# Exercise 4.1

**Q #01** Calculate the first, second and third order derivatives of

$$y = \cos^2 x.$$

$$y' = 2\cos x\,(\cos x)' = 2\cos x\,(-\sin x)$$

$$y' = -2\sin x\cos x = -\sin 2x$$

$$y'' = -2\cos 2x$$

$$y''' = -2(-\sin 2x)\,2 = 4\sin 2x.$$

**Q# 02** Find the 2nd order derivative of

$$f(x) = \frac{\cos x}{1+\sin x}$$

$$f'(x) = \frac{(1+\sin x)(-\sin x) - \cos x\,(\cos x)}{(1+\sin x)^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$f'(x) = \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2} = \frac{-\sin x - 1}{(1+\sin x)^2}$$

$$= -\frac{\cancel{(1+\sin x)}}{(1+\sin x)^{\cancel{2}}} = -(1+\sin x)^{-1}$$

$$f''(x) = -\left[-(1+\sin x)^{-2}\cos x\right] = \frac{\cos x}{(1+\sin x)^2}$$

**Q # 03** Find the fourth order derivatives of

(i) $h(t) = 3t^7 - 6t^4 + 8t^3 - 12t + 18$

$$h'(t) = 21t^6 - 24t^3 + 24t^2 - 12$$

$$h''(t) = 126t^5 - 72t^2 + 48t$$

$$h'''(t) = 630t^4 - 144t + 48$$

$$h^{(4)}(t) = 2520t^3 - 144.$$

(ii) $f(x) = \sqrt[3]{x} - \frac{1}{8x^2} - \sqrt{x}$

$$f(x) = x^{1/3} - \frac{1}{8}x^{-2} - x^{1/2}$$

$$f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} - \frac{1}{8}\left(-2x^{-3}\right) - \frac{1}{2}x^{\frac{1}{2}-1}$$

$$= \frac{1}{3}x^{-\frac{2}{3}} + \frac{1}{4}x^{-3} - \frac{1}{2}x^{-\frac{1}{2}}$$

$$f''(x) = \frac{1}{3}\left(-\frac{2}{3}x^{-\frac{2}{3}-1}\right) + \frac{1}{4}\left(-3x^{-4}\right) - \frac{1}{2}\left(-\frac{1}{2}x^{-\frac{1}{2}-1}\right)$$

$$= \frac{-2}{9}x^{-5/3} - \frac{3}{4}x^{-4} + \frac{1}{4}x^{-3/2}$$

$$f'''(x) = -\frac{2}{9}\left(-\frac{5}{3}x^{-5/3-1}\right) - \frac{3}{\cancel{4}}\left(-\cancel{4}x^{-5}\right) + \frac{1}{4}\left(-\frac{3}{2}x^{-\frac{3}{2}-1}\right)$$

$$= \frac{10}{27}x^{-8/3} + 3x^{-5} - \frac{3}{8}x^{-\frac{5}{2}}$$

$$f^{(4)}(x) = \frac{10}{27}\left(-\frac{8}{3}x^{-\frac{8}{3}-1}\right) + 3\left(-5x^{-6}\right) - \frac{3}{8}\left(-\frac{5}{2}x^{-\frac{5}{2}-1}\right)$$

$$= \frac{-80}{81}x^{-\frac{11}{3}} - 15x^{-6} + \frac{15}{16}x^{-7/2}$$

$$= \frac{-80}{81}\cdot\frac{1}{x^{\frac{11}{3}}} - \frac{15}{x^6} + \frac{15}{16}\cdot\frac{1}{x^{\frac{7}{2}}}$$

$$= \frac{-80}{81\sqrt[3]{x^{11}}} - \frac{15}{x^6} + \frac{15}{16\sqrt{x^7}}$$

$\boxed{Q\ \#\ 04}$ Determine the fourth order derivative of

(i) $r(t) = 3t^2 + 8\sqrt{t}$

$$r'(t) = 3(2t) + 8\left(\frac{1}{2}t^{\frac{1}{2}-1}\right) = 6t + 4t^{-\frac{1}{2}}$$

