from ⊛ & ⊛' $a + 0 = a + 1$

$\not\Rightarrow 0 = 1$

$\Rightarrow$ Cancellation Law do not hold for Cordinal Addition

2)

$N \times N \sim N \quad \Rightarrow \#(N \times N) = \#(N)$

$\Rightarrow aa = a \longrightarrow (1)$

Also $N \times \{1\} \sim N$ by $f(n,1) = n$

$\Rightarrow \#(N \times \{1\}) = \#(N)$

$\Rightarrow a \cdot 1 = a \longrightarrow (2)$

from equ (1) and (2) $a \cdot a = a \cdot 1$

$\not\Rightarrow a = 1$

$\Rightarrow$ Cancellation Law do not hold for Cordinal Multiplication.

** ***** **

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CH#2 :- Partially & Totally Ordered Sets

⇒ Cartesian Product:-

Let A and B be two non-empty sets, then cartesian product of A & B is denoted & defined by

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}$$

⇒ Binary Relation:-

Any subset of $A \times B$ is said to be binary relation from A to B. It is usually denoted by R.

Any subset of $A \times A$ is said to be binary relation or simple relation on A instead of "from A to A"

Example:-

Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $B = \{1, 2, 3, 4\}$

Then

$$R_1 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\} \subseteq B \times B$$

$$R_2 = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\} \subseteq B \times B$$

$$R_3 = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,2), (2,4), (2,6), (2,8), (3,3), (3,6), (4,4), (4,8), (5,5), (6,6), (7,7), (8,8)\} \subseteq A \times A$$

Then R_1 and R_2 are binary relation on B and R_3 is a binary relation on A.

Note that we can also write

$$R_1 = \{(x, y) \in B \times B : x \leq y\}$$

$$R_2 = \{(x, y) \in B \times B : x < y\}$$

$R_3 = \{(x,y) \in A \times A : x \text{ divides } y\}$

If R is a relation on A and ordered pair $(x,y) \in R$. Then we write xRy and read it as "x is related to y"

⇒ Partial Order :-

If A is a non-empty set and R be a relation defined in A s.t

(i) R is reflexive i.e xRx $\forall$ $x \in A$

(ii) R is ~~any~~ anti-symmetric i.e if xRy & yRx $\Rightarrow x = y$

(iii) R is transitive i.e if xRy & $yRz \Rightarrow xRz$

Then R is called a partial order in A and A is called a partially ordered set (Poset).

Example 1:- R_1 and R_3 defined above are partial order relations.

Example 2:- Let S be the collection of sets. Define a relation R in S as ARB if $A \subseteq B$. Then R is a partial order on S.

(i) Reflexive :- $A \subseteq A$ $\forall A \in S$

$\Rightarrow ARA$ $\forall A \in S$

(ii) Anti-Symmetric :- Let ARB and BRA

$\Rightarrow A \subseteq B$ & $B \subseteq A$

$\Rightarrow A = B$

(iii) Transitive :- Let ARB & BRC

$\Rightarrow A \subseteq B$ and $B \subseteq C$

$\Rightarrow A \subseteq C \Rightarrow ARC$

Hence R is reflexive anti symmetric and transitive, so R is partial order on S. and $(S, \subseteq)$ is a partially ordered set or poset.

Example 3:- In set of natural numbers N. Define a relation R on N as
xRy if $x \leq y$ $\forall\, x, y \in N$. Then R is partial order on N and $(N, \leq)$ is a Poset.

(i) Reflexive:- $x \leq x \quad \forall\, x \in N$
$\Rightarrow xRx \quad \forall\, x \in N$

(ii) Anti-Symmetric:- Let xRy & yRx
$\Rightarrow x \leq y$ and $y \leq x$
$\Rightarrow x = y$

(iii) Transitive:- If xRy and yRz
$\Rightarrow x \leq y$ and $y \leq z$
$\Rightarrow x \leq z \Rightarrow xRz$

So R is Reflexive, Anti-symmetric, Transitive so R is partial order on N and $(N, \leq)$ is a Poset.

Example 4:- Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and R is defined as xRy if $x|y$
i.e $R = \{(2,2), (3,3), (4,4), (5,5), (6,6), (7,7), (8,8), (2,4), (2,6), (2,8), (3,6), (4,8)\}$

(i) Reflexive:- $xRx \ \forall\, x$ because every element divides itself.

(ii) Anti-Symmetric:- If xRy & yRx
$\Rightarrow x|y$ and $y|x$
$\Rightarrow x = y$

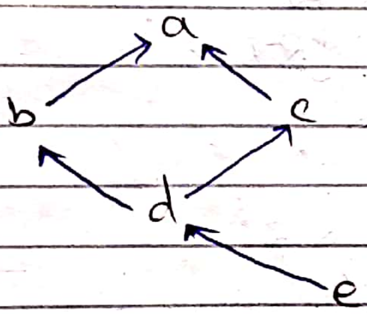
(iii) Transitive:- Let xRy and yRz

$\Rightarrow x|y$ & $y|z$

$\Rightarrow x|z \Rightarrow xRz$

R is reflexive, anti-symmetric and Transitive. So R is partial order on A. and $(A, |)$ is partially ordered set (or poset)

Example 5:- Let $W = \{a, b, c, d, e\}$ and consider the diagram



, where for $x, y \in W$, xRy if either $x = y$ or one can go from x to y through indicated direction. Then This relation defines partial order on W.

★ **Remark:-**

(i) If xRy then we write $x \preceq y$ and read it as x preceeds y

(ii) If $x \preceq y$ i.e x preceeds y then we say y dominates x.

(iii) $x \prec y$ means x strictly preceeds y i.e x preceeds y but $x \neq y$

(iv) $x \npreceq y$ means x does not preceeds y.

★★★

⇒ Total Order:-

Let R be a partial order in a set A then R is said to be total order if for all $a, b \in A$ either $a \precsim b$ or $b \precsim a$ or $a = b$ OR

we say that "a comparable b" write it as $a \precsim b$ and "b comparable a" write it as $b \precsim a$.

Remark:- 1) In a partially ordered set, every element of A is related to some element in A where as in total order every two elements of A are related (comparable) under relation R.

2) The word "Partial" is used in defining a partial order in a set "A" because some elements in A need not be comparable. If, on the other hand, every two elements in a partially ordered set A are comparable, then the partial order in A is called a total order in A.

*———**———*

Example:- $A = \{1, 2, 3, 4, 5, 6\}$ xRy if $x|y$

Then R is a partial order but not total order.

Solution:-

$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (1,2), (1,3), (1,4), (1,5), (1,6), (2,4), (2,6), (3,6)\}$

(i) Reflexive:- $x|x \quad \forall\, x \in A$

$\Rightarrow xRx \quad \forall\, x$

$\Rightarrow R$ is reflexive.

(ii) Anti-Symmetric:- If $x R y$ & $y R x$

$\Rightarrow x | y$ & $y | x$

$\Rightarrow x = y$

$\Rightarrow R$ is anti-symmetric.

(iii) Transitive:- If $x R y$ and $y R z$

$\Rightarrow x | y$ and $y | z$

$\Rightarrow x | z \Rightarrow x R z$

$\Rightarrow R$ is Transitive.

So R is partial order in A. But R is not total order in A. since 3 and 5 are not comparable. i.e 3 does not divides 5 or $3 \not R 5$.

———*———

Example:- $M = \{2, 4, 8, 16, 32, \ldots\}$

$x R y$ if $x | y$

Then R is a total order and M is totally ordered set.

———

* **Remark:-** A set X is said to be ordered set if it is either partial ordered set or totally ordered set.

*———***———*

⇒ First Element:-

Let A be an ordered set. An element $a \in A$ is said to be the first element of A if $a \le x \ \forall x \in A$ i.e "a" preceeds every element of set A.

Example 1:- The set of natural numbers with order "$\le$" i.e $(N, \le)$ is a partially ordered set.

Here 1 is its first element.

Example 2:- $M = \{2, 4, 6, 8, 16, 32, \ldots\}$ is a totally order set by divisibility. Here 2 is the first element.

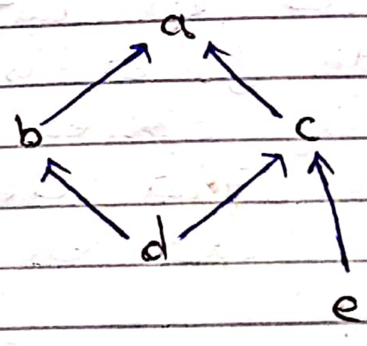
*———***———*

⇒ Last Element:-

Let A be an ordered set. An element $b \in A$ is called last element of A if $x \precsim b \; \forall \; x \in A$. i.e every element of set A preceeds "b" OR "b" dominates (Proceed) every element of A.

Example 1:- $A = \{1, 3, 6\}$ is totally ordered set by divisibility. Here 6 is last element. $\{x \precsim y$ if $x|y\}$

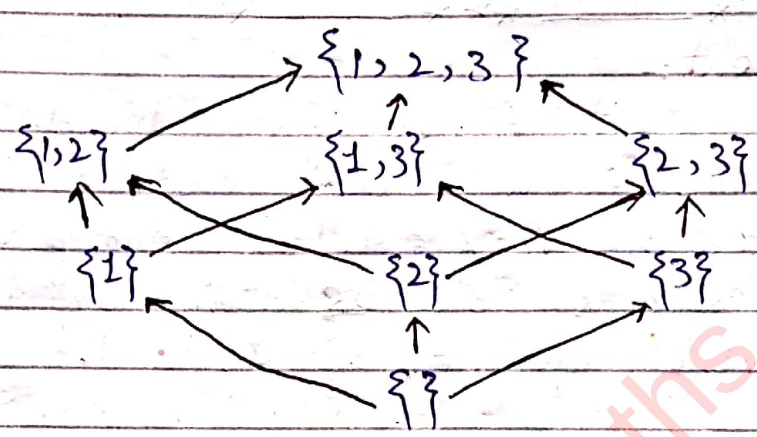
Example 2:- Let $W = \{a, b, c, d, e\}$ be ordered by the following diagram.



Then "a" is a last in W. Since "a" dominates every element. Note that W has no first element. The element d is not a first element since d does not preceeds "e"

Example 3:- Consider N, The set of natural numbers (With the natural order i.e "≤") Then 1 is first element but there is no last element.

Example 4:- $A = \{1, 2, 3\}$ and
$P(A) = 2^A = \{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$



(Hasse Diagram)

$(P(A), \subseteq)$ is partially ordered set.

First Element $= \{\}$ or ϕ
Last Element $= \{1,2,3\}$

Example 5:- $M = \{2, 4, 8, 16, 32, \ldots\ldots\}$ be an ordered set with order defined as x divides y. It is obvious that M is totally ordered set.

First element $= 2$
Last element $=$ Does not exist.

Example 6:- $S = \{x \in Q : 0 \leq x \leq 1\}$ is an ordered set with order "$\leq$" then
First element $= 0$
Last element $= 1$

Example 7:- Let $X = \{x \in Q : 0 < x < 1\}$ with natural order "$\leq$" then 1st and last elements do not exist because $0 \notin S$ & $1 \notin S$.

★ **Remark:-** First and last elements of a set must belong to the set.

★★ —— ★★ —— ★★

⇒ Maximal Element:-

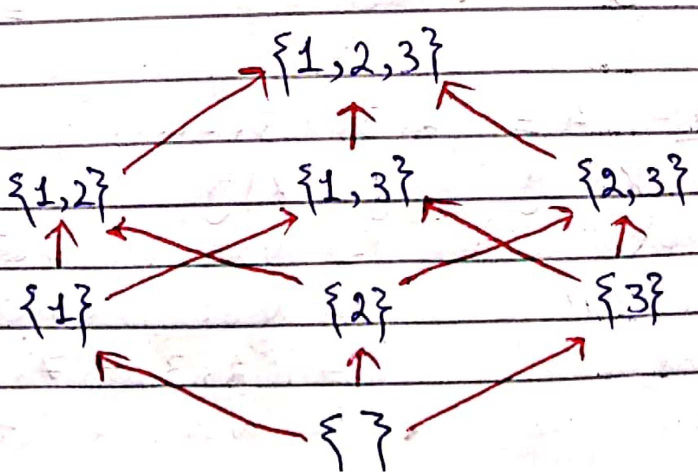
If A is an ordered set then an element $a \in A$ is said to be maximal element of A if for $x \in A$ s.t $a \leq x \Rightarrow a = x$ i.e "a" never preceds any other element of A. OR No element of A dominates "a"

⇒ Minimal Element:-

If A is an ordered set then an element $b \in A$ is said to be minimal element of A if $x \in A$ such that $x \leq b \rightarrow x = b$ i.e no other element of A preceeds "b". OR "b" never dominates any element of A.

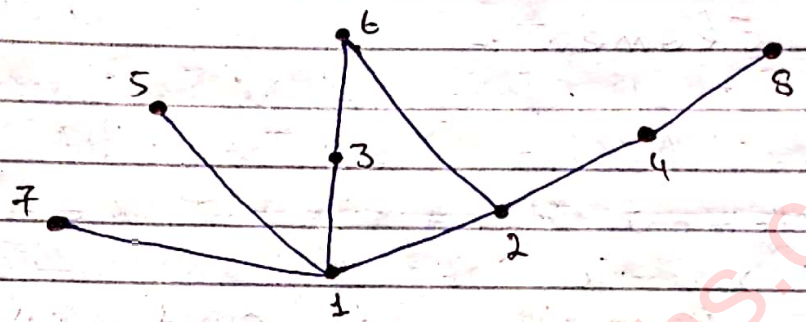
Example 1:- $A = \{1, 2, 3\}$

$(P(A), \subseteq)$ is partially ordered set.



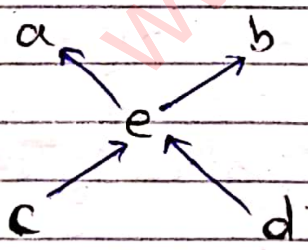
Maximal element $= \{1, 2, 3\}$
Minimal element $= \{\}$

Example 2:- $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$
xRy if $x | y$



Maximal element $= 5, 6, 7, 8$
Minimal element $= 1$
First element $= 1$
Last element $=$ Does not exist.

Example 3:- Let $W = \{a, b, c, d, e\}$ and order in W is defined by the following Hasse diagram



Maximal element $= a, b$
Minimal " $= c, d$
Last " $=$ Does not exist
First " $=$ " " "

Example 4:- $A = \{2, 3, 4, 5, \ldots\ldots\}$
xRy if $x | y$
First element $=$ Not exist
Last " $=$ Not exist
Minimal " $= \{2, 3, 5, 7, 11, \ldots\ldots\}$
Maximal " $=$ Not exist.

★ **Remark:-** 1) The first and last element of a set (if exist) are unique but maximal and minimal elements may be more than one.

2) First and last element of an ordered set are also minimal and maximal elements respectively but converse is not true i.e Maximal and Minimal elements of an ordered set may not be the last and first elements respectively.

***——*

Theorem:- In an ordered set first and last elements are unique.

Proof (a) For First Element:-

Let "a" and "b" be the two first elements of an ordered set A.

As "a" is the first element of A then by definition $a \lesssim x \quad \forall\ x \in A$

In particular $a \lesssim b \longrightarrow ①$

Also as "b" is the first element of A then by definition $b \lesssim x \quad \forall\ x \in A$

In Particular $b \lesssim a \longrightarrow ②$

From ① & ② $a = b$

$\Rightarrow$ First element is unique.

(b) For Last Element:-

Let "c" and "d" be the two last elements of an ordered set A

As "c" is the last element of A then by definition

$x \lesssim c \quad \forall \; x \in A$

In particular $d \lesssim c \longrightarrow ①$

Also as "d" is the last element of A then by definition $x \lesssim d \; \forall \; x \in A$

In particular

$c \lesssim d \longrightarrow ②$

From ① & ② $c = d$

Hence First and Last elements of an ordered set are unique.

*———***———

Theorem:- In an ordered set A if "a" is the first element of A then "a" is the only minimal element.

Proof: Let A be an ordered set and $a \in A$ be the first element of A and moreover "a" is not the minimal element of A but b is the minimal element of A.

As "a" is the first element of A

$\Rightarrow a \lesssim x \quad \forall \; x \in A$

In particular for $b \in A$

$a \lesssim b \longrightarrow ①$

But "b" is suppose to be the minimal element of A therefore by definition of minimal element if $a \lesssim b \Rightarrow a = b$

$\Rightarrow a$ is the only minimal element of A.

*———***———*

Theorem:- In an ordered set A if "a" is the last element of A then "a" is the only maximal element of A.

Proof

Let A be an ordered set and $a \in A$ be the last element of A and moreover "a" is not the maximal element of A but "b" is the maximal element of A.

As "a" is the last element of A so $x \lesssim a \quad \forall \; x \in A$

In particular for $b \in A$

$$b \lesssim a$$

Since b is suppose to be the maximal element of A so $b \lesssim a \Rightarrow a = b$

So "a" is the only maximal element.

* ——— * ——— *

Theorem:- In a totally ordered set the maximal element is unique.

Proof

Let A be a totally ordered set and also $a, b \in A$ be two maximal elements.

Since A is totally ordered, then (i) $a \lesssim b$ or (ii) $b \lesssim a$ or (iii) $a = b$

Case I:-

If $a \lesssim b$

Since "a" is the maximal element of A so $a \lesssim b \Rightarrow a = b$ (by definition of maximal element)

Case II:- If $b \lesssim a$

Since "b" is the maximal element of A so $b \lesssim a \Rightarrow a = b$ (by def of maximal element)

Case III:- If $a = b$

then there is nothing to prove.

Hence in all cases $a = b$

Therefore in a totally ordered set the maximal element is unique.

* ——— ** ——— *

Theorem:- In a totally ordered set the minimal element is unique.

Proof

Let A be the totally ordered set and also $a, b \in A$ be two minimal elements.

Since A is totally ordered then

(i) $a \lesssim b$ or (ii) $b \lesssim a$ or

(iii) $a = b$

Case I:- If $a \lesssim b$

As "b" is the minimal element of A, so $a \lesssim b \Rightarrow a = b$ (by def. of minimal element)

Case II:- If $b \lesssim a$

As "a" is minimal element, so $b \lesssim a \Rightarrow a = b$ (by def of minimal element)

Case III:- If $a = b$

Then there is nothing to prove.

Hence in all cases $a = b$ Therefore in a totally ordered set the minimal element is unique

Theorem:- Every finite partially ordered set have atleast one minimal element.

Proof

Let A be a finite partially ordered set and suppose A has no minimal element.

Let $a_1 \in A$, As A has no minimal element then there exist $a_2 \in A$ such that $a_2 \precneqq a_1$

Again as by supposition of A has no minimal element so a_2 is not minimal element of A then $\exists$ an element $a_3 \in A$ such that $a_3 \precneqq a_2$

Continuing this process we have

$$a_{i+1} \precneqq a_i$$

where neither a_{i+1} nor a_i is the minimal element of A.

Since A is finite so our process terminate after a finite number of steps i.e there exist an element $a_n \in A$ such that no element of A preceeds "a_n".

So "a_n" will be the minimal element of A.

Hence every finite partially ordered set has atleast one minimal element.

——————**

Theorem:- Every finite partially ordered set has atleast one maximal element.

Proof

Let A be a finite partially ordered set and suppose A has no maximal element.

Let $a_1 \in A$, as A has no maximal element so $a_2 \in A$ such that $a_1 \precsim a_2$

Again as by supposition of A has no maximal element so a_2 is not maximal element of A then there exist an element $a_3 \in A$ s.t $a_2 \precsim a_3$.

Continuing this process we have

$$a_i \precsim a_{i+1}$$

where neither a_{i+1} nor a_i is the maximal element of A.

Since A is finite, so our process terminates after a finite no. of steps i.e there exist an element $a_n \in A$ such that no element of A dominates "a_n"

So "a_n" will be the maximal element of A.

Hence every finite partially ordered set has atleast one maximal element.

★——★★★——★

⇒ Similar Sets :-

Two ordered sets A and B are said to be similar sets if there exist a mapping $f: A \longrightarrow B$ such that

(i) f is bijective

(ii) for $x, y \in A$, $x \leq y \Leftrightarrow f(x) \leq f(y)$

It may be noted that the function defined above is called similarity mapping OR order preserving mapping.

If A is similar to B then it is denoted $A \simeq B$.

Example 1:- Let $N = \{1, 2, 3, 4, \ldots\}$

and $E = \{2, 4, 6, 8, \ldots\}$

Also $(N, \leq)$ and $(E, \leq)$ are ordered sets

Define a function $f: N \longrightarrow E$ by $f(x) = 2x$

Obviously f is bijective mapping.

Now if $x \leq y$ $x, y \in N$

$\Rightarrow 2x \leq 2y$

$\Rightarrow f(x) \leq f(y)$, $f(x), f(y) \in E$

So f is order preserving mapping

$\Rightarrow$ f is similarity mapping

$\Rightarrow N \simeq E$

Example 2:- Let $N = \{1, 2, 3, \ldots\}$ and

$M = \{-1, -2, -3, \ldots\}$

Define $f: N \longrightarrow M$ by $f(x) = -x$

f is bijective but f is not order preserving. As $1 < 2$ $1, 2 \in N$

but $f(1) \not< f(2)$ by def of f.

$\Rightarrow -1 \nless -2 \quad -1, -2 \in M.$

***

Theorem Let $f: A \to B$ be a similarity mapping from an ordered set A to a ordered set B then an element $a \in A$ is the first element of A iff $f(a)$ is the first element of B.

Proof

Given $f: A \to B$ is similarity mapping and "a" is the first element of A.

To prove $f(a)$ is the first element of B.

As "a" is the first element of A

so $a \lesssim x \quad \forall\ x \in A$

Since f is similarity mapping

so $f(a) \lesssim f(x) \quad \forall\ f(x) \in B$

$\Rightarrow f(a)$ is the first element of B.

Conversely,

assume $f(a)$ is the first element of B then $f(a) \lesssim y \quad \forall\ y \in B$.

As $f: A \to B$ is bijection so, for all $y \in B$ we have

$x \in A$ s.t $f(x) = y$

$\Rightarrow f(a) \lesssim f(x) \quad \forall\ f(x) \in B$

Since f is similarity mapping so f^{-1} exists and

$f^{-1}(f(a)) \lesssim f^{-1}(f(x))$

$\Rightarrow a \lesssim x \quad \forall\ x \in A$

$\Rightarrow a$ is the first element of A.

*——***——*

Theorem: If $f: A \to B$ be a similarity mapping from an ordered set A to an ordered set B then an element $a \in A$ is the last element of A iff $f(a)$ is the last element of B.

Proof

Given $f: A \to B$ is similarity mapping and "a" is the last element of A.

To prove $f(a)$ is the last element.

As "a" is the last element of A so

$$x \lesssim a \quad \forall \; x \in A$$

Since f is similarity mapping so

$$f(x) \lesssim f(a) \quad \forall \; f(x) \in B$$

$\Rightarrow f(a)$ is the last element of B.

Conversely,

Given $f(a)$ is the last element of B. To prove "a" is the last element of A.

As $f(a)$ is the last element of B

So $\quad f(x) \lesssim f(a) \quad \forall \; f(x) \in B$

Since f is similarity mapping so

$$x \lesssim a \quad \forall \; x \in A$$

$\Rightarrow$ "a" is the last element of A.

** ———— ** ———— **

Theorem:- Let $f: A \to B$ be a similarity mapping then an element $a \in A$ is minimal element of a iff $f(a)$ is the minimal element of B.

Proof Given $a \in A$ be the minimal element of A. To prove $f(a)$ is the minimal element of B.

Since a is the minimal element of A then

$$x \lesssim a \Rightarrow x = a \text{ (by def. of minimal element)}$$

In other words

$$x \not\lesssim a \quad \forall\, x \in A$$

Since f is similarity mapping so

$$f(x) \not\lesssim f(a) \quad \forall\, f(x) \in B$$

i.e $f(x) \lesssim f(a) \Rightarrow f(x) = f(a)$

$\Rightarrow$ $f(a)$ is minimal element of B.

Conversely,

Given $f(a)$ is minimal element of B. To prove "a" is minimal element of A.

Since $f(a)$ is minimal element of B then $f(x) \lesssim f(a) \Rightarrow f(x) = f(a)$

In other words

$$f(x) \not\lesssim f(a) \quad \forall\, f(x) \in B$$

Since f is similarity mapping so

$$x \not\lesssim a \quad \forall\, x \in A$$

i.e $x \lesssim a \Rightarrow x = a$

$\Rightarrow$ "a" is minimal element of A.

** ——— *** ——— **

Theorem:- Let $f: A \longrightarrow B$ be a similarity mapping then an element $a \in A$ is maximal element of A iff $f(a)$ is the maximal element of B.

Proof

Given $a \in A$ be the maximal element of A. To prove, $f(a)$ is the maximal element of B.

Since "a" is the maximal element of A then $a \leq x \Rightarrow a = x$ (by def. of maximal element)

In other words

$$a \nless x \quad \forall \; x \in A \longrightarrow ①$$

Since f is similarity mapping so

$① \Rightarrow f(a) \nless f(x) \quad \forall \; f(x) \in B$

i.e $f(a) \leq f(x) \Rightarrow f(a) = f(x)$

$\Rightarrow f(a)$ is maximal element of B.

Conversely,

Given $f(a)$ is the maximal element of B. To prove "a" is the maximal element of A. Since $f(a)$ is the maximal element of B then

$$f(a) \leq f(x) \Rightarrow f(a) = f(x)$$

In other words $f(a) \nless f(x) \quad \forall \; f(x) \in B$

Since f is similarity mapping

So $a \nless x \quad \forall \; x \in A$

i.e $a \leq x \Rightarrow a = x$

$\Rightarrow$ a is maximal element of A.

Theorem If A is totally ordered set and $B \simeq A$ then B is also totally ordered set.

Proof:- As $B \simeq A$, so let $f : B \longrightarrow A$ be a similarity mapping.

To prove B is totally ordered.

Suppose on the contrary B is not totally ordered then there exist $b_1, b_2 \in B$

s.t b_1 and b_2 are not comparable
i.e $b_1 \nleq b_2$ and $b_2 \nleq b_1$

As f is similarity mapping so
$f(b_1) \nleq f(b_2)$ and $f(b_2) \nleq f(b_1)$

But $b_1, b_2 \in B \Rightarrow f(b_1), f(b_2) \in A$ and
$f(b_1), f(b_2)$ are not comparable.
$\Rightarrow$ A is not totally ordered which is a contradiction. So our supposition is wrong and hence B is totally ordered.

* *

Theorem:- The relation of similarity is an equivalence relation.

Proof

Reflexive:- Let A be any ordered set then $I: A \longrightarrow A$ defined by $I(x) = x$ is a similar mapping
$\Rightarrow A \simeq A$
$\Rightarrow \simeq$ is reflexive.

Symmetric:- Let $A \simeq B$ then there exist $f: A \longrightarrow B$ a similarity mapping.
As $f: A \longrightarrow B$ is bijective so $f^{-1}: B \longrightarrow A$ exists and bijective.
Now we prove $f^{-1}: B \longrightarrow A$ is also similarity mapping.
Now as $f: A \longrightarrow B$ is similarity mapping so is bijection, then f is onto. so there exist $a_1, a_2 \in A$ s.t
$f(a_1) = b_1$ & $f(a_2) = b_2$ then let as
$b_1 \leq b_2$ means $f(a_1) \leq f(a_2)$

Also As f is similarity mapping. So
$a_1 \lesssim a_2 \Rightarrow f^{-1}(b_1) \lesssim f^{-1}(b_2)$
$\because f(a_1) = b_1 \Rightarrow f^{-1}(b_1) = a_1$ &
$f(a_2) = b_2 \Rightarrow f^{-1}(b_2) = a_2$

Then obvious $b_1 \lesssim b_2$ iff $f^{-1}(b_1) \lesssim f^{-1}(b_2)$
$\Rightarrow f^{-1} : B \longrightarrow A$ is similarity mapping.

Transitive:- Let $A \simeq B$ and $B \simeq C$, then there exists $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be similarity mapping. As f, g are bijection so $g \circ f : A \longrightarrow C$ is also bijection

Now we prove $g \circ f$ is similarity mapping.

Let $a_1, a_2 \in A$ s.t $a_1 \lesssim a_2$

Now $a_1 \lesssim a_2$ iff $f(a_1) \lesssim f(a_2)$ [$\because f$ is similarity mapping
iff $g(f(a_1)) \lesssim g(f(a_2))$ [$\because g$ is similarity mapping

$\Leftrightarrow (g \circ f)(a_1) \lesssim (g \circ f)(a_2)$

$\Rightarrow g \circ f : A \longrightarrow C$ is similarity mapping.
$\Rightarrow A \simeq C \Rightarrow \simeq$ is Transitive.

Hence similarity of sets is an equivalence relation.

★————★

⇒ Inverse Order:-

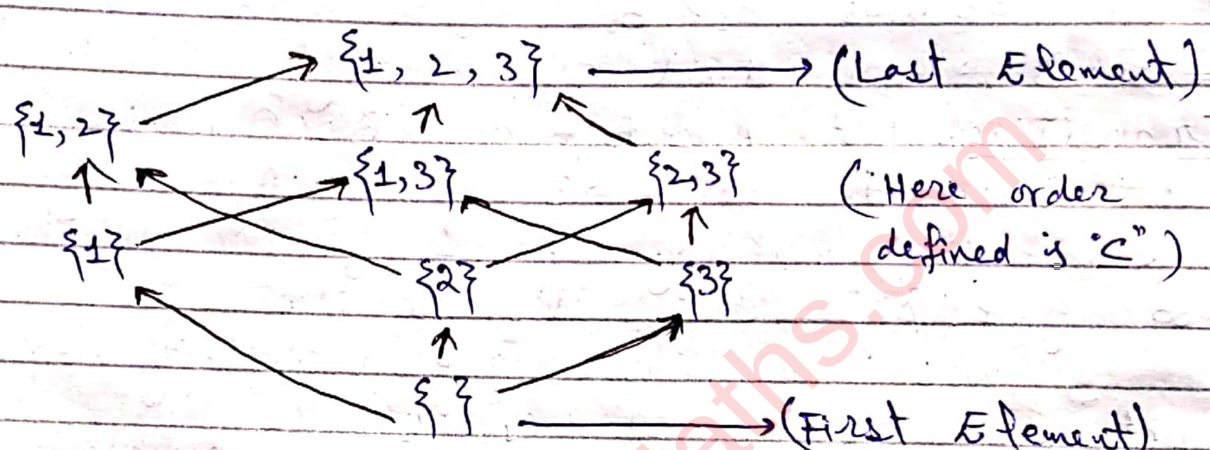
Let A be an ordered set and R be an ordered defined in A. Then inverse order is denoted by R^{-1} and defined as

$$x R^{-1} y \Leftrightarrow y R x \quad x, y \in A$$

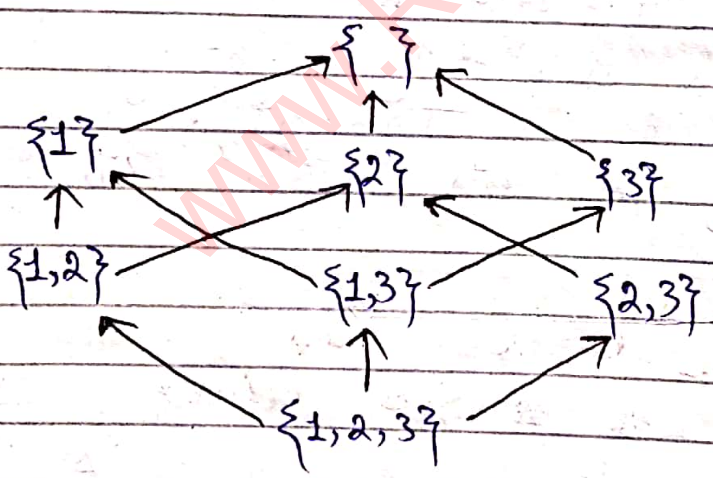
or $R^{-1} = \{(x, y) : x, y \in A \wedge (y, x) \in R\}$

Note:- "$x R^{-1} y$" read as "x inverse related to y"

Example:- $A = \{1, 2, 3\}$ xRy if $x \subseteq y$ and $(2^A, \subseteq)$ is partially ordered set.



