Exercise 6.1

$\frac{d}{dx}(x^3) = 3x^2 \Rightarrow \int 3x^2 dx = x^3$

$\frac{d}{dx}(x^3+1) = 3x^2 \Rightarrow \int 3x^2 dx = x^3+1$

$\frac{d}{dx}(x^3+c) = 3x^2 \Rightarrow \int 3x^2 dx = x^3+c.$

In $\int f(x)\,dx = F(x) + C$,

- $\int$ is the integral sign (elongated S) which is used to represent the process of integration.
- $f(x)$ is the integrand; a function which is to be integrated or under the effect of integral sign.
- dx, x is the variable of integration that tells integrand is to be integrated w.r.t x.
- C is the integral constant or constant of integration.
- $F(x) + C$ represents family of integrals or antiderivatives or primitives whose derivatives are $f(x)$.

1- $\int dx = x + C$

2- $\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$

3- $\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$

4- $\int [f(x)]^n \cdot f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + C \qquad (n \neq -1)$

5- $\int [f(x)]^{-1} \cdot f'(x)\,dx = \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C.$

6- $\int e^{ax}[a f(x) + f'(x)]\,dx = e^{ax} f(x) + C.$

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(i) $\int 9x^5\,dx$

$$= 9\int x^5\,dx$$

$$= 9\,\frac{x^{5+1}}{5+1} + C$$

$$= \cancel{9}^{3}\cdot\frac{x^6}{\cancel{6}_{2}} + C = \frac{3}{2}x^6 + C.$$

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C}$$

(ii) $\int \frac{15}{x^3}\,dx$

$$= 15\int \frac{1}{x^3}\,dx = 15\int x^{-3}\,dx$$

$$= 15\cdot\frac{x^{-3+1}}{-3+1} + C = 15\cdot\frac{x^{-2}}{-2} + C = -\frac{15}{2x^2} + C.$$

(iii) $\int \frac{a}{\sqrt{bx}}\,dx$

$$= \int \frac{a}{\sqrt{b}\sqrt{x}}\,dx = \frac{a}{\sqrt{b}}\int \frac{1}{x^{1/2}}\,dx$$

$$= \frac{a}{\sqrt{b}}\int x^{-1/2}\,dx$$

$$= \frac{a}{\sqrt{b}}\cdot\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \frac{a}{\sqrt{b}}\cdot\frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + C$$

$$= \frac{2a}{\sqrt{b}}\sqrt{x} + C.$$

(iv) $\int b\,y^{\frac{2}{3}}\,dy$

$$= b\int y^{\frac{2}{3}}\,dy$$

$$= b\,\frac{y^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C = b\cdot\frac{y^{\frac{5}{3}}}{\left(\frac{5}{3}\right)} + C$$

$$= \frac{3b}{5}\,y^{\frac{5}{3}} + C.$$

(v) $\int (3x^2 - 9x + 5)\,dx$

$$= \int 3x^2\,dx - \int 9x\,dx + \int 5\,dx$$

$$= 3\int x^2\,dx - 9\int x\,dx + 5\int dx$$

$$= \cancel{3}\cdot\frac{x^3}{\cancel{3}} - 9\,\frac{x^2}{2} + 5\,x + c$$

$$= x^3 - \frac{9}{2}x^2 + 5x + C.$$

(vi) $\int (2x^{-5} + 3x^{-2})\,dx$

$$= \int 2x^{-5}\,dx + \int 3x^{-2}\,dx$$

$$= 2\int x^{-5}\,dx + 3\int x^{-2}\,dx$$

$$= 2\cdot\frac{x^{-5+1}}{-5+1} + 3\,\frac{x^{-2+1}}{-2+1} + c = \cancel{2}\cdot\frac{x^{-4}}{\cancel{-4}\,2} + 3\cdot\frac{x^{-1}}{-1} + c$$

$$= -\frac{1}{2}\cdot\frac{1}{x^4} + 3\left(-\frac{1}{x}\right) + C = -\frac{1}{2x^4} - \frac{3}{x} + C.$$

