SPECTRAL METHODS IN FLUID DYNAMICS.

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⇒ **Introduction:-** Spectral Approximation:

The Fourier system, Orthogonal polynomials in $(-1,1)$, Lengendre polynomials, Chebyshev polynomials, Jacobi polynomials, Fundamentals of spectral methods for PDE's: spectral projection of Burger's Equation, Convolution sum, Boundary conditions, Coordinate singularities, Temporal Discretization: The Eigen values of basic spectral Operators, Some standard schemes, Conservation forms, Global Approximation Results: Fourier Approximation, Normal Sturm-Liouville Expansions, Discrete Legendre Approximations, Chebyshev Approximations, Jacobi Approximations, Theory of Stability and Convergence for Spectral Methods: Fourier Galerken method for wave equation, Chebysheve Collocation method for heat equation, Legendre Tau method for the Poisson equation, General Formulation of spectral Approximations to Linear study problems, Galerkin, Collocation and Tau methods, Condition for stability and Convergence: The parabolic Case, Condition for stability and convergence: The Hyperbolic Case.

⇒ **Books:-**

1:- Claudion Canuto, M.Y. Hussaini, Alfio Quarteroni and T.A, Zang, Spectral Methods in Fluid Dynamics, Springer-Verlag, 1988.

2:- D. Gottlieb and S.A. Orszag, Numerical Analysis of spectral methods: Theory and Applications, SIAM-CBMS, Philadelphia, 1977.

3:- Lloyd N. Trefethen, Spectral Methods in MATLAB, SIAM-Philadelphia, 2000.

4:- Spectral Methods for Time-dependent Problems, by David Gottlieb, Jan S. Hesthaven, and Sigal Gottlieb, 2007.

5:- Spectral Methods, Fundamentals in Single Domains by Canuto, C. Hussaini, M.Y. Quarteroni, A. Zang, Th. A. Springer-Verlag, 2006.

⇒ Introduction to Spectral Methods:-

Suppose we have an equation for some vector function $u(x)$, $x \in \Omega \subseteq \mathbb{R}^1$

$$\mathcal{L} u = f \longrightarrow (1)$$

with boundry conditions

$$B u = 0 \longrightarrow (2) \quad x \in \partial\Omega$$

where $\mathcal{L}$ and B are some linear operator. How can we find the best approximation for the unknown function u.

One of the possible method is based on the wild class of discretization scheme known as method of weighted residuals (MWR). The idea of the method is to approximate the unknown function $u(x)$ by a sum of so called trial or basis function $\phi_n(x)$

$$\tilde{u}(x) = \sum_{n=0}^{N} a_n \phi_n(x) \longrightarrow (3)$$

Where a_n are unknown coefficients and

$\phi_n(x) = e^{ikx}$ (Fourier spectral Method)
(Periodic Domain)

$\phi_n(x) = T_k(x)$ (Chebyshev Spectral Method)
(Bounded Domain)

$\phi_n(x) = L_n(x)$ (Legendre spectral Method)
Bounded Domain

Put equ (3) in (1) we get

$\boxed{\text{Putting } u = \tilde{u} \text{ i.e. } R = \mathcal{L}\tilde{u} - f}$

$$\text{Error} \longleftarrow R = \mathcal{L}\tilde{u} - f$$

Due to the fact that $\tilde{u}$

different from the exact solution "u" the residual R does not vanish for $x \in \Omega$.

The next step is to find the unknown co-efficient a_n, so that the choosen function approximate the exact solution in the best way. To this end, test or weighted functions.

$\psi_n(x)$ or $\chi_n(x)$; $n = 0, 1, 2, \cdots, N$ are selected. so that the residual over the domain of interest is set to zero, i.e.

$$(\chi_n, R) = 0 \Rightarrow \int_\Omega \chi_n(x) R \, dx = 0$$

The choice of test function χ_n distinguishes between the three most commonly used spectral methods.

جو کہ Trial Function Boundary Conditions کو satisfy کرتے ہیں.

1:- Galerkin Method:- The test function are same as trial function and each $\phi_n(x)$ satisfies the boundary conditions $B\phi_n = 0$, i.e.

$$\phi_n(x) = \chi_n(x)$$

$$\int_\Omega \phi_n R \, dx = 0 \Rightarrow \int_\Omega \phi_n (\mathcal{L}\hat{u} - f) dx = 0$$

$$\Rightarrow \int_\Omega \phi_n \left(\mathcal{L} \sum_{k=0}^{N} a_k \phi_k\right) dx = \int_\Omega \phi_n f \, dx$$

$$\Rightarrow \boxed{\sum_{k=0}^{N} \mathcal{L}_{nk} a_k = \int_\Omega \phi_n f}$$

where $\mathcal{L}_{nk} = \int_\Omega \phi_n \mathcal{L} \phi_k$

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2:- Tau Method:- The test function are same as trial functions, but ϕ_n do not need to satisfy the boundry conditions, i.e. $B\phi_n \neq 0$

3:- Collocation Method OR Speudospectral Method:- The test function are represented by a delta function at special points x_n, called collocation points.

$$\int_\Omega x_n R = 0 \Rightarrow \int \delta(x - x_n) R = 0$$

$$\Rightarrow R(x_n) = 0 \qquad \because \int \delta(x - x_n) R\, dx = R(x_n)$$

$$\Rightarrow \mathcal{L}\hat{u}(x_n) = f(x_n) \qquad \because R(x_n) = \mathcal{L}\hat{u}(x_n) - f(x_n)$$

$$\Rightarrow \sum_{k=0}^{N} a_k \mathcal{L}\phi_k(x_n) = f(x_n)$$

⇒ **Fourier Series:-** A Fourier series is an expression of a periodic function in term of an infinite sum of sine and cosine.

Consider a periodic function (Integrable) $f(x)$, The Fourier Series of $f(x)$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right],$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \ ; \quad n \geq 0$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt \ ; \quad n \geq 1$$

If $f(x)$ is periodic on some interval $[-L, L]$, a simple change of variables $x' = \frac{xL}{\pi} \Rightarrow x = \frac{\pi x'}{L}$ can be used to transform the interval of integration. In this case the fourier series is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x'}{L}\right) + b_n \sin\left(\frac{n\pi x'}{L}\right)\right]$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x')\cos\left(\frac{n\pi x'}{L}\right)dx' \ ; \quad n \geq 0$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x')\sin\left(\frac{n\pi x'}{L}\right)dx' \ ; \quad n \geq 1$$

⇒ **Exponential Fourier Series:-** The notion of Fourier series can also be extended to complex coefficients.

Consider a real valued function $f(x)$, Then using Euler formula ($e^{i\theta} = \cos\theta + i\sin\theta$) we can write,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \quad , \text{ where}$$

Fourier coefficients are given by

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}$$

For a function which is periodic in $\left[-\frac{L}{2}, \frac{L}{2}\right]$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{\frac{in2\pi x}{L}}$$

with

$$C_n = \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{\frac{-in2\pi x}{L}}\, dx$$

⇒ Discrete Finite Fourier Transform:

Assume that we have a set of uniformly spaced grid points in 1-D.

$$x_j = j\Delta x \quad ; \quad j = 1, 2, 3, \cdots, N$$

$$\text{where } M\Delta x = 1$$

We assume that the function we deal with are periodic with period L. which implies that $x = 0$ and $x = L = N\Delta x$

To write the fourier series for a function whose values are given only on N grid points. For generally $f(x)$ is allowed to be complex, The series is

$$f(x_j) = f_j = \sum_{n=1}^{N} F_n\, e^{ik_n x_j} \quad ,$$

where F_n are the coefficients of Fourier components or spectral coefficients, we choose

$$k_n = \frac{2\pi n}{i\Delta x} \quad ; \quad n = 1, 2, \cdots, N$$

$$\Rightarrow f(x_j) = f_j = \sum_{n=1}^{N} F_n \, e^{\frac{i2n\pi}{n\Delta x} j\Delta x}$$

$$= \sum_{n=1}^{N} F_n \, e^{\frac{i2\pi nj}{n}} \longrightarrow (1)$$

$$j = 1, 2, 3, \ldots N$$

Example: Solve 1-D advection equation using Fourier Spectral Method.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \longrightarrow (1)$$

Solution

$$\hat{U}(x,t) = \sum_{n=1}^{N} U_n(t) \, e^{inkx} \longrightarrow (*)$$

Now doing term by term differentiation

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{N} \frac{dU_n}{dt} e^{inkx} \quad \text{and}$$

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{N} U_n(t)(ink) e^{inkx}$$

So equ (1) becoms

$$\sum_{n=1}^{N} \left[\frac{dU_n}{dt} e^{inkx} + C\,ink\,U_n e^{inkx} \right] = 0$$

As we know that the error be orthogonal to each basis function, we have

$$\frac{dU_n}{dt} e^{inkx} + C\,ink\,U_n e^{inkx} = 0$$

$$\Rightarrow \frac{dU_n}{dt} + C\,ink\,U_n = 0 \longrightarrow (2)$$

Characteristic equation is

$$D + icnk = 0 \Rightarrow \boxed{D = -icnk}$$

$$\Rightarrow U_n(t) = A_n e^{-icnkt}$$

$$\Rightarrow \hat{U}(x,t) = \sum_{n=1}^{N} A_n e^{-icnkt} e^{inkx}$$

$$\Rightarrow \hat{U}(x,t) = \sum_{n=1}^{N} A_n e^{-ink(ct-x)}$$

$$= \sum_{n=1}^{N} A_n e^{ink(x-ct)}$$

Where A_n are coefficients and we find it if we have one initial condition.

⇒ Orthogonal Projection:- Let us consider an interval $\Omega = [x_{min}, x_{max}]$. In order to talk about basis, one need to define a scaler product on Ω. If ω is a positive function on Ω, one can define a scaler product of two functions f and g w.r.t weight function ω as

$$(f,g)_\omega = \int_\Omega f(x)\, g(x)\, \omega(x)\, dx$$

Using this scaler product, one can find a set of orthogonal polynomials P_n, each of degree n.

In fact the spectral representation of any function U is

its orthogonal projection on the space of polynomials of degree $\leq N$.
One can hope to represent any function U on Ω by its projection on the polynomials P_n. Doing so, we define the orthogonal projection U simplify by

$$P_N U = \sum_{n=0}^{\infty} \hat{U}_n P_n(x)$$

Where the coefficients of the projection are given by $\hat{U}_n = \frac{(U, P_n)}{(P_n, P_n)} \longrightarrow (1)$

$(\cdot,\cdot)$ inner product

The difference between U and its projection $P_N U$ is called Truncation error

$$\Rightarrow |U - P_N U| = T_n \longrightarrow 0 \text{ when } N \longrightarrow \infty$$

→ **Gauss Quadrature:-** The solution of (1) is given by Gauss Quadrature.
The theorem can be stated as follows
"If there exist $N+1$ +ve real ω_n and $N+1$ real x_n in Ω s.t

$$\forall f \in P_{2N+\varepsilon} \int_{\Omega} f(x)\omega(x)dx = \sum_{n=0}^{N} f(x_n)\omega_n$$

Then ω_n are called weights and x_n are called collocation points.

There are three Gauss Quadrature methods.

* Gauss, $\varepsilon = 1$

* Gauss Radau: $\delta = 0$ and $x_0 = x_{min}$
* Gauss Lobato: $\delta = -1$ and $x_0 = x_{min}$, $x_n = x_{max}$

⇒ **Interpolation:-** If one applies Gauss quadrature to approximate the coefficients of the expansion, one obtain

$$\tilde{U}_n = \frac{1}{\gamma_n} \sum_{j=0}^{N} U(x_j) P_n(x_j) \omega, \text{ with}$$

$$\gamma_n = \sum_{j=0}^{N} P_n^2(+x_j) \omega_j$$

Let us precise that this is not exact in the sence that $\hat{U}_n \neq \tilde{U}_n$.
The actual representation of a function U is then the polynomial from the discrete coefficients

$$I_N U = \sum_{n=0}^{\infty} \tilde{U}_n P_n(x)$$

The difference b/w $P_N U$ and $I_N U$ is called aliasing error

$$\Rightarrow \text{Aliasing Error} = |P_N U - I_N U|$$

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* Usual Families of Polynomials:-

⇒ Legendre Polynomial:-

The Legendre polynomial, denoted by P_n, constitute a family of orthogonal polynomials on $[-1, 1]$ with a measure $\omega = 1$

The scaler product of two P_N is given by

$$\int_{-1}^{1} P_n P_m \, dx = \frac{2}{2n+1} \delta_{mn}$$

The successive polynomials can be constructed by recurrence. Indeed given $P_0 = 1$ and $P_1 = x$

$$(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$$

* P_n is polynomial of degree n.
* $P_n(\pm 1) = (\pm 1)^n$
* P_n has exactly n zeros on $[-1, 1]$
* The value of the weight and collocation points can be written for the three usual quadrature.

1) **Legendre Gauss:-** x_i are the nodes of P_{N+1} and

$$\omega_i = \frac{2}{(1-x_i)^2 \left[P'_{N+1}(x_i)\right]^2}$$

2) **Legendre Gauss Raduu:-** $x_0 = -1$ and x_i are the nodes of $P_N + P_{N+1}$,

The weight are given by

$$\omega_0 = \frac{2}{(N+1)^2} \quad \text{and} \quad \omega_i = \frac{1}{(N+1)^2}$$

3) Gauss - Legendre - Lobato:- $x_0 = -1$, $x_N = 1$ and x_i are the nodes of P_N'. The weight are

$$\omega_i = \frac{2}{N(N+1)} , \frac{1}{[P_N(x_i)]^2}$$

⇒ **Chebyshev Polynomials:-** The Chebyshev polynomials "T_n" are orthogonal set on $[-1, 1]$ for the measure

$$\omega = \frac{1}{\sqrt{1-x^2}} ,$$

More perecisely one has

$$\int_{-1}^{1} \frac{T_n T_m}{\sqrt{1-x^2}} dx = \frac{\pi}{2}(1+\delta_{0n})\delta_{mn}$$

Chebyshev polynomial can be computed by knowing $T_0 = 1$, $T_1 = x$ and by measuring the recurrence relation

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$$

Put $n=1$

$$\Rightarrow T_2(x) = 2x T_1(x) - T_0(x)$$

$$= 2x(x) - 1 \qquad \because T_0 = 1 \ \& \ T_1 = x$$

$$\Rightarrow T_2(x) = 2x^2 - 1$$

Put $n=2$ ⇒ $T_3(x) = 2x T_2(x) - T_1(x)$

$$= 2x(2x^2-1) - x$$

$$= 4x^3 - 2x - x$$

$$\Rightarrow T_3(x) = 4x^3 - 3x$$

$n=3 \Rightarrow T_4(x) = 2x\,T_3(x) - T_2(x)$

$$= 2x(4x^3 - 3x) - (2x^2 - 1)$$

$$= 8x^4 - 6x^2 - 2x^2 + 1$$

$$\Rightarrow T_4(x) = 8x^4 - 8x^2 + 1$$

and so on

- ★ T_n is of degree 'n'.
- ★ $T_n(\pm 1) = (\pm 1)^n$
- ★ T_n has exactly 'n' zeros on $[-1, 1]$

The weight and collocation points associated with chebyshev polynomial can be computed

1) Chebysheve - Gauss:-

$$x_i = \cos\left(\frac{2i+1}{2N+2}\right) \text{ and } \omega_i = \frac{\pi}{N+1}$$

2) Chebyshev Gauss Raduu:-

$$x_i = \frac{2\pi i}{2N+1}, \quad \omega_0 = \frac{\pi}{2N+1}, \quad \omega_i = \frac{2\pi}{2N+1}$$

3) Chebyshev Gauss-Lobato:-

$$x_i = \frac{\cos \pi i}{N}, \quad \omega_0 = \omega_N = \frac{\pi}{2N} \quad \text{and}$$

$$\omega_i = \frac{\pi}{N}$$

$\mathcal{L}V = f$; $BV = 0$
$x \in \Omega$; $x \in \partial\Omega$
$\hat{U}(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$
$\mathcal{L}V - f = R$
$\int_{\Omega} (\chi_n, R) dx = 0$;
$\chi_n = \delta(x - x_n)$
$\Rightarrow \int_{\Omega} \delta(x - x_n), R) dx = 0$