$$r''(t) = 6 + 4\left(-\frac{1}{2}t^{-\frac{1}{2}-1}\right) = 6 - 2t^{-3/2}$$

$$r'''(t) = -\cancel{2}\left(\frac{-3}{\cancel{2}}t^{-\frac{3}{2}-1}\right) = 3t^{-\frac{5}{2}}$$

$$r^{(4)}(t) = 3\left(-\frac{5}{2}t^{-\frac{5}{2}-1}\right) = -\frac{15}{2}t^{-\frac{7}{2}}$$

$$= \frac{-15}{2t^{\frac{7}{2}}} = \frac{-15}{2\sqrt{t^7}}$$

(ii) $y = \cos x$

$y' = -\sin x$

$y'' = -\cos x$

$y''' = -(-\sin x) = \sin x$

$y^{(4)} = \cos x$

(iii) $f(y) = \sin 3y + e^{-2y} + \ln(7y).$

$f'(y) = 3\cos 3y - 2e^{-2y} + \frac{\cancel{7}}{\cancel{7}y}$

$= 3\cos 3y - 2e^{-2y} + y^{-1}$

$f''(y) = 3(-\sin 3y)(3) - 2(-2e^{-2y}) + (-y^{-2})$

$= -9\sin 3y + 4e^{-2y} - y^{-2}$

$f'''(y) = -9(\cos 3y)(3) + 4(-2e^{-2y}) - (-2y^{-3})$

$= -27\cos 3y - 8e^{-2y} + 2y^{-3}$

$f^{(4)}(y) = -27(-\sin 3y)(3) - 8(-2e^{-2y}) + 2(-3y^{-4})$

$= 81\sin 3y + 16e^{-2y} - 6y^{-4}$

$\boxed{Q\,\#05}$ If $x^2+y^2=10$, find $y''$.

$$x^2+y^2=10$$

$$(x^2+y^2)' = 10'$$

$$2x + 2yy' = 0$$

$$2yy' = -2x$$

$$y' = \frac{-\cancel{2}x}{\cancel{2}y}$$

$$y' = -\frac{x}{y}$$

$$y'' = -\left[\frac{yx' - xy'}{y^2}\right]$$

$$= -\frac{1}{y^2}\left(y - x\left(\frac{-x}{y}\right)\right) = \frac{-1}{y^2}\left(y + \frac{x^2}{y}\right)$$

$$= \frac{-1}{y^2}\left(\frac{y^2+x^2}{y}\right) = \frac{-(x^2+y^2)}{y^3}$$

Q # 06 Find $\frac{d^2y}{dx^2}$ if

(i) $2y^2 + 6x^2 = 76$

$$\frac{d}{dx}\left(2y^2 + 6x^2\right) = \frac{d}{dx}(76)$$

$$4y\frac{dy}{dx} + 12x = 0$$

$$\frac{dy}{dx} = \frac{-\cancel{12}^{3}x}{\cancel{4}y} = -3\frac{x}{y}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-3\frac{x}{y}\right)$$

$$\frac{d^2y}{dx^2} = -3\left(\frac{yx' - xy'}{y^2}\right) = \frac{-3}{y^2}\left(y - x\left(\frac{-3x}{y}\right)\right)$$

$$= -\frac{3}{y^2}\left(y - \frac{3x^2}{y}\right) = \frac{-3}{y^2}\left(\frac{y^2 - 3x^2}{y}\right)$$

$$= \frac{-3\left(y^2 - 3x^2\right)}{y^3} = \frac{3\left(3x^2 - y^2\right)}{y^3}.$$