As $(2^A, \subseteq)$ is partially ordered set then also $(2^A, \supseteq)$ is partially ordered set, consider its Hasse Diagram.



From above we see that by definition of inverse order, we get

$$x \subseteq y \iff y \supseteq x$$

i.e x is subset of $y \iff y$ is superset of x

***

Theorem:- If "a" and "b" are the first and last element respectively in partially ordered set A, then "a" and "b" will be the last & 1st elements respectively in the inverse order in A.

Proof

Let A be the set with partial order R in it and "a" be the first element of A w.r.t order R.

$\Rightarrow a \lesssim x \quad \forall x \in A$ with order R

$\Rightarrow (a, x) \in R \quad \forall x \in A$

$\Leftrightarrow (x, a) \in R^{-1} \quad \forall x \in A$

$\Rightarrow x \lesssim a \quad \forall x \in A$ with order R^{-1}

$\Rightarrow$ "a" is last element of A w.r.t order R^{-1}.

Let "b" be the last element in A then

$x \lesssim b \quad \forall x \in A$ with order R

$\Rightarrow (x, b) \in R \quad \forall x \in A$

$\Leftrightarrow (b, x) \in R^{-1} \quad \forall x \in A$

$\Rightarrow b \lesssim x \quad \forall x \in A$ with order R^{-1}

$\Rightarrow$ b is the first element w.r.t inverse order in A.

————————*

Theorem:- If A is totally ordered and $A \simeq B$ then B is totally ordered.

Proof

Let A be a totally ordered set and $A \sim B$.

To prove B is totally ordered.

As $A \simeq B$ So there exists a mapping $f: A \longrightarrow B$ which is bijective and

Preserve order.

Let $a, b \in B$ and $a \neq b$

Since f is onto so there exist $a', b' \in A$ such that

$$f(a') = a \quad , \quad f(b') = b$$

Since A is totally ordered, so either $a' < b'$ or $b' < a'$. In both the cases

$$f(a') < f(b') \quad \text{or} \quad f(b') < f(a')$$

$\Rightarrow \quad a < b \quad \text{or} \quad b < a$

$\Rightarrow$ "a" and "b" are comparable.

Hence B is totally ordered.

———————

⇒ Lower Bound:-

Let S be an ordered set (Partially or totally) and $A \subseteq S$ then an element $x \in S$ is said to be lower bound of A if $x \leq a \quad \forall \; a \in A$.
i.e $x \in S$ preceeds every element of A.

⇒ Infimum:-

An element x of an ordered set S is said to be infimum of $A \subseteq S$ if x is the last element of the set of lower bounds of A.

⇒ Upper Bound:-

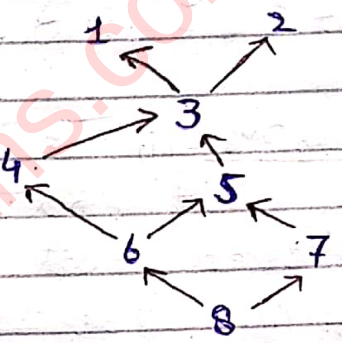
If S is an ordered set (partially or totally) and $A \subseteq S$ then an element $x \in S$ is said to be upper bound of A if $a \leq x \quad \forall \; a \in A$

⇒ Supremum:-

Let S be an ordered set and $A \subseteq S$. An element $x \in S$ is said to be supremum if "x" is the first element of the set of upper bounds of A.

Example:- $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$
$A = \{4, 5, 6\}$

The order in S is defined by the Hass diagram



Now

Lower Bound of A
$= \{6, 8\}$

Upper Bound of $A = \{1, 2, 3\}$

Infimum $= 6$

Supremum $= 3$

★———★★———★

⇒ Lexicographical Order:-

Let A and B be two totally ordered sets then the cartesian product $A \times B$ is

$A \times B = \{(a, b) : a \in A, b \in B\}$ can be totally ordered set as $(a, b) \leq (a', b')$ if $a < a'$ or if $a = a'$ then $b < b'$

The order defined above is called Lexicographical order.

Example:- Let $A = \{1, 2, 3\}$ & $B = \{4, 5\}$ and ordered defined on A & B is natural order. Now

$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$

Now $(1,4) < (2,5)$ because $1<2$

Also $(1,5) < (2,4)$

and $(1,4) < (1,5)$ because $4<5$

*———***———*

⇒ **Well ordered Set:-**

If A is an ordered set then A is said to be well ordered if every subset of A has the first element.

Example:- (i) The set of natural numbers $\mathbb{N}$ with natural order is well ordered set because every subset of $\mathbb{N}$ has first element.

ii) The set of integers $\mathbb{Z}$ with natural order is not well ordered

i.e $\{-1,-2,-3,\ldots\}$ is subset of $\mathbb{Z}$ which does not have first element.

———*———

Theorem: Every well ordered set is totally ordered.

Proof

Let A be a well ordered set,

To prove A is totally ordered.

Let $a,b \in A$ s.t $a \neq b \Rightarrow \{a,b\} \subseteq A$

As A is well ordered, so

$\Rightarrow \{a,b\}$ has first element

$\Rightarrow$ either a is the first element of $\{a,b\}$ or b is the first element of $\{a,b\}$

$\Rightarrow$ either $a \leq b$ or $b \leq a$

$\Rightarrow$ A is totally ordered

** ——— ** ——— **

Theorem:- Every subset of a well ordered set is well ordered.

Proof:- Let A be well ordered set and $B \subseteq A$.

To prove B is well ordered set.

Let $C \subseteq B$.

To prove C has 1st element.

Now $C \subseteq B \subseteq A \Rightarrow C \subseteq A$

and given that A is well ordered so C has first element.

Now $C \subseteq B$

$\Rightarrow$ Every subset of B has first element.

$\Rightarrow$ B is well ordered.

So, every subset of well ordered set is well ordered.

*** ——— ** ——— *

Theorem:- If A is well ordered set and B is similar to A then B is well ordered.

Proof:- Given, A is well ordered set

$\Rightarrow$ Every subset of A has first element

Also $A \simeq B$

$\Rightarrow$ There exist a mapping $f: A \longrightarrow B$ which is bijective and order preserving.

To prove B is well ordered.

For this we have to prove every

subset of B has first element.

Let $B' \subseteq B$.

Since A is well ordered so every subset of A say A' has first element say α.

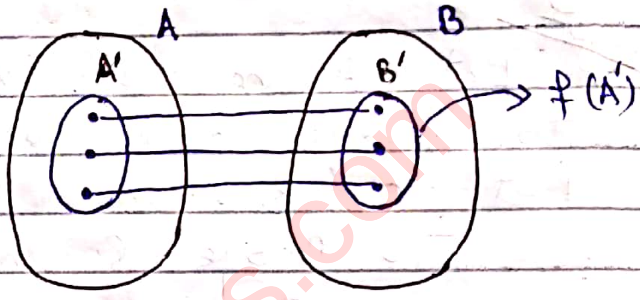
Let $f(A') = B'$

As α is the first element of A' and since f is similarity mapping so $f(a)$ is the first element of B'.

$\Rightarrow$ B is well order.

** ——— *** ——— **

$\Rightarrow$ Initial Segment:-



Let A be a well ordered set and $a \in A$, then initial segment of the element "a" is denoted by $S(a)$ and it is consists of all elements which strictly preceeds "a"

i.e $S(a) = \{x \in A : x < a\}$

It is clear that initial segment is a proper subset of A i.e

$$S(a) \subseteq A.$$

✗ wrong

Example 1:- Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

xRy if $x|y$; A is well ordered set.

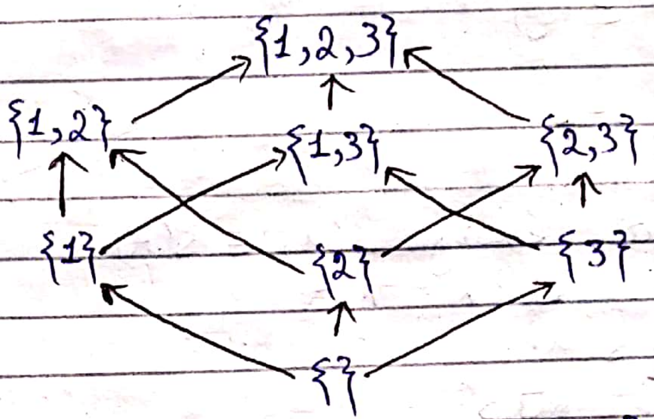
$S(4) = \{1, 2\} \subset A$, $S(1) = \phi \subset A$

$S(2) = \{1\} \subset A$, $S(3) = \{1\} \subset A$

$s(5) = \{1\} \subset A$, $s(6) = \{1,2,3\} \subset A$

$s(7) = \{1\} \subset A$, $s(8) = \{1,2,4\} \subset A$

* **Example 2:-** $A = \{1,2,3\}$ $\quad xRy$ if $x \subseteq y$



$s(\{1,2\}) = \{\phi, \{1\}, \{2\}\}$, $s(\{\}) = \phi$

$s(\{2,3\}) = \{\phi, \{2\}, \{3\}\}$, $s(\{2\}) = \{\{\}\}$ or ϕ

*** ———— * ———— ***

Theorem:- Let A be a well ordered set and $s(A)$ be the collection of all initial segments of A. Then the mapping $f: A \to s(A)$ defined as $f(a) = s(a)$ is a similarity mapping.

Proof: Given A is well ordered set and $f: A \to s(A)$ defined by $f(a) = s(a)$ for all $a \in A$.

To prove f is similarity mapping.

(i) f is one-one:-

Let $x, y \in A$ such that $x \neq y$. Consider the initial segment $s(x)$ and $s(y)$. Since A is well ordered so A is totally ordered

$\Rightarrow x$ and y are comparable.

$\Rightarrow \quad x < y \quad$ or $\quad y < x$

If $\quad x < y \Rightarrow x \in S(y)$ but $x \notin S(x)$

$\Rightarrow S(x) \neq S(y)$

$\Rightarrow f(x) \neq f(y)$

$\Rightarrow f$ is 1-1

ii) <u>f is onto:-</u>

Since for every $S(a) \in S(A)$ there exist $a \in A$ such that

$f(a) = S(a)$

$\Rightarrow f$ is onto

iii) <u>f is order preserving:-</u>

for this we have to prove if

$x \lesssim y \Leftrightarrow f(x) \lesssim f(y)$

Let $\quad a \in S(x)$ then

$a < x$ but $x \lesssim y$

$\Rightarrow a \in S(y)$

$\Rightarrow S(x) \subseteq S(y)$

$\Rightarrow f(x) \lesssim f(y)$

similarly

$f(x) \lesssim f(y) \Rightarrow x \lesssim y$

$\Rightarrow f$ is order preserving

$\Rightarrow f$ is similarity mapping.

★★———★———★———★———★★

V. Imp
Theorem:- Let A be well ordered set and B is subset of A. If $f: A \to B$ is similarity mapping then $a \leq f(a)$ $\forall$ $a \in A$.

Proof
Suppose there exist some elements of A which do not preceeds there images i.e
Let $D = \{x \in A : f(x) < x\}$
If D is empty then theorem is proved
Suppose $D \neq \phi$.
Obviously $D \subseteq A$. As A is well ordered
So D has 1st element say d_o, then
$f(d_o) < d_o$ (By structre of D)
Also f is similarity mapping
$\Rightarrow f(f(d_o)) < f(d_o)$
Put $f(d_o) = y$ $\Rightarrow f(y) < y$ & $y < d_o$
$\Rightarrow y \in D$ and $y < d_o$
which is not possible because d_o is the first element of D.
So our supposition is wrong and D is empty.
Hence $a \leq f(a)$ $\forall$ $a \in A$.

———*———**

V. Imp
Theorem:- Let A and B be well ordered similar sets then there exist only one similarity mapping from A to B.

Proof
Let f and g be two similarity mappings from A to B.
i.e $f: A \to B$, $g: A \to B$ and $f \neq g$
Since $f \neq g$ so There exist an element $a \in A$ such that $f(a) \neq g(a)$

As $a \in A \Rightarrow f(a), g(a) \in B$
Since B is well ordered $\Rightarrow B$ is totally ordered.

$\Rightarrow f(a)$ and $g(a)$ are comparable
$\Rightarrow f(a) < g(a)$ or $g(a) < f(a)$

Now

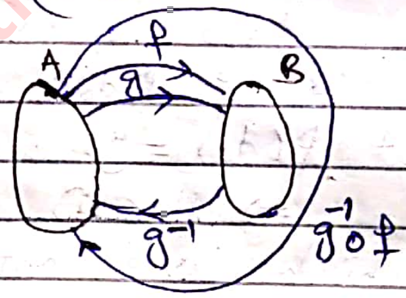
$f(a) < g(a)$

$\Rightarrow g^{-1}(f(a)) < g^{-1}(g(a))$ (since g is similarity mapping so g^{-1} exists)

$\Rightarrow (g^{-1} \circ f)(a) < (g^{-1} \circ g)(a)$

$\rightarrow (g^{-1} \circ f)(a) < a \longrightarrow (1)$ $(g^{-1} \circ g(a) = I(a) = a$

But $g^{-1} \circ f$ being the composition of two similarity mappings is a similarity mapping from A to A. and we know that



$a \leq f(a) \quad \forall\, a \in A$

if f is similarity mapping

Thus (1) is contradiction to an established result.

So our supposition is wrong.

Hence there is only one similarity mapping from A to B.

*** ———— ** ———— ***

Theorem– A well ordered set can not be similar to any of its initial segment.

Proof Let A be a well ordered set and suppose $f: A \longrightarrow S(a)$ be a similarity mapping from A into $S(a)$ then $\forall\, a \in A$

$f(a) \in S(a)$ [$\because f$ is bijective]

$\Rightarrow f(a) < a$ ($\because$ by def of $S(a)$)

which is contradiction to an established result which is $a \leq f(a)$ $\forall\, a \in A$. in a well ordered.

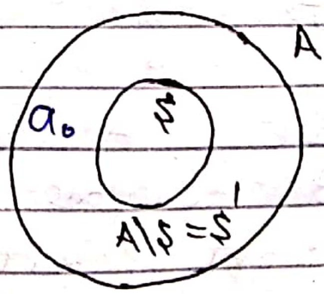
So our supposition is wrong.

Hence a well ordered set cannot be similar to any of its initial segment.

Theorem– Let A be a well ordered set and S be a subset of A with the property that if $a, b \in A$ and $a \leq b$ and $b \in S \Rightarrow a \in S$. Then $S = A$ or S is an initial segment of A.

Proof If $S = A$, Then there is nothing to prove. If $S \neq A$ i.e $S' = A \setminus S$ is non empty.

Also $A \setminus S \subset A$



Since A is well ordered and $S' = A \setminus S$ is a subset of A. So $S' = A \setminus S$ has a first element say a_0 we show that $S = S(a_0)$.

For this we have to prove
(i) $S(a_0) \subseteq S$
(ii) $S \subseteq S(a_0)$

Let $x \in S(a_0) \Rightarrow x < a_0$ (by def of $S(a_0)$)
$\Rightarrow x \notin A \setminus S = S'$ [$\because a_0$ is first element of S']
$\Rightarrow x \in S$
$\Rightarrow S(a_0) \subseteq S \longrightarrow (1)$

For (ii) we use contrapositive statement.
i.e if $y \notin S(a_0) \Rightarrow y \notin S$

Now if $y \notin S(a_0) \Rightarrow a_0 \leq y$
If $y \in S$ then $a_0 \in S$ (by given condition) which is a contradiction because a_0 is the 1st element of $A \setminus S = S'$
Hence $y \notin S$
$\Rightarrow S \subseteq S(a_0) \longrightarrow (2)$

From (1) & (2) $S = S(a_0)$
So S is an initial segment of A.

Theorem: Prove that two different initial segements of a well ordered set can not be similar.

Proof: Let A be a well ordered set and suppose $S(a)$ and $S(b)$ be two different initial segments of A such that $S(a) \simeq S(b)$, $a \neq b$
Since A is well ordered so is totally ordered.

either $a < b$ or $b < a$, $a, b \in A$

Let $a < b$ then $a \in S(b)$

If $x \in S(a)$ then

$$x < a \text{ and } a < b$$

$$\Rightarrow x \in S(b)$$

$$\Rightarrow S(a) \subseteq S(b) \subset A$$

Now $S(b)$ is well ordered being the subset of a well ordered set A. But we know a well ordered set can not be similar to its initial segment

So our supposition is wrong

So $S(a) \not\sim S(b)$

Hence two different initial segments of a well ordered set can not be similar.

S(b) well ordered ہے اور S(a) اس کا سیگمنٹ ہے
اس لیے S(b) اور S(a) similar نہیں ہوں

Note:- This Theorem can also be stated as follows "If two initial segments of a well ordered set are similar then they must be equal.

Theorem:- Let A and B be two well ordered sets and let an initial segment $S(a)$ of A is similar to an initial segment of B, then $S(a)$ is similar to a unique initial segment of B.

Proof Suppose $S(a)$ is not similar to a unique initial segment of B. i.e there exists two initial segements $S(b)$ & $S(b')$ such that

$s(a) \simeq s(b)$

$s(a) \simeq s(b')$

Since similarity is equivallence relation so it is transitive so

$s(b) \simeq s(a)$ and $s(a) \simeq s(b')$

$\Rightarrow s(b) \simeq s(b')$

which is a contradiction to the fact that "Two different initial segments of a well ordered set cannot be similar"

So our supposition is wrong.

Hence $s(b) = s(b')$

So $s(a) \simeq s(b)$ which is unique i.e $s(a)$ is similar to a unique initial segment of B.

***

Theorem :- Let A and B be well ordered sets such that an initial segment $s(a)$ of A is similar to an initial segment $s(b)$ of B. Then each initial segment of $s(a)$ is similar to an initial segment of $s(b)$ that is

$a' \lesssim a \Rightarrow s(a') \simeq s(b')$ where $b' \lesssim b$

Also $f[s(a')] = s(b')$

Proof

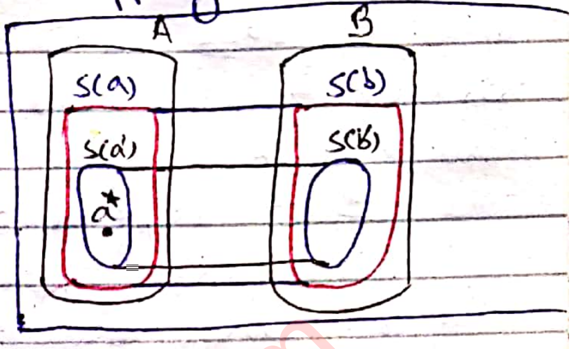
Since $s(a) \simeq s(b)$, therefore there exists a mapping $f: s(a) \longrightarrow s(b)$ which is a similarity mapping.

$\Rightarrow f(a') = b'$, where $a' \in s(a)$ & $b' \in s(b)$

Let $a^{*} \in s(a')$ ~~to~~

$t \lessdot x \Rightarrow s(t) \subseteq s(x) \Rightarrow s(t)$ is initial segment of $s(x)$

$\Rightarrow a^* < a'$ (by def of initial segment)
Since f is similarity mapping so
$f(a^*) < f(a')$



$\Rightarrow f(a^*) < b'$ ($\because f(a') = b'$)

$\Rightarrow f(a^*) \in s(b')$ [by def. of initial seg

which shows that every element of $s(a')$ is mapped to an element of $s(b')$.
Now f restricted on $s(a')$ to $s(b')$ is both one-one and onto and preserves order.

$$\Rightarrow s(a') \simeq s(b')$$

Since f is similarity mapping so f is bijective and hence

$$f[s(a')] = s(b')$$

————*

Theorem:- Let A and B be two well ordered set and let $S = \{x \in A : s(x) \simeq s(y), y \in B\}$ then $S = A$ or S is an initial segment of A.

Proof) Let $x \in S$ and $t \lessdot x$ then $s(t)$ is an initial segment of $s(x)$.
Since $s(x) \simeq s(y)$ therefore $s(t)$ is similar to an initial segment of $s(y)$ (By Previous Theorem)

$\Rightarrow t \in S$ [By construction of S]

In other words if $x \in S$ and $t \lessdot x$

$$\Rightarrow t \in S$$

If $S = A$ then there is nothing to prove.
If $S \neq A$ i.e $A \setminus S = S'$ is non-empty
Also $A \setminus S \subset A$.
Since A is well ordered and $S' = A \setminus S$ is a subset of A, so $S' = A \setminus S$ has the first element say a_0.
We show that $S = S(a_0)$.
For this we have to prove
i) $S(a_0) \subseteq S$ (ii) $S \subseteq S(a_0)$

Let $x \in S(a_0) \Rightarrow x < a_0$
$\Rightarrow x \notin A \setminus S = S'$ [$\because a_0$ is first element of S']
$\Rightarrow x \in S$
$\Rightarrow S(a_0) \subseteq S \longrightarrow ①$

For (ii) we use contrapositive definition
Now if $y \notin S(a_0)$
$\Rightarrow a_0 \leq y$
If $y \in S$ then
$a_0 \in S$ (by given condition)
which is a contradiction
because a_0 is first element of $A \setminus S = S'$

$y \notin S(a_0) \Rightarrow y \notin S$
$\Rightarrow S \subseteq S(a_0)$
Mean کہ اگر y ، $S(a_0)$ میں نہیں
تو S میں بھی نہیں ہو گا مطلب کہ S
میں $S(a_0)$ کے element ہی ہیں گے

Hence $y \notin S \Rightarrow S \subseteq S(a_0) \longrightarrow ②$
from ① & ② $S = S(a_0)$

** ——————— **

Theorem:- Let A and B be well ordered set
and $S = \{x \in A : S(x) \simeq S(y) \text{ where } y \in B\}$
$T = \{y \in B : S(y) \simeq S(x) \text{ where } x \in A\}$
Then $S \simeq T$

vice versa :- For each $y \in T$ $\exists$ $x \in S$ s.t $s(y) \simeq s(x)$

Proof Define a mapping $f: S \longrightarrow T$ by
$f(x) = y$ if $s(x) \simeq s(y)$, $y \in T$
Let $x \in S$ then $s(x)$ is similar to a unique initial segment $s(y)$ of B. Therefore to each $x \in S$ there exists a $y \in T$ st
$s(x) \simeq s(y)$ and vice versa
Therefore f is both one-one and onto.
Next we show that f is order preserving
Let $x', x \in S$ s.t $x' < x$
$\Rightarrow x' \in s(x)$ [by def. of initial segment

Therefore $s(x')$ is initial segment of $s(x)$
$\therefore$ $s(x')$ is similar to an initial segment $s(y')$ of $s(y)$ i.e
$s(x') \simeq s(y') \Rightarrow f(x') = y'$
As $y' \in s(y) \Rightarrow y' < y$
$\Rightarrow f(x') < f(x) \Rightarrow f$ is order preserving
$\Rightarrow f$ is similarity mapping
$\Rightarrow S \simeq T$

****

⇒ Principle of Transfinite Induction :-

Statement:-
Set S be a subset of a well ordered set A such that
(i) $a_0 \in S$ (ii) $s(a) \subseteq S \Rightarrow a \in S$
Then $S = A$
Here a_0 is the first element of A.

Proof
suppose $S \neq A$ then $A \setminus S = T$ is non-empty. Also $A \setminus S = T \subset A$

Since A is well ordered set Then T must have first element say t_0.
Now Consider $S(t_0) = \{x \in A : x < t_0\}$
Let $x \in S(t_0)$ then $x < t_0$
since t_0 is suppose to be first element of T. so $x \notin T = A \setminus S$
$\Rightarrow x \in S \Rightarrow S(t_0) \subseteq S$
$\Rightarrow t_0 \in S$ (by iii)
which is a contradiction because t_0 is first element of T and $t_0 \notin S$
Hence $S = A$.

*——**——*

⇒ **Definition:-**
If A and B are well ordered sets and A is similar to an initial segment of B then A is said to be shorter than B and in this case B is said to be longer than A.

——*——*

Theorem:- Let A and B be two well ordered sets then A is shorter than B or $A \simeq B$ or A is longer than B.

Proof Let S and T be defined as follows
$$S = \{x \in A : S(x) \simeq S(y) \text{ where } y \in B\}$$
$$T = \{y \in B : S(y) \simeq S(x) \text{ where } x \in A\}$$
Then $S \simeq T$

Case I: If $S = A$ & $T = B$
Since $S \simeq T \Rightarrow A \simeq B$

Case II :- If $S = A$ & T is an initial segment of B i.e $T = s(b)$, $b \in B$

Since $S \simeq T \Rightarrow A \simeq s(b)$

$\Rightarrow A$ is similar to an initial segment of B

$\Rightarrow A$ is shorter than B.

Case III :- If S is an initial segment of A i.e $S = s(a)$, $a \in A$ and $T = B$

Since $S \simeq T \Rightarrow s(a) \simeq B$

$\Rightarrow B$ is similar to an initial segment of $A \Rightarrow B$ is shorter than A

or A is longer than B.

Case IV If $S = s(a)$, $a \in A$ and $T = s(b)$, $b \in B$

Since $S \simeq T \Rightarrow s(a) \simeq s(b)$

$\Rightarrow a \in S$ (by def. of S)

$\Rightarrow a \in s(a)$ $[\because S = s(a)]$

which is a contradiction because an element itself can not belong to its initial segment.

So this case is not possible.

Hence in all cases.

A is shorter than B or $A \simeq B$ or A is Longer than B.

** ———— * ———— **

Theorem Let $\mathcal{A}$ be the family of initial segments of a well ordered set A. Then There ~~exist~~ is an initial segment $s(a) \in \mathcal{A}$ which is shorter than every other initial segment in $\mathcal{A}$.

Proof Since we know that $A \simeq S(A)$,

where $S(A)$ is the family of all initial segments of A.
ordered by set inclusion.
Since A is well ordered so $S(A)$ is well ordered.

Also $A \subseteq S(A)$

Since $S(A)$ is well ordered therefore A has first element say $s(a)$

$$s(a) \subset s(x) \quad \forall\ s(x) \in A$$

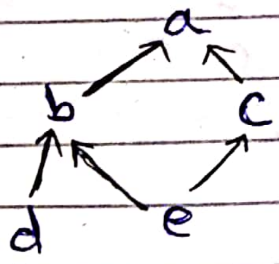
$\Rightarrow$ $s(a)$ is shorter than every other initial segment in A.

————**

⇒ Immediate Successor And Immediate Predecessor:-

If A is well ordered set then we say that any element $a \in A$ is the immediate successor of an element $b \in A$ if there does not exist an element $c \in A$ s.t $b < c < a$ and in such case b is said to be immediate predecessor of a.

Example 1:- Let $A = \{a, b, c, d, e\}$, The order is shown by Hasse diagram



Here "b" is an immediate successor of both "d" and "e" and "e" is immediate predecessor of both "b" and "c".

Example 2:- Consider the set Q of rational numbers. No element of Q has an immediate successor or an immediate predecessor because the set Q of rationals is a dense set.

** ——— *** ——— **

Theorem:- Every element in a well ordered set has an immediate successor except the last element.

Proof Let A be a well ordered set and $a \in A$. Consider the set

$$M = \{ x : x > a \}$$

If $M = \phi$, then 'a' is the last element of A. we consider $M \neq \phi$ then M is being the subset of well ordered set A has a first element say α.

Then α is an immediate successor of 'a' because $a < \beta < \alpha$

$\Rightarrow \beta$ is the first element of M

Which is contradiction because 'α' is the first element of M.

——— *** ———

⇒ Ordinal Numbers:-

If A is well ordered set and let λ denote the family of all well ordered sets which are similar to A then λ is called ordinal number of A. and written as $\lambda = \text{ord}(A)$

⇒ Definition:-

The ordinal numbers of each of the following well ordered sets $\{\}, \{1\}, \{1,2\}, \{1,2,3\}, \ldots$ are denoted by $0, 1, 2, 3, \ldots$ respectively. These are called finite ordinals.

Note, $2 = \{A : A \simeq \{1,2\}\}$

$3 = \{A : A \simeq \{1,2,3\}\}$

⇒ Definition:-

The ordinal number of the set N of natural numbers is denoted by ω. i.e $\operatorname{ord}(N) = \omega$

⇒ Definition:-

If $\lambda = \operatorname{ord}(A)$, $\mu = \operatorname{ord}(B)$ then

(i) $\lambda < \mu$ if A is shorter than B.

(ii) $\lambda = \mu$ if A is similar to B.

(iii) $\lambda > \mu$ if A is longer than B.

⇒ Addition of Ordinals:-

Let λ and μ be the ordinal numbers such that $\operatorname{ord}(A) = \lambda$, $\operatorname{ord}(B) = \mu$ where A and B are disjoint then

$$\lambda + \mu = \operatorname{ord}(A \cup B) = \operatorname{ord}(A;B)$$

Example:- Let $A = \{1,2,3\}$, $B = \{a, b\}$

$\operatorname{ord}(A) = 3$, $\operatorname{ord}(B) = 2$

$A \cup B = \{1,2,3,a,b\} \Rightarrow \operatorname{ord}(A \cup B) = 5$

$\Rightarrow \operatorname{ord}(A \cup B) = 5 = 3 + 2 = \operatorname{ord}(A) + \operatorname{ord}(B)$

Example:- $M = \{1, 3, 5, 7, \ldots\}$

$N = \{2, 4, 6, 8, \ldots\}$

$(M;N) = \{1, 3, 5, 7, \ldots, 2, 4, 6, 8, \ldots\}$

$S(5) = \{1, 3\}$, $S(7) = \{1, 3, 5\}$

$S(2) = \{1, 3, 5, 7, \ldots\}$ and

$S(4) = \{1, 3, 5, 7, \ldots, 2\}$

* * *

Theorem: Show that ordinal addition is not commutative.

Proof:

Let $\omega = \text{ord}(N)$ & $n = \text{ord}(A)$, where $A = \{a_1, a_2, a_3, \ldots, a_n\}$

Then $n + \omega = \text{ord}(\{A;N\})$ and

$\omega + n = \text{ord}(\{N;A\})$

$(A;N) = \{a_1, a_2, a_3, \ldots, a_n, 1, 2, 3, \ldots\}$

Then $\{A;N\} \simeq N$ under $f : \{A;N\} \longrightarrow N$

defined by
$$f(x) = \begin{cases} i & \text{if } x = a_i \quad , i = 1, 2, 3, \ldots \\ n+i & \text{if } x = i \quad , i = 1, 2, 3, \ldots \end{cases}$$

$\Rightarrow \text{ord}(\{A;N\}) = \text{ord}(N)$

$\Rightarrow n + \omega = \omega \longrightarrow (1)$

$(N;A) = \{1, 2, 3, \ldots, a_1, a_2, a_3, \ldots, a_n\}$

$S(a_1) = \{1, 2, 3, \ldots\} = N \subseteq \{N;A\}$

Now $N \simeq S(a_1) \Rightarrow N$ is shorter than $\{N;A\}$

$\Rightarrow \text{ord}(N) < \text{ord}\{N;A\} \Rightarrow \omega < \omega + n \longrightarrow (2)$

from (1) & (2) $n + \omega = \omega < \omega + n$

$\Rightarrow n + \omega < \omega + n$

Thus addition of ordinal numbers is not commutative.

