Date:

Advanced Set Theory

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Lecture # 01:

→ Advanced Set theory

Set theory is a branch of mathematical logics which deals with the sets.

→ **Mathematical form of Sets theory:-** was presented by "G. Contor (in 1870)"

→ Set :-

- A set is any well defined list, collection or class of objects.

→ well defined collection :-

A set is well defined, meaning That If "S" is a set and "a" is some objects then either "a" is definitely in "S" or "a" is definitely not in "S".

(well defined means ky JS mai sbh k opinion same ho).

→ Example :-

(i) : Set of numbers 1, 2, 3 and 4.

(ii) : Set of Cuntries of world.

(iii) : Set containing elements a, e, i, o, u.

(iv) : set of Square matrices of order 2.
(v) : Set of roots of equation $x^2 - 3x + 2 = 0$.
(All examples are well defined.)

→ Notation :-

set will usually denoted by "capital letters A, B, C, ------.
The elements in sets are usually denoted by lower case letters a, b, c, ------. A set consists of elements "notation" $x \in N$ means that x is an element of M (x belongs to M).

→ Representation of Sets :-

1. Descriptive method :-

A set may be described in words.

Examples :-

i) The set of all vowels of the English alphabets.
ii) The set of six natural number.
iii) The set of positive integers less than 100.

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2. Tabular method:-

A set may be described by listing its elements with in brackets.

Examples:-

i) $A = \{a, e, i, o, u\}$ (set of vowels alphabets).

ii) $B = \{2, 4, 6, 8\}$ (set of four even numbers)

iii) $C = \{1, 2, 3, \ldots 99\}$ (set of positive integers less than 100).

3. Set-builder method:-

Set builder method can be used to specify a set by describing the properties of its elements. It is sometimes more convenient or useful to employ the form.

Examples:-

i) The set of all odd positive integers less than 10 can be written as $O = \{x \mid x$ is an odd positive integers less than $10\}$.

ii) set of rational number (Q^+) of all positive number can be written as

$$Q^+ = \{x \in R \mid x = P/q \text{, for some Positive numbers } P \text{ and } q\}.$$

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iii) $B = \{x \mid x \in N : 1 \leq x \leq 4\}$.

Note :-

★ Empty set is well defined set.
(bcz we can't add any other objects or elements).

→ **Finite sets and Infinite sets :-**

A set is finite if it consists of a specific number of different elements, i.e if in counting the different members of set counting process can come to an end, other wise a set is infinite.

Examples :- (Finite)

i) $P = \{0, 3, 6, 9, \ldots\ldots 99\}$

ii) $Q = \{x \mid x$ is an integer, $1 < x < 10\}$.

A set of months in a year.

iii) M = { January, February, March, April, ... November December }

These sets are all finite because the number of all elements is countable.

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Examples:- (Infinite)

i) A set of all whole numbers, $W = \{0, 1, 2, 3, \ldots\}$

ii) A set of all points on a line.

iii) The set of all integers.

→ Equal Sets :-

Two sets A and B are equal i.e $A = B$ iff they have the same elements that is iff if every element of each set is an element of the other set.

Examples:-

i) The sets $A = \{1, 2, 3\}$ and $B = \{2, 1, 3\}$ are equal (bcz they have the same elements and order).

ii) The sets $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ is not equal set (bcz nature of both sets are different and we have to prove: That $A \subseteq B$, $B \subseteq A$ $\Rightarrow A = B$).

* if $\forall x \; (x \in A \leftrightarrow x \in B) \Rightarrow A = B$.

→ Main Notes :-

- The order in which the elements of a set are listed does not matter.

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- It does not matter if an element of a set is listed more than once. So, $\{1,3,3,3,5,5,5,5\}$ is the same as the set $\{1,3,5\}$ bcz they have the same element.

→ Subset :-

If every element of a set A is an element of set B. Then A is a Subset of B.

Symbolically this is written as

$A \subseteq B$ (A is subset of B).

In such a case we say B is a super set of A, written as

$B \supseteq A$ (B is a super set of A).

* $A \subseteq B$ iff $x \in A \Rightarrow x \in B$.

Examples :-

i) If sets $A = \{x, y\}$ and $B = \{x, y, z\}$ then A is the subset of B bcz elements of A are also present in set B.

$A = \{x, y\}$, $B = \{x, y, z\}$

$A \subseteq B$.

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ii) If sets $A=\{1,2,3\}$, $B=\{1,2,3,4,5,6\}$ then $\therefore A\subseteq B$.
(we can say that set A is a subset of set B as all elements in set A are present in set B.)

iii) The set $A=\{a,e,i,o,u\}$ and $B=\{$ All alphabets $\}$.
$\Rightarrow A\subseteq B$ (A contains all vowels elements which are the part of alphabets.
$\Rightarrow B\supseteq A$ (B is superset of A).

→ proper Subset :-

If A is a subset of B and B contains at least one element which is an element of A. Then A is said to be a proper subset of B, written as $A\subset B$ (A is proper subset of B).

Examples :-

i) let $A=\{a,b,c\}$, $B=\{c,a,b\}$ and $C=\{a,b,c,d\}$ then clearly
$A\subset C$, $B\subset C$ but $A=B$ and $B=A$.
Notice that each of A and B is an improper subset of the other become $A=B$.

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1st Criteria :-

$\forall\ x \in A \Rightarrow x \in B$

Then $A \subseteq B$

2nd Criteria :- (Contrapositive)

If $x \notin B \Rightarrow x \notin A$

Then $A \subseteq B$.

(agr koi aik element B mai ho lykn A mai na ho then wo B ka subset nhi hoga) A ka

Theorem:-

Null set is subset of every set.

Proof:-

let A is any set and $x \notin A$, clearly $x \notin \phi$

$\Rightarrow \phi \subseteq A$

Theorem:-

$A \subseteq B$ and $B \subseteq C$ we have to prove that $A \subseteq C$.

Proof:-

let us assume an element in A as

$\Rightarrow x \in A$

By using def of Subset (If an element of set A is an element of set B) then

$\Rightarrow x \in B \;\forall\; x \in A$ — (i) $\Rightarrow A \subseteq B$

Now Given That B is a subset of C.

Here we have

$\Rightarrow$ "x" $\in B$

by Same def of subset

$\Rightarrow x \in C \;\forall\; x \in B$ — (ii) $\Rightarrow B \subseteq C$.

From (i) and (ii)

$\Rightarrow x \in C \;\forall\; x \in A$

(all the elements in set A will exist in set C.)

By using def of subset we conclude that

A is a subset of C.

$\Rightarrow A \subseteq C$.

→ Set of sets :-

If elements in a sets are also sets Then set is called Set of sets.

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* Set of sets denoted by Greek letters.

Example:-

$$\beta = \{\{1\},\{2\}\}$$

$$\gamma = \{\{1,2\},\{1,5\},\{1\},\{\}\}.$$

→ Power set

A set may obtain elements which are sets themselves.

Example:-

If C = set of classes of a certain school, then elements of C are sets themselves bcz each classes is a set of students.

An important set of sets is the power set of a given set.

* The empty set has exactly one subset.

$P(\phi) = \{\phi\}$.

* The set $\{\phi\}$ has exactly two subsets.

$P(\{\phi\}) = \{\phi, \{\phi\}\}$.

* The power set of a set S denoted by P(S) is the set containing all the possible subset of S.

Examples :-

i) $A = \{a, b\}$

$P(A) = \{\phi, \{a\}, \{a,b\}, \{b\}\}$

$2^2 = {}^2C_0 + {}^2C_1 + {}^2C_2$

(ii) $B = \{1, 2, 3\}$

$P(B) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

$2^3 = {}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3$

→ Key Note :-

(1 elements kitny honge)
→ 1 ordered elements

$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n$

↳ 0th ordered element
(0 elements kitny honge)

→ Venn Euler diagram :-

- **Venn diagram :** Venn diagrams are very useful in depicting visually the basic concepts of sets and relationship b/w sets.

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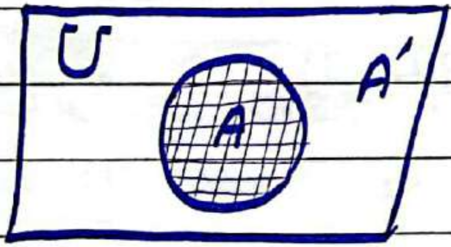
They were first used by an English logician and mathematician "John venn (1834 To 1883 A.D)"

Representation :-

In a venn diagram, a rectangular region represents the universal set and regions bounded by simple closed curve represent other sets, which are subsets of the universal set.

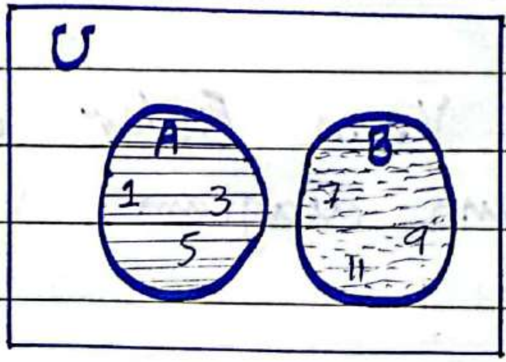


* In the adjoing figures the shaded circular region represent a set A and the remaining portion of rectangle representing the universal set U represent A'. or U–A.

$U = \{1,3,5,7,9\}$, $A = \{1,5,9\}$, $B = \{5,9,11,13\}$.

Disjoint Sets :-

In the given figure, no element of set A and set B is common. Therefore, these sets are disjoint sets. So, the lined portion and the dotted portion represent A ∪ B.



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In the Same way the dotted portion and the lined portion represents $B \cup A$.

For disjoint set $A \cap B = \phi$.

→ **Over lapping Sets :-**

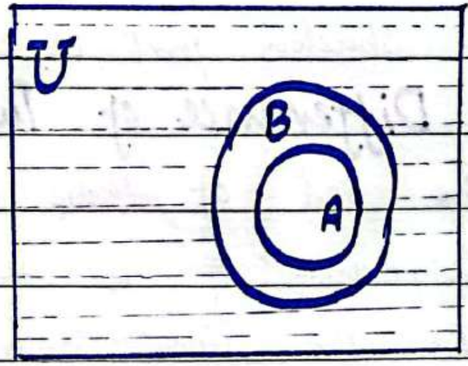
In the given figure, only a Small portion is common in both the set A and B. So, the doted and the Lined common portion b/w set A and set B is called over lapping sets.

This portion is also called $A \cap B$ or $B \cap A$.

→ **Subsets of a Set :-**

In the given figure, the rectangular region represent U (universal set) and set A and set B represent its Subsets.

Here $A \subset B$ means all the elements of set A are present in set B.



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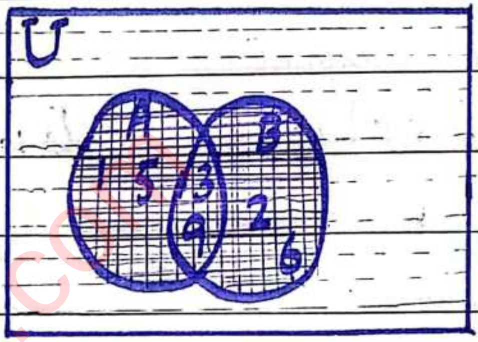
→ Union of Two Sets :-

In the given figure, the Total region bounded by set A and Set B represent $A \cup B$.

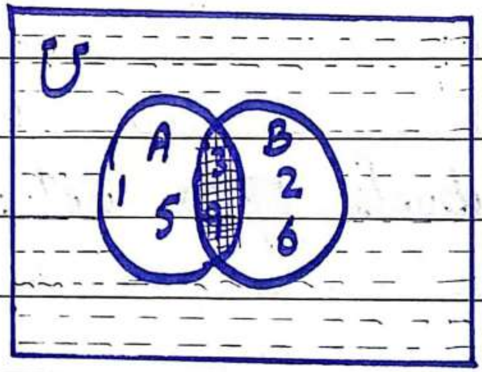


$$A \cup B = \{1,3,5,9\} \cup \{2,3,6,9\}$$
$$= \{1,2,3,5,6,9\}$$

The shaded part represent $A \cup B$.

→ Intersection of Two sets :-

In the given figure, the common region b/w the two sets A and B represent $A \cap B$.



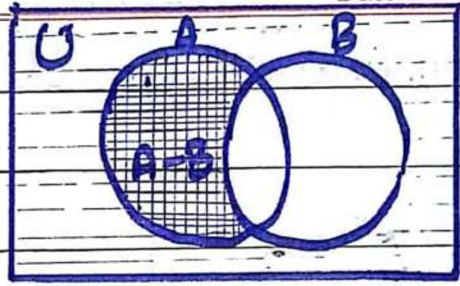
$$A \cap B = \{1,3,5,9\} \cap \{2,3,6,9\}$$
$$= \{3,9\}$$

The shaded part represent $A \cap B$.

→ Difference of Two Sets :-

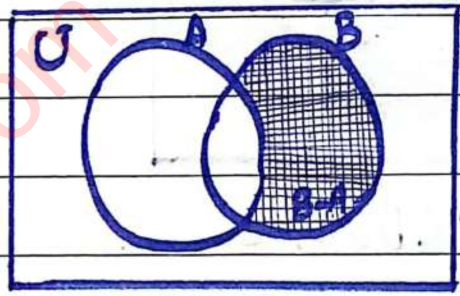
- The set of those elements of A which are not in B is called difference of Two sets A and B. It is denoted by $A-B$.

In the given figure, the shaded portion represents $A-B$.



- The set of those elements of B which are not in set A is denoted by B-A.

In the given figure, the shaded portion represent B-A.



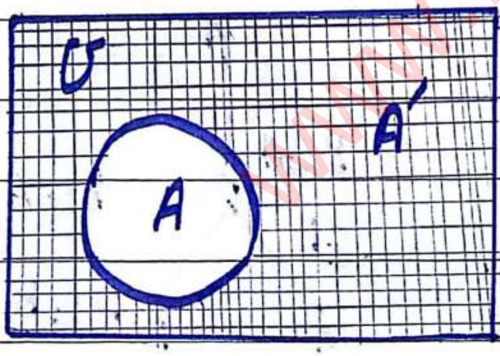
→ Complement of a set :-

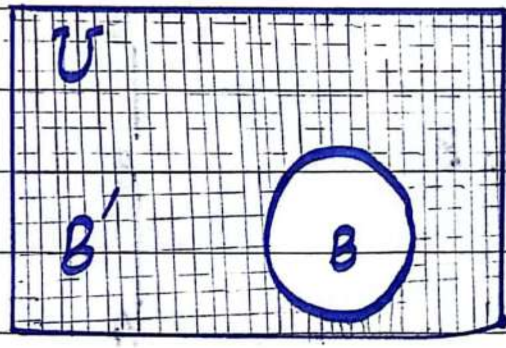
- The given figure represent complement of set A. $A^c = U - A$

The shaded part represent A'.

- The given figure represents complement of set B. $B' = U - B$

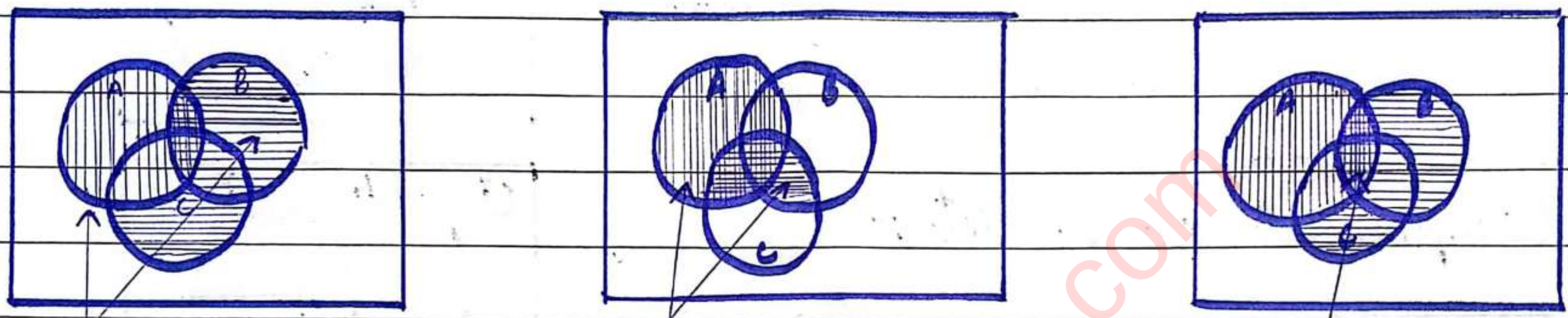
The shaded part represent B'.





Lecture #02.

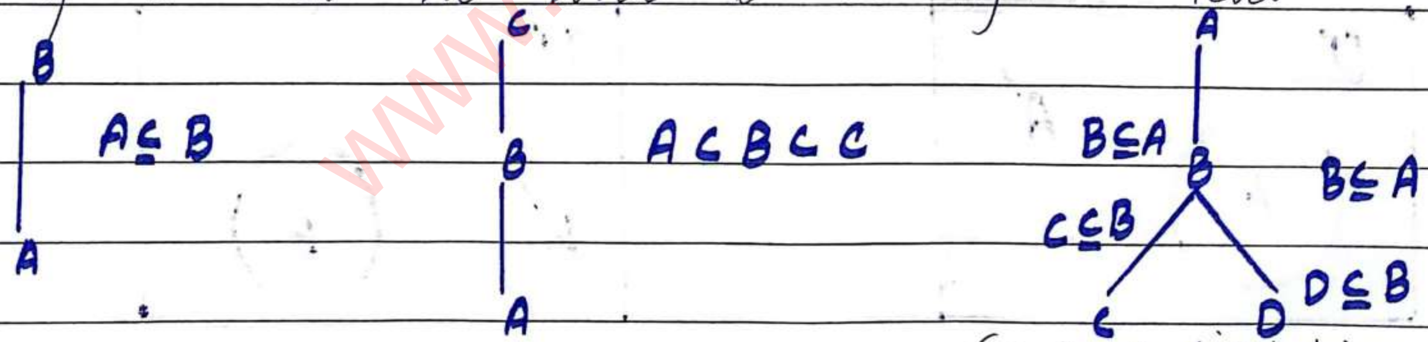
→ Demonstration of Union and Intersection of three overlapping sets:



$A \cup (B \cup C)$ $A \cup (B \cap C)$ $A \cap (B \cup C)$

imp → Line diagram:-

Another usefull and instructive way of illustrating relationship b/w sets in the use of so called line diagram, if "A" is a subset of B then we write B on higher level.



(C, D are disjoint).
and we don't know how?

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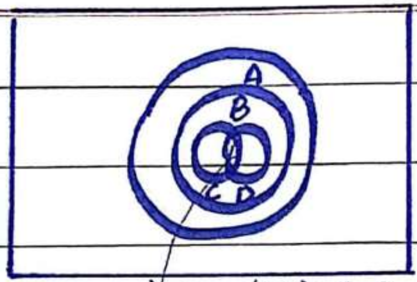

overlapp C, D.

→ Basic Set operations :-

1: Union of Sets :

Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B or in both.

An element x belongs to the union of the sets A and B iff x belongs to A or x belongs to B. This tell us that

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

→ General proofs :-

(i) $A \cup B = B \cup A$

proof :-

Let us assume that x be an arbitrary any element of the set $A \cup B$.

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Then $x \in A \cup B$

$\Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ or $x \in A$

$\Rightarrow x \in B \cup A$

$\therefore A \cup B \subseteq B \cup A$ — (i)

Similarly it can be proved that

$B \cup A \subseteq A \cup B$ — (ii)

So from (i) and (ii) we can write as

$A \cup B = B \cup A$. (commutative law).

(ii) $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $A \subseteq B$ Then $A \cup B = B$.

proof:-

Given that $A \subseteq B$

let $x \in A \cup B$

$x \in A$ or $x \in B$

If $x \in B$, Then $A \cup B \subseteq B$

If $x \in A \subseteq B$ Then $x \in B$

$\Rightarrow A \cup B \subseteq B$ — i

let $y \in B$

$\Rightarrow y \in A \cup B$

$\Rightarrow B \subseteq A \cup B$ — (ii)

From (i) and (ii) we can write as

$A \cup B = B$ C

iii) $A \cup A = A$

proof:-

let $x \in A \cup A$

$\Rightarrow x \in A$ or $x \in A$

If $x \in A$ then $A \cup A \subseteq A$ — (i)

let $y \in A$

$y \in A$ or $y \in A$

$\Rightarrow y \in A \cup A$

$\Rightarrow A \subseteq A \cup A$ — (ii)

From (i) and (ii)

$A \cup A = A$ (Idempotent property).

2. Intersection of sets:

let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

→ General proofs:-

(i) $A \cap B = B \cap A$

proof:-

let $x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in B$ and $x \in A$

$\Rightarrow x \in B \cap A$.

$\therefore A \cap B \subseteq B \cap A$ —— (i)

Similarly it can be shown that

$B \cap A \subseteq A \cap B$ —— (ii)

$x \in B \cap A$

$\Rightarrow x \in B$ and $x \in A$

$\Rightarrow$ ~~$x \in A$ and $x \in B$~~ $\Rightarrow x \in A \cap B$

$\Rightarrow B \cap A \subseteq A \cap B$

From (i) and (ii)

$A \cap B = B \cap A$ (commutative law).

(ii) $A \subseteq B \Rightarrow A \cap B = A$

proof:-

let any $x \in A \cap B$

Then $x \in A$ and $x \in B$

If $x \in A$ then $A \cap B \subseteq A$ —— (i)

If $x \in A$

Since $A \subseteq B$ and $x \in A$

So $x \in B$ Thus we have

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in A \cap B$

$\Rightarrow A \subseteq A \cap B$ —— (ii)

From (i) and (ii) we have

$A \cap B = A$.

(iii) $A \cap B \subseteq A$ and $A \cap B \subseteq B$

(i) $A \cap B \subseteq A$

let $x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B$

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If $x \in A$ then $A \cap B \subseteq A$ ($A \cap B$ being a subset of A).

(ii) $A \cap B \subseteq B$

let $y \in A \cap B$

$\Rightarrow y \in A$ and $y \in B$

$\because$ If $y \in B$ then $A \cap B \subseteq B$. ($A \cap B$ being a subset of B).

3: Difference of Set :-

Let A and B be sets. The difference of A and B, denoted by $A-B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called complement of B w.r.t A.

The difference of A and B is sometimes denoted by A/B.

$$A - B = \{x \,/\, x \in A \text{ and } x \notin B\}$$

(aisy elements jo A mai ho lykn B mai nahi)

Ex:- $A = \{1,3,5\}$, $B = \{1,2,3\}$

$A - B = \{5\}$, $B - A = \{2\}$.

General proofs:-

(i) $A - B \subseteq A$

Proof:-

let $x \in A - B$

$\Rightarrow x \in A$ and $x \notin B$ (By def of difference of sets).

If $x \in A$ then $A - B \subseteq A$ ($A-B$ is a subset of A).

$\Rightarrow A-B \subseteq A$.

(ii) $(A-B) \cap B = \phi$

Proof:-

let $x \in (A-B) \cap B$

This means $x \in A-B$ and $x \in B$

By def of $A-B$

If $x \in A-B$, it must belong to A not in B.

$\Rightarrow x \in A$ and $x \notin B$. but we assume that $x \in B$

which is contradiction to our Supposition Then

$(A-B) \cap B = \phi$.

→ Properties :-

i) $A-B \neq B-A$ (Ya us time satisfy hoga Jbh set same ho ya phr empty).

$\Rightarrow A-B \subseteq A$ (as prooved) For example:

$\Rightarrow A-B \nsubseteq B$ $U = \{1, 2, \ldots, 20\}$, $A = \{2, 3, 7, 9\}$, $B = \{5, 3, 7, 1$

$\Rightarrow B-A \nsubseteq A$ $A-B = \{2, 9\} \Rightarrow A-B \neq B-A$

$\Rightarrow B-A \subseteq B$ $B-A = \{5, 10\}$

$\Rightarrow A-B \neq B-A$

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ii) $A-B$, $A\cap B$ and $B-A$ are mutually Disjoint :-

$(A-B)\cap(B-A)=\phi$

let $x \in (A-B)\cap(B-A)$.

This means That $x\in(A-B)$ and $x\in(B-A)$

but by def of $(A-B)$ (bcz agr x A mai kr rhai ho B mai nhi krng ga.

If $x \in A-B$ then $x\in A$ and $x \notin B$. $\Rightarrow$ If $x\in A \Rightarrow A-B\subseteq A$

and by def of $(B-A)$ (qk $A-B$ mai $x\notin B$ nhi kr rha lykn $B-A$ mai kr rha To ans ϕ aygi)

If $x \in B-A$ then $x\in B$ and $x\notin A$ $\Rightarrow$ If $x\in B \Rightarrow B-A\subseteq B$

$\Rightarrow (A-B)\cap(B-A)=\phi$.

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4: Universal Set:

A Universal set is a set which contain all the elements or objec of other sets including its own elements. It is denoted by U.

Example: $U = \{1, 2, 3, \ldots 10\}$ If $A = \{1, 3, 5, 9\}$, $B = \{2, 4, 6, 8, 10\}$.

5: Complement Set

let U be the Universal set. The complement of the set A denoted by $\bar{A}$, A', is the complement of A w.r.t U. Therefore the complement of the set A is $U - A$.

An element belongs to A' iff $x \notin A$. This tells us that

$$A' = \{x \in U \mid x \notin A\}$$

Example:-

let $A = \{a, e, i, o, u\}$ and Universal set $U = \{$set of all English alphabets$\}$.

Then $A' = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$.

v.imp **Generally proofs:-**

(i) $A \subseteq B \Rightarrow B' \subseteq A'$

proof:-

$A \subseteq B$

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if $x \in B'$ but $x \notin B$ (x B mai nhi hai To usky subset A mai bh nhi hoga).

$x \notin A \Rightarrow x \in A'$

$\Rightarrow B' \subseteq A'$

Conversely:-

If $B' \subseteq A'$.

If $x \in A$ Then $x \notin A'$ (agr x A' mai nhi hai To B' mai bh nhi hoga)

$x \notin B'$

$\Rightarrow x \in B$

$\Rightarrow A \subseteq B$.

ii) $A - B = A \cap B'$

proof:-

let $x \in A - B$

by def of $(A - B)$

$\Rightarrow x \in A$ and $x \notin B$

$\Rightarrow x \in A$ and $x \in B'$

$\Rightarrow x \in A \cap B'$

$\Rightarrow A - B \subseteq A \cap B'$ ——— (i)

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let $y \in A \cap B'$

$\Rightarrow y \in A$ and $y \in B'$

$\Rightarrow y \in A$ and $y \notin B$

$\Rightarrow y \in A - B$

$\Rightarrow A \cap B' \subseteq A - B$ —— (ii)

From (i) and (ii) we have

$A \cap B' = A - B \Rightarrow A - B = A \cap B'$ (Proved).

iii) $A \cup A' = U$

proof:-

let $x \in A \cup A'$

$\Rightarrow x \in A$ or $x \in A'$

$\Rightarrow x \in A$ or $x \in (U - A)$ $\because A' = U - A$

$\Rightarrow x \in A$ or $\{x \in U$ and $x \notin A\}$

$\Rightarrow x \in A$ or $x \in U$ and $x \notin A$

$\Rightarrow x \in A$ and $x \notin A$ or $x \in U$

$\Rightarrow x \in U$

$\because A \cup A' \subseteq U$ —— (i)

let $x \in U$

$\Rightarrow x \in U$ and $x \in A$ or $x \notin A$

$\Rightarrow x \in U$ and $x \notin A$ or $x \in A$

$\Rightarrow x \in (U - A)$ or $x \in A$

$\Rightarrow x \in A'$ or $x \in A$

$\Rightarrow x \in (A' \cup A) \Rightarrow x \in (A \cup A')$

Then $U \subseteq A \cup A'$ — (ii).

From (i) and (ii) we have

$A \cup A' = U$. (Proved).

iv) $A \cap A' = \phi$

Proof:-

let $x \in A \cap A' \Rightarrow x \in A$ and $x \in A'$

$\Rightarrow x \in A$ and $x \notin A$

$\Rightarrow x \in \phi$

$\Rightarrow A \cap A' \subset \phi$ — (i)

let $x \in \phi$

$\Rightarrow x \in A$ and $x \notin A$

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$\Rightarrow x \in A$ and $x \in A'$

$\Rightarrow x \in (A \cap A')$

$\Rightarrow \phi \subset (A \cap A')$ —— (ii)

– From (i) and (ii)

$A \cap A' = \phi$. (Proved).

v) $(A \cup B)' = A' \cap B'$

let $x \in (A \cup B)'$

$\Rightarrow x \notin A \cup B$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$ (by def of complement).

$\Rightarrow x \in A' \cap B'$

$\because x \in (A \cup B)' \Rightarrow x \in A' \cap B'$

Thus $(A \cup B)' \subseteq A' \cap B'$ —— (i)

let $y \in A' \cap B'$

$\Rightarrow y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ or $y \notin B$

$\Rightarrow y \notin A \cup B$

Date:

$\Rightarrow y \in (A \cup B)'$

$\Rightarrow A' \cap B' \subseteq (A \cup B)'$ — (ii)

From (i) and (ii) we have

$(A \cup B)' = A' \cap B'$.

vi) $(A \cap B)' = A' \cup B'$

proof:-

let $x \in (A \cap B)'$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in A' \cup B'$

$\Rightarrow (A \cap B)' \subseteq A' \cup B'$ — (i)

let $y \in A' \cup B'$

$\Rightarrow y \in A'$ or $y \in B'$

$\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin A \cap B$

$\Rightarrow y \in (A \cap B)'$

 Date:

$\Rightarrow A'\cup B' \subseteq (A\cap B)'$ — (ii) by combining

$(A\cap B)' = A'\cup B'$ (Proved).

Symmetric difference of sets:-

Symmetric difference of sets A and B denoted by $A\Delta B$, is the set of all elements which are in either set A or set B but not in both A and B.

$A\Delta B = \{x \mid x\in A \text{ or } x\in B\}$

$A\Delta B = (A-B)\cup(B-A)$

$B\Delta A = (B-A)\cup(A-B)$.

Theorems :-

$A\Delta B = B\Delta A$ (Commutative).

Proof:-

$A\Delta B = B\Delta A$.

We have $A\Delta B = (A-B)\cup(B-A)$ $\because A-B = A\cap B'$

$= (A\cap B')\cup(B\cap A')$ $\because B-A = B\cap A'$

$= (B\cap A')\cup(A\cap B')$ union is commutative $\therefore A\cup B = B\cup A$

$= (B-A)\cup(A-B)$

$A\Delta B = B\Delta A$. (Proved.) (Commutative)

 Date:

$(A \Delta B) \Delta C = A \Delta (B \Delta C)$

proof:-

$A \Delta (B \Delta C) = [A - (B \Delta C)] \cup [(B \Delta C) - A]$

$= [A \cap (B \Delta C)'] \cup [(B \Delta C) \cap A']$ $\because A - B = A \cap B'$

$= [A \cap ((B-C) \cup (C-B))'] \cup [((B-C) \cup (C-B)) \cap A']$

$\because A \Delta B = (A \cup B) \cap (A' \cup B')$

$= [A \cup ((B \cup C) \cap (C' \cup B'))'] \cup [(B \cap C') \cup (C \cap B') \cap A']$

$\because (A \cap B)' = A' \cup B'$

$= [A \cup ((B \cup C)' \cup ((C')' \cup (B')'))] \cup [(B \cap C') \cap A' \cup (C \cap B') \cap A']$

$\because (A \cup B)' = A' \cap B'$, $\because (A')' = A$

$= [A \cup ((B' \cap C') \cup (C \cup B))] \cup [(B \cap C' \cap A') \cup (C \cap B' \cap A')]$

$= [(A \cup B' \cap C') \cup (A \cup C \cup B)] \cup [(B \cap C' \cap A') \cup (C \cap B' \cap A')]$ — ①

$(A \Delta B) \Delta C = [(A \Delta B) - C] \cup [(C - A \Delta B)]$

$= [(A \Delta B) \cap C'] \cup [C \cap (A \Delta B)']$ $\because A - B = A \cap B'$

$= [((A-B) \cup (B-A)) \cap C'] \cup [C \cap ((A-B) \cup (B-A))']$

$= [((A \cap B') \cup (B \cap A')) \cap C'] \cup [C \cap ((A \cup B) \cap (A' \cup B'))']$

$\because A \Delta B = (A \cup B) \cap (A' \cup B')$

33- $\because (A \cap B)' = A' \cup B'$ Date:

$= [(A \cap \bar{B}' \cap \bar{C}') \cup (B \cap A') \cap C')] \cup [C \cap ((A \cup \bar{B})' \cup (A' \cup B')')]$

$= [(A \cap B' \cap C') \cup (B \cap C' \cap A')] \cup [C \cap (A' \cap B') \cup (A \cap B)]$ $(A' \cup B')' = (A'' \cap B'')$

$= [(A \cap B' \cap C') \cup (B \cap C' \cap A')] \cup [(C \cap A' \cap B') \cup (C \cap A \cap B)]$

$= [(A \cap B' \cap C') \cup (B \cap C' \cap A')] \cup [(C \cap A' \cap B') \cup (C \cap A \cap B)]$ — (ii)

$\therefore$ From (i) and (ii),

$A \Delta (B \Delta C) = (A \Delta B) \Delta C.$ (Proved).

→ Infinite Sets :-

N : A positive whole number (1, 2, 3, ...) Sometimes with the inclusion of zero.

All +ve integers from 1 to ∞.

$N = \{1, 2, 3, ...\}$

Z : An integer is a whole number (not a fractional number that can be positive, negative or zero.

$Z = \{....., -2, -1, 0, 1, 2, ...\}$

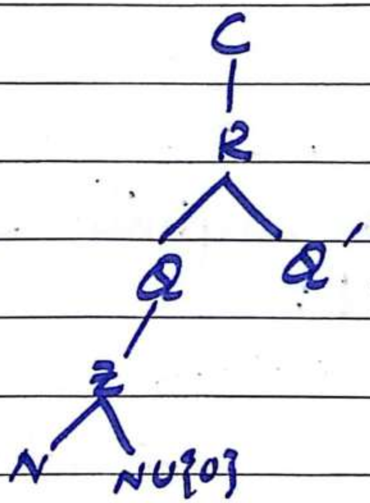
Rational number: A rational number is any number that can be written as the ratio of two integers.

$Q = \{ p/q \mid p \in Z, q \in Z \text{ and } q \neq 0 \}$

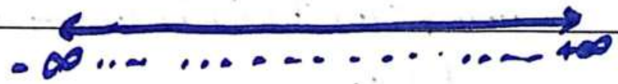
Irrational number: An irrational number is a number that cannot be expressed as a fraction for any integers.

$Q' =$

Rational	Irrational
• It is expressed in the ratio, where both numerator and denominator are the whole numbers.	• It is impossible to express irrational numbers as fractions or in a ratio of two integers.
• The decimal expansion for rational numbers executes finite or recurring decimals.	• Here, non-terminating and non-recurring decimal are executed.



Real numbers:- Real numbers included rational numbers like positive and negative integers, fractions, irrational numbers.



Complex numbers: The numbers that expressed in the form of $a+ib$ where a, b are real numbers and i is an imaginary number.

Date: ______

→ Interval on Real Line

$[a,b] = \{x : a \le x \le b\}$ (close interval)

$(a,b] = \{x : a < x \le b\}$ (open close interval)

$[a,b) = \{x : a \le x < b\}$ (close open interval)

$(a,b) = \{x : a < x < b\}$ (open interval).

→ Set of Taps :-

- $A = \{A_1, A_2, A_3\}$ (agr set finite ho to ya form likh skty hai).

$= \{A_\alpha, \alpha = 1, 2, 3\} \ldots$

$= \{A_\alpha : \alpha \in I\}$ → Index set $= \{1, 2, 3\}$

↳ indices

- If set is Infinite set

$\beta = \{A_1, A_2, \ldots\}$ (and we can't count this set).

$\beta = \{A_\alpha : \alpha \in I\}$ (we can't write this set in real no bcz we don't know whats the next number so)

where $I = \{1, 2, \ldots\}$

(I can be finite, Infinite countable and Uncountable.)

$$\bigcup_{\alpha \in I} A_\alpha \neq A_1 \cup A_2 \cup \ldots$$

(bcz we don't know about I, k I ks ky equal hai).