(vii) $\int \left(\frac{x^5 + 3x^3 - 5x + 6}{x^4}\right) dx$

$$= \int \left(\frac{x^5}{x^4} + \frac{3x^3}{x^4} - \frac{5x}{x^4} + \frac{6}{x^4}\right) dx = \int \left(x + \frac{3}{x} - \frac{5}{x^3} + \frac{6}{x^4}\right) dx$$

$$= \int x\,dx + \int \frac{3}{x}\,dx - \int \frac{5}{x^3}\,dx + \int \frac{6}{x^4}\,dx$$

$$= \int x\,dx + 3\int \frac{1}{x}\,dx - 5\int \frac{1}{x^3}\,dx + 6\int \frac{1}{x^4}\,dx$$

$$= \int x\,dx + 3\int \frac{1}{x}\,dx - 5\int x^{-3}\,dx + 6\int x^{-4}\,dx$$

$$= \frac{x^2}{2} + 3\ln|x| - 5\,\frac{x^{-2}}{-2} + \overset{2}{\cancel{6}}\,\frac{x^{-3}}{\cancel{-3}} + C$$

$$= \frac{x^2}{2} + 3\ln|x| + \frac{5}{2x^2} + \frac{2}{x^3} + C.$$

(viii) $\int (\cos x + 3\sin x)\,dx$

$= \int \cos x\,dx + \int 3\sin x\,dx$

$= \sin x + 3(-\cos x) + C$

$= \sin x - 3\cos x + C$

$\boxed{\begin{aligned}\int \sin x\,dx &= -\cos x + c\\ \int \cos x\,dx &= \sin x + c\end{aligned}}$

(ix) $\int (3\sec x - \operatorname{cosec} x)\,dx$

$= 3\int \sec x\,dx - \int \operatorname{cosec} x\,dx$

$= 3\ln|\sec x + \tan x| - \ln|\operatorname{cosec} x - \cot x| + C.$

$\boxed{\begin{aligned}\int \sec x\,dx &= \ln|\sec x + \tan x| + C\\ \int \operatorname{cosec} x\,dx &= \ln|\operatorname{cosec} x - \cot x| + C.\end{aligned}}$

(x) $\int (2\tan x - \sec x)\,dx$

$= 2\int \tan x\,dx - \int \sec x\,dx$

$= 2\ln|\sec x| - \ln|\sec x + \tan x| + C.$

$\boxed{\begin{aligned}\int \tan x\,dx &= -\int -\frac{\sin x}{\cos x}\,dx\\ &= -\ln|\cos x| + c\\ &= \ln|(\cos x)^{-1}| + c\\ &= \ln\left|\frac{1}{\cos x}\right| + c\\ &= \ln|\sec x| + c.\end{aligned}}$

(xi) $\int (9e^x - 3\cos x - 5\sin x)\,dx$

$= 9\int e^x\,dx - 3\int \cos x\,dx - 5\int \sin x\,dx$

$= 9e^x - 3\sin x + 5\cos x + C.$

$\boxed{\int e^x\,dx = e^x + C.}$

(xii) $\int (\sec^2 x + \operatorname{cosec}^2 x)\,dx$

$= \int \sec^2 x\,dx + \int \operatorname{cosec}^2 x\,dx$

$= \tan x - \cot x + C.$

$\boxed{\begin{aligned}\int \sec^2 x\,dx &= \tan x + c\\ \int \operatorname{cosec}^2 x\,dx &= -\cot x + c.\end{aligned}}$

2 (i) $\int (3x^2+9x+3)^{\frac{1}{2}} (6x+9)\, dx$

$$\boxed{\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + C \quad (n \neq -1)}$$

$$= \frac{(3x^2+9x+3)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C.$$

$$= \frac{(3x^2+9x+3)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + C = \frac{2}{3}(3x^2+9x+3)^{\frac{3}{2}}.$$

(ii) $\int \sqrt{ax^2+2bx+C}\, (ax+b)\, dx$

$$= \frac{1}{2}\int (ax^2+2bx+c)^{\frac{1}{2}}\, 2(ax+b)\, dx$$

$$= \frac{1}{2}\int (ax^2+2bx+c)^{\frac{1}{2}} (2ax+2b)\, dx$$

$$= \frac{1}{2} \frac{(ax^2+2bx+c)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{1}{\cancel{2}} \frac{(ax^2+2bx+c)^{\frac{3}{2}}}{\left(\frac{3}{\cancel{2}}\right)} + C = \frac{(ax^2+2bx+c)^{\frac{3}{2}}}{3} + C.$$

