Important Formulas for Exercise 3.4

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\operatorname{cosec} x)' = -\operatorname{cosec} x \cot x$

$(\sec x)' = \sec x \tan x$

$(\cot x)' = -\operatorname{cosec}^2 x$

$(\sin^{-1} x)' = \dfrac{1}{\sqrt{a^2 - x^2}}$

$(\cos^{-1} x)' = \dfrac{-1}{\sqrt{a^2 - x^2}}$

$(\tan^{-1} x)' = \dfrac{1}{1 + x^2}$

$(\operatorname{cosec}^{-1} x)' = \dfrac{-1}{x\sqrt{x^2-1}} \quad , \quad x > 0$

$(\sec^{-1} x)' = \dfrac{1}{x\sqrt{x^2-1}} \quad , \quad x > 0$

$(\cot^{-1} x)' = \dfrac{-1}{1 + x^2}$

$(\sin ax)' = (\cos ax)\,(ax)'$

Chain Rule

$y = f(x)$

$y \xrightarrow{\frac{dy}{dx}} x$

$y = y(u), \quad u = u(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$y \xrightarrow{\frac{dy}{du}} u \xrightarrow{\frac{du}{dx}} x$

Exercise 3.4

1. Find the derivative using chain rule.

(i) $y = (x^4 + 5x^2 + 6)^{\frac{3}{2}}$

Let $u = x^4 + 5x^2 + 6$, $y = u^{3/2}$

$\frac{du}{dx} = 4x^3 + 5(2x) + 0$, $\frac{dy}{du} = \frac{3}{2} u^{\frac{3}{2}-1}$

$= 4x^3 + 10x$, $= \frac{3}{2} u^{\frac{1}{2}}$

$= \frac{3}{2}(x^4 + 5x^2 + 6)^{\frac{1}{2}}$

So $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \frac{3}{2}(x^4 + 5x^2 + 6)^{\frac{1}{2}} \cdot (4x^3 + 10x)$

$= \frac{3}{\cancel{2}}(x^4 + 5x^2 + 6)^{\frac{1}{2}} \cancel{2}x(2x^2 + 5)$

$= 3x(x^4 + 5x^2 + 6)^{\frac{1}{2}}(2x^2 + 5)$ Ans.

(ii) $y = \left(\frac{x-1}{x+1}\right)^{\frac{3}{4}}$

Let $u = \frac{x-1}{x+1}$, then $y = u^{\frac{3}{4}}$

$\frac{du}{dx} = \frac{(x+1)(x-1)' - (x-1)(x+1)'}{(x+1)^2}$ $\qquad$ $\frac{dy}{du} = \frac{3}{4} u^{\frac{3}{4}-1}$

$= \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2}$ $\qquad$ $= \frac{3}{4} u^{-\frac{1}{4}}$

$= \frac{\cancel{x}+1 - \cancel{x}+1}{(x+1)^2} = \frac{2}{(x+1)^2}$ $\qquad$ $= \frac{3}{4}\left(\frac{x-1}{x+1}\right)^{-\frac{1}{4}}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$-\frac{1}{4} + 2 = \frac{-1+8}{4} = \frac{7}{4}$

$= \frac{3}{\cancel{4}_{2}}\left(\frac{x-1}{x+1}\right)^{-\frac{1}{4}} \cdot \frac{\cancel{2}}{(x+1)^2} = \frac{3}{2} \frac{(x-1)^{-\frac{1}{4}}}{(x+1)^{-\frac{1}{4}}(x+1)^2}$

$= \frac{3}{2(x+1)^{\frac{1}{4}+2}(x-1)^{\frac{1}{4}}} = \frac{3}{2(x+1)^{7/4}(x-1)^{1/4}}$. Ans

(iii) $y = \sqrt{\dfrac{2+x}{3+x}}$

Let $u = \dfrac{2+x}{3+x}$, then $y = \sqrt{u} = u^{1/2}$

$$\frac{du}{dx} = \frac{(3+x)(1) - (2+x)(1)}{(3+x)^2}$$

$$= \frac{3+\cancel{x}-2-\cancel{x}}{(3+x)^2}$$

$$= \frac{1}{(3+x)^2}$$

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1} = \frac{1}{2}u^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(\frac{2+x}{3+x}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\,\frac{(2+x)^{-\frac{1}{2}}}{(3+x)^{-1/2}}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{2}\,\frac{(2+x)^{-\frac{1}{2}}}{(3+x)^{-\frac{1}{2}}}\cdot\frac{1}{(3+x)^2}$$

$$= \frac{1}{2(3+x)^{-\frac{1}{2}+2}(2+x)^{\frac{1}{2}}} = \frac{1}{2(3+x)^{3/2}(2+x)^{1/2}}$$ Ans.

(iv) $y = \left(x+\sqrt{x^2-1}\right)^n$

$(f^n)' = nf^{n-1}\cdot f'$

Let $u = x + \sqrt{x^2-1}$, $y = u^n$

$$\frac{du}{dx} = 1 + \frac{1}{2}(x^2-1)^{\frac{1}{2}-1}\cdot(x^2-1)'$$

$$= 1 + \frac{1}{\cancel{2}}(x^2-1)^{-\frac{1}{2}}(\cancel{2}x)$$

$$= 1 + \frac{x}{\sqrt{x^2-1}}$$

$$= \frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}$$

$$\frac{dy}{du} = nu^{n-1}$$

$$= n\left(x+\sqrt{x^2-1}\right)^{n-1}$$

So $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= n\left(x+\sqrt{x^2-1}\right)^{n-1}\cdot\left(\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}\right)$$

$$= \frac{n\left(x+\sqrt{x^2-1}\right)^n}{\sqrt{x^2-1}}$$ Ans.