★————★

Alternate Question: Prove that $n+\omega < \omega+n$

OR $n+\omega \neq \omega+n$

★————★————★

Theorem: i) Addition of ordinal number satisfies the associative law i.e

$$(\lambda+\mu)+\eta = \lambda+(\mu+\eta)$$

ii) The ordinal number 0 is an additive identity in the set of ordinals.

Proof

i) Let λ, μ, η be the ordinal numbers of mutually disjoint well ordered set A, B and C respectively then

$$\lambda + (\mu+\eta) = \text{ord}(A) + \text{ord}(\{B; C\})$$

$$= \text{ord}(\{A; \{B; C\}\})$$

$$= \text{ord}(\{\{A; B\}; C\}) \quad \because \text{union of sets is associative}$$

$$= \text{ord}(\{A; B\}) + \text{ord}(C)$$

$$\Rightarrow \lambda+(\mu+\eta) = (\lambda+\mu)+\eta$$

ii)

Considered a well ordered set A s.t $\text{ord}(A) = \lambda$ and $\text{ord}(\{\}) = 0$. Then

$$\lambda + 0 = \text{ord}(A; \phi) = \text{ord}(A) = \lambda$$

similarly $0+\lambda = \text{ord}(\phi; A) = \text{ord}(A) = \lambda$

$$\Rightarrow 0+\lambda = \lambda+0 = \lambda$$

$\Rightarrow$ 0 is an additive identity in the set of ordinals.

——★★——

⇒ Ordinal Multiplication:-

Let λ and μ be the ordinal numbers and let A and B be well ordered sets s.t $\lambda = ord(A)$ & $\mu = ord(B)$ then $\lambda\mu = ord(A \times B)$, where $A \times B$ is ordered reverse lexicographically. The product $A \times B$ is ordered reverse Lexicographically mean

$$(a, a') < (b, b') \begin{cases} \text{if } a' < b' \text{ or} \\ \text{if } a' = b' \text{ then } a < b \end{cases}$$

***

Theorem- i) The Associative law holds for multiplication in ordinal numbers.

ii) The left distributive law of multiplication over addition holds i.e $\lambda(\mu + \eta) = \lambda\mu + \lambda\eta$

iii) The ordinals 1 is multiplicative identity element i.e $1 \cdot \lambda = \lambda \cdot 1 = \lambda$

Proof

i) Let $\lambda = ord(A)$, $\mu = ord(B)$, $\eta = ord(C)$

$$\begin{aligned}
\lambda(\mu\eta) &= ord(A) \cdot ord(B \times C) \\
&= ord(A \times (B \times C)) \\
&= ord((A \times B) \times C) \quad \because A \times (B \times C) \simeq (A \times B) \times C \\
&= ord(A \times B) \cdot ord(C) \\
&= (\lambda\mu)\eta
\end{aligned}$$

ii) Let A, B, C be well ordered sets s.t $B \cap C = \phi$.

$ord(A) = \lambda$, $ord(B) = \mu$, $ord(C) = \eta$

$$\begin{aligned}
\lambda(\mu + \eta) &= ord(A) \cdot ord(B; C) \\
&= ord(A \times (B; C))
\end{aligned}$$

$\lambda(\mu+\eta) = \text{ord}((A\times B);(A\times C))$ $\left[\because A\times(B\cup C) = (A\times B)\cup(A\times C)\right.$

$= \text{ord}(A\times B) + \text{ord}(A\times C)$

$= \lambda\mu + \lambda\eta$

iii) Let $\lambda = \text{ord}(A)$ and let $B = \{1\}$,

$\text{ord}(B) = 1$

$\lambda\cdot 1 = \text{ord}(A\times B)$ but $A\times B \simeq B\times A$

$\Rightarrow \lambda\cdot 1 = \text{ord}(B\times A) = 1\cdot\lambda$

Moreover $A\times B \simeq A$ so $\lambda\cdot 1 = 1\cdot\lambda = \lambda$

*———**———*

Question:- Show by an example that multiplication of ordinals is not commutative in general OR Show that $2\omega < \omega 2$, where ω has its usual meaning or show that $2\omega \neq \omega 2$.

Solution

$N = \{1,2,3,\ldots\}$

Let $\omega = \text{ord}(\{1,2,3,\ldots\})$ and

$2 = \text{ord}(\{a,b\})$

$\omega 2 = \text{ord}(\{(1,a),(2,a),(3,a),\ldots\ldots,(1,b),(2,b),(3,b),\ldots\ldots\})$

$= \text{ord}(A)$

where

$A = \{(1,a),(2,a),(3,a),\ldots\ldots,(1,b),(2,b),(3,b),\ldots\}$

It can be observed that the set of natural numbers is similar to the initial segment $S(1,b)$ of A. which shows

$\text{ord}(N) < \text{ord}(A)$

$\Rightarrow \omega < \omega 2 \longrightarrow (1)$

Now $2\omega = \text{ord}(\{(a,1),(b,1),(a,2),(b,2),(a,3),(b,3),\ldots\})$

$= ord(B)$, where

$B = \{(a,1),(b,1),(a,2),(b,2),(a,3),(b,3),\ldots\}$

Define $f: N \longrightarrow B$ by

$$f(n) = \begin{cases} (a, \frac{n}{2}) & \text{if } n \text{ is even} \\ (b, \frac{n+1}{2}) & \text{if } n \text{ is odd} \end{cases}$$

The mapping is clearly a similarity mapping

$\Rightarrow N \simeq B$

$\Rightarrow ord(N) = ord(B) \Rightarrow \omega = 2\omega \longrightarrow ②$

from ① & ② $2\omega = \omega < \omega 2$

$\Rightarrow 2\omega < \omega 2$

So multiplication is not commutative in ordinals in general.

** —— ** —— **

Question:- Prove by giving a countre example that the right distributive law of multiplication over addition for ordinals is not true in general. i.e

$(\lambda + \mu)\eta \neq \lambda\eta + \mu\eta$ in general.

Solution

Since $2\omega = \omega$

$\Rightarrow (1+1)\omega = \omega \longrightarrow ①$

Consider $1.\omega + 1.\omega = \omega.1 + \omega.1$ [$\because$ 1 is multiplicative identity in ordinals]

$= \omega(1+1)$ [$\because$ left distributive law holds]

$\Rightarrow 1.\omega + 1.\omega = \omega 2 > \omega$

$\Rightarrow 1.\omega + 1.\omega > \omega \longrightarrow ②$

from ① & ②

$(1+1)\omega \neq 1.\omega + 1.\omega$

$\Rightarrow$ Right distributive law of multiplication over addition for ordinals does not hold.

Theorem:- If $\lambda = ord(A)$ and $\mu < \lambda$ then $\exists$ a unique initial segment say $s(a)$ of A such that $\mu = ord(s(a))$

Proof

Let $s(a)$ and $s(b)$ be two initial segments s.t $\mu = ord(s(a))$ and $\mu = ord(s(b))$.
As $ord(s(a)) = ord(s(b)) \Rightarrow s(a) \simeq s(b)$
which is contradiction to the fact that "No two initial segments of a well ordered set are similar".
So there exist a unique initial segment $s(a)$ of A s.t

$$\mu = ord(s(a))$$

—★——★★——★—

Theorem:- Let $s(\lambda)$ be the set of ordinals less than the ordinal λ. Then

$$\lambda = ord(s(\lambda))$$

Proof

Let $\lambda = ord(A)$ and let $S(A)$ denotes the family of all initial segments of A.
Then

$$A \simeq S(A) \Rightarrow ord(A) = ord(S(A))$$

$$\Rightarrow \lambda = ord(S(A)) \quad \Rightarrow ord(S(A)) = \lambda$$

It is sufficient to show that

$$s(\lambda) \simeq S(A)$$

Let $\mu \in s(\lambda) \Rightarrow \mu < \lambda$ then by previous theorem there exist a unique initial segment $s(a)$ of A s.t $\mu = ord(s(a))$
Define a mapping $f: s(\lambda) \longrightarrow S(A)$ by

$$f(\mu) = s(a) \quad , \mu \in s(\lambda) \quad , s(a) \in S(A)$$

Now we show that f is similarity

mapping

<u>f is 1-1:-</u> Let $\mu, \nu \in S(\lambda)$ s.t

$f(\mu) = f(\nu)$

$\Rightarrow S(a) = S(b) \Rightarrow \mu = \nu$

$\Rightarrow f$ is 1-1

Initial segment 2 برابر ہوں تو ان کی ordinality بھی برابر ہوگی

<u>f is onto:-</u> for each $S(a) \in S(A)$ there exist some ordinal μ (say) s.t $f(\mu) = S(a)$ for $a \in A$ $\Rightarrow f$ is onto.

<u>f is order preserving:-</u> Let $\mu, \eta \in S(\lambda)$ s.t $\mu < \eta$ and

$\mu = \text{ord}(S(a))$ and $\eta = \text{ord}(S(b))$,

where $S(a), S(b) \in S(A)$

Then by the definition of inequality of ordinal number $S(a)$ must be shorter than $S(b)$ $\Rightarrow S(a) < S(b)$

$\Rightarrow f(\mu) < f(\eta)$

$\Rightarrow f$ is order preserving.

Hence f is similarity mapping

$\Rightarrow S(\lambda) \simeq S(A) \Rightarrow \text{ord}(S(\lambda)) = \text{ord}(S(A)) = \lambda$

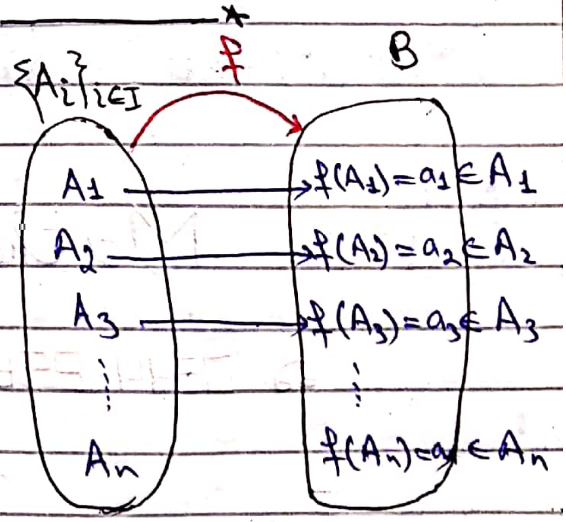
$\Rightarrow \text{ord}(S(\lambda)) = \lambda$

*———***———*

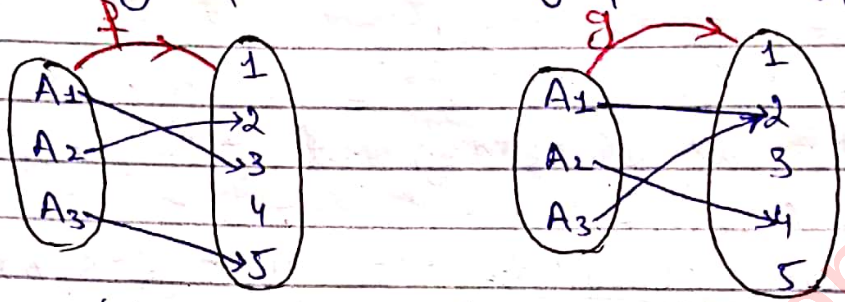
⇒ Choice Function:-

Let $\{A_i\}_{i \in I}$ be a family of non-empty subsets of B then a function $f : \{A_i\}_{i \in I} \longrightarrow B$ is called a choice function if for every $i \in I$ $f(A_i) \in A_i$.

i.e if the image of each set is an element in the set



Example: Consider the subsets
$A_1 = \{1,2,3\}$, $A_2 = \{1,3,4\}$, $A_3 = \{2,5\}$
of $B = \{1,2,3,4,5\}$ and consider the following functions of $\{A_1, A_2, A_3\}$ into B



Note that f is not a choice function because $f(A_2) = 2 \notin A_2$.
Note further that g is a choice function since $g(A_1) = 2 \in A_1$
$g(A_2) = 4 \in A_2$ and $g(A_3) = 2 \in A_3$

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MSc. Mathematics (P.U)

0344-8563284

ASSIGNMENT

Theorem:- To prove $N \times N$ is Denumerable.

Proof Every $n \in N$ can be uniquely written as
$$n = 2^{r-1}(2s-1) \quad , \text{where } r, s \in N$$

Define $f: N \to N \times N$ by
$$f(n) = (r, s)$$

f is 1-1:-
$$f(n_1) = f(n_2)$$
$$(r_1, s_1) = (r_2, s_2)$$
$$\Rightarrow r_1 = r_2 \quad \& \quad s_1 = s_2$$
$$\Rightarrow 2^{r_1} = 2^{r_2} \quad \& \quad 2s_1 = 2s_2$$
$$\Rightarrow 2^{r_1 - 1}(2s_1 - 1) = 2^{r_2 - 1}(2s_2 - 1)$$
$$\Rightarrow n_1 = n_2$$
$$\Rightarrow f \text{ is } 1-1$$

f is onto:- since every ~~no~~ $(r,s) \in N \times N$ is the image of some $n \in N$ i.e $f(n) = (r,s)$
$\therefore$ f is onto & Hence
f is bijective
$$\Rightarrow N \times N \sim N$$
$$\Rightarrow N \times N \text{ is Denumerable.}$$

**

Theorem:- The relation of being Equivalent is an equivalance Relation.

Proof Let '$\mathcal{C}$' be the class of all sets and '$\sim$' denotes the relation of being equivalent among sets.

Then we prove that the relation '$\sim$' is an equivallence relation.

(i) '$\sim$' is Reflexive:-

we have to prove that $A \sim A \ \forall A \in \mathcal{C}_S$

Now define a mapping $I: A \to A$ s.t $I(x) = x \ \forall x \in A$. Sine I is bijective

So $A \sim A$

$\Rightarrow \sim$ is Reflexive.

(ii) "$\sim$" is Symmetric:-

Now we prove that if $A \sim B$ then $B \sim A$, where $A, B \in \mathcal{C}_S$

Suppose that $A \sim B$ then by definition there exist a mapping $f: A \to B$ which is bijective.

Since inverse of bijective mapping is bijective therefore $f: B \to A$ is bijective

Thus $B \sim A$

$\Rightarrow \sim$ is symmetric.

(iii) "$\sim$" is Transitive:-

To show that if $A \sim B$ & $B \sim C$ then $A \sim C$, where $A, B, C \in \mathcal{C}_S$

Let $A \sim B$, then there exist a bijective mapping $f: A \to B$.

Also $B \sim C$, then there exist a bijective mapping $g: B \to C$. Now define a mapping $gof: A \to C$

Since composition of two bijective mapping is bijective.

Therefore $gof: A \to C$ is bijective

Thus $A \sim C$

$\Rightarrow \sim$ is Transitive.

Conversely '$\sim$' is an equivalence relation.

***

Exercise:- Show that $\mathbb{R} \sim \mathbb{R}^+$

Solution

Define a mapping $f: \mathbb{R} \to \mathbb{R}^+$ by $f(x) = e^x \quad \forall\ x \in \mathbb{R}$ ($\because e^x$ has image in $\mathbb{R}^+$)

<u>f is 1-1:-</u> Let $f(x) = f(y)$

$\Rightarrow e^x = e^y \quad \Rightarrow e^{x-y} = 1 = e^0$

$\Rightarrow x - y = 0 \quad \Rightarrow x = y$

<u>f is onto:-</u>

Since any $x \in \mathbb{R}^+$ is the image of $\ln x \in \mathbb{R}$ under f

i.e $f(\ln x) = e^{\ln x} = x \in \mathbb{R}^+$ for $\ln x \in \mathbb{R}$

Hence f is bijective

$\Rightarrow \mathbb{R} \sim \mathbb{R}^+$

*

Exercise: Show that $[0,1) \sim (0,1]$

Proof: Define a mapping $f: [0,1) \to (0,1]$ by $f(x) = 1 - x \quad \forall\ x \in [0,1)$

<u>f is 1-1:-</u> $f(x) = f(y)$ for some $x, y \in [0,1)$

$\Rightarrow 1 - x = 1 - y$

$\Rightarrow x = y$

f is onto:- Since $x \in (0,1]$ is the image of $(1-x) \in [0,1)$ under f

i.e $f(1-x) = 1-(1-x) = x$

Hence f is bijective

$\Rightarrow$ $[0,1) \sim (0,1]$

Theorem: Suppose A, B, C, D are sets with $A \sim C$ and $B \sim D$ then prove that

i) $A \times B \sim C \times D$

ii) if A & B are disjoint and C & D also then $A \cup B \sim C \cup D$

Proof: 1) Since $A \sim C$ so there exist a bijective function $f: A \to C$

Also $B \sim D$ so there exist a bijective function $g: B \to D$.

We have to prove that

$$A \times B \sim C \times D$$

For this define a function $H: A \times B \to C \times D$ by $H(a,b) = (f(a), g(b))$

H is 1-1:-

$$H(a_1, b_1) = H(a_2, b_2)$$

$$\Rightarrow (f(a_1), g(b_1)) = (f(a_2), g(b_2))$$

$$\Rightarrow f(a_1) = f(a_2) \text{ \& } g(b_1) = g(b_2) \quad \because f \text{ \& } g \text{ both are 1-1}$$

$$\Rightarrow a_1 = a_2 \text{ \& } b_1 = b_2$$

$$\Rightarrow (a_1, b_1) = (a_2, b_2)$$

$\Rightarrow$ H is 1-1

H is onto:-

We have to prove for each

$y \in C \times D$ there exist $x \in A \times B$ s.t $H(x) = y$ where $x = (a,b)$, $y = (c,d)$.

Since $g: B \to D$ and $f: A \to C$ are onto so for each $c \in C$ $\exists$ $a \in A$ and for each $d \in D$ $\exists$ $b \in B$, So for each $(c,d) \in C \times D$ $\exists$ $(a,b) \in A \times B$

$\therefore$ for each $y = (c,d) \in C \times D$ there exist $x = (a,b) \in A \times B$.

Hence H is onto.

Consequently $A \times B \sim C \times D$

2) As A and B are disjoint and C and D are disjoint we have to prove that $A \cup B \sim C \cup D$.

Now for this define a mapping

$$f \cup g : A \cup B \longrightarrow C \cup D$$

by

$$(f \cup g)(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

Also given that $f: A \to C$ and $g: B \to D$ are bijective i.e $A \sim C$ & $B \sim D$

<u>$(f \cup g)$ is 1-1:-</u>

Let $(f \cup g)(x_1) = (f \cup g)(x_2) \longrightarrow (1)$

$$(f \cup g)(x_1) = f(x_1) \text{ or } g(x_1)$$

$$(f \cup g)(x_2) = f(x_2) \text{ or } g(x_2)$$

<u>Case I:-</u> If $(f \cup g)(x_1) = f(x_1)$

& $(f \cup g)(x_2) = f(x_2)$

then from (1) $f(x_1) = f(x_2)$

$\Rightarrow \quad x_1 = x_2 \quad \because f$ is bijective

$\Rightarrow f \cup g$ is 1-1

<u>Case II</u>

If $(f \cup g)(x_1) = g(x_1)$

& $(f \cup g)(x_2) = g(x_2)$

then from ① $\quad g(x_1) = g(x_2)$

$\Rightarrow x_1 = x_2 \quad (\because g$ is bijective

$\Rightarrow f \cup g$ is 1-1

<u>Case III</u> :-

If $(f \cup g)(x_1) = f(x_1)$

& $(f \cup g)(x_2) = g(x_2)$

then from ① $\quad f(x_1) = g(x_2)$

$\because f(x_1) \in C$ & $g(x_2) \in D$ but $C \cap D = \phi$

$\Rightarrow f(x_1) \in \phi$ & $g(x_2) \in \phi$

As ϕ is an empty set so this case is not possible.

Similarly for Case IV

<u>$f \cup g$ is onto:-</u>

We have to prove for each $y \in C \cup D \ \exists \ x \in A \cup B$ s.t

$y = (f \cup g)(x)$

$\Rightarrow y = f(x)$ or $g(x)$

Since f and g are onto so for each $y \in C \cup D \ \exists \ x \in A \cup B$

Hence $A \cup B \sim C \cup D$

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Theorem:- Give an example of two sets A & B s.t both A & B have cordinality 'c' but

(i) $A \setminus B$ is non empty set
(ii) $A \setminus B$ is countable (Denumerable)
(iii) $A \setminus B$ is uncountable.

Proof) 1) Let $A = \mathbb{R}$

$B = \{x \in \mathbb{R}$ except the roots of equation $x^3 - 6x^2 + 11x - 6 = 0\}$

Now both A and B have cordinality c. i.e $|A| = |\mathbb{R}| = c = |B|$

and $A \setminus B = \{1, 2, 3\}$
which is non-empty & finite and having cordinality 3.

Let $A = \mathbb{R}$ and $B = \mathbb{R} \setminus \mathbb{Q} = \mathbb{Q}'$

Now $|A| = |\mathbb{R}| = c$
& $|B| = |\mathbb{R} \setminus \mathbb{Q}| = |\mathbb{Q}'| = c$

Hence $|A| = c = |B|$

& $A \setminus B = \mathbb{R} \setminus \mathbb{Q}' = \mathbb{Q}$
which is countable.

"MEASURE THEORY"

CH-1: MEASURE THEORY & LEBESGUE INTEGRATION**

⇒ **Ring of Sets:-** Let R be a non-empty family of subsets of set X. We say that R is a ring of sets if $A, B \in R$ then $A \cup B \in R$ and $A \setminus B \in R$

σ-Ring (Sigma Ring):- A ring of sets R is said to be σ-Ring if for any sequence of sets $\{A_n\}$ in R $\bigcup_{i=1}^{\infty} A_n \in R$

Example:- Let R be the collection of all finite subsets of N. Then R is a ring of sets but not σ-Ring.

Solution

$X = N$

$R = \{A : A \subseteq N \text{ and } A \text{ is finite}\}$

Let $A, B \in R$ Then $A, B \subseteq N$ and A, B are finite

$\Rightarrow A \cup B \subseteq N$ & $A \cup B$ is finite

$\Rightarrow A \cup B \in R$

Further $A \setminus B$ is finite & $A \setminus B \subseteq N$

$\Rightarrow A \setminus B \in R$

$\Rightarrow R$ is a ring of sets

Let $A_n = \{n\} \in R$

Then $\bigcup_{n=1}^{\infty} A_n = \{1, 2, 3, 4, \ldots\ldots\} \notin R$

So R is not σ-Ring (Sigma Ring)

—— ** ——

⇒ Algebra of Sets (Boolean Algebra):- Let X be a non-empty set then a collection $\mathcal{A}$ of some subsets of X is said to be Boolean Algebra on X if

(i) for any $A \in \mathcal{A} \Rightarrow A' \in \mathcal{A}$

(ii) for any $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$

Example:- Let $X = \{1, 2, 3\}$ and

$\mathcal{A} = \{\{1\}, \{2, 3\}, \phi, X\}$.

Clearly $\mathcal{A}$ is an algebra on X.

σ-Algebra:- An Algebra of sets $\mathcal{A}$ on X is said to be σ-Algebra on X if for any sequence $\{A_n\}$ in $\mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} A_n \in \mathcal{A}$.

The pair $(X, \mathcal{A})$ is said a measurable space and $A \in \mathcal{A}$ is called measurable set

Example:- Let $X = \{a, b, c, d\}$

$\mathcal{A}_1 = \{\phi, X, \{a\}, \{b, c, d\}\}$

$\mathcal{A}_2 = \{\phi, X, \{b\}, \{a, c, d\}\}$

Here $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ-Algebra.

—— ** ——

Remark:- (i) For any set X. $\{\phi, X\}$ is an Algebra on X. (ii) For any set X. $P(X)$ is an Algebra on X. (iii) When X is finite then Algebra and σ-Algebra are same iv) Let $\mathcal{A}$ be the collection of all subsets of A of X such that either A or A' is countable then $\mathcal{A}$ is σ-Algebra.

———*———

Theorem:- The intersection of any collection of σ-Algebra on X is itself a σ-Algebra.

Proof Let $\{\mathcal{A}_\alpha : \alpha \in I\}$ be a collection of σ-Algebra defined on X.

To prove $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is again a σ-Algebra.

Let $A, B \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha$

$$\Rightarrow A, B \in \mathcal{A}_\alpha \quad \forall \alpha \in I$$

As $\mathcal{A}_\alpha$ is σ-Algebra $\forall \alpha \in I$

$$\Rightarrow A \cup B \in \mathcal{A}_\alpha \quad \forall \alpha \in I$$

$$\Rightarrow A \cup B \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha$$

Let $A \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha \Rightarrow A \in \mathcal{A}_\alpha \; \forall \alpha \in I$

$\Rightarrow A' \in \mathcal{A}_\alpha \; \forall \alpha \in I$ [$\because \mathcal{A}$ is σ-Algebra

$$\Rightarrow A' \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha$$

Let $\{A_n\} \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha \Rightarrow \{A_n\} \in \mathcal{A}_\alpha \; \forall \alpha \in I$

As $\mathcal{A}_\alpha$ is σ-algebra $\Rightarrow \bigcup_{i=1}^{\infty} A_n \in \mathcal{A}_\alpha \; \forall \alpha \in I$

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \bigcap_{\alpha\in I} \mathcal{A}_\alpha$$

$\Rightarrow \bigcap_{\alpha\in I} \mathcal{A}_\alpha$ is σ-Algebra.

———

Note:- Union of σ-Algebras need not to be σ-Algebra.

Example:- Let $X=\{a,b,c,d\}$

$\mathcal{A}_1 = \{\phi, \{a\}, \{b,c,d\}, X\}$

$\mathcal{A}_2 = \{\phi, \{b\}, \{a,c,d\}, X\}$

Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ-Algebra

But $\mathcal{A}_3 = \mathcal{A}_1 + \mathcal{A}_2 = \{\phi, \{a\}, \{b\}, \{b,c,d\}, \{a,c,d\}, X\}$ is not σ-algebra

because $\{a\}\cup\{b\}$ is not belong to $\mathcal{A}_3 \Rightarrow \mathcal{A}_3$ is not a σ-Algebra

———***———

⇒ Corollary:- If $\mathcal{A}$ is a σ-Algebra on X then for any $A, B \in \mathcal{A} \Rightarrow A\cap B \in \mathcal{A}$

Proof:-

Let $A, B \in \mathcal{A} \Rightarrow A', B' \in \mathcal{A}$ [$\because \mathcal{A}$ is σ-Algebra

$\Rightarrow A'\cup B' \in \mathcal{A}$ ($\because \mathcal{A}$ is σ-Algebra)

$\Rightarrow (A'\cup B')' \in \mathcal{A}$ ($\because \mathcal{A}$ is σ-Algebra)

$\Rightarrow A\cap B \in \mathcal{A}$ by Demovers Law

——————*

⇒ Proposition:- Let G be a family of subsets of X then there is a smallest sigma-algebra containing G.

Proof:-

Let $\tau = \{\mathcal{A} : \mathcal{A}$ is σ-algebra on X and $G \subseteq \mathcal{A}\}$

Then the family τ is non-empty because $P(X)$ is σ-Algebra and $G \subseteq P(X)$,

$$P(X) \in \tau$$

Put $\mathcal{U} = \bigcap_{\mathcal{A} \in \tau} \mathcal{A}$ Then $\mathcal{U}$ is also a σ-Algebra

As $G \subseteq \mathcal{A} \quad \forall \; \mathcal{A} \in \tau \Rightarrow G \subseteq \bigcap_{\mathcal{A} \in \tau} \mathcal{A}$

$$\Rightarrow G \subseteq \mathcal{U}$$

So $\mathcal{U}$ is a sigma algebra containing G.

So $\mathcal{U} \in \tau$

Next we prove that $\mathcal{U}$ is smallest σ-algebra containing G.

Let $\mathcal{U}^*$ is any σ-algebra such that $G \subseteq \mathcal{U}^*$ then $\mathcal{U}^* \in \tau$

So $\bigcap_{\mathcal{A} \in \tau} \mathcal{A} \subseteq \mathcal{U}^* \Rightarrow \mathcal{U} \subseteq \mathcal{U}^*$

$\Rightarrow \mathcal{U}$ is the smallest σ-algebra containing G.

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⇒ **Proposition:-** Let $\{A_i\}$ be a sequence of sets in an algebra $\mathcal{A}$. Then there is a sequence $\{B_i\}$ of pairwise disjoint sets such that

$$\bigcup_n B_n = \bigcup_n A_n$$

Proof Define $B_1 = A_1$, $B_2 = A_2 \setminus A_1$

$B_3 = A_3 \setminus A_1 \cup A_2$, $B_4 = A_4 \setminus A_1 \cup A_2 \cup A_3$

and so on

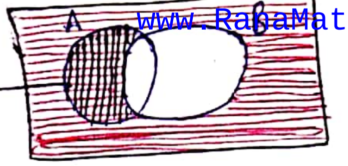
$B_n = A_n \setminus A_1 \cup A_2 \cup A_3 \cup \cdots\cdots \cup A_{n-1} \quad \forall \; n \geq 1$

Now

$B_n = A_n \setminus A_1 \cup A_2 \cup A_3 \cup \cdots\cdots \cup A_{n-1}$

$= A_n \cap (A_1 \cup A_2 \cup A_3 \cup \cdots\cdots \cup A_{n-1})'$

$A \cap B' = A \setminus B$ ←



$\Rightarrow B_n = A_n \cap (A_1' \cap A_2' \cap A_3' \cap \cdots \cap A_{n-1}')$

Then by definition of $\mathcal{A}$

$$B_n \in \mathcal{A} \quad \forall\, n$$

Now we show that $B_n \cap B_m = \phi$ for $m \neq n$

Assume $m < n$.

Now by definition of B_m

$B_m \subseteq A_m$

$$\Rightarrow B_m \cap B_n \subseteq A_m \cap B_n$$

$\Rightarrow B_m \cap B_n \subseteq A_m \cap (A_n \cap A_1' \cap A_2' \cap \cdots \cap A_{n-1}')$

$$= A_m \cap (A_n \cap A_1' \cap A_2' \cap \cdots \cap A_{m-1}' \cap A_m' \cap A_{m+1}' \cap \cdots \cap A_{n-1}')$$

$$= (A_m \cap A_m') \cap (A_n \cap A_1' \cap A_2' \cap \cdots \cap A_{m-1}' \cap A_{m+1}' \cap \cdots \cap A_{n-1}')$$

$$= \phi \cap (A_n \cap A_1' \cap A_2' \cap \cdots \cap A_{n-1}')$$

$$= \phi$$

$\Rightarrow B_m \cap B_n \subseteq \phi \quad \Rightarrow B_m \cap B_n = \phi$

$\Rightarrow B_m$ are disjoint

Now we prove $\bigcup_n B_n = \bigcup_n A_n$

obviously

$$\bigcup_n B_n \subseteq \bigcup_n A_n \longrightarrow (1)$$

Let $x \in \bigcup_n A_n \Rightarrow x \in A_n$ for some n

Let m be the least +ve integer s.t $x \in A_m$ and $x \notin A_1, A_2, \ldots, A_{m-1}$

$\Rightarrow x \in A_m \setminus A_1 \cup A_2 \cup \cdots \cup A_{m-1} \Rightarrow x \in B_m$

$\Rightarrow x \in \bigcup_n B_n \Rightarrow \bigcup_n A_n \subseteq \bigcup_n B_n \longrightarrow (2)$

from (1) & (2) $\bigcup_n A_n = \bigcup_n B_n$

⇒ **Definition:-**

Let $\mathcal{A}$ be a sigma Algebra of subsets of set X and μ is a real valued function on $\mathcal{A}$. Let $\{A_i\}$ be a sequence of set $\mathcal{A}$, Then μ is

(i) finitely subaddition if $\mu\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mu(A_i)$

(ii) Countabily subaddition if $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$

(iii) Finitely additive if $\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$ where A_i are pairwise disjoint

(iv) Countably additive or σ additive if $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ where A_i are pairwise disjoint.