(iii) $\int \frac{(6x+5)\, dx}{\sqrt{3x^2+5x+2}}$

$$= \int (3x^2+5x+2)^{-\frac{1}{2}} (6x+5)\, dx$$

$$= \frac{(3x^2+5x+2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{(3x^2+5x+2)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + C = 2\sqrt{3x^2+5x+2} + C.$$

(iv) $$\int (x^2+4x+3)^{-9}(2x+4)\,dx$$

$$= \frac{(x^2+4x+3)^{-9+1}}{-9+1} + C$$

$$= \frac{(x^2+4x+3)^{-8}}{-8} + C = -\frac{1}{8(x^2+4x+3)^8} + C.$$

(v) $$\int (x-2)(x-3)(x-4)\,dx$$

$$= \int (x^2-3x-2x+6)(x-4)\,dx$$

$$= \int (x^2-5x+6)(x-4)\,dx$$

$$= \int (x^3-4x^2-5x^2+20x+6x-24)\,dx$$

$$= \int (x^3-9x^2+26x-24)\,dx$$

$$= \int x^3 dx - 9\int x^2 dx + 26\int x\,dx - 24\int dx$$

$$= \frac{x^4}{4} - \overset{3}{\cancel{9}}\frac{x^3}{\cancel{3}} + \overset{13}{\cancel{26}}\frac{x^2}{\cancel{2}} - 24x + C$$

$$= \frac{x^4}{4} - 3x^3 + 13x^2 - 24x + C.$$

(vi) $$\int (2x^2-3)^2\,dx$$

$$= \int (4x^4+9-12x^2)\,dx$$

$$= 4\int x^4 dx + 9\int dx - 12\int x^2 dx$$

$$= 4\frac{x^5}{5} + 9x - \overset{4}{\cancel{12}}\frac{x^3}{\cancel{3}} + C$$

$$= \frac{4}{5}x^5 - 4x^3 + 9x + C.$$

(vii) $$\int (x^3-3x^2+9)^{\frac{7}{2}}(x^2-2x)\,dx$$

$$= \frac{1}{3}\int (x^3-3x^2+9)^{\frac{7}{2}}\,3(x^2-2x)\,dx = \frac{1}{3}\,\frac{(x^3-3x^2+9)^{\frac{7}{2}+1}}{\frac{7}{2}+1} + C$$

$$= \frac{1}{3}\,\frac{(x^3-3x^2+9)^{\frac{9}{2}}}{\left(\frac{9}{2}\right)} + C = \frac{2}{27}(x^3-3x^2+9)^{\frac{9}{2}} + C.$$

(viii) $\int (x^2-5)^3\,dx$

$\boxed{(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2}$

$$= \int \left((x^2)^3 - 5^3 - 3(x^2)^2 5 + 3x^2(5)^2\right) dx$$

$$= \int (x^6 - 125 - 15x^4 + 75x^2)\,dx$$

$$= \int x^6 dx - 125\int dx - 15\int x^4 dx + 75\int x^2 dx$$

$$= \frac{x^7}{7} - 125x - \overset{3}{\cancel{15}}\frac{x^5}{\cancel{5}} + \overset{25}{\cancel{75}}\frac{x^3}{\underset{1}{\cancel{3}}} + C.$$

$$= \frac{x^7}{7} - 125x - 3x^5 + 25x^3 + C.$$

(ix) $\int (\cos x + \sin x)^{\frac{3}{2}} (\cos x - \sin x)\,dx$

$$= \int (\cos x + \sin x)^{\frac{3}{2}} (-\sin x + \cos x)\,dx$$

$$= \frac{(\cos x + \sin x)^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C$$

$$= \frac{(\cos x + \sin x)^{\frac{5}{2}}}{\frac{5}{2}} + C = \frac{2}{5}(\cos x + \sin x)^{\frac{5}{2}} + C.$$

(x) $\int (\tan x + \sin x)(\sec^2 x + \cos x)\,dx$

$$= \frac{(\tan x + \sin x)^2}{2} + C.$$

③ (2)

