Vector Valued Functions

$f : \mathbb{R} \longrightarrow \mathbb{R}$ scalar function

$\vec{r} : \mathbb{R} \longrightarrow \{\text{vectors}\}$ vector function.

$$\vec{r}(t) = \underbrace{f(t)\,\hat{i} + g(t)\,\hat{j}}_{\text{Components.}}$$

✓ $\vec{r}(t) = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$ ✓

$$\text{Dom } \vec{r}(t) = \text{Dom}(f) \cap \text{Dom}(g) \cap \text{Dom}(h).$$

$$\lim_{t \to a} \vec{r}(t) = \lim_{t \to a} \left[f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k} \right]$$

$$= \left(\lim_{t \to a} f(t)\right)\hat{i} + \left(\lim_{t \to a} g(t)\right)\hat{j} + \left(\lim_{t \to a} h(t)\right)\hat{k}$$

$\vec{r}(t)$ is continuous at $t = a$ if

$$\vec{r}(a) = \lim_{t \to a} \vec{r}(t) = L$$

For videos, visit YouTube

Suppose Math.

0332-6297570

Akhtar Abbas.

Exercise 5.1

① Find the domain of the following functions:

(i) $\vec{r}(t) = 2t\,\hat{i} - 3t\,\hat{j} + \frac{1}{t}\,\hat{k}.$

Let $f(t) = 2t$, $g(t) = -3t$, $h(t) = \frac{1}{t}$

$$\text{Dom}\,\vec{r}(t) = \text{Dom}(f) \cap \text{Dom}(g) \cap \text{Dom}(h)$$

$$= \mathbb{R} \cap \mathbb{R} \cap \mathbb{R} - \{0\}$$

$$= \mathbb{R} - \{0\}.$$

(ii) $\vec{r}(t) = \sin t\,\hat{i} + \cos t\,\hat{j} + \tan t\,\hat{k}.$

Let $f(t) = \sin t$, $g(t) = \cos t$, $h(t) = \tan t = \frac{\sin t}{\cos t}$

$\text{Dom}(f) = \mathbb{R}$, $\text{Dom}(g) = \mathbb{R}$, $\text{Dom}(h) = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$

$$\text{Dom}\,\vec{r}(t) = \text{Dom}(f) \cap \text{Dom}(g) \cap \text{Dom}(h)$$

$$= \mathbb{R} \cap \mathbb{R} \cap \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$$

$$= \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}.$$

(iii) $\vec{r}(t) = (1-t)\,\hat{i} + \sqrt{t}\,\hat{j} + \frac{1}{t^2}\,\hat{k}.$

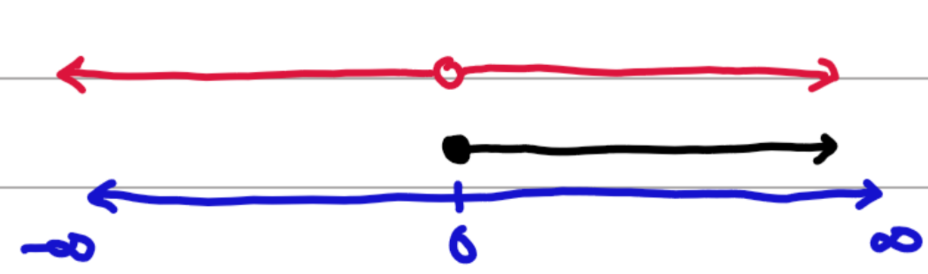
Let $f(t) = 1-t$, $g(t) = \sqrt{t}$, $h(t) = \frac{1}{t^2}$

$\text{Dom}(f) = \mathbb{R}$, $\text{Dom}(g) = \mathbb{R}^+ \cup \{0\}$, $\text{Dom}(h) = \mathbb{R} - \{0\}.$

$$\text{Dom}\,\vec{r}(t) = \text{Dom}(f) \cap \text{Dom}(g) \cap \text{Dom}(h)$$

$$= \mathbb{R} \cap \mathbb{R}^+ \cup \{0\} \cap \mathbb{R} - \{0\}$$

$$= \left(\mathbb{R}^+ \cup \{0\}\right) \cap \left(\mathbb{R} - \{0\}\right)$$



$$= \mathbb{R}^+.$$

(iv) $\vec{g}(t) = \cos t\,\hat{i} - \cos t\,\hat{j} + \operatorname{cosec} t\,\hat{k}.$

Let $f(t) = \cos t$, $h(t) = -\cos t$, $k(t) = \operatorname{cosec} t = \dfrac{1}{\sin t}$