Date:

Proof:- infinite or uncountable

$$\left(\bigcup_{\alpha \in I} A_\alpha\right)' = \bigcap_{\alpha \in I} A'_\alpha$$

let $x \in \left(\bigcup_{\alpha \in I} A_\alpha\right)'$

$\Rightarrow x \notin \bigcup_{\alpha \in I} A_\alpha$

$\Rightarrow x \notin A_\alpha, \ \forall \alpha \in I$

$\Rightarrow x \in A'_\alpha ; \ \forall \alpha \in I$

$\Rightarrow x \in \bigcap_{\alpha \in I} A'_\alpha$

$\Rightarrow \left(\bigcup_{\alpha \in I} A_\alpha\right)' \subseteq \bigcap_{\alpha \in I} A'_\alpha$ — (i)

let $y \in \bigcap_{\alpha \in I} A'_\alpha$

$\Rightarrow y \notin \bigcap_{\alpha \in I} A_\alpha$

$\Rightarrow y \notin A_\alpha ; \ \forall \alpha \in I$

$\Rightarrow y \notin \bigcup_{\alpha \in I} A_\alpha$

$\Rightarrow y \in \left(\bigcup_{\alpha \in I} A_\alpha\right)'$

$\Rightarrow \bigcap_{\alpha \in I} A'_\alpha \subseteq \left(\bigcup_{\alpha \in I} A_\alpha\right)'$ — (ii)

From (i) and (ii)

$\left(\bigcup_{\alpha \in I} A_\alpha\right)' = \bigcap_{\alpha \in I} A'_\alpha$. (Proved)

(agr $\bigcup_{\alpha \in I} A_\alpha$ countable hota to us ko open kr sktey They lykn hmy nhi pta k α countable hai ya uncountable).

(ii) $\left(\bigcap_{\alpha \in I} A_\alpha\right) \cup B = \bigcap_{\alpha \in I} (A_\alpha \cup B)$

Let $x \in \left(\left(\bigcap_{\alpha \in I} A_\alpha\right) \cup B\right)$

then $x \in \bigcap_{\alpha \in I} A_\alpha$ or $x \in B$

$\Rightarrow x \in A_\alpha, \forall \alpha \in I$ or $x \in B$

$\Rightarrow \forall \alpha \in I, x \in A_\alpha$ or $x \in B$

$\Rightarrow \forall \alpha \in I, x \in (A_\alpha \cup B)$

$\Rightarrow x \in \bigcap_{\alpha \in I} (A_\alpha \cup B)$

$\Rightarrow \left(\bigcap_{\alpha \in I} A_\alpha\right) \cup B \subseteq \bigcap_{\alpha \in I} (A_\alpha \cup B)$ — (i)

let $y \in \bigcap_{\alpha \in I} (A_\alpha \cup B)$

$\Rightarrow y \in (A_\alpha \cup B) \ \forall \alpha \in I.$

$\Rightarrow y \in A_\alpha$ or $y \in B$. $\forall \alpha \in I$

$\Rightarrow y \in A_\alpha$, for some $\alpha \in I$ or $y \in B$

$\Rightarrow y \in \bigcap_{\alpha \in I} A_\alpha$ or $y \in B$

$\Rightarrow y \in \left(\bigcap_{\alpha \in I} A_\alpha\right) \cup B$

$\Rightarrow \bigcap_{\alpha \in I} (A_\alpha \cup B) \subseteq \left(\bigcap_{\alpha \in I} A_\alpha\right) \cup B$ — (ii)

From (i) and (ii)

$\left(\bigcap_{\alpha \in I} A_\alpha\right) \cup B = \bigcap_{\alpha \in I} (A_\alpha \cup B).$

iii) $\left(\bigcup_{\alpha\in I} A_\alpha\right)\cap B = \bigcup_{\alpha\in I}(A_\alpha\cap B)$.

let $x \in \left(\bigcup_{\alpha\in I} A_\alpha\right)\cap B$

$\Rightarrow x \in \bigcup_{\alpha\in I} A_\alpha$ and $x\in B$

$\Rightarrow x\in A_\alpha$ for some $\alpha\in I$ and $x\in B$

$\Rightarrow \forall \alpha\in I,\ x\in A_\alpha$ and $x\in B$

$\Rightarrow x\in (A_\alpha\cap B)\ \forall \alpha\in I$

$\Rightarrow x\in \bigcup_{\alpha\in I}(A_\alpha\cap B)\ \forall \alpha\in I$

$\Rightarrow \left(\bigcup_{\alpha\in I} A_\alpha\right)\cap B \subseteq \bigcup_{\alpha\in I}(A_\alpha\cap B)$ — (i)

let $y\in \bigcup_{\alpha\in I}(A_\alpha\cap B)$ for some $\alpha\in I$

$\Rightarrow y\in A_\alpha\cap B$

$\Rightarrow y\in A_\alpha$ and $y\in B$

$\Rightarrow y\in A_\alpha\ \forall \alpha\in I$ and $y\in B$

$\Rightarrow y\in \bigcup_{\alpha\in I} A_\alpha$ and $y\in B$

$\Rightarrow y\in \left(\bigcup_{\alpha\in I} A_\alpha\right)\cap B$

$\Rightarrow y\bigcup_{\alpha\in I}(A_\alpha\cap B) \subseteq \left(\bigcup_{\alpha\in I} A_\alpha\right)\cap B$ — (ii

From (i) and (ii),

$\left(\bigcup_{\alpha\in I} A_\alpha\right)\cap B = \bigcup_{\alpha\in I}(A_\alpha\cap B)$.

(proved).

(repeated Theorems) Date:

Theorems:-

(i) $A \cup (\cap_i B_i) = \cap_i (A \cup B)$, $i \in I$.

Proof:-

let any $x \in A \cup (\cap_i B_i)$

$\Rightarrow x \in A$ or $x \in \cap_i B_i$

$\Rightarrow x \in A$ or $x \in B_i$ $\forall i \in I$

$\Rightarrow x \in (A \cup B_i)$, $\forall i \in I$.

$\Rightarrow x \in \bigcap_{i \in I} (A \cup B_i)$

$\Rightarrow A \cup (\cap_i B_i) \subset \cap_i (A \cup B_i)$ —— (i)

consider any $x \in \cap_i (A \cup B_i)$

$\Rightarrow x \in A \cup B_i$, $\forall i \in I$

$\Rightarrow x \in A$ or $x \in B_i$ $\forall i \in I$

$\Rightarrow x \in A$ or $x \in (\cap_i B_i)$ $\forall i \in I$

$\Rightarrow x \in A \cup (\cap_i B_i)$

$\Rightarrow \cap_i (A \cup B_i) \subseteq A \cup (\cap_i B_i)$ —— (ii)

From (i) and (ii)

$A \cup (\cap_i B_i) = \cap_i (A \cup B)$ (proved)

Date:

(ii) $A \cap (\cup_i B_i) = \cup_i (A \cap B_i)$.

proof:-

let $x \in A \cap (\cup_i B_i)$

$\Rightarrow x \in A$ and $x \in \cup_i B_i$, $\forall i \in I$

$\Rightarrow x \in A$ and $x \in B_i$, $\forall i \in I$

$\Rightarrow x \in (A \cap B_i)$ for some $i \in I$

$\Rightarrow x \in \cup_i (A \cap B_i)$

$\Rightarrow A \cap (\cup_i B_i) \subseteq \cup_i (A \cap B_i)$ — (i)

let $x \in \cup_i (A \cap B_i)$

$\Rightarrow x \in A \cap B_i$, $\forall i \in I$

$\Rightarrow x \in A$ and $x \in B_i$, $\forall i \in I$

$\Rightarrow x \in A$ and $x \in \cup_i B_i$, $\forall i \in I$

$\Rightarrow x \in A \cap (\cup_i B_i)$

$\cup_i (A \cap B_i) \subseteq A \cap (\cup_i B_i)$ — (ii)

From (i) and (ii)

$A \cap (\cup_i B_i) = \cup_i (A \cap B_i)$. (Proved).

Date:

(iii) $(\cup_i A_i)' = \cap_i A_i'$

proof:-

let $x \in (\cup_i A_i)'$

$\Rightarrow x \in U - \cup_i A_i \Rightarrow x \in U$ but $x \notin \cup_i A_i \quad \forall i \in I.$

$\Rightarrow x \in U$ and $x \notin A_i \quad \forall i \in I.$

$\Rightarrow x \in A_i' \, , \forall i \in I \Rightarrow x \in \cap_i A_i'$

$\Rightarrow (\cup_i A_i)' \subseteq \cap_i A_i'$ —— (i)

let $y \in \cap_i A_i'$

$\Rightarrow y \in A_i' \, , \forall i \in I.$

By def of compliment

If $y \in A_i' \, , \forall i \in I$ then $y \notin A_i \, , \forall i \in I$

$\Rightarrow y \notin \cup_i A_i \, , \forall i \in I.$

$\Rightarrow y \in U - \cup_i A_i \, , \forall i \in I \Rightarrow x \in (\cup_i A_i)'$

$\Rightarrow \cap_i A_i' \subseteq (\cup_i A_i)'$ —— (ii)

From (i) and (ii)

$(\cup_i A_i)' = \cap_i A_i'$ (proved).

Date:

iv) $(\cap_i A_i)' = \cup_i A_i'$

Proof:-

let $x \in (\cap_i A_i)' \Rightarrow x \in U - \cap_i A_i \; , \forall i \in I$

$\Rightarrow x \in U$ but $x \notin \cap_i A_i \; , \forall i \in I$

$\Rightarrow x \in U$ and $x \notin A_i \; , \forall i \in I.$

$\Rightarrow x \in A_i' \; , \forall i \in I. \Rightarrow x \in \cup_i A_i'$, for some $i \in I$.

$\Rightarrow (\cap_i A_i)' \subseteq \cup_i A_i'$ —— (i)

let $y \in \cup_i A_i'$ for some $i \in I$

$\Rightarrow y \in A_i' \; , \forall i \in I.$

By def of compliment

If $y \in A_i' , \forall i \in I$ then $y \notin A_i , \forall i \in I$

$\Rightarrow y \notin \cap_i A_i \; , \forall i \in I.$

$\Rightarrow y \in U - \cap_i A_i \; , \forall i \in I$

$\Rightarrow y \in (\cap_i A_i)' \; , \forall i \in I.$

$\Rightarrow \cup_i A_i' \subseteq (\cap_i A_i)'$ —— (ii)

From (i) and (ii)

$(\cap_i A_i)' = \cup_i A_i'$ (Proved).

43- Lecture # 03.

Date:

Bounded Set :-

A set "A" is said to be bounded if $\exists$ a +ve real constant M such that

$$|x| < M \quad \forall \; x \in A \quad \Rightarrow \quad -M < x < M, \; \forall \; x \in A$$

(M → fixed real constant / no.)

Example

$A = \{1, 2, 3\}$ is bounded bcz the elements of set A is less than 4.

$$|A| < 4 \quad \forall \; x \in A.$$

Cartesian product :-

The cartesian product of A and B, written as $A \times B$, is the set consisting of all such pairs.

let A and B be two sets;

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

Example:-

If $A = \{1, 2\}$; $B = \{3, 4\}$

$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

$B \times A = \{(3,1), (3,2), (4,1), (4,2)\}$

Date:

→ Commutative property:

It cannot be hold in Cartesian product bcz ordered pair is not Same (as above example).

- So $A \times B \neq B \times A$ (It can only hold when $(a,b) = (c,d)$ order pairs are same i.e $a = c$; $b = d$).
- If $A = \phi$ and $B = \phi$ Then $A \times B = \phi$.

→ Rectangular cartesian product:-

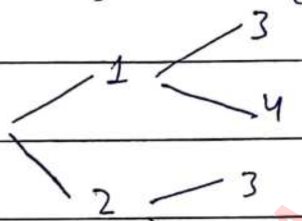
$R^2 = R \times R = \{(a,b) ; a,b \in R\}$ → ordered pair.

$R^3 = R \times R \times R = \{(a,b,c) ; a,b,c \in R\}$ → order triplet.

→ Tree diagram:-

let $A = \{1,2\}$, $B = \{3,4\}$, $C = \{5,6,7\}$

$A \times B =$



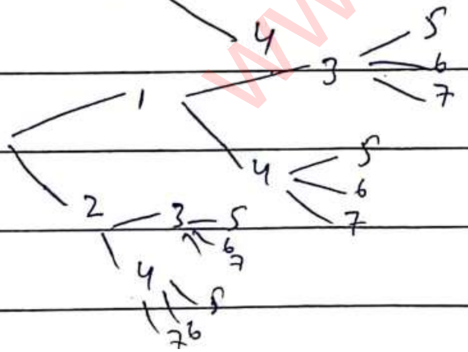
(1,3)
(1,4)
(2,3)
(2,4)

$n(A) = 2$, $n(B) = 2$, $n(A \times B) = 4$

$n(A) = 2$; $n(B) = 2$; $n(C) = 3$

$n(A \times B \times C) = 12$.

$A \times B \times C =$



(1,3,5) (1,3,6), (1,3,7), (1,4,5), (1,4,6), (1,4,7), (2,3,5), (2,3,6), (2,3,7), (2,4,5), (2,4,6), (2,4,7).

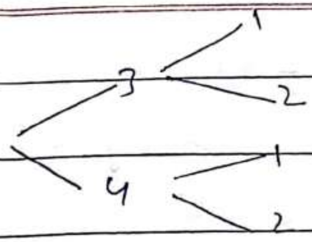
no. of an element

$\#(A) = n$

$\#(B) = m$

$\#(A \times B) = n \times m$

$B \times A =$



$(3,1), (3,2), (4,1), (4,2)$.

→ Associative property:-

$A \times (B \times C) = (A \times B) \times C$

$= [(1,3),(1,4),(2,3),(2,4)] \times (5,6,7)$

$= [(1,3,5),(1,3,6),(1,3,7),(1,4,5),(1,4,6),(1,4,7),(2,3,5),(2,3,6),(2,3,7),$
$(2,4,5),(2,4,6),(2,4,7)] = (A \times B) \times C.$ — (i)

$A \times (B \times C) = (1,2) \times [(3,5),(3,6),(3,7),(4,5),(4,6),(4,7)]$

$= [(1,3,5),(1,3,6),(1,3,7),(1,4,5),(1,4,6),(1,4,7),(2,3,5),$
$(2,3,6),(2,3,7),(2,4,5),(2,4,6),(2,4,7)]$ — (ii)

Associative From (i) and (ii) $A \times (B \times C) = (A \times B) \times C$ (Proved).
law holds in cartesian product.

(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof:-

let $(x, y) \in A \times (B \cap C)$

Date:

$\Rightarrow x \in A$ and $y \in (B \cap C)$.

$\Rightarrow x \in A$ and $y \in B$ and $y \in C$.

$\Rightarrow (x,y) \in A \times B$ —— (1)

and $x \in A$, $y \in C$.

$(x,y) \in A \times C$ —— (2)

$\Rightarrow$ From (1) and (2)

$(x,y) \in (A \times B) \cap (A \times C)$

$\Rightarrow A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ —— (3).

Conversely:-

$(x',y') \in (A \times B) \cap (A \times C)$.

$\Rightarrow (x',y') \in (A \times B)$ and $(x',y') \in A \times C$.

$\Rightarrow x' \in A$ and $y' \in B$; and also $x' \in A$ and $y' \in C$.

$\Rightarrow (y') \in B \cap C$, $x' \in A$

$\Rightarrow (x',y') \in A \times (B \cap C)$

$\Rightarrow (A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$. —— (4)

From (3), (4)

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

Date:

$A \times (B \cup C) = (A \times B) \cup (A \times C)$

Proof:-

let $(x, y) \in A \times (B \cup C)$.

$\Rightarrow x \in A$ and $y \in B \cup C$.

$\Rightarrow x \in A$ and $y \in B$ or $y \in C$.

$\Rightarrow (x, y) \in A \times B$

$\Rightarrow (x, y) \in A \times C$

$\Rightarrow (x, y) \in (A \times B) \cup (A \times C)$ — (1).

Conversely:-

let $(x', y') \in (A \times B) \cup (A \times C)$.

$\Rightarrow (x', y') \in (A \times B)$ or $(x', y') \in A \times C$.

$\Rightarrow x' \in A$ and $y' \in B$ or $x' \in A$ and $y' \in C$.

$\Rightarrow x' \in A$ and $y' \in B \cup C$.

$\Rightarrow (x', y') \in A \times (B \cup C)$ —

$\Rightarrow (A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$ — (2)

From (1) and (2)

$A \times (B \cup C) = (A \times B) \cup (A \times C)$.

1) $A \times \left(\bigcap_{\alpha \in I} B_\alpha\right) = \bigcap_{\alpha \in I} (A \times B_\alpha)$

Proof:-

let $(x, y) \in A \times \left(\bigcap_{\alpha \in I} B_\alpha\right)$

$\Rightarrow x \in A$ and $y \in \bigcap_{\alpha \in I} B_\alpha$

$\Rightarrow x \in A$ and $y \in B_\alpha \ \forall \alpha \in I$.

$\Rightarrow (x, y) \in (A \times B_\alpha) \ \forall \alpha \in I$

$\Rightarrow (x, y) \in \bigcap_{\alpha \in I} (A \times B_\alpha) \ \forall \alpha \in I$.

$\Rightarrow A \times \left(\bigcap_{\alpha \in I} B_\alpha\right) \subseteq \bigcap_{\alpha \in I} (A \times B_\alpha)$ — (1)

Conversely:-

Let $(x', y') \in \bigcap_{\alpha \in I} (A \times B_\alpha)$.

$\Rightarrow (x', y') \in (A \times B_\alpha)$

$\Rightarrow x' \in A$ and $y' \in B_\alpha, \ \forall \alpha \in I$.

$\Rightarrow x' \in A$ and $y' \in \bigcap_{\alpha \in I} B_\alpha$

$\Rightarrow (x', y') \in A \times \left(\bigcap_{\alpha \in I} B_\alpha\right)$

$\bigcap_{\alpha \in I} (A \times B_\alpha) \subseteq A \times \left(\bigcap_{\alpha \in I} B_\alpha\right)$ — (2)

From (1) and (2)

$A \times \left(\bigcap_{\alpha \in I} B_\alpha\right) = \bigcap_{\alpha \in I} (A \times B_\alpha)$

48- # Main Topics

Date:

→ Functions

→ Binary Relation :-

Any subset of $A \times B$ is called Binary relation from A into B.

$R_1 = \{\phi\}$ (hr empty set set ka subset hota hai)

$R_2 = \{(1,3), (2,3)\}$

$R_3 = \{(2,4)\}$

Example:-

$A = \{1,2\}$, $B = \{3,4\}$

$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

$R_4 = \{(x,y) : x-y \text{ is positive}\}$

$1-3 = -2$ ✗, $2-4 = -2$ ✗ $= \{\phi\}$.

$R_5 = \{(x,y) : x/y\}$

$= \{(1,3), (1,4), (2,4)\}$

$R_6 = \{(x,y) : y/x\} = \{\phi\}$.

∵ a/b integer

$b/a = k \Rightarrow b = ak$

Divisiblity (Tbh divide hoga Jbh A multiple ho B ka)

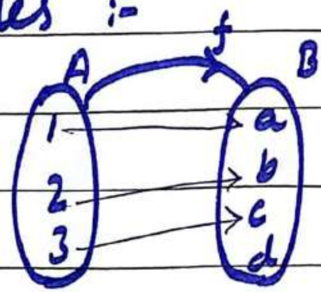
→ Function (Binary relation b/w Two sets).

Let A and B be non-empty sets. A Function f from A To B is an assignment of exactly one element of B To each element of A. we write $f(a) = b$ if b is the unique element of B assigned by the function f To the elements a of A. If f is a function from A To B we write as $f: A \to B$.

(1st Set ka hr element 2nd set ky elements ky sth unique mapping kry.)

Examples :-

1)

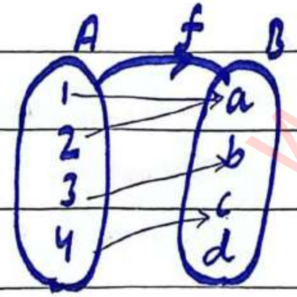


$f(1) = a$, $f(2) = b$, $f(3) = c$.

$f: A \to B$ (hr element ky correspond Unique out put mil rhi hai).

Domain = A   Co-domain = (a, b, c, d),   Range = (a, b, c).

2)



$f(1) = a$, $f(2) = a$, $f(3) = b$, $f(4) = c$.

(First element ky correspond hmy 2 out put mill rhi hai).

a function

Criteria :- (For checking Function).

$f : X \to Y$

If $x \in X$ then $\overset{\text{There exist an element}}{\nearrow}$ $y \in Y$ such that $f(x) = y$ $\longrightarrow$ Range.

3): $f(x) = 2x+1$ $\quad f : \{1,2,3,4\} \to \{3,5,7,9,11\}$

$\because f(x) = y$

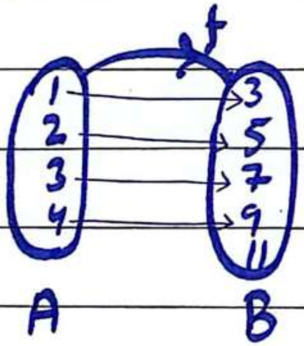
$\Rightarrow y = 2x+1 = f(x)$.

let $f(1) = 2(1)+1 = 3$

$f(2) = 2(2)+1 = 5$

$f(3) = 2(3)+1 = 7$

$f(4) = 2(4)+1 = 9$



Domain $\overset{\text{input}}{=} A$, co-domain $\overset{\text{output}}{=} B$, Range $= (3,5,7,9)$.

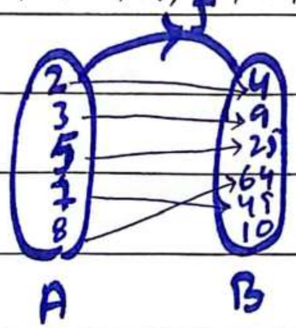
$f : \{2,3,5,7,8\} \to \{4,9,10,25,64,49\}$.

4) $f(x) = x^2$

let $f(2) = 2^2 = 4$ $\qquad f(7) = 49$

$f(3) = 3^2 = 9$ $\qquad f(8) = 64$

$f(5) = 5^2 = 25$



Domain $= A$

Co-domain $= B$

Range $= (4,9,25,64,49)$

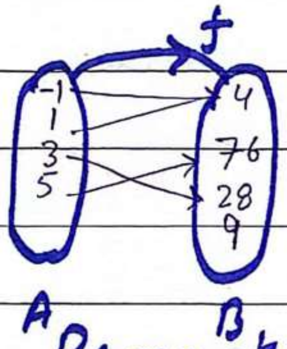
51- $f: \{-1, 1, 3, 5\} \to \{4, 28, 76, 9\}$ Date:

5) $f(x) = 3x^2+1$ $y = f(x) \Rightarrow y = 3x^2+1$.

Sol:-

let $f(1) = 3(1)+1 = 4$, $f(3) = 3(3)^2+1 = 3(9)+1 = 27+1 = 28$

$f(5) = 3(25)+1 = 75+1 = 76$, $f(-1) = 3(-1)^2+1 = 3(1)+1 = 4$.



a function

→ Remark:-

If A and B are sets in general not necessarily sets of numbers then a function is called mapping. (set ki nature ka pta ho to usy function kehty hain).

→ Types of Function :-

1. Into Function:

If Range is proper subset (Improper) of co-domain.

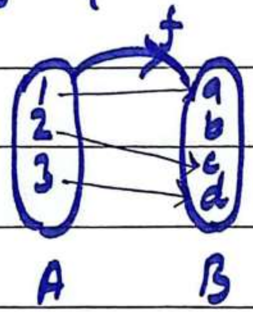
If at least one element in co-domain has no Inverse Image / pre Image then f is called Into Function (range co-domain ky equal na ho.

Date:

Examples :-

1. $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$

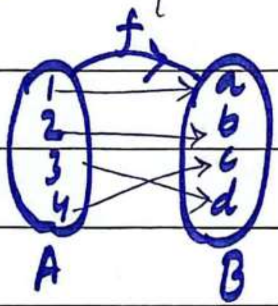
$f = \{(1, a), (2, c), (3, d)\}$



Co-domain = $\{a, b, c, d\}$ Into Function

Range = $\{a, c, d\}$

2. $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $f = \{(1, a), (2, b), (3, d), (4, c)\}$.



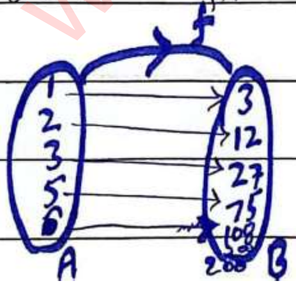
Co-domain = $\{a, b, c, d\}$ Not Into Function

Range = $\{a, b, c, d\}$

3. $f(x) = 3x^2$ $f: A \to B$ $f: \{1, 2, 3, 5, 6\} \to \{3, 12, 27, 75, 108, 200\}$

let $f(1) = 3$, $f(2) = 12$, $f(3) = 27$, $F(5) = 75$

$f(6) = 108$



Co-domain = $\{3, 12, 27, 75, 108, 200\}$

Range = $\{3, 12, 27, 75, 108\}$

Into Function.

$x \to y$
x is preimage of y Date:
y is image of x.

→ Onto Function (Surjective)

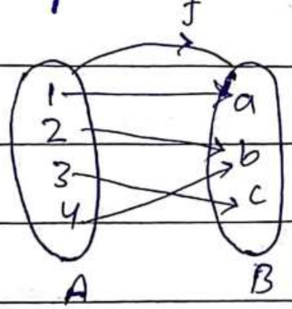
- If Range = co-domain.
- If each elements in co-domain has at least one inverse Image then f is called onto function.

def → A function f from A to B is called onto function. iff for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function is called surjective.

criteria: If any $b \in B$, then $\exists$ an element $a \in A$ s.t $f(a) = b$

Examples :-

1.



Domain = A

Co-domain = $\{a, b, c\}$

Range = $\{a, b, c\}$

Range = co-domain

Onto Function.

2. let $f: R \to R$ is defined by $f(x) = 2x+1$

$f(x) = 2x+1 \Rightarrow f(x) = y$

$y = 2x+1 \Rightarrow 2x = y-1 \Rightarrow x = \frac{y-1}{2}$ $\therefore f(x) = y$

$\Rightarrow 2\left[\frac{y-1}{2}\right]+1$ $\Rightarrow \boxed{f(x) = y}$

$\Rightarrow \frac{2(y-1)+2}{2} \Rightarrow \frac{2y-2+2}{2} \Rightarrow \frac{2y}{2} \Rightarrow = y$ $\boxed{\text{Onto}}$

Date:

3. $f(x) = x+1$

This Function is onto, because for every integer y there is an integer x such that $f(x)=y$

$\Rightarrow f(x) = x+1 \quad \because f(x)=y$

$\Rightarrow y = x+1 \Rightarrow x = y-1$

$\Rightarrow f(x) = (y-1)+1 \Rightarrow f(x) = y$ (holds) onto Function.

4. $f(x) = \dfrac{x+1}{x-1}$ let $f: R-\{1\} \to R-\{1\}$ be defined by

For every Real y $\exists$ an Real x s. that $f(x)=y$

$\Rightarrow f(x) = \dfrac{x+1}{x-1} \quad \because f(x) = y$

$\Rightarrow y = \dfrac{x+1}{x-1} \Rightarrow y(x-1) = x+1 \Rightarrow yx - y = x+1$

$\Rightarrow yx - x = y+1 \Rightarrow x(y-1) = y+1 \Rightarrow x = \dfrac{y+1}{y-1}$

$\Rightarrow f(x) = \dfrac{\frac{y+1}{y-1}+1}{\frac{y+1}{y-1}-1} \Rightarrow \dfrac{\frac{y+1+y-1}{y-1}}{\frac{y+1-y+1}{y-1}} \Rightarrow \dfrac{y+\cancel{1}+y-\cancel{1}}{\cancel{y}+1-\cancel{y}+1} \Rightarrow \dfrac{2y}{2}$

$\Rightarrow y \Rightarrow f(x) = y$ onto Function.

Date:

5) $f: N \to N \quad f(x) = x$

Let $f(x) = y$ $\quad \because y = f(x)$

$f(1) = 1, \quad f(2) = 2, \quad f(3) = 3 \quad f(4) = 4 \ldots$

$y \in N$

Preimage is the number itself.

6) $f: Z \to Z \quad f(x) = |x|$

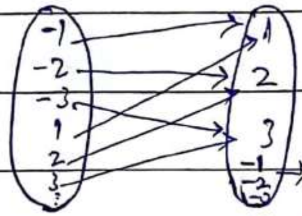
let $f(-1) = 1, \quad f(-2) = 2, \quad f(-3) = 3 \ldots$

$f(1) = 1, \quad f(2) = 2, \quad f(-3) = 3.$

Range $= \{1, 2, 3, \ldots\} = N.$ $\quad \because Z \neq N.$

$\therefore$ f is not into.

Range $\neq$ Co-domain



Natural nos

A   B

Z   Z

(lykn Co-domain hmary pass integers hai).

→ One One Function :- (Injective)

A Function f is said to be one one or an injective iff $f(a) = f(b)$
$\Rightarrow a = b \quad \forall\ a, b$ in the domain of A.

Criterias of one one Function

1st criteria

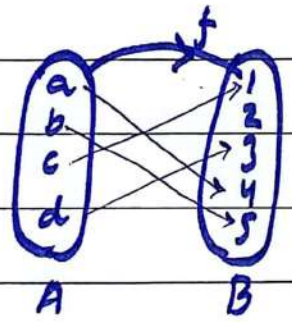
let $x_1, x_2 \in A$ and $x_1 \neq x_2$
$\Rightarrow f(x_1) \neq f(x_2)$
Then f is one one.

2nd criteria

let $x_1, x_2 \in A$
If $f(x_1) = f(x_2)$
$\Rightarrow x_1 = x_2$ Then f is one-one.

Examples:-

1: Function from $\{a, b, c, d\}$ To $\{1, 2, 3, 4, 5\}$
$f: A \to B$



$f(a) = 4, \; f(b) = 5 \quad \Rightarrow f(a) \neq f(b)$
$f(c) = 1, \; f(d) = 3 \quad \Rightarrow a \neq b$ one-one Function

57- Date: ______

2) $f: Z \to R$ $f(x) = x^2$

$f(-1) = 1$, $f(-2) = 4$, $f(-3) = 9$, ...

$f(1) = 1$, $f(2) = 4$, $f(3) = 9$, ...

but $f(-1) = f(1)$ and $f(-2) = f(2)$

$\Rightarrow -1 \neq 1$ $\Rightarrow -2 \neq 2$

This means $f(x) = x^2$ is not one - one function.

3) $f(x) = x+1$

$f(x) = x+1$ replace y into x then

$f(y) = y+1$

then by def of one - one

$f(x) \neq f(y) \Rightarrow x+1 \neq y+1$

$\Rightarrow x \neq y$ (contrapositive condition)

one - one function.

Date:

Question :- How many functions can be defined b/w A and B?

suppose $\#(A) = m$ and $\#(B) = n$

Then each element of set A can associate with elements of set B in n ways. So

The total number of functions from set A to set B $= n^m$

For Example:-

$A = \{3, 4, 5\}$, $B = \{a, b\}$

$\#(A) = 3$, $\#(B) = 2$

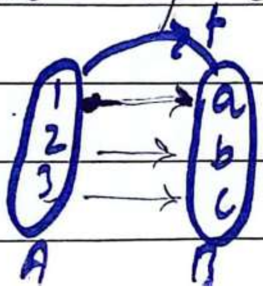
$2^3 = 8$.

Question:- If two sets are finite the possible bijective function.

If there is a bijective function b/w two sets A and B then both sets will have the same number of elements

$\#(A) = \#(B) = m$ then

number of bijective function $= m!$



$\#(A) = \#(B) = 3$

$\#(\text{bijective function}) = 3! = 3 \times 2 \times 1 = 6$

imp for paper

Lecture #04

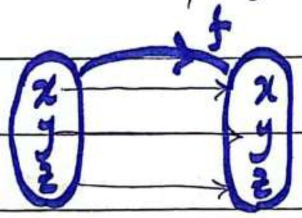
Date: 59-

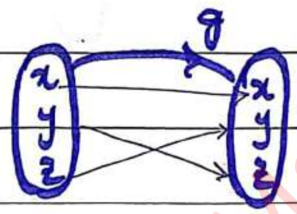
→ Identity Function :-

let A be any set and I be any function defined on set A, i.e $I: A \to A$ and $I(x) = x$ $\forall\ x \in A$ Then I is known as identity mapping.

Example:-

let $A = \{x, y, z\}$ (Ju input hoji whi output hoji).





f is identity mapping g is not identity mapping.

→ Constant Function :-

A function whose value is the same for every input value.

Example:-

$f: A \to B$, $f(x) = k$, $x \in R$

The function $y(x) = 4$ is a constant function bcz the value of $y(x)$ is 4 regardless of the input value, x.

(Ju bh out put dy wby bdly hmy constant no mly).

→ Equal Function:-

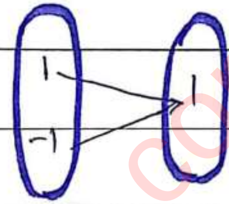
If f and g are two functions defined on same domain D and same co-domain iff $f(x) = g(x)$, $\forall\ x, y \in D$ Then f, g are equal functions-

(i) (Domain ki har value pr f, g k range equal ani chahiy. or (ii) out put (Range) same ho)

Example:-

$f: \{1, -1\} \to \mathbb{R}$ $\quad$ $g: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$ $\quad$ $g(x) = 1$.



$f(x) \neq g(x)$

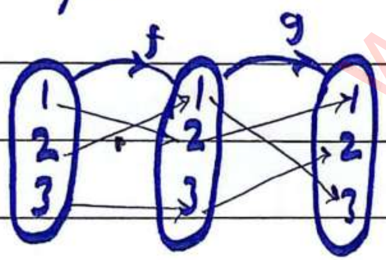
$x^2 \neq 1$

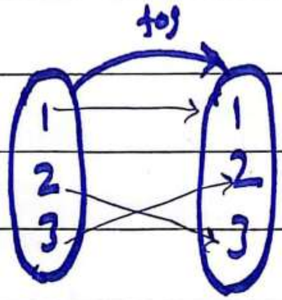
→ **Composition of Functions :-** $f: A \to B$, $g: B \to C$ f, g have same Range.

let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g denoted for all $a \in A$ by $f \circ g$ is defined by $\quad$ $f \circ g: A \to C$.

$f \circ g(x) = g(f(x))$. composition

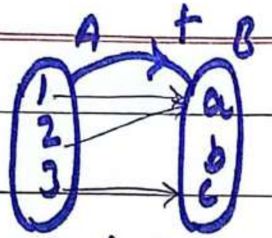
Examples:-

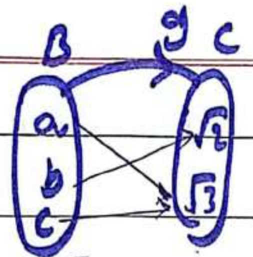




61.

Date:





$\therefore fog : A \longrightarrow C$ → f k range g ki domain ky equal ani chahiy.

$\because fog(x) = g(f(x))$

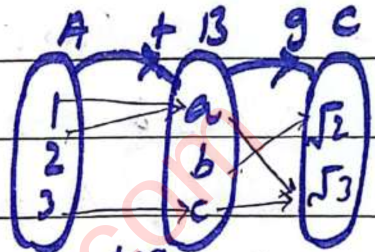
$f(1) = a$, $f(2) = a$
$f(3) = c$

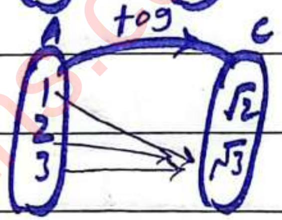
$g(a) = \sqrt{3}$, $g(b) = \sqrt{2}$
$g(c) = \sqrt{3}$

$fog(1) = g(f(1)) = g(a) = \sqrt{3}$

$fog(2) = g(f(2)) = g(a) = \sqrt{3}$

$fog(3) = g(f(3)) = g(c) = \sqrt{3}$





$fog : A \longrightarrow C.$

Proof :-

$fog = gof$ is commutative ?

Let $f(x) = 2x+3$ and $g(x) = 3x+2$.

$fog(x) = g(f(x))$

$fog(x) = g(2x+3) = 3(2x+3)+2 = 6x+9+2 = 6x+11$

$gof(x) = f(g(x))$

$gof(x) = f(3x+2) = 2(3x+2)+3 = 6x+4+3 = 6x+7$

$6x+11 \neq 6x+7$

$\Rightarrow fog \neq gof$ (is not commutative).

let $g(x) = y$

$\rightarrow$ $f\circ(g\circ h) = (f\circ g)\circ h$

Proof :-

(range or domain bhi same hai or value bhi.)

let $f : A \rightarrow B$, $g : B \rightarrow C$, $h : C \rightarrow D$

$f\circ(g\circ h) = (g\circ h) : B \rightarrow D$

$\therefore f\circ(g\circ h) : A \rightarrow D$

$f\circ(g\circ h) = (g\circ h)(f(x)) = h(g(f(x)))$ — (i)

$(f\circ g)\circ h = (f\circ g) : A \rightarrow C$

$(f\circ g)\circ h : A \rightarrow D$

$(f\circ g)\circ h = h(f\circ g)(x) = h(g(f(x)))$ — (ii)

From (i) and (ii)

$f\circ(g\circ h) = (f\circ g)\circ h$

associative.

→ Extension and Restriction of Function :-

Let A and B be sets, let $f: A \to B$, let $g: C \to B$ with $C \subseteq A$, Then g is **restriction** of f if and only if

$$f(x) = g(x) \quad , \forall \; x \in C$$

Let A and B be sets, and let $f: A \to B$. A function g is called an **extension** of g is

(i) $Dom(f) \subseteq Dom(g)$ (ii) $g|_{Dom(f)} = f$.