Example:- Solve

$$\frac{d^2u}{dx^2} - 4\frac{du}{dx} + 4u = e^x + c$$

where $c = \frac{-4e}{1+e^2}$

$$u(-1) = u(1) = 0$$

Exact Solution $\Rightarrow u(x) = e^x - \frac{\sinh(1)}{\sinh(2)} \exp(2) + \frac{c}{4}$

Solution

$$\hat{u} = \sum_{i=1}^{n} \hat{u}_i T_i(x) .$$

for $n=4$

$$\hat{u} = \sum_{i=1}^{4} \hat{u}_i T_i(x)$$

Now $\mathcal{L}u = \sum_{i=0}^{4} \sum_{j=0}^{4} L_{ij} \hat{u}_j T_i(x) = f(x_n)$

$$\therefore \mathcal{L} = \frac{d^2}{dx^2} - 4\frac{d}{dx} + 4Id$$

& $f(x) = e^x - \frac{4e}{1+e^2}$

Collocation points $x_i = -\cos\left(\frac{i\pi}{4}\right)$; $0 \le i \le 4$

$\Rightarrow x_0 = \quad ,$ $\quad x_1 = -1/\sqrt{2}$, $\quad x_2 = 0$

$x_3 = 1/\sqrt{2}$ & $x_4 = 1$

So matrix of $\frac{d}{dx}(T_i(x))$

$$\frac{d}{dx} = \begin{bmatrix} T_0'(x_0) & T_1'(x_0) & \cdots & T_4'(x_0) \\ T_0'(x_1) & T_1'(x_1) & \cdots & T_4'(x_1) \\ \vdots & & & \\ T_0'(x_4) & T_1'(x_4) & \cdots & T_4'(x_4) \end{bmatrix}$$

Example:- Solve wave equation via spectral method.

$$\left.\begin{aligned} u_{tt}(x,t) &= u_{xx}(x,t) + f(x,t) \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= u_0(x), \quad u_t(x,0) = u_0(x) \end{aligned}\right\} \longrightarrow (1)$$

Solution Replace

$$u_{tt} = -Lu \longrightarrow (*)$$

where L is symmetric linear operator with eigen values and eigen functions λ_j and ψ_j respectively with

$$L\psi_j = \lambda_j \psi_j \longrightarrow (2)$$

$$\Rightarrow u(x,t) = \sum_{j=1}^{\infty} a_j(t)\,\psi_j(x)$$

Putting in $(*)$ implies

$$\frac{\partial^2}{\partial t^2}\sum_{j=1}^{\infty} a_j(t)\psi_j(x) = -L\sum_{j=1}^{\infty} a_j(t)\psi_j(x)$$

$$\Rightarrow \sum_{j=1}^{\infty} a_j''(t)\,\psi_j(x) = \sum_{j=1}^{\infty} a_j(t)(-\lambda_j)\,\psi_j(x)$$

Now taking innerproduct of both sides with $\psi_k(x)$

$$\Rightarrow \sum_{j=1}^{\infty} a_j''(t)\left(\psi_j(x),\psi_k(x)\right) = \sum_{j=1}^{\infty} a_j(t)(-\lambda_j)\left(\psi_j(x),\psi_k(x)\right)$$

$$\text{where } \psi_j(x),\psi_k(x) = \begin{cases} 0 & \text{if } j\neq k \\ 1 & \text{if } j=k \end{cases}$$

$$\Rightarrow a_j''(t) = -\lambda_j a_j(t)$$

$$\Rightarrow a_j(t) = A\sin(\sqrt{\lambda_j}\,t) + B\cos(\sqrt{\lambda_j}\,t) \longrightarrow (3)$$

Now

$$a_j(0) = A\sin(0) + B\cos(0) = u_0(x)$$

$$\Rightarrow \boxed{a_j(0) = B = u_0(x)}$$

$$a_j'(t) = \sqrt{\lambda_j}\left(+A\cos(\sqrt{\lambda_j}\,t) - B\sin(\sqrt{\lambda_j}\,t)\right)$$

$$\Rightarrow \boxed{a_j'(0) = A\sqrt{\lambda_j}} = u_0(x)$$

$$\Rightarrow \boxed{A = u_0(x)/\sqrt{\lambda_j}}$$

Now

$$u(x,t) = \sum_{j=1}^{\infty} a_j(t)\,\psi_j(x)$$

$$\Rightarrow u(x,0) = \sum_{j=1}^{\infty} a_j(0)\,\psi_j(x) = u_0(x)$$

Now taking the inner product with $\psi_k(x)$

$$\Rightarrow (u_0(x), \psi_k(x)) = \sum_{j=1}^{\infty} a_j(0)\,(\psi_j(x), \psi_k(x))$$

$$\Rightarrow a_k(0) = \frac{u_0(x), \psi_k(x)}{(\psi_k, \psi_k(x))} = B \quad \text{for } K = 1,2,3,\ldots$$

Now $\dfrac{\partial}{\partial t} u(x,t) = \displaystyle\sum_{j=1}^{\infty} \frac{\partial}{\partial t} a_j(t)\,\psi_j(x)$

$$\Rightarrow u_t(x,0) = \sum_{j=1}^{\infty} \frac{\partial}{\partial t} a_j(0)\,\psi_j(x)$$

Now taking innerproduct with $\psi_k(x)$

$$\Rightarrow (U_0(x), \psi_k(x)) = \sum_{j=1}^{\infty} \frac{\partial}{\partial t} a_j(0) (\psi_j(x), \psi_k(x))$$

$$\Rightarrow \frac{\partial}{\partial t} a_j(0) = \frac{U_0(x), \psi_k(x)}{\psi_k(x), \psi_k(x)} = \sqrt{\lambda_j}\, A$$

$$\Rightarrow A = \frac{1}{\sqrt{\lambda_j}} \frac{U_0(x), \psi_k(x)}{\psi_k(x), \psi_k(x)}$$

⇒ General Spectral Method for Reaction Diffusion Equation:-

An equation of the form

$$U_t = d\Delta U + f(U) \longrightarrow ①$$

is called reaction diffusion equation, where d is diffusion coefficient and is constant (real number) Δ is Laplacian operator and $f(u)$ is non-linear function.

Equation for 1-D:-

$$U_t = dU_{xx} + f(U)$$

Boundary Conditions $U(0,t) = U(1,t) = 0$ and

Initial condition $U(x,0) = g(x)$

Step I:- Choose basis function $\phi_n(x)$, Then the solution is of the form

$$\tilde{U}(t,x) = \sum_{n=1}^{N} a_n(t)\phi_n(x) \longrightarrow ②$$

Let $\phi_n(x) = \sin(n\pi x)$

Step II. Substitute step-I into differential operator and obtain a set of ODE's in time

Our differential equation in this case

$$L(u) = u_t - d u_{xx} - f(u) \longrightarrow ③$$

$\Rightarrow L(u) = 0$

$$L(\tilde{u}) = \sum_{n=1}^{N} a_n'(t)\sin(n\pi x) + d\left[\sum_{n=1}^{N} n^2\pi^2 a_n(t)\sin(n\pi x)\right]$$

$$- f\left[\sum_{n=1}^{N} a_n(t)\sin(n\pi x)\right] \longrightarrow ④$$

In order to obtain $a_n(t)$, we take inner product of $\phi_n(x)$ with ④ and using $(\phi_i(x), L(\tilde{u})) = 0$, we get

$$\sum_{n=1}^{N} a_n'(t)(\sin(m\pi x), \sin(n\pi x)) + \pi^2 d \sum_{n=1}^{N} a_n(t) n^2 (\sin(m\pi x), \sin(n\pi x))$$

$$- \left(\sin(m\pi x), f\left(\sum_{n=1}^{N} a_n(t)\sin(n\pi x)\right)\right) = 0 \longrightarrow ⑤$$

$$; m = \{1, 2, \dots, N\}$$

Note that the inner product operation and the summation operation commute, because integration is linear operation.

<u>When $m = n$</u>

$$(\sin(m\pi x), \sin(n\pi x)) = \frac{1}{2}$$

<u>otherwise</u> $= 0$

$$⑤ \Rightarrow \frac{a_n'(t)}{2} = \frac{-\pi^2 m^2 d}{2} a_m(t) - \left(\phi_m(x), f\left(\sum_{m=1}^{N} a_n(t)\phi_n(x)\right)\right)$$

Step III:- Find the initial condition for the ODE

$$a_m(0) = \frac{\sin(m\pi x), g(x)}{\sin(m\pi x), \sin(m\pi x)} = 2(\sin(m\pi x), g(x))$$

$$m = 1, 2, \dots, N$$

Step IV:- Solve the initial value problem for time.

Example:- Solve Heat equation using spectral method.

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) + f(x,t)$$

For a homogeneous Bar, with Drichlet boundry conditions $u(0,t) = u(1,t) = 0$
And initial condition, $u(x,0) = u_0(x)$

Solution

More generally we think of problem of the form

$$\frac{\partial}{\partial t} u(x,t) = -L\,u(x,t) + f(x,t) \longrightarrow (*)$$

Where L is a symmetric linear operator with eigen values and eigen functions λ_j and ψ_j:

$$L\,\phi_j = \lambda_j \psi_j$$

for which we can expand the solution $u(x,t)$ as

$$u(x,t) = \sum_{j=1}^{\infty} a_j(t)\, \psi_j(x)$$

Substitute this form into $(*)$, we obtain

$$\frac{\partial}{\partial t}\sum_{j=1}^{\infty} a_j(t)\,\psi_j(x) = -L\sum_{j=1}^{\infty} a_j(t)\,\psi_j(x) + f(x,t)$$

Take the time derivative and linear operator L under the sums to get

$$\sum_{j=1}^{\infty} a_j'(t)\,\psi_j(x) = -\sum_{j=1}^{\infty} a_j(t)\,\underbrace{L\,\psi_j(x)}_{=\lambda_j \psi_j(x)} + f(x,t)$$

$$\Rightarrow \sum_{j=1}^{\infty} a_j'(t)\,\psi_j(x) = \sum_{j=1}^{\infty} -\lambda_j\, a_j(t)\,\psi_j(x) + f(x,t)$$

Take the inner product with ψ_k to obtain

$$\left[\sum_{j=1}^{\infty} a_j'(t)\left(\psi_j(x), \psi_k(x)\right)\right] = \left[\sum_{j=1}^{\infty} -\lambda_j\, a_j(t)\,\psi_j(x), \psi_k(x)\right] + (f, \psi_k)$$

Use linearity of the inner product

$$\Rightarrow \sum_{j=1}^{\infty} a_j'(t)(\psi_j, \psi_k) = \sum_{j=1}^{\infty} -\lambda_j a_j(\psi_k, \psi_j) + (f, \psi_k)$$

use orthogonality of the eigen functions

$$(\psi_j, \psi_k) = \begin{cases} 0, & j \neq k \\ (\psi_j, \psi_j), & j = k \end{cases}$$

To simplify to an equation for $a_k'(t)$

$$a_k'(t) = -\lambda_k a_k(t) + \frac{(f, \psi_k)}{(\psi_k, \psi_k)}$$

Note That $(f, \psi_k) = \int_0^1 f(x,t)\,\psi_k(x)\,dx$; for fixed t,

So $\dfrac{(f, \psi_k)}{(\psi_k, \psi_k)} = c_k(t)$ is a function of t.

We thus have

$$a_k'(t) = -\lambda_k a_k(t) + c_k(t)$$

For similicity take $f(x,t) = 0$,

$$\Rightarrow \quad a_k'(t) = -\lambda_k a_k(t)$$

Whose solution is

$$a_k(t) = e^{-\lambda_k t} a_k(0)$$

To find $a_k(0)$,
We have $u(x,t)$ satisfying the initial condition $u(x,0) = u_0(x)$

At $t=0$ we thus want

$$u(x,t) = \sum_{j=1}^{\infty} a_j(0) \psi_j(x) = u_0(x) \quad \because \text{at } t=0$$

Take the inner product with ψ_k to get

$$\sum_{j=1}^{\infty} a_j(0) (\psi_j, \psi_k) = (u_0, \psi_k)$$

And using orthoganality of the eigen-functions to get

$$a_k(0) = \frac{(u_0, \phi_k)}{(\psi_k, \psi_k)}$$

Thus $$a_k(t) = e^{-\lambda_k t} a_k(0)$$

$$= e^{-\lambda_k t} \frac{(u_0, \psi_k)}{(\psi_k, \psi_k)}$$

Putting the pieces together we have

$$u(x,t) = \sum_{i=1}^{\infty} e^{-\lambda_j t} \frac{(u_0, \psi_k)}{(\psi_k, \psi_k)} \psi_j(x)$$

Motivating Problem:-

$u_t = u_{xx}$, $x \in [0,1]$, $u(0,t) = u(1,t) = 0$

with $Lu = -u_{xx}$, $x \in C_D^2[0,1]$

So $\lambda_j = j^2\pi^2$, $\psi_j(x) = \sqrt{2}\sin(j\pi x)$,

we have $u(x,t) = \sum_{j=1}^{\infty} e^{-j^2\pi^2 t} \frac{(u_0, \psi_j)}{(\psi_j, \psi_j)} (\sqrt{2}\sin(j\pi x))$.

As $t \to \infty$, we note $e^{-j^2\pi^2 t} \to 0$, so

- ★ $u(x,t) \to 0$ as $t \to \infty$
- ★ As t increases, $e^{-\pi^2 t}$ $(j=1)$ decays to 0 quite a bit more slowly than $e^{-4\pi^2 t}$ $(j=2)$, $e^{-9\pi^2}$ $(j=3)$, etc, So as $u(x,t) \to 0$, it will assume the shape of $\psi_1(x)$ (Assuming that $(u_0, \psi_1) \neq 0$.

Example:- Heat Equation with Inhomogeneous Forcing

$u_t = u_{xx} + f$; $u(0,t) = u(1,t) = 0$, $u(x,0) = u_0(x)$

Solution Begin with the general operator setting $u_t = -Lu + f$,

where $L : C_D^2[0,1] \to C[0,1]$, $Lu = -u_{xx}$

with eigen values and eigen functions $\lambda_j = j^2\pi^2$, $\psi_j(x) = \sqrt{2}\sin(j\pi x)$.

At every fixed t, we can write the solution as

$$u(x,t) = \sum_{j=1}^{\infty} a_j(t)\,\psi_j(x) \longrightarrow (*)$$

where $a_1(t), a_2(t), \dots$ give the best approximation/projection coefficient for $u(o,t)$ onto the subspace span $\{\psi_j\}$

Plug the expansion $(*)$ into $u_t = -Lu + f$ we obtain

$$\frac{\partial}{\partial t}\left(\sum_{j=1}^{\infty} a_j(t)\,\psi_j(x)\right) = -L\sum_{j=1}^{\infty} a_j(t)\,\psi_j(x) + f(x,t)$$

$$\Rightarrow \sum_{j=1}^{\infty} a_j'(t)\,\psi_j(x) = -\sum_{j=1}^{\infty} a_j(t)\,L\psi_j(x) + f(x,t)$$

$$= -\sum_{j=1}^{\infty} a_j(t)\,\lambda_j\,\psi_j(x) + f(x,t)$$

since $L\psi_j = \lambda_j \psi_j$

Take inner product with ψ_k

$$\left[\sum_{j=1}^{\infty} a_j'(t)\,\psi_j, \psi_k\right] = -\left[\sum_{j=1}^{\infty} a_j(t)\,\lambda_j\,\psi_j, \psi_k\right] + (f, \psi_k)$$

$$\Rightarrow \sum_{j=1}^{\infty} a_j'(t)(\psi_j, \psi_k) = -\sum_{j=1}^{\infty} \lambda_j a_j(t)(\psi_j, \psi_k) + (f, \psi_k)$$

Use orthogonality of Eigen functions to simplify

$$a_k'(t)(\psi_k, \psi_k) = -\lambda_k a_k(t)(\psi_k, \psi_k) + (f, \psi_k)$$

giving the ODE for $a_k(t)$

$$a_k'(t) = -\lambda_k a_k(t) + \frac{(f, \psi_k)}{(\psi_k, \psi_k)}$$

Note that

$$\frac{(f,\psi_k)}{(\psi_k,\psi_k)} = \frac{\int_0^1 f(x,t)\,\psi_k(x)\,dx}{\int_0^1 \psi_k(x)\,\psi_k(x)\,dx} = c_k(t)$$

gives the best approximation coefficients for $f(\cdot,t)$, the function f with t frozen:

$$f(\cdot,t) = \sum_{j=1}^{\infty} \frac{(f(\cdot,t),\psi_j)}{(\psi_j,\psi_j)}\psi_j(x) = \sum_{j=1}^{\infty} c_j(t)\,\psi_j(x)$$

So we must now solve

$$a_k'(t) = -\lambda_k a_k(t) + c_k(t)$$

This first order inhomogenous linear equation is a staple of first courses in the solution of ODEs.