$\text{Dom}(f) = \mathbb{R}$, $\text{Dom}(h) = \mathbb{R}$, $\text{Dom}(k) = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\}.$

$$\text{Dom}(\vec{g}(t)) = \text{Dom}(f) \cap \text{Dom}(h) \cap \text{Dom}(k)$$

$$= \mathbb{R} \cap \mathbb{R} \cap \mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$$

$$= \mathbb{R} \cap \mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$$

$$= \mathbb{R} - \{n\pi : n \in \mathbb{Z}\}.$$

(2) Find the limit of the vector function

$$\vec{r}(t) = (e^{3t} - 1)\hat{i} + \frac{\sqrt{3+t} - \sqrt{3}}{3t}\hat{j} + \frac{1}{9t+1}\hat{k} \quad \text{at} \quad t = 0.$$

$$\lim_{t\to 0} \vec{r}(t) = \lim_{t\to 0}\left[(e^{3t}-1)\hat{i} + \frac{\sqrt{3+t}-\sqrt{3}}{3t}\hat{j} + \frac{1}{9t+1}\hat{k}\right]$$

$$= \left(\lim_{t\to 0} e^{3t} - 1\right)\hat{i} + \left(\lim_{t\to 0}\frac{\sqrt{3+t}-\sqrt{3}}{3t}\right)\hat{j} + \left(\lim_{t\to 0}\frac{1}{9t+1}\right)\hat{k}$$

$$= (e^{0} - 1)\hat{i} + \left(\lim_{t\to 0}\frac{\sqrt{3+t}-\sqrt{3}}{3t} \times \frac{\sqrt{3+t}+\sqrt{3}}{\sqrt{3+t}+\sqrt{3}}\right) + \left(\frac{1}{9(0)+1}\right)\hat{k}$$

$$= (1-1)\hat{i} + \left(\lim_{t\to 0}\frac{\cancel{3}+\cancel{t}-\cancel{3}}{3\cancel{t}(\sqrt{3+t}+\sqrt{3})}\right)\hat{j} + \left(\frac{1}{1}\right)\hat{k}$$

$$= 0\hat{i} + \left(\frac{1}{3(\sqrt{3}+\sqrt{3})}\right)\hat{j} + \hat{k}$$

$$= 0\hat{i} + \frac{1}{6\sqrt{3}}\hat{j} + \hat{k}$$

$$= \frac{1}{6\sqrt{3}}\hat{j} + \hat{k}.$$

For videos, visit YouTube
Suppose Math.
0332-6297570
AKHTAR ABBAS.

③ If $\vec{u} = t^2\hat{i} - 2\hat{j}$; $\vec{v} = 2t\hat{i} - 5\hat{k}$ are vector functions and $h = 3t$ is a scalar function, then find the following:

(i) $\lim\limits_{t\to 0} [\vec{u}(t) + \vec{v}(t)]$ (ii) $\lim\limits_{t\to 1} [\vec{u}(t) \cdot \vec{v}(t)]$

(iii) $\lim\limits_{t\to 1} [\vec{u}(t) \times \vec{v}(t)]$ (iv) $\lim\limits_{t\to 5} [h\vec{u}(t)]$.

(i)
$$\lim_{t\to 0}\left[\vec{u}(t) + \vec{v}(t)\right] = \lim_{t\to 0} \vec{u}(t) + \lim_{t\to 0} \vec{v}(t)$$
$$= \lim_{t\to 0} (t^2\hat{i} - 2\hat{j}) + \lim_{t\to 0} (2t\hat{i} - 5\hat{k})$$
$$= (0^2\hat{i} - 2\hat{j}) + (0\hat{i} - 5\hat{k})$$
$$= -2\hat{j} - 5\hat{k}.$$

(ii)
$$\lim_{t\to 1}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \lim_{t\to 1} \vec{u}(t) \cdot \lim_{t\to 1} \vec{v}(t)$$
$$= \lim_{t\to 1} (t^2\hat{i} - 2\hat{j}) \cdot \lim_{t\to 1} (2t\hat{i} - 5\hat{k})$$
$$= (\hat{i} - 2\hat{j}) \cdot (2\hat{i} - 5\hat{k})$$
$$= (\hat{i} - 2\hat{j} + 0\hat{k}) \cdot (2\hat{i} + 0\hat{j} - 5\hat{k})$$
$$= 1(2) + (-2)(0) + 0(-5) = 2+0+0 = 2.$$

(iii)
$$\lim_{t\to 1}\left[\vec{u}(t) \times \vec{v}(t)\right] = \lim_{t\to 1} \vec{u}(t) \times \lim_{t\to 1} \vec{v}(t)$$
$$= \lim_{t\to 1} (t^2\hat{i} - 2\hat{j}) \times \lim_{t\to 1} (2t\hat{i} - 5\hat{k})$$
$$= (\hat{i} - 2\hat{j}) \times (2\hat{i} - 5\hat{k})$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 0 \\ 2 & 0 & -5 \end{vmatrix}$$
$$= \hat{i}\begin{vmatrix} -2 & 0 \\ 0 & -5 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 0 \\ 2 & -5 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -2 \\ 2 & 0 \end{vmatrix}$$
$$= \hat{i}(10-0) - \hat{j}(-5-0) + \hat{k}(0+4) = 10\hat{i} + 5\hat{j} + 4\hat{k}.$$

(iv)
$$\lim_{t\to 5}\left[h\,\vec{u}(t)\right] = \left(\lim_{t\to 5} h\right)\left(\lim_{t\to 5} \vec{u}(t)\right) = \left(\lim_{t\to 5} 3t\right)\left(\lim_{t\to 5} t^2\hat{i} - 2\hat{j}\right)$$
$$= (3(5))(5^2\hat{i} - 2\hat{j}) = 15(25\hat{i} - 2\hat{j})$$
$$= 375\hat{i} - 30\hat{j}.$$

(4) Show that the function

$$\vec{R}(t) = \sin^2 t\,\hat{i} + \tan t\,\hat{j} + \frac{1}{t}\hat{k}$$

is continuous at $t = \frac{\pi}{4}$.