(i) $$\int \frac{x}{x^2+3}\,dx$$

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c.}$$

$$= \frac{1}{2}\int \frac{2x}{x^2+3}\,dx$$

$$= \frac{1}{2}\ln|x^2+3| + c.$$

$$= \frac{1}{2}\ln(x^2+3) + c.$$

(ii) $$\int \frac{\sec^2 x + \cos x}{\tan x + \sin x}\,dx$$

$$\boxed{(\tan x + \sin x)' = \sec^2 x + \cos x}$$

$$= \ln|\tan x + \sin x| + c.$$

(iii) $$\int \frac{5x^4 + 4x^3 - 3x^2 + 2x}{x^5 + x^4 - x^3 + x^2}\,dx$$

$$\boxed{(x^5 + x^4 - x^3 + x^2)' = 5x^4 + 4x^3 - 3x^2 + 2x}$$

$$= \ln|x^5 + x^4 - x^3 + x^2| + c.$$

(iv) $$\int \frac{\left(e^x + \frac{1}{x}\right)}{e^x + \ln x}\,dx$$

$$\boxed{(e^x + \ln x)' = e^x + \frac{1}{x}}$$

$$= \ln|e^x + \ln x| + c.$$

(v) $$\int (5x^3 - 3x^2 + 6x - 9)^{-1}(5x^2 - 2x + 2)\,dx$$

$$\boxed{\begin{aligned}(5x^3 - 3x^2 + 6x - 9)' &= 15x^2 - 6x + 6\\ &= 3(5x^2 - 2x + 2)\end{aligned}}$$

$$= \frac{1}{3}\int \frac{3(5x^2 - 2x + 2)}{5x^3 - 3x^2 + 6x - 9}\,dx$$

$$= \frac{1}{3}\ln|5x^3 - 3x^2 + 6x - 9| + c.$$

(vi) $$\int \frac{1}{x + \sqrt{x}}\,dx$$

$$= \int \frac{1}{\sqrt{x}\sqrt{x} + \sqrt{x}}\,dx = \int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx = \int \frac{2\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}+1}\,dx$$

$$\boxed{(\sqrt{x}+1)' = \frac{1}{2\sqrt{x}}}$$

$$= 2\int \frac{\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}+1}\,dx = 2\ln|\sqrt{x}+1| + c.$$

$$\boxed{\int e^{ax}\left(a f(x)+f'(x)\right)dx = e^{ax} f(x) + C.}$$

4 (i) $\int e^{x}(\sin x+\cos x)\,dx$

Let $a=1$, $f(x)=\sin x$, $f'(x)=\cos x$

$$\int e^{ax}\left(a f(x)+f'(x)\right)dx = e^{ax} f(x) + C$$

$$\int e^{x}(\sin x+\cos x)\,dx = e^{x}\sin x + C.$$

(ii) $\int e^{1x}\left(\overset{1f}{\sin^{-1}x}+\overset{f'}{\frac{1}{\sqrt{1-x^2}}}\right)dx$ $\quad a=1,\ f(x)=\sin^{-1}x$

$$= e^{x}\sin^{-1}x + C.$$

(iii) $\int e^{x}\left(\tan^{-1}x+\frac{1}{1+x^2}\right)dx$

$$= e^{x}\tan^{-1}x + C.$$

(iv) $\int e^{x}(\sec^2 x+\tan x)\,dx$

$$= \int e^{x}(\tan x+\sec^2 x)\,dx$$

$$= e^{x}\tan x + C.$$

(v) $\int e^{x}\left(\frac{1}{x}+\ln x\right)dx$

$$= \int e^{x}\left(\ln x+\frac{1}{x}\right)dx$$

$$= e^{x}\ln x + C.$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C.$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C.$$

5 (i) $\displaystyle\int \frac{dx}{x^2+9}$

$$= \int \frac{dx}{x^2+3^2}$$

$$= \frac{1}{3}\tan^{-1}\frac{x}{3} + C.$$

(ii) $\displaystyle\int \frac{dt}{\sqrt{4-t^2}}$

$$= \int \frac{dt}{\sqrt{2^2-t^2}} = \sin^{-1}\frac{x}{2} + C$$

(iii) $\displaystyle\int \frac{dy}{y\sqrt{y^2-9}}$

$$= \int \frac{dy}{y\sqrt{y^2-3^2}} = \frac{1}{3}\sec^{-1}\frac{x}{3} + C.$$

(iv) $\displaystyle\int \frac{dt}{4t^2-9}$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$= \int \frac{dt}{4\left(t^2-\frac{9}{4}\right)} = \frac{1}{4}\int \frac{dt}{t^2-\left(\frac{3}{2}\right)^2}$$