(v) $y = \sqrt[3]{\dfrac{x^3+1}{x^3-1}}$

Let $u = \dfrac{x^3+1}{x^3-1}$, then $y = \sqrt[3]{u} = u^{1/3}$

$$\frac{du}{dx} = \frac{(x^3-1)(3x^2)-(x^3+1)(3x^2)}{(x^3-1)^2}$$

$$= \frac{3x^2(\cancel{x^3}-1-\cancel{x^3}-1)}{(x^3-1)^2}$$

$$= \frac{3x^2(-2)}{(x^3-1)^2}$$

$$= \frac{-6x^2}{(x^3-1)^2}$$

$$\frac{dy}{du} = \frac{1}{3}u^{\frac{1}{3}-1} = \frac{1}{3}u^{-2/3}$$

$$\frac{dy}{du} = \frac{1}{3}\left(\frac{x^3+1}{x^3-1}\right)^{-2/3}$$

$$= \frac{1}{3}\,\frac{(x^3+1)^{-2/3}}{(x^3-1)^{-2/3}}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{\cancel{3}}\,\frac{(x^3+1)^{-2/3}}{(x^3-1)^{-2/3}}\cdot\frac{-\overset{2}{\cancel{6}}x^2}{(x^3-1)^2}$$

$$-\frac{2}{3}+2 = \frac{-2+6}{3} = \frac{4}{3}$$

$$= \frac{-2x^2}{(x^3-1)^{-\frac{2}{3}+2}\,(x^3+1)^{2/3}}$$

$$= \frac{-2x^2}{(x^3-1)^{4/3}\,(x^3+1)^{2/3}} \quad \text{Ans.}$$

② Differentiate $\frac{x^3}{1+x^3}$ w.r.t. x^3.

$\boxed{\frac{du}{dv} = ?}$

Let $u = \frac{x^3}{1+x^3}$ and $v = x^3$

$$\frac{du}{dx} = \frac{(1+x^3)(3x^2) - x^3(3x^2)}{(1+x^3)^2} \qquad \frac{dv}{dx} = 3x^2$$

$$= \frac{3x^2(1+\cancel{x^3}-\cancel{x^3})}{(1+x^3)^2}$$

$$= \frac{3x^2}{(1+x^3)^2}$$

$$\frac{du}{dv} = \frac{du}{dx} \cdot \frac{dx}{dv}$$

$$= \frac{\cancel{3x^2}}{(1+x^3)^2} \cdot \frac{1}{\cancel{3x^2}} = \frac{1}{(1+x^3)^2}$$ Ans.

Another Method.

Differentiate $\frac{x^3}{1+x^3}$ w.r.t. x^3.

Let $u = \frac{x^3}{1+x^3}$, $v = x^3$

$$\Rightarrow \quad u = \frac{v}{1+v}$$

$$\frac{du}{dv} = \frac{(1+v)(1) - v(1)}{(1+v)^2} = \frac{1+\cancel{v}-\cancel{v}}{(1+v)^2}$$

$$= \frac{1}{(1+x^3)^2}$$

③ Find $\frac{dy}{dx}$

(i) $y - xy - \sin y = 0$

$$\frac{d}{dx}(y - xy - \sin y) = \frac{d}{dx}(0)$$

$\frac{d}{dx}(\sin x) = \cos x$

$$\frac{dy}{dx} - \frac{d}{dx}(xy) - \frac{d}{dx}(\sin y) = 0$$

$$\frac{dy}{dx} - \left[x\frac{dy}{dx} + y\frac{dx}{dx}\right] - \cos y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} - x\frac{dy}{dx} - y - \cos y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} - x\frac{dy}{dx} - \cos y\frac{dy}{dx} = y$$

$$\frac{dy}{dx}(1 - x - \cos y) = y$$

$$\frac{dy}{dx} = \frac{y}{1 - x - \cos y}$$

Ans.

(ii) $y^3 - 3y + 2x = 0$

$$\frac{d}{dx}(y^3 - 3y + 2x) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(y^3) - \frac{d}{dx}(3y) + \frac{d}{dx}(2x) = 0$$

$$3y^2\frac{dy}{dx} - 3\frac{dy}{dx} + 2 = 0$$

$$3y^2\frac{dy}{dx} - 3\frac{dy}{dx} = -2$$

$$\frac{dy}{dx}(3y^2 - 3) = -2$$

$$\frac{dy}{dx} = \frac{-2}{3y^2 - 3} = \frac{-2}{3(y^2-1)}.$$

(iii) $x^2 + y^2 + 4x + 6y - 12 = 0$

$$\frac{d}{dx}\left(x^2 + y^2 + 4x + 6y - 12\right) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) + \frac{d}{dx}(4x) + \frac{d}{dx}(6y) - \frac{d}{dx}(12) = 0$$

$$2x + 2y\frac{dy}{dx} + 4 + 6\frac{dy}{dx} - 0 = 0$$

$$2y\frac{dy}{dx} + 6\frac{dy}{dx} = -2x - 4$$

$$\frac{dy}{dx}(2y + 6) = -2x - 4$$

$$\frac{dy}{dx} = \frac{-2x-4}{2y+6} = \frac{-\cancel{2}(x+2)}{\cancel{2}(y+3)} = \frac{-(x+2)}{y+3}$$

(iv) $\sin xy + \sec x = 2$

$$\frac{d}{dx}\left(\sin xy + \sec x\right) = \frac{d}{dx}(2)$$

$$\frac{d}{dx}(\sin xy) + \frac{d}{dx}(\sec x) = 0$$

$$(\cos xy)\frac{d}{dx}(xy) + \sec x \tan x = 0$$

$$(\cos xy)\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + \sec x \tan x = 0$$

$$x(\cos xy)\frac{dy}{dx} + y\cos xy + \sec x \tan x = 0$$

$$x\cos xy \cdot \frac{dy}{dx} = -y\cos xy - \sec x \tan x$$

$$\frac{dy}{dx} = \frac{-y\cos xy - \sec x \tan x}{x\cos xy}$$

Ans.

