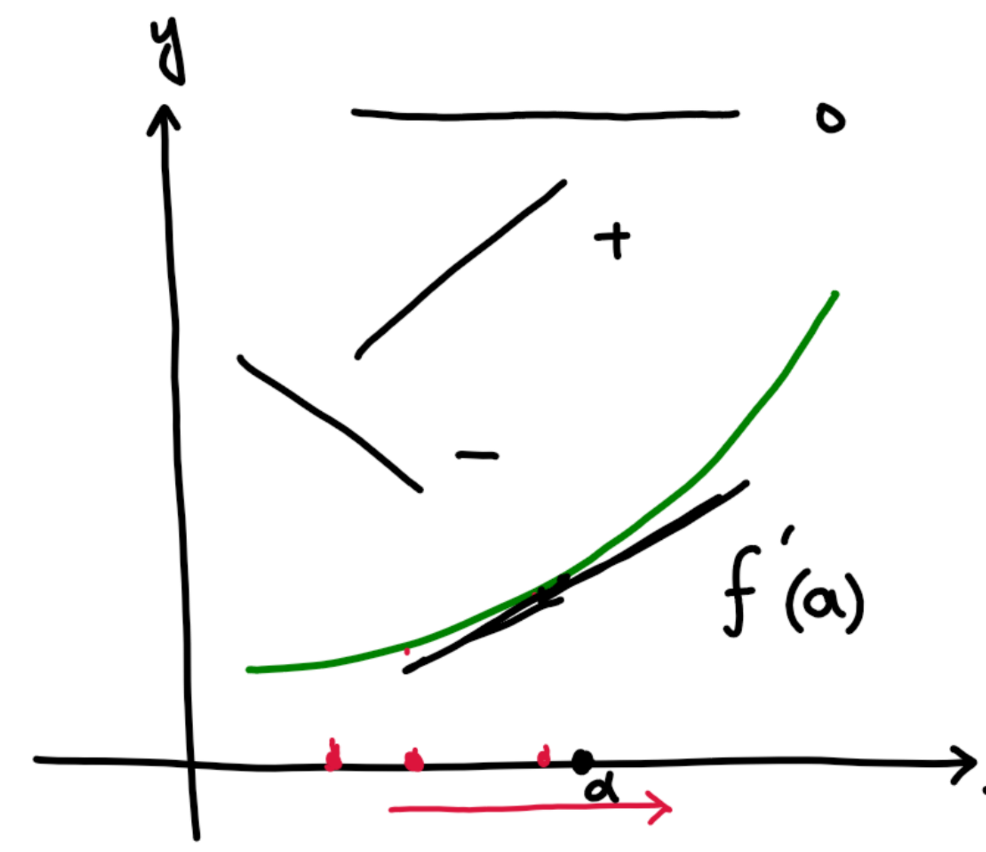
# Important Points for Exercise 4.4




If $f(x)$ is differentiable function on the open interval $(a, b)$ then

- $f(x)$ is increasing on $(a, b)$ if $f'(x) > 0 \ \forall x \in (a, b)$
- $f(x)$ is decreasing on $(a, b)$ if $f'(x) < 0 \ \forall x \in (a, b)$

maximum / minimum

**State the second derivative rule to find the extreme values of a function at a point**

Let $y = f(x)$ be a function.

(i) Differentiate w.r.t '$x$' and obtain $f'(x)$.

(ii) Put $\boxed{f'(x) = 0}$, solve it and obtain critical ~~function~~. point.