We shall quickly recap the solution.

Multiply both sides $e^{\lambda_k t}$, and

$$e^{\lambda_k t} a_k'(t) + \lambda_k e^{\lambda_k t} a_k(t) = e^{\lambda_k t} c_k(t)$$

Integrate both sides

$$\int_0^t \left[e^{\lambda_k s} a_k'(s) + \lambda_k e^{\lambda_k s} a_k(s)\right] ds = \int_0^t e^{\lambda_k s} c_k(s)\,ds$$

The integrand on the left is a derivative (Product rule)

$$\Rightarrow \int_0^t \frac{d}{ds}\left(e^{\lambda_k s} a_k(s)\right) ds = \int_0^t e^{\lambda_k s} c_k(s)\,ds$$

So by the fundamental theorem of calculus $\left[e^{\lambda_k s} a_k(s)\right]_{s=0}^{s=t} = \int_0^t e^{\lambda_k s} c_k(s)\,ds$

$$\Rightarrow e^{\lambda_k t} a_k(t) - a_k(0) = \int_0^t e^{\lambda_k s} G_k(s)\, ds$$

$$\Rightarrow a_k(t) = a_k(0) e^{-\lambda_k t} + \int_0^t e^{\lambda_k (s-t)} G_k(s) ds$$

Hence the solution of the inhomogeneous PDE is

$$\boxed{\begin{aligned} u(x,t) &= \sum_{j=1}^{\infty} a_j(t)\, \varphi_j(x) \\ &= \sum_{j=1}^{\infty} \left[a_j(0) e^{-\lambda_j t} + \int_0^t e^{\lambda_j (s-t)} G_j(s) ds \right] \varphi_j(x) \end{aligned}}$$

Recall that the coefficients $a_k(0)$ come from the best approximation expansion of the initial condition

$$u(x,0) = (\cdot) = \sum_{j=1}^{\infty} a_j(0) \varphi_j(x)$$

given by $\quad a_j(0) = \dfrac{(u_0, \varphi_j)}{(\varphi_j, \varphi_j)}$

Special Case:- Time independent f

If $f(x,t) = f(x)$ has no dependence on t, then the coefficients $G_j(t)$ are constant, $G_j(t) = G_j$.

Then we can significantly simplify the formulas for $a_j(t)$

$$a_j(t) = a_j(0) e^{-\lambda_j t} + \int_0^t e^{\lambda_j (s-t)} G_j ds$$

$$= a_j(0) e^{-\lambda_j t} + G_j e^{-\lambda_j t} \int_0^t e^{\lambda_j s} ds \quad \left(\text{Assuming } \lambda_j \neq 0\right)$$

$$= a_j(0) e^{-\lambda_j t} + G_j e^{-\lambda_j t} \underbrace{\left[\frac{e^{\lambda_j s}}{\lambda_j}\right]_0^t}_{= \frac{e^{\lambda_j t} - 1}{\lambda_j}}$$

{need to handle $\lambda_j = 0$ separately if the problem has a zero eigen value

$$\Rightarrow a_j(t) = a_j(0)e^{-\lambda_j t} + c_j\left(\frac{1-e^{-\lambda_j t}}{\lambda_j}\right)$$

For the operator $Lu = -u$ on $C_0^2[0,1] \to C[0,1]$, $\lambda_j = j^2\pi^2$; $j = 1, 2, \dots$, so

$$a_j(t)e^{-\lambda_j t} \longrightarrow 0, \quad \frac{1-e^{-\lambda_j t}}{\lambda_j} \longrightarrow \frac{1}{\lambda_j}$$

Hence $a_j(t) \longrightarrow c_j/\lambda_j$ as $t \to \infty$

And the solution

$$u(x,t) = \sum_{j=1}^{\infty}\left(a_j(0)e^{-\lambda_j t} + c_j\left[\frac{1-e^{-\lambda_j t}}{\lambda_j}\right]\right)\varphi_j(x)$$

tends to $\displaystyle\sum_{j=1}^{\infty}\frac{c_j}{\lambda_j}\varphi_j = \sum_{j=1}^{\infty}\frac{1}{\lambda_j}\frac{(f,\varphi_j)}{(\varphi_j,\varphi_j)}\varphi_j$

Note:- This is exactly the solution to $Lu = f$ we obtain from the spectral method! We have just confirmed that the best equation with ~~Dt~~ Drichlet boundary conditions and time-independent f tends to a steady state, justifying as earlier study of $Lu = f$.

Question:- How general was this Analysis? Will this apply to other boundary conditions?

What do you need to require of the eigen values.

Example:- Handling Inhomogeneous Boundary Condition.

Consider the heat equation

$$u_t = u_{xx} + f \quad , \quad u(x,0) = u_0(x)$$

with inhomogeneous boundary conditions

$$u(0,t) = g(t) \quad , \quad u(1,t) = h(t)$$

For function $g(t)$ and $h(t)$. (Thus the boundary conditions can vary in time)

How can we incorporate these boundary conditions?

As usual L is defined on a domain with <u>Homogeneous Boundry</u> conditions of the same kind.

$$LV = -V_{xx} \quad \text{for } V \in C_0^2[0,1]$$

We seek to build a solution

$$u(x,t) = V(x,t) + w(x,t)$$

with $w(x,t)$ engineered to satisfy the Inhomogeneous boundary conditions, and $V(x,t)$ the solution of a related differential equation with Homogeneous boundary conditions.

Following the lead of the approach we took for $-u_{xx} = 0$

$$w(x,t) = f(t) + x(g(t) - f(t))$$

So that $w(0,t) = f(t)$ and $w(1,t) = g(t)$.

Now consider $u(x,t) = V(x,t) + w(x,t)$. Plug this into the PDE to see what V must satisfy:

$$u_t(x,t) = u_{xx}(x,t) + f(x,t)$$

$$\Rightarrow V_t(x,t) + \omega_t(x,t) = V_{xx}(x,t) + \omega_{xx}(x,t) + f(x,t)$$

Now $\omega_t(x,t) = f'(t) + x\left(g'(t) - f'(t)\right)$

And $\omega_{xx}(x,t) = 0$

Thus $\boxed{V_t(x,t) = V_{xx}(x,t) + f(x,t) - \omega_t(x,t)}$

Define $\tilde{f}(x,t) = f(x,t) - \omega_t(x,t)$

Then find $V(x,t)$ to solve

$$\boxed{\begin{aligned} V_t(x,t) &= V_{xx}(x,t) + \tilde{f}(x,t) \\ \Rightarrow V_t(x,t) &= -LV(x,t) + \tilde{f}(x,t) \end{aligned}}$$

Don't forget above initial condition

$$\begin{aligned} u_0(x) = u(x,0) &= V(x,0) + \omega(x,0) \\ &= V(x,0) + f(0) + x\left(g(0) - f(0)\right) \end{aligned}$$

$$\Rightarrow \boxed{V_0(x) = V(x,0) = u_0(x) - \left[f(0) + x\left(g(0) - f(0)\right)\right]}$$

Solve for V, Build ~~$u(x,t)$~~

$$u(x,t) = V(x,t) + \omega(x,t)$$

Heat Equation With Periodic Boundry Conditions:- (The Fourier Series)

Consider the heat equation posed on a bar, where the ends are bent around so that they join together and form a ring. At the points where the ends meet, we will require that $u(x,t)$ and $u_x(x,t)$ be continuous. For convenience, we shall use the physical domain $x \in [-1,1]$.

$$u_t(x,t) = u_{xx}(x,t)$$

$$u(x,0) = u_0(x)$$

$$\left.\begin{aligned} u(-1,t) &= u(1,t) \\ u_x(-1,t) &= u_x(1,t) \end{aligned}\right\} \text{Periodic boundry condition}$$

As usual, we pose this problem as a linear operator equation

$$u_t = -Lu \quad , \quad u(x,0) = u_0(x) \quad , \text{ where}$$

$$L : C_p^2[-1,1] \longrightarrow C[0,1]$$

$$Lu = -u_{xx}$$

for $C_p^2[-1,1] = \{ u \in C[-1,1] : u(-1) = u(1), u_x(-1) = u_x(1) \}$

One can show, using the usual techniques, that L is symmetric.

What are the eigenvalues and eigen functions of L?

Eigen Values & Eigen Functions of L.

We see $\psi \neq 0$ such that

- $\psi \in C_p^2[-1,1]$
- $L\psi = \lambda \psi$

The second requirment implies
$-\psi'' = \lambda\psi$ giving the general solution

$$\psi(x) = A\sin(\sqrt{\lambda}\,x) + B\cos(\sqrt{\lambda}\,x)$$

where A, B, λ give non-zero ψ in $C^2[-1,1]$?

$$\psi(-1) = \psi(1) \quad , \quad \psi'(-1) = \psi'(1)$$

$$\psi(-1) = A\sin(-\sqrt{\lambda}) + B\cos(-\sqrt{\lambda})$$
$$= -A\sin(\sqrt{\lambda}) + B\cos(\sqrt{\lambda})$$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos$

$$\psi(1) = A\sin(\sqrt{\lambda}) + B\cos(\sqrt{\lambda})$$

Equating $\psi(-1)$ and $\psi(1)$ gives

$$-A\sin(\sqrt{\lambda}) + B\cos(\sqrt{\lambda}) = A\sin(\sqrt{\lambda}) + B\cos(\sqrt{\lambda})$$

$$\Rightarrow 2A\sin(\sqrt{\lambda}) = 0$$

$$\Rightarrow \begin{cases} A = 0 \Rightarrow \psi(x) = B\cos(\sqrt{\lambda}\,x) \\ \text{or} \\ \sin(\sqrt{\lambda}) = 0 \Rightarrow \sqrt{\lambda} = 0, \pi, 2\pi \end{cases}$$

Now compute, in general

$$\psi'(x) = \sqrt{\lambda}\left(A\cos(\sqrt{\lambda}\,x) - B\sin(\sqrt{\lambda}\,x)\right)$$

Hence $\psi'(-1) = \sqrt{\lambda}\left(A\cos(-\sqrt{\lambda}) - B\sin(-\sqrt{\lambda})\right)$

$$= \sqrt{\lambda}\left\{A\cos(\sqrt{\lambda}) + B\sin(\sqrt{\lambda})\right\}$$

$$\psi'(1) = \sqrt{\lambda}\left\{A\cos(\sqrt{\lambda}) - B\sin(\sqrt{\lambda})\right\}$$

Equating $\psi'(-1)$ and $\psi'(1)$ gives

$$\sqrt{\lambda}\left\{A\cos(\sqrt{\lambda}) + B\sin(\sqrt{\lambda})\right\} = \sqrt{\lambda}\left\{A\cos\sqrt{\lambda} - B\sin\sqrt{\lambda}\right\}$$

$\Rightarrow 2B\sqrt{\lambda}\sin(\sqrt{\lambda}) = 0$

$\Rightarrow \begin{cases} B = 0 & \Rightarrow \psi(x) = A\sin(\sqrt{\lambda}x) \\ \text{or} & \\ \sqrt{\lambda}\sin(\sqrt{\lambda}) = 0 & \Rightarrow \sqrt{\lambda} = 0, \pi, 2\pi, --- \end{cases}$

So $\psi(-1) = \psi(1)$ and $\psi'(-1) = \psi'(1)$ each gives 2 series. We must thus analyze $2\times 2 = 4$ possibilities

① $A = 0$, $B = 0$	② $A = 0$, $\sqrt{\lambda} = 0, \pi, 2\pi, --$	③ $\sqrt{\lambda} = 0, \pi, 2\pi, -$, $B = 0$

④
$\sqrt{\lambda} = 0, \pi, 2\pi, --$
$\sqrt{\lambda} = 0, \pi, 2\pi, ---$

① $A = 0$, $B = 0 \Rightarrow \psi(x) = 0$: Not an eigen function.

② $A = 0$, $\sqrt{\lambda} = 0, \pi, 2\pi, -- \Rightarrow \psi(x) = B\cos(\sqrt{\lambda}x)$, $\sqrt{\lambda} = 0, \pi, 2\pi,$

If $\sqrt{\lambda} = 0$, Pick $B = \frac{1}{2}$, so $\psi_0(x) = \frac{1}{2}\cos(0) = 1$, $\|\psi_0\| = 1$

If $\sqrt{\lambda} = \pi, 2\pi, ---$, Pick $B = 1$,

so $\psi_n(x) = \cos(n\pi x)$, $\|\psi_n\| = 1$,
for $n = 1, 2, 3, ---$

③ $\sqrt{\lambda} = 0, \pi, 2\pi, ---$, $B = 0$

$\Rightarrow \psi(x) = A\sin(\sqrt{\lambda}x)$

If $\sqrt{\lambda} = 0$, $\psi(x) = 0 \Rightarrow$ Not an eigen function

If $\sqrt{\lambda} = \pi, 2\pi, ---$ Pick $A = 1$, so $\psi_{-n}(x) = \sin(n\pi x)$, $\|\psi_{-n}\| = 1$
for $n = 1, 2, 3, ---$

Note:- Indexing here — the negative index is used to distinguish these eigenfunctions from the eigenfunctions $\psi_n(x) = \cos(n\pi x)$ found earlier.

Note that both eigen functions have the same eigen values

$$\lambda_{-n} = \lambda_n = n^2\pi^2$$

$$\text{yet } (\psi_n, \psi_{-n}) = \int_{-1}^{1} \cos(n\pi x)\sin(n\pi x)\,dx = 0$$

So these eigen functions are orthogonal

(4) $\sqrt{\lambda} = 0, \pi, 2\pi, \cdots$

$\sqrt{\lambda} = 0, \pi, 2\pi, \cdots$

$\Rightarrow \psi(x) = A\sin(n\pi x) + B\cos(n\pi x)$

Note that this function is a linear combination of $\psi_n(x)$, $\psi_{-n}(x)$:

$$\psi(x) = \cancel{A\sin}\ A\psi_{-n}(x) + B\psi_n(x)$$

So this solution does not lead to any new eigen functions.

In summary

The eigen values and eigen functions of L are

$\lambda_0 = 0$, $\psi_0(x) = 1/2$

$\lambda_{-n} = n^2\pi^2$, $\psi_{-n}(x) = \sin(n\pi x)$, $n = 1, 2, 3, \cdots$

$\lambda_n = n^2\pi^2$, $\psi_n(x) = \cos(n\pi x)$, $n = 1, 2, 3, \cdots$

Using the techniques described earlier we can easily write down a solution to the heat equation

$$u(x,t) = e^{-\lambda_0 t} a_0(0)\psi_0(x) + \sum_{n=1}^{\infty} e^{-\lambda_n t} a_n(0)\psi_n(x) + \sum_{n=1}^{\infty} e^{-\lambda_n t} a_{-n}(0)\psi_{-n}(x)$$

$$= \frac{1}{2}a_0(0) + \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} a_n(0)\cos(n\pi x) + \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} a_{-n}(0)\sin(n\pi x)$$

where $a_k(0) = \dfrac{(u_0, \psi_k)}{(\psi_k, \psi_k)}$ for $k = 0, \pm 1, \pm 2, -$

But there is another interesting point. We can expand a generic 2-periodic function $f \in C_p^2[-1,1]$ in this basis of eigen functions using the usual best approximation technique.

$$f = \sum_{n=-\infty}^{\infty} C_n \psi_n(x), \quad C_n = \frac{(f, \psi_n)}{(\psi_n, \psi_n)}$$

This gives

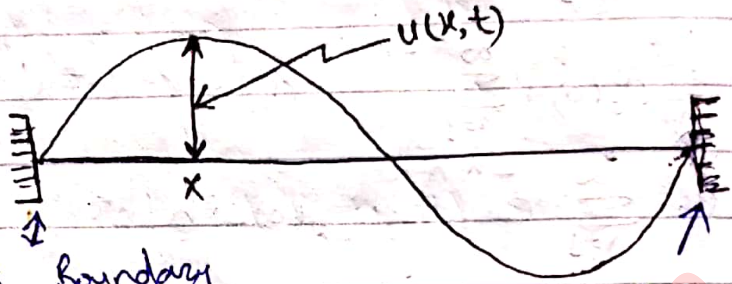
$$\boxed{f(x) = \frac{1}{2}\int_{-1}^{1} f(x)\,dx + \sum_{n=1}^{\infty}\left(\int_{-1}^{1} f(x)\cos(n\pi x)\,dx\right)\cos(n\pi x) + \sum_{n=1}^{\infty}\left(\int_{-1}^{1} f(x)\cos(n\pi x)\,dx\right)\sin(n\pi x)}$$

This expansion is often simply called "The Fourier Series" for a function on $x \in [-1,1]$.