(v) $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$$\frac{d}{dx}\left(x(1+y)^{\frac{1}{2}} + y(1+x)^{\frac{1}{2}}\right) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}\left[x\cdot(1+y)^{\frac{1}{2}}\right] + \frac{d}{dx}\left[y\,(1+x)^{\frac{1}{2}}\right] = 0$$

$$x\frac{d}{dx}(1+y)^{\frac{1}{2}} + (1+y)^{\frac{1}{2}}\cdot 1 + y\frac{d}{dx}(1+x)^{\frac{1}{2}} + (1+x)^{\frac{1}{2}}\cdot\frac{dy}{dx} = 0$$

$$x\cdot\frac{1}{2}(1+y)^{\frac{1}{2}-1}\frac{d}{dx}(1+y) + (1+y)^{\frac{1}{2}} + y\cdot\frac{1}{2}(1+x)^{\frac{1}{2}-1}\cdot\frac{d}{dx}(1+x) + (1+x)^{\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\frac{x}{2}(1+y)^{-\frac{1}{2}}\frac{dy}{dx} + (1+y)^{\frac{1}{2}} + \frac{y}{2}(1+x)^{-\frac{1}{2}}\cdot 1 + (1+x)^{\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\frac{x}{2}\cdot\frac{1}{\sqrt{1+y}}\frac{dy}{dx} + \sqrt{1+x}\frac{dy}{dx} = -\sqrt{1+y} - \frac{y}{2}\cdot\frac{1}{\sqrt{1+x}}$$

$$\frac{dy}{dx}\left[\frac{x}{2\sqrt{1+y}} + \sqrt{1+x}\right] = -\left[\sqrt{1+y} + \frac{y}{2\sqrt{1+x}}\right]$$

$$\frac{dy}{dx}\left[\frac{x+2\sqrt{1+x}\sqrt{1+y}}{2\sqrt{1+y}}\right] = -\left[\frac{2\sqrt{1+x}\sqrt{1+y}+y}{2\sqrt{1+x}}\right]$$

$$\frac{dy}{dx} = -\left[\frac{\cancel{2}\sqrt{1+y}}{x+2\sqrt{1+x}\sqrt{1+y}}\right]\left[\frac{y+2\sqrt{1+x}\sqrt{1+y}}{\cancel{2}\sqrt{1+x}}\right]$$

$$\frac{dy}{dx} = -\frac{\sqrt{1+y}\left[y+2\sqrt{1+x}\sqrt{1+y}\right]}{\sqrt{1+x}\left[x+2\sqrt{1+x}\sqrt{1+y}\right]}$$ Ans.

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(vi) $$y(x^2+1) = x(y^2+1)$$

$$x^2y + y = xy^2 + x$$

$$\frac{d}{dx}(x^2y+y) = \frac{d}{dx}(xy^2+x)$$

$$\frac{d}{dx}(\underline{x^2}\,\underline{y}) + \frac{d}{dx}(y) = \frac{d}{dx}(\underline{x}\,\underline{y^2}) + \frac{d}{dx}(x)$$

$$x^2\frac{dy}{dx} + y\frac{d}{dx}(x^2) + \frac{dy}{dx} = x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x) + 1$$

$$x^2\frac{dy}{dx} + y(2x) + \frac{dy}{dx} = x\left(2y\frac{dy}{dx}\right) + y^2 + 1$$

$$x^2\frac{dy}{dx} + 2xy + \frac{dy}{dx} = 2xy\frac{dy}{dx} + y^2 + 1$$

$$x^2\frac{dy}{dx} + \frac{dy}{dx} - 2xy\frac{dy}{dx} = y^2 + 1 - 2xy$$

$$\frac{dy}{dx}(x^2+1-2xy) = y^2+1-2xy$$

$$\frac{dy}{dx} = \frac{y^2+1-2xy}{x^2+1-2xy}$$

Ans.

4. If $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$, where a and b are nonzero constants, find $\frac{du}{dv}$ and $\frac{dv}{du}$.

Given $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$

$$\frac{d}{dv}\left(\frac{u^2}{a^2} + \frac{v^2}{b^2}\right) = \frac{d}{dv}(1)$$

$$\frac{d}{dv}\left(\frac{u^2}{a^2}\right) + \frac{d}{dv}\left(\frac{v^2}{b^2}\right) = 0$$

$$\frac{1}{a^2}\frac{d}{dv}(u^2) + \frac{1}{b^2}\frac{d}{dv}(v^2) = 0$$

$$\frac{1}{a^2}\cdot 2u\frac{du}{dv} + \frac{1}{b^2}\cdot 2v = 0$$

$$\frac{2u}{a^2}\frac{du}{dv} = \frac{-2v}{b^2}$$

$$\frac{du}{dv} = -\frac{\cancel{2}v}{b^2}\cdot\frac{a^2}{\cancel{2}u} = -\frac{a^2 v}{b^2 u}$$

$$\frac{dv}{du} = -\frac{b^2 u}{a^2 v}$$

⑤ Find the slope of the tangent to the curve

$$3x^2 - 7y^2 + 14y - 27 = 0$$

at the point $(-3,0)$.

$\left.\frac{dy}{dx}\right|_{(a,b)}$ = slope of tangent at (a,b).

$$3x^2 - 7y^2 + 14y - 27 = 0$$

$$\frac{d}{dx}\left(3x^2 - 7y^2 + 14y - 27\right) = \frac{d}{dx}(0)$$

$$6x - 7\left(2y\frac{dy}{dx}\right) + 14\frac{dy}{dx} - 0 = 0$$

$$6x - 14y\frac{dy}{dx} + 14\frac{dy}{dx} = 0$$

$$-14y\frac{dy}{dx} + 14\frac{dy}{dx} = -6x$$

$$\frac{dy}{dx}\left(-14y + 14\right) = -6x$$

$$\frac{dy}{dx} = \frac{-6x}{-14y+14}$$

Slope of tangent at $(-3,0) = \left.\frac{dy}{dx}\right|_{(-3,0)} = \frac{-6(-3)}{-14(0)+14} = \frac{\cancel{18}^{\,9}}{\cancel{14}_{\,7}} = \frac{9}{7}$

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⑥ Differentiate by using first principle.