$$= \frac{1}{4}\cdot\frac{1}{\cancel{2}\left(\frac{3}{\cancel{2}}\right)}\ln\left|\frac{t-\frac{3}{2}}{t+\frac{3}{2}}\right| + C$$

$$= \frac{1}{12}\ln\left|\frac{\left(\frac{2t-3}{\cancel{2}}\right)}{\left(\frac{2t+3}{\cancel{2}}\right)}\right| + C$$

$$= \frac{1}{12}\ln\left|\frac{2t-3}{2t+3}\right| + C.$$

(v) $$\int \frac{dx}{\sqrt{9x^2+16}}$$

$$\boxed{\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right|+C}$$

$$= \int \frac{dx}{\sqrt{9\left(x^2+\frac{16}{9}\right)}} = \frac{1}{3}\int \frac{dx}{\sqrt{x^2+\left(\frac{4}{3}\right)^2}}$$

$$= \frac{1}{3}\ln\left|x+\sqrt{x^2+\left(\frac{4}{3}\right)^2}\right|+C_1$$

$$= \frac{1}{3}\ln\left|x+\sqrt{\frac{9x^2+16}{9}}\right|+C_1$$

$$= \frac{1}{3}\ln\left|x+\frac{\sqrt{9x^2+16}}{3}\right|+C_1$$

$$= \frac{1}{3}\ln\left|\frac{3x+\sqrt{9x^2+16}}{3}\right|+C_1$$

$$= \frac{1}{3}\left[\ln\left|3x+\sqrt{9x^2+16}\right|-\ln 3\right]+C_1$$

$$= \frac{1}{3}\ln\left|3x+\sqrt{9x^2+16}\right|-\frac{1}{3}\ln 3+C_1$$

$$= \frac{1}{3}\ln\left|3x+\sqrt{9x^2+16}\right|+C.$$

(vi) $$\int \frac{dx}{\sqrt{16x^2-9}}$$

$$= \int \frac{dx}{\sqrt{16\left(x^2-\frac{9}{16}\right)}} = \int \frac{dx}{4\sqrt{x^2-\left(\frac{3}{4}\right)^2}}$$

$$= \frac{1}{4}\int \frac{dx}{\sqrt{x^2-\left(\frac{3}{4}\right)^2}}$$

$$\boxed{\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right|+C.}$$

$$= \frac{1}{4}\ln\left|x+\sqrt{x^2-\left(\frac{3}{4}\right)^2}\right|+C_1$$

$$= \frac{1}{4}\ln\left|x+\sqrt{x^2-\frac{9}{16}}\right|+C_1$$

$$= \frac{1}{4}\ln\left|x+\sqrt{\frac{16x^2-9}{16}}\right|+C_1$$

$$= \frac{1}{4}\ln\left|x+\frac{\sqrt{16x^2-9}}{4}\right|+C_1 = \frac{1}{4}\ln\left|\frac{4x+\sqrt{16x^2-9}}{4}\right|+C_1$$

$$= \frac{1}{4}\left[\ln\left|4x+\sqrt{16x^2-9}\right|-\ln 4\right]+C_1 = \frac{1}{4}\ln\left|4x+\sqrt{16x^2-9}\right|-\frac{1}{4}\ln 4+C_1$$

$$= \frac{1}{4}\ln\left|4x+\sqrt{16x^2-9}\right|+C.$$

(vii) $\int \frac{dx}{9-x^2}$

$$\boxed{\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C.}$$

$$= \int \frac{dx}{3^2 - x^2}$$

$$= \frac{1}{2(3)} \ln\left|\frac{3+x}{3-x}\right| + C$$

$$= \frac{1}{6} \ln\left|\frac{3+x}{3-x}\right| + C.$$

(viii) $\int \frac{dx}{x\sqrt{4x^2-16}}$

$$= \int \frac{dx}{x\sqrt{4\left(x^2 - \frac{16}{4}\right)}} = \int \frac{dx}{x(2)\sqrt{x^2-4}} = \frac{1}{2}\int \frac{dx}{x\sqrt{x^2-2^2}}$$

$$\boxed{\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C.}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \sec^{-1}\frac{x}{2} + C$$