In the context of our class, it is simply an eigenfunction expansion associated with $L u = -u_{xx}$ imposing periodicity in u & u_x.

⇒ Wave Equation Exact Solution Via Spectral Method:-

We seek to model the transverse vibration of a taut string.



$u(x,t)$ describes the distance of the string from its equilibrium position ($u(x,t)=0$) at the point $x \in [0,1]$ at time $t>0$. We shall not derive the wave equation from first principles (See the book "Non linear problems in Elasticity" by "Stuart Antman" for a masterful derivation), but merely state the simplest version

$$u_{tt}(x,t) = u_{xx}(x,t) + f(x,t) \longrightarrow (1)$$

$$u(0,t) = u(1,t) = 0$$

Since the equation is of 2nd order in time, we need initial conditions for both position and velocity.

$$u(x,0) = u_0(x) = \text{Initial displacement of string}$$

$$u_t(x,0) = v_0(x) = \text{〃 velocity of string}$$

More generally, we shall replace eqn (1) by

$$u_{tt} = -Lu + f \longrightarrow ②$$

where L is a symmetric linear operator whose eigenfunctions $\psi_1, \psi_2, \ldots$ allows us to write

$$u(x,t) = \sum_{j=1}^{\infty} a_j(t)\,\psi_j(x) \longrightarrow ③$$

for any fixed time, we write

$$f(x,t) = \sum_{j=1}^{\infty} c_j(t)\,\psi_j(x)$$

But in this lecture forces on time case of $f(x,t)=0$.
We follow the same strategy we used for the heat equation.
Use the PDE ② to derive the ordinary differential equations that govern the coefficients $a_j(t)$.
Substitute equ ③ in equ ② to obtain

$$\frac{\partial^2}{\partial t^2}\sum_{j=1}^{\infty} a_j(t)\,\psi_j(x) = -L\sum_{j=1}^{\infty} a_j(t)\,\psi_j(x)$$

$$\Rightarrow \sum_{j=1}^{\infty} a_j''(t)\,\psi_j(x) = \sum_{j=1}^{\infty} a_j(t)\left(-L\,\psi_j(x)\right)$$

use $L\psi_j = \lambda_j \psi_j$ to obtain

$$\sum_{j=1}^{\infty} a_j''(t)\,\psi_j(x) = \sum_{j=1}^{\infty} a_j(t)\left(-\lambda_j\,\psi_j(x)\right)$$

Take the inner product of both sides with ψ_k to get

$$\left[\sum_{j=1}^{\infty} a_j''(t)\,\psi_j, \psi_k\right] = \left[\sum_{j=1}^{\infty} a_j(t)(-\lambda_j)\,\psi_j, \psi_k\right]$$

$$\Rightarrow \sum_{j=1}^{\infty} a_j''(t)\,(\psi_j, \psi_k) = \sum_{j=1}^{\infty} a_j(t)(-\lambda_j)(\psi_j, \psi_k)$$

use orthogonality of the eigen functions

$$(\psi_j, \psi_k) = \begin{cases} 0 & \text{if } j \neq k \\ \neq 0 & \text{if } j = k \end{cases}$$

to obtain

$$a_k''(t) = -\lambda_k a_k(t)$$

(Compare this to the equation $(a_k'(t) = -\lambda_k a_k(t))$ obtained for the heat equation)

This ODE for $a_k(t)$ has the same form as the ODE we have been using all semesters to find eigen-functions (in-space, x). It has the general solution

$$a_k(t) = A\sin(\sqrt{\lambda_k}\,t) + B\cos(\sqrt{\lambda}\,t)$$

But Now we use initial conditions to determine A and B. use

$$a_k(0) = A\sin(0) + B\cos(0) = B$$

$$a_k'(t) = \sqrt{\lambda_k}\left[A\cos\sqrt{\lambda_k}\,t - B\sin(\sqrt{\lambda}\,t)\right]$$

$$\Rightarrow a_k'(0) = \sqrt{\lambda_k}\left[A\cos(0) - B\sin(0)\right] = A\sqrt{\lambda_k}$$

If we expand the initial condition as

$$u(x,0) = \sum_{j=1}^{\infty} a_j(0)\,\psi_j(x) = u_0(x).$$

Take an inner product with ψ_k

to find
$$a_k(0) = \frac{(U_0, \psi_k)}{(\psi_k, \psi_k)} \quad ; k = 1, 2, 3, \ldots$$

Similarly expand the initial condition
$$U_t(x, 0) = \sum_{j=1}^{\infty} a_j'(0)\, \psi_j(x) = V_0(x)$$

and take the inner product with ψ_k to obtain
$$a_k'(0) = \frac{(V_0, \psi_k)}{(\psi_k, \psi_k)}$$

These formulas for $a_k(0)$ and $a_k'(0)$ are of course, just the best approximation for U_0 and V_0.

Now we can identify the A and B in the formulas for $a_k(t)$

$$a_k(0) = B \Rightarrow B = \frac{(U_0, \psi_k)}{(\psi_k, \psi_k)}$$

$$a_k'(0) = \sqrt{\lambda_k}\, A \Rightarrow A = \frac{1}{\sqrt{\lambda_k}} \frac{(V_0, \psi_k)}{(\psi_k, \psi_k)} \quad (\text{Provide } \lambda_k \neq 0)$$

Thus, presuming L has no zero eigen values we can express
$$a_k(t) = \frac{(U_0, \psi_k)}{(\psi_k, \psi_k)} \cos(\sqrt{\lambda_k}\, t) + \frac{1}{\sqrt{\lambda_k}} \frac{(V_0, \psi_k)}{(\psi_k, \psi_k)} \sin(\sqrt{\lambda_k}\, t)$$

with
$$U(x,t) = \sum_{k=1}^{\infty} \underbrace{\left[\frac{(U_0, \psi_k)}{(\psi_k, \psi_k)} \cos(\sqrt{\lambda_k}\, t) + \frac{1}{\sqrt{\lambda_k}} \frac{(V_0, \psi_k)}{(\psi_k, \psi_k)} \sin(\sqrt{\lambda_k}\, t)\right]}_{} \psi_k(x)$$

Note that this term oscillates in t. Fundamentally different behaviour than for the heat equation.

In particular, note that these oscillations occur as a result of the $\partial^2/\partial t^2$ term, independent of the eigenfunctions $\varphi_k(x)$.

For a vibrating string, This suggests that an initial pluck will cause the string to vibrate forever. To get around this apparently unrealistic behaviour, we can add damping to the model

$$u_{tt}(x,t) = u_{xx}(x,t) - \underbrace{\gamma\, u_t(x,t)}$$

"Viscous Damping"
In proportional to the string velocity.

Instead of persuing this direction here, we shall instead consider in the next lecture what happens when we drive the string with a forcing function.

MUHAMMAD TAHIR WATTOO

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⇒ Chebyshev Differentiation Matrices:-

1) Discretize the interval $[-1, 1]$ using chebychev points

$$x_j = \cos\left(\frac{j\pi}{n}\right) ; \; j = 0, 1, \cdots, N$$

$$y = [y(x_0), y(x_1), \cdots, y(x_N)]$$

Guass Lobato Points
$x_j = \cos\left(\frac{j\pi}{n}\right)$
$j = 0, 1, \cdots, N$

2) Find the algebraic polynomials P of degree at most N that interpolate the data, such that $P(x_i) = y_i$

3) Obtain spectral derivative vector y' by differentiating P and evaluating at the grid points.

$$y'_i = P'(x_i) \quad , i = 0, 1, \cdots, N$$

This procedure will give us D_n and $\boxed{y' = D_N y}$

For Chebyshev Polynomials
$D^2 = (D)(D)$
$\frac{d^2}{dx^2} = \left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right)$

First row at x_0 & 2nd row at x_1

For $N=1$ We have two points

$$x_0 = 1 \quad , \quad x_1 = -1$$

$$P(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

$$\Rightarrow P(x) = \frac{x+1}{1+1} y_0 + \frac{x-1}{-1-1} y_1$$

$$= \frac{x+1}{2} y_0 + \frac{1-x}{2} y_1$$

$$P'(x) = \frac{1}{2} y_0 - \frac{1}{2} y_1$$

$$\Rightarrow y' = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = D_1 y$$

For $n=2$:- $x_0 = 1$, $x_1 = 0$, $x_2 = -1$

$$P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$\Rightarrow P(x) = \frac{(x-0)(x+1)}{(1-0)(1+1)} y_0 + \frac{(x-1)(x+1)}{(0-1)(0+1)} y_1 + \frac{(x-1)(x-0)}{(-1-1)(-1-0)} y_2$$

$$= \frac{x(x+1)}{2} y_0 - (x-1)(x+1) y_1 + \frac{(x-1)(x)}{2} y_2$$

$$\Rightarrow P'(x) = \left(x+\frac{1}{2}\right) y_0 - 2x y_1 + \left(x-\frac{1}{2}\right) y_2$$

$$\Rightarrow P'(x_1) = \left(x_1+\frac{1}{2}\right) y_0 - 2x_1 y_1 + \left(x_1-\frac{1}{2}\right) y_2$$

$$= \left(0+\frac{1}{2}\right) y_0 - 2(0) y_1 + \left(0-\frac{1}{2}\right) y_2$$

$$= \frac{1}{2} y_0 - 0 - \frac{1}{2} y_2$$

$$P'(x_0) = \left(x_0+\frac{1}{2}\right) y_0 - 2x_0 y_1 + \left(x_0-\frac{1}{2}\right) y_2$$

$$= \left(1+\frac{1}{2}\right) y_0 - 2(1) y_1 + \left(1-\frac{1}{2}\right) y_2$$

$$= \frac{3}{2} y_0 - 2y_1 + \frac{1}{2} y_2$$

$$P'(x_2) = \left(x_2+\frac{1}{2}\right) y_0 - 2x_2 y_1 + \left(x_2-\frac{1}{2}\right) y_2$$

$$= \left(-1+\frac{1}{2}\right) y_0 - 2(-1) y_1 + \left(-1-\frac{1}{2}\right) y_2$$

$$= -\frac{1}{2} y_0 + 2y_1 - \frac{3}{2} y_2$$

$$\Rightarrow y' = \begin{bmatrix} \frac{3}{2} & -2 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \Rightarrow D_2 = \begin{bmatrix} \frac{3}{2} & -2 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

$$\Rightarrow y' = D_2 y$$

* For $N \geq 1$, set the rows and columns of $(N+1) \times (N+1)$
* Chebyshev spectral differentiation matrix D_N be the indexed from 0 to N, The entries of this matrix are

$$(D_N)_{00} = \frac{2N^2+1}{6}, \quad (D_N)_{NN} = \frac{-(2N^2+1)}{6}$$

$$(D_N)_{jj} = \frac{-x_j}{2(1-x_j^2)} ; j = 1, 2, \cdots, N-1 \quad \left\{ \begin{array}{l} \text{Diagonal} \\ \text{Entries} \end{array} \right.$$

$$(D_N)_{ij} = \frac{c_i (-1)^{i+j}}{c_j (x_i - x_j)}, \; i \neq j \quad \left\{ \text{Non-Diagonal} \right.$$

where $c_i = \begin{cases} 2 & ; \; i = 0, N \\ 1 & ; \; \text{otherwise} \end{cases}$

Example:- From above relation find matrix for $N=2$

Solution:

$$D_2 = \begin{bmatrix} D_{00} & D_{01} & D_{02} \\ D_{10} & D_{11} & D_{12} \\ D_{20} & D_{21} & D_{22} \end{bmatrix} \quad \left\{ \begin{array}{l} x_0 = 1 \\ x_1 = 0 \\ x_2 = -1 \end{array} \right.$$

$$(D_2)_{00} = \frac{2(2)^2+1}{6} = \frac{9}{6} = \frac{3}{2}$$

$$(D_2)_{22} = \frac{-[2(2)^2+1]}{6} = \frac{-9}{6} = \frac{-3}{2}$$

$$(D_2)_{11} = \frac{-x_1}{2(1-x_1^2)} = \frac{0}{2(1-0)} = 0$$

$$(D_2)_{01} = \frac{c_0(-1)^{0+1}}{c_1(x_0-x_1)} = \frac{2(-1)}{1(1-0)} = -2$$

$$(D_2)_{02} = \frac{c_0(-1)^{0+2}}{c_2(x_0-x_2)} = \frac{2(+1)}{2(1+1)} = \frac{1}{2}$$

$$(D_2)_{10} = \frac{c_1(-1)^{1+0}}{c_0(x_1-x_0)} = \frac{1(-1)}{2(0-1)} = \frac{1}{2}$$

$$(D_2)_{12} = \frac{c_1(-1)^{1+2}}{c_2(x_1-x_2)} = \frac{1(-1)}{2(0+1)} = \frac{-1}{2}$$

$$(D_2)_{20} = \frac{c_2(-1)^{2+0}}{c_0(x_2-x_0)} = \frac{2(1)}{2(-1-1)} = \frac{-1}{2}$$

$$(D_2)_{21} = \frac{c_2(-1)^{2+1}}{c_1(x_2-x_1)} = \frac{2(-1)}{1(-1-0)} = 2$$

$$\Rightarrow D_2 = \begin{bmatrix} \frac{3}{2} & -2 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{-1}{2} & 2 & \frac{-3}{2} \end{bmatrix}$$

Exercise:- Solve $y'' = e^{4x}$; $N=4$

$y(-1) = y(1) = 0$

Solution

$\because y'' = D^2 y$

first of all we have to find D.
For this we can have Gauss Lobato points

$$x_i = \cos\left(\frac{\pi i}{N}\right)$$

Here we have $N=4$ i.e $0 \le i \le 4$
So the grid points will be given as (Galerkin Lobato points)

$x_0 = \cos(0) = 1$, $x_1 = \cos\left(\frac{\pi}{4}\right) = 0.7071$

$x_2 = \cos\left(\frac{\pi}{2}\right) = 0$, $x_3 = \cos\left(\frac{3\pi}{4}\right) = -0.7071$