(i) $\sin 4x$

Let $f(x) = \sin 4x$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\sin 4(x+\delta x) - \sin 4x\right]$$

$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\sin(4x+4\delta x) - \sin 4x\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\, 2\cos\left(\frac{4x+4\delta x+4x}{2}\right)\sin\left(\frac{\cancel{4x}+4\delta x-\cancel{4x}}{2}\right)$$

$$= \lim_{\delta x \to 0} 2\cos(4x+2\delta x)\,\frac{\sin(2\delta x)}{2\delta x}\cdot 2$$

$$= \left[\lim_{\delta x \to 0} 2\cos(4x+2\delta x)\right]\left[\lim_{\delta x \to 0}\frac{\sin(2\delta x)}{2\delta x}\right]\left[\lim_{\delta x \to 0} 2\right]$$

$$= 2\cos(4x+0)\,(1)\cdot 2$$

$$= 4\cos 4x \qquad \text{Ans.}$$

(ii) Let $f(x) = \cos^2 2x$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cos^2 2(x+\delta x) - \cos^2 2x\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cos^2(2x+2\delta x) - \cos^2 2x\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cos(2x+2\delta x)+\cos 2x\right]\left[\cos(2x+2\delta x)-\cos 2x\right]$$

$\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cos(2x+2\delta x)+\cos 2x\right]\left[-2\sin\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{\cancel{2x}+2\delta x-\cancel{2x}}{2}\right)\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cos(2x+2\delta x)+\cos 2x\right]\left[-2\sin(2x+\delta x)\sin\delta x\right]$$

$$= \lim_{\delta x \to 0}\left[\cos(2x+2\delta x)+\cos 2x\right]\left[-2\sin(2x+\delta x)\,\frac{\sin\delta x}{\delta x}\right]$$

$$= \left[\cos 2x + \cos 2x\right]\left[-2\sin 2x\cdot 1\right]$$

$$= -2\sin 2x\,(2\cos 2x)$$

$$= -4\sin 2x\cos 2x \qquad \text{Ans.}$$

(iii) Let $f(x) = \sec\sqrt{x}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\sec\sqrt{x+\delta x} - \sec\sqrt{x}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{1}{\cos\sqrt{x+\delta x}} - \frac{1}{\cos\sqrt{x}}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{\cos\sqrt{x} - \cos\sqrt{x+\delta x}}{\cos\sqrt{x+\delta x}\,\cos\sqrt{x}}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x\left(\cos\sqrt{x+\delta x}\,\cos\sqrt{x}\right)}\left[\cos\sqrt{x} - \cos\sqrt{x+\delta x}\right]$$

$$\boxed{\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x\left(\cos\sqrt{x+\delta x}\cos\sqrt{x}\right)}\left[-2\sin\left(\frac{\sqrt{x}+\sqrt{x+\delta x}}{2}\right)\sin\left(\frac{\sqrt{x}-\sqrt{x+\delta x}}{2}\right)\right]$$

$$= \lim_{\delta x \to 0} \frac{+2\sin\left(\frac{\sqrt{x}+\sqrt{x+\delta x}}{2}\right)}{\cos\sqrt{x+\delta x}\,\cos\sqrt{x}} \cdot \frac{\sin\left(\frac{\sqrt{x}-\sqrt{x+\delta x}}{2}\right)}{x-x-\delta x}$$

$$x - x - \delta x$$
$$= x - (x+\delta x)$$
$$= (\sqrt{x})^2 - (\sqrt{x+\delta x})^2$$
$$= (\sqrt{x}+\sqrt{x+\delta x})(\sqrt{x}-\sqrt{x+\delta x})$$

$$= \lim_{\delta x \to 0}\left[\frac{\cancel{2}\sin\left(\frac{\sqrt{x}+\sqrt{x+\delta x}}{2}\right)}{\cos\sqrt{x+\delta x}\,\cos\sqrt{x}\left(\sqrt{x}+\sqrt{x+\delta x}\right)\cancel{2}}\right]\left[\frac{\sin\left(\frac{\sqrt{x}-\sqrt{x+\delta x}}{2}\right)}{\left(\frac{\sqrt{x}-\sqrt{x+\delta x}}{2}\right)}\right]$$

$$= \frac{\sin\left(\frac{\sqrt{x}+\sqrt{x}}{2}\right)}{\cos\sqrt{x}\,\cos\sqrt{x}\left(\sqrt{x}+\sqrt{x}\right)} \cdot 1$$

$$= \frac{\sin\sqrt{x}}{2\sqrt{x}\,\cos\sqrt{x}\cdot\cos\sqrt{x}} = \frac{1}{2\sqrt{x}}\tan\sqrt{x}\cdot\sec\sqrt{x} \quad \text{Ans.}$$

(iv) Let $f(x) = \sqrt{\tan x}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\sqrt{\tan(x+\delta x)} - \sqrt{\tan x}\right] \times \frac{\left(\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}\right)}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x\left(\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}\right)}\left[\tan(x+\delta x) - \tan x\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x\left(\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}\right)}\left[\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x\left(\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}\right)}\left[\frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{\cos(x+\delta x)\cos x}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\left(\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}\right)} \cdot \frac{1}{\cos(x+\delta x)\cos x} \cdot \frac{\sin(\cancel{x} + \delta x - \cancel{x})}{\delta x}$$

$\boxed{\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)}$

$$= \frac{1}{\sqrt{\tan x} + \sqrt{\tan x}} \cdot \frac{1}{\cos x \cdot \cos x} \cdot 1$$

$$= \frac{1}{2\sqrt{\tan x}} \cdot \frac{1}{\cos^2 x} = \frac{1}{2}\sqrt{\cot x}\ \sec^2 x.$$