Value:

$$R\left(\frac{\pi}{4}\right) = \sin^2\frac{\pi}{4}\,\hat{i} + \tan\frac{\pi}{4}\,\hat{j} + \frac{1}{(\pi/4)}\hat{k}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2\hat{i} + 1\,\hat{j} + \frac{4}{\pi}\hat{k}$$

$$= \frac{1}{2}\hat{i} + \hat{j} + \frac{4}{\pi}\hat{k}.$$

Limit

$$\lim_{t\to\pi/4}\vec{R}(t) = \left(\lim_{t\to\pi/4}\sin^2 t\right)\hat{i} + \left(\lim_{t\to\pi/4}\tan t\right)\hat{j} + \left(\lim_{t\to\pi/4}\frac{1}{t}\right)\hat{k}$$

$$= \left(\sin^2\frac{\pi}{4}\right)\hat{i} + \left(\tan\frac{\pi}{4}\right)\hat{j} + \left(\frac{1}{\pi/4}\right)\hat{k}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2\hat{i} + 1\,\hat{j} + \frac{4}{\pi}\hat{k}$$

$$= \frac{1}{2}\hat{i} + \hat{j} + \frac{4}{\pi}\hat{k}.$$

So $\vec{R}(t)$ is continuous at $t = \frac{\pi}{4}$.

(5) Show that the function

$$\vec{r}(t) = \frac{2}{t}\hat{i} + \frac{t^3}{2t^3-5}\hat{j} + \frac{1}{e^t}\hat{k}$$

is continuous at $t\to\infty$.

$$\lim_{t\to\infty}\vec{r}(t) = \lim_{t\to\infty}\left[\frac{2}{t}\hat{i} + \frac{t^3}{2t^3-5}\hat{j} + \frac{1}{e^t}\hat{k}\right]$$

$$= \left(\lim_{t\to\infty}\frac{2}{t}\right)\hat{i} + \left(\lim_{t\to\infty}\frac{t^3}{2t^3-5}\right)\hat{j} + \left(\lim_{t\to\infty}\frac{1}{e^t}\right)\hat{k}$$

$$= 0\,\hat{i} + \lim_{t\to\infty}\frac{\left(\frac{t^3}{t^3}\right)}{\left(\frac{2t^3-5}{t^3}\right)}\hat{j} + 0\,\hat{k}$$

$$= 0\hat{i} + \lim_{t\to\infty}\frac{1}{2-\frac{5}{t^3}}\hat{j} + 0\hat{k}$$

$$= 0\hat{i} + \frac{1}{2-0}\hat{j} + 0\hat{k} = 0\hat{i} + \frac{1}{2}\hat{j} + 0\hat{k}.$$

6) For what values of t, the following vector functions are continuous?

(i) $\vec{r}(t) = \ln(t+3)\,\hat{i} + \frac{1}{t-1}\,\hat{j} + \frac{t+2}{t^2-4}\,\hat{k}$

(ii) $\vec{r}(t) = \frac{1}{3t+1}\,\hat{i} + \frac{1}{t}\,\hat{j}$.

(i) $\vec{r}(t) = \ln(t+3)\,\hat{i} + \frac{1}{t-1}\,\hat{j} + \frac{t+2}{t^2-4}\,\hat{k}$.

Let $f(t) = \ln(t+3)$, $g(t) = \frac{1}{t-1}$, $h(t) = \frac{t+2}{t^2-4}$

$f(t)$ is continuous at $t+3>0$, $t>-3$.

$g(t)$ is continuous at $t-1 \neq 0$, $t \neq 1$.

$h(t)$ is continuous at $t^2-4 \neq 0$, $t^2 \neq 4$, $t \neq \pm 2$.

So $\vec{r}(t)$ is continuous at

$$\{t \in \mathbb{R} : t > -3,\ t \neq 1 \text{ and } t \neq \pm 2\}.$$

(ii) $\vec{r}(t) = \frac{1}{3t+1}\,\hat{i} + \frac{1}{t}\,\hat{j}$

Let $f(t) = \frac{1}{3t+1}$, $g(t) = \frac{1}{t}$

$f(t)$ is continuous at $3t+1 \neq 0$, $t \neq -\frac{1}{3}$

$g(t)$ is continuous at $t \neq 0$.

So $\vec{r}(t)$ is continuous at

$$\{t \in \mathbb{R} : t \neq -\frac{1}{3},\ t \neq 0\}.$$

For videos, visit YouTube
Suppose Math.
0332-6297570
AKHTAR ABBAS.