$x_4 = \cos(\pi) = -1$

Now we find D_4 (Chybeshev spectral differential matrix)

$$D_4 = \begin{bmatrix} (D_4)_{00} & (D_4)_{01} & (D_4)_{02} & (D_4)_{03} & (D_4)_{04} \\ (D_4)_{10} & (D_4)_{11} & (D_4)_{12} & (D_4)_{13} & (D_4)_{14} \\ (D_4)_{20} & (D_4)_{21} & (D_4)_{22} & (D_4)_{23} & (D_4)_{24} \\ (D_4)_{30} & (D_4)_{31} & (D_4)_{32} & (D_4)_{33} & (D_4)_{34} \\ (D_4)_{40} & (D_4)_{41} & (D_4)_{42} & (D_4)_{43} & (D_4)_{44} \end{bmatrix}$$

$$(D_4)_{00} = \frac{2(4)^2+1}{6} = \frac{33}{6} = 5.5$$

$$(D_4)_{44} = -\frac{2N^2+1}{6} = \frac{-11}{6} = -5.5$$

$$(D_4)_{11} = \frac{-x_1}{2(1-x_1^2)} = \frac{-0.7071}{2(1-(0.7071)^2)} = -0.7071$$

$$(D_4)_{22} = \frac{-x_2}{2(1-x_2^2)} = \frac{0}{2(1-0^2)} = 0$$

$$(D_4)_{33} = \frac{-x_3}{2(1-x_3^2)} = \frac{0.7071}{2(1-(0.7071)^2)} = 0.7071$$

$$(D_4)_{10} = \frac{c_1(-1)^{1+0}}{c_0(x_1-x_0)} = \frac{1(-1)}{2(0.7071-1)} = 1.7071$$

$$(D_4)_{01} = \frac{c_0(-1)^{0+1}}{c_1(x_0-x_1)} = \frac{2(-1)}{1(1-0.7071)} = -6.8283$$

$$(D_4)_{02} = \frac{c_0(-1)^{0+2}}{c_2(x_0-x_2)} = \frac{2(1)}{1(1-0)} = 2$$

$$(D_4)_{03} = \frac{c_0(-1)^{0+3}}{c_3(x_0-x_3)} = \frac{2(-1)}{1(1+0.7071)} = -1.1716$$

$$(D_4)_{04} = \frac{c_0(-1)^{0+4}}{c_4(x_0-x_4)} = \frac{2(1)}{2(1+1)} = 0.5$$

$$(D_4)_{12} = \frac{c_1(-1)^{1+2}}{c_2(x_1-x_2)} = \frac{1(-1)}{1(0.7071-0)} = -1.4142$$

$$(D_4)_{13} = \frac{c_1(-1)^{1+3}}{c_3(x_1-x_3)} = \frac{1(1)}{1(0.7071+0.7071)} = 0.7071$$

$$(D_4)_{14} = \frac{c_1(-1)^{1+4}}{c_4(x_1-x_4)} = \frac{1(-1)}{2(0.7071+1)} = -0.2929$$

$$(D_4)_{20} = \frac{C_2(-1)^{2+0}}{C_0(x_2-x_0)} = \frac{1(1)}{2(0-1)} = -0.5$$

$$(D_4)_{21} = \frac{C_2(-1)^{2+1}}{C_1(x_2-x_1)} = \frac{1(-1)}{1(0-0.7071)} = 1.4142$$

$$(D_4)_{23} = \frac{C_3(-1)^{2+3}}{C_3(x_2-x_3)} = \frac{1(-1)}{1(0+0.7071)} = -1.4142$$

$$(D_4)_{24} = \frac{C_2(-1)^{2+4}}{C_4(x_2-x_4)} = \frac{1(1)}{2(0+1)} = 0.5$$

similarly

$(D_4)_{30} = 0.2929$, $(D_4)_{31} = -0.7071$

$(D_4)_{32} = 1.4142$, $(D_4)_{34} = -1.7071$

$(D_4)_{40} = -0.5$, $(D_4)_{41} = 1.1716$

$(D_4)_{42} = -2$ & $(D_4)_{43} = 6.8283$

$$\Rightarrow D = \begin{bmatrix} 5.5 & -6.8283 & 2 & -1.1716 & 0.5 \\ 1.7071 & -0.7071 & -1.4142 & 0.7071 & -0.2929 \\ -0.5 & 1.4142 & 0 & -1.4142 & 0.5 \\ 0.2929 & -0.7071 & 1.4142 & 0.7071 & -1.7071 \\ -0.5 & 1.1716 & -2 & 6.8283 & -5.5 \end{bmatrix}$$

As we know $y'' = D^2 y$

So

$$y'' = \begin{bmatrix} 17 & -28.4845 & 18 & -11.5147 & 5 \\ 9.2426 & -14 & 6 & -2 & 0.7574 \\ -1 & 4 & -6 & 4 & -1 \\ 0.7574 & -2 & 6 & -14 & 9.2426 \\ 5 & -11 & 18 & -28.4853 & 17 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Now from the boundary conditions

$$y(-1) = y(1) = 0$$

So that we have

$$y'' = \begin{bmatrix} -14 & 6 & -2 \\ 4 & -6 & 4 \\ -2 & 6 & -14 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

We have $f(x) = e^{4x}$

$$\Rightarrow f(x_1) = e^{4x_1} = e^{4(0.7071)} = 16.9198$$

$$f(x_2) = e^{4x_2} = e^{4(0)} = 1$$

$$f(x_3) = e^{4x_3} = e^{4(-0.7071)} = 0.591$$

So that

$$\begin{bmatrix} -14 & 6 & -2 \\ 4 & -6 & 4 \\ -2 & 6 & -14 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 16.9198 \\ 1 \\ 0.591 \end{bmatrix}$$

$$AY = f(x) \quad \text{where } A = D^2$$

$$\Rightarrow Y = A^{-1} f(x)$$

$$\Rightarrow Y = \begin{bmatrix} -0.1042 & -0.125 & -0.0208 \\ -0.0833 & -0.333 & -0.0833 \\ -0.208 & -0.1250 & -0.1042 \end{bmatrix} \begin{bmatrix} 16.9198 \\ 1 \\ 0.591 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1.8886 \\ -1.7482 \\ -0.4836 \end{bmatrix}$$

<u>Exact Solution</u>:-

$$y(x) = \frac{e^{4x} - x\sinh(4) - \cosh(4)}{16}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1.8554 \\ -1.6443 \\ -0.4970 \end{bmatrix}$$

Assignment No. 2.

Question 1:- Solve $u'' = e^{4x}$, $u(\pm 1) = 0$ by Chebysheve differential matrix D_N method for $N = 4$.

Exact solution $u(x) = \dfrac{e^{4x} - x\sinh(4) - \cosh(4)}{16}$

(Already solved)

Question 2:- Consider the Non-Linear ODE

$$u_{xx} = e^{u} \quad \text{with} \quad u(-1) = u(1) = 0$$

Solve it by chebyshev differentiation matrix D_N method for $N=4$.

Solution

For discretization of interval, we will use chebyshev Gauss-Lobatto formula $x_j = \cos\left(\frac{\pi j}{N}\right)$; $j = 0, 1, \cdots, N$

for $N=4$ $\quad x_j = \cos\left(\frac{\pi j}{4}\right)$; $j = 0, 1, 2, \cdots, 4$

So we have $x_0 = 1$, $x_1 = 0.707$, $x_2 = 0$

$x_3 = -0.707$ & $x_4 = -1$

As in previous problem we have already calculated chebyshev differentiation matrix D_N and consequently found D_4^2

i.e

$$D_4^2 = \begin{bmatrix} 17 & -28.4853 & 18 & -11.5147 & 5 \\ 9.2426 & -14 & 6 & -2 & 0.7574 \\ -1 & 4 & -6 & 4 & -1 \\ 0.7574 & -2 & 6 & -14 & 9.2426 \\ 5 & -11.5147 & 18 & -28.4853 & 17 \end{bmatrix}$$

Now we discard boundaries because of given boundary conditions $u(\pm 1) = 0$.

So we have

$$D_4^2 = \begin{bmatrix} -14 & 6 & -2 \\ 4 & -6 & 4 \\ -2 & 6 & -14 \end{bmatrix}$$

As we have $D_4^2 u = u_{xx}$ or $D_4^2 u = u''$

so ODE becomes

$$D_4^2 u = e^{u}$$

i.e
$$\begin{bmatrix} -14 & 6 & -2 \\ 4 & -6 & 4 \\ -2 & 6 & -14 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} e^{u_1} \\ e^{u_2} \\ e^{u_3} \end{bmatrix}$$

or
$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -14 & 6 & -2 \\ 4 & -6 & 4 \\ -2 & 6 & -14 \end{bmatrix}^{-1}\begin{bmatrix} e^{u_1} \\ e^{u_2} \\ e^{u_3} \end{bmatrix}$$

As in previous problem we have calculated the inverse of above 3x3 matrix. So we have

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -0.1042 & -0.125 & -0.208 \\ -0.0833 & -0.333 & -0.0833 \\ -0.0208 & -0.125 & -0.1042 \end{bmatrix}\begin{bmatrix} e^{u_1} \\ e^{u_2} \\ e^{u_3} \end{bmatrix}$$

Now we will solve it by iterative method

$$\Rightarrow \begin{bmatrix} u_1^{k+1} \\ u_2^{k+1} \\ u_3^{k+1} \end{bmatrix} = \begin{bmatrix} -0.1042 & -0.125 & -0.208 \\ -0.0833 & -0.333 & -0.0833 \\ -0.0208 & -0.125 & -0.1042 \end{bmatrix}\begin{bmatrix} e^{u_1^k} \\ e^{u_2^k} \\ e^{u_3^k} \end{bmatrix} \longrightarrow (1)$$

For $k=0$ taking initial guess ~~at~~ $u=(0,0,0)$

$$(1) \Rightarrow \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = \begin{bmatrix} \text{Same as Above} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = \begin{bmatrix} -0.25 \\ -0.4996 \\ -0.25 \end{bmatrix}$$

for $K=1$ $\begin{bmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \end{bmatrix} = \begin{bmatrix} 0.0937 \\ 0.20801 \\ 0.0937 \end{bmatrix}$ & for $K=2$ $\begin{bmatrix} u_1^3 \\ u_2^3 \\ u_3^3 \end{bmatrix} = \begin{bmatrix} -0.0377 \\ -0.0771 \\ -0.0377 \end{bmatrix}$

And by using MATLAB program we get the result upto certain accuracy

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -0.1899 \\ -0.3684 \\ -0.1899 \end{bmatrix}$$

Question 3:- Consider the ODE

$$u'' = e^{4x} \quad \text{with} \quad u'(-1) = 0 \;\&\; u(1) = 0$$

Solve it by chebyshev differentiation matrix D_N method for $N=4$.

Solution

Like previous two questions we can calculate D_4 and D_4^2. so from previous problem we have

$$D_4 = \begin{bmatrix} 5.5 & -6.8284 & 2 & -1.1716 & 0.5 \\ 1.7071 & -0.7071 & -1.4142 & 0.7071 & -0.2929 \\ -0.5 & 1.4142 & 0 & -1.4142 & 0.5 \\ 0.2929 & -0.7071 & 1.4142 & 0.7071 & -1.7071 \\ -0.5 & 1.1716 & -2 & 6.8284 & -5.5 \end{bmatrix}$$

And

$$D_4^2 = \begin{bmatrix} 17 & -28.4853 & 18 & -11.5147 & 5 \\ 9.2426 & -14 & 6 & -2 & 0.7574 \\ -1 & 4 & -6 & 4 & -1 \\ 0.7574 & -2 & 6 & -14 & 9.2426 \\ 5 & -11.5147 & 18 & -28.4853 & 17 \end{bmatrix}$$

Now we discard 1st row and 1st column of D_4 and D_4^2. also we we replace 5th row of D_4^2 with 5th row of D_4 Because of B.Cs $u'(-1)=0$ and $u(1)=0$.
So we have

$$D_4^2 = \begin{bmatrix} -14 & 6 & -2 & 0.7574 \\ 4 & -6 & 4 & -1 \\ -2 & 6 & -14 & 9.2426 \\ 1.1716 & -2 & 6.8284 & -5.5 \end{bmatrix} \longrightarrow (1)$$

Now as $u'' = D_4^2 u$ so we have
$D_4^2 u = e^{4x}$, $f(x) = e^{4x}$

$$\Rightarrow D_4^2 u = f(x)$$

i.e
$$\begin{bmatrix} -14 & 6 & -2 & 0.7574 \\ 4 & -6 & 4 & -1 \\ -2 & 6 & -14 & 9.2426 \\ 1.1716 & -2 & 6.8284 & -5.5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -0.1328 & -0.2226 & -0.1875 & -0.2929 \\ -0.1810 & -0.6667 & -0.6524 & -1 \\ -0.1875 & -0.6940 & -1.7056 & -1.7071 \\ -0.1953 & -0.667 & -1.1381 & -2 \end{bmatrix} \begin{bmatrix} 18.9188 \\ 1 \\ 0.0591 \\ 0.01831 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} -2.4853 \\ -3.7852 \\ -3.9611 \\ -4.0741 \end{bmatrix}$$

Question 4:- Consider the ODE

$$u'' = e^{4x} \quad \text{with} \quad u(-1) = 0 \ \& \ u(1) = 1$$

Solve it by Chebyshev differentiation matrix D_4 method for $N=4$.

Solution

Like previous problems, we will calculate D_4 and D_4^2. Then we will discard 1st row and 1st column of D_4^2 because of B.Cs $u(-1) = 0$ and $u(1)=1$, so we will have a 4×4 matrix D_4^2 as

$$D_4^2 = \begin{bmatrix} -14 & 6 & -2 & 0.7574 \\ 4 & -6 & 4 & -1 \\ -2 & 6 & -14 & 9.2426 \\ -11.5147 & 18 & -28.4853 & 17 \end{bmatrix}$$

Now as $D_4^2 u = u''$, so we have

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} \\ \text{Same as Above} \\ \\ \end{bmatrix}^{-1} \begin{bmatrix} 16.9188 \\ 1 \\ 0.0591 \\ 0.0183 \end{bmatrix}$$

⇒ **Spectral Methods in Multi-Dimentional Domain:**— Consider a poisson equation

$$u_{xx} + u_{yy} = f(x, y)$$ with appropriate boundary conditions

★ The solution is written in terms of product of two functions

$$u(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} a_{mn} \phi_m(x) \phi_n(y)$$

In periodic function problem:—

$$\phi_m(x) = e^{-imx}, \quad \phi_n(x) = e^{-iny}$$

Non periodic

$$\left.\begin{aligned} \phi_m(x) &= T_m(x) \\ \phi_n(y) &= T_n(y) \end{aligned}\right\} \longrightarrow \text{Cheb}$$

$$\left.\begin{aligned} \phi_m(x) &= L_m(x) \\ \phi_n(y) &= L_n(y) \end{aligned}\right\} \longrightarrow \text{Legender}$$

★ Multidimentional problem require a larger compotational effect.

★ Here we have to compute the sum

$$u(x_i, y_i) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} a_{mn} \phi_m(x_i) \phi_n(y_i)$$ on a grid with $N \times M$ collocation points.

$(i = 0, 1, \ldots, M-1)$, $(j = 0, 1, \ldots, N-1)$

★ The easiest way to solve a problem on a tensor product spectral grid is to use tensor product in linear algebra, also known as kronecker product.

Definition:- The kronecker product of two matrix A and B is denoted by $A \otimes B$. If A and B are of dimension $p \times q$ and $r \times s$ respectively then $A \otimes B$ is the matrix of dimension $pr \times qs$ with $p \times q$ block forms.

Example 1

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow A \otimes B = \left[\begin{array}{cc|cc} a & b & 2a & 2b \\ c & d & 2c & 2d \\ \hline 3a & 3b & 4a & 4b \\ 3c & 3d & 4c & 4d \end{array}\right]$$

Example 2

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix}$$

$$\Rightarrow A \otimes B = \left[\begin{array}{cc|cc} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ \hline 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{array}\right]$$

* Our Laplacian will be the kronecker

Sum $\quad L_N = I \otimes \tilde{D}_N^2 + \tilde{D}_N^2 \otimes I,$

where I is identity matrix of same order as $\tilde{D}_N^2$.

★ Now the 3x3 differentiation matrix with N=4 in one dimension is given by D=cheb(4); D2=D^2;
D2 = D2(2:4,2:4);

$$\tilde{D}_4^2 = \begin{bmatrix} -14 & 6 & -2 \\ 4 & -6 & 4 \\ -2 & 6 & -14 \end{bmatrix}$$

If I denotes the 3x3 identity, then the 2nd derivative with respect to x will accordingly by computed by the matrix kron(I,D2);

$$I \otimes \tilde{D}_N^2 = \left[\begin{array}{ccc|ccc|ccc} -14 & 6 & -2 & & & & & & \\ 4 & -6 & 4 & & & & & & \\ -2 & 6 & -14 & & & & & & \\ \hline & & & -14 & 6 & -2 & & & \\ & & & 4 & -6 & 4 & & & \\ & & & -2 & 6 & -14 & & & \\ \hline & & & & & & -14 & 6 & -2 \\ & & & & & & 4 & -6 & 4 \\ & & & & & & -2 & 6 & -14 \end{array}\right]$$

The 2nd derivative w.r.t y will be computed by kron(D_2, I);

$$\tilde{D}_N^2 \otimes I = \left[\begin{array}{ccc|ccc|ccc}
-14 & & & 6 & & & -2 & & \\
 & -14 & & & 6 & & & -2 & \\
 & & -14 & & & 6 & & & -2 \\
\hline
4 & & & -6 & & & 4 & & \\
 & 4 & & & -6 & & & 4 & \\
 & & 4 & & & -6 & & & 4 \\
\hline
-2 & & & 6 & & & -14 & & \\
 & -2 & & & 6 & & & -14 & \\
 & & -2 & & & 6 & & & -14
\end{array}\right]$$

Question: Solve the following 2D ODE by Chebyshev differentiation matrix for $N=3$

$$u_{xx} + u_{yy} = 10\sin(8x(y-1)) \,;$$

$$-1 < x, y < 1 \;;\; u = 0 \text{ on boundaries}$$

Solution For discretization of interval we will use Chebyshev Gauss Lobato formula

$$x_j = \cos\left(\frac{\pi j}{N}\right) \;;\; j = 0, 1, 2, \cdots, N$$

For $N=4$

$$x_j = \cos\left(\frac{\pi j}{4}\right) \;;\; j = 0, 1, \cdots, 4$$

So we have

$x_0 = 1$, $x_1 = 0.707$, $x_2 = 0$, $x_3 = -0.707$

$x_4 = -1$ And $y_0 = 1$, $y_1 = 0.707$

$y_2 = 0$, $y_3 = -0.707$ & $y_4 = -1$

Now we find D_4 (Chebyshev diffirentiation matrix.