Ans

(v) Let $f(x) = \operatorname{cosec} 3x$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\operatorname{Cosec}(3x+3\delta x) - \operatorname{Cosec} 3x\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{1}{\sin(3x+3\delta x)} - \frac{1}{\sin 3x}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{\sin 3x - \sin(3x+3\delta x)}{\sin(3x+3\delta x)\sin 3x}\right]$$

$$\boxed{\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x\,(\sin(3x+3\delta x)\sin 3x)} \cdot 2\cos\left(\frac{3x+3x+3\delta x}{2}\right)\sin\left(\frac{\cancel{3x}-\cancel{3x}-3\delta x}{2}\right)$$

$$= \lim_{\delta x \to 0} \left(\frac{2\cos\left(3x+\frac{3}{2}\delta x\right)}{\sin(3x+3\delta x)\sin 3x}\right) \cdot \left(\frac{\sin\left(-\frac{3}{2}\delta x\right)}{-\frac{3}{2}\delta x}\right) \cdot \left(-\frac{3}{2}\right)$$

$$= \frac{\cancel{2}\cos 3x}{\sin 3x \cdot \sin 3x} \cdot 1 \cdot \left(-\frac{3}{\cancel{2}}\right)$$

$$= -3\,\frac{\cos 3x}{\sin 3x} \cdot \frac{1}{\sin 3x} = -3\cot 3x \cdot \operatorname{Cosec} 3x$$

Ans.

(vi) Let $f(x) = \cot 2x$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cot(2x+2\delta x) - \cot 2x\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{\cos(2x+2\delta x)}{\sin(2x+2\delta x)} - \frac{\cos 2x}{\sin 2x}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{\sin 2x\cos(2x+2\delta x) - \cos 2x\sin(2x+2\delta x)}{\sin(2x+2\delta x)\sin 2x}\right]$$

$$= \lim_{\delta x \to 0} \left(\frac{1}{(\sin(2x+2\delta x)\sin 2x)}\right) \cdot \left(\frac{\sin(\cancel{2x}-\cancel{2x}-2\delta x)}{-2\delta x}\right) \cdot \left((-2)\right)$$

$$= \frac{1}{\sin 2x \cdot \sin 2x} \cdot 1 \cdot (-2)$$

$$= -2\operatorname{Cosec}^2 2x$$

Ans.

7. Using differentiation rules, differentiate w.r.t. involved variables:

(i) $f(x) = (x+2)\cdot \sin x$

$$f'(x) = \left((x+2)\cdot \sin x\right)'$$

$$= (x+2)(\sin x)' + \sin x \cdot (x+2)'$$

$$= (x+2)\cos x + \sin x \cdot (1+0)$$

$$= (x+2)\cos x + \sin x \quad \text{Ans.}$$

(ii) $f(\theta) = \tan^2\theta \cdot \sec^3\theta$

$$\frac{d}{d\theta} f(\theta) = \frac{d}{d\theta}\left(\tan^2\theta \cdot \sec^3\theta\right)$$

$$= \tan^2\theta \cdot \frac{d}{d\theta}\left(\sec^3\theta\right) + \sec^3\theta \cdot \frac{d}{d\theta}\left(\tan^2\theta\right)$$

$$= \tan^2\theta \cdot 3\sec^2\theta \cdot \frac{d}{d\theta}(\sec\theta) + \sec^3\theta \cdot 2\tan\theta \cdot \frac{d}{d\theta}(\tan\theta)$$

$$= 3\tan^2\theta \sec^2\theta \cdot \sec\theta\tan\theta + 2\sec^3\theta \tan\theta \sec^2\theta$$

$$= 3\tan^3\theta \sec^3\theta + 2\tan\theta \sec^5\theta$$

$$= \tan\theta \sec^3\theta \left(3\tan^2\theta + 2\sec^2\theta\right)$$

Ans.

(iii) $f(t) = \sin^2 3t \cdot \cos^3 t$

$$\frac{d}{dt} f(t) = \frac{d}{dt}\left(\sin^3 3t \cdot \cos^3 t\right)$$

$$= \sin^3 3t \cdot \frac{d}{dt}\left(\cos^3 t\right) + \cos^3 t \cdot \frac{d}{dt}\left(\sin^3 3t\right)$$

$$= \sin^3 3t \cdot 3\cos^2 t \cdot \frac{d}{dt}(\cos t) + \cos^3 t \cdot 3\sin^2 3t \cdot \frac{d}{dt}(\sin 3t)$$

$$= 3\sin^3 3t \cos^2 t\,(-\sin t) + 3\cos^3 t \sin^2 3t\,(\cos 3t)\frac{d}{dt}(3t)$$

$$= -3\sin^3 3t \cos^2 t \sin t + 3\cos^3 t \sin^2 3t \cos 3t\,(3)$$

$$= -3\sin^3 3t \cos^2 t \sin t + 9\cos^3 t \sin^2 3t \cos 3t$$

(iv) $f(x) = \sqrt{\dfrac{\sin 2x}{\cos x}}$

$$f(x) = \sqrt{\frac{2\sin x \cancel{\cos x}}{\cancel{\cos x}}}$$

$$f(x) = \sqrt{2}\ \sqrt{\sin x}$$

$$f'(x) = \sqrt{2}\left((\sin x)^{\frac{1}{2}}\right)'$$

$$= \sqrt{2}\cdot\frac{1}{2}(\sin x)^{\frac{1}{2}-1}\cdot(\sin x)'$$

$$= \frac{\sqrt{2}}{2}(\sin x)^{-\frac{1}{2}}\cos x = \frac{\sqrt{2}\cos x}{2\sqrt{\sin x}}$$ Ans.