Example:- Define $f: [0,\infty) \to \mathbb{R}$, $x \to x$. Now define g_1 and g_2 by $g_1: \mathbb{R} \to \mathbb{R}$
$g_2: \mathbb{R} \to \mathbb{R}$.

Then both g_1 and g_2 are extension of f.

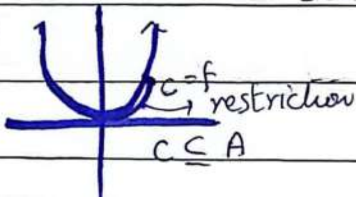
Example:- (1) Define $f: (0,\infty) \to \mathbb{R}$, $x \to 1/x$.

f is Unbounded, when x gets close to 0, $1/x$ takes arbitrary large values.

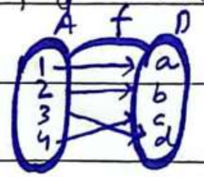
2) Let $g = f|_{[1/2,\infty)}$. Then g is a restriction of f and g is bounded.

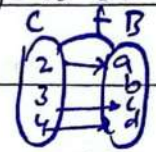
The function g is called restriction of f on X, and f is called an extension to g to Y.

$g = A$



f is restriction of g and g is extension of f.





$C \subseteq A$

F is extention of g and g restriction of f.

 Date:

Theorem :-

If f and g are bijective functions then $f \circ g$ is bijective.

Proof:-

let $f: A \to B$ and $g: B \to C$ be bijective function

As we know that $f \circ g: A \to C$

* fog is one-one :-

let $x_1, x_2 \in A$

$x_1 \neq x_2$ and we have to find that $f \circ g(x_1) \neq f \circ g(x_2)$

As $f: A \to B$ is one-one.

$f(x_1) \neq f(x_2)$ (by using contrapositive criteria)

As $f(x_1), f(x_2) \in B$ and $g: B \to C$ is one-one function

$g(f(x_1)) \neq g(f(x_2)) \Rightarrow f \circ g(x_1) \neq f \circ g(x_2)$

* fog is onto :-

let $c \in C$ and we have to prove that $f \circ g(a) = c$.

As $g: B \to C$ is onto and $c \in C$ then $\exists\ b \in B$ s.that $g(b) = c$

As $f: A \to B$ is onto and $b \in B$ then $\exists\ a \in A$ s.that $f(a) = b$

$g(f(a)) = g(b) = c. \Rightarrow f \circ g(a) = c.$

Lecture # 05

Date: 05

Def: Call a subset of B of set A is co-finite if the compliment of B in A is finite. If "B" and "C" are co-finite subsets of A. Then $B \cap C$ is co-finite.

Proof:-

Given that B and C are co-finite in A. Then $B' = A - B$ and $C' = A - C$ are finite set. Since B' and C' are finites then $B' \cup C'$ also finite.

$$B' \cup C' = (B \cap C)'$$

$\Rightarrow$ $B \cap C$ is co-finite.

(Similarly we can find for $B \cup C$ is co-finite)

Proof:-

Given that B and C are co-finite in A. Then $B' = A - B$ and $C' = A - C$ are finite set. Since B' and C' are finites then $B' \cap C'$ also finite

$$B' \cap C' = (B \cup C)'$$

$\Rightarrow$ $B \cup C$ is co-finite.

(co-finite ka mltb k complement finite ho).

$\because B' = A - B$

$E'^{N} = N - E = \emptyset$ ↳ Infinite

$A \subseteq N$

$A = \{10, 11, \ldots\}$ graph set

$N - A$

$A' = \{1, 2, \ldots 9\}$ ↳ cofinite

$A = \{10, 11, 12, \ldots\}$

$A' = R - A$

$= (-\infty, 0) \cup (10, 11) \cup (11, 12) \cup \ldots$

Compact Form

$A' = R - A$

$= (-\infty, 0) \cup \{\bigcup_{i=10}^{\infty} (i, i+1)\}$

$\because (B \cap C)' := B' \cup C'$

let $\#(A) = t$, $\#(B) = s$

Then $\#(A \cup B) = t + s$ ya equal ay ga ya less.

Results :- (One - one)

If f and g are one to one then fog is one to one.

Proof:-

$f: A \to B$, $g: B \to C$, $fog: A \to C$

Given f and g are one-one. and we have to find that fog is one-one.

let $x_1, x_2 \in A$ such that

$$fog(x_1) = fog(x_2) \quad \because fog = g(f(x))$$

$$g(f(x_1)) = g(f(x_2))$$

($f: A \to B$ one-one ki condition ya thi k koi 2ish elements ly $x_1, x_2 \in A$ and f apply krway to wo $f(x_1) = f(x_2)$ $\Rightarrow$ then $x_1 = x_2$.)

Since g is one-one.

$$\Rightarrow f(x_1) = f(x_2)$$

Since f is one-one

$$\Rightarrow x_1 = x_2$$

$\Rightarrow$ fog is one-one

2: If fog is one-one then f is one-one.

Proof:-

let $a_1, a_2 \in A$ such that

$$f(a_1) = f(a_2)$$

Date:

$\Rightarrow g(f(a_1)) = g(f(a_2))$

$\Rightarrow fog(a_1) = fog(a_2)$ (f ki range g ki domain ky equal hai).

$\Rightarrow fog$ is one-one.

Then $a_1 = a_2$

$\Rightarrow f$ is one-one.

3: If f is onto and fog is one-one. Then g is one-one.

Proof:-

$f: A \to B$, $g: B \to C$ Then $fog: A \to C$.

Let $b_1, b_2 \in B$ such that

$g(b_1) = g(b_2)$ ——— ①

$\hookrightarrow$ elements of C

Since $b_1, b_2 \in B$ and $f: A \to B$ is onto $\exists$ $a_1, a_2 \in A$ such that

$\Rightarrow f(a_1) = b_1$

$\Rightarrow f(a_2) = b_2$ Put in (1)

$\Rightarrow g(f(a_1)) = g(f(a_2))$

$\Rightarrow fog(a_1) = fog(a_2)$

Since fog is one-one

$\Rightarrow a_1 = a_2$

By apply f on b.s

$\Rightarrow f(a_1) = f(a_2)$

$b_1 = b_2$

$\Rightarrow g$ is one-one.

Date:

$\Rightarrow$ ~~By apply f on b's~~

$\Rightarrow$ ~~$f(a_1) = f(a_2)$~~

Results :- (Onto)

1: If f and g are onto, then $f\circ g$ is onto.

proof:-

$f: A \to B$, $g: B \to C$, $f\circ g: A \to C$:- start from here Given that f, g are onto.

let $c \in C$ and we have to Show $f\circ g$ is onto. $\because f\circ g(a) = c$.

As $g: B \to C$ is onto and $c \in C$ then $\exists\ b \in B$ Such that

$g(b) = c$.

As $f: A \to B$ is onto and $b \in B$ then $\exists\ a \in A$ Such that $f(a) = b$

$\Rightarrow g(f(a)) = g(b) = c$.

$\Rightarrow f\circ g\,(a) = c$.

2: If $f\circ g$ is onto, then g is onto.

proof:-

let any $c \in C$ then $\exists\ a \in A$ Such that

Given that $f\circ g$ is onto, then

$f\circ g\,(a) = c$

 Date: ______

$g(f(a)) = c$ —— (1).

As $a \in A$ then $b \in B$ Such that (by using def of function)

$f(a) = b$

Put in (1) $\Rightarrow g(b) = c \Rightarrow g$ is onto.

3: If $f \circ g$ is onto and g is one-one then f is onto.

Proof:-

$f : A \to B$, $g : B \to C$, $f \circ g : A \to C$

let any $b \in B$, g is function Then $\exists$ $c \in C$ such that $g(b) = c$.

Since $c \in C$ and $f \circ g : A \to C$ is onto Then $\exists$ $a \in A$ Such That

$f \circ g\,(a) = c \Rightarrow g(f(a)) = c = g(b)$

$\Rightarrow g(f(a)) = g(b) \Rightarrow g$ is one-one.

$\Rightarrow f(a) = b \Rightarrow f$ is onto.

→ Inverse function :-

A Function from set A to the set B, represented by $f : A \to B$ is a relation from the set A (a set of inputs/domain) to the set B (a set of possible outputs) such that every element in A is related to exactly one element from the set B.

Date: $f: A \to B$ (bijective) $f: B \to A$ onto,

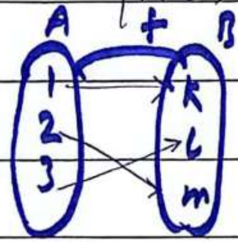
When Inverse of function is exist.

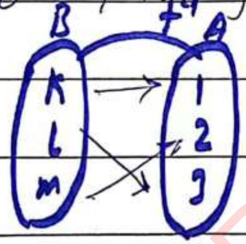
inverse function exist only when function is one-one and bijective.

Def:- let f be a one-one correspondence from the set A to B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A s.that $f(a) = b$. The inverse function f is denoted by f^{-1}. Hence $f^{-1}(b) = a$ when $f(a) = b$.

Examples:- (The domain of f^{-1} is Range of f and the range of f^{-1} is Domain of f.)

1) $A = \{1, 2, 3\}$, $B = \{k, l, m\}$





2) $f(x) = x+1$

$f(x) = x+1$

replace $f(x)$ by y

$f(x) = y$

$y = x+1$

interchange x, y

$x = y+1$

$y = x-1$

replace y by $f^{-1}(x)$

$f^{-1}(x) = x-1$

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$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

3) $f(x) = x$

replace $f(x)$ by y

$y = f(x)$

$y = x$

$x = y$ → interchange x, y

$y = x$

replace y by $f^{-1}(x)$

$f^{-1}(x) = x.$

The inverse of an identity function is the identity function itself.

4) $g(x) = 5(x+3)$

$\cancel{y = g(x)}$

$\cancel{y = 5(x+3)}$

$\cancel{y = 5x+15}$

$\cancel{y - 15 = 5x}$

$\cancel{x = \frac{y-15}{5}}$

replace $g(x)$ by y

$g(x) = y$

$y = 5(x+3)$

interchange x, y.

$x = 5(y+3)$

$x = 5y + 15$

$5y = x - 15$

$y = \frac{x-15}{5}$

replace y by $g^{-1}(x)$

$g^{-1}(x) = \frac{x-15}{5}.$

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4) $y = \frac{1}{2}x + 1$, expressed as a function of x.

- Solve for x in term of y:

$$y = \frac{1}{2}x + 1$$
$$2y = x + 2$$
$$x = 2y - 2$$

- interchange x and y:

$$y = 2x - 2.$$

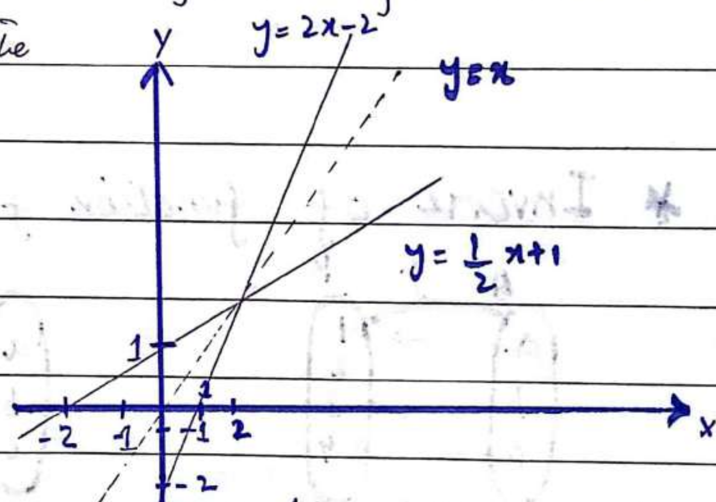
- The inverse of the function $f(x) = \frac{1}{2}x + 1$ is the function $f^{-1}(x) = 2x - 2$.

To check, we varify the both composites give the identity function.

$$f \circ f^{-1} = f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + \cancel{2} - \cancel{2} = x$$

$$f^{-1} \circ f = f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x \cancel{-1} \cancel{+1} = x.$$

$$f \circ f^{-1} = f^{-1} \circ f = x$$



- Graphing $f(x) = \frac{1}{2}x + 1$ and $f^{-1}(x) = 2x - 2$ Together shows The graph's symmetry w.r.t to line $y = x$.

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5) Suppose a one-to-one function $y = f(x)$ is given by a table of values.

x	1	2	3	4	5	6	7
$f(x)$	3	4.5	7	10.5	15	20.5	27

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in the columns (or rows) of the table for f:

y	3	4.5	7	10.5	15	20.5	27
$f^{-1}(y)$	1	2	3	4	5	6	7

If we apply f to send an input x to the output $f(x)$ and follow by applying f^{-1} to $f(x)$, we get right back to x, just where we started. Similarly, if we take some number y in the range of f, apply f^{-1} to it, and then apply f to the resulting value $f^{-1}(y)$, we get back the value y which we begun.

Composing a function and its inverse has the same effect as doing nothing.

- $(f^{-1} \circ f)(x) = x$ for all x in the domain of f.
- $(f \circ f^{-1})(y) = y$ for all y in the domain of f^{-1} (or range of f).

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★ only a one-one function can have an inverse.

The reason is that if "$f(x_1) = y$" and "$f(x_2) = y$" for two distinct inputs "x_1" and "x_2", then there is no way to assign a value to $f^{-1}(y)$ that satisfies both $f^{-1}(f(x_1)) = x_1$ and $f^{-1}(f(x_2)) = x_2$.

★ A function that is increasing on an interval satisfies the inequality $f(x_2) > f(x_1)$ when $x_2 > x_1$. so it is one-one and has an inverse. Decreasing function also have an inverse.

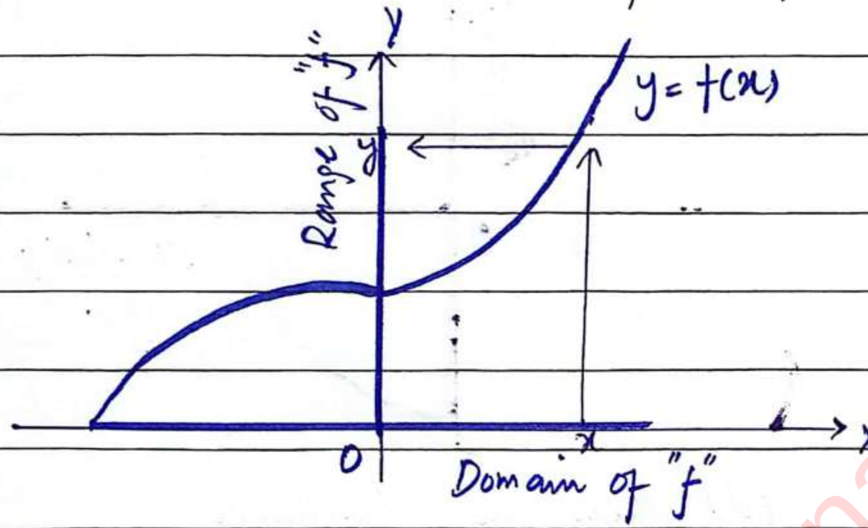
Functions that are neither increasing or decreasing may still be one-one and have an inverse, as with the function $f(x) = \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$ defined on $(-\infty, \infty)$ and passing through horizontal line test.

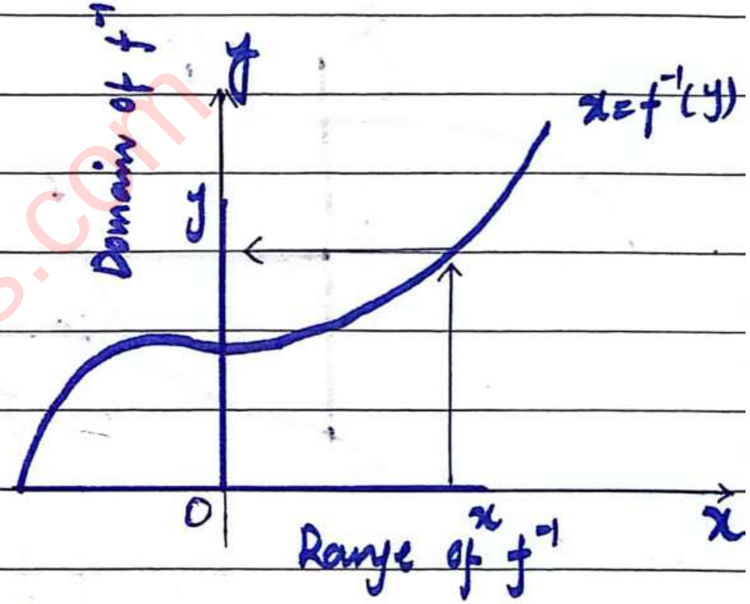
→ Graph representation of Inverse function

→ The graph of a function and its inverse are closely related. we start at a point x on the x-axis, go vertically to the graph and then moving horizontally to the y-axis to read the value of y. The inverse function can be read from the graph by

Date:

reversing this process. Start with a point y on the y-axis, go horizontally to the graph of $y = f(x)$ and then move vertically to the x-axis to read the value of $x = f^{-1}(y)$.





(a) To find the value of f at x, we start at x go up to the curve and then over to the y-axis.

(b) The graph of f^{-1} is the graph of f, but with x and y interchanged. To find x that gave y we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.

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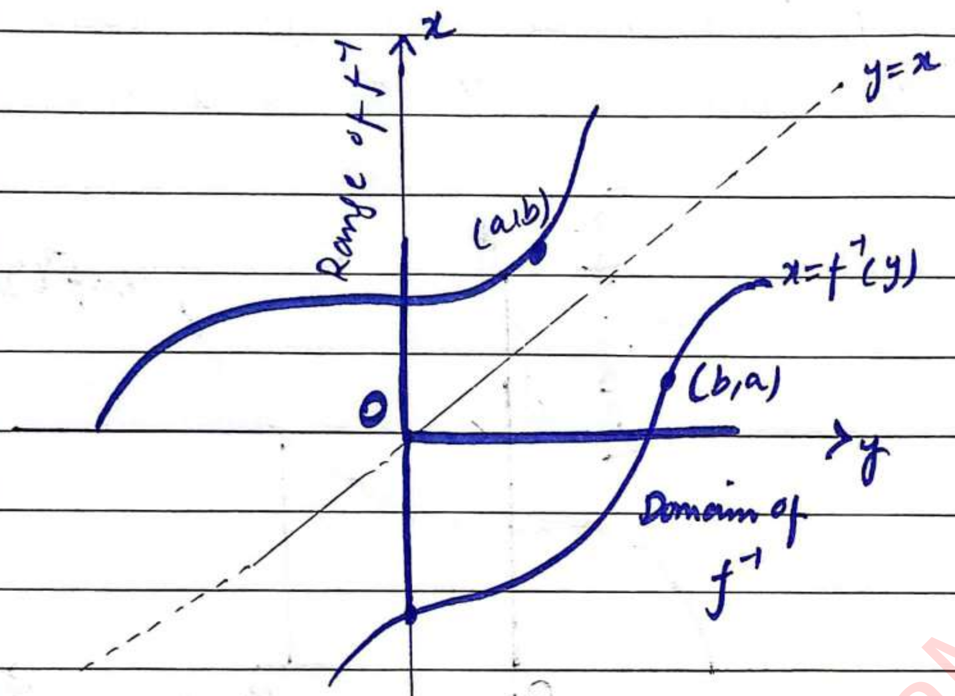
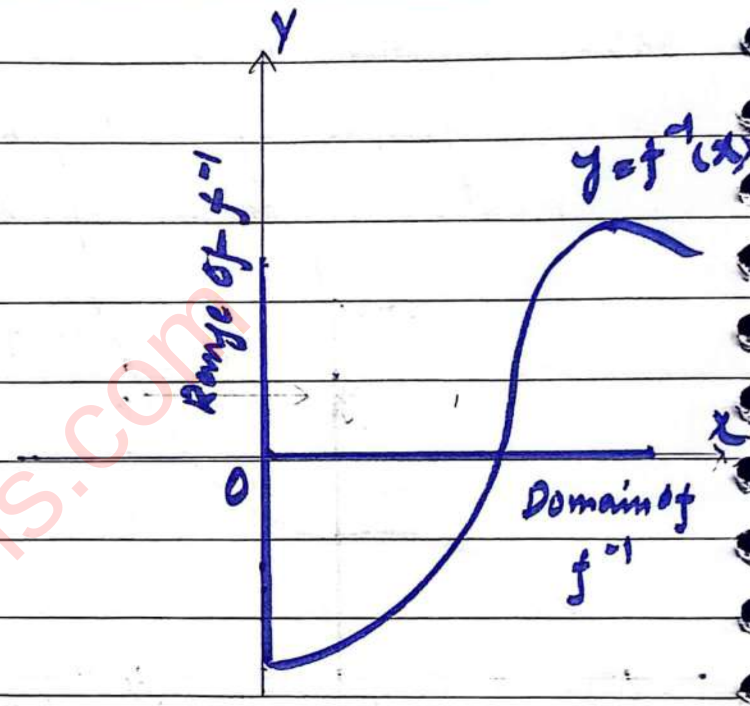




(c) To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y=x$.

(d) Then we interchange the letters x and y, we now have a normal-looking graph of f^{-1} as a function of x.

→ Inverse of element :-

Def: An element a^{-1} is called an inverse of a under the binary operation $*$ if
$$a * a^{-1} = a^{-1} * a = e.$$
where e is the identity element under the operation $*$.

Inverse of an element is one-one:-

The inverse of an element in a mathematical set (such as group) is said to be "one-one" if it has a unique inverse.

In other words, for each element "a" in the set ∃ a unique element "b" (the inverse of a) such that product of "a, b" is the identity element ensuring that every element has a well defined and unique inverse.

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

Proof:-

$f : A \to B$
$g : B \to C$ $\Rightarrow$ $f \circ g : A \to C$ bijective

$(f \circ g)^{-1} : C \to A$ exist

$f^{-1} : B \to A$
$g^{-1} : C \to B$ $\Rightarrow$ $g^{-1} \circ f^{-1} : C \to A$ bijective

Let $x \in A$, $y \in B$, $z \in C$ such that

$f(x) = y$, $g(y) = z$
$f^{-1}(y) = x$, $g^{-1}(z) = y$

Date:

$\rightarrow$ $f \circ g(x) = g(f(x)) = g(y) = z$ $\because f(x) = y$, $g(y) = z$

$(f \circ g)(x) = z$

Then inverse of $(f \circ g)$

$(f \circ g)^{-1}(z) = x$ —— (1)

$\rightarrow$ $g^{-1} \circ f^{-1}(z) = f^{-1}(g^{-1}(z)) = f^{-1}(y) = x$

$g^{-1} \circ f^{-1}(z) = x$ —— (2)

From (1) and (2)

$(f \circ g)^{-1}(z) = g^{-1} \cdot f^{-1}(z) = x.$ (Proved).

Question:-

Let N be set of +ve Z. Let $f: N \rightarrow N$ and $g: N \rightarrow N$ be defined by $f(x) = x^2$ and $g(x) = x+1$. Give formulas for $f \circ g$ and $g f$. Are these two functions equal? How do they compare with the function given by $f(x)\, g(x)$.

Solution:-

Given that $f: N \rightarrow N$ and $g: N \rightarrow N$ are defined as

$f(x) = x^2$ and $g(x) = x+1$

we have to show that gf and fg.

For gf:-

$g \circ f(x) = f(g(x)) = f(x+1) = (x+1)^2$ —— (1)

For fg :-

$f \circ g(x) = g(f(x)) = g(x^2) = x^2+1$ —— (2)

Clearly From (1) and (2)

$f \circ g(x) \neq g \circ f(x)$

$x^2+1 \neq (x+1)^2$

How do They Compare with f(x) g(x).

$f(x)\,g(x) = (x^2)(x+1)$

$= x^3 + x^2$ —— (3)

Now comparing f(x) g(x), fog and gof:

$f \circ g < g \circ f < f(x)\,g(x)$

$x^2+1 < (x+1)^2 < x^3+x^2$

Since $f: N \to N$, $g: N \to N$

let $x = 3$

$(3)^2+1 < (3+1)^2 < (3)^3+(3)^2 \Rightarrow 10 < 16 < 36$

$I: X \to X$ $I(x) = x$ same input same output.

Date:

Imp:- Lecture #06

Prove that a function $f: A \to B$ is bijective if and only if $g: B \to A$ with $f \circ g = i_A$ and $g \circ f = i_B$ → identity from B To B.

Proof:-

$f: A \to B$
$f: B \to A$

let $f: A \to B$ is bijective.

$\because$ f^{-1} exist and $f^{-1}: B \to A$ is bijective function.

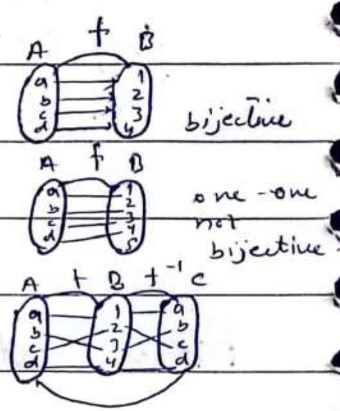
let $g = f^{-1}$

Then $f \circ g = f \circ f^{-1} : A \to A$

If any $a \in A$

$f \circ g(a) = f \circ f^{-1}(a) = f^{-1}(f(a)) = a = i_A$ ⟶ Because f is one one

$$\boxed{f \circ g(a) = i_A}$$



$f \circ f^{-1}: A \to A$

$f^{-1}: B \to A$

$f: A \to B$

$f^{-1} f: B \to B.$

and $g \circ f = f^{-1} \circ f : B \to B$

If any $b \in B$

$g \circ f(b) = f^{-1} \circ f(b) = f(f^{-1}(b)) = b = i_B$ ⟶ Because f is one-one.

$$\boxed{g \circ f(b) = i_B}$$

81- $\because$ identity mapping is bijective. Date:

Conversely :-

Given That fog is identity mapping from A To A and gof is identity mapping from B To B.

$\Rightarrow$ fog and gof are bijective (one-one, onto)

$\Rightarrow$ g is one-one

$\Rightarrow$ f is onto. > (If fog is onto Then g is one-one and f is onto,

and we have to show that f is one-one

$\rightarrow$ If fog is one-one then f is one-one

Let $a_1, a_2 \in A$

Such That

$$f(a_1) = f(a_2)$$

$$\Rightarrow g(f(a_1)) = g(f(a_2))$$

$$\Rightarrow fog(a_1) = fog(a_2)$$

$\Rightarrow$ fog is one-one

Then $a_1 = a_2$

$\Rightarrow$ f is one-one.

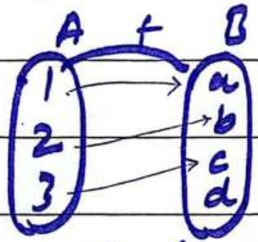
Then f is bijective

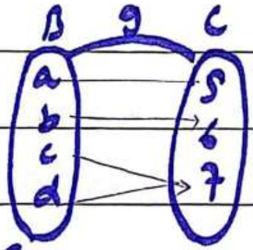
→ If fog is one-one Then g is not necessary one-one.

Given That fog is one-one Then we have To show That g is not one-one.

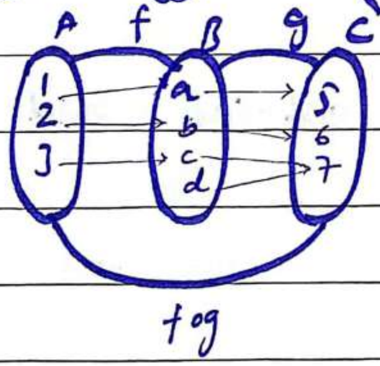
let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ and $C = \{5, 6, 7\}$

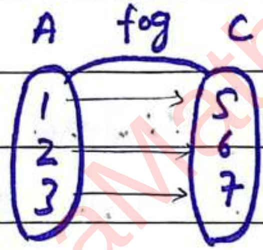
we defined $f : A \to B$, $g : B \to C$ and $fog : A \to C$.





Since f is one-one but g is not one-one.





If fog is one-one Then g is not.

→ If fog is onTo Then f is not necessary onTo.

Given That fog is onTo Then we have To show That f is not onTo.

let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ and $C = \{5, 6, 7\}$.

$f : A \to B$ and $g : B \to C$ and $fog : A \to C$.

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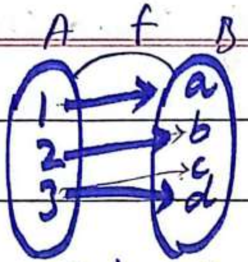
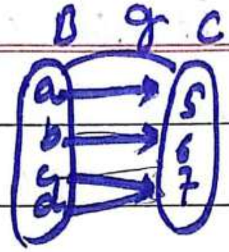
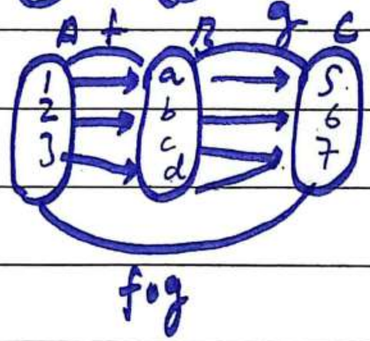
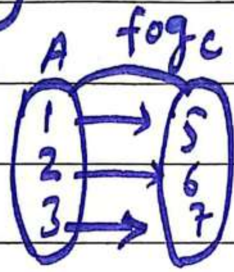




Since f is not onto but g is onto.





If fog is onto then f is not.

→ Suppose that $f: A \to B$, $g: B \to C$ are both one-one and onto. Prove that $g \circ f$ is one-one and onto.

Given that $f: A \to B$ and $g: B \to C$ are bijective, we have to show that $g \circ f$ is bijective.

$$g \circ f : A \longrightarrow C.$$

$$g \circ f(x_1) = g \circ f(x_2), \quad \forall\, x_1, x_2 \in A$$

$$\Rightarrow \quad f(g(x_1) = f(g(x_2)$$

$\because$ f is one-one, bijetive

$$g(x_1) = g(x_2)$$

Since g is bijective / one-one. $\Rightarrow$ $x_1 = x_2$

Date:

$\Rightarrow$ $g \circ f$ is one-to-one

For onto:-

Given that g is onto, then for any $b \in B \ \exists \ a \in A$ such that

$g(a) = b$ —— (1)

g is onto, for any $c \in C \ \exists \ b \in B$

$g(b) = c$ —— (2)

using eq (1)

$g(a) = b$

Taking composition f on both sides

$f(g(a)) = f(b)$ $\quad \because$ since f is onto

$g \circ f(a) = c.$

$\Rightarrow$ $g \circ f$ is onto.

$\Rightarrow$ $g \circ f$ is bijective.

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Theorem:-

$\phi \subseteq A$

Proof:-

let $B \subseteq A$ iff

every element of set B is also an element of A.

If $B = \phi$

So

$\Rightarrow \quad B = \phi \subseteq A.$

Date:

→ Propositional Function :-

A propositional function defined on cartesian product $A \times B$ of two sets A and B is an expression denoted by $p(x,y)$ which have the property that $p(a,b)$ is True or False.

Example:-

$A = R \ , \ B = R$

$p(x,y)$: x is less then y. (by putting the values of x, y)

$p(1,3)$: 1 is less then 3. (True.)

$p(2,1)$: 2 is less then 1. (False)

(Variable ki value fix krdy to wo propositional statement mai change ho jata hai).

→ Relation :-

A relation R consists of the following

(i) a set A. (ii) a set B.

(iii) An open statement $p(x,y)$ in which $p(a,b)$ is true or False.

→ Notation :-

If $p(a,b)$ is true we write $a \, R \, b$ and If $p(a,b)$ is False then we write $a \not{R} \, b$.

$R_1 = \{(a,b) : a < b\}$

$R_2 = \{(x,y) : p(x,y) : x < y\}$.

Example:- relation denoted by $R = (A, B, P(a,b))$

$A = \{1, 2\}$, $B = \{3, 4\}$

$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

$P(x,y)$: x/y

$P(2,3)$: $2/3$

$P(1,2)$: $1/2$

$1 R 2$

$R_1 = \{(x,y) : P(x,y) : x/y\}$

$R_1 = \{(1,2), (1,4), (2,4)\}$

$R_1^{-1} = \{(2,1), (4,1), (4,2)\}$

→ **Reflexive Relation:-** (dono set ky elements ka same hona zaruri hai).

Let $R = (A, A, P(x,y))$ then R is reflexive if every element in A is reflected to itself.

E.g:- $A = \{1, 2, 3\}$

$R_1 = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$

$\because$ $(1,1), (2,2), (3,3) \rightarrow$ reflexive. $1 \to 1$, $2 \to 2$, $3 \to 3$.

 Date:

$R_2 = \{(x,y) : P(x,y) : x < y\}$

$= \{(1,2), (1,3), (2,3)\}$ → Not reflexive.

$R_3 = \{(x,y) : P(x,y) : x | y\}$

$= \{(1,1), (2,2), (3,3)\}$ → reflexive.

2) A = set of all Triangles in plane.

$P(x,y)$: x is Similar to y.

$R_1 = \{(x,y) : x$ is Similar to $y\}$ → reflexive.

(reflexive hai qk hr triangle Similar zrur hoti hai).

(x bh triangle hai or y bh triangle dono set A mai hai).

3) $A = Z^+$

$P(x,y)$: $x-y$ is divisible by 5.

$R_1 = \{(0,0), (1,1), (2,2), (3,3), (4,4), ----\}$

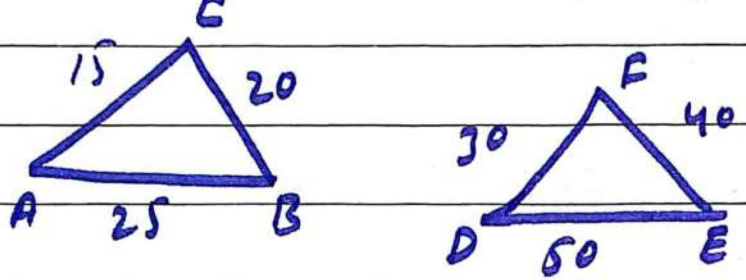
$\frac{0}{a} = 0 = z$

0 ko hr integer divide kr skta hai.

R_1 is reflexive.

→ When 2 Triangles are Similar?

Two triangles are similar if They meet one of the following criteria. Two pairs of corresponding angles are equal. Three pairs of corresponding sides are proportional. Two pairs of corresponding sides are proportional and the corresponding angles b/w them are equal.



$\frac{15}{30} = \frac{1}{2}$

$\frac{20}{40} = \frac{1}{2}$

$\frac{25}{50} = \frac{1}{2}$.

$a \leq a$.
$(R, \leq)$ reflexive
But $(R, <)$ $a < a$ is not reflexive.

→ Symmetric Relation:-

let $R = (A, A, D(x,y))$ Then R is symmetric relation if "$a R b$" then "$b R a$".

(i) if aRb
(ii) then bRa.

Example:-

(1) $A = \{1, 2, 3\}$

$R_1 = \{(1,1), (1,2), (2,1)\}$ symmetry.

$\because$ if $1R1$ Then $1R1$. , if $1R2$ Then $2R1$.

$R_2 = \{(x, y) : x/y\}$.

$\because$ if $1/2$ $\because$ (is not symmetry). $1 \not R 2$. reflexive but not symmetry.

$R_3 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$. Symmetry

2) $A = \mathbb{Z}$
$R_4 = \{(x, y) : x = y$ or $x = -y\}$

R_4 is symmetric for if $x = y$ or $x = -y$ then $y = x$ or $y = -x$.

 $A = \{1, 2, 3\}$ Date:

$R_S = \{(x, y) : x + y \leq 3\}$

R_S is symmetric if $x + y \leq 3$ Then $y + x \leq 3$.

$\because R_S = \{(1, 2) : 1 + 2 \leq 3\}$

if $1 + 2 \leq 3$ Then $2 + 1 \leq 3$. $\Rightarrow$ if $3 \leq 3$ Then $3 \leq 3$.

→ Anti-Symmetric Relation :-

Let $R = (A, A, P(x, y))$ Then R is anti symmetric relation if aRb and bRa Then $a = b$. i.e if $(x, y) \in R$ and $(y, x) \in R$ then $x = y$. $\forall x, y \in A$.

Example :-

(i) if aRb and bRa

(ii) Then $a = b$.

$A = \{1, 2, 3\}$

$R_1 = \{(1, 1), (1, 2), (2, 1)\}$

is not anti-symmetric because $1 \neq 2$. of $1R2$ and $2R1$ then $1 \neq 2$.
(bcz $(1, 2), (2, 1)$ are present).

$A = \mathbb{Z}$

$R_2 = \{(x, y) : x / y\}$

is not anti symmetric relation because if $-2/2$ and $2/-2$ Then $-2 \neq 2$.

 Date:

Examples:-

$A = \{1, 2, 3, 4\}$

$R_3 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

is anti symmetric relation

$R_4 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (2,3), (3,3), (3,4), (4,4)\}$

$R_5 = \{(3,4)\}$.

R_4 and R_5 are also anti symmetric relation.