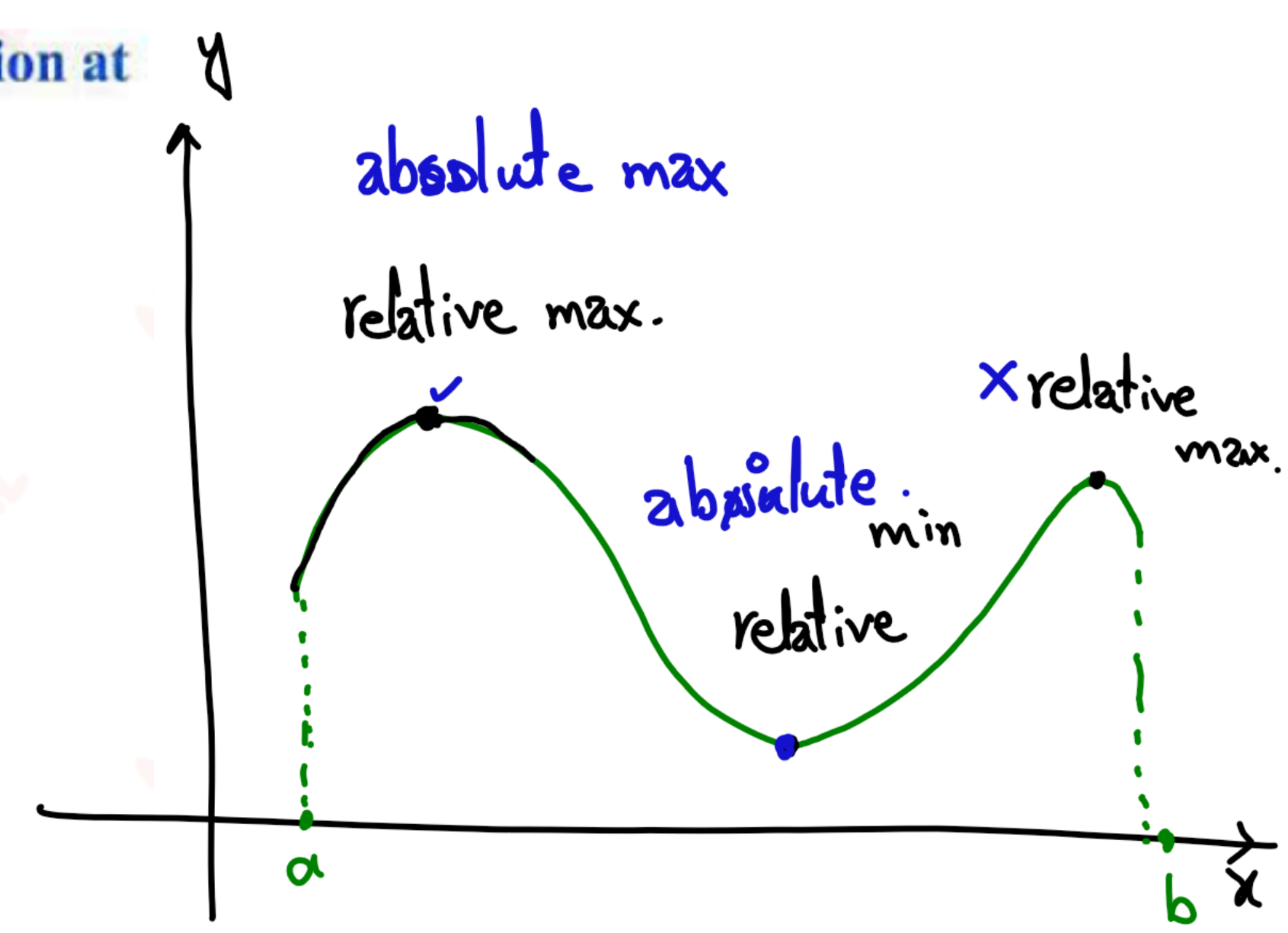
(iii) Differentiate again w.r.t '$x$' of obtain $f''(x)$.

(iv) Let $x = a$ be a critical point.

If the $f''(a) < 0 \Rightarrow x = a$ is a point of maxima.

If the $f''(a) > 0 \Rightarrow x = a$ is point of minima.

If the $f''(a) = 0 \Rightarrow$ test fails.

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Suppose Math.

0332-6297570

Akhtar Abbas.

# Exercise 4.4

① Show that function $f(x) = -x^2 + 10x + 9$ is increasing at $x = 4$.