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(v) $f(\theta) = \dfrac{\tan\theta - 1}{\sec\theta}$

$$\frac{d}{d\theta}f(\theta) = \frac{d}{d\theta}\left(\frac{\tan\theta - 1}{\sec\theta}\right)$$

$$= \frac{(\sec\theta)\frac{d}{d\theta}(\tan\theta - 1) - (\tan\theta - 1)\frac{d}{d\theta}(\sec\theta)}{\sec^2\theta}$$

$$= \frac{\sec\theta(\sec^2\theta - 0) - (\tan\theta - 1)\sec\theta\tan\theta}{\sec^2\theta}$$

$1+\tan^2\theta = \sec^2\theta$
$1 = \sec^2\theta - \tan^2\theta$

$$= \frac{\sec^3\theta - \tan^2\theta\sec\theta + \sec\theta\tan\theta}{\sec^2\theta}$$

$$= \frac{\cancel{\sec\theta}(\sec^2\theta - \tan^2\theta + \tan\theta)}{\sec^{\cancel{2}}\theta}$$

$$= \frac{1+\tan\theta}{\sec\theta}$$ Ans.

(vi) $f(x) = \sin x^2 + \sin^2 x$

$$f'(x) = (\sin x^2)' + (\sin^2 x)'$$

$$= (\cos x^2)(x^2)' + 2\sin x\,(\sin x)'$$

$$= (\cos x^2)(2x) + 2\sin x\cos x$$

$$= 2x\cos x^2 + 2\sin x\cos x.$$

8. Differentiate $\dfrac{1+\tan^2 x}{1-\tan^2 x}$ w.r.t. $\tan^2 x$.

Let $u = \dfrac{1+\tan^2 x}{1-\tan^2 x}$, $v = \tan^2 x$

$$u = \frac{1+v}{1-v}$$

$$\frac{du}{dv} = \frac{d}{dv}\left(\frac{1+v}{1-v}\right)$$

$$= \frac{(1-v)\frac{d}{dv}(1+v) - (1+v)\frac{d}{dv}(1-v)}{(1-v)^2}$$

$$= \frac{(1-v)(0+1) - (1+v)(0-1)}{(1-v)^2} = \frac{1-\cancel{v}+1+\cancel{v}}{(1-v)^2}$$

$$\frac{du}{dv} = \frac{2}{(1-\tan^2 x)^2} \quad \text{Ans}$$

Alternative

$u = \dfrac{1+\tan^2 x}{1-\tan^2 x}$, $v = \tan^2 x$

$$\frac{du}{dx} = \frac{(1-\tan^2 x)(0+2\tan x\sec^2 x) - (1+\tan^2 x)(0-2\tan x\sec^2 x)}{(1-\tan^2 x)^2}$$

$$\frac{du}{dx} = \frac{1}{(1-\tan^2 x)^2}\left[2\tan x\sec^2 x - \cancel{2\tan^3 x\sec^2 x} + 2\tan x\sec^2 x + \cancel{2\tan^3 x\sec^2 x}\right]$$

$$\frac{du}{dx} = \frac{4\tan x\sec^2 x}{(1-\tan^2 x)^2}$$

$v = \tan^2 x$

$\dfrac{dv}{dx} = 2\tan x\sec^2 x$

$$\frac{du}{dv} = \frac{du}{dx}\cdot\frac{dx}{dv}$$

$$= \frac{\overset{2}{\cancel{4\tan x\sec^2 x}}}{(1-\tan^2 x)^2}\cdot\frac{1}{\cancel{2\tan x\sec^2 x}}$$

$$= \frac{2}{(1-\tan^2 x)^2} \quad \text{Ans.}$$

(9) Find $\frac{dy}{dx}$ of the following:

(i) $y = \sin^{-1}\sqrt{\frac{1-\cos x}{2}}$

$$2\sin^2 x = 1-\cos 2x$$
$$2\sin^2\frac{x}{2} = 1-\cos x$$
$$\sin^2\frac{x}{2} = \frac{1-\cos x}{2}$$

$$y = \sin^{-1}\sqrt{\sin^2\frac{x}{2}}$$
$$= \sin^{-1}\left(\sin\frac{x}{2}\right)$$
$$y = \frac{x}{2}$$
$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}\frac{d}{dx}(x) = \frac{1}{2}$$ Ans.

(ii) $y = \cot^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right)$

$$2\cos^2 x = 1+\cos 2x$$
$$2\cos^2\frac{x}{2} = 1+\cos x$$

$$2\sin^2 x = 1-\cos 2x$$
$$2\sin^2\frac{x}{2} = 1-\cos x$$

$$y = \cot^{-1}\left(\sqrt{\frac{\cancel{2}\cos^2\frac{x}{2}}{\cancel{2}\sin^2\frac{x}{2}}}\right)$$
$$y = \cot^{-1}\sqrt{\cot^2\frac{x}{2}}$$
$$y = \cot^{-1}\left(\cot\frac{x}{2}\right)$$
$$y = \frac{x}{2}$$
$$\frac{dy}{dx} = \frac{1}{2}$$ Ans.

(iii) $y = \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)$

$$y = \tan^{-1}\left(\frac{\cancel{2}\sin x\,\cancel{\cos x}}{\cancel{2}\cos^{\cancel{2}} x}\right)$$
$$y = \tan^{-1}(\tan x)$$
$$y = x$$
$$\frac{dy}{dx} = 1$$ Ans.

(iv) $y = x + (\cos^{-1}x)\cdot\left(\sqrt{1-x^2}\right)$

$$\frac{dy}{dx} = \frac{d}{dx}\left[x + (\cos^{-1}x)\left(\sqrt{1-x^2}\right)\right]$$

$$= \frac{d}{dx}(x) + \frac{d}{dx}\left[(\cos^{-1}x)\left(\sqrt{1-x^2}\right)\right]$$

$$= 1 + \cos^{-1}x \cdot \frac{1}{2}(1-x^2)^{\frac{1}{2}-1}\frac{d}{dx}(1-x^2) + \cancel{\sqrt{1-x^2}} \cdot \frac{-1}{\cancel{\sqrt{1-x^2}}}$$

$$= \cancel{1} + \cos^{-1}x \cdot \frac{1}{\cancel{2}}\frac{1}{\sqrt{1-x^2}}(-\cancel{2}x) \cancel{-1}$$

$$= \frac{-x\cos^{-1}x}{\sqrt{1-x^2}}$$

(v) $y = \dfrac{x\tan^{-1}x}{1+x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x\tan^{-1}x}{1+x^2}\right)$$

$$= \frac{(1+x^2)\frac{d}{dx}(x\tan^{-1}x) - (x\tan^{-1}x)\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{1}{(1+x^2)^2}\left[(1+x^2)\left(x\cdot\frac{1}{1+x^2} + \tan^{-1}x\right) - (x\tan^{-1}x)(2x)\right]$$

$$= \frac{1}{(1+x^2)^2}\left[\frac{x\,\cancel{(1+x^2)}}{\cancel{1+x^2}} + (1+x^2)\tan^{-1}x - 2x^2\tan^{-1}x\right]$$

$$= \frac{1}{(1+x^2)^2}\left[x + (1+x^2-2x^2)\tan^{-1}x\right]$$

$$= \frac{x + (1-x^2)\tan^{-1}x}{(1+x^2)^2}$$ Ans.