$R_6 = \{(x,y) : x = y+1\}$ is anti symmetric

because it is possible that $x = y+1$ and $y = x+1$ then $x = y$.

$R_7 = \{(x,y) : x \leq y\}$

R_7 is anti symmetric because the inequalities $x \leq y$ and $y \leq x$

$\Rightarrow x = y$.

Transitive Relation:-

Let $R = (A, A, P(x,y))$ then R is Transitive relation if aRb and bRc then aRc.

(i) if aRb and bRc

(ii) then aRc.

i.e $(x,y) \in R$, $(y,z) \in R$ then $(x,z) \in R$

$\forall\ x, y, z \in A$.

Date:

Examples:-

$A = \{1, 2, 3\}$

$R_1 = \{(1,1), (2,2), (1,2), (2,1)\}$

is Transitive relation.

$\because$ $(\underset{x}{1}, \underset{y}{2}), (\underset{y}{2}, \underset{z}{1})$, $1R2$, $2R1$. $\Rightarrow$ $1R1$. $(1,1)$ is also in R_1.

$R_2 = \{(1,2), (2,3)\}$

is not transitive relation bcz $1R2$, $2R3$ but $1 \not R 3$ $(1,3)$ is not in R_2.

$A = \{1, 2, 3, 4\}$

$R_3 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

R_3 is Transitive relation because $(3,2)$ and $(2,1)$, $(4,2)$, $(2,1)$, $(4,3)$, $(3,1)$ and $(4,3)$ and $(3,2)$.

$\because$ $(\underset{x}{3}, \underset{y}{2}), (\underset{y}{2}, \underset{z}{1})$ $3R2$, $2R1$ $\Rightarrow$ $3R1$. $(3,1)$ is in R_3.

Note:-

IS empty set is transitive, Symmetric and anti symmetric.

It's true that the empty set (relation) is transitive and Symmetric (also antisymmetric) on every set.

Date:

→ Equivalence Relation

A relation R in "A" is called an equivalence relation if

* R is reflexive
* R is Symmetric
* R is Transitive.

→ partition :-

Let $\{B_i\}_{i \in I}$ be a family of non-anti subsets of a set A. Then $\{B_i\}_{i \in I}$ is called an partition of A if

(i) $\bigcup_{i \in I} B_i = A$ (ii) $\bigcap_{i \in I} B_i = \phi$

→ Fundamental Theorem of Equivalence :-

Let R be an equivalence Relation in a set A and "B_α" for every $\alpha \in A$, let $B_\alpha = \{x : x R a\}$ or $B_\alpha = \{x : (x, a) \in R\}$. Then the set $\{B_\alpha\}_{\alpha \in A}$ is a partition of A.

(A mai sy elements lyy ju relate krty gy then un ka aik set bnaty jay gy).

Date:

Example:-

Let $A = Z$, $\Leftrightarrow$ $x-y$ is divisible by 5. is equivalence relation.

Solution:-

$R = \{(x,y) : 5 | x-y\}$.

(i) → **Reflexive:**

Let any $a \in Z$ Then

$a-a = 0$ $\because$ $a-a$ is divisible by 5.

$\because (\{(0,0), (1,1), (2,2), (3,3) \ldots\})$ reflexive. $(2,-2),(-1,-1)$

$\Rightarrow 5 | 0 = a-a$

$\Rightarrow a R a$. (a is related to it self).

(ii) → **Symmetry:-**

$x R y$ if and only if $5 | x-y$

Suppose that $x R y$

$\Rightarrow 5 | x-y$

$\Rightarrow \frac{x-y}{5} = t$, $t \in Z$ $\because$ by def of divisiblity $a/b = k$ $\because k$ is some integer.

$\Rightarrow x-y = 5t$, $t \in Z$

$\Rightarrow -y-x = -5t$, $t \in Z$

$\Rightarrow y-x = 5(-t)$, $t \in Z$

Date:

$\Rightarrow y - x = 5t' \qquad \because (-t) = t'$

$\Rightarrow \frac{y-x}{5} = t'$

$\Rightarrow 5 \mid y - x$

$\Rightarrow y R x$ (Symmetric).

(iii) Transitive :-

If xRy and yRz Then xRz.

if $x-y/5$ and $y-z/5$ Then 5 divides both $x-y$ and $y-z$

$\frac{x-y}{5} = k \quad ; \quad \frac{y-z}{5} = l \qquad \because k, l \in Z$

$x - y = 5k \;,\; k \in Z \quad ; \quad y - z = 5l \;,\; l \in Z$

$-y = 5k - x \;,\; k \in Z \quad ; \quad y = 5l + z \;,\; l \in Z$

$y = x - 5k \;,\; k \in Z$

$\Rightarrow x - 5k = 5l + z \;,\; k, l \in Z$

$\Rightarrow x - z = 5(k + l) \qquad \because k + l = t \in Z$

$\Rightarrow \frac{x - z}{5} = t \;,\; t \in Z$

$\Rightarrow 5 \mid x - z \Rightarrow xRz$ (Transitive).

Date:

→ Partitions :-

$B_0 = \{x : 0 R x\}$ $\qquad \because 5 / x-0 = 5 | x$.

$B_0 = \{0, \pm 5, \pm 10, \pm 15, \ldots\ldots\}$ $\qquad 5 | x$

$B_1 = \{x : 1 R x\}$ $\qquad 5 | x-1$

subtract 5 $\quad$ add 5

$B_1 = \{\ldots\ldots, -9, -4, 1, 6, 11, 16, \ldots\ldots\}$

$B_2 = \{x : 2 R x\}$ $\qquad 5 | x-2$

$B_2 = \{\ldots\ldots, -8, -3, 2, 7, 12, \ldots\ldots\}$

$B_3 = \{x : 3 R x\}$ $\qquad 5 | x-3$

$B_3 = \{\ldots -7, -2, 3, 8, 13, \ldots\ldots\}$

$B_4 = \{x : 4 R x\}$ $\qquad 5 | x-4$

$B_4 = \{\ldots\ldots -6, -1, 4, 9, 14, \ldots\ldots\}$

$B_5 = \{x : 5 R x\}$ $\qquad 5 | x-5$

$B_5 = \{0, \pm 5, \pm 10, \pm 15, \ldots\}$ → Same as B_0.

(Sirf 5 partitions bny gy ek 5 partition pr values repeat ho rhi hai).

$\Rightarrow \bigcup_{i=I} B_i = A$

$\Rightarrow \bigcap_{i=I} B_i = \phi$

Date: ______

→ Example:-

Let R be a relation in the set $N \times N$. defined by $(a,b) R (c,d)$ iff $ad = bc$ show that R is an equivalence relation.

Solution :-

$A = N$

$N \times N = \{(a,b) : a,b \in N\}$

$(a,b) R (c,d) \iff ad = bc$

$R_{(a,b)} = \{(x,y) : (x,y) R (a,b) \iff xb = ya\}$.

* Reflexive :-

for any $(a,b) \in N \times N$

$(a,b) R (a,b)$ if $ab = ba = ab$

which is True since multiplication is commutative and hold in N because $a,b \in N$.

* Symmetric:-

Let $(a,b) R (c,d)$

$\Rightarrow$ $ad = bc$ (commutative law hold).

$da = cb$

Date:

$\Rightarrow$ $cb = da$

$\Rightarrow$ $(c,d) R (a,b)$ (Symmetric : if aRb then bRa)

★ Transitive:-

Let $(a,b) R (c,d)$ and then $(c,d) R (e,f)$

$\Rightarrow$ $ad = bc$ and $cf = de$

multiply these equation then

$(ad).(cf) = (bc).(de)$

$(a.c).(d.f) = (b.c).(d.e)$ (since multiplication is associative)

$(a.f).(c.d) = (b.e).(c.d)$

Since $(c.d)$ cancelled from both sides

$(a.f) = (b.e)$

$\Rightarrow$ $af = be$

$\Rightarrow$ $(a,b) R (e,f)$. (Transitive : if aRb and bRc then aRc).

Since R is equivalence relation.

→ Partitions:-

$B_{(1,1)} = \{(x,y) : (x,y) R (1,1)\}$

Date:

Since $N \times N = \{(a,b) : a,b \in N\}$

$= \left\{ \begin{matrix} (1,1), (1,2), (1,3) \ldots \\ (2,1), (2,2), (2,3) \ldots \\ \vdots \quad \vdots \quad \vdots \end{matrix} \right\}$

$B_{(1,1)} = \{(1,1), (2,2), (3,3) \ldots\}$

$= \{(a,a) : a \in N\}$

$B_{(1,2)} = \{(x,y) : (x,y) R (1,2)\}$

$= \{(1,2), (2,4), (3,6), (4,8), \ldots\}$

$= \{(a,2a) ; a \in N\}$

$B(1,3) = \{(1,3), (2,6), (3,9), \ldots\}$

$= \{(a,3a) ; a \in N\}$

$B(1,4) = \{(1,4), (2,8), (3,12), \ldots\}$

$B(1,4) = \{(a,4a) ; a \in N\}$

$B(1,5) = \{(x,y) : (x,y) R (1,5)\}$

$= \{(1,5), (2,10), (3,15), \ldots\}$

$= \{(a,5a) ; a \in N\}$

$\vdots$

and so on. . .

$\therefore$ $ad = bc$.

$(1,1) R (1,3)$

$1 \cdot 3 = 1 \cdot 1$

$3 \neq 1$

$(1,2) R (2,1)$

$1 \cdot 1 = 2 \cdot 2$

$\therefore 1 \neq 4$

$(2,4) R (1,2)$

$\therefore 4 = 4$

$(4,8) R (1,4)$

$8 = 8$

$(3,6) R (1,2)$

$6 = 6$.

$\cup B_{(x,y)} = A$

$\cap B(x,y) = \varphi$

Date:

→ Example:-

Let R be a relation in a set $A = [0,1]$ defined by

$R = \{(x,y) : x - y \text{ is a rational no.}\}$.

Show that R is equivalence Relation.

Solution :-

★ Reflexive:-

For any x in A, (x,x) must be in R.

Let x be any element in the set $A = [0,1]$.

$\Rightarrow \quad x - x = 0$,

which is rational number. (x,x) is in R. (reflexive).

★ Symmetry:-

For any x, y in A, if (x,y) is in R then (y,x) must also be in R.

Let (x,y) be in R. meaning that $x-y$ is rational number.

$\Rightarrow \quad x - y$ is a rational no.

$\Rightarrow \; -(x-y)$ is also rational no.

$\Rightarrow \; (y-x)$ is rational no.

Then (y,x) be in $R \Rightarrow yRx$. (Symmetric)

Date: ______

★ Transitive:-

For any x, y, z in A.

if (x, y) and (y, z) in R then (x, z) must be in R.

If xRy and yRz then xRz.

let (x, y) and (y, z) be in R.

$\Rightarrow$ $x-y$ and $y-z$ are rational numbers.

$\Rightarrow$ The sum of two rational numbers is also a rational number.

$(x-y)+(y-z) = x-\cancel{y}+\cancel{y}-z = x-z$

$\Rightarrow$ $x-z$ is a rational no.

$\Rightarrow$ xRz. (transitive).

Lecture #08

Date:

$B_{0.15} = \{0.15, 0.2, 0.25, \ldots\}$

$B_{0.2} = \{0.2, 0.25, 0.3, \ldots\}$

$B_{0.25} = \{0.25, 0.3, 0.35, \ldots\}$ $\qquad \cup B_{(x,y)} = A$

$B_{0.3} = \{0.3, 0.35, 0.4, \ldots\}$ $\qquad \cap B_{(x,y)} = \phi$

$B_{0.4} = \{0.4, 0.45, 0.5, \ldots\}$

$\vdots$

Inverse Relation:-

Suppose R is a relation of the form $\{(x,y) : x \in A \text{ and } y \in B\}$ such that the inverse relation of R is denoted by R^{-1} and $R^{-1} = \{(y,x) : y \in B \text{ and } x \in A\}$. If R is form A to B, then R^{-1} is from B to A.

In other words if $(x,y) \in R$ then $(y,x) \in R^{-1}$ and vice versa.

Also we know that relation in sets is a subset of the cartesian product of sets i.e R is a subset of $A \times B$, and R^{-1} is a subset of $B \times A$.

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Example:-

Find The inverse of a relation $R = \{(15, -4), (-18, 8), (-6, 2), (-12.55, 3)\}$.

Sol:-

$R = \{(-15, -4), (-18, 8), (-6, 2), (-12.55, 3)\}$

Domain $= \{-15, -18, -6, -12.55\}$

Range $= \{-4, -8, 2, 3\}$

$R^{-1} = \{(-4, -15), (-8, -18), (2, -6), (3, -12.55)\}$

Domain of $R^{-1} = \{-4, -8, 2, 3\}$

Range of $R^{-1} = \{-15, -18, -6, -12.55\}$

Example:-

Find The Inverse of a relation $R = \{(1, 2), (3, 4), (5, 7), (9, 8)\}$

Sol:-

$R = \{(1, 2), (3, 4), (5, 7), (9, 8)\}$

Domain $= \{1, 3, 5, 9\}$

Range $= \{2, 4, 7, 8\}$

$R^{-1} = \{(2, 1), (4, 3), (7, 5), (8, 9)\}$

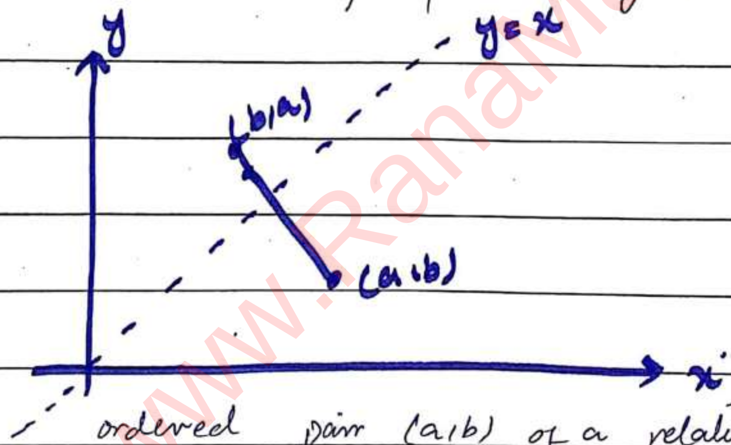
Date:

Domain of $R^{-1} = \{2, 4, 7, 8\}$

Range of $R^{-1} = \{1, 3, 5, 9\}$.

Graph representation of Inverse Relation :-

If an equation describes the relation in the variables x and y, the equation of the Inverse relation is obtained by replacing every x in the equation with y and y in the equation with x. Their graphs are mirror images over the line of reflection $y = x$.



ordered pair (a,b) of a relation such that the inverse relation of equation contains ordered pair as (b,a) about the line $y = x$.

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Vimp

Equivalence Set :-

Let A and B be Two sets Then A is equivalent To B. if $\exists$ a bijective function A To B.

$\Rightarrow$ **If $\#(A) = \#(B)$ Then A is equivalent To B.**

Finite case of A & B :-

$\Rightarrow$ If $\#(A) = m$ and $\#(B) = n$

Then A is equivalent To B.

iff $m = n$.

(2 set equivalent tbh hogy jbh 2 set mai no. of an element equal hogy) or bijective function bhi mil jay ga).

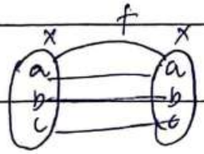
$\Rightarrow$ We can represent equivalent sets by $A \sim B$, $A \simeq B$

$\Rightarrow$ Two equal sets are always equivalent but converse may not True bcz $\mathbb{N}$ and $\mathbb{E}$ are not equal but equivalent.

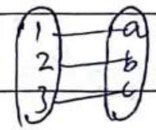
bcz $\exists$ a bijective mapping.

$\mathbb{Z} \sim \mathbb{Z}$, $N \sim N$

$\hookrightarrow$ (identity mapping ki wja sy)



identity Function is bijective.



(equal set nhi hai lykn equivalent set hai)

Date:

Theorem :-

The relation equivalent b/w sets is an equivalent relation.

proof :-

→ Reflexive :-

let any set "A", Then we want to show That $A \sim A$.

let $I : A \to A$ defined by $I(x) = x$ → same Input same out put. $f : A \to A$, A is element To A.

Since I is identity function The I is bijective.

(If we Take from A To the inverse image of A itself.)

Then $A \sim A$.

→ Symmetric :-

let $A \sim B$ Then $\exists$ a bijective function from A To B.

Such That

$f : A \to B$ → bijective.

and we have To show That $B \sim A$ is bijective function.

Then f^{-1} is exist and f^{-1} is also bijective function from B To A.

Moreover

 Date:

$g: B \to A$

$g = f^{-1}: B \to A$ will also be bijective.

Therefore,

$\Rightarrow$ B is equivalent to A.

$\Rightarrow$ $B \sim A$.

Transitive:-

Let $A \sim B$ and $B \sim C$ then $A \sim C$. and we have to show that $A \sim C$. is the defined function from A to C. That is bijective then $\exists$ a bijective function b/w A and B and B and C.

let $f: A \to B$ and $g: B \to C$. are bijective function.

we want to show that $A \sim C$. we defined a function b/w A and C.

Such that $f \circ g: A \to C$. is bijective.

One-one: consider $x_1, x_2 \in A$.

$$f \circ g(x_1) = f \circ g(x_2)$$

$$g(f(x_1)) = g(f(x_2))$$

Since g is one-one (bijective)

$$f(x_1) = f(x_2).$$

Since f is one-one (bijective)

$\Rightarrow x_1 = x_2$

$\Rightarrow fog$ is one-one.

Onto:-

$f: A \to B$, $g: B \to C$, $fog: A \to C$.

let $c \in C$ and we have to show that fog is onto. $\because fog(a) = c$

As $g: B \to C$ is onto and $c \in C$ then $\exists$ $b \in B$ such that:

$g(b) = c$

As $f: A \to B$ is onto and $b \in B$ then $\exists$ $a \in A$ such that $f(a) = b$

$\Rightarrow g(f(a)) = g(b) = c$

$\Rightarrow fog(a) = c$

$\Rightarrow fog$ is onto.

So fog is bijective function

Hence $A \sim C$.

Lecture #09

Date:

Example 01:- (Part 1)

Show that the set of positive rational number is equivalent to $N \times N$.

Solution:-

Let $f : Q^+ \longrightarrow N \times N$ defined by

$$f(p/q) = (p, q) \qquad p/q \in Q^+,\ p, q \in Z^+$$

→ equivalent set proof krny ky liy function define krway gy or un ky b/w bijective function ~~define~~ hoga.

→ f is one-one:-

Let $\frac{p_1}{q_1}, \frac{p_2}{q_2} \in Q^+$

Images: and $f(p_1/q_1) = f(p_2/q_2)$

$\Rightarrow (p_1, q_1) = (p_2, q_2)$

and corresponding elements are equal.

$\Rightarrow p_1 = p_2, \quad q_1 = q_2$

$\Rightarrow p_1/q_1 = p_2/q_2$

$\Rightarrow f$ is one-one.

→ $\overleftarrow{Q^-} \quad 0 \quad \overrightarrow{Q^+}$

$Q^+ = \{x = p/q_1 : p, q \in Z^+\}$

$N \times N = \{(1,1), (1,2), (1,3) \ldots$
$(2,1), (2,2), (2,3) \ldots$
$\vdots \quad \vdots \quad \vdots \ldots$

→ 2 ordered pair us ~~time equal hogy jab~~ un ky no of an element (corresponding) equal hogy.

→ $f : Q \to N \times N$
$1/2 \quad (1,2)$
$3/5 \quad (3,5)$
hm 2 elements $N \times N$ sy ly raw ki Inverse Image Q mai hogi.

→ f is onto:-

Since for each $(p, q) \in N \times N$ $\exists$ $p/q \in Q^+$ such that

$$f(p/q) = (p, q)$$

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$\Rightarrow$ f is onto.

$\Rightarrow$ f is bijective.

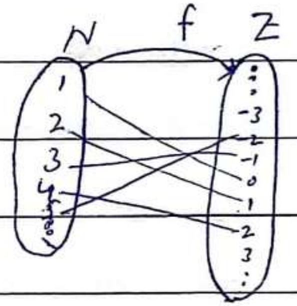
$\Rightarrow$ $Q^+ \sim N \times N$.

Imp for subject.

Example #02:-

Show that the set of N is equivalent to Z.

Solution:-



let $f: N \to Z$ defined by

(piecewise function is defined)

Images

$$f(x) = \begin{cases} n/2 & , \text{if } n \text{ is even} \\ \frac{-n+1}{2} & , \text{if } n \text{ is odd.} \end{cases}$$

$\frac{-3+1}{2}$

$-\frac{2}{2} = -1$

$4/2 = 2$

$\frac{-5+1}{2} = -4/2 = -2$

co-domain ky correspond pre-Image hai or unique.

→ f is one-one :-

let $n_1, n_2 \in N$ and

$f(n_1) = f(n_2)$

• Case 01 :-

If n_1 and n_2 are even.

$\frac{n_1}{2} = \frac{n_2}{2}$

$\Rightarrow n_1 = n_2$

Date:

- **Case 02:-**

If n_1 and n_2 are odd

$$\frac{-n_1+1}{2} = \frac{-n_2+1}{2}$$

$$-n_1+1 = -n_2+1$$

$$-n_1+n_2 = 1-1$$

$$-n_1+n_2 = 0$$

$$n_2 = n_1$$

$\Rightarrow n_1 = n_2$

$\Rightarrow f$ is one-one.

- **Case 03:-**

If n_1 is even and n_2 is odd, So it is not posisible for given condition..

(Third case bn skta hai js mai hm b/w even or both odd ka case lg lgky yah piecewise function mai Images same (equal) nhi hai To yah hm Third case nhi lg gy.)

→ Images equal nhi hai To pre Images bhi nhi hogi.)

→ f is onto:-

Since every $n/2$ or $-n+1/2 \in \mathbb{Z}$ is Image of some $n \in \mathbb{N}$ Under f

∵ Since f is onto

$\Rightarrow f$ is bijective.

$\Rightarrow \mathbb{N} \sim \mathbb{Z}$.

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→ Characteristic function :-

let A be any Subset of Universal set U Then real valued function

$$\chi_A : U \rightarrow \{0,1\}$$

(Kaii) In other words

A characteristics function is a function defining membership in a set That assigns the value 1 to the member of a given set and the value 0 to its non-members.

* let U be the Universal Set and let $A \subseteq U$.
* The characteristics function of A, denoted by χ_A is defined for each $x \in U$ by.

(A ka hr element U mai hota hai lykn ya zruri nhi k U ka hr element A mai bhi ho).

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

The domain of χ_A is U and range is $(0,1)$.

→ ground set ← $X = \{1,2\}$ contain 4 Subsets.

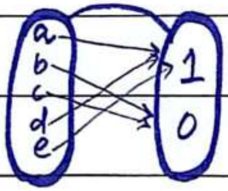
$= \underset{A_1}{\phi}, \underset{A_2}{\{1\}}, \underset{A_3}{\{2\}}, \underset{A_4}{\{1,2\}}. \Rightarrow \chi_{A_1}, \chi_{A_2}, \chi_{A_3}, \chi_{A_4}$

(* Subset ki tadad or χ_A set ki tadad equal hogi (* both subset and characte function are equal in number)

Date:

Examples:-

→ let $U = \{a, b, c, d, e\}$ and $A = \{a, d, e\}$ Then characteristics function of A defined as.



(onto Function) $\chi_A(a) = 1$, $\chi_A(b) = 0$, $\chi_A(c) = 0$, $\chi_A(d) = 1$

$\chi_A(e) = 1$.

→ If The Universal set is finite. let $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2\}$, $B = \{2, 4, 6\}$ and $C = \{1, 2, 3, 4, 5, 6\}$.

- $\chi_A(1) = 1$, $\chi_A(2) = 1$, $\chi_A(3) = 0$, $\chi_A(4) = 0$, $\chi_A(5) = 0$, $\chi_A(6) = 0$.
- $\chi_B(1) = 0$, $\chi_B(2) = 1$, $\chi_B(3) = 0$, $\chi_B(4) = 1$, $\chi_B(5) = 0$, $\chi_B(6) = 1$.
- $\chi_C(1) = 1$, $\chi_C(2) = 1$, $\chi_C(3) = 1$, $\chi_C(4) = 1$, $\chi_C(5) = 1$, $\chi_C(6) = 1$.

→ **Theorem:-**

let X be any set and let $C(X)$ be The family / class of all characteristics function of X Then

$$P(X) \sim C(X) \quad \text{or we can write as} \quad 2^X \sim C(X).$$

proof:-

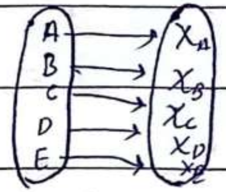
set ki form — function k form

let $f : P(X) \to C(X)$

($P(X)$ mai sy subset ly gy wo χ_A mai mapp kry gy).

defined by

$f(A) = \chi_A$



→ F is one-one:-

Let $A_1, A_2 \in P(X)$ and

$f(A_1) = f(A_2)$

$\chi_{A_1} = \chi_{A_2}$ — (1) → by def. of Equal Function (Same output same Input).

(P(X) mai sy 2 eleme lyg or check kry g. k 2 set ky correspo C(X) mai mapp kry

Let $x \in A_1$

$\Rightarrow \chi_{A_1}(x) = 1$ By def, 1 if $x \in A$.

from (1) $\Rightarrow \chi_{A_2}(x) = 1$.

$\Rightarrow x \in A_2$

$\Rightarrow A_1 \subseteq A_2$ —— (2)

Conversely:-

Let $y \in A_2$

$\Rightarrow \chi_{A_2}(x) = 1$. → by def

From (1)

$\chi_{A_1}(x) = 1$

$\Rightarrow y \in A_1 \Rightarrow A_2 \subseteq A_1$ — (3)

From (2) and (3)

$A_1 = A_2$.

$\Rightarrow f$ is one-one.

→ f is onto :-

Since for every $\chi_A \in C(X)$ $\exists$ $A \in P(X)$ such that

$f(A) = \chi_A$

$\Rightarrow f$ is onto

$\Rightarrow f$ is bijective

$\Rightarrow C(X) \sim P(X)$

OR $\Rightarrow P(X) \sim C(X)$.

Lecture # 10

Date:

Example :-

$N \sim O^+$

(prove That natural number is equivalent to positive odd number).

Solution :-

let $f : N \to O^+$ defined by

$f(n) = 2n - 1$. , $n \in N$.

→ f is one-one :-

To prove f is one-one.

let $n_1, n_2 \in N$ and

$f(n_1) = f(n_2)$

$\Rightarrow$ $2n_1 - 1 = 2n_2 - 1$

$\Rightarrow$ $2n_1 = 2n_2 - 1 + 1$

$\Rightarrow$ $2n_1 = 2n_2$ (This shows That for every n_1 and n_2

$\Rightarrow$ Divided by 2 on b.s in N if $f(n_1) = f(n_2)$ Then $n_1 = n_2$).

$\Rightarrow$ $n_1 = n_2$

$\Rightarrow$ f is one-one.

→ **f is onto :-**

To prove f is onto. we need to show that for each $o \in O^+$ there exists an $n \in N$ such that

$f(n) = o$

∵ if $o = 2n-1$
$o+1 = 2n \Rightarrow n = \frac{o+1}{2}$
$o \in O^+$ natural
(for each positive odd number no. we can find natural no).

$\Rightarrow$ f is onto.

$\Rightarrow$ f is bijective.

$\Rightarrow$ $N \sim O^+$.

→ **Example :-**

$N \sim E^+$

(prove that natural number is equivalent to positive even number).

Solution :-

let $f: N \to E^+$ defined as

$f(n) = 2n$. for $n \in N$

→ **f is one-one :-**

To prove f is one-one.

let $n_1, n_2 \in N$ and

$f(n_1) = f(n_2)$

$\Rightarrow 2n_1 = 2n_2$

$\Rightarrow$ divided by 2 on b.s.

$\Rightarrow n_1 = n_2$

$\Rightarrow f$ is one-one.

→ f is onto :-

To prove f is onto.

Since for each $e \in E^+$ There exists $n \in N$ such That

$f(n) = e$.

$\because e = 2n$

$n = \frac{e}{2}$

$\because f(n) = 2n$

$\because = 2\left(\frac{e}{2}\right)$

$\because f(n) = e$.

(For any positive even number we can find a natural no).

$\Rightarrow f$ is onto.

$\Rightarrow f$ is bijective

$\Rightarrow N \sim E^+$

Imp (Any closed Interval is equivalent to other closed Interval). Date: ______

$[a,b] \sim [c,d]$, $a,b,c,d \in R$

Proof :-

Since $a<b$ and $c<d$

let $f: [a,b] \to [c,d]$

defined by

$$f(x) = \frac{d-c}{b-a}(x-a)+c$$

→ **well define :-**

let $x \in [a,b]$.

$\Rightarrow a \le x \le b$

By adding $-a$ on b.s.

$\Rightarrow a-a \le x-a \le b-a$

$\Rightarrow 0 \le x-a \le b-a$

If we multiply any positive no in equality then inequality does not change.

as $d-c$ is +ve and also $b-a$ is +ve then their division is also +ve.

$\because d-c \ge 0$ $b-a \ge 0$ $\therefore \frac{d-c}{\;} \ge 0$.

- $[1,2]$, $[17,99]$ no of points equal hai in mai ~~kyke~~ or in dono mai infinite points hai.
- $[1,2]$ is mai sy 2 points ly gy to un ky b/w infinite points hogy.
- $Z = \{0, \pm1, \pm2, ---\}$ in mai sy 2 points ly gy to in mai koi infinite points nhi hogy.
- Domain mai sy aik element lygy f apply krygy to element co-domain mai sy hoga.
- $f: A \to B$ $a \in A$ and $f(a) \in B$ if $f(a) \notin B$ then f is not

Date:

$$\Rightarrow \quad 0 \le \frac{d-c}{b-a}(x-a) \le \cancel{(b-a)}\frac{d-c}{\cancel{b-a}}$$

By adding c

$$\Rightarrow \quad c \le \frac{d-c}{b-a}(x-a)+c \le d-\cancel{c}+\cancel{c}.$$

$$\Rightarrow \quad c \le f(x) \le d.$$

$\Rightarrow$ f is well-defined.

→ f is one-one:-

let $x_1, x_2 \in [a,b]$ and

$f(x_1) = f(x_2)$

$$\frac{d-c}{b-a}(x_1-a)+c = \frac{d-c}{b-a}(x_2-a)+c.$$

By multiply $\frac{b-a}{d-c}$ on b.s.

$$(x_1-a)+c = (x_2+a)+c.$$

$$x_1 - x_2 = \cancel{a}-\cancel{a}+\cancel{c}-\cancel{c}$$

$$x_1 - x_2 = 0$$

$$\Rightarrow \quad x_1 = x_2$$

Date:

$\Rightarrow$ f is one-one.

$\rightarrow$ f is onTo :-

we have to find Inverse of f. $\because f(x) = y$

as $y = \frac{d-c}{b-a}(x-a)+c$. (range ka aik element ly or uska inverse agr domain mai exist kr jay to wo function onto hoga).

$y-c = \frac{d-c}{b-a}(x-a)$

$\frac{b-a}{d-c}(y-c) = x-a$

$\frac{b-a}{d-c}(y-c)+a = x$ $\rightarrow$ Inverse Image.

let $y \in [c,d]$.

$\Rightarrow c \le y \le d$.

$\Rightarrow c-c \le y-c \le d-c$ $\because$ by adding $-c$.

$\Rightarrow 0 \le y-c \le d-c$

$\Rightarrow 0 \le \frac{y-c}{d-c} \le 1$ $\because$ by dividing $d-c$

$\Rightarrow 0 \le \frac{y-c}{d-c}(b-a) \le (b-a)$ $\because$ by multiply $b-a$

Date:

$\Rightarrow 0 \le \frac{(b-a)}{d-c}(y-c) \le b-a$ $\quad \because f(x) = \frac{d-c}{b-a}(x-a)+c$

$= \frac{d-c}{b-a}\left(\frac{b-a}{d-c}\right)(y-\cancel{c}+\cancel{a})+c$

$\Rightarrow a \le \frac{b-a}{d-c}(y-c) \le b$ $\quad \because$ by adding a. $\quad \because f(x) = y$

$\Rightarrow a \le x \le b$

$\Rightarrow f$ is onto.

$\Rightarrow f$ is bijective.

$\Rightarrow [a,b] \sim [c,d]$.

(Any open Interval is equivalent to other open Interval).

V.Imp $\quad (0,1) \sim (a,b) \quad a,b \in \mathbb{R} \quad a<b$

proof:-

let $f : (0,1) \to (a,b)$

defined by

$f(x) = (b-a)x + a. \quad \text{——— } ①$

→ **Well-defined :-**

let $x \in (0,1)$

$\Rightarrow 0 < x < 1$

Date:

$\Rightarrow f(0) < f(x) < f(1)$ $\quad \because f(0) = (b-a)0 + a$

$f(0) = a.$

$\Rightarrow a < f(x) < b.$ $\quad \because f(1) = (b-a)(1) + a$

$= b - a + a$

$\Rightarrow f$ is well-defined. $\quad f(1) = b$

$\because f(x) = (b-a)x + a$

→ f is one-one :-

let $x_1, x_2 \in (0,1)$

and $f(x_1) = f(x_2)$

$\Rightarrow (b-a)x_1 + a = (b-a)x_2 + a.$ $\quad \because$ by subtracting a.

$\Rightarrow (b-a)x_1 = (b-a)x_2$ $\quad \because$ by dividing $(b-a)$.

$\Rightarrow x_1 = x_2$

$\Rightarrow f$ is one-one.

→ f is onto :-

First we have to find inverse of y.

let $y = (b-a)x + a$ $\quad \because f(x) = y$

$\Rightarrow y - a = (b-a)x$

$\Rightarrow \dfrac{y-a}{b-a} = x$ $\quad \rightarrow$ Inverse of y.

124- let $y \in (a,b)$ Date:

let $a < y < b$

$\Rightarrow \quad a-a < y-a < b-a$ $\quad \because$ by adding $-a$

$\Rightarrow \quad 0 < y-a < b-a$

$\Rightarrow \quad 0 < \dfrac{y-a}{b-a} < \dfrac{b-a}{b-a}$ $\quad \because$ by dividing $b-a$.

$\Rightarrow \quad 0 < \dfrac{y-a}{b-a} < 1$ $\quad \because f(x) = (b-a)x + a$

$= \cancel{(b-a)} \dfrac{(y-a)}{\cancel{(b-a)}} + a$

$= y - \cancel{a} + \cancel{a}$

$\Rightarrow \quad 0 < x < 1.$ $\quad f(x) = y.$

$\Rightarrow$ f is onto.

$\Rightarrow$ f is bijective.

$\Rightarrow \quad (0,1) \sim (a,b).$

Infinite Set :-

A set is said to be an infinite if it is equivalent to a proper subset of itself.

$\because \quad N \sim Z$

Proper Subset

Date:

→ $\mathbb{R}$ is infinite set

$(0,1) \sim \mathbb{R}$

proof:-

let $f : (0,1) \to \mathbb{R}$ is defined as

$$f(x) = \ln\left(\frac{x}{1-x}\right)$$

→ f is one-one :-

let $x_1, x_2 \in (0,1)$ and $f(x_1) = f(x_2)$

$\Rightarrow \ln\left(\frac{x_1}{1-x_1}\right) = \ln\left(\frac{x_2}{1-x_2}\right)$

$\because$ taking exponential to B.S.

$\Rightarrow e^{\ln\left(\frac{x_1}{1-x_1}\right)} = e^{\ln\left(\frac{x_2}{1-x_2}\right)}$ $\qquad \because e^{\ln x} = x.$

$\Rightarrow \frac{x_1}{1-x_1} = \frac{x_2}{1-x_2}$

$\Rightarrow x_1(1-x_2) = x_2(1-x_1)$ $\qquad \because$ cross multiplication

$\Rightarrow x_1 - x_1x_2 = x_2 - x_1x_2$

$\Rightarrow x_1 - x_2 = x_1x_2 - x_1x_2$

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$

Date:

$\Rightarrow$ f is one-one.

→ f is onto :-

for each $y \in R$ Then There exist $x \in (0,1)$ Such That

$f(x) = y$

Now we will check That inverse of y is exist.