(v) Monotone if $A, B \in \mathcal{A}$ and $A \subseteq B$. then $\mu(A) \leq \mu(B)$

⇒ **Definition:-**

Let $\mathcal{A}$ be a sigma algebra of subsets of X and μ be a extended real valued function on $\mathcal{A}$, we say μ is a measure on $\mathcal{A}$ if

(i) $\mu(\phi) = 0$

(ii) $\mu(A) \geq 0$ for every $A \in \mathcal{A}$

(iii) μ is countably additive i-e

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Example:- Let $X = N$, $\mathcal{A} = 2^N$

Define $\mu : \mathcal{A} \longrightarrow R$ by

$$\mu(A) = \begin{cases} \text{no. of elements in } A & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite.} \end{cases}$$

Solution

i) $\mu(\phi) = 0$ $\because$ By definition of μ

ii) $\mu(A) \geq 0 \quad \forall A \in \mathcal{A}$
$\because$ By definition of μ.

iii) Now to prove μ is countably additive. Let $\{A_n\}$ be sequence of pairwise disjoint sets in $\mathcal{A}$. Then there exists two cases.

Case I Suppose there exist an integer n_0 s.t A_{n_0} is infinite set. Then

$$\mu(A_{n_0}) = \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \mu(A_n) = \infty \longrightarrow (1)$$

Also $\bigcup_{i=1}^{\infty} A_n$ is infinite

$$\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \infty \longrightarrow (2)$$

from (1) & (2) $\mu\left(\bigcup_{i=1}^{\infty} A_n\right) = \sum_{i=1}^{\infty} \mu(A_n)$

Similar if the case if for all n, A_n is infinite.

Case II If for all n, A_n is finite then clearly

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \quad \because (A_n \text{ are disjoint})$$

***

Example:- (Case II)

$A_1 = \{2,3,4\} \Rightarrow \mu(A_1) = 3$

$A_2 = \{1,5\} \Rightarrow \mu(A_2) = 2$

$A_3 = \{7,8,9\} \Rightarrow \mu(A_3) = 3$

$\bigcup_{i=1}^{3} A_i = \{2,3,4,1,5,7,8,9\}$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{3} A_i\right) = 8$$
$$= 3+2+3$$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{3} A_i\right) = \mu(A_1) + \mu(A_2) + \mu(A_3)$$
$$= \sum_{i=1}^{3} \mu(A_i)$$

Note:- If μ is a measure on σ-Algebra $\mathcal{A}$ Then the triplet $(X, \mathcal{A}, \mu)$ is called measure space.

*———***———*

Proposition:- Let $(X, \mathcal{A}, \mu)$ be a measure space

1) if $\exists$ A in $\mathcal{A}$ s.t $\mu(A) < \infty$ then $\mu(\phi) = 0$
2) μ is monotonic
3) If $\{A_n\}$ be a sequence of pairwise disjoint sets in $\mathcal{A}$ then

$$\mu(T) = \sum_{n=1}^{\infty} \mu(A_n \cap T) + \mu\left[T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)'\right]$$

for any $T \in \mathcal{A}$

Proof i)

$$\mu(A) = \mu(A \cup \phi)$$

$$= \mu(A) + \mu(\phi) \quad [\because \mu \text{ is measure space}$$

$$\Rightarrow \mu(A) - \mu(A) = \mu(\phi) \quad [\because \mu(A) < \infty$$

$$\Rightarrow \mu(\phi) = 0$$

ii) Let $A, B \in \mathcal{A}$ s.t $A \subseteq B$.

Now $B = (B \setminus A) \cup A$

Also $B \setminus A$ and A are disjoint

$$\Rightarrow \mu(B) = \mu[(B \setminus A) \cup A]$$

$$= \mu(B \setminus A) + \mu(A) \quad [\because \mu \text{ is measure space}$$

$$\geq \mu(A) \quad (\because \mu(B \setminus A) \geq 0$$

$$\Rightarrow \mu(A) \leq \mu(B)$$

$$\Rightarrow \mu \text{ is monotonic.}$$

iii) $T = T \cap X$

$$\Rightarrow T = T \cap \left[\left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} A_n\right)'\right]$$

$$= \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)\right] \cup \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)'\right]$$

$$\Rightarrow \mu(T) = \mu\left[T\cap\left(\bigcup_{n=1}^{\infty} A_n\right)\right] + \mu\left[T\cap\left(\bigcup_{n=1}^{\infty} A_n\right)'\right]$$

$$= \mu\left[\bigcup_{n=1}^{\infty}(T\cap A_n)\right] + \mu\left[T\cap\left(\bigcup_{n=1}^{\infty} A_n\right)'\right]$$

$$= \sum_{n=1}^{\infty} \mu(T\cap A_n) + \mu\left[T\cap\left(\bigcup_{n=1}^{\infty} A_n\right)'\right]$$

{$\because \mu$ is measure so it is countably additive

*———**———*

⇒ Definition:-

A measure space $(X, \mathcal{A}, \mu)$ is said to be finite measure space if $\mu(X) < \infty$. In general a set A is said to be of finite measure if $\mu(A) < \infty$

⇒ Definition:-

A measure space $(X, \mathcal{A}, \mu)$ is called σ-finite if $\exists$ a sequence $\{A_n\}$ of sets in $\mathcal{A}$ with

$$X = \bigcup_n A_n \;\&\; \mu(A_n) < \infty \;\forall\, n$$

e.g The counting measure μ is σ-finite because

$$N = \bigcup_{n\in N} \{n\} \text{ with } \mu(\{n\}) = 1 < \infty$$

——————*

⇒ Outer Measure:-

Outer Measure μ^* is a non-negative extended real valued function defined on 2^X with the following properties.

i) $\mu^*(\phi) = 0$ ii) μ^* is monotone

iii) μ^* is countably subadditive i.e

$$\mu^*\left(\bigcup_n A_n\right) \leq \sum_n \mu^*(A_n)$$

Example:- Define $\mu^*: 2^X \longrightarrow [0,\infty]$ by

$$\mu^*(E) = \begin{cases} 0 & \text{if } E = \phi \\ 1 & \text{if } E \neq \phi \end{cases}$$

Then μ^* is an outer measure.

i) $\mu^*(\phi) = 0$ ($\because$ By Definition of μ^*

ii) Let $A, B \in 2^X$ such that $A \subseteq B$

(a) If $A = \phi$ and $B = \phi$

$\Rightarrow \mu^*(A) = 0$ & $\mu^*(B) = 0$

$\Rightarrow \mu^*(A) = \mu^*(B)$

(b) If $A = \phi$ & $B \neq \phi$

$\Rightarrow \mu^*(A) = 0$ & $\mu^*(B) = 1$

$\Rightarrow \mu^*(A) < \mu^*(B)$

(c) If $A \neq \phi$ & $B \neq \phi$

$\Rightarrow \mu^*(A) = 1$ and $\mu^*(B) = 1$

$\Rightarrow \mu^*(A) = \mu^*(B)$

Combining all cases we have

$\mu^*(A) \leq \mu^*(B)$

$\Rightarrow \mu^*$ is monotone.

iii) μ^* is countably sub additive.

Let $\{E_n\}$ be pairwise disjoint sequence of sets in 2^X.

In case $E_n \neq \phi$ for each n

$\Rightarrow \bigcup_{n=1}^{\infty} E_n \neq \phi$ so $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = 1$

Also $\mu^*(E_n) = 1 \; \forall n$ given $\sum_{n=1}^{\infty} \mu^*(E_n) = \infty$

So $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) < \sum_{n=1}^{\infty} \mu^*(E_n)$

Similarly in all other cases

$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$

$\Rightarrow \mu^*$ is an outer Measure.

⇒ Lebesgue Outer Measure:-

Lebesgue outer measure m^* is a set function defined on $2^{\mathbb{R}} \longrightarrow [0,\infty]$. i.e $m^*: 2^R \longrightarrow [0,\infty]$ and is defined as

$$m^*(A) = \text{Inf}\left\{ \sum_n l(I_n) : A \subseteq \bigcup_n I_n \right\}$$

$A \subseteq R$ and hence infimum is taken over finite or countable sequence $\{I_n\}$ of open intervals and l stands for length of open interval.

Example 1:- $A = \{1\}$

Sequence 1:- $I_1 = (0,2)$, $I_2 = (0.5, 1.5)$

$$A \subseteq \bigcup_{i=1}^{2} I_i = (0,2) \cup (0.5, 1.5)$$

$$l(I_1) = l(0,2) = 2 - 0 = 2$$

$$l(I_2) = l(0.5, 1.5) = 1.5 - 0.5 = 1$$

$$\sum_{i=1}^{2} l(I_i) = 2 + 1 = 3$$

Sequence 2:- $I_1 = (-1, 0)$, $I_2 = (-2, 4)$

$$\bigcup_{n=1}^{2} I_n = (-1,0) \cup (-2,4)$$

$$=]-2, 4[$$

$$\Rightarrow A \subseteq \bigcup_{n=1}^{2} I_n$$

Now $l(I_1) = l(-1,0) = 0 - (-1) = 1$

$$l(I_2) = l(-2,4) = 4 - (-2) = 6$$

$$\sum_{n=1}^{2} l(I_n) = 1 + 6 = 7$$

$$m^*(A) = \text{Inf}\{3, 7\} \Rightarrow m^*(A) = 3$$

$A \subseteq R$ لینا ہے اور پھر open
Intervals کے ایسے Finite یا Count
able-sequences لیتے ہیں جن میں
Open Interval کی یونین A کو Containg
کرے۔ پھر ہر sequence میں
Open Intervals کی Length کا مجموعہ
لینا ہے اور ان کو ایک set میں
رکھ دینا ہے۔ اور ان میں جو
سب سے Infimum ویلیو ہو گی
وہ m^* یعنی کہ Lebesgue outer
Measure کہلائے گی

Example 2:- $A = \{0, 1\}$

$A \subseteq \{(-1,1) \cup (0,1) \cup (0,2)\}$
$\quad\quad I_1 \quad\quad I_2 \quad\quad I_3$

$\sum_{i=1}^{3} \ell(I_n) = 2 + 1 + 2 = 5$

$A \subseteq \{(-3,3) \cup (5,10)\}$, $\sum_{i=1}^{2} \ell(I_n) = 6 + 5 = 11$
$\quad\quad I_1 \quad\quad I_2$

$m^*(A) = \inf\{6, 11\} = 6$

**

Remark:- 1) For every subset A of real numbers $m^*(A) \geq 0$. Because length of any interval is non negative.

2) $m^*(\phi) = 0$

Proof for this take $\varepsilon > 0$, clearly $\phi \subseteq \left(-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}\right)$

Now $\ell\left(-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}\right) = \frac{\varepsilon}{4} - \left(-\frac{\varepsilon}{4}\right) = \frac{\varepsilon}{2}$

By definition of m^*

$m^*(\phi) \leq \frac{\varepsilon}{2} < \varepsilon$ $\quad$ $\boxed{\because m^*(\phi) \text{ is infimum}}$

$\Rightarrow m^*(\phi) < \varepsilon \Rightarrow m^*(\phi) = 0$

3) Lebesgue Outer Measure of a singleton is zero.

Proof Let $x \in \mathbb{R}$, we have to show

$m^*(\{x\}) = 0$

Since for every $\varepsilon > 0$

$\{x\} \subseteq \left]x - \frac{\varepsilon}{4}, x + \frac{\varepsilon}{4}\right[$

Now $\ell\left(\left]x - \frac{\varepsilon}{4}, x + \frac{\varepsilon}{4}\right[\right) = x + \frac{\varepsilon}{4} - x + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$

$\Rightarrow m^*(\{x\}) \leq \frac{\varepsilon}{2} < \varepsilon$

$\Rightarrow m^*(\{x\}) < \varepsilon$ for every +ve real ε

$$\Rightarrow m^*(\{x\}) = 0$$

4) m^* is monotone

Proof Let $A, B \subseteq \mathbb{R}$ s.t $A \subseteq B$

To prove $m^*(A) \leq m^*(B)$

Now any countable union of open intervals $\bigcup_n I_n$ containing B also containing A.

So $m^*(A) \leq \sum_n \ell(I_n)$ for each collection $\{I_n\}$ of open intervals for which $B \subseteq \bigcup_n I_n$

$$\Rightarrow m^*(A) \leq \text{Inf}\left\{\sum_n \ell(I_n) : \text{for each collection } I_n \text{ s.t } B \subseteq \bigcup_n I_n\right\}$$

$$\Rightarrow m^*(A) \leq m^*(B)$$

$\Rightarrow m^*$ is monotone.

*——***——*

Theorem:- Lebesgue outer Measure of a countable set is zero.

Proof Let A be a countable set

To prove $m^*(A) = 0$

Since A is countable so then

$$A = \bigcup_n \{x_n\}$$

$$m^*(A) = m^*\left(\bigcup_n \{x_n\}\right)$$

Now $m^*\left(\bigcup_n \{x_n\}\right) \leq \sum_n m^*(\{x_n\})$ $\because m^*$ is countably subadditive.

$$\Rightarrow m^*(A) \leq \sum_n m^*(\{x_n\})$$

$$\Rightarrow m^*(A) \leq \sum_n (0) \Rightarrow m^*(A) \leq 0$$

Now as m^* is non-negative.
So $m^*(A) = 0$
$\Rightarrow$ Lebesgue outer measure of countable set is zero.

Paper میں 10/16 آیا ہے Remark 3 Prove کرنا

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Theorem:- Lebesgue outer measure m^* is an outer measure.

Proof: To prove Lebesgue outer measure is an outer measure we have to prove

i) $m^*(\phi) = 0$ (Already Proved)
ii) m^* is monotone (Already Proved)
iii) m^* is countably subadditive i-e
$m^*(\bigcup_n A_n) \leq \sum_n m^*(A_n)$ where $\{A_n\}$ is sequence of sets of real numbers.

(iii) **Case I:-** If $m^*(A_n) = \infty$ for some n. Then required result holds as equality with common value ∞.
Same is result if $m^*(A_n) = \infty \; \forall n$.

Case II:-
Now suppose that for each "n" $m^*(A_n)$ is finite. Take $\varepsilon > 0$ then for each n $\frac{\varepsilon}{2^n} > 0$. Then by definition of m^* there is a countable collection $\{I_{n,i}\}$ of open intervals such that

$$A_n \subseteq \bigcup_i I_{n,i} \quad \text{and}$$

$$\sum_i l(I_{n,i}) < m^*(A_n) + \frac{\varepsilon}{2^n} \quad \text{for some } i \in \text{Indexing set.} \longrightarrow (1)$$

Now as countable union of countable set is countable so.

$$\bigcup_n \left(\bigcup_i I_{n,i}\right) \text{ is countable.}$$

So $\bigcup_n A_n \subseteq \bigcup_n \left(\bigcup_i I_{n,i}\right)$ (by def m^*

$$m^*\left(\bigcup_n A_n\right) \leq \sum_{n,i} \ell(I_{n,i})$$

$$= \sum_n \left[\sum_i \ell(I_{n,i})\right]$$

$$\Rightarrow m^*\left(\bigcup_n A_n\right) < \sum_n \left[m^*(A_n) + \frac{\varepsilon}{2^n}\right] \quad \text{using } ①$$

$$= \sum_n m^*(A_n) + \varepsilon$$

$$\Rightarrow m^*\left(\bigcup_n A_n\right) < \sum_n m^*(A_n) + \varepsilon$$

Since ε is arbitrary so,

$$m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n)$$

$\Rightarrow m^*$ is countably subadditive.

Hence Lebesgue outer measure is outer measure.

*——— *** ———*

⇒ Translation:-

By the translation of set E of real numbers by any real number 'y' we means that the set $E+y = \{x+y : x \in E\}$

——

Theorem:- Show that m^* is Translation Invariant OR $m^*(E) = m^*(E+y)$

Proof

Let E be the subset of R and $y \in R$

To prove m^* is Translation invariant we have to prove $m^*(E) = m^*(E+y)$

Case I :-

If E is countable set then $E+y$ is also countable. so

$m^*(E+y) = 0$ & $m^*(E) = 0$ [$\because$ Lebesgue outer mease of countable set is zero]

Therefore $m^*(E+y) = m^*(E)$

Case II :-

If E is uncountable. Then by definition of m^*

$$m^*(E) = \text{Inf}\left\{\sum_n \ell(I_n) : E \subseteq \bigcup_n I_n\right\}$$

where I_n is a sequence of open intervals.

Put $I_n + y = J_n = \{x+y : x \in I_n\}$

As I_n is an open interval so J_n is also an open interval. Now we prove

$E+y \subseteq \bigcup_n J_n$

Let $x+y \in E+y \Rightarrow x \in E \subseteq \bigcup_n I_n$

$\Rightarrow x \in I_i$ for some i

$\Rightarrow x+y \in I_i + y = J_i$ for some i

$\Rightarrow x+y \in \bigcup_n J_n \Rightarrow E+y \subseteq \bigcup_n J_n$

Further $\ell(I_n) = \ell(J_n)$

$$\Rightarrow \sum_n \ell(I_n) = \sum_n \ell(J_n)$$

$$\Rightarrow \text{Inf}\left\{\sum_n \ell(I_n) : E \subseteq \bigcup_n I_n\right\} = \text{Inf}\left\{\sum_n \ell(J_n) : E+y \subseteq \bigcup_n J_n\right\}$$

$$\Rightarrow m^*(E) = m^*(E+y)$$

Hence m^* is translation Invariant.

*———————**———————*

Remark:- m^* is not 1-1 because $m^*(E) = m^*(E+y)$ but $E \neq E+y$ $\forall y \in \mathbb{R}$

Theorem:- Lebesgue Outer Measure of an interval is its length.

Proof: We divide the proof into different cases.

Case I:- When the interval is closed

Let $I = [a,b]$ for some $a,b \in \mathbb{R}$

we have to prove $m^*(I) = \ell(I) = b-a$

For each $\varepsilon > 0$, $[a,b] \subseteq (a-\varepsilon, b+\varepsilon)$

and $\ell(a-\varepsilon, b+\varepsilon) = b+\varepsilon-(a-\varepsilon) = b-a+2\varepsilon$

Now by definition of Lebesgue outer measure

$$m^*([a,b]) \leq b-a+2\varepsilon$$

Put $A = \{b-a+2\varepsilon : \varepsilon > 0\}$ then clearly $\text{Inf}(A) = b-a$. Hence it follows that

$$m^*([a,b]) \leq b-a \longrightarrow (1)$$

Next we show that $m^*([a,b]) \geq b-a$

Consider a sequence of open intervals $\{I_m\}$ s.t $I \subseteq \bigcup_n I_m \Rightarrow \{I_m\}$ is an open cover for I.

Since I is closed and bounded so by Heine boral theorem I is compact.

So This open cover I_m has finite subcover say $\{I_1, I_2, \dots, I_n\}$ s.t

$$I \subseteq \bigcup_{i=1}^{n} I_i$$

Now for $a \in \bigcup_{i=1}^{n} I_i$ $\exists$ an open interval $I_i^* = (a_1, b_1)$ in $\{I_1, I_2, \dots, I_n\}$ such that

$$a \in (a_1, b_1) \Rightarrow a_1 < a < b_1$$

If $b \notin (a_1, b_1)$ then $b_1 \leq b$.

as $a < b_1 \Rightarrow a < b_1 \leq b \Rightarrow b_1 \in [a,b] \subseteq \bigcup_{i=1}^{n} I_i$

$\Rightarrow b_1 \in \bigcup_{i=1}^{n} I_i$ Then There exist an open interval $(a_2, b_2) \in \{I_1, I_2, \dots, I_n\}$ s.t $b_1 \in (a_2, b_2)$. Now if $b \notin (a_2, b_2)$

Then $b_2 \leq b \Rightarrow a < b_2 \leq b$

$\Rightarrow b_2 \in [a,b] \subseteq \bigcup_{i=1}^{n} I_i$

$\Rightarrow b_2 \in \bigcup_{i=1}^{n} I_i$ Then there exist an open interval (a_3, b_3) in $\{I_1, I_2, I_3, \dots, I_n\}$ such that $b_2 \in (a_3, b_3)$.

Continuing this process we get intervals $(a_1, b_1), (a_2, b_2), \dots\dots, (a_k, b_k)$ in $\{I_1, I_2, \dots, I_n\}$ such that $a_i < b_{i-1} < b_i$, $i = 1, 2, 3, \dots, n$

Since $\{I_1, I_2, \dots, I_n\}$ is finite so our process must terminate with some interval (a_k, b_k), $k \leq n$ and in this case $b \in (a_k, b_k)$

$\Rightarrow b_k > b$ Also $a_1 < a$

$\Rightarrow b_k - a_1 > b - a \longrightarrow (*)$

Now $\sum_{i=1}^{n} \ell(I_i) \geq \sum_{i=1}^{k} \ell(a_i, b_i)$

$$\Rightarrow \sum_{i=1}^{n} \ell(I_i) \geq \ell(a_1, b_1) + \ell(a_2, b_2) + \dots + \ell(a_k, b_k)$$

$$= (b_1 - a_1) + (b_2 - a_2) + \dots + (b_k - a_k)$$

$$= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \dots + (b_2 - a_2) + (b_1 - a_1)$$

$$= b_k - (a_k - b_{k-1}) - (a_{k-1} - b_{k-2}) - \dots - (a_2 - b_1) - a_1$$

$> b_k - a_1$ $\because a_k - b_{k-1} < 0$.

$> b - a$ using $(*)$

$$\Rightarrow \sum_{n} \ell(I_n) > \sum_{i=1}^{n} \ell(I_i) > b - a$$

$\Rightarrow \sum_n l(I_n) > b-a$

$\inf\{\sum_n l(I_n) : I \subseteq \bigcup_n I_n\} \geq b-a$

$\Rightarrow m^*([a,b]) \geq b-a \longrightarrow (2)$

from (1) & (2) $m^*([a,b]) = b-a$

So Lebesgue outer measure of a closed interval is its length.

Case II :-

When Interval is open

If $I = (a,b)$ is any finite open interval for $\varepsilon > 0$, there is a closed interval $J \subset I$ such that $l(I) - \varepsilon < l(J) = m^*(J) \longrightarrow (*)$

As $J \subset I$ so

$m^*(J) < m^*(I)$ [$\because m^*$ is monotone

$l(I) - \varepsilon < m^*(J) < m^*(I)$ using $(*)$

$l(I) - \varepsilon < m^*(I)$

Since ε is arbitrary so

$l(I) \leq m^*(I) \Rightarrow b-a \leq m^*(I) \longrightarrow (1)$

Now as $I \subseteq \bar{I}$

$\Rightarrow m^*(I) \leq m^*(\bar{I})$ (m^* is monotone)

$\Rightarrow m^*(I) \leq b-a \longrightarrow (2)$

from (1) and (2) $m^*(I) = b-a$

Case III :-

If I is an infinite interval then for a given non-negative real number α there is a closed interval J, $J \subset I$

s.t $\ell(J) = \alpha$

As $J \subset I \Rightarrow m^*(J) < m^*(I)$ ($\because m^*$ is monotone)

$\Rightarrow m^*(I) > m^*(J) = \alpha$ ($\because \ell(J) = m^*(J) = \alpha$

$\Rightarrow m^*(I) > \alpha$ for every real α

$\Rightarrow m^*(I) = \infty = \ell(I)$

$\Rightarrow m^*(I) = \ell(I)$

Hence in all cases Lebesgue outer measure of an interval is its length.

***

Theorem:- 1) If $m^*(A) = 0$ then $m^*(A \cup B) = m^*(B)$

2) If A is a set of rational numbers b/w '0' and '1' and $\{I_n\}$ is the finite collection of open intervals s.t. $A \subseteq \bigcup_n I_n$ then

$$\sum_n \ell(I_n) \geq 1$$

Proof

1) By subadditivity of m^*

$m^*(A \cup B) \leq m^*(A) + m^*(B)$

$\Rightarrow m^*(A \cup B) \leq 0 + m^*(B)$ ($\because m^*(A) = 0$)

$\Rightarrow m^*(A \cup B) \leq m^*(B) \longrightarrow ①$

Also $B \subseteq A \cup B$

$\Rightarrow m^*(B) \leq m^*(A \cup B) \longrightarrow ②$ ($\because m^*$ is monotone)

from ① & ②

$m^*(A \cup B) = m^*(B)$

2) Let I_n be a finite sequence of open intervals such that $A \subseteq \bigcup_n I_n$.

Obviously $]0,1[$ is the smallest open interval such that $A \subseteq (0,1)$ and $\bigcup_n I_n$ is any other collection such that $A \subseteq \bigcup_n I_n$ So

$(0,1) \subseteq \bigcup_n I_n \Rightarrow m^*(0,1) \leq m^*(\bigcup_n I_n)$

$\Rightarrow 1 \leq \sum_n m^*(I_n)$ ($\because m$ is subadditive)

$\Rightarrow 1 \leq \sum_n m^*(I_n) = \sum_n \ell(I_n)$

$\Rightarrow 1 \leq \sum_n \ell(I_n) \quad \Rightarrow \sum_n \ell(I_n) \geq 1$

————*

Corollary: The set $[0,1]$ is uncountable

Proof

Suppose $[0,1]$ is countable. Then

$m^*([0,1]) = 0 \longrightarrow ①$ $\because$ m^* of countable set is zero

Also the Lebesgue outer measure of an interval is its length. So

$m^*([0,1]) = \ell[0,1] = 1 \longrightarrow ②$

By ① & ② $0 = 1$ which is not true

So our supposition is wrong.

Hence $[0,1]$ is uncountable.

(Imp) ——***——

Proposition:- Given any set A and any $\varepsilon > 0$ there is an open set O s.t $A \subseteq O$ and

$m^*(O) < m^*(A) + \varepsilon$

Proof

By definition of $m^*(A)$, for $\varepsilon > 0$ there exist a countable collection I_n of open intervals such that $A \subseteq \bigcup_n I_n$ and

$\sum_n \ell(I_n) < m^*(A) + \varepsilon \longrightarrow ①$

Put $O = \bigcup_n I_n$.

As countable union of open sets is open. So O is open set

Now $m^*(O) = m^*(\bigcup_n I_n) \leq \sum_n m^*(I_n)$ [$\because m^*$ is countably subadditive]

$\Rightarrow m^*(O) \leq \sum_n \ell(I_n)$ [$\because m^*$ of interval is its length]

$\Rightarrow m^*(O) \leq \sum_n \ell(I_n) < m^*(A) + \varepsilon$

$\Rightarrow m^*(O) < m^*(A) + \varepsilon$

*——**——*

⇒ Lebesgue Measurable OR Measurable Set:-

A set "E" of real numbers is said to be Lebesgue measurable or simply measurable set if for each sub set A of $\mathbb{R}$ we have $m^*(A) = m^*(A \cap E) + m^*(A \cap E')$

———————

Remark:- 1) The set A is often called test set since it is used to test the measurability of E

2) Since $A = A \cap \mathbb{R} \Rightarrow A = A \cap (E \cup E')$

$\Rightarrow A = (A \cap E) \cup (A \cap E')$

$\Rightarrow m^*(A) \leq m^*(A \cap E) + m^*(A \cap E')$ [$\because m^*$ is subadditive

The above result always hold. So to check measurability of E it is sufficient to verify $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E')$

3) E is measurable iff E' is measurable.

proof $m^*(A) = m^*(A \cap E) + m^*(A \cap E')$

$= m^*(A \cap (E')') + m^*(A \cap E')$

$= m^*(A \cap E') + m^*(A \cap (E')')$

$\Rightarrow E'$ is measurable.

4) ϕ and $\mathbb{R}$ are measureable sets.

proof (a) To prove ϕ is measurable we have to prove

$$m^*(A) = m^*(A \cap \phi) + m^*(A \cap \phi')$$

R.H.S $= m^*(A \cap \phi) + m^*(A \cap \phi')$

$= m^*(\phi) + m^*(A \cap \mathbb{R})$

$= 0 + m^*(A) = m^*(A) = $ L.H.S

$\Rightarrow \phi$ is measureable.

b) As ϕ is measurable

$\Rightarrow \phi'$ is measurable

$\Rightarrow \mathbb{R}$ is measurable.

***

(V. Imp)

Lemma:- If $m^*(E)=0$ Then E is measurable.

Proof

For any subset A of $\mathbb{R}$

$A\cap E \subseteq E$

$\Rightarrow m^*(A\cap E) \leq m^*(E)$ [$\because m^*$ is monotone

$\Rightarrow m^*(A\cap E) \leq 0$ ($\because m^*(E=0)$

$\Rightarrow m^*(A\cap E)=0$ ($\because m^*$ is non negative $\longrightarrow (1)$

Also $(A\cap E') \subseteq A \Rightarrow m^*(A\cap E') \leq m^*(A)$

$\Rightarrow m^*(A) \geq m^*(A\cap E') + 0$

$\Rightarrow m^*(A) \geq m^*(A\cap E') + m^*(A\cap E)$ using (1)

$\Rightarrow m^*(A) \geq m^*(A\cap E) + m^*(A\cap E')$

$\therefore E$ is measurable.

*

(V.V. Imp)

Theorem:- Union and Intersection of two measurable sets is measurable set.

Proof

Let E_1 and E_2 be two measurable sets. To prove $E_1 \cup E_2$ and $E_1 \cap E_2$ are measurable.

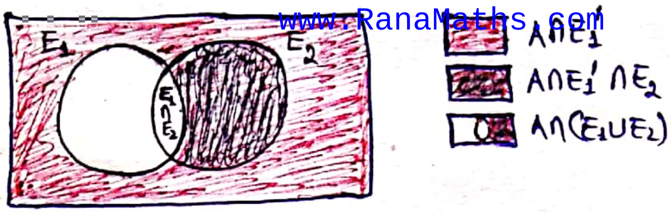
1) Let $A \subseteq \mathbb{R}$ be a test set

Then we have to prove

$m^*(A) \geq m^*[A\cap(E_1\cup E_2)] + m^*[A\cap(E_1\cup E_2)']$

As E_2 is measurable so for $A\cap E_1'$ we have

$m^*(A\cap E_1') = m^*(A\cap E_1' \cap E_2) + m^*(A\cap E_1' \cap E_2') \rightarrow (1)$



Also as $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_1' \cap E_2)$ ⊛

$\Rightarrow m^*[A \cap (E_1 \cup E_2)] \leq m^*(A \cap E_1) + m^*[A \cap E_1' \cap E_2] \longrightarrow ②$

$\because m^*$ is countably subadd.

Consider

$m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)']$

$= m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap E_1' \cap E_2']$

$\leq m^*(A \cap E_1) + m^*(A \cap E_1' \cap E_2) + m^*[A \cap E_1' \cap E_2']$ using ②

$= m^*(A \cap E_1) + m^*(A \cap E_1')$ using ①

$= m^*(A)$ ($\because E_1$ is measurable)

$\Rightarrow m^*(A) \geq m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)']$

$\Rightarrow E_1 \cup E_2$ is measurable

2)

Since E_1 and E_2 are measurable then E_1' and E_2' are measurable

$\Rightarrow E_1' \cup E_2'$ is measurable

$\Rightarrow (E_1' \cup E_2')'$ is measurable

$\Rightarrow (E_1' \cup E_2')' = E_1 \cap E_2$ is mearable.