(vi) $y = \tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$$y = \tan^{-1}\left(\frac{(\sqrt{1+x}-\sqrt{1-x})}{\sqrt{1+x}+\sqrt{1-x}} \times \frac{(\sqrt{1+x}-\sqrt{1-x})}{\sqrt{1+x}-\sqrt{1-x}}\right)$$

$$y = \tan^{-1}\left(\frac{(\sqrt{1+x})^2+(\sqrt{1-x})^2-2\sqrt{1+x}\sqrt{1-x}}{(\sqrt{1+x})^2-(\sqrt{1-x})^2}\right)$$

$$y = \tan^{-1}\left(\frac{1+\cancel{x}+1-\cancel{x}-2\sqrt{1-x^2}}{\cancel{1}+x-\cancel{1}+x}\right)$$

$$y = \tan^{-1}\left(\frac{\cancel{2}(1-\sqrt{1-x^2})}{\cancel{2}x}\right)$$

$\boxed{\left(\tan^{-1}f(x)\right)' = \dfrac{1}{1+f(x)^2}\cdot f'(x)}$

$$y = \tan^{-1}\left(\frac{1-(1-x^2)^{1/2}}{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{1-(1-x^2)^{1/2}}{x}\right)^2}\cdot\frac{d}{dx}\left(\frac{1-(1-x^2)^{1/2}}{x}\right)$$

$$= \frac{1}{1+\frac{(1-\sqrt{1-x^2})^2}{x^2}}\cdot\frac{x\left(0-\frac{1}{\cancel{2}}(1-x^2)^{\frac{1}{2}-1}(-\cancel{2}x)\right)-\left(1-(1-x^2)^{\frac{1}{2}}\right)\cdot 1}{x^2}$$

$$= \frac{1}{\frac{x^2+(1-\sqrt{1-x^2})^2}{\cancel{x^2}}}\cdot\frac{\left[\frac{x^2}{\sqrt{1-x^2}}-(1-\sqrt{1-x^2})\right]}{\cancel{x^2}}$$

$$= \frac{1}{x^2+(1-\sqrt{1-x^2})^2}\left[\frac{x^2-\sqrt{1-x^2}(1-\sqrt{1-x^2})}{\sqrt{1-x^2}}\right]$$

$$= \frac{1}{\cancel{x^2}+1+1-\cancel{x^2}-2\sqrt{1-x^2}}\left[\frac{\cancel{x^2}-\sqrt{1-x^2}+1-\cancel{x^2}}{\sqrt{1-x^2}}\right]$$

$$= \frac{1}{2\cancel{(1-\sqrt{1-x^2})}}\left[\frac{\cancel{1-\sqrt{1-x^2}}}{\sqrt{1-x^2}}\right]$$

$$= \frac{1}{2\sqrt{1-x^2}}.$$

(10) If $y = \tan\left(2\tan^{-1}\frac{x}{2}\right)$, then prove that $\frac{dy}{dx} = 4\left(\frac{1+y^2}{4+x^2}\right)$.

$$y = \tan\left(2\tan^{-1}\frac{x}{2}\right)$$

$$\frac{dy}{dx} = \sec^2\left(2\tan^{-1}\frac{x}{2}\right) \cdot \frac{d}{dx}\left(2\tan^{-1}\frac{x}{2}\right)$$

$$= \sec^2\left(2\tan^{-1}\frac{x}{2}\right) \cdot 2\,\frac{1}{1+\left(\frac{x}{2}\right)^2} \cdot \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$= \sec^2\left(2\tan^{-1}\frac{x}{2}\right) \cdot \frac{\cancel{2}}{\left(1+\frac{x^2}{4}\right)} \cdot \frac{1}{\cancel{2}}$$

$$= \left[1+\tan^2\left(2\tan^{-1}\frac{x}{2}\right)\right] \cdot \frac{1}{\left(\frac{4+x^2}{4}\right)}$$

$$= \frac{4\left(1+y^2\right)}{4+x^2} = 4\left(\frac{1+y^2}{4+x^2}\right).$$

(11) If $\frac{y}{x} = \tan^{-1}\left(\frac{x}{y}\right)$, show that $\frac{dy}{dx} = \frac{y}{x}$.

$$\boxed{(\tan^{-1}x)' = \frac{1}{1+x^2}}$$

$$\frac{y}{x} = \tan^{-1}\frac{x}{y}$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{d}{dx}\left(\tan^{-1}\frac{x}{y}\right)$$

$$\frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \qquad \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\frac{\left(x\frac{dy}{dx} - y\right)}{x^2} = \frac{1}{\left(1+\frac{x^2}{y^2}\right)} \cdot \frac{y \cdot 1 - x\frac{dy}{dx}}{y^2}$$

$$\frac{\left(x\frac{dy}{dx} - y\right)}{x^2} = \frac{1}{\left(\frac{y^2+x^2}{\cancel{y^2}}\right)} \cdot \frac{\left(y - x\frac{dy}{dx}\right)}{\cancel{y^2}}$$

$$\frac{\left(x\frac{dy}{dx} - y\right)}{x^2} = \frac{\left(y - x\frac{dy}{dx}\right)}{x^2+y^2}$$

$$(x^2+y^2)\left(x\frac{dy}{dx} - y\right) = x^2\left(y - x\frac{dy}{dx}\right)$$

$$x^3\frac{dy}{dx} - x^2y + xy^2\frac{dy}{dx} - y^3 = x^2y - x^3\frac{dy}{dx}$$

$$x^3\frac{dy}{dx} + xy^2\frac{dy}{dx} + x^3\frac{dy}{dx} = x^2y + x^2y + y^3$$

$$\frac{dy}{dx}\left(x^3 + xy^2 + x^3\right) = 2x^2y + y^3$$

$$\frac{dy}{dx} = \frac{2x^2y + y^3}{2x^3 + xy^2} = \frac{y\,\cancel{(2x^2+y^2)}}{x\,\cancel{(2x^2+y^2)}}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

