Derivative of a Function

If $\quad y = f(x)$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$y = f(x)$$

$$y', \ \frac{dy}{dx}, \ f', \ Dy, \ \dot{f}$$

Exercise 3.2

Q# 01 Find by definition (ab-initio) the derivatives w.r.t. x of the following functions defined as:

(i) $f(x) = 2x$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2(x+\delta x) - 2x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\cancel{2x} + 2\delta x - \cancel{2x}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cancel{\delta x}}{\cancel{\delta x}} = 2 \quad \text{Ans}$$

(ii) $f(x) = 1 - \sqrt{x}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[1 - \sqrt{x+\delta x}] - [1 - \sqrt{x}]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{1} - \sqrt{x+\delta x} - \cancel{1} + \sqrt{x}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[(\sqrt{x} - \sqrt{x+\delta x}) \times \frac{(\sqrt{x} + \sqrt{x+\delta x})}{\sqrt{x} + \sqrt{x+\delta x}}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{(\sqrt{x})^2 - (\sqrt{x+\delta x})^2}{\sqrt{x} + \sqrt{x+\delta x}}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{\cancel{x} - \cancel{x} - \delta x}{\sqrt{x} + \sqrt{x+\delta x}}\right]$$

$$= \lim_{\delta x \to 0} \frac{-1}{\sqrt{x} + \sqrt{x+\delta x}} = \frac{-1}{\sqrt{x} + \sqrt{x}} = \frac{-1}{2\sqrt{x}}$$

Ans

(iii) $f(x) = \dfrac{1}{\sqrt{x}}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$f'(x) = \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{1}{\sqrt{x+\delta x}} - \frac{1}{\sqrt{x}}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{(\sqrt{x} - \sqrt{x+\delta x})}{\sqrt{x+\delta x}\,\sqrt{x}} \times \frac{(\sqrt{x} + \sqrt{x+\delta x})}{\sqrt{x} + \sqrt{x+\delta x}}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{\cancel{x} - \cancel{x} - \delta x}{\sqrt{x+\delta x}\,\sqrt{x}\,(\sqrt{x} + \sqrt{x+\delta x}}\right]$$

$$= \lim_{\delta x \to 0} \frac{-1}{\sqrt{x+\delta x}\,\sqrt{x}\,(\sqrt{x} + \sqrt{x+\delta x})}$$

$$= \frac{-1}{\sqrt{x}\,\sqrt{x}\,(\sqrt{x} + \sqrt{x})} = \frac{1}{x\,(2\sqrt{x})} = \frac{1}{2x\sqrt{x}}$$

(iv) $f(x) = 3 - x^2$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left\{3 - (x+\delta x)^2\right\} - \left\{3 - x^2\right\}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[3 - (x^2 + \delta x^2 + 2x\,\delta x) - 3 + x^2\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{3} - \cancel{x^2} - \delta x^2 - 2x\,\delta x - \cancel{3} + \cancel{x^2}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[-\delta x - 2x\right]$$

$$= 0 - 2x = 2x$$ <u>Ans</u>

(v) $f(x) = x(x+1) = x^2 + x$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left\{(x+\delta x)^2 + (x+\delta x)\right\} - \left\{x^2 + x\right\}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{x^2} + \delta x^2 + 2x\delta x + \cancel{x} + \delta x - \cancel{x^2} - \cancel{x}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[\delta x + 2x + 1\right] = 0 + 2x + 1$$

$$= 2x + 1.$$

(vi) $f(x) = x^2 - 3$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left\{(x+\delta x)^2 - 3\right\} - \left\{x^2 - 3\right\}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{x^2} + \delta x^2 + 2x\delta x - \cancel{3} - \cancel{x^2} + \cancel{3}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[\delta x + 2x\right] = 0 + 2x$$

$$= 2x.$$

(vii) $f(x) = x^3 + 5$ ✓

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left\{(x+\delta x)^3 + 5\right\} - \left\{x^3 + 5\right\}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{x^3} + \delta x^3 + 3x^2\delta x + 3x\delta x^2 + \cancel{5} - \cancel{x^3} - \cancel{5}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[\delta x^2 + 3x^2 + 3x\delta x\right]$$

$$= 0 + 3x^2 + 0 = 3x^2.$$

(viii) $f(x) = 4x^2 - 3x$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\{4(x+\delta x)^2 - 3(x+\delta x)\} - \{4x^2 - 3x\}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[4(x^2+\delta x^2+2x\delta x) - 3x - 3\delta x - 4x^2 + 3x\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{4x^2} + 4\delta x^2 + 8x\delta x - \cancel{3x} - 3\delta x - \cancel{4x^2} + \cancel{3x}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[4\delta x + 8x - 3\right] = 0 + 8x - 3$$

$$= 8x - 3.$$

(ix) $f(x) = \frac{1}{x+2}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{1}{x+\delta x+2} - \frac{1}{x+2}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{\cancel{x}+\cancel{2}-\cancel{x}-\delta x-\cancel{2}}{(x+\delta x+2)(x+2)}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[\frac{-1}{(x+\delta x+2)(x+2)}\right] = \frac{-1}{(x+2)(x+2)}$$

$$= \frac{-1}{(x+2)^2}$$ Ans.