And

$$D_4 = \begin{bmatrix} 5.5 & -6.8283 & 2 & -1.1716 & 0.5 \\ 1.7071 & -0.7071 & -1.4142 & 0.7071 & -0.2929 \\ -0.5 & 1.4142 & 0 & -1.4142 & 0.5 \\ 0.2929 & -0.7071 & 1.4142 & 0.7071 & -1.7071 \\ -0.5 & 1.1716 & -2 & 6.8283 & -5.5 \end{bmatrix}$$

And

$$D_4^2 = \begin{bmatrix} 17 & -28.4845 & 18 & -11.5147 & 5 \\ 9.2426 & -14 & 6 & -2 & 0.7574 \\ -1 & 4 & -6 & 4 & -1 \\ 0.7574 & -2 & 6 & -14 & 9.2426 \\ 5 & -11.5147 & 18 & -28.4845 & 17 \end{bmatrix}$$

Now from boundary conditions –
Since $u=0$ on boundaries. So we have

$$\tilde{D}_4^2 = \begin{bmatrix} -14 & 6 & -2 \\ 4 & -6 & 4 \\ -2 & 6 & -14 \end{bmatrix}$$

As in 2D our Laplacian will be

$$L_N = I \otimes \tilde{D}_4^2 + \tilde{D}_4^2 \otimes I \longrightarrow (1)$$

Where $I \otimes \tilde{D}_4^2$ is 2nd derivative with respect to x and $\tilde{D}_4^2 \otimes I$ is 2nd derivative with respect to y.
And $\otimes$ denotes the Kronecker product

Now

$$I\otimes\tilde{D}_4^2 = \left[\begin{array}{ccc|ccc|ccc}
-14 & 6 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & -6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 6 & -14 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -14 & 6 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & -6 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 6 & -14 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -14 & 6 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & -6 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 6 & -14
\end{array}\right]$$

And

$$\tilde{D}_4^2\otimes I = \left[\begin{array}{ccc|ccc|ccc}
-14 & 0 & 0 & 6 & 0 & 0 & -2 & 0 & 0 \\
0 & -14 & 0 & 0 & 6 & 0 & 0 & -2 & 0 \\
0 & 0 & -14 & 0 & 0 & 6 & 0 & 0 & -2 \\
\hline
4 & 0 & 0 & -6 & 0 & 0 & 4 & 0 & 0 \\
0 & 4 & 0 & 0 & -6 & 0 & 0 & 4 & 0 \\
0 & 0 & 4 & 0 & 0 & -6 & 0 & 0 & 4 \\
\hline
-2 & 0 & 0 & 6 & 0 & 0 & -14 & 0 & 0 \\
0 & -2 & 0 & 0 & 6 & 0 & 0 & -14 & 0 \\
0 & 0 & -2 & 0 & 0 & 6 & 0 & 0 & -14
\end{array}\right]$$

$$\Rightarrow L_4 = \begin{bmatrix}
-28 & 6 & -2 & 6 & 0 & 0 & -2 & 0 & 0 \\
4 & -20 & 4 & 0 & 6 & 0 & 0 & -2 & 0 \\
-2 & 6 & -28 & 0 & 0 & 6 & 0 & 0 & -2 \\
4 & 0 & 0 & -20 & 6 & -2 & 4 & 0 & 0 \\
0 & 4 & 0 & 4 & -12 & 4 & 0 & 4 & 0 \\
0 & 0 & 4 & -2 & 6 & -20 & 0 & 0 & 4 \\
-2 & 0 & 0 & 6 & 0 & 0 & -28 & 6 & -2 \\
0 & -2 & 0 & 0 & 6 & 0 & 4 & -20 & 4 \\
0 & 0 & -2 & 0 & 0 & 6 & -2 & 6 & -28
\end{bmatrix}$$

As we know that

$$L_4 u = f(x, y)$$

where $f(x,y) = 10 \sin[8x(y-1)]$

$$\Rightarrow \begin{bmatrix} \text{Same} & & \\ & \text{Previous} & \\ & & \text{Matrix} \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{31} \\ u_{32} \\ u_{33} \end{bmatrix} = \begin{bmatrix} f(x_1,y_1) \\ f(x_1,y_2) \\ f(x_1,y_3) \\ f(x_2,y_1) \\ f(x_2,y_2) \\ f(x_2,y_3) \\ f(x_3,y_1) \\ f(x_3,y_2) \\ f(x_3,y_3) \end{bmatrix}$$

Now $f(x_1,y_1) = 10 \sin[8(0.707)(0.707-1)]$

$= -9.9627$

$f(x_1,y_2) = 10 \sin[8(0.707)(0-1)]$

$= 5.8687$

$f(x_1,y_3) = 10 \sin[8(0.707)(-0.707-1)]$

$= 2.2799$

$f(x_2,y_1) = 10 \sin[8(0)(0.707-1)]$

$= 0$

$f(x_2,y_2) = 10 \sin[8(0)(0-1)]$

$= 0$

$f(x_2,y_3) = 10 \sin[8(0)(-0.707-1)]$

$= 0$

$$f(x_3, y_1) = 10\sin[8(-0.707)(0.707-1)]$$
$$= 9.9627$$

$$f(x_3, y_2) = 10\sin[8(-0.707)(0-1)]$$
$$= -5.8687$$

$$f(x_3, y_3) = 10\sin[8(-0.707)(-0.707-1)]$$
$$= -2.2799$$

$$\Rightarrow \begin{bmatrix} -28 & 6 & -2 & 6 & 0 & 0 & -2 & 0 & 0 \\ 4 & -20 & 4 & 0 & 6 & 0 & 0 & -2 & 0 \\ -2 & 6 & -28 & 0 & 0 & 6 & 0 & 0 & -2 \\ 4 & 0 & 0 & -20 & 6 & 2 & 4 & 0 & 0 \\ 0 & 4 & 0 & 4 & -12 & 4 & 0 & 4 & 0 \\ 0 & 0 & 4 & -2 & 6 & -20 & 0 & 0 & 4 \\ -2 & 0 & 0 & 6 & 0 & 0 & -28 & 6 & -2 \\ 0 & -2 & 0 & 0 & 6 & 0 & 4 & -20 & 4 \\ 0 & 0 & -2 & 0 & 0 & 6 & -2 & 6 & -28 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{31} \\ u_{32} \\ u_{33} \end{bmatrix} = \begin{bmatrix} -9.9627 \\ 5.8687 \\ 2.2799 \\ 0 \\ 0 \\ 0 \\ 9.9627 \\ -5.8687 \\ -2.2799 \end{bmatrix}$$

$$AU = B \quad \Rightarrow \quad U = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} U_{11} \\ U_{12} \\ U_{13} \\ U_{21} \\ U_{22} \\ U_{23} \\ U_{31} \\ U_{32} \\ U_{33} \end{bmatrix} = \begin{bmatrix} -0.401 & -0.0151 & 0.0005 & -0.0151 & -0.0170 \\ -0.0101 & -0.0648 & -0.0101 & -0.0114 & -0.0455 \\ -0.0005 & -0.0151 & -0.0401 & -0.0019 & -0.0170 \\ -0.0101 & -0.0114 & -0.0013 & -0.0648 & -0.0455 \\ -0.0076 & -0.0303 & -0.0076 & -0.0303 & -0.1439 \\ -0.0013 & -0.0114 & -0.0101 & -0.0034 & -0.0455 \\ 0.0005 & -0.0019 & -0.006 & -0.051 & -0.0170 \\ -0.0013 & -0.0034 & -0.0013 & -0.0114 & -0.0455 \\ -0.0006 & -0.019 & 0.0005 & -0.0019 & -0.0170 \end{bmatrix}$$

$$\begin{bmatrix} -0.0019 & 0.0005 & -0.0019 & -0.0006 \\ -0.0114 & -0.0013 & -0.0034 & -0.0013 \\ -0.0151 & -0.006 & -0.0019 & 0.0005 \\ -0.0034 & -0.0101 & -0.0114 & -0.0013 \\ -0.0303 & -0.0076 & -0.0303 & -0.0076 \\ -0.0648 & -0.0013 & -0.0114 & -0.0101 \\ -0.0019 & -0.0401 & -0.0151 & 0.0005 \\ -0.0114 & -0.0101 & -0.0648 & -0.0101 \\ -0.0151 & 0.0005 & -0.0151 & -0.0401 \end{bmatrix} \begin{bmatrix} -9.9627 \\ 5.8687 \\ 2.2799 \\ 0 \\ 0 \\ 0 \\ 9.9627 \\ -5.8687 \\ -2.2799 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{31} \\ u_{32} \\ u_{33} \end{bmatrix} = \begin{bmatrix} 0.3295 \\ -0.2930 \\ -0.1806 \\ 0 \\ 0 \\ 0 \\ -0.3295 \\ 0.2930 \\ 0.1806 \end{bmatrix}$$

Sturm Liouville Expansion (S.L Expansion

$$y'' - \omega^2 y = 0$$

Solution will be in 'sin' and cos

*Legendre Polynomial:-

$$P_e(x) = (1-x^2)y'' - 2xy' + e(e+1)y = 0$$

*Hermite Polynomial:-

$$H_n(x) = y'' - 2xy' + 2ny = 0$$

*Bessel Function:-

$$J_p(x) = x^2y'' + xy' + (x^2 - p^2)y = 0$$

These all are spacial cases of S.L Theory

⇒ **S.L. Theory:-** Let $y(x)$ be the solution to the differential equation

$$\frac{d}{dx}\left\{P(x)\frac{dy}{dx}\right\} + \omega(x)y'(x) + \lambda\, r(x)\, y(x) = 0 \longrightarrow \circledast$$

$a \leq x \leq b$ subject to boundary conditions

$$c_1 y(a) + c_2 y'(a) = 0$$

$$c_3 y(b) + c_4 y'(b) = 0$$

Then by S.L Theorem, there is a set of $y_\lambda(x)$ called eigen functions, which depends on λ called eigen values. This solution is complete and orthogonal i.e

$$f(x) = \sum^{\infty} A_n y_\lambda(x)$$

and $\int_a^b r(x) y_n(x) y_m(x)\, dx = 0$ if $n \neq m$

Put $P(x) = 1 = r(x)$, $w(x) = 0$

Then ⊛ becomes

$$\frac{d^2y}{dx^2} + \lambda y(x) = 0 \qquad \begin{array}{l} 0 \le x \le L \\ y(0) = y(L) = 0 \end{array}$$

$\Rightarrow$ $y_n = \sin\left(\frac{n\pi x}{L}\right)$ eigen function

& $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ eigen Values.

Norm:- A norm is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ that satisfies following

* Vector Norm:-

1) $\|x\| \geq 0$, and $\|x\| = 0$ only if $x = 0$
2) $\|x + y\| \leq \|x\| + \|y\|$
3) $\|\alpha x\| = |\alpha| \|x\|$, where α is scaler

* P-Norms:-

l_1 Norm:- $\|x\|_1 = \sum_{i=1}^{m} |x_i|$

l_2 Norm:- $\|x\|_2 = \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2}$

l_∞ Norm:- $\max_{1 \le i \le m} |x_i|$

* Matrix Norm:-

1) $\|A\| \geq 0$
2) $\|A + B\| \leq \|A\| + \|B\|$
3) $\|\alpha A\| = |\alpha| \|A\|$

* P-Norms:-

l_1 Norm:- $\|A\|_1 = \max\limits_{1\le j\le n} \sum\limits_{i=1}^{m} |a_{ij}|$

l_2 Norm (Spectral Norm):- $\|A_2\| = \sqrt{\text{maximum eigen value of } A^TA}$

l_∞ Norm:- $\max\limits_{1\le i\le m} \sum\limits_{j=1}^{n} |a_{ij}|$

Example:-

$$x = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} \quad \text{vector}$$

$$\|x\|_1 = \sum_{i=1}^{3} |x_i| = |2| + |5| + |-3|$$
$$= 2 + 5 + 3$$
$$= 10$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{3} |x_i|^2} = \sqrt{|2|^2 + |5|^2 + |-3|^2}$$
$$= \sqrt{4 + 25 + 9} = \sqrt{38}$$
$$= 6.1644$$

$$\|x\|_\infty = \max(|2|, |5|, |-3|)$$
$$= \max(2, 5, 3)$$
$$= 5$$

Example:-

$$A = \begin{bmatrix} 3 & 9 & 5 \\ 7 & 2 & 4 \\ 6 & 8 & 1 \end{bmatrix} \quad \text{matrix}$$

$$\|A\|_1 = \max_{1\le j\le 3} \left[|3| + |7| + |6|, |9| + |2| + |8|, |5| + |4| + |1| \right]$$

$$\|A\|_1 = \max [16, 19, 10]$$
$$= 19$$

$$\|A\|_\infty = \max_{1 \le i \le m} [|3|+|9|+|5|, |7|+|2|+|4|, |6|+|8|+|1|]$$

$$= \max [17, 13, 15]$$
$$= 17$$

* **Convolution Sum:-** Suppose we have two functions f and g, the complex fourier series for the two functions is given by

$$f(x) = \sum_{j=-\infty}^{\infty} a_j e^{2\pi i j x/L}$$

$$g(x) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi i k x/L}$$

To find the fourier series of the sum of two functions we simply calculate

$$f(x) + g(x) = \sum_{k=-\infty}^{\infty} (a_k + b_k) e^{2\pi i k x/L}$$

And the product of the functions

$$f(x) \times g(x) = \sum_{j,k=-\infty}^{\infty} a_j b_k e^{2\pi i (j+k) x/L}$$

if we put $j+k = l$

$$\Rightarrow f(x) \times g(x) = \sum_{l=-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} a_j b_{l-j} \right) e^{2\pi i l x/L}$$

Convolution sum of $O(N^2)$

$N \log N$

Algorithm:- ★ Choose the Chebyshev Gauss Lobato points given by

$x_j = \cos\theta_j$, where $\theta_j = \frac{j\pi}{N}$, $j = 0, 1, \dots, N$

$$\Rightarrow x_j = \cos\frac{j\pi}{N} \; ; \; j = 0, 1, 2, \dots, N$$

★ Given data $v_0, v_1, \dots, v_N$ at Chebyshev points $x_0 = 1, \dots, x_N = -1$, extend this data to a vector V of length $2N$ with $V_{2N-j} = v_j$, $j = 1, 2, \dots, N-1$

★ Using FFT, calculate

$$\hat{V}_K = \frac{\pi}{N}\sum_{j=1}^{2N} e^{-ik\theta_j} V_j , \quad K = -N+1, \dots, N$$

$\boxed{\because V_N = V_0}$

★ Define $\hat{W}_k = ik\hat{V}_k$, except $\hat{W}_N = 0$

★ Compute the derivative of the trignometric interpolant Q on the equispaced grid by the inverse FFT

$$W_j = \frac{1}{2\pi}\sum_{K=-N+1}^{N} e^{ik\theta_j}\hat{W}_K , \quad j = 1, 2, \dots, 2N$$

★ Calculate the derivative of the algebraic polynomial interpolant q on the interior grid points by

$$\omega_j = -\frac{W_j}{\sqrt{1 - x_j^2}} , \quad j = 1, 2, \dots, N-1$$

With the special formulas at the end points

$$\omega_0 = \frac{1}{2\pi}\sum_{n=0}^{N}{}' n^2\hat{V}_n , \quad \omega_N = \frac{1}{2\pi}\sum_{n=0}^{N}{}' (-1)^{n+1} n^2\hat{V}_n$$

Prime indicates that the terms $n = 0, N$ are multiplied by $1/2$

⇒ Solution of Non-Homogeneous BVP:-

$$u_{xx} + x u_x - u = (24+5x^2)e^{5x} + (2+2x^2)\cos x^2 - (4x^2+1)\sin x^2 \longrightarrow ①$$

$$u(-1) = e^{-5} + \sin(1) = g-$$

$$u(1) = e^{5} + \sin(1) = g+ \qquad u(x) = ?$$

Let $x = (x_0, x_1, \dots, x_N)^T$,

where $x_j = \cos\left(\frac{\pi j}{N}\right)$; $j = 0, 1, \dots, N$

Let $f = f(x_j) = (f(x_0), f(x_1), \dots, f(x_N))^T$

Let $u = (g+, u(x_1), u(x_2), \dots, u(x_{N-1}), g-)^T$

$U_m = (u(x_1), u(x_2), \dots, u(x_{N-1}))^T$ be the vector of unknowns to be determined

D_{ij} = Chebyshev differentiation matrix

$$D^2 = DD$$

① implies

$$D_m^2 U_m + X_m \otimes D_m U_m - U_m = F;$$

$$X_m = x(1:N-1)$$

$$D_m = D(1:N-1, 1:N-1)$$

$$D_m^2 = D^2(1:N-1, 1:N-1)$$

$$F = f(1:N-1) - [D^2(1:N-1, 0) + X_m \otimes D(1:N-1, 0)]g+ - [D^2(1:N-1, N) + X_m \otimes D(1:N-1, N)]g-$$

$\Rightarrow AU_m = F$, where $A = D^2 + \Lambda D - I$

where Λ is the matrix whose values at diagonal are X_m.