(12) If $y = \tan(a\tan^{-1}x)$, show that $(1+x^2)\frac{dy}{dx} - a(1+y^2) = 0$.

$$y = \tan(a\tan^{-1}x)$$

$$\tan^{-1}y = a\tan^{-1}x$$

$$\frac{d}{dx}(\tan^{-1}y) = a\frac{d}{dx}(\tan^{-1}x)$$

$$\frac{1}{1+y^2}\cdot\frac{dy}{dx} = a\cdot\frac{1}{1+x^2}$$

$$(1+x^2)\frac{dy}{dx} = a(1+y^2)$$

$$(1+x^2)\frac{dy}{dx} - a(1+y^2) = 0$$

as required.

(13) Find $\frac{dy}{dx}$:

(i) $x = a\sin\theta$, $y = a\cos\theta$

$\frac{dx}{d\theta} = a\cos\theta$, $\frac{dy}{d\theta} = -a\sin\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = (-\cancel{a}\sin\theta) \cdot \left(\frac{1}{\cancel{a}\cos\theta}\right)$$

$$= -\frac{\sin\theta}{\cos\theta} = -\tan\theta .$$

(ii) $x = t + \frac{1}{t}$, $y = t + 1$

$x = t + t^{-1}$, $y = t + 1$

$\frac{dx}{dt} = 1 - t^{-2}$

$= 1 - \frac{1}{t^2}$, $\frac{dy}{dt} = 1$

$\frac{dx}{dt} = \frac{t^2 - 1}{t^2}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= 1 \cdot \frac{t^2}{t^2 - 1} = \frac{t^2}{t^2 - 1}$$ Ans.

(iii) $x = \dfrac{a(1-t^2)}{1+t^2}$, $y = \dfrac{2bt}{1+t^2}$

$$\frac{dx}{dt} = a\frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$$

$$= a\left[\frac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2}\right]$$

$$= a\left[\frac{-2t-\cancel{2t^3}-2t+\cancel{2t^3}}{(1+t^2)^2}\right]$$

$$\frac{dx}{dt} = \frac{-4at}{(1+t^2)^2}$$

$$\frac{dy}{dt} = 2b\frac{d}{dt}\left(\frac{t}{1+t^2}\right)$$

$$= 2b\left[\frac{(1+t^2)(1)-t(2t)}{(1+t^2)^2}\right]$$

$$= 2b\left[\frac{1+t^2-2t^2}{(1+t^2)^2}\right]$$

$$\frac{dy}{dt} = \frac{2b(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \frac{\cancel{2}b(1-t^2)}{\cancel{(1+t^2)^2}}\cdot\frac{\cancel{(1+t^2)^2}}{-\cancelto{2}{4}at}$$

$$= \frac{-b(1-t^2)}{2at}$$

Ans

(iv) $x = a\theta^2$, $y = 2a\theta$

$$\frac{dx}{d\theta} = 2a\theta, \qquad \frac{dy}{d\theta} = 2a$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$\frac{dy}{dx} = \cancel{2a}\cdot\frac{1}{\cancel{2a}\theta} = \frac{1}{\theta}.$$

(14) If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \cdots + \infty}}}$, prove that $(2y-1)\frac{dy}{dx} = \sec^2 x$.

$$y^2 = \tan x + \left(\sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \cdots + \infty}}}\right)$$

$$y^2 = \tan x + y$$

$$y^2 - y = \tan x$$

$$\frac{d}{dx}\left(y^2 - y\right) = \frac{d}{dx}(\tan x)$$

$$2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

$$(2y-1)\frac{dy}{dx} = \sec^2 x$$

as required.

(15) Find the derivatives of $\cos^{-1}x$, $\csc^{-1}x$ and $\cot^{-1}x$ by using differentiation formulas.

Let $y = \cos^{-1}x$

$$\cos y = x$$

$$\frac{d}{dx}(\cos y) = \frac{d}{dx}(x)$$

$$-\sin y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{d}{dx}\left(\cos^{-1}x\right) = \frac{-1}{\sqrt{1-x^2}}.$$

Let $y = \operatorname{cosec}^{-1}x$

$$\operatorname{cosec} y = x$$

$$\frac{d}{dx}(\operatorname{cosec} y) = \frac{d}{dx}(x)$$

$$-\operatorname{cosec} y \cot y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\operatorname{cosec} y \cot y} = \frac{-1}{\operatorname{cosec} y\sqrt{\operatorname{cosec}^2 y - 1}}$$

$$\frac{d}{dx}\left(\operatorname{cosec}^{-1}x\right) = \frac{-1}{x\sqrt{x^2-1}}$$

$$1+\cot^2 y = \operatorname{cosec}^2 y$$

$$\cot^2 y = \operatorname{cosec}^2 y - 1$$

Let $y = \cot^{-1}x$

$$\cot y = x$$

$$\frac{d}{dx}(\cot y) = \frac{d}{dx}(x)$$

$$-\operatorname{cosec}^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\operatorname{cosec}^2 y} = \frac{-1}{1+\cot^2 y}$$

$$\frac{d}{dx}\left(\cot^{-1}x\right) = \frac{-1}{1+x^2}.$$