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(x) $$f(x) = \frac{3}{2x+5}$$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{3}{2(x+\delta x)+5} - \frac{3}{2x+5}\right]$$

$$= \lim_{\delta x \to 0} \frac{3}{\delta x}\left[\frac{1}{2x+2\delta x+5} - \frac{1}{2x+5}\right]$$

$$= \lim_{\delta x \to 0} \frac{3}{\delta x}\left[\frac{\cancel{2x}+\cancel{5} - \cancel{2x} - 2\delta x - \cancel{5}}{(2x+2\delta x+5)\,(2x+5)}\right]$$

$$= \lim_{\delta x \to 0} 3\,\frac{\cancel{\delta x}}{\cancel{\delta x}}\left[\frac{-2}{(2x+2\delta x+5)(2x+5)}\right]$$

$$= 3\left[\frac{-2}{(2x+5)(2x+5)}\right]$$

$$= \frac{-6}{(2x+5)^2}$$ Ans

Q #02 Find $f'(x)$ for the following functions using definition:

(i) $f(x) = \sqrt[3]{2x+1} = (2x+1)^{1/3}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x} \left[(2(x+\delta x)+1)^{1/3} - (2x+1)^{1/3} \right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x} \left[(2x+2\delta x+1)^{1/3} - (2x+1)^{1/3} \right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x} \left[(\underline{2x+1} + \underline{2\delta x})^{1/3} - (2x+1)^{1/3} \right]$$

Formula $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \cdots$

$$f'(x) = \lim_{\delta x \to 0} \frac{1}{\delta x} \left[\cancel{(2x+1)^{1/3}} + \frac{1}{3}(2x+1)^{\frac{1}{3}-1}(2\delta x) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(2x+1)^{\frac{1}{3}-2}(2\delta x)^2 + \cdots - \cancel{(2x+1)^{1/3}} \right]$$

$$f'(x) = \lim_{\delta x \to 0} \frac{1}{\delta x} \left[\frac{1}{3}(2x+1)^{-\frac{2}{3}} 2\,\delta x + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(2x+1)^{-\frac{5}{3}} 4\,\delta x^2 + \cdots \right]$$

$$= \lim_{\delta x \to 0} \cancel{\frac{\delta x}{\delta x}} \left[\frac{1}{3}(2x+1)^{-\frac{2}{3}} 2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(2x+1)^{-\frac{5}{3}} 4\,\delta x + \cdots \right]$$

$$= \frac{1}{3}(2x+1)^{-\frac{2}{3}} 2 + 0 + 0 + \cdots$$

$$= \frac{2}{3}(2x+1)^{-2/3} = \frac{2}{3(2x+1)^{2/3}}$$ Ans

(ii) $f(x) = (2x-1)^{-\frac{1}{2}}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$f'(x) = \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left(2(x+\delta x)-1\right)^{-1/2} - (2x-1)^{-1/2} \right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[(2x-1+2\delta x)^{-1/2} - (2x-1)^{-1/2} \right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{(2x-1)^{-1/2}} + \left(-\frac{1}{2}\right)(2x-1)^{-\frac{1}{2}-1}(2\delta x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(2x-1)^{-\frac{1}{2}-2}(2\delta x)^2 + \cdots - \cancel{(2x-1)^{-1/2}} \right]$$

$$f'(x) = \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[-\frac{1}{2}(2x-1)^{-\frac{3}{2}}2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(2x-1)^{-\frac{5}{2}}4\delta x + \cdots \right]$$

$$= -\frac{1}{\cancel{2}}(2x-1)^{-\frac{3}{2}}\cancel{2} + 0 + 0 + \cdots$$

$$= \frac{-1}{(2x-1)^{3/2}} \qquad \underline{\text{Ans}}$$

(iii) $f(x) = (6x+7)^{\frac{5}{2}}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left(6(x+\delta x)+7\right)^{\frac{5}{2}} - (6x+7)^{\frac{5}{2}}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left(\underline{6x+7} + 6\delta x\right)^{\frac{5}{2}} - (6x+7)^{\frac{5}{2}}\right]$$

$$f'(x) = \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{(6x+7)^{\frac{5}{2}}} + \frac{5}{2}(6x+7)^{\frac{5}{2}-1} 6\delta x + \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}(6x+7)^{\frac{5}{2}-2}(6\delta x)^2 + \cdots - \cancel{(6x+7)^{\frac{5}{2}}}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[\frac{5}{2}(6x+7)^{\frac{3}{2}} 6 + \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}(6x+7)^{\frac{1}{2}} 36\,\delta x + \cdots\right]$$

$$= \frac{5}{\cancel{2}}(6x+7)^{\frac{3}{2}} \cancelto{3}{6} + 0 + 0 + 0 + \cdots$$

$$= 15(6x+7)^{\frac{3}{2}}$$ Ans.

(iv) $f(x) = (3x-5)^{-\frac{3}{2}}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left(3(x+\delta x) - 5\right)^{-3/2} - (3x-5)^{-3/2}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left(3x-5+3\delta x\right)^{-3/2} - (3x-5)^{-3/2}\right]$$

$$f'(x) = \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{(3x-5)^{-3/2}} + \left(\frac{-3}{2}\right)(3x-5)^{-\frac{3}{2}-1} 3\delta x + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)}{2!}(3x-5)^{-\frac{3}{2}-2}(3\delta x)^2 + \cdots - \cancel{(3x-5)^{-3/2}}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[-\frac{3}{2}(3x-5)^{-\frac{5}{2}} 3 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)}{2!}(3x-5)^{-\frac{7}{2}} 9\delta x + \cdots\right]$$

$$= -\frac{3}{2}(3x-5)^{-\frac{5}{2}} 3 + 0 + 0 + \cdots$$

$$= -\frac{9}{2}(3x-5)^{-5/2} = \frac{-9}{2(3x-5)^{5/2}}.$$

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