$\Rightarrow y = \ln\left(\frac{x}{1-x}\right)$

$\Rightarrow e^y = e^{\ln\left(\frac{x}{1-x}\right)}$

$\Rightarrow e^y = \left(\frac{x}{1-x}\right) \Rightarrow e^y(1-x) = x \Rightarrow e^y - xe^y = x$ $\quad \because e^y$ is +ve

$\Rightarrow e^y = x + xe^y \Rightarrow e^y = x(1+e^y) \Rightarrow \frac{e^y}{1+e^y} = x$ → Inverse of y

$\Rightarrow$ f is onto

$\Rightarrow$ f is bijective

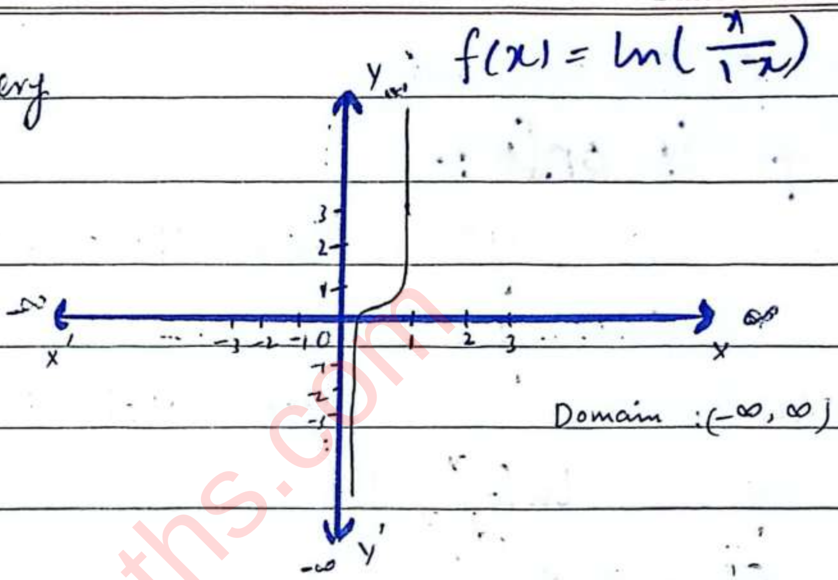
$\Rightarrow (0,1) \sim R$.

→ Graph representation :-

From This graph we can see That This function is bijective because it passes the horizontal line test and also vertical line Test That is also function and It is one-one and onto.

Date:

(function is onto because we will get every point of y-axis line). It generates the real line by putting the values b/w (0,1) gives the line of y-axis.



→ $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \sim R$

Proof:-

let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ is defined as

$f(x) = \tan x$ for x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

→ **f is one-one:-**

let $x_1, x_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $f(x_1) = f(x_2)$

$\Rightarrow \tan(x_1) = \tan(x_2)$

$\Rightarrow \tan^{-1}\tan(x_1) = \tan^{-1}\tan(x_2)$ ∵ by multiply. $\tan^{-1}$ on b.s.

$\Rightarrow x_1 = x_2$

$\Rightarrow f$ is one-one.

Date:

→ f is onto :-

for each $y \in R$ Then There exist $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ Such That

$$f(x) = y$$

as $y = \tan(x)$

$\Rightarrow$ $x = \tan^{-1}(y)$

$\because$ Since the range of the $\tan^{-1}$ function is all real numbers x can be in $(-\frac{\pi}{2}, \frac{\pi}{2})$ for any y in R.

$\Rightarrow$ f is onto.

$\Rightarrow$ f is bijective

$\Rightarrow$ $(-\frac{\pi}{2}, \frac{\pi}{2}) \sim R$.

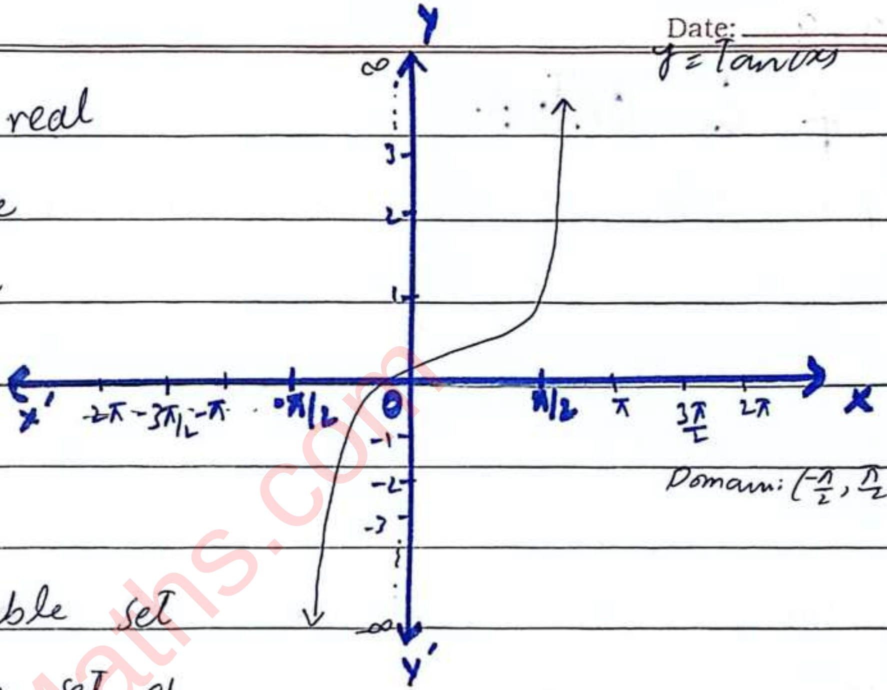
→ Graph representation :-

From This graph we can see That $f(x) = \tan(x)$ is bijective because It passes the horizontal line Test and also vertical line Test. It is one-one and onto.

(function is onto because we will get every point of

Date:

y-axis line). It generates the real line (x-axis line) by putting the values between $(-\frac{\pi}{2}, \frac{\pi}{2})$ gives the line of y-axis.



Domain: $(-\frac{\pi}{2}, \frac{\pi}{2})$.

→ Denumerable Sets :-

A set A is called Denumerable set If it is equivalent to the set of natural numbers.

- $N \sim N$ (reflexive)
  N is denumerable.
- $N \sim Z$, Z is denumerable.
- O, E are denumerable.

→ Show that an infinite sequence of distinct elements is denumerable.

(Ksi set ko sequence k form mai likh ly wo denumerable hai).

proof:-

Date:

let $(a_n)_{n=1}^{\infty}$ be an infinite sequence of distinct elements.

let $A = \{a_1, a_2, \ldots\ldots\}$ $\rightarrow$ distinct elements.

be the set of points of sequence and also $a_i \neq a_j$ if $i \neq j$; $i, j \in N$ (Input).

let $f : N \rightarrow A$ defined by

$$f(n) = a_n \quad , \quad n \in N$$

$\rightarrow$ **f is one-one :-**

$n = 1, 2, \ldots$; $A = \{a_1, a_2, \ldots\}$.

$1 \rightarrow a_1$, $2 \rightarrow a_2$

$3 \rightarrow a_3$ ------

let $n_1, n_2 \in N$ and

$$f(n_1) = f(n_2)$$

$$\Rightarrow \quad a_{n_1} = a_{n_2}$$

$$\Rightarrow \quad n_1 = n_2$$ (Index ka equal hona zruri hai).

$\rightarrow$ **f is onto :-**

Since for each $a_n \in A$ $\exists$ an element in $n \in N$ such that

$$f(n) = a_n$$

Note:- (An other def of Denumerable).

A set is denumerable if its elements can be written in the form of sequence.

Date:

Eg:- $Z = \{ \ldots -3, -2, -1, 0, 1, 2, 3, \ldots \}$, $A = \{ 0, -1, 1, 2, 2, \ldots \}$.

A is Denumerable , B is denumerable. $A \cup B$ is denumerable , $A \times B$ is denumerable.

→ Theorem :-

If A and B are denumerable sets then $A \times B$ is denumerable.

Proof:-

Given that A and B are denumerable sets Now

let $A = \{ a_1, a_2, a_3, \ldots \}$ and $B = \{ b_1, b_2, b_3, \ldots \}$

Then

$$A \times B = \left\{ \begin{array}{l} (a_1, b_1), (a_1, b_2), \ldots \\ (a_2, b_1), (a_2, b_2), \ldots \\ (a_3, b_1), (a_3, b_2), \ldots \end{array} \right\}$$

As

$A \times B = (a_1, b_1), (a_2, b_1), (a_1, b_2), \ldots$

is sequence of distinct element.

Thus $A \times B$ is denumerable.

Date:

Theorem:-

Every infinite set has denumerable subset.

Proof:-

(Infinite set hai To zruri nhi ai set ka phla element ho.)

Let A be an infinite set.

Pick any element $a_1 \in A$

Then $A - \{a_1\}$ is still infinite

Pick any element $a_2 \in A - \{a_1\}$

Then $a_2 \neq a_1$ and $A - \{a_1, a_2\}$ is still infinite.

Pick any $a_3 \in A - \{a_1, a_2\}$

Then $a_3 \neq a_2 \neq a_1$ and $A - \{a_1, a_2, a_3\}$ is infinite.

By continuing, we form a set $B = \{a_1, a_2, a_3, \text{----}\} = \{a_n\}_{n \in N}$

Such that $a_i \neq a_j$, if $i \neq j$.

Thus $B = \{a_n\}_{n \in N}$ is sequence of distinct element.

$\therefore$ B is denumerable.

Date:

→ Theorem:-

A subset of a denumerable set is either finite or denumerable.

proof:-

let A be a denumerable and $B \subseteq A$.

If $B = \phi$, then B is finite.

If $B \neq \phi$, let a_{n_1} be first element of A which belongs to B.

let a_{n_1} be 1^{st} element which belongs to B.

let a_{n_2} be 2^{nd} element followed by a_{n_1} which belongs to B.

If a_{n_j} is last element in B then

$$B = \{a_{n_1}, a_{n_2}, \ldots\ldots a_{n_j}\}$$

$\therefore$ B is finite.

If not, then $a_{n_1}, a_{n_2}, \ldots a_{n_j}, \text{——}$ is infinite sequence of distinct elements belong to B.

Thus B is denumerable.

→ Note:-

- $N \times N$ is denumerable.
- $N \times Z$ is denumerable.

$N = \{1, 2, 3, \ldots\}$ and $A = \{1, 10, 11, 101, 105, \ldots\}$

$A \subseteq N$ (If a denumerable hai 1ykn bijective function define krwana mushkil hai)

- $N \times N \times N$ is denumerable.
- $\underbrace{N \times N \times \ldots \times N}_{n\text{-times}}$ is denumerable.

→ Countable set :-

A set is said to be countable if it is either finite or denumerable.

→ Remark :-

A subset of a countable set is countable.

proof :-

Let A be a countable set and $B \subseteq A$. To prove B is countable. Then by def of countable set, If A is countable then A is finite or denumerable

Case I :-

If A is finite then B is also.

$\Rightarrow$ B. is countable.

Case II :-

If A is not finite then A is denumerable

Date:

$\Rightarrow$ B is finite or denumerable because subset of denumerable set is either finite or infinite (Denumerable) then B is countable.

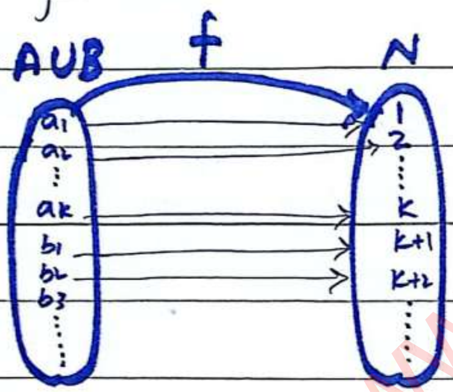
Theorem:-

If A is finite and B is denumerable then $A \cup B$ is denumerable.

Proof:-

let $A = \{a_1, a_2, \ldots\ldots a_k\}$ and $B = \{b_1, b_2, \ldots\ldots\}$

let $f : A \cup B \longrightarrow N$ defined by



Thus from above f is bijective.

$\Rightarrow$ $A \cup B \sim N$

$\Rightarrow$ $A \cup B$ is denumerable.

 Date:

→ Theorem :-

A_i's are infinite or denumerable. let $\{A_i\}_{i=1}^{\infty}$ be a Sequence of sets then $\exists$ $\{B_i\}_{i=1}^{\infty}$ a Sequence of sets Such that.

(i) $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$

(ii) $B_i \cap B_j = \phi$ if $i \neq j$.

Proof:- Given that

$\{A_i\}_{i=1}^{\infty} = \{A_1, A_2, A_3, \ldots\ldots\}$ be a Sequence of set.

let $B_1 = A_1$, $B_2 = A_2 - A_1$

$B_3 = A_3 - \{A_1 \cup A_2\}$

$\vdots$

$B_i = A_i - \{A_1 \cup A_2 \cup \cdots \cdots A_{i-1}\}$

$\vdots$

(i) $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$

let $x \in \bigcup_{i=1}^{\infty} B_i$

$\Rightarrow x \in B_i$, for Some $i \in N$

 Date:

$\Rightarrow x \in A_i - \{A_1 \cup A_2 \cup \cdots\cdots \cup A_{i-1}\}$, for some $i \in N$ ∵ By def of difference of set $A - B = \{x \mid x \in A$ and $x \notin B\}$ ∵ $x \in A_i$ But $x \notin \{A_1 \cup A_2 \cdots \cup A_{i-1}$

$\Rightarrow x \in A_i$, for some $i \in N$

$\Rightarrow x \in \bigcup_{i=1}^{\infty} A_i$

$\Rightarrow \bigcup_{i=1}^{\infty} B_i \subseteq \bigcup_{i=1}^{\infty} A_i$ ——— (i)

NOW:-

let $y \in \bigcup_{i=1}^{\infty} A_i$

non-empty

$\Rightarrow y \in A_i$, for some $i \in N$

if $y \in A_j$, $j < i$, $i, j \in N$ $\Rightarrow y \notin A_i - \{A_1 \cup A_2 \cup \cdots \cup A_{i-1}\}$

Then y vanished.

Then $\bigcup_{i=1}^{\infty} A_i$ is empty.

Which is contradiction to our supposition.

$\Rightarrow y \notin A_j$, $j < i$, $i, j \in N$

$\Rightarrow y \in A_i - \{A_1 \cup A_2 \cup \cdots\cdots \cup A_{i-1}\}$, for some $i \in N$.

$\Rightarrow y \in B_i$, for some $i \in N$

$\Rightarrow y \in \bigcup_{i=1}^{\infty} B_i$, for some $i \in N$

 Date:

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} B_i \quad \text{—— (ii)}$$

From (i) and (ii), we get

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i \quad \text{(proved).}$$

(ii) $B_i \cap B_j = \phi$, if $i \neq j$

Let $i < j$

$$B_i \cap B_j = \{A_i - \{A_1 \cup A_2 \cup \cdots \cup A_{i-1}\}\} \cap \{A_j - \{A_1 \cup A_2 \cup \cdots \cup A_{i-1} \cup \cdots \cup A_{j-1}\}\}$$

$\because A - B = A \cap B^c$

$$= \{A_i \cap \{A_1 \cup A_2 \cup \cdots \cup A_{i-1}\}^c\} \cap \{A_j - \{A_1 \cup A_2 \cup \cdots \cup A_{i-1} \cup \cdots \cup A_{j-1}\}^c\}$$

$\because (\cup A_i)^c = \cap A_i^c$ $\because$ De-Morgan's law

$$= \{\overset{A}{A_i} \cap \overset{B}{\{A_1' \cap A_2' \cap \cdots \cap A_{i-1}'\}}\} \cap \{\overset{C}{A_j} \cap \overset{D}{\{A_1' \cap A_2' \cap \cdots \cap A_{i-1}' \cap \cdots \cap A_{j-1}'\}}\}$$

$\because (A \cap B) \cap (C \cap D) = (A \cap C) \cap (A \cap D) \cap (B \cap C) \cap (B \cap D)$

$$= \{A_i \cap A_j\} \cap \{A_i \cap \{A_1' \cap A_2' \cap \cdots \cap A_{i-1}' \cap \cdots \cap A_{j-1}'\}\} \cap \{\{A_1' \cap A_2' \cap \cdots \cap A_{i-1}'\} \cap A_j\} \cap$$
$$\{\{A_1' \cap A_2' \cap \cdots \cap A_{i-1}'\} \cap \{A_1' \cap A_2' \cap \cdots \cap A_{i-1}' \cap \cdots \cap A_{j-1}'\}\}.$$

$\because A \cap B = B \cap A$

$$= \{A_i \cap \{A_1' \cap A_2' \cap \cdots \cap A_{i-1}' \cap \cdots \cap A_{j-1}'\}\} \cap \{A_i \cap A_j\} \cap \{\{A_1' \cap A_2' \cap \cdots \cap A_{i-1}'\} \cap A_j\} \cap$$
$$\{\{A_1' \cap A_2' \cap \cdots \cap A_{i-1}'\} \cap \{A_1' \cap A_2' \cap \cdots \cap A_{i-1}' \cap \cdots \cap A_{j-1}'\}\}.$$

$\because \quad A \cap A' = \phi$

$\because \quad A_i \cap A_i' = \phi$

$= \phi \cap \{A_i \cap A_j\} \cap \{\{A_1' \cap A_2' \cap \cdots \cap A_{i-1}'\} \cap A_j\} \cap \{\{A_1' \cap A_2' \cap \cdots \cap A_{i-1}\} \cap$
$\{A_1' \cap A_2' \cap \cdots \cap A_{i-1}' \cap \cdots \cap A_{j-1}'\}\}.$

(agr him set ka intersection empty set ky sth lgy to ans empty set ay ga).

$= \quad \phi$

$\Rightarrow \quad B_i \cap B_j = \phi$

 Date:

V.imp

→ Theorem:-

let $A_1, A_2, \dots$ be a denumerable family of pairwise disjoint sets Then $\cup A_i$ is denumerable.

OR

A countable infinite (Denumerable) Union of denumerable sets is denumerable.

proof:-

Suppose That $\{A_1, A_2, \dots\}$ is countably infinite (denumerable) collection of denumerable Sets.

we have to prove that

$A = \bigcup_{i \in N} A_i$ is denumerable.

we may assume That A_i's are disjoint otherwise we can make

$B_1 = A_1$, $B_2 = A_2 - A_1$, $B_3 = A_3 - (A_1 \cup A_2)$

$\Rightarrow$ $B_j \cap B_i = \phi$, $j \neq i$ and $\bigcup_{i \in N} A_i = \bigcup_{i \in N} B_i$,

So if $A_1, A_2, \dots$ be a Sequence of pair wise disjoint sets.

yah pr B_i bh ly skty hai.

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let $A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\}$

$A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\}$

$A_3 = \{a_{31}, a_{32}, a_{33}, \ldots\}$

$\vdots$

$\because$ $A_1, A_2, A_3 \ldots$ elements are denumerable So we can write Them in sequence of set.

$\because$ we want To prove That $\cup B_i$ is Denumerable and it automatically show that $\cup A_i$ is also denumerable.

$$\bigcup_{i \in N} A_i = \left\{ \begin{array}{cccc} a_{11}, & a_{12}, & a_{13} & \cdots \\ a_{21}, & a_{22}, & a_{23} & \cdots \\ a_{31}, & a_{32}, & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{array} \right\}$$

As we know That "N" is denumerable Then $N \times N$ is also denumerable.

Now

let $f : \bigcup_{i \in N} A_i \to N \times N$

defined by

$f(a_{ij}) = (i, j)$

$\Rightarrow$ f is one - one :-

let $a_{ij}, a_{i'j'} \in \bigcup_{i \in N} A_i$ and

$f(a_{ij}) = f(a_{i'j'})$

$\Rightarrow (i, j) = (i', j')$

$\Rightarrow i = i'$ Or. $j = j'$ $\because$ Corresponding elements are equal to each other).

 Date:

$\Rightarrow a_{ij} = a_{i'j'}$

$\Rightarrow f$ is one-one.

→ f is onto:-

For every $(i,j) \in N \times N$ Then $\exists$ $a_{ij} \in \cup A_i$ Such That

$f(a_{ij}) = (i,j)$

$\Rightarrow f$ is onto.

$\Rightarrow f$ is bijective.

$\Rightarrow \bigcup_{i\in N} A_i \sim N\times N$. and $N\times N \sim N$ (By Transitive Property of equivalence set)

Thus $\bigcup_{i\in N} A_i$ is denumerable. $\because \bigcup_{i\in N} A_i \sim N\times N \sim N$

because $N\times N \sim N$. $\bigcup_{i\in N} A_i \sim N$

Note :-

- $A\cup B$ is denumerable. if A and B are denumerable.

Date:

Question :-

Prove that the set of rational number is denumerable

Solution :-

The set of rational number Q can be written as

$$Q = Q^+ \cup Q^- \cup \{0\}.$$

We defined a function

$f : Q^+ \longrightarrow N \times N$ such that

$f(p/q) = (p, q)$.

f is one-one :-

for $\frac{p}{q}, \frac{p_1}{q_1} \in Q^+$

$f(p/q) = f(p_1/q_1)$

$(p, q) = (p_1, q_1)$

$\Rightarrow \; p = p_1 \;,\; q = q_1$

$\Rightarrow \; \frac{p}{q} = \frac{p_1}{q_1}$

Hence f is one-one.

Part 1 $\quad Q^+ \sim N \times N$

and $\quad N \times N \sim N$

$Q^+ \sim N \times N \sim N$

By Transitive property of equivalence set

$\Rightarrow Q^+ \sim N$

Part 2 $\quad Q^+ \sim Q^-$

$N \sim Q^+ \sim Q^-$

$\Rightarrow N \sim Q^-$

If Q^+ is Denumerable then Q^- is also Denumerable.

If a finite set and any set is Denumerable Then Union of both set is Denumerable

$Q^+ \cup Q^- \sim N$

$\Rightarrow Q^+ \cup Q^- \cup \{0\} \sim N$

$Q = Q^+ \cup Q^- \cup \{0\}$ is Denumerable

$\Rightarrow Q \sim N$.

 Date:

→ **f is onto:-**

for every $(p,q) \in N \times N$ then $\exists$ $f(p/q) \in Q^+$ such that

$$f(p/q) = (p,q)$$

$\Rightarrow$ f is onto.

$\Rightarrow$ f is bijective.

so $Q^+ \sim N \times N$

Also $N \times N \sim N$

There by trancition property

$$Q^+ \sim N.$$

Hence Q^+ is denumerable.

→ Again we define a function

$$g : Q^+ \longrightarrow Q^-$$

by $g(p/q) = -p/q$

→ **g is one-one:-**

let $g(p_1/q_1) = g(p_2/q_2)$ for $(p_1/q_1),(p_2/q_2) \in Q^+$

$$\Rightarrow -\frac{p_1}{q_1} = \frac{-p_2}{q_2}$$

Date:

$\Rightarrow \dfrac{p_1}{q_1} = \dfrac{p_2}{q_2}$

$\Rightarrow$ g is one-one.

→ g is onto:-

Since every $-p/q \in Q^-$ is image of some $p/q \in Q^+$. This g is onto.

$\Rightarrow$ g is onto.

$\Rightarrow$ g is bijective.

$\Rightarrow Q^+ \sim Q^-$

$\Rightarrow Q^+ \cup Q^- \sim N$

$\Rightarrow Q^+ \cup Q^- \cup \{0\} \sim N$ ($\{0\}$ is finite)

Thus Q is denumerable set.

Date:

→ Non-Denumerable Sets :-

A set "A" is called Non-Denumerable set If it is not equivalent to the set of natural numbers. (and If it not denumerable).

→ Un-Countable Set :-

A set that is not countable is called uncountable set.

→ Theorem :-

Show that $(0,1)$ is non-denumerable.

in def of ε we can write as $1 = 0.999\ldots$

Proof :-

We will show that it by contradiction. let if $(0,1)$ is denumerable.

"infinite sequence of distinct elements is denumerable."

Then $\exists$ a sequence of distinct elements $a_1, a_2, \ldots$ of all real number between 0 and 1.

$\because$ hum esa sequence define kr skty hai JS mai $(0,1)$ ky b/w sary real nos hogy.

Date:

sequence elements (0,1)

let

$a_1 = 0.a_{11}a_{12}a_{13}\ldots\ldots$

$a_2 = 0.a_{21}a_{22}a_{23}\ldots\ldots$

$a_3 = 0.a_{31}a_{32}a_{33}\ldots\ldots$

$\vdots$

$a_i = 0.a_{i1}a_{i2}a_{i3}\ldots\ldots$

$\vdots$

So every number between 0 and 1 is the above list.

Now define

$b_1 = 2$ if $a_{11} = 1$ $\quad \because b_1 \neq a_{11}$

otherwise $b_1 = 1$. if $a_{11} \neq 1$

$b_2 = 2$ " if $a_{22} = 1$ " $\quad \because b_2 \neq a_{22}$

otherwise $b_2 = 1$ if $a_{22} \neq 1$

and similarly

$b_3 \neq a_{33}$

$b_4 \neq a_{44}$

$\vdots$

we can write b in compact form

$$\therefore \quad b_i = \begin{cases} 2 & \text{if } a_{ii} = 1 \\ 1 & \text{if } a_{ii} \neq 1 \end{cases}$$

a) (0,1) mai sy aik esa element mil jay ga jo in sequence mai sy hoga.

=) 0 or 1 ky b/w element decimal ki form mai hai

0.5789 b/w 0 and 1

$5 \to a_{11}$

$7 \to a_{12}$

$8 \to a_{13}$

$9 \to a_{14}$

Date: ____

Now we define a new number

$b = 0 \cdot b_1 b_2 b_3 \ldots\ldots\ldots$

$\Rightarrow b \in (0,1)$

So it must be in $\{a_i\}_{i \in N}$

but $b \neq a_i$, $i \in N$

This is contradiction to the fact

$(0,1)$ is denumerable.

Thus $(0,1)$ is not denumerable.

$\Rightarrow a_1 = 0 \cdot a_{11} a_{12} \ldots \neq b = 0 \cdot b_1 b_2 \ldots$

$b_1 \neq a_{11}$, $\overset{1st}{a_{12}} = b_1 \ldots$

$a_2 = 0 \cdot a_{21} a_{22} \ldots \neq b = 0 \cdot b_1 b_2 \ldots$

1st $a_{21} = b_1$, $a_{22} \neq b_2$

$a_3 = 0 \cdot a_{31} a_{32} \ldots = b \doteq 0 \cdot b_1 b_2 b_3 \ldots$

1st $a_{31} = b_1$, $a_{32} = b_2$

but $a_{33} \neq b_3$.

$\Rightarrow$ if decimal elements are equal then corresponding elements are also equal.

$\Rightarrow$ $0 = 0$

$a_{11} \neq b_1$

$a_{22} = b_2$

$a_{33} \neq b_3$

agr decimal ki form mai elements equal na ho to baki bhi elements equal nhi hogy.

$(0,1)$ mai diagonal elements hi equal nhi hai.

$\Rightarrow \because$ Diagonal element $(0,1)$ mai nhi hai to wo b ky equal nhi hogy.

$\therefore$ Result.

v.imp

→ Theorem :-

Prove that set of irrational number is non-denumerable.

$[0,1] \sim \mathbb{R}$

Proof:-

Suppose that contrary

Q' is denumerable

$\Rightarrow Q$ is denumerable.

$\because Q = Q^+ \cup Q^- \cup \{0\}$

$Q \sim N$.

$\Rightarrow Q' \cup Q$ is denumerable

(bcz union of 2 set is denumerable)

Date:

But $Q' \cup Q = \mathbb{R}$

$\Rightarrow$ $\mathbb{R}$ is denumerable.

but R is not denumerable.

which is contradiction to our supposition that

$\Rightarrow$ Q' is not denumerable.

$\Rightarrow$ $(0,1) \sim R \sim [c,d]$

proof:-

$(0,1) \sim R$ (proved)

$R \sim [c,d]$

let $f: R \rightarrow [c,d]$ is defined as

$$f(x) = c + \frac{x-c}{d-c}$$

→ f is One-one :-

let $x_1, x_2 \in R$ and $f(x_1) = f(x_2)$

$$\Rightarrow \cancel{c} + \frac{x_1 - c}{d-c} = \cancel{c} + \frac{x_2 - c}{d-c}$$

Date:

$$\frac{x_1 - c}{d-c} = \frac{x_2 - c}{d-c}$$

$\Rightarrow (x_1 - c)(d-c) = (x_2 - c)(d-c)$ ∵ cross multiplying

$\Rightarrow x_1 - \cancel{c} = x_2 - \cancel{c}$ ∵ by dividing $d-c$

$\Rightarrow x_1 = x_2$

$\Rightarrow f$ is one-one.

→ **f is onto:-**

for each $y \in [c,d]$ then there exist $x \in R$ such that $f(x) = y$

we can do this by solving for x:

$$y = c + \frac{x-c}{d-c}$$

$\Rightarrow y - c = \frac{x-c}{d-c}$ $\Rightarrow (d-c)(y-c) = x-c$

$\Rightarrow x = c + (d-c)(y-c)$ Inverse of y

$\Rightarrow f$ is onto.

$\Rightarrow f$ is bijective.

Date:

$\Rightarrow \quad (0,1) \sim R$

$R \sim [c,d)$ $\quad \Rightarrow \quad (0,1) \sim R \sim [c,d)$

by transitive property of equivalence set

$\Rightarrow \quad (0,1) \sim [c,d)$ If we have to prove

Date:

→ Note :-

- If any set is equivalent to natural number then is Denumerable.

Examples : $N \sim N$, $Z \sim N$, $Q \sim N$, $E \sim N$ are Denumerable.

- If any set is not denumerable then is non-denumerable.

Examples : Q', R and every interval of real line is equivalent and non-Denumerable.

we proved that

- $(0,1)$ is non-denumerable
- $(0,1) \sim R$ That means R is non-denumerable.
- $R \sim [c,d]$ That means $[c,d]$ is non-denumerable.

From here we find that

- If we take any Interval from the real line is non-Denumerable.

Date:

→ If $A \sim B$ and $C \sim D$ Then $(A \times C) \sim (B \times D)$.

proof:-

- Given That $A \sim B$ Then There exist a bijective function $f: A \to B$
- Given That $C \sim D$ then There exist a bijective function $g: C \to D$.

Now we want to show That $(A \times C) \sim (B \times D)$.

let $h: (A \times C) \to (B \times D)$ defined by

$h(a, c) = (f(a), g(c)) \quad \forall \ (a, c) \in (A \times C)$

(A×C) ky elements cartesian form mai hogy-

→ h is one-one:

let (a_1, c_1) and $(a_2, c_2) \in (A \times C)$ and

$h(a_1, c_1) = h(a_2, c_2)$

and we want to show that

$(a_1, c_1) = (a_2, c_2)$

$\Rightarrow \quad h(a_1, c_1) = h(a_2, c_2)$

$\Rightarrow \quad (f(a_1), g(c_1)) = (f(a_2), g(c_2))$

Date:

$\because$ f is one-one.

$\Rightarrow f(a_1) = f(a_2)$

$\Rightarrow a_1 = a_2$

$\because$ Similarly g is one-one

$\Rightarrow g(c_1) = g(c_2)$

$\Rightarrow c_1 = c_2$

$\Rightarrow (a_1, c_1) = (a_2, c_2)$

$\Rightarrow$ h is one-one.

$\rightarrow$ h is onto:

Since for any $(b,d) \in (B \times D)$ and we need to show that there exist $(a,c) \in (A \times C)$ such that

$h(a,c) = (b,d)$

$\because$ f is onto.

There exist an element $a \in A$ such that

$\Rightarrow f(a) = b$

$\because$ g is onto.

Date:

There exist an element $c \in C$ such that

$\Rightarrow \quad g(c) = d$

Therefore we can define $(a,c) \in (A \times C)$ such that

$h(a,c) = (f(a), g(c)) = (b,d)$

$\Rightarrow \quad h(a,c) = (b,d).$

$\Rightarrow \quad h$ is onto.

$\Rightarrow \quad h$ is bijective.

$\Rightarrow \quad (A \times C) \sim (B \times D).$

Prove that $[0,1] \sim (0,1)$.

Proof:-

The sets $[0,1]$ and $(0,1)$ can be written as

$[0,1] = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \cup A$

where $A = [0,1] - \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

$(0,1) = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \cup A$

Date:

where $A = (0,1) - \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\}$

Define a mapping $f: [0,1] \to (0,1)$ by

$$f(x) = \begin{cases} x & \text{if } x \in A \\ \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+2} & \text{if } x = \frac{1}{n} \end{cases} \quad ; \quad n = 1, 2, 3, \ldots\ldots$$

Now we have to show that f is bijective.

→ **f is one-one:**

For $x, y \in [0,1]$ and $x, y \in A$

let $f(x) = f(y)$

$\Rightarrow x = y$ [by def of f]

For $x = 0$

$\Rightarrow f(0) = \frac{1}{2}$ [by def of f] $\because x = 0$ then $\frac{1}{2}$

- For $x, y \in [0,1]$ and x, y are of the form $x = \frac{1}{n}$ and $y = \frac{1}{m}$

let $f(x) = f(y)$

$\Rightarrow f(\frac{1}{n}) = f(\frac{1}{m})$

Date:

$\Rightarrow \dfrac{1}{n+2} = \dfrac{1}{m+2}$ $\because x = \dfrac{1}{n}$ then $\dfrac{1}{n+2}$

$\Rightarrow n+\cancel{2} = m+\cancel{2}$ $\because$ by cross multiply

$\Rightarrow n = m$

$\Rightarrow \dfrac{1}{n} = \dfrac{1}{m}$

$\Rightarrow x = y$

From the above discussion it is clear that in all cases

$f(x) = f(y)$

$\Rightarrow x = y$

$\Rightarrow f$ is one - one.

→ f is onto:

For any $x \in (0,1)$ and $x \in A$ then we have to show that there exist $x' \in [0,1]$ such that

$f(x') = x'$

For $x = \frac{1}{2} \in (0,1)$ then there exist $0 \in [0,1)$

such that $f(0) = \frac{1}{2}$ $\because x = 0$ then $\frac{1}{2}$

Date:

For $x = \frac{1}{n} \in (0,1)$; $n \in 3, 4, 5, \ldots\ldots$

Then There exist $\frac{1}{n-2} \in [0,1)$ Such That

$f\left(\frac{1}{n-2}\right) = \frac{1}{n}$

$\Rightarrow$ f is onto.

$\Rightarrow$ f is bijective.

$\Rightarrow$ $[0,1] \sim (0,1)$.

$\because$ If $x = \frac{1}{n}$ Then $\frac{1}{n+2}$

$\because$ if $x = \frac{1}{n-2}$ Then

$\frac{1}{n-2+2} = \frac{1}{n}$.

$\frac{1}{3} \to \frac{1}{1}$ (Inverse Image)

$\frac{1}{4} \to \frac{1}{2}$

$\frac{1}{5} \to \frac{1}{3} \ldots\ldots$

Date:

→ Cardinal Number

Let A be any set and let "α" denote the family of sets which are equivalent to A. Then "α" is called cardinal number of A and is denoted by

$$\#(A) = \alpha$$

cardinal no. of A.

$$\alpha = \{B : B \sim A\}$$

$$\beta = \{N, O, E, Q, Z, B \sim N\}$$

→ Cardinality of finite and Infinite sets:

→ cardinality of finite:

If a set is finite, then the cardinality of set is equal to size of set.

(The Total distinct element of set).

$$\#(\varphi) = 0$$

$$\#(\{1\}) = 1$$

$$\#(\{2\}) = 1.$$

Date:

→ Cardinality of Infinite Set:

If A is denumerable, then let

$$\#(A) = a$$

If B is uncountable then let

$$\#(B) = c$$

$\#(Z) = a$, $\#(Q) = a$, $\#(N) = a$

$\#(R) = c$, $\#(Q') = c$, $\#(0,1) = c$.

Set
- finite
- Infinite
  - denumerable: N, O, set β
  - Non-Denumerable / uncountable
    - $\downarrow$ R
    - Complex no is super set of R.

Note :-

If α and β are cardinal numbers of A and B respectively then

$$\alpha = \beta \iff A \sim B.$$

$\because$ agr 2 cardinals equal hogy then A equivalent hoga B ky.

→ Arithmetic of Cardinals:

let $\alpha = \#(A)$ and $\beta = \#(B)$ then

(i) $\alpha + \beta = \#(A \cup B)$ if $A \cap B = \phi$

(ii) $\alpha\beta = \#(A \times B)$.

$\Rightarrow$ $A = \{1, 2\}$, $B = \{2, 3\}$

$A \cup B = \{1, 2, 3\}$

$\#(A) = 2$, $\#(B) = 2$

$\#(A \cup B) = 3$

$\#(A) + \#(B) \neq \#(A \cup B)$

bcz A or B are not disjoint.

agr disjoint hogy tbh $A \cup B$ ki cardinality equal hogi.

Date:

Theorem :

For any cardinals α, β and γ

(i) $\alpha+\beta = \beta+\alpha$ (Commutative under addition).

proof:-

let $\alpha = \#(A)$ and $\beta = \#(B)$ and $A \cap B = \phi$

$\#(A \cup B) = \alpha+\beta$

$\because A \cup B = B \cup A$

$\#(B \cup A) = \alpha+\beta$

$\because \#(B \cup A) = \beta+\alpha$

$\beta+\alpha = \alpha+\beta$

$\Rightarrow \alpha+\beta = \beta+\alpha$

(ii) $\alpha\beta = \beta\alpha$ (Commutative Under multiplication)

let $\alpha = \#(A)$ and $\beta = \#(B)$

$\#(A \times B) = \alpha\beta$

$\#(B \times A) = \beta\alpha$

Date:

We want to show that

$A \times B \sim B \times A$

let $f: A \times B \to B \times A$ defined by

$f(a, b) = (b, a)$

→ f is one-one:

let $(a_1, b_1), (a_2, b_2) \in A \times B$ and

$f(a_1, b_1) = f(a_2, b_2)$

$(b_1, a_1) = (b_2, a_2)$

$\Rightarrow$ $b_1 = b_2$, $a_1 = a_2$ $\because$ corresponding elements are equals.