Given $f(x) = -x^2 + 10x + 9$.

$$f'(x) = -2x + 10$$

$$f'(4) = -2(4) + 10$$
$$= -8 + 10 = 2 > 0$$

So $f(x)$ is increasing at $x = 4$.

② Show that $f(x) = \tan^2 x$ is decreasing at $x = \frac{3\pi}{4}$.

$$f(x) = \tan^2 x$$

$$f'(x) = 2\tan x \sec^2 x$$

$\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$



$$f'\left(\frac{3\pi}{4}\right) = 2\tan\frac{3\pi}{4} \cdot \sec^2\frac{3\pi}{4}$$
$$= 2(-1) \cdot (-\sqrt{2})^2$$
$$= -2 \cdot (2) = -4 < 0$$

So $f(x)$ is decreasing at $x = \frac{3\pi}{4}$.

③ Find the maximum and minimum values, if any, of the function $f: \mathbb{R} \to \mathbb{R}$ in the following cases:

(i) $f(x) = x^2 - 2x + 3$

$$f'(x) = 2x - 2$$

Set $f'(x) = 0$

$$2x - 2 = 0$$
$$2x = 2$$
$$\boxed{x = 1} \quad \text{critical value.}$$

$$f''(x) = 2$$
$$f''(1) = 2 > 0$$

So $f(x)$ has minimum value at $x = 1$, and minimum value is

$$f(1) = 1^2 - 2(1) + 3 = 1 - 2 + 3 = 2.$$

(iii) $f(x) = x^3 - 9x^2 + 15x + 3$

$f'(x) = 3x^2 - 18x + 15$

Critical value

$f'(x) = 0$

$3x^2 - 18x + 15 = 0$

$3(x^2 - 6x + 5) = 0$

$x^2 - 6x + 5 = 0$

$x^2 - 5x - x + 5 = 0$

$x(x-5) - 1(x-5) = 0$

$(x-1)(x-5) = 0$

$x - 1 = 0$ , $x - 5 = 0$

$\boxed{x = 1}$ $\boxed{x = 5}$

$f''(x) = 6x - 18$

At $x = 1$

$f''(1) = 6(1) - 18 = 6 - 18 = -12 < 0$

So $f$ has maximum value at $x = 1$ and max. value is

$f(1) = 1 - 9 + 15 + 3 = 10.$

At $x = 5$

$f''(5) = 6(5) - 18 = 30 - 18 = 12 > 0$

So $f$ has min. value at $x = 5$ and min. value is

$f(5) = 5^3 - 9(5)^2 + 15(5) + 3$

$= 125 - 225 + 75 + 3$

$= -100 + 78 = -22.$

(iii) $f(x) = -x^4 + 2x^2$

$f'(x) = -4x^3 + 4x$

Critical value

Put $f'(x) = 0$

$-4x^3 + 4x = 0$

$-4x(x^2 - 1) = 0$

$x(x+1)(x-1) = 0$

$x = 0$, $x+1 = 0$, $x-1 = 0$

$\boxed{x = 0}$, $\boxed{x = -1}$, $\boxed{x = 1}$

$f''(x) = -12x^2 + 4$

At $x = 0$

$f''(0) = -12(0)^2 + 4 = 4 > 0$

So $f$ has min. value at $x = 0$

and min. value is

$f(0) = -0^4 + 2(0)^2 = 0.$

At $x = -1$

$f''(-1) = -12(-1)^2 + 4 = -12 + 4 = -8 < 0$

So $f$ has max. value at $x = -1.$

and max. value is

$f(-1) = -(-1)^4 + 2(-1)^2 = -1 + 2 = 1.$

At $x = 1$

$f''(1) = -12(1)^2 + 4 = -12 + 4 = -8 < 0$

So $f$ has max. value at $x = 1$

and max. value is

$f(1) = -1^4 + 2(1^2) = -1 + 2 = 1.$

(iv) $f(x) = e^x \sin x$

$$f'(x) = e^x \cos x + e^x \sin x$$

<u>Critical value</u>

Put $f'(x) = 0$

$$e^x \cos x + e^x \sin x = 0$$