Example:- Solve for $N=4$

$$u_{xx} + xu_x - u = (24+5x^2)e^{5x} + (2+2x^2)\cos x^2 - (4x^2+1)\sin x^2 \longrightarrow ①$$

$$u(-1) = e^{-5} + \sin(1) = g-$$
$$u(+1) = e^{5} + \sin(1) = g+$$

Solution

First of all we calculate the Shebyshev Gauss Lobato points by the formula $x_j = \cos\left(\frac{\pi j}{N}\right)$, $j = 0, 1 \ldots, 4$

$$\Rightarrow x_0 = 1, \quad x_1 = 0.707, \quad x_2 = 0, \quad x_3 = -0.707$$
$$x_4 = -1$$

$$f(x) = (24+5x^2)e^{5x} + (2+2x^2)\cos x^2 - (4x^2+1)\sin x^2$$

$$\Rightarrow f(x_0) = 4301.9355, \quad f(x_2) = 26$$
$$f(x_1) = 909.987, \quad f(x_3) = 1.9678$$
$$f(x_4) = -1.8507$$

Now

$$D_4 = \begin{bmatrix} 5.5 & -6.8284 & 2 & -1.1716 & 0.5 \\ 1.7071 & -0.7071 & -1.4142 & 0.7071 & -0.2929 \\ -0.5 & 1.4142 & 0 & -1.4142 & 0.5 \\ 0.2929 & -0.7071 & 1.4142 & 0.7071 & -1.7071 \\ -0.5 & 1.1716 & -2 & 6.8284 & -5.5 \end{bmatrix}$$

$$D_4^2 = \begin{bmatrix} 17 & -28.8445 & 18 & -11.5147 & 5 \\ 9.2426 & -14 & 6 & -2 & 0.7574 \\ -1 & 4 & -6 & 4 & -1 \\ 0.7574 & -2 & 6 & -14 & 9.2426 \\ 5 & -11.5147 & 18 & -28.4845 & 17 \end{bmatrix}$$

$$\Rightarrow \tilde{D}^4 = \begin{bmatrix} -0.7071 & -1.4142 & 0.7071 \\ 1.4142 & 0 & -1.4142 \\ -0.7071 & 1.4142 & 0.7071 \end{bmatrix}$$

$$\tilde{D}_4^2 = \begin{bmatrix} -14 & 6 & -2 \\ 4 & -6 & 4 \\ -2 & 6 & -14 \end{bmatrix}$$

So our Laplacian will be of the form

$$\tilde{D}_4^2 U_m + X_m \otimes \tilde{D}_4 U_m - U_m = F \longrightarrow ⊛$$

where $F = f(1:N-1) - [D^2(1:N-1,0) + X_m \otimes D(1:N-1,0)]g_+$
$\qquad - [D^2(1:N-1,N) + X_m \otimes D(1:N-1,N)]g_-$

∴ we have non homogenous boundary conditions, so we adjust our right hand side according to the boundary condition and name it as F

کیونکہ یہاں پر U_0 اور U_N zero نہیں ہیں اس لیے ہم D Matrix کے U_0 میں اور U_N والی انٹریز کو Discard نہیں کرنا بلکہ ان کو (B.C) جو کہ $U_0 = g_+$ اور $U_N = g_-$ ہیں کی انٹریز کے ساتھ correspond ملٹی پلائی کر کے R.H.S پر لے جا کر ٹرانسفر کرنا ہے کیونکہ یہ ہمارے پاس Known انٹریز ہیں

$$\Rightarrow F = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix} - \left\{ \begin{bmatrix} 9.2426 \\ -1 \\ 0.7574 \end{bmatrix} + \begin{bmatrix} 0.7071 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.707 \end{bmatrix} \begin{bmatrix} 1.7071 \\ -0.5 \\ 0.2929 \end{bmatrix} \right\} 149.25 - \left\{ \begin{bmatrix} 0.7574 \\ -1 \\ 9.2426 \end{bmatrix} + \begin{bmatrix} 0.707 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.707 \end{bmatrix} \begin{bmatrix} -0.2929 \\ 0.5 \\ -1.7071 \end{bmatrix} \right\} 0.8482$$

$$\Rightarrow F = \begin{bmatrix} 909.987 \\ 26 \\ 1.9678 \end{bmatrix} - \left\{ \begin{bmatrix} 9.2426 \\ -1 \\ 0.7574 \end{bmatrix} + \begin{bmatrix} 1.2069 \\ 0 \\ -0.2071 \end{bmatrix} \right\} 149.25$$

$$- \left\{ \begin{bmatrix} 0.7574 \\ -1 \\ 9.2426 \end{bmatrix} + \begin{bmatrix} -0.2071 \\ 0 \\ 1.2069 \end{bmatrix} \right\} 0.8482$$

$$= \begin{bmatrix} 909.987 \\ 26 \\ 1.9678 \end{bmatrix} - \begin{bmatrix} 1556.8506 \\ -149.25 \\ 82.1323 \end{bmatrix} - \begin{bmatrix} 0.4668 \\ -0.8482 \\ 8.8633 \end{bmatrix}$$

$$\Rightarrow F = \begin{bmatrix} -647.3304 \\ 176.0982 \\ -89.0278 \end{bmatrix}$$

So (*) becomes

$$\left\{ \begin{bmatrix} -14 & 6 & -2 \\ 4 & -6 & 4 \\ -2 & 6 & -14 \end{bmatrix} + \begin{bmatrix} 0.707 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.707 \end{bmatrix} \begin{bmatrix} -0.7071 & -1.4142 & 0.7071 \\ 1.4142 & 0 & -1.4142 \\ -0.7071 & 1.4142 & 0.7071 \end{bmatrix} \right.$$

$$\left. - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F \end{bmatrix}$$

$$\Rightarrow \left\{ \begin{bmatrix} -14 & 6 & -2 \\ 4 & -6 & 4 \\ -2 & 6 & -14 \end{bmatrix} + \begin{bmatrix} -0.5 & -1 & 0.5 \\ 0 & 0 & 0 \\ 0.5 & -1 & -0.5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -647.3304 \\ 176.0982 \\ -89.0278 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -15.5 & 5 & -1.5 \\ 4 & -7 & 4 \\ -1.5 & 5 & -15.5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -647.3304 \\ 176.0982 \\ -89.0278 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \; & \; & \; \\ & & \\ & & \end{bmatrix}^{-1} \begin{bmatrix} \; \\ \\ \\ \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -0.08 & -0.0633 & -0.0086 \\ -0.0506 & -0.2152 & -0.0506 \\ -0.0086 & -0.0633 & -0.08 \end{bmatrix} \begin{bmatrix} -647.3304 \\ 176.0982 \\ -89.0278 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 41.4051 \\ -0.6366 \\ 1.5422 \end{bmatrix}$$

⇒ Solution of Biharmonic Problem:- (Fourth Order Problem)

Suppose we wish to solve a Biharmonic equation of the form

$$u_{xxxx} = f(x) \quad ; \quad -1 \le x \le 1$$

$$u(\pm 1) = u_x(\pm 1) = 0$$

To compute the spectral approximation to u_{xxxx}, Let v_j be the $(N-1)$ vector of values u sampled at $x_1, x_2, \dots, x_{N-1}$. Then imposing the boundary conditions suggest the following

Let p be the unique polynomial of degree $\le N+2$ with $p(\pm 1) = 0$ and $p(x_j) = v_j$

Set $w_j = p_{xxxx}(x_j)$ $\quad \boxed{W = D_N V}$

∵ $p(x_j) = v_j$, so we have to find v_j at the end

For Chebyshev differentiation matrix D_N

Let $P(x) = (1-x^2)\,q(x) \longrightarrow ⊛$

$$\Rightarrow P'(x) = (-2x)q(x) + (1-x^2)q'(x)$$

$$P''(x) = (1-x^2)q''(x) - 2x\,q'(x) - 2x\,q'(x) - 2q(x)$$

$$= (1-x^2)q''(x) - 4x\,q'(x) - 2q(x)$$

$$P'''(x) = (1-x^2)q'''(x) - 2x\,q''(x) - 4x\,q''(x) - 4q'(x) - 2q'(x)$$

$$= (1-x^2)q'''(x) - 6x\,q''(x) - 6q'(x)$$

$$P^{(4)}(x) = (1-x^2)q^{(4)}(x) - 2x\,q'''(x) - 6x\,q'''(x) - 6q''(x) - 6q''(x)$$

$$\Rightarrow p_{xxxx}(x) = (1-x^2)q_{xxxx}(x) - 8xq_{xxx}(x) - 12q_{xx}(x) \longrightarrow (A)$$

A polynomial q of degree $\le N$ with $q(\pm 1) = 0$ corresponds to a polynomial p of degree $\le N+2$ with

$$p(\pm 1) = p_x(\pm 1) = 0$$

Carry the required spectral differentiation like this,

Let q be the unique polynomial of degree $\le N$ with $q(\pm 1) = 0$ and

$$q(x_j) = \frac{V_j}{1-x_j^2} \qquad \because \text{from } (*) \text{ \& } p(x_j) = V_j$$

$$\Rightarrow w_j = (1-x_j^2)\underbrace{q_{xxxx}(x_j)}_{D_N^4} - 8x_j\underbrace{q_{xxx}(x_j)}_{D_N^3} - 12\underbrace{q_{xx}(x_j)}_{D_N^2}$$

Our spectral biharmonic operator in this case

$$L = [\text{diag}(1-x_j^2)\tilde{D}_N^4 - 8\,\text{diag}\,x_j\,\tilde{D}_N^3 - 12\tilde{D}_N^2] \times \text{diag}\frac{1}{1-x_j^2} \longrightarrow (B)$$

$$\Rightarrow LV = f$$

$$\because w_j = p_{xxxx}(x_j)$$

$$\Rightarrow p_{xxxx}(x_j) = (1-x_j^2)q_{xxxx}(x_j) - 8x_jq_{xxx}(x_j) - 12q_{xx}(x_j)$$

$$= [\text{diag}(1-x_j^2)\tilde{D}_N^4 - 8\,\text{diag}(x_j)\tilde{D}_N^3 - 12\tilde{D}_N^2]\,q(x_j)$$

$$= [\text{———}\,''\,\text{———}]\frac{V_j}{1-x_j^2}$$

$$= LV, \quad \text{where } L \text{ given in eqn } (B)$$

$$\Rightarrow p_{xxxx}(x_j) = LV$$

So solution is $LV = f$

⇒ Shebyshev Spectral Differentiation via DFT :-

Solve $f(x) = e^x \sin 5x$

Consider $N = 2$, $x_j = \cos \frac{j\pi}{N}$, $j = 0, 1, .. N$

⇒ $x_0 = 1$, $x_1 = 0$, $x_2 = -1$

⇒ $f(x_0) = -2.6066 = V_0$

$f(x_1) = 0 = V_1$

$f(x_2) = 0.3528 = V_2$

Extend above data to a vector V of length $2N (=4)$ with

$V_{2N-j} = V_j$; $j = 1, 2, \dots, N-1$

i.e $V_3 = V_1$ (Here $N=2$, $V_3 = V_1$)

Now we have

$$V = \begin{matrix} -2.6066 \\ 0 \\ 0.3528 \\ 0 \end{matrix}$$

★ DFT of vector V.

$$\hat{V}_k = \sum_{j=1}^{2N} e^{-2\pi i (j-1)(k-1)/2N}$$

$2N = 4$ (length of V)

for $K=1$

$$\hat{V}_1 = \sum_{j=1}^{4} e^{-2\pi i (j-1)(0)/4}$$

$= -2.2538$

for K=2 $\hat{V}_2 = \sum_{j=1}^{4} e^{-2\pi i(j-1)(1)/4}$

$= -2.9594$

for K=3 $\hat{V}_3 = \sum_{j=1}^{4} e^{-2\pi i(j-1)(2)/4}$

$= -2.2538$

for K=4 $\hat{V}_4 = \sum_{j=1}^{4} e^{-2\pi i(j-1)(3)/4}$

$= -2.9594$

So $U =$

$-2.2539 = \hat{V}_0$
$-2.9594 = \hat{V}_1$
$-2.2539 = \hat{V}_2$
$-2.9594 = \hat{V}_3$

* Define $\hat{W}_K = ik\hat{V}_K$, except $\hat{W}_N = 0$ since matlab stores these numbers in the order $K = 0, 1, \ldots, N, -N+1, -N+2, \cdots$

Here $\hat{W} = 0$ $K=0$
$= -2.9594\,i$ $K=1$
$= 0$ $K=2$
$= 2.9594\,i$ $K=2-1$

Now we find the IFFT/IDFT

Since $V(j) = \frac{1}{N} \sum_{K=1}^{2N} \hat{V}(K)\, e^{2\pi i(j-1)(K-1)/2N}$

$1 < j \leq 2N$

IDFT of $\hat{W} = [0 \quad -2.9594i \quad 0 \quad 2.9594i]$

for $j = 0, 1, 2, 3$

$$W = 0 = W_0$$

$$1.4794 = W_1$$

* Calculate the Derivative

For end points we have

$$w_0 = \frac{1}{N} \sum_{n=0}^{N-1} n^2 \hat{V}_n + \frac{1}{2} \times N^2 \times \hat{V}_N \longrightarrow (a)$$

$$w_N = \frac{1}{N} \sum_{n=0}^{N-1} (-1)^{n+1} \hat{V}_n + \frac{1}{2} (-1)^{N+1} \times N^2 \times \hat{V}_N \longrightarrow (b)$$

For interior points

$$w_j = \frac{-W_j}{\sqrt{1-x_j^2}} \qquad j = 1, 2, \ldots N-1 \longrightarrow (c)$$

By using (a)

$$w_0 = \frac{1}{N}[0 + \hat{V}_1] + 0.5 \times N \times \hat{V}_2$$

$$= -3.7336$$

(b) $\Rightarrow$ $w_2 = \frac{1}{2}[0 + (1)(1)^2 \hat{V}_1] + (0.5)(-1)(2) \times \hat{V}_2$

$$= 0.7742$$

(c) $\Rightarrow$ $w_1 = \frac{-W_1}{\sqrt{1-x_1^2}} = \frac{-1.4797}{\sqrt{1-0}}$

$$= -1.4797$$

Hence $f'(x_0) = -3.7336$

$f'(x_1) = -1.4797$

$f'(x_2) = 0.7742$

Chebyshev Differentiation matrix

$N=2$ $\quad x_j = \cos\left(\frac{j\pi}{N}\right)$

$x_0 = 1 \quad , \quad x_1 = 0 \quad , \quad x_2 = -1$

$$D_2 = \begin{bmatrix} 3/2 & -2 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/2 & 2 & -3/2 \end{bmatrix}$$