$\Rightarrow$ $(a_1, b_1) = (a_2, b_2)$

$\Rightarrow$ f is one-one.

→ f is onto:

Since for any $(b, a) \in B \times A$ then there exist $(a, b) \in A \times B$ such that

$f(a, b) = (b, a)$

$\Rightarrow$ f is onto.

$\Rightarrow$ f is bijective.

Date: ______

$\Rightarrow \quad A \times B \sim B \times A$

$\#(A \times B) = \#(B \times A)$

$\alpha\beta = \beta\alpha$ (Proved).

iii) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ (Associative property under addition).

let $\alpha = \#(A)$, $\beta = \#(B)$ and $\gamma = \#(C)$ and $A \cap B \cap C = \varphi$

$\because \; A \cup (B \cup C) = (A \cup B) \cup C.$

$\#(A \cup (B \cup C)) = \alpha + (\beta + \gamma)$

$\because \; A \cup (B \cup C) = (A \cup B) \cup C$

$\#((A \cup B) \cup C) = \alpha + (\beta + \gamma)$

$\because \; (A \cup B) \cup C = (\alpha + \beta) + \gamma$

$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

$\Rightarrow \quad \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ (Proved)

(iv) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ (Associative property under multiplication)

let $\alpha = \#(A)$, $\beta = \#(B)$ and $\gamma = \#(C)$

$\#[A \times (B \times C)] = \alpha(\beta\gamma)$

$\#[(A \times B) \times C] = (\alpha\beta)\gamma$

Date:

let $f: A\times(B\times C) \to (A\times B)\times C$ defined by

$f(a,(b,c)) = ((a,b),c)$

→ **f is one-one:**

let $(a,(b,c)), (a',(b',c')) \in A\times(B\times C)$ and

$f(a,(b,c)) = f(a',(b',c'))$

$\Rightarrow ((a,b),c) = ((a',b'),c') \Rightarrow (a,b) = (a',b'), \text{or } c = c' \Rightarrow (a,b,c) = (a',b',c')$

$\Rightarrow (a,(b,c)) = (a',(b',c'))$

$\Rightarrow f$ is one-one.

→ **f is OnTo:**

Since for any $((a,b),c) \in (A\times B)\times C$ then there exist $(a,(b,c)) \in A\times(B\times C)$

Such that $f(a,(b,c)) = ((a,b),c)$

$\Rightarrow f$ is onto.

$\Rightarrow f$ is bijective.

$\Rightarrow A\times(B\times C) \sim (A\times B)\times C$

$\#(A\times(B\times C)) = \#((A\times B)\times C)$

$\alpha(\beta\gamma) = (\alpha\beta)\gamma$ (Proved)

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imp

→ proposition

$a^c = c$

proof:-

let Z and $A = [0,1]$

$\Rightarrow \#(Z) = a \quad , \quad \#(A) = c$

$\#(Z \times A) = ac$

$\#(R) = c$

Now let

$f : Z \times A \to R.$ defined by

$f(n, b) = n + b \quad , \quad n \in Z, b \in A$

$2 \cdot 13$
$= 2 + 0 \cdot 13$
$\quad n + b$

$= -2 \cdot 13$

$= -3 + 0 \cdot 97$
$\quad Z + b$

$b \in [0,1]$

→ f is one-one:

let $(n_1, b_1), (n_2, b_2) \in Z \times A$

and $(n_1, b_1) \neq (n_2, b_2)$

$n_1 \neq n_2$ or $b_1 \neq b_2$

$\Rightarrow (n_1 + b_1) \neq (n_2, b_2)$

$\Rightarrow f(n_1, b_1) \neq f(n_2, b_2)$

$\Rightarrow f$ is one-one.

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→ f is onto:

For every $x \in R$, $x = n+b$ then there exist (n, b) is inverse image under f.

$\Rightarrow$ f is onto.

$\Rightarrow$ f is bijective.

$\Rightarrow$ $Z \times A \sim R$.

$\Rightarrow$ $ac \sim c$

$\Rightarrow$ $ac = c$.

→ Inequalities in cardinal numbers:

Let $\alpha = \#(A)$ and $\beta = \#(B)$ then $\alpha < \beta$ if every one-one function $f: A \to B$ is not onto. or, If A is equivalent to some subset of B and A is not equivalent to B. Then we write $A \prec B$ and read it as

"A strictly precedes B."

Further $\alpha \leq \beta$, if A is equivalent to some subset of B. Then we write $A \preceq B$ and read it as

" A precedes B.

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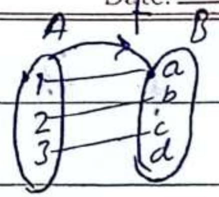
→ Remarks :-

(i) Cancellation laws do not hold for cardinal addition.

(ii) cancellation laws do not hold for cardinal multiplication.



A or B ky b/w 1-1 function hai lykn onto nhi hai..

$\#(A) = 3$ or

$\#(B) = 4$

① A is strictly less then B-

$A < B$.

② A kei subset of B ky equivalent hai lykn A B ky equivalent nhi hai:

proof:

(i) $\#(N) = a$

$\#(\varphi) = 0$

$\#(\{x\}) = 1$:

$\because N \cup \varphi = N$

$\#(N \cup \varphi) = N$

$\#(N \cup \varphi) = \#(N)$

$a + 0 = a$ —— ①

Then by statement

"If A is finite and B is denumerable then A∪B is denumerable."

Date:

$N \cup \{x\} \sim N$

$\#(N \cup \{x\}) = \#(N)$

$a+1 = a \quad \text{——} (2)$

From (1) and (2)

$\cancel{a}+0 = \cancel{a}+1$

but $0 \neq 1$ So

cancellations laws do not hold for cardinal addition.

(ii) we know that

$N \times N \sim N$

$\#(N \times N) = \#(N)$

$a \cdot a = a \quad \text{——} (1)$

"If A and B are denumerable then product $A \times B$ is denumerable."

$\therefore (N \times \{x\}) \sim N$

$\#(N \times \{x\}) = \#(N)$

$a \cdot 1 = a \quad \text{——} (2)$

from (1) and (2)

$a \cdot a = a \cdot 1$ but $a \neq 1$

$\#(N) = a \rightarrow$ finite
wo 1 ky equal
nhi ho skta

Date:

So cancellation laws do not hold for cardinal multiplication.

imp

Theorem:

If β be any infinite cardinal number then $\alpha+\beta=\beta$

Proof:-

let $A=\{a_1, a_2, \ldots\}$ be a denumberable set and B be any infinite set such that

$$A\cap B=\phi \quad \text{and} \quad \#(A)=\alpha, \ \#(B)=\beta$$

we want to show that

$$A\cup B \sim B.$$

Since $\#(A\cup B)=\alpha+\beta$ and $\#(B)=\beta$

$\because$ B is infinite set then B has denumberable subset. Say D.

" Every infinite set has a denumerable subset."

$$D\subseteq B.$$

and we can write D in sequence $\because$ (by def of Denumberable set.)

$\Rightarrow$ $D=\{d_1, d_2, d_3, \ldots\}$

let $f: A\cup B \longrightarrow B$ defined by

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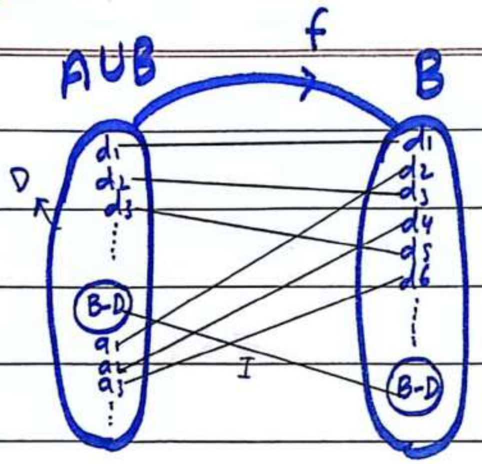
$\because B$ ky under D hai

$B = (B-D) \cup D$

$B = (B \cap D') \cup D$



$f: A \cup B \to B$ is defined by

$$f(x) = \begin{cases} x & \text{if } x \in B-D \\ d_{2i-1} & \text{if } x = d_i \in D, \ i \in N \\ d_{2i} & \text{if } x = a_i \in A, \ i \in N \end{cases}$$

$d_1 \to d_1$
$d_2 \to d_3$
$d_3 \to d_5$
$\vdots$
$a_1 \to d_2$
$a_2 \to d_4$
$a_3 \to d_6$
$a_4 \to d_8$
$\vdots$

from the above, we conclude that there is a bijection function exist.

$\Rightarrow A \cup B \sim B$

$\Rightarrow \alpha + \beta \sim \beta$

$\Rightarrow \alpha + \beta = \beta.$

imp lecture: 22-11-2023.

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→ Theorem:-

Suppose that S and T are sets and that $T \subseteq S$.

a) If S is a countable set, then T is countable.

b) If T is uncountable then S is uncountable.

Proof:-

(a) If S is a countable set and $T \subseteq S$ and we have to show that T is countable.

Then by def of countable set.

"If A is countable then A is finite or denumerable"

Case I:

If S is finite then T is also finite.

$\Rightarrow$ T is countable

Case II:

If S is not finite then S is denumerable.

$\Rightarrow$ T is finite or denumerable because subset of denumerable set is either finite or denumerable then T is countable.

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(b) Assume T is Uncountable and $T \subseteq S$. To prove S is Uncountable.

Suppose contrary, S is countable.

Since $T \subseteq S$ and S is countable. by statement "If S is countable then T is countable". So T must also be countable.

This contradicts our assumption that T is uncountable.

Hence by contradiction S must be Uncountable. if T is uncountable.

→ **Theorem:-**

The following Statements are equivalent.

a) S is a countable.

b) There exist a surjection of N onto S.

c) There exist a injection of S into N.

Proof:-

(a) ⇒ (b) (countable) ⇒ (Surjection).

we have S is countable.

by def "S is finite or denumerable."

of countable.

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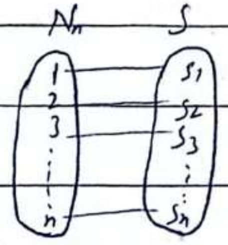
Case I: If S is finite then there exist a bijection h of some set N_n onto S. $h: N_n \to S$ for some $n \in N$. $N_n = 1, 2, \ldots n$

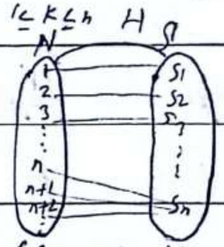
and we define H on N by. $H: N \to S$ by



$$H(k) = \begin{cases} h(k) & \text{for } k = 1, 2, \ldots n \ ; \ 1 \le k \le n \\ h(n) & \text{for } k > n \ ; \ k = n+1, n+2, \ldots \end{cases}$$

$H(k) = h(k)$ for $1 \le k \le n$

Then H is a surjection of N onto S.



Case II:- If S is denumerable then there exists a bijection H of N onto S. which is also a surjection of N onto S.

(Corresponding to every elements in range we can find an elements in domain).

(b) $\Rightarrow$ (c) (Surjection) $\Rightarrow$ (injection).

If H is a surjection of N onto S.

we define $H_1: S \to N$ defined by $H_1(S) = n_s$

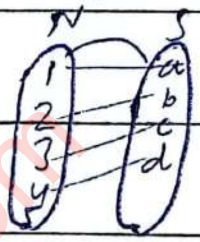
by letting $H_1(S)$ be the least element in the set

$H^{-1}(S) = \{n \in N : H(n) = S\}$.

Note:-

⇒ If there exist a Surjection from N onto S.

It means the cardinality of N is greater than or equal to the cardinality of S.

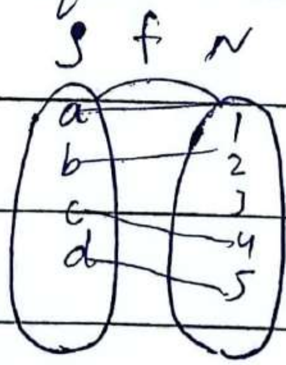


$$\#(S) \leq \#(N).$$

⇒ If there exist an injection from S into N.

it means that the cardinality of S is less Then or equal to cardinality of N.

$$\#(S) \leq \#(N).$$



$\#(S) = 4$

$\#(N) = 5$

$\#(S) \leq \#(N).$

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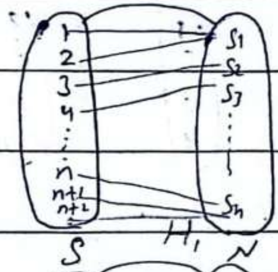
To see that H_1 is an injection of S into N.



onto

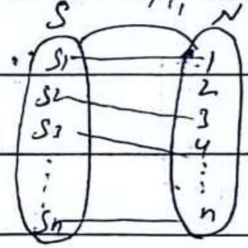
If $s, t \in S$ and $H_1(s) = H_1(t)$

$\Rightarrow H_1(s) = H_1(t)$.

$\Rightarrow n_s = n_t$

$\Rightarrow H(n_s) = H(n_t)$ by apply H on b.s.

$\Rightarrow s = t$ $\quad \because H(n) = s$ by set $H^{-1}(s)$



one-one.

(c) $\Rightarrow$ (a) (Injection) $\Rightarrow$ (countable).

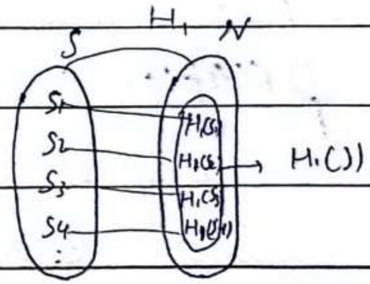
$H_1(S)$ is image of H_1 then clear $H_1(S)$ subset of N.

consider an image set of S in N. let it be $H_1(S)$.

Clearly this $H_1(S) \subseteq N$.

(corresponding to every element there may exist exactly one element).

Also define $H : S \to H_1(S)$ then it is a bijection of S onto $H_1(S) \subseteq N$.



clearly H is a bijection.

$\rightarrow H_1(S)$ is a subset of a countable set.

$\Rightarrow H_1(S)$ is countable. $\quad \because H_1(S) \subseteq N$

by using ee If S is a countable then T is countable with $T \subseteq S$.

$\Rightarrow$ whence the set S is countable

(a) $\Rightarrow$ (b) $\Rightarrow$ (c).

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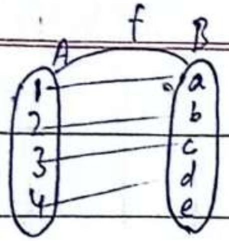
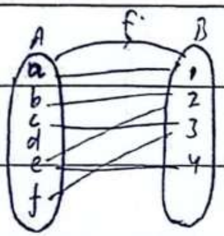


one-one, not onto

$\#(A) = 4$, $\#(B) = 5$.

$\#(A) < \#(B)$. $A < B$.



onto

$\#(A) = 6$, $\#(B) = 4$ $\#(B) < \#(A)$.

imp → **Contor Theorem:**

If A is any set, then there is no Surjection of A onto the $p(A)$ of all Subsets of A.

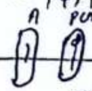
proof:-

Suppose that $\phi : A \to p(A)$ is a surjection of A.

Since $\phi(a)$ is a subset of A, either a belong to $\phi(a)$ or it does not to this set.

$A_1 = \{1\}$

$p(A) = \{\phi, \{1\}\}$



one-one, not onto

we let

$$D = \{a \in A \; ; \; a \notin \phi(a)\}$$

(is set mai wo tmam elements A ky hogy jo P(A) mai map kry gy to $\phi(a)$ mai map nhi kry gy).

Date:

Since D is a subset of A.

If ϕ is surjection then $D = \phi(a_0)$ for some $a_0 \in A$

↳ by using def of onto.

we must have either $a_0 \in D$ or $a_0 \notin D$.

Case I :-

If $a_0 \in D$, $\Rightarrow a_0 \in \phi(a_0)$ $\because D = \phi(a_0)$

contrary to the def of D.

Case II :-

Similarly

If $a_0 \notin D$

$\Rightarrow a_0 \notin \phi(a_0)$

$\Rightarrow a_0 \in D$

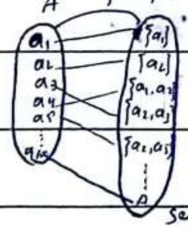
which is also a contradiction.

Therefore ϕ cannot be a surjection.

Cantor's Theorem implies that there is an unending progression of larger and larger sets. In particular, it implies that the collection $P(N)$ of all subsets of the natural numbers N is uncountable.

A ky subsets $P(A)$ mai hai to D bh $P(A)$ mai hoga.

To $P(A)$ ky corresponde hmy A mai preimage mil jay gi



$\phi(A_{100}) = A$

set wha ja k map kry gy Jha wo khud bhi ho sky ho or nhi bhi ho sky.

$P(A) \to A$ ky subsets

$\phi(a_1) = \{a_1\}$

$\phi(a_2) = \{a_2\}$

$\vdots$

$\Rightarrow$ agr aik set D mai hai wo $\phi(a)$ mai hona chahiye lykn yah nhi hai.

$\Rightarrow$ R sbh sy braa read no hai is sy braa bhi mil skta hai $P(R)$ R sy bra hoga $P(R)$ sy koi or nmbr braa ho skta hai or usy larger bhi mil sky ham or yh silsla chalta jay ga.

$P(N)$ $\cup$ R uncountable or a or c cardinality sy larger cardinality mil skti hai ju ky infinite hogi.

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imp

→ Schroder - Bernstein Theorem :-

If $X_1 \subseteq Y \subseteq X$ and $X \sim X_1$ Then $X \sim Y$.

proof:-

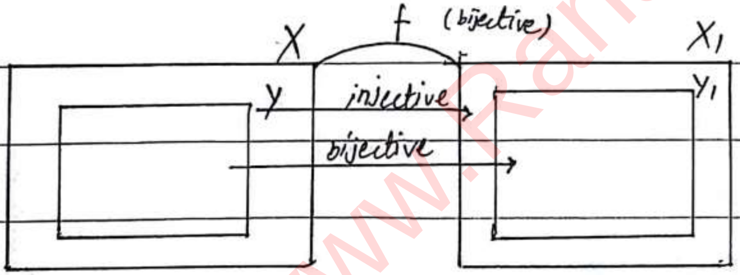
Given that $X \sim X_1$.

$\Rightarrow \exists$ a bijective function $f : X \to X_1$.

Further given $Y \subset X$, So restriction of f To Y is one-one (injective) $\because Y \subseteq X$.

let us denote the restriction of f to Y by

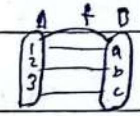
$f : Y \to X_1$ is one-one.



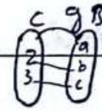
$\Rightarrow Y$ is equivalent to Some Subset of X_1 say Y_1 $(Y_1 \subseteq X_1)$.

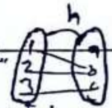
$\Rightarrow Y \sim Y_1$ $(f : Y \to Y_1$ is bijective$)$

$f : A \to B$ or
$g : C \to B$
$C \subseteq A$
$f(c) = g(c) \ \forall c \in C$
$\therefore g$ is restriction of f
f is extension of g.



$C \subseteq A$
$c = \{2, 3\}$





f is extension of g
g is restriction of f
and h is restriction of f.

Note:-
$\Rightarrow$ bijective function ki restriction injective hai.
$\Rightarrow$ agr function injective hai to zruri nhi ky function bijective ho.

$Y \subseteq X$.
$Y_1 \subseteq X_1$

$\Rightarrow$ Jo $f : X \to X_1$ pr elements map kr rhy hai whi $f : Y \to X_1$ pr bhi same hogy. or Then $f : Y \to Y_1$ pr bh same hogy.

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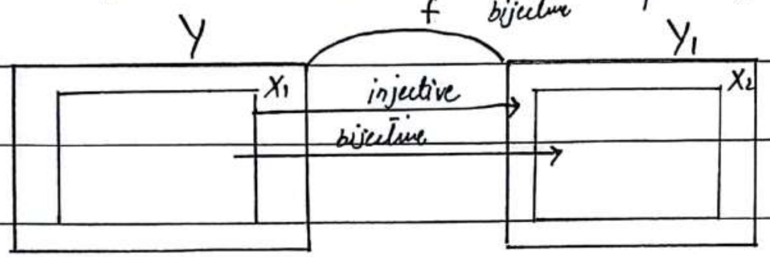
As $X_1 \subseteq Y_1$, So restriction of f to X_1 is also one-one.



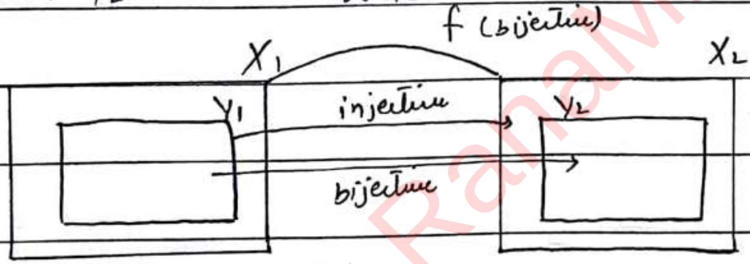
⇒ Yah X_1 or Y_1 ka subset X_2 ky equivalent hoga.

⇒ X_1 is equivalent to some subset of Y_1 say X_2 under f.

⇒ $X_1 \sim X_2$ ($f: X_1 \to X_2$ is bijective)

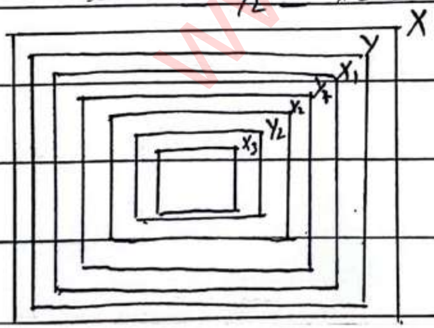
As $Y_1 \subset X_1$

⇒ $Y_1 \sim Y_2$ where $Y_2 \subseteq X_2$



and continues this process

⇒ $\cdots \subset Y_2 \subseteq X_2 \subseteq Y_1 \subset X_1 \subset Y \subseteq X$



Date:

Accordingly $\exists$ equivalent sets

$X \sim X_1 \sim X_2 \sim X_3 \sim \ldots$

and $Y \sim Y_1 \sim Y_2 \sim Y_3 \sim \ldots$

So that

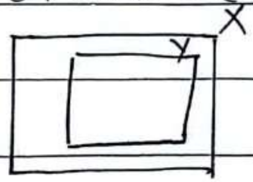
$f: X_k \to X_{k+1}$ and

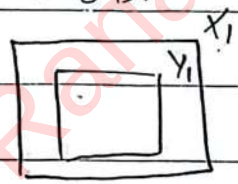
$f: Y_k \to Y_{k+1}$ are bijective mappings, $k = 0, 1, 2, \ldots$

let $B = X \cap Y \cap X_1 \cap Y_1 \cap \ldots$

$X = (X - Y) \cup (Y - X_1) \cup (X_1 - Y_1) \cup \ldots \cup B$.

and $Y = (Y - X_1) \cup (X_1 - Y_1) \cup \ldots \cup B$.





$\Rightarrow X \sim X_1$

$Y \sim \cancel{X_1}$

$(X - Y) \sim (X_1 - Y_1)$.

clearly $X - Y, X_1 - Y_1, X_2 - Y_2, \ldots$

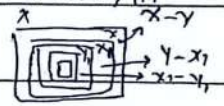
are all equivalents.

$\ldots Y_2 \subseteq X_2 \subseteq Y_1 \subseteq X_1 \subseteq Y \subseteq X$

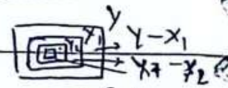
by Taking intersection

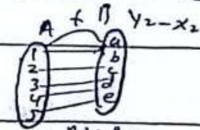
$B = X \cap Y \cap X_1 \cap Y_1 \cap \ldots$

$X =$

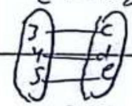


$X =$





$C \quad A \sim B \quad D$



$C \subseteq A$ and

$D \subseteq B$

$C \sim D$

baki jo elements

Peechy wo

bhi equivalent

hogy.

Date:

Infact

$f: X_K \to Y_K \to X_{K+1} - Y_{K+1}$, $k = 0, 1, 2, \ldots$

are bijective mappings.

$X = (X-Y) \cup (Y-X_1) \cup (X_1 - Y_1) \cup (Y_1 - X_2) \cup (X_2 - Y_2) \cup \cdots \cup B$.

$Y = (Y - X_1) \cup (X_1 - Y_1) \cup (Y_1 - X_2) \cup (X_2 - Y_2) \cup (Y_2 - X_3) \cup \cdots \cup B$.

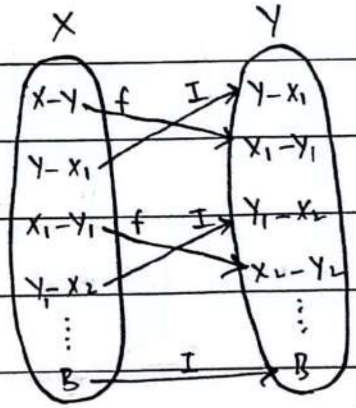
$(Y-X_1) \sim (Y-X_1)$ are same so also equivalent under Identity mapping. or baki different f ky under equivalent hogy.

$\Rightarrow (X-Y) \sim (X_1 - Y_1)$ and $(Y-X_1) \sim (Y-X_1)$ and $(X_1-Y_1) \sim (X_2 - Y_2)$

$\ldots\ldots\ldots$ $B \sim B$, So $X \sim Y$.

To prove $X \sim Y$ are defined a mapping $g: X \to Y$ such that

$$g(x) = \begin{cases} f(x) & \text{if } x \in X_K - Y_K \text{ or } X-Y \\ x & \text{if } x \in B \text{ or } Y_K - X_{K+1} \end{cases}$$



$\Rightarrow f: X_K - Y_K \to X_{K+1} - Y_{K+1}$ are bijective.

$\Rightarrow I: Y_K - X_{K+1} \to Y_K - X_{K+1}$ are bijective.

$\Rightarrow g$ is bijective.

$\Rightarrow X \sim Y$.

CamScanner

Date:

→ Cantor - Shroder - Bernstein's Theorem:

If $X \preceq Y$ and $Y \preceq X$ then $X \sim Y$.

OR If $\alpha \leq \beta$ and $\beta \leq \alpha$ then $\alpha = \beta$.

proof:-

let X and Y be the sets with $\#(X) = \alpha$ and $\#(Y) = \beta$

if $\alpha \leq \beta \Rightarrow X \preceq Y$ then $X \sim Y_1 \subseteq Y \Rightarrow Y_1 \sim X$ — (1)

if $\beta \leq \alpha \Rightarrow Y \preceq X$ then $Y \sim X_1 \subseteq X \Rightarrow Y \sim X_1$ — (2)

As $Y_1 \subseteq Y$ and $Y \sim X_1 \Rightarrow Y_1$ is equivalent to a subset of X_1

i.e $Y_1 \sim X_2 \subseteq X_1 \Rightarrow Y_1 \sim X_2$ — (3)

By (1) and (3)

$X \sim Y_1$ and $Y_1 \sim X_2 \Rightarrow X \sim X_2$

Now $X_2 \subset X_1 \subset X$ and $X \sim X_2$

So by schroder-bernstein's theorem that is i.e if $X_1 \subseteq Y \subseteq X$ and $X \sim X_1$ then $X \sim Y$.

$X \sim X_1$ — (4)

Also by (2), $X_1 \sim Y$ — (5) using transitive property for (4) and (5) we get

$X \sim Y$

Hence $\#(X) = \#(Y)$ i.e $\alpha = \beta$.

Date:

V. imp

→ Applications of Cantor. Schroder Berstein's Theorem,

(i) $c^2 = c$ (ii) $2^a = c$.

$a \to$ cardinality of Denumerable set

$c \to$ cardinality of uncountable set.

Target

$c^2 \le c$ — ①

$c \le c^2$ — ②

$c = c^2$ — ③

$1 = 0.999\ldots$

$1/2 = 0.5$

$1/2 = 0.4999\ldots$

$0 = 0.000\ldots$

(0, 1) key b/w no ko decimal ki form mai likh sakty hai.

imp

proof :- (i)

let $A = [0, 1]$ → Uncountable set $\Rightarrow \#(A) = c$.

$\#(A \times A) = c \cdot c = c^2$

Part of Theorem

let $x, y \in A$ such That

$x = 0.x_1x_2x_3\ldots$

and

$y = 0.y_1y_2y_3\ldots$

$x_i, y_i \in \{0, 1, 2, \ldots, 9\}$

Define $f : A \times A \to A$ such That

$$f(x, y) = 0.x_1 y_1 x_2 y_2 \ldots\ldots$$

→ f is one-one :

let $(x, y), (x', y') \in A \times A$

$\Rightarrow f(x, y) = f(x', y')$

$\Rightarrow 0.x_1y_1x_2y_2\ldots = 0.x_1'y_1'x_2'y_2'\ldots$

$\Rightarrow 0 = 0,\ x_1 = x_1',\ y_1 = y_1',\ x_2 = x_2' \ldots$

$x_1 = x_1'$	$y_1 = y_1'$
$x_2 = x_2'$	$y_2 = y_2'$
$x_3 = x_3'$	$y_3 = y_3'$
$\vdots \quad \vdots$	$\vdots \quad \vdots$
$\Rightarrow 0.x_1x_2x_3\ldots = 0.x_1'x_2'\ldots$	$\Rightarrow 0.y_1y_2y_3\ldots = 0.y_1'y_2'y_3'\ldots$
$\Rightarrow x = x'$	$\Rightarrow y = y'$

Date:

corresponding elements are always equal.

$\Rightarrow$ $(x, y) = (x', y')$

$\Rightarrow$ f is one-one.

$\Rightarrow$ $(A \times A) \preceq A$ $\quad$ ($(A\times A)$ is strictly precedes To A).

$\Rightarrow$ $\#(A \times A) \leq \#(A)$

$\Rightarrow$ $c^2 \leq c$ — ①

Again define: $\quad \boxed{\text{Target } \; c \leq c^2}$

$f: A \to A \times A$, $\quad A = [0, 1]$.

defined by $\quad f(x) = (x, 0)$

→ f is one-one:

let $x_1, x_2 \in A$

$\Rightarrow$ $f(x_1) = f(x_2)$

$\Rightarrow$ $(x_1, 0) = (x_2, 0)$

$\Rightarrow$ $x_1 = x_2$, $0 = 0$ $\quad$ (Any fix no is equal to other fix no).

$\Rightarrow$ $x_1 = x_2$

$\Rightarrow$ f is One-one

$\Rightarrow$ $A \preceq A \times A$

$\Rightarrow$ $\#(A) \leq \#(A \times A)$

$\Rightarrow$ $c \leq c^2$ — ②

From ① and ②

$c = c^2$. (Proved).

$\Rightarrow$ agr $(x, 0)$ mai 0 ki jgha y_0 ly wo general hoga wo nhi ly skty.

$\Rightarrow$ $(x, 0)$ mai 0 fix no hai or 0 A mai belong krrha hai. Iykn yah 0 ki jgha y_0 ly skty hai

$\Rightarrow$ (where $y_0 \in [0,1]$ and any fix element of A.

imp

Date:

Proof:- (ii)

let $\mathbb{R}$ be the set of real number then

$\#(R) = c$

let $P(Q)$ be the power set of set Q then

$\#(P(Q)) = 2^a$

Now we have to show that $c \leq 2^a$.

For this we defined a function

$f : R \to P(Q)$ given by

$f(r) = \{x : x \in Q, x < r\} \quad \forall\ r \in R$

$\Rightarrow$ That is if f maps each real number r into set of rational numbers less than r.

Now we show that f is one-one.

→ f is one-one:

let $r_1, r_2 \in R$ such that $r_1 \neq r_2$.

let $r_1 < r_2$

By property of real numbers $\exists$ a rational number "y" such that

2^a Power set of denumerable set

$2a : P(Q)$

$c = R$. (rational ko irrational)

in ky darmiyan mapping one-one karway gy.

Target

① $c \leq 2^a$

② $2^a \leq c$

(3) $2^a = c$.

Date:

$r_1 < y < r_2$

$\Rightarrow r_1 < y$ and $y < r_2$

$\Rightarrow y \notin f(r_1)$ { by def of f }

and $y \in f(r_2)$ [$\because$ by def of f].

$\Rightarrow f(r_1) \neq f(r_2)$ $\because$ by any contrapositive condition.

$\Rightarrow f$ is one-one

$\Rightarrow \mathbb{R} \sim P(Q)$

$\Rightarrow \#(\mathbb{R}) \leq \#(P(Q))$

$\Rightarrow c \leq 2^a$ — (1)

Now we have to show that $2^a \leq c$.

let N be the set of +ve natural nos and let C(N) be the class of characteristics function of N.

$f : N \to \{0, 1\}$

$\Rightarrow C(N) \sim 2^n$

Then $C(N) \sim P(N)$ where $P(N)$ is power set of N.

$\Rightarrow \#(C(N)) = \#(P(N))$

$\Rightarrow \#(C(N)) = 2^a$

Date:

Now let $I = [0,1]$ is the closed unit Interval then

$\# (I) = c$.

Now, to prove $2^a \leq c$ we define a function.

$g : C(N) \to I$.

given by

$g(f) = 0.f(1)f(2)f(3)\ldots\ldots \quad , \forall f \in C(N)$

Therefore the image of f under g is an infinite decimal consisting of 0 and 1.

Now we show that g is one-one.

→ g is one-one:

let $f_1, f_2 \in C(N)$ such that

$\Rightarrow f_1 \neq f_2$ where f_1, f_2 are characteristics function from N to $\{0,1\}$

$\Rightarrow f_1(n) \neq f_2(n)$ for some $n \in N$

$\Rightarrow 0 \cdot f_1(1) f_1(2) \ldots\ldots \neq 0 \cdot f_2(1) f_2(2) \ldots\ldots$

$\Rightarrow g(f_1) \neq g(f_2) \quad , \forall f_1, f_2 \in C(N)$

$\Rightarrow g$ is one-one.

Date:

$\Rightarrow \quad C(N) \leq [0,1] = I. \qquad \because I = [0,1].$

$\Rightarrow \quad \#(C(N)) \leq \#(I)$

$\Rightarrow \quad 2^a \leq c \quad \text{--- (2)}$

From (1) and (2)

$\Rightarrow \quad 2^a = c$ (Proved).

(0,1) is non-denumerable

$R \sim (0,1)$ is non-den...

cardinality of $R = c$

or cardinality of $(0,1) = c$.

R and $(0,1)$ have same cardinality.

(iii) $R \sim R^2$

proof :-

we have shown that $R \sim (0,1)$

we have also shown that for any sets A, B, C, D.

if we have $A \sim C$ and $B \sim D$ then we have that

$A \times B \sim C \times D$.

So if we can prove that $(0,1) \times (0,1) \sim (0,1)$

then the above result shown that

By Transitive property of equivalence set

$R \times R \sim (0,1) \times (0,1) \sim (0,1) \sim R$.

So it remains only to prove that

$(0,1) \times (0,1) \sim (0,1)$.

In order to do this by Cantor-Schroder Bernstein theorem Date:

For this it is enough to prove that $\exists$ an injection from $(0,1)$ into $(0,1)\times(0,1)$ and an injection from $(0,1)\times(0,1)$ into $(0,1)$.

Target
① $\mathbb{R} \leq \mathbb{R}^2$
② $\mathbb{R}^2 \leq \mathbb{R}$
③ $\mathbb{R}^2 = \mathbb{R}$.

Define a function

$f: (0,1) \to (0,1)\times(0,1)$ for injection. defined by

$f(x) = (x, y_0)$ where $y_0 \in (0,1)$ and any fix element of $(0,1)$

→ f is one-one:

let $x_1, x_2 \in (0,1)$

$\Rightarrow f(x_1) = f(x_2)$

$\Rightarrow (x_1, y_0) = (x_2, y_0)$

$\Rightarrow x_1 = x_2 \;,\; y_0 = y_0$

$\Rightarrow x_1 = x_2$

$\Rightarrow f$ is one-one. $\Rightarrow$ $\Rightarrow (0,1) \leq ((0,1)\times(0,1))$ — ①

Define another function

$g: (0,1)\times(0,1) \to (0,1)$ by

$g(x,y) = (0.x_1x_2x_3\ldots\ldots\;0.y_1y_2y_3\ldots) = 0.x_1y_1x_2y_2\ldots$

Date:

g is one-one:

let $(x,y), (x', y') \in (0,1)\times(0,1)$

$\Rightarrow f(x,y) = f(x', y')$

$\Rightarrow 0.x_1y_1x_2y_2\ldots = 0.x_1'y_1'x_2'y_2'\ldots$

$\Rightarrow 0.0, \; x_1 = x_1' \ldots$

$\Rightarrow \begin{array}{ll} x_1 = x_1' & y_1 = y_1' \\ x_2 = x_2' & y_2' = y_2 \\ x_3 = x_3' & y_3 = y_3' \\ \vdots \quad \vdots & \vdots \quad \vdots \end{array}$

$\Rightarrow 0.x_1x_2x_3\ldots = 0.x_1'x_2'\ldots \qquad \Rightarrow 0.y_1y_2y_3\ldots = 0.y_1'y_2'\ldots$

$\Rightarrow x = x' \qquad \Rightarrow y = y'$

$\Rightarrow (x,y) = (x',y')$

$\Rightarrow g$ is one-one.