(ii) $x^3 + y^3 = 1$

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(1)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\cancel{3}x^2}{\cancel{3}y^2} = -\frac{x^2}{y^2}$$

$$\frac{d^2y}{dx^2} = -\left(\frac{y^2(2x) - x^2(2yy')}{(y^2)^2}\right)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{y^4}\left(2xy^2 - 2x^2y\left(-\frac{x^2}{y^2}\right)\right)$$

$$= \frac{-1}{y^4}\left(2xy^2 + \frac{2x^4y}{y^2}\right)$$

$$= \frac{-1}{y^4}\left(\frac{2xy^4 + 2x^4y}{y^2}\right)$$

$$= -\frac{2xy\left(y^3 + x^3\right)}{y^6}$$

$$= \frac{-2x\left(x^3 + y^3\right)}{y^5}$$

$x = f(t)$ , $y = g(t)$

Q # 07 Find $\frac{d^2y}{dx^2}$ if

(i) $x = -5t^3 - 7$ , $y = 3t^2 + 16$

$$\frac{dx}{dt} = -15t^2 \quad , \quad \frac{dy}{dt} = 6t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\boxed{\frac{dy'}{dx} = \frac{dy'}{dt} \cdot \frac{dt}{dx}}$$

$$y' = \frac{dy}{dx} = {}^{2}\cancel{6t} \cdot \left(\frac{1}{-\cancel{15}t^{\cancel{2}}}\right) = -\frac{2}{5}t^{-1}$$

$$\frac{dy'}{dt} = \frac{-2}{5}\left(-t^{-2}\right) = \frac{2}{5t^2}$$

$$\frac{dy'}{dx} = \frac{dy'}{dt} \cdot \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{2}{5t^2} \cdot \left(\frac{1}{-15t^2}\right) = \frac{-2}{75t^4}.$$

(ii) $x = \cos\theta$ , $y = \sin\theta$

$$x^2 = \cos^2\theta \quad , \quad y^2 = \sin^2\theta$$

$$x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$$

$$x^2 + y^2 = 1$$

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(1)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y} = \frac{-x}{y}$$

$$\frac{d^2y}{dx^2} = -\left(\frac{yx' - xy'}{y^2}\right) = -\frac{1}{y^2}\left(y - x\left(\frac{-x}{y}\right)\right)$$

$$= -\frac{1}{y^2}\left(y + \frac{x^2}{y}\right) = -\frac{1}{y^2}\left(\frac{y^2 + x^2}{y}\right)$$

$$= -\frac{\left(x^2 + y^2\right)}{y^3}$$

$$= -\frac{1}{\sin^3\theta}$$

Q # 08 The derivative of the function $r(t)$ is given by

$$r'(t) = 6t + 4t^{-1/2} + e^t \;;$$

find $r''(t)$, $r'''(t)$ and $r^{(4)}(t)$.

$$r'(t) = 6t + 4t^{-1/2} + e^t$$

$$r''(t) = 6 + 4\left(-\frac{1}{2}t^{-\frac{1}{2}-1}\right) + e^t$$

$$r''(t) = 6 - 2t^{-\frac{3}{2}} + e^t$$

$$r'''(t) = 0 - 2\left(-\frac{3}{2}t^{-\frac{3}{2}-1}\right) + e^t$$

$$r'''(t) = 3t^{-\frac{5}{2}} + e^t$$

$$r^{(4)}(t) = 3\left(-\frac{5}{2}t^{-\frac{5}{2}-1}\right) + e^t$$

$$= -\frac{15}{2}t^{-\frac{7}{2}} + e^t .$$


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Q # 09 If $x^2+y^2=25$, then find $\frac{d^2y}{dx^2}$ at $(4,3)$.

$$x^2+y^2=25$$

$$(x^2+y^2)' = 25'$$

$$2x+2yy' = 0$$

$$y' = -\frac{\cancel{2}x}{\cancel{2}y} = -\frac{x}{y}$$

$$y'' = -\left(\frac{yx' - xy'}{y^2}\right) = -\frac{1}{y^2}\left(y - x\left(\frac{-x}{y}\right)\right)$$

$$= -\frac{1}{y^2}\left(y+\frac{x^2}{y}\right) = -\frac{1}{y^2}\left(\frac{y^2+x^2}{y}\right)$$

$$\frac{d^2y}{dx^2} = y'' = -\frac{25}{y^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(4,3)} = \frac{-25}{3^3} = \frac{-25}{27} \quad \text{Ans.}$$

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