Rough⊛

$E_1 \cup E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1) \cup (E_1 \cap E_2)$

$A \cap (E_1 \cup E_2) = A \cap [(E_1 \setminus E_2) \cup (E_2 \setminus E_1) \cup (E_1 \cap E_2)]$

$= A \cap [(E_1 \cap E_2') \cup (E_2 \cap E_1') \cup (E_1 \cap E_2)]$

$= [A \cap (E_1 \cap E_2')] \cup [A \cap (E_2 \cap E_1')] \cup [A \cap (E_1 \cap E_2)]$

$= [A \cap (E_1 \cap E_2')] \cup [A \cap (E_1 \cap E_2)] \cup [A \cap (E_2 \cap E_1')]$

$= A \cap \{(E_1 \cap E_2') \cup (E_1 \cap E_2)\} \cup [A \cap (E_2 \cap E_1')]$

$= A \cap [E_1 \cap (E_2' \cup E_2)] \cup [A \cap (E_1' \cap E_2)]$

$= [A \cap (E_1 \cap R)] \cup [A \cap (E_1' \cap E_2)]$

$\Rightarrow A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_1' \cap E_2)$

⇒ Symmetric Difference of A & B:-

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

**

Theorem:- If F is measurable set and $m^*(F \Delta G) = 0$ then G is measurable set.

Proof: Since $F \Delta G = (F \setminus G) \cup (G \setminus F)$

$\Rightarrow F \setminus G \subseteq F \Delta G$

$\Rightarrow m^*(F \setminus G) \leq m^*(F \Delta G)$ ($\because m^*$ is monotone.

$\Rightarrow m^*(F \setminus G) \leq 0$ $\because m^*(F \Delta G) = 0$

Since m^* is ~~monotone~~ non-negative so

$m^*(F \setminus G) = 0$, similarly $m^*(G \setminus F) = 0$

$\Rightarrow G \setminus F$ is measurable

Since $m^*(F \setminus G) = 0$

$\Rightarrow F \setminus G$ is measurable set

$\Rightarrow (F \setminus G)'$ " " "

$\Rightarrow F \cap (F \setminus G)'$ " " " ($\because F$ is measurable

Now as $F \cap G = F \cap (F \setminus G)'$

$\Rightarrow F \cap G$ is measurable set

Further $G = (F \cap G) \cup (G \setminus F)$

$\Rightarrow G$ is measurable being union of two measurable sets.

⇒ G_δ - Set :-

A set G is said to be G_δ set if it is the countable intersection of open sets i.e $G = \bigcap_{i=1}^{\infty} G_i$, where each G_i is open set.

⇒ F_σ - Set:-

A set F which is countable union of closed sets is called F_σ - set i.e $F = \bigcup_{i=1}^{\infty} F_i$, where each F_i is closed.

*———**———*

Theorem:- Given any $A \subseteq R$ and $\varepsilon > 0$, there is a G_δ-set with $A \subseteq G$ and $m^*(A) = m^*(G)$

Proof:

As we know that for any set A and for each $\varepsilon > 0$ there exist an open set O such that $A \subseteq O$ and

$$m^*(O) < m^*(A) + \varepsilon \longrightarrow ①$$

choose $\varepsilon = 1/n$

$$① \Rightarrow m^*(O_n) < m^*(A) + \frac{1}{n} \longrightarrow ②, \quad A \subseteq O_n$$

Put $G = \bigcap_{n=1}^{\infty} O_n$. Then G being the intersection of countable open sets is G_δ - set.

Now $A \subseteq \bigcap_{n=1}^{\infty} O_n = G \quad \Rightarrow A \subseteq G$

$$\Rightarrow m^*(A) \leq m^*(G) \longrightarrow ③ \quad (\because m \text{ is monotone})$$

Also $m^*(G) = m^*(\bigcap_{n=1}^{\infty} O_n) \leq m^*(O_n) < m^*(A) + \frac{1}{n} \;\; \forall n$

$$\Rightarrow m^*(G) < m^*(A) + 1/n$$

Taking n sufficiently large we get

$$m^*(G) \leq m^*(A) \longrightarrow ④$$

from ③ & ④

$$m^*(A) = m^*(G)$$

★★★

Theorem:- Let A be any set of real numbers and $\{E_1, E_2, E_3, \ldots, E_n\}$ be a finite family of pairwise disjoint measurable sets then

$$m^*\left[A\cap\left(\bigcup_{i=1}^{n} E_i\right)\right] = \sum_{i=1}^{n} m^*(A\cap E_i)$$

Proof:-

We prove this result by mathematical induction

For $n=1$:- $\quad m^*(A\cap E_1) = m^*(A\cap E_1)$

So given result is true for $n=1$

Suppose given result is true for $n=K-1$

i.e

$$m^*\left[A\cap\left(\bigcup_{i=1}^{K-1} E_i\right)\right] = \sum_{i=1}^{K-1} m^*(A\cap E_i) \longrightarrow ⓐ$$

As E_i are pairwise disjoint so we have

$$A\cap\left(\bigcup_{i=1}^{K} E_i\right)\cap E_K = A\cap\left[\bigcup_{i=1}^{K}(E_i\cap E_K)\right] = A\cap E_K \longrightarrow ①$$

$$A\cap\left(\bigcup_{i=1}^{K} E_i\right)\cap E_K' = A\cap\left[\bigcup_{i=1}^{K}(E_i\cap E_K')\right] = A\cap\left(\bigcup_{i=1}^{K-1} E_i\right) \rightarrow ②$$

The set E_K is measurable so we have

$$m^*\left[A\cap\left(\bigcup_{i=1}^{K} E_i\right)\right] = m^*\left[A\cap\left(\bigcup_{i=1}^{K} E_i\right)\cap E_K\right] + m^*\left[A\cap\left(\bigcup_{i=1}^{K} E_i\right)\cap E_K'\right]$$

$$= m^*(A\cap E_K) + m^*\left[A\cap\left(\bigcup_{i=1}^{K-1} E_i\right)\right] \quad \text{using ① \& ②}$$

$$= m^*(A\cap E_K) + m^*\left[\bigcup_{i=1}^{K-1}(A\cap E_i)\right]$$

$$\Rightarrow m^*\left[A\cap\left(\bigcup_{i=1}^{k} E_i\right)\right] = m^*(A\cap E_k) + \sum_{i=1}^{k-1} m^*(A\cap E_i) \quad \text{using } (*)$$

($\because m^*$ is countably subadditive)

$$= \sum_{i=1}^{k} m^*(A\cap E_i)$$

So result is true for $n=k$

Hence the proof.

**

Theorem:- Let A be any set and $\{E_i\}$ be a sequence of pairwise disjoint measurable sets then

$$m^*\left(A\cap\left(\bigcup_{i=1}^{\infty} E_i\right)\right) = \sum_{i=1}^{\infty} m^*(A\cap E_i)$$

Proof: As $A\cap\left(\bigcup_{i=1}^{n} E_i\right) \subseteq A\cap\left(\bigcup_{i=1}^{\infty} E_i\right)$

$$\Rightarrow m^*\left(A\cap\left(\bigcup_{i=1}^{n} E_i\right)\right) \le m^*\left[A\cap\left(\bigcup_{i=1}^{\infty} E_i\right)\right] \quad \because m^* \text{ is monotone}$$

$$\Rightarrow \sum_{i=1}^{n} m^*(A\cap E_i) \le m^*\left(A\cap\left(\bigcup_{i=1}^{\infty} E_i\right)\right)$$

As $n \longrightarrow \infty$

$$\sum_{i=1}^{\infty} m^*(A\cap E_i) \le m^*\left(A\cap\left(\bigcup_{i=1}^{\infty} E_i\right)\right) \longrightarrow (1)$$

As m^* is countably subadditive

So $$m^*\left(A\cap\left(\bigcup_{i=1}^{\infty} E_i\right)\right) \le \sum_{i=1}^{\infty} m^*(A\cap E_i) \longrightarrow (2)$$

from (1) & (2)

$$m^*\left[A\cap\left(\bigcup_{i=1}^{\infty} E_i\right)\right] = \sum_{i=1}^{\infty} m^*(A\cap E_i)$$

Theorem:- The interval (a,∞) is a measurable set for each $a \in \mathbb{R}$

Proof:-

Let $A \subseteq \mathbb{R}$ be any test set
To prove (a,∞) is measurable, we prove
$m^*(A) \geq m^*(A\cap(a,\infty)) + m^*(A\cap(-\infty,a])$

Put $A\cap(a,\infty) = A_1$
and $A\cap(-\infty,a] = A_2$

Now we show that
$$m^*(A) \geq m^*(A_1) + m^*(A_2) \longrightarrow (A)$$

Case I:- If $m^*(A) = \infty$ Then result is obvious
Case II:- If $m^*(A)$ is finite then for any $\varepsilon>0$ by definition of $m^*(A)$ there exist a sequence $\{I_n\}$ of open intervals such that
$A \subseteq \bigcup_n I_n$ and $\sum_n \ell(I_n) < m^*(A)+\varepsilon \longrightarrow (1)$

Put $J_n = I_n\cap(a,\infty)$ & $K_n = I_n\cap(-\infty,a]$

Then it is clear that
$I_n = J_n \cup K_n$ and $J_n\cap K_n = \phi$
$\ell(I_n) = \ell(J_n\cup K_n)$ {$\because J_n$ & K_n are disjoint

$\Rightarrow \ell(I_n) = m^*(J_n) + m^*(K_n) \longrightarrow (2)$

Now as $A \subseteq \bigcup_n I_n$

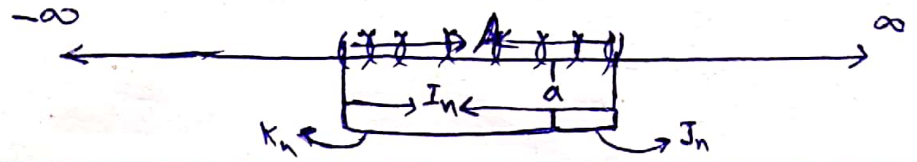
$A\cap(a,\infty) \subseteq (\bigcup_n I_n)\cap(a,\infty)$

$\Rightarrow A_1 \subseteq \bigcup_n [I_n\cap(a,\infty)]$

$\Rightarrow A_1 \subseteq \bigcup_n J_n \Rightarrow m^*(A_1) \leq m^*(\bigcup_n J_n)$ $\because m^*$ is monotone

$\Rightarrow m^*(A_1) \leq \sum_n m^*(J_n)$ ($\because m^*$ is countably subadditive

Similarly



$$m^*(A_2) \leq \sum_n m^*(K_n)$$

Now $$m^*(A_1) + m^*(A_2) \leq \sum_n m^*(J_n) + \sum_n m^*(K_n)$$

$$= \sum_n [m^*(J_n) + m^*(K_n)]$$

$$= \sum_n m^*(J_n \cup K_n)$$

$$= \sum_n m^*(I_n)$$

$$= \sum_n \ell(I_n) \quad \text{using } ②$$

$$\Rightarrow m^*(A_1) + m^*(A_2) < m^*(A) + \varepsilon \quad \text{using } ①$$

Since ε is arbitrary so

$$m^*(A_1) + m^*(A_2) \leq m^*(A)$$

$$\Rightarrow m^*(A) \geq m^*(A_1) + m^*(A_2)$$

$$\Rightarrow m^*(A) \geq m^*[A \cap (a,\infty)] + m^*[A \cap (-\infty, a])$$

$\Rightarrow (a,\infty)$ is measurable.

Lemma:- For any a, b in $\mathbb{R}$ the interval (a,b) is a measurable set.

Proof:

As we know that

(a,∞) is measurable for any $a \in \mathbb{R}$

$\Rightarrow (a,\infty)'$ is measurable for any $a \in \mathbb{R}$

$\Rightarrow (-\infty, a]$ is " " " " "

In particular

$(-\infty, b-\frac{1}{n}]$ is measurable for each n

Now we can write

$$(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b-\tfrac{1}{n}]$$

each set on right side is measurable

thus $(-\infty, b)$ is measurable

Now $(a,b) = (-\infty, b) \cap (a, \infty)$
$\Rightarrow (a,b)$ is measurable being the intersection of two measurable sets.

***

Lemma:- Any open set O in $\mathbb{R}$ is measurable

Proof

Since every open set can be expressed as a countable union of pairwise disjoint open intervals I_n and since every open interval I_n is measurable.
Also countable union of measurable ~~sets~~ intervals is measurable
$\Rightarrow$ O being an open set can be expressed as a countable union of pairwise disjoint open intervals and therefor O is measurable.

* * *

Corollary. Any G_δ-set is measurable.

Proof

Since every open set is measurable and countable intersection of measurable sets is measurable. So G_δ being countable intersection of open sets is measurable.

Corollary:- Any closed set in $\mathbb{R}$ is measurable.

Proof

The compliment of an open set is always a closed set.
Since every open set is measurable and compliment of any measurable

set is measurable

⇒ Any closed set (being the compliment of open set) in $\mathbb{R}$ is measurable.

Theorem Any open set can be expressed as a countable union of open intervals

Proof ? ? ? ? ? ?

Corollary:- Any F_σ-set is measurable.

Proof:

Since every closed set is measurable and countable union of measurable sets is measurable. So F_σ-set being the countable union of closed sets is measurable.

—————— ★★★ ——————

⇒ **Proposition:-** A subset E of $\mathbb{R}$ is measurable iff for given $\varepsilon>0$, there is an open set $O \supseteq E$ with $m^*(O \setminus E) < \varepsilon$

Proof:

Given E is measurable.

To prove, for given $\varepsilon > 0$ there is an open set $O \supseteq E$ with $m^*(O\setminus E) < \varepsilon$

As E is measurable then there arises two cases.

Case I:- If $m^*(E)$ is finite i.e $m^*(E) < \infty$ then there exist an open set $O \supseteq E$ s.t

$$m^*(O) < m^*(E) + \varepsilon$$

$$\Rightarrow m^*(O) - m^*(E) < \varepsilon \longrightarrow (1)$$

Further as $E \subseteq O$ we can write

$$O = E \cup (O \setminus E)$$

$$\Rightarrow m^*(O) = m^*(E) + m^*(O \setminus E)$$

$$\Rightarrow m^*(O) - m^*(E) = m^*(O \setminus E)$$

$$\Rightarrow m^*(O \setminus E) = m^*(O) - m^*(E)$$

$$< \varepsilon \quad \text{using } ①$$

$$\Rightarrow m^*(O \setminus E) < \varepsilon$$

Case II:- If $m^*(E)$ is infinite i.e $m^*(E) = \infty$.

Now we write

$\mathbb{R} = \bigcup_{n=1}^{\infty} I_n$, where I_n is a finite open interval and $I_n \cap I_m = \phi$ for $m \neq n$.

Put $E_n = E \cap I_n$, Then each E_n is measurable being the intersection of two measurable sets.

Since each I_n is finite therefore

$E_n = E \cap I_n$ is finite

$\Rightarrow m^*(E_n) < \infty$. So by case I there is an open set O_n containing E_n such that

$$m^*(O_n \setminus E_n) < \varepsilon/2^n \longrightarrow ②$$

Put $O = \bigcup_{n=1}^{\infty} O_n$, Then O being the countable union of open sets is open

Also $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E \cap I_n) = E \cap \left(\bigcup_{n=1}^{\infty} I_n \right)$

$$= E \cap \mathbb{R} = E$$

$$\Rightarrow \bigcup_{n=1}^{\infty} E_n = E$$

As $O_n \supseteq E_n$

$\Rightarrow \bigcup_n O_n \supseteq \bigcup_n E_n$

$\Rightarrow O \supseteq E$ $\quad \because \bigcup_{n=1}^{\infty} O_n = O$ & $\bigcup_{n=1}^{\infty} E_n = E$

$O \setminus E = \bigcup_{n=1}^{\infty} O_n \setminus \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} (O_n \setminus E_n)$ ⊛

$\Rightarrow O \setminus E \subseteq \bigcup_{n=1}^{\infty} (O_n \setminus E_n)$

$\Rightarrow m^*(O \setminus E) \leq m^*\left[\bigcup_{n=1}^{\infty} (O_n \setminus E_n)\right]$ $\quad \because m^*$ is monote

$\Rightarrow m^*(O \setminus E) \leq \sum_{n=1}^{\infty} m^*(O_n \setminus E_n)$ $\quad (\because m^*$ is countably subadditive

$< \sum_{n=1}^{\infty} \varepsilon/2^n$

$\Rightarrow m^*(O \setminus E) < \varepsilon$

$E_1 = \{1,2,3\}$, $E_2 = \{4,5\}$, $E_3 = \{9,10\}$

$O_1 = \{1,2,3,4\}$; $O_2 = \{4,5,7\}$ ⊛

$O_3 = \{9,10,11\}$

$\bigcup_{n=1}^{3} E_n = \{1,2,3,4,5,9,10\}$

$\bigcup_{n=1}^{3} O_n = \{1,2,3,4,5,7,9,10,11\}$

$\bigcup_{n=1}^{3} O_n \setminus \bigcup_{n=1}^{3} E_n = \{7,11\}$

& Now $O_1 \setminus E_1 = \{4\}$, $O_2 \setminus E_2 = \{7\}$

$O_3 \setminus E_3 = \{11\}$

$\Rightarrow \bigcup_{n=1}^{3} (O_n \setminus E_n) = \{4,7,11\}$

$\Rightarrow \bigcup_{n=1}^{3} O_n \setminus \bigcup_{n=1}^{3} E_n \subseteq \bigcup_{n=1}^{3} (O_n \setminus E_n)$

Conversely,

given for $\varepsilon = \frac{1}{n}$ $(n \in N)$ there is an open set $O_n \supseteq E$ for all n such that

$m^*(O_n \setminus E) < \frac{1}{n} \;\forall\, n$

To prove E is measurable

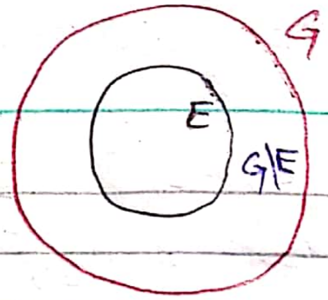
Put $G = \bigcap_{n=1}^{\infty} O_n$ then G is measurable being the countable intersection of open sets. Since $E \subseteq O_n \;\forall\, n$

$\Rightarrow E \subseteq \bigcap_{n=1}^{\infty} O_n \Rightarrow E \subseteq G$

Now $G = \bigcap_{n=1}^{\infty} O_n \Rightarrow G \subseteq O_n$

$\Rightarrow G \setminus E \subseteq O_n \setminus E \Rightarrow m^*(G \setminus E) \leq m^*(O_n \setminus E)$ $\quad \because m^*$ is monotone

$G\setminus(G\setminus E)' = E$



$\Rightarrow m^*(G\setminus E) < \frac{1}{n} \quad \forall\, n$

$\Rightarrow m^*(G\setminus E) < \frac{1}{n}$

when ~~n~~ 'n' is sufficiently large

then $m^*(G\setminus E) = 0$

$\Rightarrow G\setminus E$ is measurable

Now $E = G\setminus(G\setminus E) = G\cap(G\setminus E)'$

As G & $(G\setminus E)'$ are measurable so E is measurable.

***

$\Rightarrow$ **Proposition:-** For a subset E of $\mathbb{R}$ and given $\varepsilon>0$ there is an open set $O\supseteq E$ and $m^*(O\setminus E)<\varepsilon$ iff there is a closed set $F\subseteq E$ and $m^*(E\setminus F)<\varepsilon$

Proof:

Given for $E\subseteq\mathbb{R}$ and $\varepsilon>0$ there is an open set $O\supseteq E$ such that

$$m^*(O\setminus E) < \varepsilon$$

To prove there is a closed set $F\subseteq E$ and $m^*(E\setminus F) < \varepsilon$

As the given result is true for any $E\subseteq\mathbb{R}$ therefore the result is true for $E'\subseteq\mathbb{R}$ and then by given condition there is an open set $O\supseteq E'$ s.t

$$m^*(O\setminus E') < \varepsilon$$

$$\Rightarrow m^*(O\cap(E')') < \varepsilon \qquad \because A\setminus B = A\cap B'$$

$$\Rightarrow m^*(O\cap E) < \varepsilon$$

$$\Rightarrow m^*(E\cap O) < \varepsilon \quad \Rightarrow m^*(E\setminus O') < \varepsilon$$

$\Rightarrow m^*(E \setminus F) < \varepsilon \qquad , F = O'$

As $\quad O \supseteq E'$ $\qquad$ If $E \subseteq O \Rightarrow O' \subseteq E'$

$\Rightarrow O' \subseteq (E')' \quad \Rightarrow O' \subseteq E$

$\Rightarrow F \subseteq E \qquad \because F = O'$

Further O is open

$\Rightarrow F = O'$ is closed

Conversly,

Given for a subset E' of $\mathbb{R}$ and $\varepsilon > 0$ there is a closed set $F \subseteq E'$ s.t

$$m^*(E' \setminus F) < \varepsilon$$

$\Rightarrow m^*(E' \cap F') < \varepsilon \quad \Rightarrow m^*(F' \cap E') < \varepsilon$

$\Rightarrow m^*(F' \setminus E) < \varepsilon \Rightarrow m^*(O \setminus E) < \varepsilon \quad , F' = O$

Further since F is closed

$\Rightarrow F' = O$ is open

As $\quad F \subseteq E' \quad \Rightarrow F' \supseteq (E')'$

$\Rightarrow F' \supseteq E \quad \Rightarrow O \supseteq E \qquad \because F' = O$

$\Rightarrow E \subseteq O$

Imp $\qquad$ ***

Theorem: Let $E \subseteq \mathbb{R}$, then the following statements are equivalant

1) There is a G_δ set G with $E \subseteq G$, $m^*(G \setminus E) = 0$
2) There is a F_σ set F with $F \subseteq E$, $m^*(E \setminus F) = 0$

Proof: $\quad 1 \Rightarrow 2$

The given statement is true for any

$E \subseteq \mathbb{R}$ so it must hold for E'. Therefore there is a G_δ set $G_1 \supseteq E'$ such that

$$m^*(G_1 \setminus E') = 0 \Rightarrow m^*(G_1 \cap E) = 0$$

$$\Rightarrow m^*(E \cap G_1) = 0$$

$G_\delta' = F_\sigma$ and $(F_\sigma)' = G_\delta$

$$\Rightarrow m^*(E \setminus G_1') = 0$$

$$\Rightarrow m^*(E \setminus F) = 0 \quad , F = G_1'$$

where F is an F_σ set being the complement of a G_δ set G_1

Also $G_1 \supseteq E'$

or $E' \subseteq G_1 \Rightarrow G_1' \subseteq (E')'$

$$\Rightarrow G_1' \subseteq E \quad \Rightarrow F \subseteq E \quad (\because F = G_1')$$

Now $2 \Longrightarrow 1$

The given statement is true for any $E \subseteq \mathbb{R}$ so it must hold for E' and then by the given condition there is an F_σ set $F \subseteq E'$ s.t

$$m^*(E' \setminus F) = 0 \Rightarrow m^*(E' \cap F') = 0$$

$$\Rightarrow m^*(F' \cap E') = 0 \Rightarrow m^*(F' \setminus E) = 0$$

$$\Rightarrow m^*(G \setminus E) = 0 \quad (\because G = F')$$

where G is a G_δ set being the compliment of an F_σ set F.

Also $F \subseteq E' \Rightarrow F' \supseteq E$

$$\Rightarrow G \supseteq E \quad (F' = G) \quad \text{or}$$

$$E \subseteq G \quad \text{As Required.}$$

Theorem:- The class $\mathcal{M}$ of Lebesgue measurable sets (or simply measurable) is a σ-Algebra

Proof:- Since union of the measurable sets is measurable and compliment of an measurable set is also measurable. Therefore the class $\mathcal{M}$ is an algebra.

Now we prove $\mathcal{M}$ is σ-Algebra. Let $\{E_i\}$ be a sequance of sets in $\mathcal{M}$ we have to show

$$E = \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$$

Since $\{E_i\}$ be a sequence of sets in $\mathcal{M}$, then there is a sequence $\{F_i\}$ of pairwise disjoint measurable sets s.t

$$\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$$

Define $H_n = \bigcup_{i=1}^{n} F_n$. Then H_n is measureable set for each n

Now $\bigcup_{i=1}^{n} F_i \subseteq \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i = E$

$$\Rightarrow H_n \subseteq E \qquad (\because H_n = \bigcup_{i=1}^{n} F_i)$$

$$\Rightarrow E' \subseteq H_n' \qquad \Rightarrow A \cap E' \subseteq A \cap H_n'$$

$$\Rightarrow m^*(A \cap E') \leq m^*(A \cap H_n') \longrightarrow (1)$$

$\because m^*$ is monotone

Further since each H_n is measurable

$$\Rightarrow m^*(A) = m^*(A \cap H_n) + m^*(A \cap H_n')$$

$(\because A$ is test set$)$

$\Rightarrow m^*(A) \geq m^*(A \cap H_n) + m^*(A \cap E')$ using ⓐ

$= m^*(A \cap (\bigcup_{i=1}^{n} F_i)) + m^*(A \cap E')$

$= \sum_{i=1}^{n} m^*(A \cap F_i) + m^*(A \cap E') \quad \forall n$

$\because m^*$ is countably subadditive, each F_i are disjoint

$\Rightarrow m^*(A) \geq \sum_{i=1}^{n} m^*(A \cap F_i) + m^*(A \cap E') \quad \forall n$

when $n \to \infty$

$m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap F_i) + m^*(A \cap E')$

$= m^*[A \cap (\bigcup_{i=1}^{\infty} F_i)] + m^*(A \cap E')$

$\because m^*$ is countably subadditive.

$\Rightarrow m^*(A) \geq m^*(A \cap E) + m^*(A \cap E')$

$\Rightarrow E$ is measurable

$\Rightarrow E \in \mathcal{M} \Rightarrow \mathcal{M}$ is σ algebra.

* ——— * * * ——— *

⇒ Lebesgue Measure:-

Let E be a measurable set then Lebesgue measure of E is defined to be Lebesgue outer measure of E and is symbolically denoted by $m(E)$. In short we can say " If E is measurable then

$$m(E) = m^*(E)$$

* ——— * ——— *

Theorem Let $\{E_i\}$ be a sequence of measurable sets then

1) m is countably subadditive.
2) m is finitly additive provided E_i are pairwise disjoint.
3) m is countably additive provided E_i are pairwise disjoint.
4) m is monotone.
5) m is translate invariant.

Proof: 1) Since E_i is a sequence of measurable sets and countable union of measurable sets is measurable. so

$$\bigcup_{i=1}^{\infty} E_i \text{ is measurable}$$

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} E_i\right) = m^*\left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$\leq \sum_{i=1}^{\infty} m^*(E_i) \quad \because m^* \text{ is countably subaditive}$$

$$= \sum_{i=1}^{\infty} m(E_i)$$

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m(E_i)$$

$\Rightarrow m$ is countably sub additive.

2) m is finitly additive provides E_i's are pairwise disjoint.

Since E_i's are pairwise disjoint and measurable, and finite union of measurable sets is measurable so

$\bigcup_{i=1}^{n} E_i$ is measurable

As we know that

$$m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

Put $A = \mathbb{R}$ then we have

$$m^*\left[\mathbb{R} \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] = \sum_{i=1}^{n} m^*(\mathbb{R} \cap E_i)$$

$$\Rightarrow m^*\left[\bigcup_{i=1}^{n} E_i\right] = \sum_{i=1}^{n} m^*(E_i) \qquad \left(\because \bigcup_{i=1}^{n} E_i \subseteq \mathbb{R}\right.$$

$$\Rightarrow m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m(E_i) \qquad \because E_i \text{ are measurable}$$

$\Rightarrow$ m is finitly additive provided E_i are pairwise disjoint.

3) To prove m is countably additive provided E_i are pairwise disjoint.

Sol Above 4 at end Apply $n \to \infty$

Since E_i's are pairwise disjoint and measurable and countable union of measurable sets is measurable. So $\bigcup_{i=1}^{\infty} E_i$ is measurable.

As we know that

$$m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

Put $A = \mathbb{R}$ then we have

$$m^*\left[\mathbb{R} \cap \left(\bigcup_{i=1}^{\infty}\right) E_i\right] = \sum_{i=1}^{\infty} m^*(\mathbb{R} \cap E_i)$$

$$\Rightarrow m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i)$$

$\Rightarrow m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty}(E_i)$ $\because \bigcup_{i=1}^{\infty} E_i$'s are measurable

$\Rightarrow$ m is countably additive provided E_i are Pairwise disjoint.

4) m is monotone:-

Let A and B be two measurable sets s.t $A \subseteq B$. then

$$B = A \cup (B \setminus A)$$

then A and $B\setminus A$ are pairwise disjoint therefore by finite additivity of m (2nd part)

$$m(B) = m(A) + m(B\setminus A) \longrightarrow \text{Ⓐ}$$

Now As $B\setminus A = B \cap A'$ is measurable so

$$m(B\setminus A) = m^*(B\setminus A) \geq 0$$

So Ⓐ $\Rightarrow$ $m(B) \geq m(A)$ $\Rightarrow m(A) \leq m(B)$

$\Rightarrow$ m is monotone

5) m is translation invariant:-

First we show that if E is measurable then $E+y$ is also measurable

As we know that $E \subseteq \mathbb{R}$ is measurable iff there exist an open set $O \supseteq E$ s.t $m^*(O\setminus E) < \varepsilon$

Now as O is an open set and translation of an open set is open set so $O+y$, $y \in \mathbb{R}$ is open

Further $E \subseteq O$

$$\Rightarrow E+y \subseteq O+y$$

Also $O+y \setminus E+y = (O\setminus E)+y$

$\Rightarrow m^*[(O+y)\setminus(E+y)] = m^*[(O\setminus E)+y]$

$= m^*(O\setminus E)$ $\because m^*$ is translation invariant.

$< \varepsilon$

$\Rightarrow m^*[(O+y)\setminus(E+y)] < \varepsilon$

$\Rightarrow E+y$ is measurable

Now as m^* is translation invariant so

$m^*(E+y) = m^*(E)$

$\Rightarrow m(E+y) = m(E)$ $\because E+y$ and E are measurable

$\Rightarrow m$ is translation invariant.

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v. Imp

Theorem:- Let $\{E_n\}$ be a sequence of measurable sets (1) if E_n is a decreasing sequence and $m(E_1) < \infty$ ($\because m(E_1)$ is finite)

$$m^*\left(\bigcap_{i=1}^{\infty} E_i\right) = m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$$

Show by an example that the condition $m(E_1) < \infty$ is necessary for.