$$e^x (\cos x + \sin x) = 0$$

$e^x \neq 0$ , $\cos x + \sin x = 0$

Divide by $\cos x$

$$1 + \tan x = 0$$

$$\tan x = -1$$

| | |
|---|---|
| ✓ S | A |
| T | C ✓ |

$x = \pi - \frac{\pi}{4}$ , $x = -\frac{\pi}{4}$

$\boxed{x = \frac{3\pi}{4}}$ , $\boxed{x = -\frac{\pi}{4}}$

$$f''(x) = \cancel{-e^x \sin x} + e^x \cos x + e^x \cos x + \cancel{e^x \sin x}$$

$$f''(x) = 2e^x \cos x$$

At $x = -\frac{\pi}{4}$

$$f''\left(-\frac{\pi}{4}\right) = 2e^{-\frac{\pi}{4}} \cos\left(-\frac{\pi}{4}\right) = 2e^{-\frac{\pi}{4}} \cos\frac{\pi}{4} = 2e^{-\pi/4} \cdot \frac{1}{\sqrt{2}} > 0$$

So $f$ has min. value at $x = -\frac{\pi}{4}$.

and the min. value is

$$f\left(-\frac{\pi}{4}\right) = e^{-\frac{\pi}{4}} \sin\left(-\frac{\pi}{4}\right) = -e^{-\frac{\pi}{4}} \sin\frac{\pi}{4} = -\frac{e^{-\pi/4}}{\sqrt{2}}$$

At $x = \frac{3\pi}{4}$

$$f''\left(\frac{3\pi}{4}\right) = 2e^{3\pi/4} \cos\left(\frac{3\pi}{4}\right) = \frac{-2}{\sqrt{2}} e^{\frac{3\pi}{4}} < 0$$

So $f$ has max. value at $x = \frac{3\pi}{4}$,

and the max. value is

$$f\left(\frac{3\pi}{4}\right) = e^{\frac{3\pi}{4}} \sin\left(\frac{3\pi}{4}\right) = \frac{e^{\frac{3\pi}{4}}}{\sqrt{2}}.$$

(v) $f(x) = 2e^{x} + e^{-x}$

$f'(x) = 2e^{x} - e^{-x}$

<u>Critical value</u>

Put $f'(x) = 0$

$$2e^{x} - e^{-x} = 0$$

$$2e^{x} - \frac{1}{e^{x}} = 0$$

$$\frac{2e^{2x} - 1}{e^{x}} = 0 \qquad , e^{x} \neq 0$$

$$2e^{2x} - 1 = 0$$

$$2e^{2x} = 1$$

$$e^{2x} = \frac{1}{2}$$

$$\ln e^{2x} = \ln\left(\frac{1}{2}\right) \qquad \boxed{m \ln a = \ln a^{m}}$$

$$2x = \ln \frac{1}{2}$$

$$x = \frac{1}{2} \ln\left(\frac{1}{2}\right) = \ln\left(\frac{1}{2}\right)^{\frac{1}{2}} = \ln\left(\sqrt{\frac{1}{2}}\right)$$

$$\boxed{x = \ln\left(\frac{1}{\sqrt{2}}\right)}$$

$$f''(x) = 2e^{x} + e^{-x} = f(x)$$

At $x = \ln\left(\frac{1}{\sqrt{2}}\right)$

$$✓ \; f''\left(\ln \frac{1}{\sqrt{2}}\right) = 2e^{\ln\left(\frac{1}{\sqrt{2}}\right)} + e^{-\ln\left(\frac{1}{\sqrt{2}}\right)}$$

$$= 2\left(\frac{1}{\sqrt{2}}\right) + e^{\ln\left(\frac{1}{\sqrt{2}}\right)^{-1}}$$

$$= \frac{2}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\right)^{-1} = \frac{2}{\sqrt{2}} + \sqrt{2}$$

$$= \frac{2+2}{\sqrt{2}} = \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancel{4}^{2}\sqrt{2}}{\cancel{2}}$$

$$= 2\sqrt{2} > 0$$

So $f$ has minimum value at $x = \ln\left(\frac{1}{\sqrt{2}}\right)$.