$\Rightarrow (0,1)\times(0,1) \preceq (0,1)$ — (2)

From (1) and (2)

$\Rightarrow (0,1)\times(0,1) \sim (0,1)$ but $R\times R \sim (0,1)\times(0,1) \sim (0,1) \sim R$

$\Rightarrow$ by Transitive property of equivalence

$R\times R \sim R$

$\Rightarrow R \sim R^2$ (Proved).

Date:

(iii) $\mathbb{R} \sim \mathbb{R}^n$

Target
$\mathbb{R} \leq \mathbb{R}^n$
$\mathbb{R}^n \leq \mathbb{R}$
$\Rightarrow \mathbb{R} \sim \mathbb{R}^n$
$\Rightarrow c = c^n$.

Proof:-

let $\#(\mathbb{R}) = c$, $\#(\mathbb{R}^n) = c^n$

Define $f: \mathbb{R} \to \mathbb{R}^n$ such that

$f(x) = (x, 0, 0, \ldots\ldots 0)$

→ **f is one one:**

let $x, x' \in \mathbb{R}$

$\Rightarrow f(x) = f(x')$

$\Rightarrow (x, 0, 0, \ldots\ldots 0) = (x', 0, 0, \ldots\ldots 0)$

$\Rightarrow x = x', \; 0 = 0, \; 0 = 0, \; \ldots\ldots \; 0 = 0$ $\because$ corresponding elements are always equals.

$\Rightarrow x = x'$

$\Rightarrow f$ is one-one.

$\Rightarrow \mathbb{R} \leq \mathbb{R}^n$ — ①

Again define $g: \mathbb{R}^n \to \mathbb{R}$ such that

$g(x_1, x_2, x_3, \ldots\ldots x_n) = x_1$

Date:

→ g is one - one :

let $(x_1, x_2, x_3, \dots x_n), (x_1', x_2', x_3', \dots x_n') \in R^n$

$\Rightarrow \quad g(x_1, x_2, x_3, \dots x_n) = g(x_1', x_2', x_3', \dots x_n')$

$\Rightarrow \quad x_1 = x_1'$

all other components must be equal

$\Rightarrow \quad x_2 = x_2', \quad x_3 = x_3', \dots \quad x_n = x_n'$

$\Rightarrow \quad x_1, x_2, x_3, \dots x_n = x_1', x_2', x_3', \dots x_n'$

$\Rightarrow \quad g$ is one - one.

$\Rightarrow \quad R^n \leq R \quad — (2)$

From (1) and (2) $\quad \because$ by using cantor schroder berstein theorem $X \leq Y$ and $Y \leq X$ then $X \sim Y$.

$\Rightarrow \quad R \sim R^n$

As $\#(R) = c$, $\#(R^n) = c^n$

$\Rightarrow \quad c \sim c^n$

$\Rightarrow \quad c = c^n$ (Proved).

As proved $R^2 \sim R$.

$\therefore R^3 \sim R^2 \times R \sim R \times R \sim R$.

Similarly by induction

$R^{n-1} \times R \sim R. \quad \Rightarrow \quad R^n \sim R.$ (Proved)

Date: ____

→ Exponents of Cardinal numbers:

If A and B are two non-empty sets then B^A denotes the collection of all function from A to B.

If $\#(A) = \alpha$ and $\#(B) = \beta$ then

$$\#(B^A) = \beta^\alpha.$$

→ Theorem:

For any cardinals α, β, and γ then $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$

Proof:-

let $\#(A) = \alpha$, $\#(B) = \beta$ and $\#(C) = \gamma$

and $(B \cap C) = \phi$

$$\alpha^\beta = \#(A^B) \qquad A^B = \{f : f : B \to A\}$$

$$\alpha^\gamma = \#(A^C) \qquad A^C = \{g : g : C \to A\}$$

$$\Rightarrow \#(A^B \times A^C) = \alpha^\beta \cdot \alpha^\gamma$$

$$\Rightarrow \#(B \cup C) = \beta + \gamma$$

$2^2 = 4$

$2^3 = 2 \times 2 \times 2 = 8$

$\alpha^\beta = ?$

$A^B = ?$

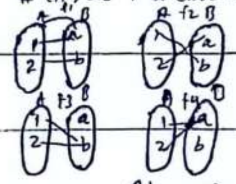
$\Rightarrow$ exponents ki cardinality check krni hai ks ky equal hai.

$\Rightarrow f : A \to B$ h mny check krna hai k A or B ky b/w kitny function define kr skty hai.

$\Rightarrow B^\alpha = \{f : f : A \to B\}$ is mai wo elements ay gy jo hmy A sy B function define kry gy wo likhy gy.

$A = \{1, 2\}$, $B = \{a, b\}$

$\#(A) = 2$, $\#(B) = 2$



$\#(B^A) = 2^2 = 4$

iska mtlb hmy A sy B 4 function mly gy.

193.

Date: ______

$\Rightarrow \#(A^{B\cup C}) = \alpha^{\beta+\gamma}$ $\quad \{ A^{B\cup C} = \{h : h : B\cup C \to A\}$

we want to show that

$A^{B\cup C} \sim A^{B} \times A^{C}$

let $f \in A^{B\cup C}$

$\Rightarrow f : B\cup C \to A$ is function.

$\Rightarrow f_B : B \to A$ and $f_C : C \to A$ are

extensions of f on B and C.

$\Rightarrow f_B = A^{B}$, $f_C = A^{C}$

$\Rightarrow$ If $(f, g) \in A^{B} \times A^{C}$

$\Rightarrow f \in A^{B}$ and $g \in A^{C}$

$\Rightarrow f : B \to A$ and $g : C \to A$

$\Rightarrow f^{*}$ is extension of f and g.

$f^{*} : B\cup C \to A$

$f^{*} \in A^{B\cup C}$

Now define $F : A^{B} \times A^{C} \to A^{B\cup C}$

by $F(f, g) = f^{*}$ (f^{*} is extension of f and g)

Target

$\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}$

let $\#(A) = 3$,

$\#(B) = 5$

$\#(B^{A}) = 5^{3} = 125$

is ka milb ky hmy

A sy B 125

function mly gy.

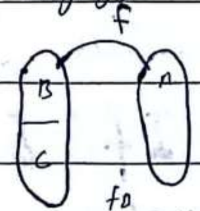
let $\#(A^{B}) = 3^{5}$

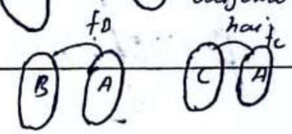
$= 243$

iska milb hmy

B sy A 243 function

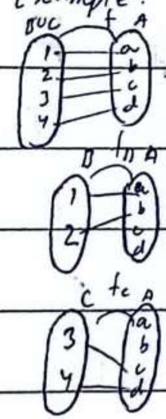
mly gy.





B or c disjoint hai

Example:



f_B, f_C are extension of f.

Date:

→ F is one-one :

let $(f_1, g_1), (f_2, g_2) \in A^B \times A^C$ such that

$(f_1, g_1) \neq (f_2, g_2)$

$\Rightarrow$ $f_1 \neq f_2$ or $g_1 \neq g_2$

As $f_1 : B \to A$ and $g_1 : C \to A$ then $f_1^* : B \cup C \to A$ is extension of f_1 and g_1.

Similarly:

$f_2^* : B \cup C \to A$ is extension of f_2 and g_2.

but $f_1 \neq f_2$ or $g_1 \neq g_2$ $\Rightarrow$ $f_1^* \neq f_2^*$ $\Rightarrow F(f_1, g_1) \neq F(f_2, g_2)$

$\Rightarrow$ F is one-one.

→ F is onto :

Since for any $f^* \in A^{B \cup C}$ then there exist $(f, g) \in A^B \times A^C$ such that

$F(f, g) = f^*$

$\Rightarrow$ F is onto.

$\Rightarrow$ F is bijective.

$\Rightarrow$ $A^{B \cup C} \sim A^B \times A^C$

$\Rightarrow$ $a^{\beta + \gamma} = a^\beta \cdot a^\gamma$.

Date:

→ Theorem:

If α, β and γ are cardinals and if $\alpha \leq \beta$ Then show That

(1) $\alpha+\gamma \leq \beta+\gamma$

(2) $\alpha\gamma \leq \beta\gamma$

(3) $\alpha^{\gamma} \leq \beta^{\gamma}$

(4) $\gamma^{\alpha} \leq \gamma^{\beta}$

Proof:

(1) $\alpha+\gamma \leq \beta+\gamma$

let $\#(A) = \alpha$, $\#(B) = \beta$ and $\#(C) = \gamma$, $A \subseteq B$ and $B \cap C = \phi$

As $A \subseteq B$

$A \cup C \subseteq B \cup C$ ∵ Taking union with "C" on b.s

$\#(A \cup C) \subseteq \#(B \cup C)$ ∵ by arithematic of cardinals, $\#(A \cup C) = \alpha+\gamma$

$\Rightarrow \alpha+\gamma \leq \beta+\gamma$ (proved). and $\#(B \cup C) = \beta+\gamma$.

(2) $\alpha\gamma \leq \beta\gamma$

let $\#(A) = \alpha$, $\#(B) = \beta$ and $\#(C) = \gamma$ and $\#(A \times C) = \alpha\gamma$

and $\#(B \times C) = \beta\gamma$.

Date:

As $A \subseteq B$

Taking cartesian product with "C" on b.s

$\Rightarrow A \times C \subseteq B \times C$.

$\Rightarrow \#(A \times C) \subseteq \#(B \times C)$

$\Rightarrow \alpha\gamma \leq \beta\gamma$ (Proved)

(3) $\alpha^\gamma \leq \beta^\gamma$

Let $\#(A) = \alpha$, $\#(B) = \beta$ and $\#(C) = \gamma$

Also $\#(A^C) = \alpha^\gamma$ and $\#(B^C) = \beta^\gamma$

Now we want to show that $A^C \subseteq B^C$, let $f \in A^C$

where A^C is the set of all function from C to A such that

$f: C \to A$ is function

As $A \subseteq B$

$\Rightarrow f: C \longrightarrow B$ is also function.

$\Rightarrow f \in B^C$ and B^C is the set of all functions from C to B

$\Rightarrow A^C \subseteq B^C$

$\Rightarrow \#(A^C) \subseteq \#(B^C)$

$\Rightarrow \alpha^\gamma \leq \beta^\gamma$ (Proved)

Date:

(4) $\gamma^{\alpha} \leq \gamma^{\beta}$

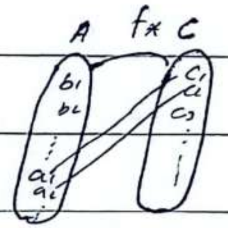
let $\#(C^A) = \gamma^{\alpha}$, $\#(C^B) = \gamma^{\beta}$

let $f \in C^A$

where C^A is the set of all functions from A to C such that

$f: A \to C$ is function.

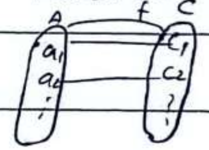
As $A \subseteq B$.



$\Rightarrow$ $f_*: B \to C$ is extension of f.

$\Rightarrow$ $f_* \in C^B$

Take $A \subseteq B$



Now define a mapping $g: C^A \to C^B$ such that

$g(f) = f_*$

where f_* is the extension of f.

Now we want to show that g is one-one.

→ g is one-one:

let $f_1, f_2 \in C^A$ and

$f_1 \neq f_2$ $\qquad$ ∵ by contrapositive criteria

$\Rightarrow f_{*_1} \neq f_{*_2}$

Date: ______

$\Rightarrow g(f_1) \neq g(f_2)$

$\Rightarrow g$ is one-one.

Hence C^A precedes C^B, such that

$$C^A \propto C^B$$

$$\#(C^A) \leq \#(C^B)$$

$$\gamma^\alpha \leq \gamma^\beta \quad \text{(Proved).}$$

→ Continum hypothesis:

There does not exist any cardinal β such that

$$\underset{\text{denumerable}}{a} < \beta < \underset{\text{uncountable}}{c}$$

⇒ If δ is any infinite cardinal number then $\delta + \delta = \delta$.

Proof:-

If δ is any infinite cardinal then $\delta + \delta = \delta$.

We will show it by two cases.

(i) $a + a = a$

(ii) $c + c = c$.

qk hmy nhi pta ky δ denumerable hai ya phr uncountable To is ko hm 2 case ky kr prove kry gy. Jis ky phly prove kr chuky hai.

Date:

Case I: $(a+a=a)$

$\because$ $a=$ cardinal of Denumerable

If $\delta=a$

let $\#(A)=a$, $\#(B)=a$ and $A\cap B=\phi$.

$\#(A\cup B)=a+a$.

$\#(C)=a$.

and we have to show that $f: A\cup B\to C$ is injective.

and $g: C\to A\cup B$ is injective.

then $\Rightarrow$ $\#(A\cup B)\le\#(C)$ $\Rightarrow$ $\because$ $(a+a)\le a$. —(i)

$\Rightarrow$ $\#(C)\le\#(A\cup B)$ $\Rightarrow$ $a\le(a+a)$ —(ii)

From (i) and (ii)

$a+a=a$ $\Rightarrow$ $\delta+\delta=\delta$.

Case II: $(c+c=c)$

$c=$ cardinal of uncountable.

If $\delta=c$

let $\#(D)=c$, $\#(E)=c$, $\#(F)=c$ and $D\cap E=\phi$

$\#(D+E)=c+c$

$\#(F)=c$.

Date:

and we have To show That $f: D\cup E \to F$ and $g: F \to D\cup E$ are injectives.

Then $\Rightarrow$ $\#(D\cup E) \leq \#(F) \Rightarrow c+c \leq c$ — (i)

$\Rightarrow$ $\#(F) \leq \#(D\cup E) \Rightarrow c \leq c+c$ — (ii)

From (i), (ii)

$\Rightarrow c+c = c \Rightarrow \delta+\delta = \delta$ (proved)

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→ Partial Order Relation:

A relation "R" on a set X is said to be partial order relation if it is

(i) reflexive (ii) Anti-Symmetric (iii) Transitive.

→ Reflexive:

For any $a \in S$, aRa

For any $a \in S$, $a \leq a$.

→ Anti-symmetric:

for any $a, b \in S$

if $a \leq b$, $b \leq a$ then $a = b$.

or if aRb and bRa then $a = b$.

→ Transitive:

For any $a, b, c \in S$

if aRb and bRc then aRc.

if $a \leq b$ and $b \leq c$ then $a \leq c$.

Then "$\preceq$" is said to be partial order or partial order relation and the set S is called partially ordered. It is denoted by $(S, \preceq)$.

Example # 01:

If $X = N$ or any subset of N or "$\leq$" (Natural order relation)

Sol:

$N \times N = \{(a,b) : a, b \in N\}$

$$N \times N = \begin{Bmatrix} (1,1), & (1,2), & (1,3), \cdots \\ (2,1), & (2,2), & (2,3), \cdots \\ (3,1), & (3,2), & (3,3) \cdots \\ \vdots & \vdots & \end{Bmatrix}$$

$R_1 = \{(a', b') ; a' \leq b'\}$

$$= \begin{Bmatrix} (1,1), & (1,2), & (1,3), \cdots \\ (2,2), & (2,3), & (2,4) \cdots \\ (3,3), & (3,4), & (3,5) \cdots \\ \vdots & \vdots & \vdots \end{Bmatrix}$$

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→ Reflexive:

For any $a \in N$, $a \leq a$ is True because any number is always less than or equal to itself.

Thus $a \leq a$.

→ Anti-symmetric:

let $(a,b) \in R$, and $(b,a) \in R$, for some $a, b \in N$ This means.

$\Rightarrow$ $a \leq b$ and $b \leq a$.

Since $a \leq b$ and $b \leq a$.

$\Rightarrow$ $a = b$.

Thus if $a \leq b$ and $b \leq a$ then $a = b$.

→ Transitive:

If $(a,b) \in R$, and $(b,c) \in R$, for some $a,b,c \in N$

$\Rightarrow$ $a \leq b$ and $b \leq c$

$\Rightarrow$ $a \leq b \leq c$.

Therefore by transitive of the order relation

$\Rightarrow$ $a \leq c$ which means $(a,c) \in R$, Then if $a \preceq b$ and $b \preceq c$ then $a \preceq c$.

Hence R_1 is partially ordered relation on $N \times N$.

→ Example #02:-

Let R be relation in the natural number N defined by "x is a multiple of y".

Sol:-

Let $X = N$

$R_1 : xRy \Leftrightarrow x$ is multiple of y.

$R_1 = \{(x,y) : ty = x\}$.

→ Reflexive :-

For any $x \in X$, $x \preceq x$ must hold.

if x is multiple of x. (any number is multiple of itself).

$\Rightarrow \quad tx = x$.

which is true for $t = 1$ (any number multiplied by 1 equals itself)

$\Rightarrow \quad 1 \cdot x = x$.

Hence xR_1x or $x \preceq x$ is true for all x.

$\Rightarrow$ R_1 is reflexive.

→ Anti-Symmetric :-

If $x, y \in X$.

If xRy and yRx both holds then

x is multiple of y and y is multiple of x.

$x = ty$ — (1) and $y = kx$ — (2) (by using def. if x is multiple of y then $\exists$ integers t, k.

put value of y (2) in eq (1)

$\Rightarrow x = t(kx)$

$\Rightarrow x = (tk)x$ where tk is integer. which means $kt = 1$.

$\Rightarrow tk = 1$ $k = 1, t = 1$

Therefore $x = y$.

(If xRy and yRx then $x = y$).

In case of $x = (tk)x$ if we take $tk = 1$ then left and right side will be equal otherwise can't be equal. where tk is integer and 1 is also integer. we can't take $tk = -1$ because the universal set is positive natural no. so $tk = 1$ by putting 1 in $x = (tk)x$ then $x = y$.

→ Transitive :-

For any $x, y, z \in X$.

If xRy and yRz both hold then

x is multiple of y and y is multiple of z

$x = ty$ — (1) and $y = kz$ — (2) $\because \exists$ some integers $t, k \in Z$.

Put eq (2) in (1) we get

$\Rightarrow$ $x = t(kz)$

$\Rightarrow$ $x = (tk)z$ $\qquad tk = l$. (multiplication of 2 integers is also Integer)

$\Rightarrow$ $x = lz$ $\qquad l \in z$.

$\Rightarrow$ x is multiple of z.

Hence $x R_1 z$. $\Rightarrow$ $x \preceq z$.

(If $x \preceq y$ and $y \preceq z$ then $x \preceq z$).

Hence R_1 is partially ordered relation on $X = N$.

→ **Example :-**

let A be a family of sets then the relation in A defined by "x is a subset of y".

Sol:-

let A be a family of sets

$R_1 : xRy \Leftrightarrow x$ is a subset of y.

$R_1 = \{(x,y) : x \subseteq y\}$.

→ Reflexive :-

For any set $x \in A$

Then xRx hold.

" x is always subset of itself "

$\Rightarrow$ $x \subseteq x$. which is True bcz

(every set is a subset of itself) by def of subset.

$\Rightarrow$ $x \preceq x$.

→ Anti-symmetric :-

If xRy and yRx for sets $x, y \in A$.

Then x is a subset of y and y is a subset of x.

$\Rightarrow$ $x \subseteq y$ and $y \subseteq x$

(every element of x is also in y) (every element of y is also in x).

if $x \subseteq y$ and $y \subseteq x$

$\Rightarrow$ $x = y$

(If $x \preceq y$ and $y \preceq x$ then $x = y$).

→ Transitive :-

If xRy and yRz for sets $x, y, z \in A$

$\Rightarrow$ x is subset of y and y is subset of z.

$x \subseteq y$ and $y \subseteq z$.

by the def of subsets

if $x \subseteq y$ and $y \subseteq z$. $\Rightarrow$ $x \subseteq y \subseteq z$ (by using Transitive ordered relation).

Then $x \subseteq z$. (x is subset of z)

$xRz \Rightarrow x \preceq z$.

(If $x \preceq y$ and $y \preceq z$ then $x \preceq z$)

Hence R. is partially ordered relation on A.

Example # 04:-

let $W = \{a, b, c, d, e\}$ Then the diagram

$$\begin{tikzcd}
a \arrow[r] & b \\
d \arrow[u] \arrow[r] & c \arrow[u] \\
& e \arrow[u]
\end{tikzcd}$$

xRy if and only if $x = y$ or one can go along indicated arrow.

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Sol:-

$W = \{a, b, c, d, e\}$

R' ; $xRy \iff x = y$

$R' = \{(a,a), (b,b), (c,c), (d,d), (e,e), (e,c), (c,b), (e,b), (d,c), (d,b),$
$(d,a), (a,b), (d,b)\}$.

* → Reflexive :-

For every element $x \in W$, xRx must hold.
by checking the line diagram we see that every element x can reach itself by following the arrows. $(a \to a, b \to b, c \leftrightarrow c, --)$.
So $xRx \Rightarrow x \alpha x$. holds for all $x \in W$.

→ Anti-Symmetric :-

If xRy and yRx hold for distinct elements $x, y \in W$.
Then $x = y$.

In this line diagram, no two distinct elements have arrows pointing both ways between them.
So $x = y$.

e.g: if $c \to b$ (c,b) $c \preceq b$ but (b,c) are not present in R'. so $c=b$.

Hence $x \preceq y$ and $y \preceq x$ then $x=y$.

→ Transitive :-

For any $x,y,z \in W$

if xRy and yRz hold then xRz must hold.

From the line diagram, following the arrows shows that if there's a path from x to y and from y to z then there is always a path from x to z.

if $x \preceq y$ and $y \preceq z$ then $x \preceq z$.

e.g $(e,c),(c,b)$. $e \preceq c$ and $c \preceq b$ then $e \preceq b$. also present in R'.

Hence R' is partial order relation on W.

→ Notation & terminology:-

(i) $xRy \Rightarrow x \preceq y$ read it as "x precedes y" or y dominent to x.

(ii) $x \prec y$ means x strictly precedes y.

(iii) $x \not\preceq y$ means x does not precedes y.

(iv) $x \not\prec y$ means x does not strictly preceeds y.

(v) If R is a partial order relation R^{-1} is also partial order relation.

(vi) (A, R) is partial order set.
Partial order relation on A.

→ Total Order relation :

A partial Order on a set X is said to be total order if for every $a, b \in X$ either $a \preceq b$ or $b \preceq a$.

OR

A partially ordered set is called totally ordered set if any two elements of the set are comparable.

→ Examples :-

$X = N$

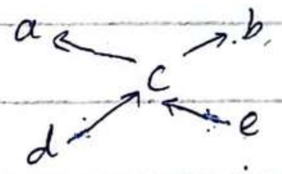
1) $R_1 = \{(a, b) : a | b\}$

2) $R_2 = \{(a,b) : a \leq b\}$.

R_1 is not totally ordered since $3 \nmid 2$ $3 \not\leq 2$ and $2 \nmid 3$ $2 \not\leq 3$.
(2, 3 are not comparable).

but R_2 is totally ordered relation: because any two elements of N are comparable.

3) $W = \{a, b, c, d, e\}$.



$R' = \{(a,a), (b,b), (c,c), (d,d), (e,e), (d,c), (c,b), (d,b), (e,c), (c,a), (e,a),$
$(d,a), (e,b)\}$

each of the set $\{a,c,d\}$, $\{b,d\}$, $\{b,c,e\}$, $\{a,c,e\}$ and $\{a,c\}$ are totally ordered subset.

The sets $\{a,b,c\}$ and $\{d,e\}$ are not totally ordered.
because $d \nleq e$, $e \nleq d$ and $a \nleq b$, $b \nleq a$.
So W is not totally ordered relation.

4) A = collection of all subset of $\{1,2,3\}$. "$\subseteq$" relation.

power sets of $\{1,2,3\}$.

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$.

A is not total order relation.

because $\{1\} \nsubseteq \{2\}$, $\{2\} \nsubseteq \{1\}$.

$\{1\} \nsubseteq \{3\}$, $\{3\} \nsubseteq \{1\}$.

Power sets of A are not relate to each others. So A is not totally ordered relation.

Note:

An ordered set is either partially ordered set or total ordered set.

→ Remark:-

A and B are totally ordered sets then $A \times B$ is totally ordered set under the relation $(a,b) \prec (a',b')$.

↳ strictly precedes

if $\begin{cases} a < a' \\ \text{or } a = a' \text{ then } b < b' \end{cases}$

lexicographical order. and show that totally ordered relation.

Proof:-

Given that A and B are totally ordered sets then we can write as "$(A, \leq_1)$, $(B, \leq_2)$" then $A \times B$ is totally ordered set $(A \times B, \leq)$ under the relation $(a, b) \propto (a', b')$ if $a \propto a'$ or $a = a'$ then $b \propto b'$.

→ Necessary condition:

let $(A, \leq_1)$ and $(B, \leq_2)$ be totally ordered sets.

let $(a, a'), (b, b') \in A \times B$.

The following cases are to be examined

(1) $a \neq b$

As "$\propto_1$" is a totally ordering it follows that either $a \propto b$ or $b \propto a$ ∵ by def of totall order relation

Thus by def of lexicographical order on $A \times B$.

either $(a, a') \propto_1 (b, b')$ or $(b, b') \propto_1 (a, a')$

(2) $a = b$

As "$\propto_2$" is a total ordering it follows that one of the following holds

$a' \propto_2 b'$ in which case $(a, a') \propto_1 (b, b')$

$b' \propto_2 a'$ in which case $(b, b') \propto_1 (a, a')$

$a = b$ in which case $(a,a') = (b,b')$

by def of lexicographic order on $A \times B$. in all cases (a,a') is comparable with (b,b'), That is α_1 is a totall ordering on $A \times B$.

→ Sufficient condition:-

let $(A \times B, \alpha_1)$ be a totally ordered set.

There are two cases

(1) let $(a,a'),(b,b') \in A \times B$ such That $a = a'$

As "α_1" is a total ordering it follows That one of the following holds:

$(a,a') \alpha_1 (b,b')$ in which case $a' \alpha_2 b'$

$(b,b') \alpha_1 (a,a')$ in which case $b' \alpha_2 a'$

$(a,a') = (b,b')$ in which case $a = b$.

by def of lexicographical order on $A \times B$. As above holds for all $a,b' \in B$ it follows that α_2 is a total ordering on B.

(2) let $(a,a'),(b,b') \in A \times B$ such That $a \neq a'$

As "α_2" is a total ordering it follows That one of the following holds

$(a,a') \alpha_2 (b,b')$ in which case $a \alpha_1 b$

$(b,b') \alpha_2 (a,a')$ in which case $b \alpha_1 a$ ∵ by def of lexicographical order on $A \times B$.

$a,b \in A$ it follows That α_1 is totall ordering on A.

Definitions:

let "A" be an ordered set. Then

means ky wo set. Partial ordered ya total order ho skta hai.

First Element :-

An element $a \in A$ is said to be first element of A if for every $x \in A$ $a \preceq x$

That is if a precedes every element in A.

Last Element :-

An element $b \in A$ is said to be last element of A if for every $x \in A$ $x \preceq b$

That is if b dominates every element belonging to A.

Minimal Element :-

An element $m \in A$ is said to be minimal element of A if $x \preceq m \Rightarrow x = m$.

That is if there is no element in A which strictly precedes m.

Js mai koi element left side sy precedes na kry agr kry to wo equal hoga

→ Maximal Element :-

An element $l \in A$ is said to be maximal element of A if $l \preceq x \Rightarrow l = x$.

that is if there is no element in A which dominents l.

→ Example # 01:-

$A = \{1, 2, 3, 4, \ldots\ldots\}$. Natural relation "$\leq$".

$\Rightarrow$ 1 is first element of A. bcz 1 is less than or equal to all elements. which is 1 precedes every element in A.

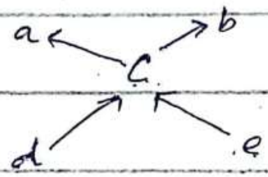
$\Rightarrow$ A has no last element.

$\Rightarrow$ 1 is minimal element of A.

$\Rightarrow$ A has no maximal element.

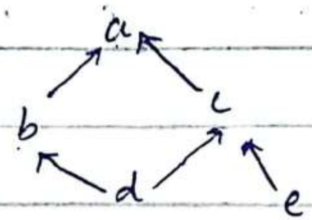
→ Example # 02:-

$W = \{a, b, c, d, e\}$.



(i) W has no first and last elements.

(ii) d, e are minimal elements and a, b are maximal element



(i) W contains no first element. Since no element of W precedes every element of W.

(ii) a is the last element of W.

(iii) d, e are two minimal elements of W. because no element of W strictly precedes d, e.

(iv) a is the only maximal element of W. because no element of W strictly dominent a.

→ Theorem :-

In an ordered set first and last elements are unique.

proof :-

(a) In an ordered set first element is unique.

let A be an ordered set.

let a and a' are Two First elements of A.

As a is first element of A.

$\Rightarrow \quad a \preceq x \quad , \forall\ x \in A$

In particular $a' \in A$

$a \preceq a' \quad — (1)$

Also a' is first element of A.

$\Rightarrow \quad a' \preceq x, \quad \forall\ x \in A$

In particular $a \in A$

$\Rightarrow \quad a' \preceq a \quad — (2)$

From (1) and (2)

$\Rightarrow \quad a \preceq a' \preceq a$ $\quad \because$ by using def of anti-symmetric
if $x \preceq y$ and $y \preceq x$ then $x = y$

$\Rightarrow \quad a = a'$.

Thus first element of ordered set is Unique (if exist.)

(b) In an ordered set last element is Unique.

let A be an ordered set.

let b and b' are Two last elements of A.

As b is last element of A.

$\Rightarrow \quad x \preceq b \quad , \forall\, x \in A$

In particular $b' \in A$

$\Rightarrow \quad b' \preceq b \quad \text{— (1)}$

Also b' is last element of A.

$\Rightarrow \quad x \preceq b' \quad , \forall\, x \in A.$

In particular $b \in A$

$\Rightarrow \quad b \preceq b' \quad \text{— (2)}$

From (1) and (2)

$\Rightarrow \quad b' \preceq b \preceq b'$ ∵ by using def of anti-symmetric if $x \preceq y$ and $y \preceq x$ then $x = y$

$\Rightarrow \quad b' = b.$

Thus last element of ordered set is Unique (If exist).

(a) **Theorem :-**

In an ordered set if "a" is the first element Then "a" is only minimal element.

Proof:-

let A be an ordered set.

let "a" is the first element of A.

$\Rightarrow a \preceq x \quad , \forall x \in A.$

Then "a" is minimal element of A.

let "a'" is another minimal element of A.

$\Rightarrow$ As "a" is first element of A.

$\Rightarrow a \preceq a'$

by def. of minimality of a'. Since a' is the minimal element of A. so,

if $a \preceq a'$ then $a = a'$

(because no element precedes the minimal element).

(b) Theorem:-

In an ordered set if "b" is the last element then "b" is the only maximal element.

proof:-

let A be an ordered set.

let "b" is the last element of A.

$\Rightarrow \quad x \preceq b \quad , \forall \; x \in A.$

then "b" is maximal element of A.

let "b′" is another maximal element of A.

$\Rightarrow$ As "b" is last element of A.

$\Rightarrow \quad b' \preceq b$

by def of maximality of b′.

since b′ is the maximal element of A then

if $b' \preceq b$ then $b' = b$

(because no element dominent the maximal element).

(a) **Theorem :-**

In a total ordered set no two minimal elements exist.

proof :-

let A be a Totally ordered set.

let a and a′ are two minimal elements of A.

As $a, a' \in A$ (A is totally ordered set, so any two elements of A must be comparable)

$\Rightarrow a \preceq a'$ or $a' \preceq a$.

Case I:-

if $a \preceq a'$

by def of minimal element of A so a' is the minimal element.

$\Rightarrow a \preceq a'$ then $a = a'$.

$\because$ If no element precedes from left side, if precedes then both elements will be equal to each other.

Case II:-

if $a' \preceq a$

by def of minimal element of A so a is the minimal element.

$a' \preceq a$ then $a' = a$.

Thus the minimal element is unique.

→ Theorem:-

In a total ordered set no two maximal elements exist.

proof:-

let A be a totally ordered set.

let b and b' are two maximal elements of A.

A) $b, b' \in A$ (A is Totally ordered set so any Two elements of A must be comparable).

$\Rightarrow$ $b \preceq b'$ or $b' \preceq b$.

Case I:-

$b \preceq b'$

by The def of maximal element of A

Since b is The maximal element of A, so

$b \preceq b'$ Then $b = b'$.

Case II:-

$b' \preceq b$

by The def of maximal element of A

Since b' is The maximal element of A, so

$b' \preceq b$ Then $b' = b$

Thus the maximal element is Unique.

(a) $\rightarrow$ **Theorem :-**

Every finite partially ordered set has at least one minimal element.

Proof:-

Let A be a finite partially ordered set of order n.
Let $a_1 \in A$.
In A, if no other element precedes a.
Then "a_1" is minimal element.
If "a_1" is not minimal element then $\exists\ a_2 \in A$.
Such that $a_2 \prec a_1$.
If "a_2" is not minimal element then $\exists\ a_3 \in A$.
Such that $a_3 \prec a_2$
If a_3 is not minimal element then $\exists\ a_4 \in A$
Such that $a_4 \preceq a_3$ $a_n \preceq a_{n-1} \preceq a_{i+1} \preceq a_i \preceq \cdots \preceq a_3 \preceq a_2 \preceq a_1$.
Continuing this process, Since A is finite.
Then $\exists$ an element $a_n \in A$ such that
no element of A precedes a_n. (i.e $a_n \preceq x \Rightarrow a_n = x$.)
Thus a_n is minimal of A.

Infinite case "$\leq$"
$Z^- = \{-1, -2, -3, \dots\}$
partial ordered set hai (ysku minimal element nhi milta
$N^+ = \{1, 2, 3, \dots\}$
Partial ordered hai iysku maximal nhi milta.

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(b) **Theorem:-**

Every finite partially ordered set has at least one maximal element.

Proof:-

let A be a finite partially ordered set of order n.
let $b_1 \in A$,
In A, If no other element dominent b.
Then b_1 is maximal element.
If b_1 is not maximal element. Then $\exists$ $b_2 \in A$.
Such that $b_1 \alpha b_2$
If b_2 is not a maximal element then $\exists$ $b_3 \in A$
such that $b_2 \alpha b_3$.
If b_3 is not a maximal element then $\exists$ $b_4 \in A$
Such that $b_3 \alpha b_4$ $\quad b_1 \alpha b_2 \alpha b_3 \alpha b_4 \dots \alpha b_i \alpha b_{i-1} \alpha b_{n-1} \alpha b_n$
continuing this process, Since A is finite then $\exists$
an element $b_n \in A$ such that

no element of A dominent b_n.

Thus b_n is the maximal element of A.

→ Now we have to check That if set is finite then can we find First and last element of set.

• Zruri nhi minimal element First ho lykn First element minimal ho skta hai.

• Zruri nhi maximal element last ho lykn last element maximal ho skta hai.

Example :-

let $X = \{2,3,4,5,6\}$ is finite partial order relation on $R_1 = \{(x,y) : x \mid y\}$.

So X have no First and last element but have minimal or maximal elements

(1) Here 2 is minimal element of X. because no element of X strictly preceds 2. As (2 does not divide 3 and 5) ($3 \not\preceq 2$, $5 \not\preceq 2$.) ∵ "$x \preceq a$. $x \in A$"

(2) Here 6 is maximal element of X because no element of X strictly dominate 6. (6 does not divide 4 and 5). ($6 \not\preceq 4$, $6 \not\preceq 5$) ∵ "$b \preceq x : x \in A$"

Now we need to find That first and last element exist in X.

(3) by def of First element : ($a \preceq x$, $x \in A$) (if a preceds every element in A).
$\Rightarrow 2 \not\preceq 3$, $2 \nmid 3$, $2 \not\preceq 5$, $2 \nmid 5$, $2 \preceq 4$ $2/4$, $2 \preceq 6$ $2/6$ (2 does not divide 3 and 5)
it means That 2 is not First element. In X no element is First element bcz no one can preceds element of X.

(4) by def of last element ($x \preceq b$, $x \in A$) (if b dominate every element in A
$6 \nmid 5$ $5 \not\preceq 6$, $6 \nmid 4$, $4 \not\preceq 6$ (it means That no element in X That is last element

→ Lower and upper bounds:

let A be an ordered set and $\phi \neq B \subseteq A$ then an element

(i) $l \in A$ is said to be lower bound of B, if for all $x \in B$,

$$l \preccurlyeq x \quad \forall\ x \in B$$

(ii) $u \in A$ is said to be upper bound of B then $x \preccurlyeq u$, $x \in B$

→ Example:-

let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be ordered set on relation "$\leq$" (Natural order relation)

Sol:-

(i) let $B = \{2, 5, 6\}$ be a subset of A.