2) If $\{E_n\}$ is an increasing sequence then

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$$

Proof Put

$$E = \bigcap_{i=1}^{\infty} E_i \quad \text{and} \quad F_i = E_i \setminus E_{i+1}$$
$$= E_i \cap E_{i+1}'$$

1) F_i is measurable for each i being

the intesection of two measurable sets

For $i \neq j$ (say $i > j$) so $i \geq j+1$

Now

$$E_i \subseteq E_{j+1} \Rightarrow E_i \setminus E_{j+1} = \phi$$

$$\Rightarrow E_i \cap E_{j+1}' = \phi$$

$$F_i \cap F_j = (E_i \cap E_{i+1}') \cap (E_j \cap E_{j+1}')$$ by construction

$$= (E_i \cap E_{j+1}') \cap (E_{i+1}' \cap E_j)$$ associative & commutative

$$= \phi \cap (E_{i+1}' \cap E_j)$$

$$= \phi$$

$\Rightarrow F_i$'s are pairwise disjoint

Now we show that

$$E_1 \setminus E = \bigcup_{i=1}^{\infty} F_i$$

Let $x \in E_1 \setminus E \Rightarrow x \in E_1$ but $x \notin \bigcap_{i=1}^{\infty} E_i$. As $\{E_i\}$ is decreasing sequence so $x \in E_j$

$$\Rightarrow x \notin E_{j+1}$$

$$\Rightarrow x \in E_j \setminus E_{j+1} = F_j$$

$$\Rightarrow x \in \bigcup_{i=1}^{\infty} F_i$$

$$\therefore E_1 \setminus E \subseteq \bigcup_{i=1}^{\infty} F_i \longrightarrow (1)$$

Reverse the argument for other inclusion we get

$$\bigcup_{i=1}^{\infty} F_i \subseteq E_1 \setminus E \longrightarrow (2)$$

from ① and ②

$$E_1 \setminus E = \bigcup_{i=1}^{\infty} F_i$$

$$\Rightarrow m(E_1 \setminus E) = m\left(\bigcup_{i=1}^{\infty} F_i\right)$$

$$= \sum_{i=1}^{\infty} m(F_i) \quad \because m \text{ is countably additive}$$

$$\Rightarrow m(E_1 \setminus E) = \sum_{i=1}^{\infty} m(E_i \setminus E_{i+1}) \longrightarrow ③$$

Now As $E \subset E_1$

$$\Rightarrow E_1 = E \cup (E_1 \setminus E)$$

$$\Rightarrow m(E_1) = m[E \cup (E_1 \setminus E)]$$

$$= m(E) + m(E_1 \setminus E)$$

$$\Rightarrow m(E_1) - m(E) = m(E_1 \setminus E) \longrightarrow ④$$

Similarly

$$E_{i+1} \subseteq E_i$$

$$\Rightarrow m(E_i) - m(E_{i+1}) = m(E_i \setminus E_{i+1}) \longrightarrow ⊛$$

from ③

$$m(E_1 \setminus E) = \sum_{i=1}^{\infty} m(E_i \setminus E_{i+1})$$

$$\Rightarrow m(E_1) - m(E) = \sum_{i=1}^{\infty} [m(E_i) - m(E_{i+1})] \quad \text{using ④ \& ⊛}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} [m(E_i) - m(E_{i+1})]$$

$$= \lim_{n \to \infty} [m(E_1) - \cancel{m(E_2)} + \cancel{m(E_2)} - \cancel{m(E_3)} + \cdots + \cancel{m(E_n)} - m(E_{n+1})]$$

$$\Rightarrow \cancel{m(E_1)} - m(E) = \cancel{m(E_1)} - \lim_{n\to\infty} m(E_{n+1})$$

$$\Rightarrow m(E) = \lim_{n\to\infty} m(E_{n+1})$$

$$\Rightarrow m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n)$$

Counter Example:-

For counter example consider μ as a counting measure on $\mathbb{N}$ and take $E_n = \{n, n+1, n+2, \dots\}$ then

$$\bigcap E_n = \phi \quad \text{but} \quad \mu(E_n) = \infty \quad \forall\, n$$

2) If E_n is an increasing sequance then

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$$

put $B_1 = E_1$

$B_2 = E_2 \setminus E_1$, $B_3 = E_3 \setminus E_2$

$\dots\dots$ $B_n = E_n \setminus E_{n-1}$, $n > 1$

clearly B_n is measurable

Moreover $B_n \cap B_m = \phi$, when $n \neq m$

$\Rightarrow \{B_n\}$ is a sequence of Pairwise disjoint measurable set and

$$E_n = \bigcup_{i=1}^{n} B_i \quad \text{for each } n$$

It follows that

$$\bigcup_n E_n = \bigcup_{i=1}^{\infty} B_i$$

$$\Rightarrow m\left(\bigcup_n E_n\right) = m\left(\bigcup_{i=1}^{\infty} B_i\right)$$
$$= \sum_{i=1}^{\infty} m(B_i)$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} m(B_i)$$
$$= \lim_{n\to\infty} m\left(\bigcup_{i=1}^{n} B_i\right)$$
$$= \lim_{n\to\infty} m(E_n)$$

* * *

⇒ **Definition :-**

A measure space $(X, \mathcal{A}, \mu)$ is said to be complete if each subset of a set of measure zero is itself measurable. That is if $A \in \mathcal{A}$ with $\mu(A) = 0$ and $B \subseteq A$, Then $B \in \mathcal{A}$

* *

Examples Imp 1) The Lebesgue measure space $(R, \mathcal{M}, m)$ is complete

Imp 2) The counting measure space (X, A, μ) is complete.

3) The measurable space $(X, P(X), \mu)$ is complete.

4) The measure space $(X, \mathcal{A}, \mu)$ where X contains more than one point, $\mathcal{A} = \{\phi, X\}$ and μ is zero measure is not complete measure space.

Solution

1) The Lebesgue measure space (or m-space) $(R, \mathcal{M}, m)$ is complete

Let $A \in \mathcal{M}$ with $m(A) = 0$
$\Rightarrow m^*(A) = 0$ $\because A \in \mathcal{M} \Rightarrow A$ is measurable

Let $B \subseteq A$
$\Rightarrow m^*(B) \leq m^*(A)$ $\because m^*$ is monotone
$\Rightarrow m^*(B) \leq 0$
$\Rightarrow m^*(B) = 0$
$\Rightarrow B$ is measurable (by Previous theorem
$\Rightarrow B \in \mathcal{M}$
So $(\mathbb{R}, \mathcal{M}, m)$ is complete.

2) The counting measure space $(X, \mathcal{A}, \mu)$ is complete.
As $\phi \in \mathcal{A}$ is the only set in $\mathcal{A}$ with $\mu(\phi) = 0$.
So the result is obvious

$\boxed{\begin{array}{l}\phi \subseteq \phi \\ \text{Also } \phi \in \mathcal{A}\end{array}}$

3) The measure space $(X, P(X), \mu)$ is complete
Let $A \in P(X)$ with $\mu(A) = 0$
Let $B \subseteq A \subseteq X$ $\Rightarrow B \subseteq X$
$\Rightarrow B \in P(X)$
$\Rightarrow (X, P(X), \mu)$ is complete

4) E be any proper subset of X
As $\mu(X) = 0$, so $\mu(E) = 0$
But clearly $E \notin \mathcal{A}$
$\Rightarrow$ The measure space $(X, \mathcal{A}, \mu)$, where X contains more than one point, $\mathcal{A} = \{\phi, X\}$ and μ is zero measure is not complete measure space.

Example – Find the Lebesgue measure of the following subsets of $\mathbb{R}$

1) $(-2,6)$, $(-2,6]$, $[-2,6)$, $[-2,6]$

2) $B = ([3,5] \cup [-4,-2])$

3) $F = \bigcup_{k=1}^{\infty} \{x \in \mathbb{R} : \frac{1}{2^k} \leq x < \frac{1}{2^{k-1}}\}$

4) Q , Q', $\mathbb{R}$

Solution

1) As we know that any interval (a,b) is measurable, $a,b \in \mathbb{R}$

Also outer measure of an interval is its length. So

$m(-2,6) = m^*(-2,6) = 6-(-2) = 8$

$m(-2,6] = m[(-2,6) \cup \{6\}]$

$= m(-2,6) + m(\{6\})$ $\left\{ \begin{array}{l} \because \text{ they are disjoint} \\ \text{\& } m \text{ is finitly} \\ \text{additive} \end{array} \right.$

$= 8 + 0$

$= 8$

Similarly

$m[-2,6) = l[-2,6) = 8$

& $m[-2,6] = l[-2,6] = 8$

2)

$B = ([3,5] \cup [-4,-2])$

the set $[3,5]$ and $[-4,-2]$ are disjoint and measurables by the finite additive property of m,

$$m(B) = m[3,5] + m[-4,-2]$$
$$= (5-3) + (-2+4)$$
$$= 2+2$$
$$= 4$$

3) As

$$F = [\tfrac{1}{2},1) \cup [\tfrac{1}{4},\tfrac{1}{2}) \cup \cdots \cup [\tfrac{1}{2^n},\tfrac{1}{2^{n-1}}) \cup \cdots$$

The set F is countable union of pairwise disjoint measurable sets. By the countable additive property of m, we have

$$m(F) = m([\tfrac{1}{2},1)) + m([\tfrac{1}{4},\tfrac{1}{2})) + m([\tfrac{1}{8},\tfrac{1}{4})) + \cdots$$
$$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

This is infinite geometric series and $S_p = \frac{a}{1-r}$

$$= \frac{1/2}{1-1/2}$$
$$= 1$$

4) As Q is countable so

$$m^*(Q) = 0 \Rightarrow Q \text{ is measurable}$$
$$\Rightarrow m(Q) = m^*(Q) = 0$$

Also $\mathbb{R} = (-\infty,\infty)$

$$m(\mathbb{R}) = \infty - (-\infty) = \infty$$

Further as Q and Q' are disjoint

$$\mathbb{R} = Q \cup Q'$$
$$\Rightarrow m(\mathbb{R}) = m(Q) + m(Q')$$
$$\Rightarrow \infty = 0 + m(Q')$$
$$\Rightarrow m(Q') = \infty$$

V.V.Imp

⇒ **Proposition:-** If E_1 and E_2 are measurable sets then
$$m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$$

Proof

Since E_1 is measurable so for any subset "A" of $\mathbb{R}$
$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1')$$
In particular
$$A = E_2$$
$$\Rightarrow m^*(E_2) = m^*(E_2 \cap E_1) + m^*(E_2 \cap E_1') \longrightarrow (1)$$
Now as E_2 is measurable so similarly
$$m^*(E_1) = m^*(E_1 \cap E_2) + m^*(E_1 \cap E_2') \longrightarrow (2)$$
Further as E_1 and E_2 are measurable so (1) and (2) takes the form
$$m(E_2) = m(E_1 \cap E_2) + m(E_1' \cap E_2) \longrightarrow (3)$$
$$m(E_1) = m(E_1 \cap E_2) + m(E_1 \cap E_2') \longrightarrow (4)$$
add (3) & (4)
$$m(E_1) + m(E_2) = m(E_1 \cap E_2) + m(E_1 \cap E_2) + m(E_1 \cap E_2') + m(E_1' \cap E_2) \longrightarrow (5)$$
As we know that
$$E_1 \cup E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1) \cup (E_1 \cap E_2)$$
$$\Rightarrow E_1 \cup E_2 = (E_1 \cap E_2') \cup (E_2 \cap E_1') \cup (E_1 \cap E_2)$$
Also as $E_1 \cap E_2$, $E_2 \cap E_1'$, $E_1 \cap E_2'$ are disjoint
$$\Rightarrow m(E_1 \cup E_2) = m(E_1 \cap E_2') + m(E_2 \cap E_1') + m(E_1 \cap E_2) \longrightarrow (6)$$
$\because m$ is finitely additive

using equ ⑥ equ ⑤ implies

$$m(E_1) + m(E_2) = m(E_1 \cap E_2) + m(E_1 \cup E_2)$$

$$\Rightarrow m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$$

* * *

⇒ **Proposition:-** If E_1, E_2, E_3 are measurable sets then

$$m(E_1 \cup E_2 \cup E_3) = m(E_1) + m(E_2) + m(E_3) - m(E_1 \cap E_2) - m(E_1 \cap E_3) - m(E_2 \cap E_3) + m(E_1 \cap E_2 \cap E_3)$$

Proof:-

Let $E_1 \cup E_2 = D$

Then $m(E_1 \cup E_2 \cup E_3) = m(D \cup E_3)$

$$\Rightarrow m(E_1 \cup E_2 \cup E_3) = m(D) + m(E_3) - m(D \cap E_3)$$

$$= m(E_1 \cup E_2) + m(E_3) - m[(E_1 \cup E_2) \cap E_3]$$

$$= m(E_1) + m(E_2) - m(E_1 \cap E_2) + m(E_3) - m[(E_1 \cap E_3) \cup (E_2 \cap E_3)]$$

$$= m(E_1) + m(E_2) - m(E_1 \cap E_2) + m(E_3) - [m(E_1 \cap E_3) + m(E_2 \cap E_3) - m(E_1 \cap E_3 \cap E_2 \cap E_3)]$$

$$= m(E_1) + m(E_2) + m(E_3) - m(E_1 \cap E_2) - m(E_1 \cap E_3) - m(E_2 \cap E_3) + m(E_1 \cap E_2 \cap E_3)$$

$$\Rightarrow m(E_1 \cup E_2 \cup E_3) = m(E_1) + m(E_2) + m(E_3) - m(E_1 \cap E_2) - m(E_1 \cap E_3) - m(E_2 \cap E_3) + m(E_1 \cap E_2 \cap E_3)$$

Proved.

Example:- Find the Lebesgue measure of

1- $A = (-3,4) \cup [1,6]$

2- $B = [-1,1] \cup (0,1)$

Solution

Since $(-3,4)$ and $[1,6]$ are not disjoint

$\Rightarrow (-3,4) \cap [1,6] = [1,4)$

$$m(A) = m[(-3,4) \cup [1,6]]$$

$$= m[(-3,4)] + m([1,6]) - m([1,4))$$

$$= 7 + 5 - 3$$

$$= 9$$

⇒ Sum Modulo 1:-

Let $A = [0,1)$ and $x, y \in A$, we denote and define the sum modulo 1 of x, y by

$$x \overset{\circ}{+} y = \begin{cases} x+y & \text{if } x+y<1 \\ x+y-1 & \text{if } x+y \geq 1 \end{cases}$$

Example

$0.5,\ 0.7 \in A$

$$0.5 \overset{\circ}{+} 0.7 = 0.5 + 0.7 - 1$$

$$= 1.2 - 1$$

$$= 0.2$$

A sum یعنی کہ
میں ہی رہے

& $0.2,\ 0.4 \in A$

$$\Rightarrow 0.2 \overset{\circ}{+} 0.4 = 0.2 + 0.4$$

$$= 0.6$$

⇒ Translate Modulo 1:-

Let $E \subseteq A = [0,1)$, we define the translate modulo 1 of E to be the set

$$E \dot{+} y = \{ x \dot{+} y : x \in E \} \;, \; y \in A$$

⇒ **Lemma:-** Let $E \subseteq A = [0,1)$ be a measurable set. for each $y \in A$, the set $E \dot{+} y$ is measurable and $m(E \dot{+} y) = m(E)$

Proof: Put $E_1 = E \cap [0, 1-y)$

$E_2 = E \cap [1-y, 1)$

Then the set E, $[0, 1-y)$, $[1-y, 1)$ are measurable. then E_1 and E_2 are also measurable. Further

$$E_1 \cup E_2 = E, \quad E_1 \cap E_2 = \phi$$

by finite additivity of m

$$m(E) = m(E_1 \cup E_2) = m(E_1) + m(E_2) \longrightarrow (1)$$

$\because E_1$ and E_2 are disjoint

Now if $y_1 \in E_1$, then $y_1 \in E \cap [0, 1-y)$

$\Rightarrow y_1 \in [0, 1-y)$

$\Rightarrow 0 \leq y_1 < 1-y \quad \Rightarrow y_1 < 1-y$

$\Rightarrow y_1 + y < 1$

$\Rightarrow y_1 \dot{+} y = y_1 + y$ (by def of sum Modulo 1

$\Rightarrow E_1 \dot{+} y = E_1 + y$

Now if $y_2 \in E_2 \Rightarrow y_2 \in E \cap [1-y, 1)$

$\Rightarrow y_2 \in [1-y, 1)$

$\Rightarrow 1-y \leq y_2 < 1$

$\Rightarrow 1-y \leq y_2 \quad \Rightarrow 1 \leq y_2+y$

or $y_2+y \geq 1$

$\Rightarrow y_2 \dot{+} y = y_2+y-1$ {by def of sum Modulo 1

$\Rightarrow E_2 \dot{+} y = E_2+(y-1)$

The sets E_1+y and $E_2+(y-1)$ are measurable by translation of a measurable set is a measurable set.

Now by translation invariant of m

$$\left.\begin{aligned} m(E_1 \dot{+} y) &= m(E_1+y) = m(E_1) \\ \& \ m(E_2 \dot{+} y) &= m(E_2+(y-1)) = m(E_2) \end{aligned}\right\} \longrightarrow ②$$

Now $E \dot{+} y = (E_1 \dot{+} y) \cup (E_2 \dot{+} y)$

$\Rightarrow m(E \dot{+} y) = m(E_1 \dot{+} y) + m(E_2 \dot{+} y)$

$= m(E_1) + m(E_2)$ using ②

$= m(E)$ using ①

$\Rightarrow m(E \dot{+} y) = m(E)$

***

Proposition Construct Cantor's set and also find Lebesgue measure.

Proof consider the closed unit interval

$[0,1]$. Let $I_{11} = (\frac{1}{3}, \frac{2}{3})$ be the middle third ~~term~~ of the closed unit interval. We remove this open interval I_{11} from $[0,1]$, we get two closed sub intervals $[0, 1/3]$, $[2/3, 1]$.

Let I_{21} and I_{22} be their middle third term i.e $I_{21} = (1/9, 2/9)$, $I_{22} = (7/9, 8/9)$

After removing these two open intervals we have four closed intervals $[0, 1/9]$, $[2/9, 3/9]$, $[6/9, 7/9]$, $[8/9, 1]$

Let $I_{31}, I_{32}, I_{33}, I_{34}$ be the middle third term of the above closed intervals.

Note that in the first step 2^{1-1} open intervals i.e I_{11} are removed.

At second step 2^{2-1} intervals i.e I_{21}, I_{22} are removed.

At third step 2^{3-1} open intervals i.e I_{31}, I_{32}, I_{33} and I_{34} are removed.

Continuing in this way at nth step 2^{n-1} pairwise disjoint open intervals I_{nm}, $m = 1, 2, 3, \dots, 2^{n-1}$ are removed

Note each I_{nm} has length equal to 3^{-n}.

If we continuing above process we have sequence of $\{I_{nm}\}$ of pairwise disjoint open sets

Put $G = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}$

Then G is the union of sequence of open sets is open and measurable. The compliment of G w.r.t $[0,1]$ is called cantor's set and it is denoted by "C" i.e

$$C = [0,1] \setminus G$$

As compliment of measurable set is measurable. So "C" is measurable. Further G is open. So $G' = C$ is closed.

Now $m(G) = m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}\right)$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} m(I_{nm})$$ $\because$ m is countably additive & I_{nm} are disjoint

$$= m(I_{11}) + [m(I_{21}) + m(I_{22})] + [m(I_{31}) + m(I_{32}) + m(I_{33}) + m(I_{34})]$$

$$= \frac{1}{3} + \left(\frac{1}{9} + \frac{1}{9}\right) + \left(\frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27}\right) + \cdots$$

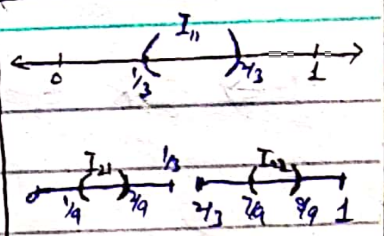
$$= \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots$$

$$= \frac{1/3}{1 - 2/3}$$

G.P
$S_\infty = \frac{a}{1-r}$
a = 1st term
r = Common ratio

$$= \frac{1/3}{1/3}$$

$$= 1$$



$l(I_{11}) = 1/3 = 3^{-1}$
$l(I_{21}) = l(I_{22}) = 1/9 = 3^{-2}$

clearly $[0,1] = C \cup G$

$$m([0,1]) = m(C \cup G)$$
$$\Rightarrow 1 = m(C) + m(G)$$
$$\Rightarrow 1 = m(C) + 1$$
$$\Rightarrow \boxed{m(C) = 0}$$

***

Proposition:- Let F be a subset of $[0,1]$ constructed in the same manner as the cantor set except that each of the interval length removed at the nth step has length $\alpha 3^{-n}$ with $0<\alpha<1$ then F is closed set and $m(F) = 1-\alpha$.

Proof

As in the construction of cantor set at the nth step we removed 2^{n-1} pairwise disjoint open intervals I_{nm}, $m = 0, 1, 2, \dots, 2^{n-1}$ and of length 3^{-n} but by given condition, here in the construction of F at the nth step we remove 2^{n-1} pairwise disjoint open intervals I_{nm}, $m = 0, 1, 2, \dots, 2^{n-1}$ and each of length $\alpha 3^{-n}$, $0 < \alpha < 1$. So then if

$$G = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm} \quad \& \quad F = [0,1] \setminus G$$

then $m(G) = m(I_{11}) + [m(I_{21}) + m(I_{22})] + [m(I_{31}) + m(I_{32}) + m(I_{33}) + m(I_{34})] + \dots$

$$= \frac{\alpha}{3} + \frac{2\alpha}{9} + \frac{4\alpha}{27} + \dots$$

$$\Rightarrow m(G) = \alpha\left[\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\cdots\right]$$

$$= \alpha\left[\frac{1/3}{1-2/3}\right] = \alpha(1)$$

$$= \alpha$$

So then

$$[0,1] = F \cup G$$

$$m([0,1]) = m(F \cup G)$$

$$\Rightarrow 1 = m(F) + m(G)$$

$$= m(F) + \alpha$$

$$\Rightarrow m(F) = 1-\alpha$$

This F is called generalized cantor set.

** ——— **

⇒ Cantor Function:-

Any $x \in [0,1]$ can be represented in a binary expansion as

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n} \quad \text{where } a_n = 1 \text{ or } 0$$

Define a function

$$f:[0,1] \longrightarrow [0,1] \text{ by}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2a_n}{3^n} \quad \text{where } a_n = 0 \text{ or } 1$$

This function takes values entirely in the cantor set C and is called cantor function

——— ★ ———

Theorem:- The cantor set C is uncountable

Proof:- Define a cantor function

$f : [0,1] \longrightarrow C$ as

$$f(x) = \sum_{n=1}^{\infty} \frac{2a_n}{3^n} \qquad a_n = 0 \text{ or } 1,$$
where $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$

f is 1-1:

Let $f(x_1) = f(x_2)$

$$\Rightarrow \cancel{\sum_{n=1}^{\infty}} \frac{\cancel{2}a_n'}{3^n} = \cancel{\sum_{n=1}^{\infty}} \frac{2a_n^2}{3^n}$$

$$\Rightarrow a_n' = a_n^2$$

$$\Rightarrow \frac{a_n'}{2^n} = \frac{a_n^2}{2^n}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{a_n'}{2^n} = \sum_{n=1}^{\infty} \frac{a_n^2}{2^n}$$

$\Rightarrow x_1 = x_2 \Rightarrow f$ is 1-1

f is onto

As for every $\sum_{n=1}^{\infty} \frac{2a_n}{3^n}$ there exist some $x \in [0,1]$ s.t

$$f(x) = \sum_{n=1}^{\infty} \frac{2a_n}{3^n}$$

$\Rightarrow f$ is onto

therefore f is bijective

$\Rightarrow [0,1] \sim C$

As $[0,1]$ is uncountable $\Rightarrow C$ is uncountable

Theorem:- Demonstrate the existance of a non-Lebesgue measure set.

Proof:- Let $A = [0, 1)$ define a relation "$\sim$" on $A = [0, 1)$ by $x \sim y$ iff $x - y$ is rational number.

The relation $\sim$ is an equivallence

i) Reflexive:-

$x \sim x \quad \forall \; x \in A$ because

$x - x = 0$ is rational number

$\Rightarrow \sim$ is reflexive.

ii) Symmetric:-

Let $x \sim y$

$\Rightarrow x - y$ is rational

$\Rightarrow -(x-y)$ is rational

$\Rightarrow y - x$ is rational $\Rightarrow y \sim x$

$\Rightarrow \sim$ is symmetric.

iii) Transitive:-

Let $x, y, z \in A$ and

$x \sim y$ & $y \sim z$

Now

$$x - z = (x-y) + (y-z)$$

Since $x \sim y \Rightarrow x - y \in \mathbb{Q}$

& $y \sim z \Rightarrow y - z \in \mathbb{Q}$

$$\Rightarrow x - z = (x-y) + (y-z) \in \mathbb{Q}$$

$$\Rightarrow x \sim z$$

$\Rightarrow \sim$ is transitive

$\Rightarrow$ "$\sim$" is equivallence on $A = [0,1)$

So "$\sim$" partitions the set $A = [0,1)$ into disjoint equivallence classes s.t any two elements of a (same) class differ by a rational number. So we can find a set P which contains exactly one element from each equivallence class. This set P is desired non measurable set.

Let $\{x_0, x_1, x_2, \ldots\}$ be the set of rational numbers belonging to $[0,1)$ with $r_0 = 0$ constant

$$P_i = P \dotplus r_i \quad , i = 1, 2, 3, \ldots$$

Then $P_0 = P$

Here all P_i's are disjoint i.e

$$P_i \cap P_j = \phi \quad , i \neq j$$

Suppose $P_i \cap P_j \neq \phi$ for $i \neq j$

$\Rightarrow x \in P_i \cap P_j$

$\Rightarrow x \in P_i$ & $x \in P_j$

$\Rightarrow x \in P \dotplus r_i$ & $x \in P \dotplus r_j$

$\Rightarrow x = p_i + r_i$ & $x = p_j + r_j$, $p_i, p_j \in P$

$\Rightarrow p_i + r_i = p_j + r_j$

$\Rightarrow p_i - p_j = r_j - r_i \in Q$

$\Rightarrow p_i - p_j$ is rational number

$\Rightarrow P_i$ & P_j belonging to same equivalent class which is a contradiction. because P contains exactly one element from each equivalance class.

So $\quad P_i \cap P_j = \phi \quad$ for $i \neq j$

Now if $x \in A$ then x is in some equivallence class. Then x is equivalent to (related to) some element P_i of P

$\Rightarrow x - P_i$ is a rational number

$\Rightarrow x - P_i = r_i$ for some rational r_i

$\Rightarrow x = P_i + r_i \in P \dot{+} r_i = P_i$

$\Rightarrow x \in P_i$ for some i

$\Rightarrow x \in \bigcup_i P_i \quad \Rightarrow A \subseteq \bigcup_i P_i$

Put $\bigcup_i P_i \subseteq A \quad \because A$ is universal set here

$$\Rightarrow A = \bigcup_i P_i$$

Now assume that P is measurable then each P_i is measurable

Now

$$m([0,1]) = m(A) = m\left(\bigcup_i P_i\right)$$

$$= \sum_i m(P_i) \quad \because m \text{ is countably additive}$$

$$= \sum_i m(P \dot{+} r_i)$$

$$\Rightarrow m([0,1)) = \sum_i m(P) \quad \because m(P+r_i) = m(P)$$
$$\longrightarrow \circledast$$

Now by assumption P is measurable

$$\Rightarrow m(P) \geq 0$$

If $m(P) > 0$ then by $\circledast$

$$1 = \infty$$

which is not possible $\Rightarrow m(P) \not> 0$

If $m(P) = 0$ then by $\circledast$

$$1 = 0$$

which is not possible

$$\Rightarrow m(P) \neq 0$$

So $m(P) \not\geq 0$ which is a contradiction.

So our assumption is wrong & Hence P is not measurable.

***

⇒ Interesting Facts:-

1) Measure of an open set is always non zero but measure of closed set might be zero e.g. cantor set C is closed and $m(C) = 0$
2) If the measure of set is non-zero then it is uncountable. But if a set is uncountable then its measure may be zero for example cantor set C is uncountable and has measure zero.

3) Note that it might be possible that $A \sim B$ but both has different measures. e.g $\mathbb{R} \sim [0,1] \sim C$ and $m(\mathbb{R}) = \infty$ and $m[0,1] = 1$ & $m(C) = 0$

4) Any finite set in $\mathbb{R}$ is closed & so is measurable.

————————

⇒ Borel Set:-

The collection β of borel sets is defined to be σ- algebra generated by the collection of the intervals of the form $]a, b]$, $a, b \in \mathbb{R}$. The existance of β is guaranteed by "Let G be a family of subsets of X, then there is a smallest σ- algebra containg G"

Since open intervals $]a, b[= \bigcup_{n=1}^{\infty}]a, b-\frac{1}{n}]$

⇒ β contains all open intervals.

Similarly we may replace $]a, b]$ in the definition of β (Borel set) by $[a, b[$, $]a, \infty[$, $]-\infty, b[$, $]-\infty, b]$ etc.

————————

Theorem:- The borel σ-Algebra β is generated by each of the following collection of sets.

1) The collection C_1 of all closed subsets of $\mathbb{R}$
2) The collection C_2 of all subintervals of the form $]-\infty, b]$
3) The collection C_3 of all subintervals of the form $]a, b]$

Proof:- Let the σ-Algebra generated by C_1, C_2, C_3 by denoted by β_1, β_2 & β_3 respectively.

Further by definition of Borel σ-algebra β contains the family of open sets and is closed under complimentation. Now To prove

$$\beta = \beta_1 = \beta_2 = \beta_3$$

As β contains the family of open sets and is closed under complimentation so β also contains closed sets.

($\because$ Compliment of an open set is closed & β is σ-Algebra $\Rightarrow$ If $A \in \beta$ so $A' \in \beta$)

$$\Rightarrow \beta_1 \subseteq \beta$$

Now $]-\infty, b] =]b, \infty['$

$\Rightarrow]-\infty, b]$ are closed sets for $b \in \mathbb{R}$ and so belonging to β_1

$$\Rightarrow \beta_2 \subseteq \beta_1$$

Now $]a, b] =]-\infty, b] \cap]a, \infty[$
$=]-\infty, b] \cap]-\infty, a]'$

$\Rightarrow]a, b] \in \beta_2$

$\Rightarrow \beta_3 \subseteq \beta_2$

Now $]a, b[= \bigcup_{n=1}^{\infty}]a, b - 1/n] \in \beta_3$

$\Rightarrow \beta \subseteq \beta_3$

$\Rightarrow \beta \subseteq \beta_3 \subseteq \beta_2 \subseteq \beta_1 \subseteq \beta$

$\Rightarrow \beta_1 = \beta_2 = \beta_3 = \beta$

——

Theorem Every Borel set is measurable.

Proof

We know that every open set is measurable, so it means $\mathcal{M}$ contains all open sets. But β is the smallest σ-Algebra containing all open sets.

Hence $\beta \subseteq \mathcal{M}$

$\Rightarrow$ Every Borel set is measurable.

——**——

⇒ Definition:-

The restriction of Lebesgue measure m to β is called Borel measure and the triplet $(\mathbb{R}, \beta, m)$ is called Borel measure space.

**

Theorem:- Any singleton set, finite set and countable set is a Borel set, with Borel measure zero.

Proof) To prove the theorem it is sufficent to prove that any singlton set is Borel set with Borel measure zero because the remaining part of the theorem is obvious by the fact that β is σ-algebra and m is finitly and countably additive.