and minimum value is

$$f\left(\ln\left(\frac{1}{\sqrt{2}}\right)\right) = f''\left(\ln\left(\frac{1}{\sqrt{2}}\right)\right) = 2\sqrt{2}.$$

(vi) $f(x) = 2x - x^2.$

$f'(x) = 2 - 2x$

Critical value

Put $f'(x) = 0$

$2 - 2x = 0$

$2 = 2x$

$\boxed{x = 1}$

$f''(x) = -2$

$f''(1) = -2 < 0$

So $f$ has maximum value at $x = 1$

and the max. value is

$f(1) = 2(1) - 1^2 = 2 - 1 = 1.$

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Suppose Math.

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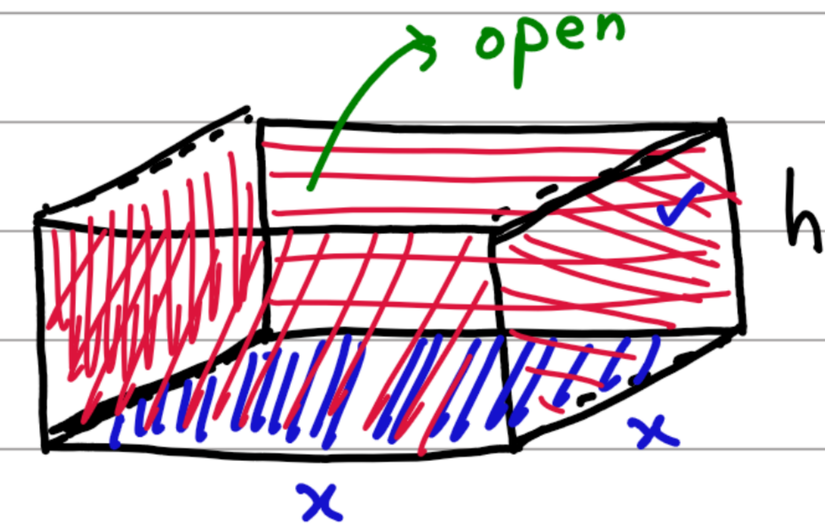
④ A rectangular reservoir with a square bottom and open top is to be lined inside with lead. Find the dimensions of the reservoir to hold $\frac{1}{2}a^3$ cubic meters, such that the lead required is minimum.

Let $x$ be the side of bottom.

and $h$ be the height.




Given

$$\text{volume} = \frac{1}{2}a^3$$

$$x \cdot x \cdot h = \frac{1}{2}a^3$$

$$x^2 h = \frac{1}{2}a^3$$

$$h = \frac{a^3}{2x^2} \quad \text{— ①}$$

Surface area $S = x^2 + xh + xh + xh + xh$

$$S = x^2 + 4xh$$

$$S = x^2 + \cancel{4}^{2}\cancel{x}\left(\frac{a^3}{\cancel{2}x^{\cancel{2}}}\right) \quad (\because \text{from ①})$$