Given $f(x) = e^x \sin 5x$

As we know that $f'(x_j) = D_N f(x)$

So

$$\begin{bmatrix} f'(x_0) \\ f'(x_1) \\ f'(x_2) \end{bmatrix} = \begin{bmatrix} 3/2 & -2 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/2 & 2 & -3/2 \end{bmatrix} \begin{bmatrix} -2.6066 \\ 0 \\ 0.3528 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} f'(x_0) \\ f'(x_1) \\ f'(x_2) \end{bmatrix} = \begin{bmatrix} -3.7336 \\ -1.4797 \\ 0.7742 \end{bmatrix}$$

Question :- Find the derivative of $f(x) = e^x \sin 5x$ at the Gauss-Lobato points $x_j = \cos\left(\frac{j\pi}{N}\right)$ via FFT

Solution

Let us take $N=3$

$$x_j = \cos\left(\frac{j\pi}{N}\right)$$

$$\Rightarrow x_0 = 1 , \quad x_1 = 0.5 , \quad x_2 = -0.5$$
$$x_3 = -1$$

Now as
$$f(x) = e^x \sin 5x , \text{ so}$$

$$f(x_0) = -2.6066 = v_0$$
$$f(x_1) = 0.9867 = v_1$$
$$f(x_2) = -0.3630 = v_2$$
$$f(x_3) = 0.3528 = v_3$$

* Now given data v_0, v_1, v_2, v_3 at the Chebyshev points x_0, x_1, x_2, x_3. We extend this data to a vector V of length $2N$ with
$$V_{2N-j} = V_j \quad , j = 1, \cdots N-1$$
Here $N=3$, so by extending data we have

$$V_0 = -2.6066$$
$$V_1 = 0.9867$$
$$V_2 = -0.3630$$
$$V_3 = 0.3528$$
$$V_4 = -0.3630 = v_2$$
$$V_5 = 0.9867 = v_1$$

$\Rightarrow \quad V = \{V_0, V_1, \dots, V_5\}$

i.e we have a vector of length $2N$

$\star$ Now, we find FFT of V

$$\Rightarrow U = \begin{matrix} -1.0064 \\ -1.6097 \\ -2.8774 \\ -5.6588 \\ -2.8774 \\ -1.6097 \end{matrix}$$

$\star$ Define $\hat{W}_K = iK\hat{U}_K$, $K = -N+1, \dots, N$

Since $N=3$ so, $K = -2, -1, \dots, 3$

except $\hat{W}_N = 0$

So, we have

$$\hat{\omega}_0 = 0 + 0i$$

$\hat{\omega}_1 = 0 - 1.6067i$ $\quad$ $\hat{\omega}_2 = 0 - 5.7552i$

$\hat{\omega}_3 = 0 + 0i$ $\quad$ $\hat{\omega}_{-2} = 0 + 5.7552i$

$$\hat{\omega}_{-1} = 0 + 1.6097i$$

$\star$ Now we find inverse FFT of $\hat{W}_K$

$$W = \begin{matrix} 0 \\ 2.1260 \\ -1.1967 \\ 0 \\ -1.1967 \end{matrix}$$

Now we calculate the derivative of the algebraic polynomial impolant q on the interior grid points by

$$\omega_j = -\frac{W_j}{\sqrt{1-x_j^2}}, \quad j = 1, 2, \dots, N-1$$

with special formulas at the end points

$$W_0 = \frac{1}{2\pi}\sum_{n=0}^{N}{}' n^2 \hat{V}_n \;, \quad W_N = \frac{1}{2\pi}\sum_{n=0}^{N}{}' (-1)^{n+1} n^2 \hat{V}_n$$

where the prime indicates that the terms $n=0, N$ are multiplies by $\frac{1}{2}$

Firstly we calculate ω_0, ω_N

$$\omega_0 = \frac{1}{N}\left[0 + \hat{V}_1 + 4\hat{V}_2\right] + \tfrac{1}{2} N \hat{V}_3$$

$$\Rightarrow \omega_0 = -12.8615$$

$$\omega_N = \frac{1}{N}\left[(-1)^{0+1} * 0 * \hat{V}_0 + (-1)^{1+1} * 1^2 * \hat{V}_1 + (-1)^{2+1} * 2^2 * \hat{V}_2\right] + \frac{1}{2} * (-1)^{3+1} * N * \hat{V}_3$$

$$\Rightarrow \omega_3 = -5.1879$$

Now we calculate ω_1 & ω_2

$$\Rightarrow \omega_1 = \frac{-W_1}{\sqrt{1-x_1^2}} = -2.4548$$

$$\omega_2 = \frac{W_2}{\sqrt{1-x_2^2}} = 1.3818$$

Hence we have	Exact Solution
$f'(x_0) = -12.8615$	$f'(x_0) = 1.2487$
$f'(x_1) = -2.4549$	$f'(x_1) = -5.6176$
$f'(x_2) = 1.3818$	$f'(x_2) = -2.7926$
$f'(x_3) = 5.1880$	$f'(x_3) = 0.8745$

Question:- Solve the following Berger Equation using Galerkin-Spectral Method.

$$u_t - \varepsilon u_{xx} + u u_x = 0 \quad , \quad x \in [0,1] \longrightarrow (1)$$

B.Cs:- $\quad u(0,t) = u(1,t) = 0$

I.C $\quad u(x,0) = g(x)$

Solution

Step I:- Choose basis functions $\phi_n(x)$, Then the solution is of the form

$$\tilde{u}(x,t) = \sum_{n=1}^{N} a_n(t)\phi_n(x) \longrightarrow (2)$$

where $\phi_n(x) = \sin(n\pi x) \longrightarrow (3)$

Step II:-

Substitute step I into differential operator and obtain a set of ODE's in time, our differential operator in this case

$$L(\tilde{u}) = \sum_{n=1}^{N} a_n'(t)\sin(n\pi x) + \varepsilon \sum_{n=1}^{N} n^2\pi^2 a_n(t)\sin(n\pi x)$$
$$+ \left[\sum_{n=1}^{N} a_n(t)\sin(n\pi x)\right]\left[\sum_{k=1}^{N} a_k(t)\, n\pi \cos(k\pi x)\right] \longrightarrow (4)$$

Taking inner product with $\phi_m(x)$ of equation (4) and using $(\phi_i(x), L(\tilde{u})) = 0$, we get

$$\sum_{n=1}^{N} a_n'(t)[\sin(m\pi x), \sin(n\pi x)] + \varepsilon \sum_{n=1}^{N} n^2\pi^2 a_n(t)$$
$$[\sin(m\pi x), \sin(n\pi x)] + \sum_{n=1}^{N}\sum_{K=1}^{N} a_n(t) a_k'(t) [\sin(m\pi x), \sin(n\pi x) \cos(K\pi x)]$$

Suppose

$$aP_m a' = \sum_{n=1}^{N}\sum_{K=1}^{N} a_n(t)\, a_k'(t) \left[\sin(m\pi x), \sin(n\pi x)\cdot \cos K\pi x\right.$$

where we have denoted as a, a' for programming purpose and a is $1\times N$ row vector whose entries are a_i, $i = 1, \dots, N$ and P is a matrix defined by

$$P_m(i,K) = (\phi_m(x), \phi_j'(x)\cdot\phi_j(x))$$

$$\text{i.e} \int_0^1 \left[\left[\int_0^1 \sin(m\pi x), \sin(n\pi x)\cdot(\cos K\pi x)\, dx\right.\right.$$

It is easy to show that

$$P_m(i,j) = \begin{cases} -i\pi/4 & , K+m = i \\ i\pi/4 & , |K-m| = i \\ 0 & , j+m \neq i \ \& \ |j-m| \neq i \end{cases}$$

Therefore the Galerkin method reduces to N differential equations in N unknowns of the form

$$a_m'(t) = -\pi^2 m^2 \Sigma a_n(t) - 2(aP_m a')$$

The initial condition for the above system is

$$a_m(0) = \frac{\sin(m\pi x), g(x)}{(\sin(m\pi x), \sin(m\pi x))}$$

Program 11:-

Given function is

$$u(x) = e^{x}\sin(5x)$$

where

$$x_i = \cos\left(\frac{\pi i}{N}\right)$$

We have to solve for $N=2$

so

$$x_0 = 1 \quad , \quad x_1 = 0 \quad , \quad x_2 = -1$$

As $v_j = f(x_j)$

$$\Rightarrow v_0 = f(x_0) = -2.6066$$

$$v_1 = f(x_1) = 0$$

$$v_2 = f(x_2) = 0.3528$$

And

$$v_{2N-j} = v_j \quad , j = 1, \dots N-1$$

$$\Rightarrow v_{4-1} = v_3 = v_1 \quad \& \quad v_4 = v_0$$

using FFT to calculate

$$\hat{v}_k = \pi/N \sum_{j=1}^{2N} e^{-ik\theta_j} v_j \quad , \quad k = -N+1, \dots, N$$

so

$$\hat{v}_{-1} = \frac{\pi}{2}\left(e^{i\theta_1}\cancel{v_1} + e^{i\theta_2}v_2 + e^{i\theta_3}\cancel{v_3} + e^{i\theta_4}v_4\right)$$

$$= \pi/2\left(e^{i\theta_2}v_2 + e^{i\theta_4}v_4\right)$$

$$\hat{v}_0 = \frac{\pi}{2}\left(e^{0}\cancel{v_1} + e^{0}v_2 + e^{0}\cancel{v_3} + e^{0}v_4\right)$$

$$= \frac{\pi}{2}\left(v_2 + v_4\right)$$

$$\hat{v}_1 = \frac{\pi}{2}\left(e^{-i\theta_1}\cancel{v_1}^{0} + e^{-i\theta_2}v_2 + \cancel{e^{-i\theta_3}v_3}^{0} + e^{-i\theta_4}v_4\right)$$

$$= \frac{\pi}{2}\left(e^{-i\theta_2}v_2 + e^{-i\theta_4}v_4\right)$$

$$\hat{v}_2 = \frac{\pi}{2}\left(\cancel{e^{-2i\theta_1}v_1}^{0} + e^{-2i\theta_2}v_2 + \cancel{e^{-2i\theta_3}v_3}^{0} + e^{-2i\theta_4}v_4\right)$$

$$= \frac{\pi}{2}\left(e^{-2i\theta_2}v_2 + e^{-2i\theta_4}v_4\right)$$

Define $\hat{W}_K = iK\hat{v}_K$ except, $\hat{W}_N = 0$

implies $\hat{W}_{-1} = -i\,\hat{v}_{-1}$

$\hat{W}_0 = 0$

$\hat{W}_1 = i\hat{v}_1$

$\hat{W}_2 = 0$

Next to compute the derivative of the trignometric interpolant q in the equispaced grid by the inverse FFT

$$W_j = \frac{1}{2\pi}\sum_{K=-N+1}^{N} e^{iK\theta_j}\hat{W}_K \; ; \; j = 1, \dots, 2N$$

$$\Rightarrow W_1 = \frac{1}{2\pi}\left\{e^{-i\theta_1}\hat{W}_{-1} + e^{i(0)}\hat{W}_0 + e^{i\theta_1}\hat{W}_1 + e^{i\theta_2}\hat{W}_2\right\}$$

$$= \frac{1}{2\pi}\left\{e^{-i\theta_1}(-i)\hat{v}_{-1} + 0 + e^{i\theta_1}i\hat{v}_1 + 0\right\}$$

$$= \frac{1i}{2\pi}\left\{e^{i\theta_1}\hat{v}_1 - e^{-i\theta_1}\hat{v}_{-1}\right\}$$

$$\Rightarrow W_1 = \frac{i}{2\pi}\left\{ e^{i\theta_1}\frac{\pi}{2}\left(e^{-i\theta_2}V_2 + e^{-i\theta_4}V_4\right) - e^{-i\theta_1}\frac{\pi}{2}\left(e^{i\theta_2}V_2 + e^{i\theta_4}V_4\right)\right\}$$

calculate the derivative of the algebraic polynomial interpolant q on the interior grid points by

$$\omega_j = \frac{-W_j}{\sqrt{1-x_j^2}} \quad ; \quad j = 1, \dots, N-1$$

$$\Rightarrow \omega_1 = \frac{-W_1}{\sqrt{1-0}} = -W_1$$

with special formulas at the end points

$$\omega_0 = \frac{1}{2\pi}\sum_{n=0}^{N}{}' n^2 \hat{V}_n \quad \& \quad \omega_N = \sum_{n=0}^{N}{}' (-1)^{n+1} n^2 \hat{V}_n$$

where the prime indicates that the term $n=0, N$ are multiplied by $1/2$

So

$$\omega_0 = \frac{1}{2\pi}\left(0 + 1\hat{V}_1 + \frac{1}{2}2^2\hat{V}_2\right)$$

$$\Rightarrow \boxed{\omega_0 = \frac{1}{2\pi}\left(\hat{V}_1 + 2\hat{V}_2\right)}$$

$$\omega_2 = \frac{1}{2\pi}\left(-\pi 0 + 1\hat{V}_1 + \frac{(-1)^3}{2}(2)^2\hat{V}_2\right)$$

$$\Rightarrow \boxed{\omega_2 = \frac{1}{2\pi}\left(\hat{V}_1 - 2\hat{V}_2\right)}$$

Now by putting the values in ω_0 and ω_2, we have

$$\omega_0 = \frac{1}{2\pi}\left\{\frac{\pi}{2}\left(e^{-i\theta_2}V_2 + e^{-i\theta_4}V_4\right) + \frac{2\pi}{2}\left(e^{-2i\theta_2}V_2 + e^{-2i\theta_4}V_4\right)\right\}$$

As $\theta_j = \pi j/N$; $\theta_1 = \frac{\pi}{2}$, $\theta_2 = \pi$, $\theta_4 = 2\pi$

$$\Rightarrow \omega_0 = \frac{1}{2\pi}\left\{\frac{\pi}{2}\left(e^{-2\pi}(0.3528) + e^{-i2\pi}(-2.6066)\right) + \pi\left(e^{-2i\pi}(0.3528) + e^{-i4\pi}(-2.6066)\right)\right.$$

$$= \frac{1}{4}\left\{0.3528\cos(\pi) + (-2.6066)\cos(2\pi)\right\} + \frac{1}{2}\left\{0.3528\cos(2\pi) - 2.6066\cos(4\pi)\right\}$$

$$= \frac{1}{4}\left\{-0.3528 - 2.6066\right\} + \frac{1}{2}\left\{0.3528 - 2.6066\right\}$$

$$\Rightarrow \boxed{\omega_0 = -1.8653}$$

And

$$\omega_2 = \frac{1}{2\pi}\left\{\frac{\pi}{2}\left(e^{-i\theta_2}V_2 + e^{-i\theta_4}V_4\right) - 2\frac{\pi}{2}\left(e^{-2i\theta_2}V_2 + e^{-2i\theta_4}V_4\right)\right\}$$

$$= \frac{1}{4}\left\{e^{-i\pi}(0.3528) + e^{-i2\pi}(-2.6066)\right\} - \frac{1}{2}\left\{e^{-2i\pi}(0.3528) + e^{-4i\pi}(-2.6066)\right\}$$

$$= \frac{1}{4}\left\{\cos(\pi)(0.3528) + \cos(2\pi)(-2.6066)\right\} - \frac{1}{2}\left\{\cos(2\pi)(0.3528) - 2.6066\cos(4\pi)\right\}$$

$$= \frac{1}{4}\left\{-0.3528 - 2.6066\right\} - \frac{1}{2}\left\{-0.3528 - 2.6066\right\}$$

$$\Rightarrow \boxed{\omega_2 = 0.3869}$$

$$\boxed{\omega_1 = 0}$$

Since the exact Solution

$u(x) = e^{x} \sin 5x$

$\Rightarrow u'(x) = e^{x} \sin(5x) + e^{x} \cos(5x) \cdot 5$

$= e^{x} \sin 5x + 5e^{x} \cos 5x$

$u'(x_0) = 1.2486$

$u'(x_1) = 5$

$u'(x_2) = 0.8745$