$$= \frac{1}{4}\sec^{-1}\frac{x}{2} + C.$$

(ix) $\int \sqrt{9-4x^2}\, dx$

$$= \int \sqrt{4\left(\frac{9}{4} - x^2\right)}\, dx = \int 2\sqrt{\left(\frac{3}{2}\right)^2 - x^2}\, dx$$

$$= 2\int \sqrt{\left(\frac{3}{2}\right)^2 - x^2}\, dx$$

$$\boxed{\int \sqrt{a^2-x^2}\, dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2 \sin^{-1}\frac{x}{a} + C.}$$

$$= \cancel{2} \cdot \frac{1}{\cancel{2}} x\sqrt{\left(\frac{3}{2}\right)^2 - x^2} + \frac{1}{2}\left(\frac{3}{2}\right)^2 \sin^{-1}\left(\frac{x}{3/2}\right) + C.$$

$$= x\sqrt{\frac{9}{4} - x^2} + \frac{1}{2}\left(\frac{9}{4}\right)\sin^{-1}\left(\frac{2x}{3}\right) + C$$

$$= x\sqrt{\frac{9-4x^2}{4}} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right) + C.$$

$$= \frac{x}{2}\sqrt{9-4x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right) + C.$$

(x) $\int \sqrt{25+9x^2}\, dx$

$$= \int \sqrt{9\left(\frac{25}{9}+x^2\right)}\, dx = \int 3\sqrt{\left(\frac{5}{3}\right)^2+x^2}\, dx$$

$$= 3\int \sqrt{\left(\frac{5}{3}\right)^2+x^2}\, dx$$

$$\boxed{(26)\quad \int \sqrt{a^2+x^2}\, dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2 \ln\left|x+\sqrt{a^2+x^2}\right| + C.}$$

$$= 3\cdot\left[\frac{1}{2}x\sqrt{\left(\frac{5}{3}\right)^2+x^2} + \frac{1}{2}\left(\frac{5}{3}\right)^2 \ln\left|x+\sqrt{\left(\frac{5}{3}\right)^2+x^2}\right|\right] + C_1$$

$$= \frac{3}{2}x\sqrt{\frac{25}{9}+x^2} + \frac{\cancel{3}}{2}\cdot\frac{25}{\cancel{9}_3}\ln\left|x+\sqrt{\frac{25}{9}+x^2}\right| + C_1$$

$$= \frac{3}{2}x\sqrt{\frac{25+9x^2}{9}} + \frac{25}{6}\ln\left|x+\sqrt{\frac{25+9x^2}{9}}\right| + C_1$$

$$= \frac{\cancel{3}}{2}x\frac{\sqrt{25+9x^2}}{\cancel{3}} + \frac{25}{6}\ln\left|\frac{3x+\sqrt{25+9x^2}}{3}\right| + C_1$$

$$= \frac{x}{2}\sqrt{25+9x^2} + \frac{25}{6}\left[\ln\left|3x+\sqrt{25+9x^2}\right| - \ln|3|\right] + C_1$$

$$= \frac{x}{2}\sqrt{25+9x^2} + \frac{25}{6}\ln\left|3x+\sqrt{25+9x^2}\right| - \frac{25}{6}\ln 3 + C_1$$

$$= \frac{x}{2}\sqrt{25+9x^2} + \frac{25}{6}\ln\left|3x+\sqrt{25+9x^2}\right| + C$$

(xi) $\int \frac{dy}{9y^2+81}$

$$= \int \frac{dy}{9(y^2+9)} = \frac{1}{9}\int \frac{dy}{y^2+3^2}$$

$$\boxed{\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.}$$

$$= \frac{1}{9}\cdot\frac{1}{3}\tan^{-1}\frac{y}{3} + C$$

$$= \frac{1}{27}\tan^{-1}\frac{y}{3} + C.$$

(xii) $\int \frac{dx}{4x^2-16}$

$\boxed{(21) \quad \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C.}$

$$= \int \frac{dx}{4(x^2-4)}$$

$$= \frac{1}{4} \int \frac{dx}{x^2-2^2}$$

$$= \frac{1}{4} \cdot \frac{1}{2(2)} \ln \left| \frac{x-2}{x+2} \right| + C.$$

$$= \frac{1}{16} \ln \left| \frac{x-2}{x+2} \right| + C.$$