$$S = x^2 + \frac{2a^3}{x} = x^2 + 2a^3x^{-1}$$

$$S' = 2x - 2a^3x^{-2} = 2x - \frac{2a^3}{x^2}$$

Put $S' = 0$

$$2x - \frac{2a^3}{x^2} = 0.$$

$$\frac{2x^3 - 2a^3}{x^2} = 0 \Rightarrow x \neq 0, \quad 2x^3 - 2a^3 = 0$$

$$2x^3 = 2a^3$$

$$\boxed{x = a}$$

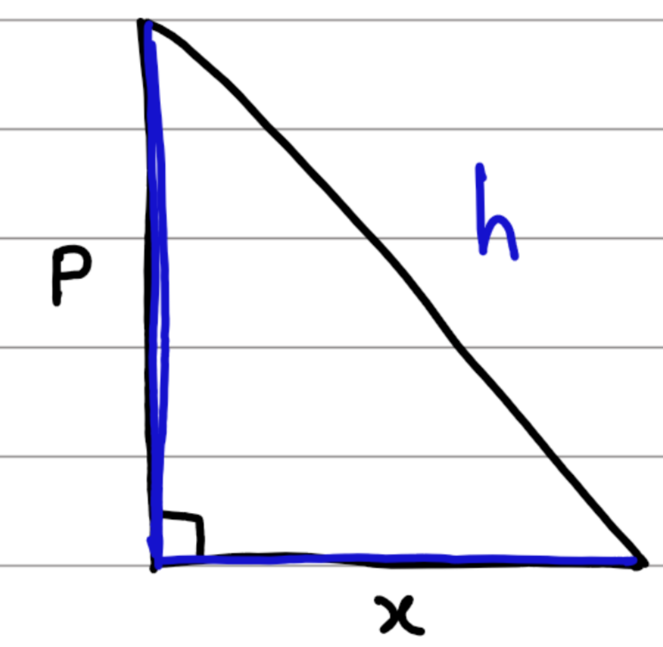
$$S'' = 2 + 4a^3x^{-3}$$

At $x = a$, $S'' = 2 + 4a^3a^{-3} = 2 + 4 = 6 > 0$

So $S$ is minimum at $x = a$.

Put $x = a$ in ① $h = \frac{a^3}{2x^2} = \frac{a^{\cancel{3}}}{2\cancel{a^2}} = \frac{1}{2}a.$

5. Find a right-angled triangle of maximum area with a hypotenuse of length h.




Let base of $\Delta = x$

then
$$p^2 + x^2 = h^2$$
$$p^2 = h^2 - x^2$$
$$p = \sqrt{h^2 - x^2}$$

So

Area $A = \frac{1}{2}(x \cdot p)$

$$A = \frac{1}{2}\left(x \cdot \sqrt{h^2 - x^2}\right)$$

$$A' = \frac{1}{2}\left[x \cdot \frac{1}{\cancel{2}\sqrt{h^2-x^2}}(-\cancel{2}x) + \sqrt{h^2-x^2} \cdot 1\right]$$

$$A' = \frac{1}{2}\left[\frac{-x^2}{\sqrt{h^2-x^2}} + \sqrt{h^2-x^2}\right] = \frac{1}{2}\left[\frac{-x^2 + h^2 - x^2}{\sqrt{h^2-x^2}}\right]$$

$$A' = \frac{1}{2}\left(\frac{-2x^2 + h^2}{\sqrt{h^2-x^2}}\right)$$

Put

$$A' = 0$$

$$\frac{1}{2}\left(\frac{-2x^2+h^2}{\sqrt{h^2-x^2}}\right) = 0 \quad \Rightarrow \quad -2x^2 + h^2 = 0$$
$$h^2 = 2x^2$$
$$\sqrt{2}\,x = h$$
$$\boxed{x = \frac{h}{\sqrt{2}}}$$

$$A = \frac{1}{2}\left[\frac{h}{\sqrt{2}}\sqrt{h^2 - \left(\frac{h}{\sqrt{2}}\right)^2}\right] = \frac{h}{2\sqrt{2}}\sqrt{h^2 - \frac{h^2}{2}}$$

$$A = \frac{h}{2\sqrt{2}}\sqrt{\frac{h^2}{2}} = \frac{h}{2\sqrt{2}} \cdot \frac{h}{\sqrt{2}} = \frac{h^2}{4}.$$

6. A particle moves so that its distance s at time t is given by

$$s = ut + \frac{1}{2}at^2$$

where u and a are fixed real numbers. Find its speed and magnitude of its acceleration at time t.

$$s = ut + \frac{1}{2}at^2$$

$$\text{Speed} = \frac{ds}{dt} = u + \frac{1}{\cancel{2}}a(\cancel{2}t) = u + at.$$

$$\text{Acceleration} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = 0 + a(1) = a.$$