Let a be the real number then we have

$$\{a\} = \bigcap_{n=1}^{\infty}]a-\tfrac{1}{n}, a]$$

then $\{a\} \in \beta$

$\Rightarrow \{a\}$ is borel set

Let $E_n =]a - 1/n, a]$

Then E_n is decreasing sequence and

$$m(E_1) = m(]a-1, a[) = a-(a-1) = 1 < \infty$$

So $$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n) = \lim_{n\to\infty} m(]a-\tfrac{1}{n}, a])$$

$$= \lim_{n\to\infty} \left[a-\left(a-\tfrac{1}{n}\right)\right]$$

$$\Rightarrow m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} (1/n) = 0$$

$$\Rightarrow m(\{a\}) = 0$$

Existance of Non-Measurable Sets:-

We have seen that

- Every finite set, every countable set, every interval is measurable.
- ϕ, $\mathbb{R}$ are measurable
- Every open set, every closed set is measurable
- If E is measurable then E' is measurable
- F_σ, G_δ set is measurable.
- Cantor set "C" is measurable
- Union & Intersection of sequence of measurable sets is measurable.

It seems from the above that every subset of $\mathbb{R}$, we think is measurable. But the fact is that it is not true and non emeasurable sets exist.

See theorem on Page 88 (back)

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CH-2: MEASURABLE FUNCTIONS.

⇒ Measurable Functions:-

Let $(X, \mathcal{A}, \mu)$ be a measure space then an extended real valued function f on E is said to be measurable function if for each $\alpha \in \mathbb{R}$, $\{x \in E \,;\, f(x) > \alpha\} \in \mathcal{A}$

In other words if $D \subseteq \mathbb{R}$ and D is measurable then the function $f: D \to \overline{\mathbb{R}}$ is measurable if for all $\alpha \in \mathbb{R}$, $\{x \in D, f(x) > \alpha\}$ is measurable.

* * *

Theorem:- Let f be an extended real valued function defined on a measurable set D, then the following are equivalent.

(i) f is measurable

(ii) $\{x \in D : f(x) \geq \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$

(iii) $\{x \in D : f(x) < \alpha\}$ " " " "

(iv) $\{x \in D : f(x) \leq \alpha\}$ " " " "

(V) Moreover (i) → (iv) implies that $\{x \in D : f(x) = \alpha \;\forall\, \alpha \in \overline{\mathbb{R}}\}$ is measurable.

Proof (i) ⇒ (ii) i.e f is measurable

To prove $\forall\, \alpha \in \mathbb{R}$, $\{x \in D : f(x) \geq \alpha\}$ is measurable.

As f is measurable so $\{x \in D : f(x) > \alpha\}$, for all $\alpha \in \mathbb{R}$ measurable.

we claim that

$$\{x\in D: f(x)\geq\alpha\} = \bigcap_{n=1}^{\infty}\{x\in D: f(x)>\alpha-\tfrac{1}{n}\}$$

Let $y\in\{x: f(x)\geq\alpha\}$

$\Rightarrow f(y)\geq\alpha \quad \Rightarrow f(y)>\alpha-\frac{1}{n} \quad \forall n\in N$

$\Rightarrow y\in\{x\in D: f(x)>\alpha-\frac{1}{n}\} \quad \forall n$

$\Rightarrow y\in\bigcap_{n=1}^{\infty}\{x\in D: f(x)>\alpha-\frac{1}{n}\}$

$\Rightarrow \{x\in D: f(x)\geq\alpha\}\subseteq\bigcap_{n=1}^{\infty}\{x\in D: f(x)>\alpha-\frac{1}{n}\}$

working backward we have

$\bigcap_{n=1}^{\infty}\{x\in D: f(x)>\alpha-\frac{1}{n}\}\subseteq\{x\in D: f(x)\geq\alpha\}$

$\Rightarrow \{x\in D: f(x)\geq\alpha\} = \bigcap_{n=1}^{\infty}\{x\in D: f(x)> \alpha-\frac{1}{n}\}$

As given $\{x\in D: f(x)>\alpha-\frac{1}{n}\}$ is measurable & countable intersection of measurable sets is measurable so

$\bigcap_{n=1}^{\infty}\{x\in D: f(x)>\alpha-\frac{1}{n}\}$ is measurable

$\Rightarrow \{x\in D: f(x)\geq\alpha\}$ is measurable

(ii) $\Rightarrow$ (iii)

Given $\{x\in D: f(x)\geq\alpha\}$ is measurable

To prove $\{x\in D: f(x)<\alpha\}$ " " $\forall\alpha\in\mathbb{R}$

Note that

$$\{x\in D: f(x)<\alpha\} = D\setminus\{x\in D: f(x)\geq\alpha\}$$

$$= D\cap\{x\in D: f(x)\geq\alpha\}'$$

As R.H.S is measurable being the

intersection of two measurable sets

$\Rightarrow$ L.H.S i.e $\{x \in D : f(x) < \alpha\}$ is measurable.

(iii) $\Rightarrow$ (iv) i.e

given $\{x \in D : f(x) < \alpha\}$ is measurable

To prove $\{x \in D : f(x) \leq \alpha\}$ " " $\forall \alpha \in \mathbb{R}$

As $\{x \in D : f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in D : f(x) < \alpha + \frac{1}{n}\}$

As $\{x \in D : f(x) < \alpha + \frac{1}{n}\}$ is measurable and countable intersection of measurable sets is measurable. So R.H.S is measurable

$\Rightarrow$ L.H.S is measurable.

(iv) $\Rightarrow$ (i)

Given $\{x \in D : f(x) \leq \alpha\}$ is measurable

To prove f is measurable

As $\{x : f(x) > \alpha\} = D \setminus \{x : f(x) \leq \alpha\}$

$= D \cap \{x \in D : f(x) \leq \alpha\}'$

As R.H.S is measurable

$\Rightarrow \{x : f(x) > \alpha\}$ is measurable

$\Rightarrow f$ is measurable.

(v) Let $\alpha \in \mathbb{R}$ then

$\{x \in D : f(x) = \alpha\} = \{x \in D : f(x) \geq \alpha\} \cap \{x \in D : f(x) \leq \alpha\}$

Since R.H.S is measurable

$\Rightarrow$ L.H.S is measurable

If $\alpha = \infty$ then $\{x \in D : f(x) = \infty\}$

$= \bigcap_{n=1}^{\infty} \{x \in D : f(x) > n\}$

If $\alpha = -\infty$ then $\{x : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) < -n\}$

$\Rightarrow \{x \in D : f(x) = \alpha\}$ is measurable for $\alpha = \infty$ & $-\infty$

$\Rightarrow \{x \in D : f(x) = \alpha\}$ is measurable for every extended real number α i.e $\alpha \in \overline{\mathbb{R}}$

* * *

Remark:- (i) It is obvious that if a result holds for extended real valued function, then it is holds in particular for real valued function.

(ii) From previous theorem (Part V) it is clear that extended real valued constant function is measurable. however, it will be interesting to note that for $f(x) = c$, where $c \in \overline{\mathbb{R}}$ then the measurability of 'f' follows from

$$\{x \in D : f(x) > \alpha\} = \begin{cases} D & \text{if } \alpha < c \\ \phi & \text{if } \alpha \geq c \end{cases}$$

and measurability of D and ϕ

(iii) In previous theorem (Part V) we have shown that if f is measurable function on D then the set $\{x \in D : f(x) = \alpha\}$ is measurable for each real number α. However, conversely if $\{x \in D : f(x) = \alpha\}$ is measurable set for each $\alpha \in \mathbb{R}$ then f is not necessarily measurable. for example

Example:-

Let P be a non measurable +ve subset of $\mathbb{R}$, then also P' is non measurable

Now Put $A = \{x \in D : x > 0\}$ and
$B = \{x \in D : x \leq 0\}$ then $A \cup B = D$
and $A \cap B = \phi$.
Now Let
$g : A \to P$ and $h : B \to P'$
be any two bijective function and define
$f : D \to \mathbb{R}$ by $f(x) = \begin{cases} g(x) & \text{if } x \in A \\ h(x) & \text{if } x \in B \end{cases}$

Then "f" is extension of both "g" and "h" also obviously bijective function. Then for any real number α, The set $\{x \in D : f(x) = \alpha\}$ contains exactly one point and so it is measurable. But note that the set $\{x \in D : f(x) > 0\}$ being the same as P is non measurable. Hence f is not a measurable function.

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⇒ Definition:-

Let f be an extended real valued function defined on any set A. Then the positive part f^+ and -ve part f^- of f are the extended real valued functions defined by

$$f^+(x) = \text{Max}\{f(x), 0\} = f \vee 0$$

and

$$f^-(x) = \text{Max}\{-f(x), 0\} = -f \vee 0$$ for all $x \in A$.

——

Theorem:- If f is an extended real valued function, then

(i) $f = f^+ - f^-$ (ii) $|f| = f^+ + f^-$

Proof) 1) Here arises the following cases

Case I

If $f(x) = 0$ for all x

then $f^+(x) = \text{Max}\{f(x), 0\} = 0$

$f^-(x) = \text{Max}\{-f(x), 0\} = 0$

$\Rightarrow f^+(x) - f^-(x) = 0 - 0 = 0 = f(x)$

$\Rightarrow f^+(x) - f^-(x) = f(x)$

$\Rightarrow f = f^+ - f^-$

Case II

If $f(x) > 0$ then

$f^+(x) = \text{Max}\{f(x), 0\} = f(x)$

$f^-(x) = \text{Max}\{-f(x), 0\} = 0$

$\Rightarrow f^+(x) - f^-(x) = f(x) - 0 = f(x)$

$\Rightarrow f^+ - f^- = f$

Case III

If $f(x) < 0$ then

$f^+(x) = \text{Max}\{f(x), 0\} = 0$

$f^-(x) = \text{Max}\{-f(x), 0\} = -f(x)$

$f^+(x) - f^-(x) = 0 - (-f(x)) = f(x)$

$\Rightarrow f^+ - f^- = f$

Hence combining all the cases

$$f = f^+ - f^-$$

2)

Case I :-

If $f(x) = 0 \Rightarrow |f(x)| = 0 \quad \forall\, x \in A$

Then $f^+(x) = \text{Max}\{f(x), 0\} = 0$

$f^-(x) = \text{Max}\{-f(x), 0\} = 0$

$\Rightarrow f^+(x) + f^-(x) = 0 + 0 = 0 = |f(x)|$

$\Rightarrow |f| = f^+ + f^-$

Case II :- If $f(x) > 0$, then $|f(x)| = f(x)$

Now $f^+(x) = \text{Max}\{f(x), 0\} = f(x)$

$f^-(x) = \text{Max}\{-f(x), 0\} = 0$

So $f^+(x) + f^-(x) = f(x) + 0 = f(x) = |f(x)|$

$\Rightarrow |f| = f^+ + f^-$

Case III If $f(x) < 0$ then $|f(x)| = -f(x)$

Now $f^+(x) = \text{Max}\{f(x), 0\} = 0$

$f^-(x) = \text{Max}\{-f(x), 0\} = -f(x)$

Now $f^+(x) + f^-(x) = 0 + (-f(x)) = -f(x)$

$= |f(x)|$

$\Rightarrow |f| = f^+ + f^-$

Hence combining all the cases

$$\Rightarrow |f| = f^+ + f^-$$

$\vee \to$ or
$\wedge \to$ and
$\forall \to$ for all

Theorem Let f and g be two real valued measurable functions on some measurable domain D and $c \in \mathbb{R}$, then

(i) $f+c$ (ii) cf (iii) $f+g$
(iv) $f-g$ (v) fg (vi) $f\backslash g$, $g \neq 0$
(vii) $f \vee g$ (viii) $f \wedge g$ (ix) $|f|$

are all measurable functions

Proof) (i) $f+c$ is measurable

Let $\alpha \in \mathbb{R}$, then as f is measurable then $\{x \in D : f(x) > \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$

Now

$$\{x \in D : (f+c)(x) > \alpha\}$$
$$= \{x \in D : f(x) > \alpha - c\}$$

$\Rightarrow \{x \in D : (f+c)(x) > \alpha\} = \{x \in D : f(x) > \alpha - c\}$

As R.H.S is measurable, so L.H.S is measurable

$\Rightarrow f+c$ is measurable

(ii)

cf is measurable

Here arises the following cases

Case I:- If $c = 0$ then $(cf)(x) = c\,f(x)$
$= 0 \cdot f(x) = 0$

$\Rightarrow cf$ is then a constant function and hence is measurable.

Case II:- If $c > 0$, then for any $\alpha \in \mathbb{R}$

$\{x \in D : (cf)(x) > \alpha\} = \{x \in D : c\,f(x) > \alpha\}$

$$= \{x \in D : f(x) > \alpha/c\}$$

$$\Rightarrow \{x \in D : (cf)(x) > \alpha\} = \{x \in D : f(x) > \alpha/c\}$$

As R.H.S is measurable $\Rightarrow$ L.H.S is measurable

$\Rightarrow$ cf is measurable.

<u>Case III</u>

If $c < 0$ then for any $\alpha \in \mathbb{R}$

$\{x \in D : (cf)(x) > \alpha\} = \{x \in D : f(x) < \alpha/c\}$

As R.H.S is measurable ($\because f$ is measurable)

So L.H.S is measurable

$\Rightarrow$ cf is measurable

Hence combining all the cases cf is measurable.

(iii) $f+g$ is measurable.

To prove $f+g$ is measurable while given f and g are measurable functions

Let $\alpha \in \mathbb{R}$, be any real number

Now

$(f+g)(x) > \alpha \Rightarrow f(x) + g(x) > \alpha$ [Point wise addition]

$\Rightarrow f(x) > \alpha - g(x)$

Then by rational density theorem of real analysis, there exist some rational number r such that

$$f(x) > r > \alpha - g(x)$$

$\Rightarrow f(x) > r$ and $r > \alpha - g(x)$

$\Rightarrow f(x) > r$ and $g(x) > \alpha - r$

Now we show that

$$\{x \in D : (f+g)(x) > \alpha\} = \bigcup_{r \in \mathbb{Q}} \left[\{x \in D : f(x) > r\} \cap \{x \in D : g(x) > \alpha - r\}\right]$$

Let $y \in \{x \in D : (f+g)(x) > \alpha\}$

$\Rightarrow (f+g)(y) > \alpha \Rightarrow f(y) + g(y) > \alpha$

$\Rightarrow f(y) > r$ & $g(y) > \alpha - r$

for some $r \in \mathbb{Q}$

$\Rightarrow y \in \{x \in D : f(x) > r\} \cap \{x \in D : g(x) > \alpha - r\}$

$\Rightarrow y \in \bigcup_{r \in \mathbb{Q}} \left[\{x \in D : f(x) > r\} \cap \{x \in D : g(x) > \alpha - r\}\right]$

$\Rightarrow \{x \in D : (f+g)(x) > \alpha\} \subseteq \bigcup_{r \in \mathbb{Q}} \left[\{x \in D : f(x) > r\} \cap \{x \in D : g(x) > \alpha - r\}\right] \longrightarrow (1)$

working backward we get the converse result. Hence

$\{x \in D : (f+g)(x) > \alpha\} = \bigcup_{r \in \mathbb{Q}} \left[\{x \in D : f(x) > r\} \cap \{x \in D : g(x) > \alpha - r\}\right]$

As R.H.S is measurable

($\because$ f and g are measurable and intersection of two measurable sets is measurable)

$\Rightarrow$ L.H.S is measurable

$\Rightarrow f+g$ is measurable.

(iv) $f-g$ is measurable.

As f and g are measurable

$\Rightarrow (-1)g$ is measurable

$\Rightarrow -g$ " "

and sum of two measurable functions is measurable

$\Rightarrow f + (-g)$ is measurable

$\Rightarrow f-g$ is measurable

(V)

f^2 is measurable.

To prove f^2 is measurable

Let $\alpha \in \mathbb{R}$. then

Case I If $\alpha < 0$ then

$\{x \in D : f^2(x) > \alpha\} = D$

and is measurable because D is measurable

$\Rightarrow f^2$ is measurable

$\boxed{x \in D : f(x) \geq \pm\sqrt{\alpha}}$

Case II :- If $\alpha \geq 0$, then

$$\{x \in D : f^2(x) \geq \alpha\} = \{x \in D : f(x) > \sqrt{\alpha}\} \cup \{x \in D : f(x) \not> -\sqrt{\alpha}\}$$

Since f is measurable so $\{x \in D : f(x) > \sqrt{\alpha}\}$ and $\{x \in D : f(x) \not> -\sqrt{\alpha}\}$ is measurable and union of measurable sets is measurable.

$\Rightarrow$ R.H.S is measurable.

So L.H.S is also measurable.

$\Rightarrow f^2$ is measurable

(Vi) fg is measurable.

As $fg = \frac{1}{4}\{(f+g)^2 - (f-g)^2\} \longrightarrow ①$

As f and g are measurable functions

$\Rightarrow f+g$ and $f-g$ are measurable,

$\Rightarrow (f+g)^2$ and $(f-g)^2$ are "

($\because f$ is measurable $\Rightarrow f^2$ is measurable

$\Rightarrow (f+g)^2 - (f-g)^2$ is measurable

($\because f$ & g are measurable

$\Rightarrow f-g$ is "

$\Rightarrow \frac{1}{4}\{(f+g)^2 - (f-g)^2\}$ is measurable
($\because$ if f is measurable
$\Rightarrow cf$ " "

$\Rightarrow$ R.H.S of equ ① " "
So L.H.S " " " "
$\Rightarrow fg$ is measurable.

(vii) f/g is measurable. $g(x) \neq 0$

As given $g(x) \neq 0$
So $\frac{1}{g(x)}$ is defined
Let $\alpha \in R$, Then

$$\{x \in D : \frac{1}{g(x)} > \alpha\} = \begin{cases} \{x \in D : g(x) > 0\}, & \text{if } \alpha = 0 \\ \{x \in D : g(x) > 0\} \cap \{x \in D : g(x) < 1/\alpha\}, & \text{if } \alpha > 0 \\ \{x \in D : g(x) > 0\} \cup \{x \in D : g(x) < 0\} \cap \{x \in D : g(x) < 1/\alpha\}, & \text{if } \alpha < 0 \end{cases}$$

As R.H.S is measurable
($\because g$ is measurable)
So L.H.S is measurable
$\Rightarrow 1/g$ is measurable.

Also f is measurable
$\Rightarrow f \cdot 1/g$ is " " ($\because$ if f & g are measurable
$\Rightarrow fg$ is ")
$\Rightarrow f/g$ is measurable.

viii) $f \vee g$ is measurable.
As $\{x : (f \vee g)(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : g(x) > \alpha\}$

Since R.H.S is measurable so L.H.S is also measurable

$\Rightarrow f \vee g$ is measurable

(ix)

$f \wedge g$ is measurable.

As $\{x : (f \wedge g)(x) > \alpha\} = \{x : f(x) > \alpha\} \cap \{x : g(x) > \alpha\}$

Since R.H.S is measurable So L.H.S is also measurable

$\Rightarrow f \wedge g$ is measurable.

(x) $|f|$ is measurable.

As $|f| = f^+ + f^-$

As $f^+ = f \vee 0$ here $0 = 0(x)$

Since f is measurable, 0 i.e zero function which maps all elements on zero i.e constant function and constant function is measurable

$\Rightarrow 0$ is measurable

$\Rightarrow f \vee 0$ is measurable

$\Rightarrow f^+$ " "

Also $(-1) f$ is measurable

$\Rightarrow -f$ " "

$\Rightarrow -f \vee 0$ is measurable

$\Rightarrow f^-$ is measurable

$\Rightarrow f^+ + f^-$ is measurable

$\Rightarrow |f|$ is measurable.

** ———— **

Remark:- Converse of the above theorem "If f is measurable, then so is $|f|$" is not true in general e.g

Example:- Let E be a non measurable set and χ_E is defined as

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x \in E \\ -\frac{1}{2} & \text{if } x \notin E \end{cases}$$

Now Let $f(x) = \chi_E(x) - 1/2$

then f is not a measurable function because $\{x : f(x) > 0\} = E$ and E is non measurable

$\Rightarrow f$ is non measurable

Now $|f(x)| = 1/2$

Then $|f|$ is measurable

($\because$ Constant function is measurable)

————————*

Theorem Let f be an extended real valued measurable function defined on measurable set D. Let A be a measurable subset of D. Then restriction of f to A is also measurable.

Proof:- Let g be the restriction of f on $A \subseteq D$ then for all $x \in A$

$$g(x) = f(x)$$

Now Let $\alpha \in \mathbb{R}$

$$\{x \in A : g(x) > \alpha\} = \{x \in A : f(x) > \alpha\}$$
$$\subseteq \{x \in D : f(x) > \alpha\}$$

$\Rightarrow \{x \in A : g(x) > \alpha\} \subseteq \{x \in D : f(x) > \alpha\}$

$\Rightarrow \{x \in A : g(x) > \alpha\} = A \cap \{x \in D : f(x) > \alpha\}$

As A is measurable and also f is measurable.

$\Rightarrow$ R.H.S is measurable
$\Rightarrow$ L.H.S " "
$\Rightarrow$ g is measurable.

***

$\Rightarrow$ **Corollary:** Let D and E be the two measurable sets and f is a function with domain $D \cup E$. Then f is measurable iff D and E are measurable.

Proof: Suppose restriction of f to D and E are measurable.

To prove f is measurable with domain $D \cup E$.

Now, $D \cup E$ is measurable being the union of two measurable sets.

Let $\{x \in D \cup E : f(x) > \alpha\} = \{x \in D : f(x) > \alpha\} \cup \{x \in E : f(x) > \alpha\}$

As R.H.S is measurable. So L.H.S is measurable

$\Rightarrow$ f is measurable with domain $D \cup E$

Conversely suppose f is measurable with domain $D \cup E$. To prove restriction of f to D and E are measurable.

As $D \subseteq D \cup E$

Then $\{x \in D : f(x) > \alpha\} \subseteq \{x \in D \cup E : f(x) > \alpha\}$

$\Rightarrow \{x \in D : f(x) > \alpha\} = D \cap \{x \in D \cup E : f(x) > \alpha\}$

As D and $\{x \in D \cup E : f(x) > \alpha\}$ are measurable $\Rightarrow$ R.H.S is measurable
So L.H.S is measurable
$\Rightarrow$ Restriction of f to D is measurable
Similarly, we can prove that restriction of f to E is measurable.

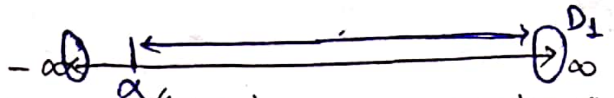
Theorem:- Let f be a function with measurable domain D then f is measurable iff the function $g(x)$ i.e

$$g(x) = \begin{cases} f(x) & , x \in D \\ 0 & , x \notin D \end{cases}$$

is measurable.

Proof:- Suppose f is measurable function
To prove g is measurable function
If $x \in D$ then $g(x) = f(x)$ and $f(x)$ is measurable so $g(x)$ is also measurable.
If $x \notin D$ then $g(x) = 0 \Rightarrow g(x)$ is constant function and constant function is measurable, so $g(x)$ is measurable.
Conversely, Suppose g is measurable.
To prove f is measurable
As g is measurable on $D \cup D'$ and $D \subseteq D \cup D'$
$\Rightarrow g$ is measurable on D because g is restriction to D.
And when $x \in D$ then $f(x) = g(x)$
$\Rightarrow f$ is measurable function.

★



greater than α سے $-\infty$ نہیں آئے گی so D_2 نہیں آئے گا

Theorem:- Let f be an extended real valued function with measurable domain D and let $D_1 = \{x \in D : f(x) = \infty\}$, $D_2 = \{x \in D : f(x) = -\infty\}$ then f is measurable iff D_1 and D_2 are measurable and the restriction of f to $D \setminus (D_1 \cup D_2)$ is measurable function.

Proof: Let us assume f is measurable, then for all $\alpha \in \mathbb{R}$ $\{x \in D : f(x) \geq \alpha\}$ is measurable.

Now $D_1 = \{x \in D : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in D : f(x) > n\}$

and

$$D_2 = \{x \in D : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in D : f(x) < -n\}$$

$\Rightarrow$ D_1 and D_2 are measurable.

Now $D_1 \cup D_2$ is measurable

$\Rightarrow$ $D \setminus (D_1 \cup D_2)$ is measurable

$\Rightarrow$ $D \setminus (D_1 \cup D_2)$ is measurable subset of D then restriction of f to $D \setminus (D_1 \cup D_2)$ is measurable.

Conversely,

Assume D_1 and D_2 are measurable and restriction of f to $D \setminus (D_1 \cup D_2)$ is measurable.

To prove f is measurable on D.

Let $\alpha \in \mathbb{R}$ then

$$\{x \in D : f(x) > \alpha\} = \{x \in D : f(x) = \infty\} \cup \{x \in D \setminus (D_1 \cup D_2) : f(x) > \alpha\}$$ (*)

As R.H.S is measurable

So L.H.S is measurable
$\Rightarrow$ f is measurable function on domain D.

***

Theorem f and g are extended real valued measurable functions and α be any fixed number then $f+g$ is measurable provided we define $f+g$ to be α whenever it is of the form $\infty-\infty$ or $-\infty+\infty$

Proof)

Let us define

$D_1 = \{x \in D : f(x) = \infty\} \cap \{x \in D : g(x) = -\infty\}$

$D_2 = \{x \in D : f(x) = -\infty\} \cap \{x \in D : g(x) = \infty\}$

Then D_1 is measurable being the intersection of two measurable sets. Similarly D_2 is measurable.

Further as $f+g$ is constant function on $D_1 \cup D_2$ with value α. Therefore $f+g$ is measurable on $D_1 \cup D_2$.

Further $f+g$ is also measurable on $D \setminus (D_1 \cup D_2)$ by the fact that $f+g$ is real valued function on $D \setminus (D_1 \cup D_2)$

Now $f+g$ is measurable function on D by the fact that $f+g$ is measurable on $D \setminus (D_1 \cup D_2)$ and $f+g$ is measurable on $D_1 \cup D_2$ and union of two measurable sets is measurable.

**

$(f+g)(x) = f(x) + g(x) = \infty - \infty$ or $-\infty + \infty$ in this case $(f+g)(x) = \alpha$

$D_1 = \infty - \infty$ form $\Rightarrow f+g = \alpha$
$D_2 = -\infty + \infty$ 〃 $\Rightarrow$ 〃 〃 〃

Theorem:- Any extended real valued function define on a set of measure zero is measurable.

Proof: Let f be defined on D, Then by given condition $m^*(D) = 0$

$\Rightarrow D$ is measurable.

Now for any real number α

$\{x \in D : f(x) > \alpha\} \subseteq D$

$\Rightarrow m^*(\{x \in D : f(x) > \alpha\}) \leq m^*(D)$

$\because m^*$ is monotone

$\Rightarrow m^*(\{x \in D : f(x) > \alpha\}) \leq 0$

$\Rightarrow m^*(\{x \in D : f(x) > \alpha\}) = 0$

$\Rightarrow \{x \in D : f(x) > \alpha\}$ is measurable

$\Rightarrow f$ is measurable.

***

⇒ Definitions:-

(i) A property is said to hold almost every where (written as a.e.) if the set of points where it does not hold has measure zero.

(ii) Two functions f and g with some domain D are equal almost every where if measure of the set of points where they are not equal has measure zero.

Obviously $f = g$ a.e $\Rightarrow g = f$ a.e

(iii) A sequence $\{f_n\}$ of functions defined on E is said to converge almost every where to a function f, if the set of points where $\{f_n\}$ does not converge to f has measure zero.

Example 1:-

Define $f: \mathbb{R} \to \{1, 2\}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}' \\ 2 & \text{if } x \in \mathbb{Q} \end{cases}$$

Then $f(x) = 1$ a.e $\because m^*(\mathbb{Q}) = 0$

Example 2:-

Define $g: \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}' \\ x & \text{if } x \in \mathbb{Q} \end{cases}$$

then $g = f$ a.e if $m^*(\mathbb{Q}) = 0$
and $f(x) \neq g(x)$ when $x \in \mathbb{Q}$

* * *

Theorem Let f and g be extended real valued measurable functions which are finite almost every where then $f+g$ is measurable.

Proof: Let $D_1 = \{x \in D : f(x) = \infty\} \cap \{x \in D : g(x) = -\infty\}$

$$D_2 = \{x \in D : g(x) = \infty\} \cap \{x \in D : f(x) = -\infty\}$$

Then as f and g are finite a.e. so $m^*(D_1) = 0$ and $m^*(D_2) = 0$ Then

$$m^*(D_1 \cup D_2) = 0$$

D_1 اور D_2 میں وہ پوائنٹس
ہیں جہاں پر finite image
نہیں ہے اور $\cap$ اس لیے لی ہے
کہ $f+g$ پر ڈیفائن ہو

Now as an extended

E پر دونوں فنکشن برابر نہیں لہذا D\E پر برابر ہونگے

real valued function defined on a set of measure zero is measurable So $f+g$ is measurable on $D_1 \cup D_2$ Then also $f+g$ is measurable on $D \setminus (D_1 \cup D_2)$

$\Rightarrow f+g$ is measurable on D.

Theorem Let f be a measurable function with $f = g$ a.e. then g is also measurable function.

Proof Put $E = \{x \in D : f(x) \neq g(x)\}$

then $m^*(E) = 0$

$\Rightarrow E$ is measurable

$\Rightarrow E' = D \setminus E$ is measurable

Now obviously for any $\alpha \in \mathbb{R}$

$$\{x \in D : g(x) > \alpha\} = \{x \in D \setminus E : g(x) > \alpha\} \cup \{x \in E : g(x) > \alpha\}$$

$$\Rightarrow \{x \in D : g(x) > \alpha\} = \{x \in D \setminus E : f(x) > \alpha\} \cup \{x \in E : g(x) > \alpha\} \longrightarrow (*)$$

Now as f is measurable on D and $D \setminus E = E'$ is measurable subset of D so restriction of f on $D \setminus E = E'$ is measurable

$\Rightarrow \{x \in D \setminus E : f(x) > \alpha\}$ is measurable

Further as a function defined on a set of measure zero is measurable So $\{x \in E : g(x) > \alpha\}$ is measurable

$\Rightarrow$ R.H.S of ⊛ is measurable being the union of two measurable sets

$\Rightarrow \{x \in D : g(x) > \alpha\}$ is measurable

$\Rightarrow g$ is measurable.

***

$\Rightarrow$ Limit Superior & Limit Inferior:-

Let $\{x_i\}$ be a sequence of real numbers and let

$a_1 = \sup\{x_1, x_2, x_3, \ldots\} = \sup\limits_{i \geq 1}\{x_i\}$

$a_2 = \sup\{x_2, x_3, x_4, \ldots\} = \sup\limits_{i \geq 2}\{x_i\}$

$a_3 = \sup\{x_3, x_4, x_5, \ldots\} = \sup\limits_{i \geq 3}\{x_i\}$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

Then $a_1 \geq a_2 \geq a_3 \geq \ldots$

and let

$b_1 = \inf\limits_{i \geq 1}\{x_i\}$

$b_2 = \inf\limits_{i \geq 2}\{x_i\}$

$b_3 = \inf\limits_{i \geq 3}\{x_i\}$ and so on

Then, Limit superior of $\{x_i\}$ is denoted & defined by

$$\overline{\lim}\{x_i\} = \inf_K a_K = \inf_K \sup_{i \geq K}\{x_i\}$$

and limit Inferior of $\{x_i\}$ is denoted and defined by

$$\underline{\lim}\{x_i\} = \sup_K b_K = \sup_K \inf_{i \geq K}\{x_i\}$$

* Remark :- Limit of a sequence $\{x_i\}$ exists if $\overline{\lim}\{x_i\} = \underline{\lim}\{x_i\}$ Then we write it $\lim\{x_i\}$

———— *** ————

Theorem :- Let $\{f_n\}$ be a sequence of extended real valued measurable functions with same domain D then

(i) $\overset{n}{\underset{i=1}{\text{Max}}}\, f_i$ is measurable, for each n

(ii) $\overset{n}{\underset{i=1}{\text{Min}}}\, f_i$ is measurable, for each n

(iii) $\underset{n\in N}{\text{Inf}}\, f_n$ is measurable.

(iv) $\underset{n\in N}{\text{Sup}}\, f_n$ is measurable.

(v) $\overline{\text{Lim}}\, f_n$ is measurable.

(vi) $\underline{\text{Lim}}\, f_n$ " " .

(vii) If $\lim_{n\to\infty} f_n(x) = f(x)$ then f is measurable.

Proof :- (i) $\overset{n}{\underset{i=1}{\text{Max}}}\, f_i$ is measurable, for each n

Put $h = \overset{n}{\underset{i=1}{\text{Max}}}\, f_i$, then for any $\alpha \in \mathbb{R}$ we prove

$$\{x\in D : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x \in D : f_i(x) > \alpha\}$$

Let $y \in \{x \in D : h(x) > \alpha\} \Rightarrow h(y) > \alpha$

Now $h(x) = \overset{n}{\underset{i=1}{\text{Max}}}\, f_i(x)$

Therefore, Then there exist a j such that

$h(x) = f_j(x) \quad \Rightarrow h(y) = f_j(y)$

Now $\quad h(y) > \alpha \quad \Rightarrow f_j(y) > \alpha$

$\Rightarrow y \in \{x : f_i(x) > \alpha\}$ for some i

$\Rightarrow y \in \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$

$\Rightarrow \{x \in D : h(x) > \alpha\} \subseteq \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\} \longrightarrow ①$

Let $y \in \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$ for some i

$\Rightarrow f_i(y) > \alpha$

Now as $h(x) = \max_{i=1}^{n} f_i(x)$ for each i

$\Rightarrow h(x) \geq f_i(x)$

$\Rightarrow h(y) \geq f_i(y) > \alpha$

$\Rightarrow h(y) > \alpha$

$\Rightarrow y \in \{x \in D : h(x) > \alpha\}$

$\Rightarrow \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\} \subseteq \{x : h(x) > \alpha\} \longrightarrow ②$

from ① and ②

$\{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$

Now as f_i is measurable and union of finite measurable sets is measurable. So R.H.S is measurable. Hence L.H.S is measurable

$\Rightarrow h$ is measurable

$\Rightarrow \max_{i=1}^{n} f_i$ is measurable.

(ii) $\underset{i=1}{\overset{n}{\text{Min}}} f_i$ is measurable, for each n

Put $g = \underset{i=1}{\overset{n}{\text{Min}}} f_i$. Then we prove that

$$\{x \in D : g(x) > \alpha\} = \bigcap_{i=1}^{n} \{x \in D : f_i(x) > \alpha\}$$

Let $y \in \{x : g(x) > \alpha\} \Rightarrow g(y) > \alpha$

Now as

$g(x) = \underset{i=1}{\overset{n}{\text{Min}}} f_i(x)$ then there exist at least one i such that

$$g(x) = f_i(x) \quad \& \quad g(x) \le f_i(x) \quad \forall i$$

$$\Rightarrow g(y) \le f_i(y) \quad \forall i$$

$$\Rightarrow f_i(y) \ge g(y) > \alpha \quad \forall i$$

$$\Rightarrow f_i(y) > \alpha \quad \forall i$$

$$\Rightarrow y \in \{x : f_i(x) > \alpha\} \; \forall i \Rightarrow y \in \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\}$$

$$\Rightarrow \{x : g(x) > \alpha\} \subseteq \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\} \longrightarrow (1)$$

Let $y \in \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\}$

$$\Rightarrow y \in \{x : f_i(x) > \alpha\} \quad \forall i$$

$\Rightarrow f_i(y) > \alpha$ for each i

As $g(x) = \underset{i=1}{\overset{n}{\text{Min}}} f_i(x)$

then there exist some j such that

$$g(x) = f_j(x)$$

$$\Rightarrow g(y) = f_j(y) > \alpha \Rightarrow g(y) > \alpha$$

$\Rightarrow y \in \{x : g(x) > \alpha\}$

$\Rightarrow \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\} \subseteq \{x : g(x) > \alpha\} \longrightarrow ②$

from equ ① & ②

$\{x : g(x) > \alpha\} = \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\}$

As R.H.S is measurable

Therefore L.H.S is also measurable

$\Rightarrow g$ is measurable

$\Rightarrow \underset{i=1}{\overset{n}{\text{Min}}}\, f_i$ is measurable.

(iii) $\underset{n \in N}{\text{Inf}}\, f_n$ is measurable.

Put $g = \underset{n \in N}{\text{Inf}}\, f_n$

Now we prove that

$\{x : g(x) > \alpha\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$

Let $y \in \{x : g(x) > \alpha\} \Rightarrow g(y) > \alpha$

Now as $g = \underset{n \in N}{\text{Inf}}\, f_n$

$\Rightarrow g(y) \leq f_n(y) \quad \forall n$

$\Rightarrow f_n(y) \geq g(y) > \alpha$

$\Rightarrow f_n(y) > \alpha$

$\Rightarrow y \in \{x : f_n(x) > \alpha\}, \forall n$

$\Rightarrow y \in \bigcap_{i=1}^{n} \{x : f_n(x) > \alpha\}$

$\Rightarrow \{x : g(x) > \alpha\} \subseteq \bigcap_{i=1}^{n} \{x : f_n(x) > \alpha\} \longrightarrow (1)$

Now Let $y \in \bigcap_{i=1}^{n} \{x : f_n(x) > \alpha\}$

$\Rightarrow y \in \{x : f_n(x) > \alpha\}, \ \forall n$

$\Rightarrow f_n(y) > \alpha$ for each n

As $g(x) = \inf_{n \in N} f_n$ ~~Then there exist some i~~

~~such that~~

$\Rightarrow g(x) \leq f_n(x) \ \forall n \Rightarrow g(y) \leq f_n(y) \ \forall n \in N$

Now as $f_n(y) \geq g(y)$ & $f_n(y) > \alpha$ then $g(y) > \alpha$

$\Rightarrow y \in \{x : g(x) > \alpha\}$

$\Rightarrow \bigcap_{i=1}^{n} \{x : f_n(x) > \alpha\} \subseteq \{x : g(x) > \alpha\} \longrightarrow (2)$

from (1) & (2)

$\{x : g(x) > \alpha\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$

as each $f_n(x)$ is measurable & countable intersection of measurable sets is measurable, So R.H.S is measurable.

Therefore L.H.S is measurable

$\Rightarrow g(x)$ is measurable

$\Rightarrow \inf_{n \in N} f_n$ is measurable.

(iv) $\sup_{n \in N} f_n$ is measurable.

Put $g = \sup_{n \in N} f_n$

Now we prove that

$$\{x: g(x) > \alpha\} = \bigcup_{i=1}^{\infty} \{x: f_n(x) > \alpha\}$$

Let $y \in \{x: g(x) > \alpha\} \Rightarrow g(y) > \alpha$

Now as $g(x) = \sup_{n \in N} f_n(x)$

$\Rightarrow g(x) \geq f_n(x) \quad \forall n \in N$

or $g(y) \geq f_n(y) \quad \forall n \in N$

Now as $g(y) \geq f_n(y)$ & $g(y) > \alpha$

Then there exist $m \in N$, such that

$$f_m(y) > \alpha$$

Then $y \in \bigcup_{n=1}^{\infty} \{x: f_n(x) > \alpha\}$

$$\Rightarrow \{x: g(x) > \alpha\} \subseteq \bigcup_{n=1}^{\infty} \{x: f_n(x) > \alpha\} \longrightarrow (1)$$

Now let

$$y \in \bigcup_{n=1}^{\infty} \{x: f_n(x) > \alpha\}$$

$\Rightarrow y \in \{x: f_n(x) > \alpha\}$ for some n

$\Rightarrow f_n(y) > \alpha$

Now as $g = \sup_{n \in N} f_n$

$\Rightarrow g(y) \geq f_n(y) > \alpha$

$\Rightarrow g(y) > \alpha \Rightarrow y \in \{x: g(x) > \alpha\}$

$$\Rightarrow \bigcup_{n=1}^{\infty} \{x: f_n(x) > \alpha\} \subseteq \{x: g(x) > \alpha\} \longrightarrow (2)$$

from (1) & (2)

$\{x : g(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$

As R.H.S is measurable
$\Rightarrow$ L.H.S is " $\Rightarrow h$ is measurable
$\Rightarrow \sup_{n \in N} f_n$ is measurable

(V) $\overline{\text{Lim}}\, f_n$ is measurable.

By definition of limit superior

$\overline{\lim} f_n = \text{Inf}_n (\sup_{i \geq n} f_i)$

Suppose $F_n = \sup_{i \geq n} f_i = \sup\{f_n, f_{n+1}, f_{n+2}, \ldots\}$

Since each f_n is measurable and we know that if $\{f_n\}$ is measurable then $\sup f_n$ is measurable, then we get $\{F_n\}$ a sequence of measurable function. Now as $\{F_n\}$ is a sequence of measurable functions so $\text{Inf}\{F_n\}$ is measurable $\Rightarrow \text{Inf} \sup_{i \geq n} f_i$ is measurable

$\Rightarrow \overline{\lim} f_n$ is measurable

(Vi) $\underline{\text{Lim}}\, f_n$ is measurable.

By definition of $\underline{\lim} f_n = \sup_n (\text{Inf}_{i \geq n} f_i)$

Let $F_n = \text{Inf}\{f_n, f_{n+1}, f_{n+2}, \ldots\}$
Then $\{F_n\}$ is a sequence of measurable functions, so $\sup_n \{F_n\}$ is measurable

$\Rightarrow \sup_n (\text{Inf}_{i \geq n} f_i) = \underline{\lim} f_n$ is measurable

(vii) If $\lim_{n\to\infty} f_n = f(x)$ then f is measurable

Given $\lim_{n\to\infty} f_n(x) = f(x)$ exists,
Then to prove f is measurable

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{so,}$$

$$\overline{\lim} f_n = \underline{\lim} f_n = f$$

$\Rightarrow$ f is measurable because $\overline{\lim} f_n$ and $\underline{\lim} f_n$ both are measurable.

Theorem:- Let f be a measurable function and G be an open set then $\{x : f(x) \in G\}$ is measurable.

Proof: Given that G is an open set. Then G can be expressed as a countable union of pairwise disjoint open intervals because every non-empty open set in $\mathbb{R}$ is the union of a countable collection of disjoint open intervals.
Therefore

$$G = \bigcup_{n=1}^{\infty} I_n, \text{ where } I_n =]a_n, b_n[$$

are pairwise disjoint open intervals. Now as each open interval is measurable and countable union of measurable sets is measurable.

$\Rightarrow$ G is measurable being the countable union of measurable sets.
Now as

$$\{x : f(x) \in G\} = \bigcup_{n=1}^{\infty} \left[\{x : f(x) > a_n\} \cap \{x : f(x) < b_n\} \right]$$

As R.H.S is measurable

$\Rightarrow \{x : f(x) \in G\}$ is measurable

* ——— ** ——— *

Theorem. Let f and g be two measurable functions defined on some domain D. Then the followings sets are measurable

(i) $\{x : f(x) < g(x)\}$ (ii) $\{x : f(x) > g(x)\}$

(iii) $\{x : f(x) \leq g(x)\}$ (iv) $\{x : f(x) \geq g(x)\}$

(v) $\{x : f(x) = g(x)\}$

Proof)

(i) Let $f(x) < g(x)$

As we know that between every two real numbers there exist a rational number, therefore $f(x) < r < g(x)$

As $\{x : f(x) < g(x)\} = \bigcup_{r \in Q} [\{x : f(x) < r\} \cap \{x : g(x) > r\}]$

As R.H.S is measurable.

Therefore L.H.S is measurable

$\Rightarrow \{x : f(x) < g(x)\}$ is measurable

(ii) Let $f(x) > g(x)$

Then by rational density theorem there exist a rational number "r" such that

$f(x) > r > g(x)$

As $\{x : f(x) > g(x)\} = \bigcup_{r \in Q} [\{x : f(x) > r\} \cap \{x : g(x) < r\}]$

As R.H.S is measurable

$\Rightarrow$ L.H.S " "

$\Rightarrow \{x : f(x) > g(x)\}$ is measurable.

(iii) $\{x : f(x) \leq g(x)\}$

As $\{x : f(x) \leq g(x)\} = D \setminus \{x : f(x) > g(x)\}$

As D and $\{x : f(x) > g(x)\}$ are measurable & the difference of two measurable sets is measurable.

Therefore, R.H.S is measurable

$\Rightarrow$ L.H.S is also measurable

$\Rightarrow \{x : f(x) \leq g(x)\}$ is measurable.

(iv) $\{x : f(x) \geq g(x)\}$

As $\{x : f(x) \geq g(x)\} = D \setminus \{x : f(x) < g(x)\}$

As D and $\{x : f(x) < g(x)\}$ are measurable

$\Rightarrow$ R.H.S is measurable

Therefore L.H.S " "

$\Rightarrow \{x : f(x) \geq g(x)\}$ is measurable

(v) $\{x : f(x) = g(x)\}$

As $\{x : f(x) = g(x)\} = \{x : f(x) \leq g(x)\} \cap \{x : f(x) \geq g(x)\}$

As R.H.S is measurable

So L.H.S is also measurable

$\Rightarrow \{x : f(x) = g(x)\}$ is measurable.

* * *

Theorem:- Let f be any real valued function defined on a measurable domain D and G is an open set in $\mathbb{R}$, then f is measurable iff $f^{-1}(G)$ is measurable.

Proof: Suppose f is measurable.
To prove $f^{-1}(G)$ is measurable
As G is an open set so there exist a sequence $\{I_n\}$ of pairwise disjoint open intervals such that $G = \bigcup_{n=1}^{\infty} I_n$

where say $I_n =]a_n, b_n[=]-\infty, b_n[\cap]a_n, \infty[$

Then $f^{-1}(G) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n)$

$$= \bigcup_{n=1}^{\infty} f^{-1}(]-\infty, b_n[\cap]a_n, \infty[)$$

$$= \bigcup_{n=1}^{\infty} [f^{-1}(]-\infty, b_n[) \cap f^{-1}(]a_n, \infty[)]$$

$$= \bigcup_{n=1}^{\infty} [\{x : f(x) < b_n\} \cap \{x : f(x) > a_n\}]$$

Since R.H.S is measurable so is L.H.S $\Rightarrow f^{-1}(G)$ is measurable.

Conversely, assume $f^{-1}(G)$ is measurable
To prove f is measurable.
As G is an open set so in particular if $G =]\alpha, \infty[$ then $f^{-1}(G) = f^{-1}(]\alpha, \infty[)$

$$= \{x : f(x) > \alpha\}$$

As given $f^{-1}(G)$ is measurable

$\Rightarrow \{x : f(x) > \alpha\}$ is measurable

$\Rightarrow f$ is measurable

** ——————— **

Remark:- Above theorem is valid if f is an extended real valued function.

Proof)

Suppose for any open set G in $\overline{\mathbb{R}}$ $f^{-1}(G)$ is measurable.

To prove f is measurable.

As G is any open set so in particular

Let $G =]\alpha, \infty[$ then

$$f^{-1}(G) = f^{-1}(]\alpha, \infty[)$$

$$= \{x : f(x) > \alpha\}$$

$\Rightarrow \{x : f(x) > \alpha\}$ is measurable

So f " "

Conversely assume f is measurable

To prove $f^{-1}(G)$ is measurable.

For any open set G in $\overline{\mathbb{R}}$

As f is measurable, so for any α

$$\{x : f(x) > \alpha\} = f^{-1}(]\alpha, \infty])$$

$$\{x : f(x) \geq \alpha\} = f^{-1}([\alpha, \infty])$$

$$\{x : f(x) < \alpha\} = f^{-1}([-\infty, \alpha[)$$

$$\{x : f(x) \leq \alpha\} = f^{-1}([-\infty, \alpha])$$

are all measurable and also

$$f^{-1}(]\alpha,\beta[) = f^{-1}(]\alpha,\infty[) \cap f^{-1}(]-\infty,\beta[)$$

is measurable.

So then if we define

$$\Omega = \{E \subseteq \bar{\mathbb{R}} : f^{-1}(E) \text{ is measurable}\}$$

Then Ω is a σ-algebra and if G is any open set in $\mathbb{R}$ then G is countable union of pairwise disjoint open intervals of the form $]\alpha,\beta[$, $]\alpha,\infty]$, $[-\infty,\alpha[\in \Omega$ etc.

Then by above argument $G \in \Omega$

$\Rightarrow f^{-1}(G)$ is measurable.

* ——— ** ——— *

Theorem: If F is closed in $\bar{\mathbb{R}}$ then $f^{-1}(F)$ is measurable iff f is measurable.

Proof: Assume f is measurable

To prove $f^{-1}(F)$ is measurable

As F is closed so $F' = G$ is open

$\Rightarrow f^{-1}(G)$ is measurable

$\Rightarrow (f^{-1}(G))'$ " "

$\Rightarrow f^{-1}(G')$ " "

$\Rightarrow f^{-1}(F)$ is measurable

Conversely assume $f^{-1}(F)$ is measurable

To prove f is measurable

As $f^{-1}(F)$ is measurable

$\Rightarrow (f^{-1}(F))'$ is measurable

$\Rightarrow f^{-1}(F')$ " "

$\Rightarrow f^{-1}(G)$ " "

where G is an open set.

Then by previous theorem f is measurable.

*———————***

⇒ Simple Function:-

A real valued function s ($s: X \to \mathbb{R}$) defined on a non-empty set X is said to be simple function if the set of all images $s(X)$ is finite. e.g.

(i) The characteristic function

(ii) Constant function.

⇒ Characteristic Function:-

Let E be a non-empty subset of a set X. Then the characteristic function of the set E is denoted & defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

*———**———*

Theorem:- Show that

(i) $\chi_{A\cap B} = \chi_A \cdot \chi_B$

(ii) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$

(iii) $\chi_{A\cup B} = \chi_A + \chi_B$, provided $A\cap B = \phi$

(iv) $\chi_{A'} = 1 - \chi_A$

Proof: (i) $\chi_{A\cap B} = \chi_A \cdot \chi_B$

Case I:- If $x \in A\cap B$

$\Rightarrow x \in A$ & $x \in B$

$\chi_{A\cap B}(x) = 1$

$\chi_A(x) = 1$ and $\chi_B(x) = 1$

$\Rightarrow \chi_A \cdot \chi_B = 1 \cdot 1 = 1 = \chi_{A\cap B}$

$\Rightarrow \chi_{A\cap B} = \chi_A \cdot \chi_B$

Case II:- If $x \notin A\cap B$

$\Rightarrow x \notin A$ or $x \notin B$

$\chi_{A\cap B}(x) = 0$ and

$\chi_A(x) = 0$ or $\chi_B(x) = 0$

$\Rightarrow \chi_A(x) \cdot \chi_B(x) = 0$

$\Rightarrow \chi_{A\cap B} = \chi_A \cdot \chi_B$

(ii) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$

Let $x \in A\cup B \Rightarrow \chi_{A\cup B}(x) = 1$

Now as $x \in A\cup B$ Then there are

Three cases

Case I:- If $x \in A$ & $x \notin B \Rightarrow x \notin A \cap B$

$\Rightarrow \chi_A(x) = 1$, $\chi_B(x) = 0$

and $\chi_{A \cap B}(x) = 0$

$\Rightarrow \chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x)$

Case II:-

If $x \notin A$ & $x \in B \Rightarrow x \notin A \cap B$

$\Rightarrow \chi_A(x) = 0$, $\chi_B(x) = 1$ & $\chi_{A \cap B}(x) = 0$

$\Rightarrow \chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$

Case III If $x \in A$ & $x \in B$

$\Rightarrow x \in A \cap B$

$\Rightarrow \chi_A(x) = 1$, $\chi_B(x) = 1$ and $\chi_{A \cap B}(x) = 1$

Now $\chi_{A \cup B}(x) = 1 \longrightarrow (1)$

$\chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x) = 1 + 1 - 1 = 1 \rightarrow (2)$

from (1) & (2)

$\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x)$

Hence for all the cases

$\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$

(iii) $\chi_{A \cup B} = \chi_A + \chi_B$, Provided $A \cap B = \phi$

As $A \cap B = \phi \Rightarrow \chi_{A \cap B} = 0$

Also we know that

$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$$

$$= \chi_A + \chi_B - 0$$

$$\Rightarrow \chi_{A\cup B} = \chi_A + \chi_B$$

(iv) $\chi_{A'} = 1 - \chi_A$

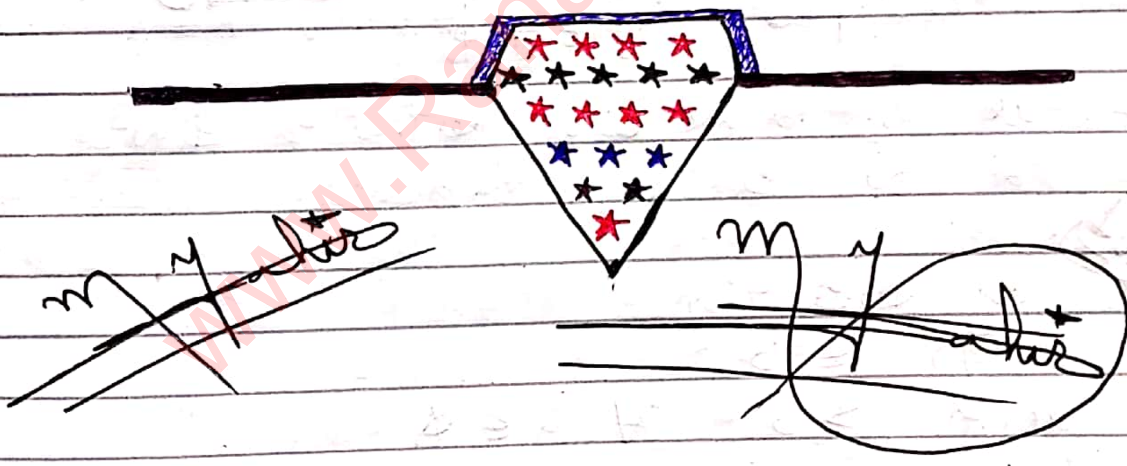
If $x \in A' \Rightarrow \chi_{A'}(x) = 1 \longrightarrow ①$

As $x \in A' \Rightarrow x \notin A \Rightarrow \chi_A(x) = 0$

$\Rightarrow 1 - \chi_A(x) = 1 - 0 = 1 \longrightarrow ②$

from equ ① & ② we have

$$\chi_{A'} = 1 - \chi_A$$



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ASSIGNMENT

Exercise- If $E, F \in \mathcal{A}$ s.t $F \supseteq E$ and $\mu(E)$ is finite then $\mu(F-E) = \mu(F) - \mu(E)$

Proof: Let $E, F \in \mathcal{A}$ s.t $E \subseteq F$. Then

$$F = E \cup (F \setminus E)$$

$$\Rightarrow \mu(F) = \mu(E) + \mu(F \setminus E) \quad (\because E \text{ \& } F\setminus E \text{ are disjoint})$$

$$\Rightarrow \mu(F) - \mu(E) = \mu(F \setminus E) \quad (\because \mu(E) < \infty)$$

or $\mu(F-E) = \mu(F) - \mu(E)$

* ——— ** ——— *

Exercise- Let $\mathcal{A}$ be a σ Algebra of all subsets of a set X. Define

$$\mu(E) = \begin{cases} 0 & \text{if } E = \phi \\ \infty & \text{if } E \neq \phi \end{cases} \longrightarrow (1)$$

then μ is a measure neither a finite not a σ finite measure.

Proof: By Definition.

(i) $\mu(\phi) = 0$

(ii) $\mu(A) \geq 0 \quad \forall A \in \mathcal{A}$

(iii) Let $\{A_n\}$ be a sequence of pairwise disjoint subsets of X in $\mathcal{A}$

If $\bigcup_{n=1}^{\infty} A_n = \phi$ then $A_n = \phi$ and

$$\mu(A_n) = 0 \quad \forall n \in \mathbb{N}$$

$$0 = \mu(\phi) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$= \sum_{n=1}^{\infty} \mu(A_n) = 0$$

$\Rightarrow$ μ is measure on $\mathcal{A}$

μ is not finite as $\mu(A) = \infty \ \forall A \neq \phi \in \mathcal{A}$

To show μ is not σ finite

since for any sequence $\{A_n\}$ of subsets of X is

$$\phi \neq X = \bigcup_{n=1}^{\infty} A_n$$

$$\Rightarrow A_{n_0} \neq \phi \text{ for some } n_0 \in \mathbb{N}$$

$$\Rightarrow \mu(A_{n_0}) = \infty \quad \text{as } A_{n_0} \neq \phi \text{ by } (1)$$

which shows μ is not σ finite.

* __________ *

Exercise- Let $X \neq \phi$ be a set and ν is defined by

$$\nu(E) = \begin{cases} 0 & \text{if } p \notin E \\ 1 & \text{if } p \in E \end{cases}$$

where p is fixed element of X. then ν is a finite measure.

Proof

By Definition of ν

(i) $\nu(\phi) = 0$ as $p \notin \phi$

(ii) $\nu(E) \geq 0 \ \forall E \in \mathcal{A}$ (as it is 0 or 1)

Now let $\{E_n\}$ be a sequence of pair-wise disjoint subsets of X in $\mathcal{A}$.

Consider the set $\bigcup_{n=1}^{\infty} E_n$,

If $p \notin \bigcup_{n=1}^{\infty} E_n$ then $\nu(\bigcup_{n=1}^{\infty} E_n) = 0$

Also $p \notin \bigcup_{n=1}^{\infty} E_n \Rightarrow p \notin E_n \ \forall n \in \mathbb{N}$

$$\Rightarrow \nu(E_n) = 0 \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \nu\left(\bigcup_{n=1}^{\infty} E_n\right) = 0 = \sum_{n=1}^{\infty} \nu(E_n)$$

If $p \in \bigcup_{n=1}^{\infty} E_n$ then

$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = 1$$

Also $p \in \bigcup_{n=1}^{\infty} E_n \Rightarrow \exists$ atleast one say $n_0 \in \mathbb{N}$ s.t $p \in E_{n_0}$

$$\Rightarrow \nu(E_{n_0}) = 1$$

then $1 = \nu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \nu(E_n)$

$\Rightarrow \nu$ is a measure (outer) in $\mathcal{A}$

If $p \in \bigcup_{n=1}^{\infty} E_n$, then since the sequence $\{E_n\}$ is pairwise disjoint, therefore $p \in E_n$ for exactly n say $n = n_0$

then $\nu(E_{n_0}) = 1$ & $\nu(E_n) = 0 \quad \forall \; n \neq n_0$

$$\Rightarrow 1 = \nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n) = 1$$

$\Rightarrow \nu$ is a measure on the σ algebra $\mathcal{A}$ on X.

Since $\nu(X) = 1 < \infty$

$\Rightarrow \nu$ is finite measure.

***

Exercise:- Let X be a set with at least two elements and let $x_0 \in X$.

Define $\mu(\phi) = 0$ and

$$\mu(A) = \begin{cases} 1 & \text{if } x_0 \notin A \\ 2 & \text{if } x_0 \in A \end{cases}$$

As A is non empty subset of X.

Prove that μ is an outer measure on X.

Proof:- By definition of μ

(i) $\mu(\phi) = 0$

Also (ii) $\mu(A) \geq 0 \quad \forall \ A \in 2^X$

(iii) Let $\{A_n\}$ be a sequence of subsets of X. if $x_0 \in \bigcup_{n=1}^{\infty} A_n$ then either

$\bigcup_{n=1}^{\infty} A_n = \phi$ or $x_0 \notin A_n \quad \forall \ n \in \mathbb{N}$

then $\mu(\bigcup_{n=1}^{\infty} A_n) = \mu(\phi) = 0$

Also $A_n = \phi \quad \forall \ n \in \mathbb{N}$

$\Rightarrow \mu(A_n) = 0 \quad \forall \ n \in \mathbb{N}$

$\Rightarrow 0 = \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$

Now if $\bigcup_{n=1}^{\infty} A_n \neq \phi$ and $x_0 \notin \bigcup_{n=1}^{\infty} A_n$

then, $\mu(\bigcup_{n=1}^{\infty} A_n) = 1$

Also $x_0 \notin \bigcup_{n=1}^{\infty} A_n \Rightarrow x_0 \notin A_n \quad \forall \ n \in \mathbb{N}$

$\mu(A_n) = 1$

Since some of A_n's are non-empty.

as $(\bigcup_{n=1}^{\infty} A_n \neq \phi)$

$\Rightarrow 1 = \mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n) = 1$

Now if $x_0 \in \bigcup_{n=1}^{\infty} A_n$ then $\exists$ at least one A_n say A_{n_0} s.t $x_0 \in A_{n_0}$

So $\mu(\bigcup_{n=1}^{\infty} A_n) = 2$.

Also $\mu(A_{n_0}) = 2$

$\Rightarrow 2 = \mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} A_n$

Hence μ is an outer measure on X.

Exercise:- Sum and Product of two outer measures is an outer measure. Whereas difference of two outer measures is not an outer measure.

Proof: Let μ and ν be two outer measures on a set X. we want to show that $\mu+\nu$ is a measure on X i-e

(i) $(\mu+\nu)\phi = \mu(\phi)+\nu(\phi)$

$(\mu+\nu)\phi = 0+0$ $\quad$ $\mu(\phi)$ & $\nu(\phi)$ are outer measure. So

$= 0$ $\quad$ $\mu(\phi)=0, \nu(\phi)=0$

(ii) $(\mu+\nu)(A) = \mu(A)+\nu(A)$

≥ 0 $\quad$ $\forall\ A\subseteq X \Rightarrow \mu(A)\geq 0$

(iii) Let $\{A_n\}$ be a sequence of subsets of X then

$$(\mu+\nu)\left(\bigcup_{n=1}^{\infty}A_n\right) = \mu\left(\bigcup_{n=1}^{\infty}A_n\right)+\nu\left(\bigcup_{n=1}^{\infty}A_n\right)$$

$$\leq \sum_{n=1}^{\infty}\mu(A_n)+\sum_{n=1}^{\infty}\nu(A_n)$$

$$= \sum_{n=1}^{\infty}\left(\mu(A_n)+\nu(A_n)\right)$$

$$= \sum_{n=1}^{\infty}(\mu+\nu)(A_n)$$

$\Rightarrow \mu+\nu$ is an outer measure on X,

Now let μ be a measure on a set X. we want to show that $a\mu$,

$(a>0)$ is a measure on X.

(i) $(a\mu)\phi = a\mu(\phi) = a\cdot 0$

$= 0$ $\because \mu$ is measure so $\mu(\phi)=0$

(ii) $(a\mu)(A) = a\mu(A) \quad \forall A \subseteq X$ $\because \mu(A) \geq 0$

≥ 0

(iii) Let $\{A_n\}$ be a sequence of subset of X then

$$(a\mu)\left(\bigcup_{n=1}^{\infty} A_n\right) = a\mu\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$\leq a\sum_{n=1}^{\infty} \mu(A_n)$$

$$= \sum_{n=1}^{\infty} (a\mu)A_n$$

$$\Rightarrow a\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} (a\mu)A_n$$

$\Rightarrow a\mu$ is a outer measure on X.

Now for difference.

Let μ & ν be two measures on a set X.

Since $(\mu-\nu)(A) = \mu(A) - \nu(A)$

which may be negative

So $\mu-\nu$ is not a outer measure on X.