Now $6, 7, 8, 9$ are upper bounds of B in A. because 6 dominate every element of B. i.e $2 \preccurlyeq 6$, $5 \preccurlyeq 6$ and $6 \preccurlyeq 6$., $7, 8, 9$ dominates every element of B. $2 \preccurlyeq 7$, $5 \preccurlyeq 8$, $6 \preccurlyeq 9$.

(ii) Now $1, 2$ are lower bounds of B in A. because 1 precedes every element of B. i.e $1 \preccurlyeq 2$, $1 \preccurlyeq 5$, $1 \preccurlyeq 6$. and 2 is also precedes every element of B. $2 \preccurlyeq 2$, $2 \preccurlyeq 5$ and $2 \preccurlyeq 6$.

→ Supremum and Infimum:-

• Infimum:-

Let A be an ordered set and $B \subseteq A$ then an element $l \in A$ is greatest lower bound (Infimum) of B if

(i) l is lower bound of B. $(\because l \preceq x \ \forall \ x \in B)$

(ii) If l_1 is any other lower bound of B then $l_1 \preceq l$.

(If a lower bound of B dominate every other lower bound of B then it is Infimum and denoted by inf(B))

• Supremum:-

Let A be an ordered set and $B \subseteq A$ then an element $u \in A$ is least upper bound (Supremum) of B if

(i) u is upper bound of B. $(x \preceq u, \ \forall \ x \in B)$.

(ii) If u_1 is any other upper bound of B then $u \preceq u_1$

(If an upper bound of B precedes every other upper bound of B then it is Supremum and denoted by $Sup(B)$.

→ Example :-

Let $A = \{1,2,3,4,5,6,7,8,9\}$ be ordered set on relation "$\leq$"

Sol:-

let $B = \{2,5,6\}$ be a subset of A.

→ Supremum:

6, 7, 8, 9 are upper bounds of B in A, and we can find an element from upper bounds that can precede any other upper bounds of B. That is 6. $\because u \preceq u_1$

$6 \preceq 6$, $6 \preceq 7$, $6 \preceq 8$ and $6 \preceq 9$. then (bcz 6 is less than or equal to all upper bounds)

$Sup(B) = 6$.

So 6 is the least upper bound of B.

→ Infimum:-

1, 2 are lower bounds of B in A, and also we can find an element from lower bounds that can dominate any other lower bounds of B. That is 2.

$1 \alpha 2$, $2 \alpha 2$. Then $\because l_1 \alpha l$

$Inf(B) = 2$. (α: 1 is related to 2. $1 \le 2$ and $2 \le 2$)

So 2 is the greatest lower bound of B.

→ Bounded below, bounded above and bounded sets:-

let A be an ordered set and $B \subseteq A$ Then

(i) B is said to be bounded below if B has an Lower bound.

(ii) B is said to be bounded above if B has an upper bound.

(iii) B is said to be bounded if B is both bounded above and bounded below.

→ Example # 01:-

Let $A = N$ is an ordered set and order is natural order.

Sol:-

let $A = N = \{1, 2, 3, 4, \ldots\ldots\}$ be an order set.

and let $B = \{4, 5, 6, 7\}$ is subset of A.

and we have to Find upper and lower bounds, and Infimum or Supremum of set B. in A. that satisfy the Natural order relation.

1) upper bound :-

7, 8, 9, 10, 11, 12...... are all upper bounds of B in A. because these upper bounds dominate every element of B like 8 dominate every element of B. according to def of upper bounds. $x \preceq u \;\forall\; x \in B$.

$\Rightarrow$ $4 \preceq 8$, $5 \preceq 8$, $6 \preceq 8$, $7 \preceq 8$ and satisfy the relation.

bcz 4, 5, 6, 7 are all less then to 8.

So 7, 8, 9, 10, ... are all upper bounds of B in A.

2) Lower bound :-

1, 2, 3, 4 are Lower bounds of B in A. because these lower bounds precede every element of B. like 1, 2 precede every element of B. by using def of lower bound. $l \preceq x \;\forall\; x \in B$.

$\Rightarrow$ $1 \preceq 4$, $1 \preceq 5$, $1 \preceq 6$, $1 \preceq 7$

$\Rightarrow$ $2 \preceq 4$, $2 \preceq 5$, $2 \preceq 6$, $2 \preceq 7$. and all satisfy the relation.

bcz 1 or 2 are less than to set B.

So. 1,2,3,4 are lower bounds of B in A. That Satisfy the relation.

3) Supremum:-

7, 8, 9, 10, 11, ... are all upper bounds of B. in A, and we can find an element from upper bounds that can precedes any other upper bounds of B. That is 7. and also Satisfy the natural order relation "$\leq$".

by using def of Supremum. $u \preceq u$, $x \preceq u$ $\forall$ $u \in A$ and $x \in B$.

$\Rightarrow 7 \preceq 7$, $7 \preceq 8$, $7 \preceq 9$, $7 \preceq 10$, $7 \preceq 11$, and so on.

7 preceding all upper bounds of B. and also satisfy relation because 7 is less then or equal to all upper bounds.

So $Sup(B) = 7$.

So 7 is least upper bound. of B.

4) Infimum:-

1,2,3,4 are lower bounds of B in A and we can find an element from lower bounds that can dominate any other

• Infimum or Supremum set B mai no bhi skta hai or nhi bh no skta.

lower bounds of B. That is 4. and also satisfy the natural order relation "$\leq$".

by using def of Infimum. $l_1 \alpha\, l$ $\forall$ $l \in A$. $l \alpha x$ $\forall$ $x \in B$.

$\Rightarrow$ $1 \alpha 4$, $2 \alpha 4$, $3 \alpha 4$, $4 \alpha 4$.

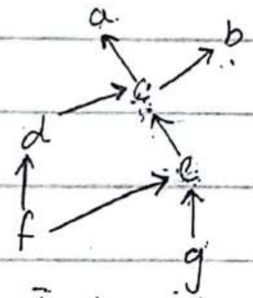
4 dominate to all lower bounds of B. and also satisfy the relation. because 1, 2, 3, 4 are all less then or equal to 4.)

So then $\text{Inf}(B) = 4$.

So 4 is greatest lower bound of B.

→ Example # 02:-

Let $V = \{a, b, c, d, e, f, g\}$ is partial order set then the line diagram.



Sol:-

let $V = \{a, b, c, d, e, f, g\}$ be partial order set. and

$W = \{c, d, e\}$ be a subset of V.

and we have to find lower & upper bounds and Infimum and Supremum.

$R' = \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(g,g),$
$(g,e),(e,c),(g,c), \ldots, (c,a),(e,a),(c,b),(e,b),(f,e),(f,c),$
$(f,d),(d,c),(d,b),(d,a),(c,b)\}.$

(1) upper bound:-

From line diagram, we can see that a, b, c are upper bounds of W in V. These upper bound dominate every element of W.

By using def of upper bound " $x \preceq u \ \forall x \in B$ ".

$\Rightarrow c \preceq a$, $d \preceq a$, $e \preceq a$

$\Rightarrow c \preceq b$, $d \preceq b$, $e \preceq b$

$\Rightarrow c \preceq c$, $d \preceq c$, $e \preceq c$.

$\because R'$ relation mein yeh element precede or Dominate kr rhy hain

its mean upper bounds (a, b, c) dominate every element of W.

So a, b, c are upper bounds of W in V.

(2) Lower bound :-

From line Diagram, we can see that f is only lower bound of w in V. This lower bound precede every element of w. by using the def of lower bound "$l \preceq x \ \forall x \in B$."

$\Rightarrow f \preceq c$, $f \preceq d$, $f \preceq e$

So f is only lower bound of W in V because f precede every element of W.

Note that e, g are not lower bounds of W.

$\Rightarrow g \preceq c$ True but $g \npreceq d$

$\Rightarrow g$ does not precede d. So g or d are not comparable

$\Rightarrow e \preceq c$ but $e \npreceq d$.

$\Rightarrow e$ does not precede d. So e or d are not comparable.

(3) → Supremum:-

a, b, c are upper bounds of w in V. and we can find an element from upper bounds that can precedes any other upper bounds. That is c.

By using def of Supremum "$u \preceq u_1$, $u \in A$."

$\Rightarrow c \preceq a$, $c \preceq b$, $c \preceq c$. $\because$ From R'

is mean: c precedes other upper bounds of W.

But $a \not\preceq b$ and $b \not\preceq a$.

because a and b are not relate to each other. and a and b are not comparable. So

$Sup(W) = c$.

c is only least upper bound of W.

(4) → Infimum:-

As f is only lower bound of W in V. So f is Infimum of W.

So by def of Infimum "$l \preceq l$ $l \in A$"

$\Rightarrow f \preceq f$ (So every element precedes to itself).

$\Rightarrow Inf(W) = f$.

So f is greatest lower bound of W in V.

⇒ W is bounded set because W is bounded below and bounded above.

→ Similar Sets:-

Two ordered set A and B are said to be Similar Sets if there exist a bijective mapping $f: A \to B$ which satisfies the following property.

For $a, a' \in A$

$a \preceq a'$ if and only if $f(a) \preceq f(a')$

→ Theorem :-

$N = \{1, 2, 3, \ldots\}$, "$\leq$"

$E = \{2, 4, 6, \ldots\}$, "$\leq$" (order of N and E can be Same and different).

Show That N and E are Similar sets..

Proof:-

let $f: N \to E$ defined as

$f(n) = 2n$ for $n \in N$.

we have to show That f is bijective.

→ f is one-one:

let $n_1, n_2 \in N$ and

⇒ hm elements ly gy Ju unka orden hoga whi un ki images ka ho To ors order preserve kry gy.

⇒ similar set wo hai. Js mai 2 elements ly unky b/w bijective function mil jay or Ju order preserve kry.

⇒ $a \preceq a'$ $f(a) \preceq f(a')$

↓ ya us order pr preserve kry gi Ju hm A pr define kry gy. ↓ ya us order ko preserve kry ga Ju B pr define kry gy

⇒ agr 2 set order preserve kry or 2 equivalent set Zruri nhi wo similar bi ho..

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$$f(n_1) = f(n_2)$$

$\Rightarrow \quad 2n_1 = 2n_2$ ∵ divided by 2.

$\Rightarrow \quad n_1 = n_2$

$\Rightarrow \quad f$ is one-one

→ **f is onto:-**

Since for each $2n \in E$ then there exist an element $n \in N$ such that $f(n) = 2n$

$\Rightarrow f$ is onto.

$\Rightarrow f$ is bijective.

→ Now we have to check that order is preserved

let $n_1, n_2 \in N$ such that

$$n_1 \preceq n_2$$

$\Leftrightarrow \quad n_1 \leq n_2$

$\Leftrightarrow \quad 2n_1 \leq 2n_2$

$\Leftrightarrow \quad f(n_1) \leq f(n_2)$

$\Leftrightarrow \quad f(n_1) \preceq f(n_2)$

$\Leftrightarrow$ f preserve the order.

$\Rightarrow$ f is Similarity mapping.

$\Rightarrow$ $N \simeq E$.

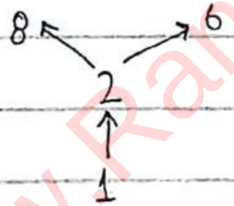
Notation:-

If A is similar to B then we write $A \simeq B$.

Example #01:-

let $V = \{1, 2, 6, 8\}$ be ordered set by "x divides y" and let $W = \{a, b, c, d\}$ be ordered by following diagram.

$R' = \{(1,1), (2,2), (6,6), (8,8), (1,2), (2,6), (1,6), (2,8), (1,8)\}$



digram of V.

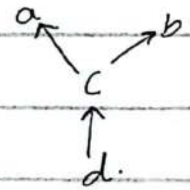
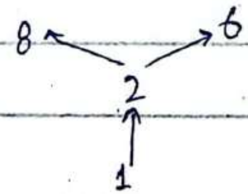


diagram of W.

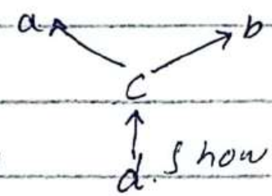
$R_1 = \{(d,d), (c,c), (b,b), (a,a), (d,c), (c,b), (d,b), (c,a), (d,a).\}$

Sol:-

let $V = \{1, 2, 6, 8\}$ be Ordered by "x divides y" and Order by line diagram.



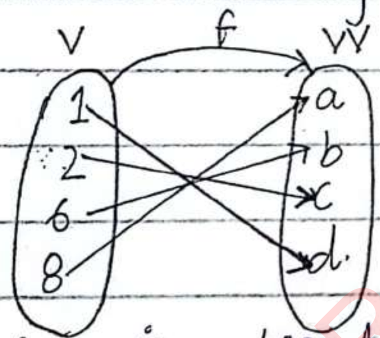
and let $W = \{a, b, c, d\}$ ordered by line diagram



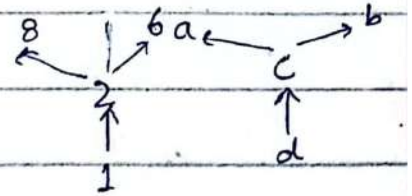
Then we want to show that $V \cong W$.

(i) First we have to find a bijective function b/w V and W.

let $f: V \to W$ is function defined by



by comparision line diagram of both V and W



clearly f is bijective.

Now we have to show that order is preserve.

let $2, 6 \in V$ such that

$\Leftrightarrow \quad 2 \leq 6 \quad \because$ 2 divides 6.

$\Leftrightarrow \quad f(2) \leq f(6)$

$\Leftrightarrow \quad c \leq b$

$\Leftrightarrow$ f preserve the order.

$\Rightarrow$ f is similarity mapping.

$\Rightarrow$ $V \simeq W$.

Theorem:-

If A is a totally ordered set and $B \simeq A$ then B is also totally ordered set.

Proof:-

Given A is totally ordered and $B \simeq A$.

Then $\exists$ $f : B \to A$ which is bijective.

To prove B is totally ordered set.

On contrary, suppose B is not totally ordered

Then $\exists$ $b_1, b_2 \in B$ such that

$\Rightarrow$ $b_1 \not\leq b_2$ and $b_2 \not\leq b_1$

$\Rightarrow$ If A not totally ordered set then $a, b \in A$ such that a and b are not comparable.

As f is similarity mapping then

$\Rightarrow$ $f(b_1) \not\leq f(b_2)$ and $f(b_2) \not\leq f(b_1)$

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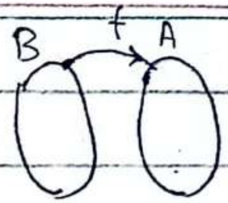

As $f(b_1), f(b_2) \in A$ and A is totally ordered set.

but

$f(b_1) \nleq f(b_2)$

and $f(b_2) \nleq f(b_1)$

$\Rightarrow$ A is not totally ordered set, a contradiction because A is totally ordered so our supposition is wrong.

$\Rightarrow$ B is totally ordered.

Remark:-

If A is similar to B then A is equivalent to B. Converse of this is not true, in general because $N \sim M$ where $M = \{-1, -2, -3, \ldots\}$ by $f(n) = -m$ but $N \not\simeq M$.

proof:-

If $f: A \to B$ is similar mapping then $\exists$ a bijective function who preserve order, So clearly A is equivalent to B.

→ Converse of This Theorem is not True:-

Consider the natural numbers $N = \{1, 2, 3, \ldots\}$ and the negative integer $M = \{-1, -2, -3, \ldots\}$ each ordered by natural order "$\leq$".

To show $N \sim M$ we need to find bijective function.

let $f : N \to M$ is function defined by

$$f(n) = -n \quad , \; n \in N$$

To show f is bijective.

→ f is one-one:-

let $n_1, n_2 \in N$ such that

$\Rightarrow \; f(n_1) = f(n_2)$

$\Rightarrow \; \cancel{-}n_1 = \cancel{-}n_2$

$\Rightarrow \; n_1 = n_2$

$\Rightarrow \; f$ is one-one.

→ f is onto:-

Since for each $-n \in M$ then there exist an element $n \in N$ such that

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$f(n) = -n$

$\Rightarrow$ f is onto.

$\Rightarrow$ f is bijective

$\Rightarrow$ $N \simeq M$.

Then for show $N \simeq M$.

we need to find only f preserves order.
because we already proof that a function $f: N \to M$ is bijective function.

So

If $n_1, n_2 \in N$ such that

$\Leftrightarrow$ $n_1 \preccurlyeq n_2$

$\Leftrightarrow$ $n_1 \leq n_2$

$\Leftrightarrow$ $-n_1 \geq -n_2$

$\Leftrightarrow$ $-n_2 \leq -n_1$

$\Leftrightarrow$ $f(n_2) \leq f(n_1)$

$\Leftrightarrow$ $f(n_2) \preccurlyeq f(n_1)$.

$\Leftrightarrow$ f is not preserve order.

$\Rightarrow$ f is not similarity mapping.

$\Rightarrow$ $N \not\simeq M$.

$\Rightarrow$ If $N \cup M$ then $N \not\simeq M$. (Proved).

ek hm $n_1 \alpha n_2$ ko n_1 precede krwa rhy thy n_2 sy. lykn hmery pass images. $f(n_2) \alpha f(n_1)$ ya ai han. Ju ky condition ko fulfille nhi krrhy.

→ **Theorem:-**

The relation of similarity "$\simeq$" is an equivalence relation.

Proof:-

→ **Reflexive:-**

Define $I: A \to A$ by $I(x) = x$.

clearly I is bijective.

let $x_1, x_2 \in A$ such that

$x_1 \alpha x_2$

$\Leftrightarrow$ $I(x_1) \alpha I(x_2)$

$\Leftrightarrow$ I preserve order.

$\Rightarrow$ I is similarity mapping.

$\Rightarrow A \simeq A$.

$\Rightarrow \simeq$ is reflexive.

→ Symmetric:

let $A \simeq B$.

$\Rightarrow f: A \to B$ be a similarity mapping.

clearly $f^{-1}: B \to A$ is bijective because $f: A \to B$ is bijective.

let $b_1, b_2 \in B$ such that

$b_1 \underset{-}{\propto} b_2$

As $f: A \to B$ is onto.

$\Rightarrow \exists\ a_1, a_2 \in A$ such that

$\Rightarrow \exists\ f(a_1) = b_1$ and $f(a_2) = b_2$

As $b_1 \underset{-}{\propto} b_2 \Rightarrow f(a_1) \underset{-}{\propto} f(a_2)$

$\Rightarrow a_1 \underset{-}{\propto} a_2$

$\therefore f^{-1}: B \to A$

$f^{-1}(b) = a$

$\Rightarrow f^{-1}(b_1) \underset{-}{\propto} f^{-1}(b_2)$

$\Rightarrow f^{-1}$ preserve order

$\Rightarrow B \simeq A$

$\Rightarrow$ $\simeq$ is Symmetric.

Transitive:-

let $A \simeq B$ and $B \simeq C$

$\Rightarrow$ $f: A \to B$ and $g: B \to C$ are Similarity mappings.

$\Rightarrow$ $fog: A \to C$. is bijective.

$\therefore$ $fog(x) = g(f(x))$

let $a_1, a_2 \in A$ such That

$a_1 \preceq a_2$

$\Leftrightarrow$ $f(a_1) \preceq f(a_2)$

$\Leftrightarrow$ $g(f(a_1)) \preceq g(f(a_2))$ $\because$ by apply g on. b.s.

$\Leftrightarrow$ $fog(a_1) \preceq fog(a_2)$

$\Leftrightarrow$ fog preserve order.

$\Rightarrow$ fog is Similarity mapping.

$\Rightarrow$ $A \simeq C$.

$\Rightarrow$ "$\simeq$" is Transitive.

$\Rightarrow$ The relation of Similarity is an equivalence relation.

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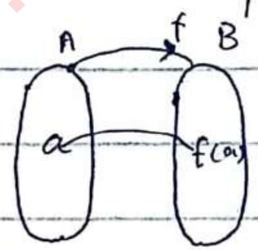
→ Theorem:-

Let $f: A \to B$ is similarity mapping Then "a" is first element of A if and only if $f(a)$ is the first element of B.

proof:-

Given that "a" is first element of A. To prove $f(a)$ is first element of B.



As "a" is first element of A.

$\Rightarrow a \precsim x \quad , \forall x \in A$ $\quad$ ∵ by def of first element

$\Rightarrow f(a) \precsim f(x) \quad , \forall f(x) \in B$ $\quad$ ∵ by apply f on b.s

∵ f is Similarity mapping)

$\Rightarrow f(a)$ is first element of B.

Conversely,

given $f(a)$ is first element of B.

$\Rightarrow f(a) \precsim y \quad , \forall y \in B$

As $y \in B$ and $f: A \to B$ is onto

$\Rightarrow \exists$ an element $x \in A$ such that "$f(x) = y$"

$f(a) \preceq f(x)$, $\forall\ f(x) \in B$

$\Rightarrow a \preceq x$, $\forall\ x \in A$

Because f is Similarity mapping (Bijective + order preserve)

$\Rightarrow a$ is the first element of A.

→ **Theorem:-**

let $f: A \to B$ is a Similarity mapping then b is the last element of if and only if $f(b)$ is last element of B.

proof:-

let b is the last element of A.

$\Rightarrow x \preceq b$, $\forall\ x \in A$ ∵ by def of last element.

$\Rightarrow f(x) \preceq f(b)$ $\forall\ f(x) \in B$ ∵ by apply f on b.s.

(∵ f is Similarity mapping)

$\Rightarrow f(b)$ is the last element of B.

Conversely,

let $f(b)$ be the last element of B.

$\Rightarrow \quad y \preceq f(b) \quad , \forall \; y \in B.$

As $f: A \to B$ is onto

$\Rightarrow \exists \; x \in A$ such that

$f(x) = y$

$\therefore \; f(x) \preceq f(b) \quad , \forall \; f(x) \in B$

$\Rightarrow \quad x \preceq b \quad , \forall \; x \in A$

Because f is similarity mapping

$\Rightarrow$ b is last element of A.

→ Theorem:-

Let $f: A \to B$ is a similarity mapping. Then "a" is the minimal element of A if and only if $f(a)$ is the minimal element of B.

proof:-

let "a" is the minimal element of A.

$\Rightarrow \quad x \not\preceq a$ for any other element "x" of A.

$\Rightarrow \quad f(x) \not\preceq f(a)$ for any other element "$f(a)$" of B.

$\Rightarrow$ $f(a)$ is minimal element of B.

Conversely,

Given $f(a)$ is minimal element of B.

Then $b \not< f(a)$ for any other element b of B.

As $f: A \to B$ is onto then $\exists$ an element $x \in A$ such that "$f(x) = b$"

$f(x) \not< f(a)$ for any other element $f(x)$ of B.

$\Rightarrow$ $x \not< a$ for any other element x of A.

($\because$ f is similarity mapping)

$\Rightarrow$ a is minimal element of A.

Theorem :-

Let $f: A \to B$ is a similarity mapping. Then "b" is the maximal element of A if and only if $f(b)$ is maximal element of B.

Proof:-

Let "b" is the maximal element of A

$\Rightarrow$ $b \not< x$ for any other element "x" of A.

$\Rightarrow$ $f(b) \not< f(x)$ for any other element $f(x)$ of B.

$\Rightarrow$ $f(b)$ is maximal element of B.

Conversely,

given $f(b)$ is maximal element of B.

Then $f(b) \not< y$ for any other element y of B.

As $f: A \to B$ is onto

Then $\exists$ an element $x \in A$ such that

$f(x) = y$

$\Rightarrow$ $f(b) \not< f(x)$ for any other element $f(x)$ of B.

$\Rightarrow$ $b \not< x$ for any other element x of A.

($\because$ f is similarity mapping)

$\Rightarrow$ b is the maximal element of A.

V. imp lecture

→ Well ordered Set :-

An ordered set is said to be well ordered if its every subset has first element.

Example # 01 :-

$N = \{1, 2, 3, \ldots\}$ Under natural order. $(N, \leq)$

Sol:-

let $N = \{1, 2, 3 \ldots\}$ be an ordered set under natural order $(N, \leq)$.

⇒ let $A = \{1, 2, 5\}$ is subset of N.

clearly A has first element. That is 1.

by def of first element $a \leq x \quad \forall \ x \in A$

So 1 precedes every element in A.

$\Rightarrow 1 \leq 1, \ 1 \leq 2, \ 1 \leq 5$ (1 is less then or equal to element of A)

So 1 is first element of A.

So N is well-ordered set.

⇒ let $B = \{100, 105, 200, 250\}$ be subset of N.

and we have to show that B has first element.

So clearly B has first element that is 100.
by def of first element " $a \preceq x \ \forall \ x \in A$."
So 100 precedes every element of B.
$\Rightarrow$ $100 \preceq 100$, $100 \preceq 105$, $100 \preceq 200$, $100 \preceq 250$.
(100 is less then or equal to all elements of B.)
So 100 is first element of B.
which is show that N is well-ordered. because subset of N that is B has first element.

→ Note :-

One of the fundamental properties of N, the set of natural numbers with the natural order is that N. $(N, \leq)$ and every subset of N does have a first element. Then N is well-ordered set.

Example #02:-

let $W = \{a, b, c, d, e\}$ is partial order under the line diagram

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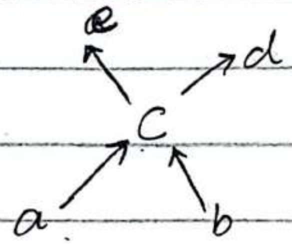
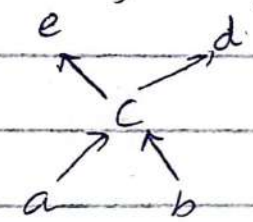


Show that W is well-order set.

Sol:-

let $W = \{a, b, c, d, e\}$ is partially order under the line diagram



→ let $A = \{a, b\}$ is subset of W and we need to show that A has first element.

By def of first element "$a \preceq x \quad \forall x \in A$"

$\Rightarrow a \preceq a$, $a \npreceq b$ (a does not precede b bcz a and b are not related to each other)

So a and b are not comparable.

So A has no first element.

which is shows that W is not well-ordered set.

$R' = \{(a,a),(b,b),(c,c),(d,d),(e,e)$
$(a,c),(c,d),(a,d),(b,c),(b,d),(c,e),(a,e),(b,e)\}.$

257.

→ let $B = \{a, c, d\}$ be subset of W. and we need to show That B has first element.

Then by def of first element "$a \preceq x$, $\forall x \in A$"

So $a \preceq a$, $a \preceq c$, $a \preceq d$

clearly a precedes every element of B.

Then a is first element of B.

* ⇒ W is not wellordered set. because we can't find the first element in every subset of W.

→ **Theorem :-**

(i) Every well ordered set is totally ordered.

(ii) Every subset of a well-ordered is well-ordered.

proof :-

~(i)~

let A be a well-ordered set.

To prove A is totally ordered set.

let $a, b \in A$

$\Rightarrow \{a,b\}$ is Subset of A. , As A is well ordered.

$\Rightarrow \{a,b\}$ has first element.

$\Rightarrow$ Thus either a is first or b is first element of $\{a,b\}$.

$\Rightarrow a \preceq b$ or $b \preceq a$.

$\Rightarrow A$ is totally ordered because a and b are comparable.

Hence A is totally ordered.

$\Rightarrow$ Every well ordered set is totally oredred.

~ (ii) ~

let A be a well-ordered set

and let $B \subseteq A$.

let $C \subseteq B$. As $B \subseteq A$

$\Rightarrow C \subseteq B \subseteq A$

$\Rightarrow C \subseteq A$

Since A is well ordered set and $C \subseteq A$. So

$\Rightarrow$ C has first element.

$\Rightarrow$ B is well-ordered set.

$\Rightarrow$ Every subset of well ordered set is well-ordered.

→ Principal of Mathematical Induction:-

Let $S \subseteq N$ with the properties:

(i) $1 \in S$

(ii) For $n \in S \Rightarrow n+1 \in S$ Then $S = N$

Rough Note:-

Let $N = \{1, 2, 3, \ldots\}$ and $S \subseteq N$

find $S = N$

(i) $1 \in S$

Then by (ii), $n+1 \in S$. For $n \in S$.

$\Rightarrow 1+1 = 2 \in S$.

Then by (ii), $n+1 \in S$ for $n \in S$.

$\Rightarrow 2+1 = 3 \in S$.

Then by (ii), $n+1 \in S$ for $n \in S$.

$\Rightarrow 3+1 = 4 \in S$.

$\vdots \quad \vdots$

its means every numbers of N is also belongs in S. $\therefore S \subseteq N$.

For $n \in S \Rightarrow n+1 \in S$.

Then $S = N$.

$\{1, 2, 3, \ldots\} = \{1, 2, 3, \ldots\}$.

→ Initial Segment:-

let A be a well ordered set and $a \in A$ then initial Segment of "a" is denoted and defined as

$S(a) = \{x : x \in A \text{ and } x \prec a\}$.

S(a) mai wo tmam elements ay gy A key elements strictly precede krg a ko.

→ Example # 01:-

let $N = \{1, 2, 3, \ldots\ldots\}$ Order Under Natural order. $(N, \leq)$.

Sol:-

let $N = \{1, 2, 3, \ldots\}$ be ordered set and well-ordered set under natural order $(N, \leq)$.

by using def of Initial Segment. $\because a \in A$ $S(a) = \{x : x \in A \text{ and } x \prec a\}$.

→ let $5 \in N$ then Initial Segment of 5 is

$\because$ 5 strictly dominates to 1, 2, 3, 4.

$\therefore$ $1 \prec 5$, $2 \prec 5$, $3 \prec 5$, $4 \prec 5$

$S(5) = \{1, 2, 3, 4\}$.

→ let $100 \in N$ Then Initial Segment of 100 is

$\because$ 100 strictly dominates to $1, 2, 3, \ldots\ldots, 99$

$\because$ $1 \alpha 100,\ 2 \alpha 100,\ 3 \alpha 100,\ \ldots\ 99 \alpha 100$

The initial segment of 100

$S(100) = \{1, 2, 3, \ldots, 99\}.$

→ Example #02:-

let $A = \{1, 3, 5, 7, \ldots, 2, 4, 6, \ldots\}$ be well ordered set.

let $3 \in A$ then Initial segment of 3

$S(3) = \{1\}$

because 3 strictly dominate 1. $1 \alpha 3$.

A can be written as
$A = \{a_1, a_2, a_3, \ldots b_1, b_2, \ldots$
hmy nhi pta ky A pr kunsa order define hai

let $2 \in A$ Then Initial segment of 2.

$S(2) = \{1, 3, 5, 7, \ldots\}.$

because 2 strictly dominate $1, 3, 5, 7, \ldots$

$1 \alpha 2,\ 3 \alpha 2,\ 5 \alpha 2,\ 7 \alpha 2, \ldots$

let $5 \in A$ then Initial segment of 5

$S(5) = \{1, 3\}$ (because 5 strictly dominate 1, 3.) $(1 \alpha 5,\ 3 \alpha 5.)$

let $4 \in A$ then Initial segment of 4

$S(4) = \{1, 3, 5, 7, \ldots, 2\}$ because 4 strictly dominate $1, 3, 5, 7, \ldots 2$.

$1 \alpha 4,\ 3 \alpha 4,\ 5 \alpha 4, \ldots 2 \alpha 4.$

let $1 \in A$ Then Initial segment of 1

$S(1) = \phi$ (because there exist no element in A which 1 strictly dominate to A)

Note :-

Any element of $a \in A$ does not belong to its initial segment.

Immediate Successor :-

An element $b \in A$ is said to be immediate successor of an element $a \in A$ if $a < b$ and there is no element $c \in A$ such that $a < c < b$.

Example :-

$(N, \leq)$

$N = \{1, 2, 3, 4, \ldots\}$ under natural order relation.

$5 < 6$ $\qquad$ $99 < 100$

6 is Immediate Successor of 5 $\qquad$ 100 is immediate successor of 99.

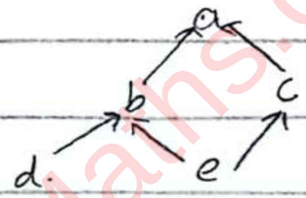
Immediate predecessor :-

If an element $b \in A$ is immediate predecessor of an element

$a \in A$ if $b \propto a$ and there is no element $c \in A$
such that $b \propto c \propto a$ (no element of A lies b/w a and b).

Example :-

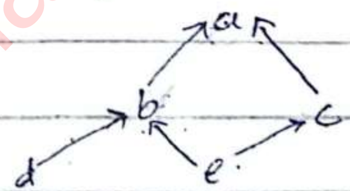
let $W=\{a,b,c,d,e\}$ is partially ordered under line diagram



Sol:-

let $W=\{a,b,c,d,e\}$ is partially ordered under line diagram.

$\because R' = \{(a,a),(b,b),(c,c),(d,d),(e,e),$
$(d,b),(b,a),(d,a),(e,b),(b,a),$
$(e,a),(e,c),(c,a)\}$



Then b is an immediate Successor of both d and e.
because b strictly dominate d and e. and there is no element of A lies b/w b and e, and b and d.

$\Rightarrow d \propto b \;,\; e \propto b.$

So b is immediate Successor of d and c.

e is an immediate predecessor of both b and c. because e strictly precede b and c. and there is no element of W lies between "e and b" and "e and c".

$\Rightarrow e \alpha b, \; e \alpha c$.

So e is immediate predecessor of b and c.

Remark :-

(PU) Every element in a well-ordered set has an unique immediate Successor except the last element.

Proof:-

let A be a well-ordered set.

let $a \in A$ and $M(a) = \{x : a \alpha x\}$ is the set of elements of A which strictly dominate a.

If a is not the last element, then

$$M(a) \neq \phi$$

Since A is well-ordered, $M(a)$ has first element b.

$$\therefore a \alpha b.$$

We claim that b is an immediate successor of a.
Otherwise, there is an element $c \in A$ such that

$a \prec c \prec b$.

Then $c \in M(a)$ and this contradicts to fact that b is the first element of $M(a)$.

Now we claim that b is only immediate successor of a.
Otherwise there is another immediate successor of a say d.
Then $d \in M(a)$ and since b is the first element of $M(a)$. $\therefore a \prec d$
we have $a \prec b \prec d$.
This contradict the assumption that d is immediate successor of a
Thus the first element b of $M(a)$ is the unique immediate successor of a.

→ Limit element:-

An element in a well-ordered set is called limit element if it does not have an immediate predecessor and if it is not the first element.

→ Theorem:-

Limit element is unique or not in the well-ordered set.

proof:-

let A be a well-ordered set.

Suppose contrary a and b are limit elements of A. such that $a \neq b$.

As $a, b \in A$

$\because$ A is well-ordered set.

So Every well-ordered set is totally ordered set.

$\Rightarrow$ A is totally ordered set.

$\Rightarrow$ $a \prec b$ or $b \prec a$.

Case I :-

if $a \prec b$.

Then b is not limit element.

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Case II :-

if $b < a$

Then a is not Limit element.

This is a contradiction to our supposition that a and b are ~~not~~ limit elements of well-ordered set. $\therefore a = b.$

So Limit element is Unique or not in the well-ordered set.

Theorem :-

Every element in well-ordered set has a unique immediate predecessor except the first element.

proof :- ⇒ let A be a well-ordered set.
let $a \in A$ and $S(a)$ be the set of all those element of A which strictly precedes a.
If a is not the 1^{st} element then

$$S(a) \neq \phi.$$

Since set A well-ordered then $S(a)$ has last element say "b"

$$\therefore \; b \prec a$$

we claim that b is an immediate predecessor of a
otherwise there exist an element $c \in A$ such that

$$b \prec c \prec a$$

$$\Rightarrow \; c \in S(a)$$

and this contradicts the fact that "b" is last element

of $S(a)$ & $b \prec c$ Hence b is immediate predecessor.
we claim that b is the unique predecessor of a.
Otherwise there is another immediate predecessor of a say d. Then $d \in S(a)$ and
Since b is the last element of $S(a)$.
we have $b \prec d \prec a$.
This contradicts that assumption that "d" is the immediate predecessor of $S(a)$.
Thus last element b is the unique immediate predecessor of a.

→ Imp Principle of transfinite Induction:-

Let A be a well ordered set and $S \subseteq A$ with the properties

(i) $a_0 \in S$ where a_0 is the first element of A.

(ii) $S(a) \subseteq S \Rightarrow a \in S$ then $S = A$

Proof:-

Given that $S \subseteq A$, A is well ordered set also

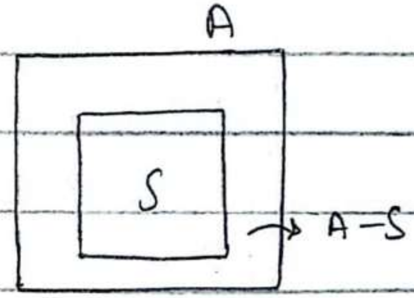
(i) $a_0 \in S$ where a_0 is the first element of A.

(ii) $S(a) \subseteq S \Rightarrow a \in S$ then we want to prove $A = S$

On contrary, suppose $A \neq S$.

$\Rightarrow S \subset A$

$\Rightarrow A - S$ is non-empty.



As $A - S$ is subset of A and A is well ordered set.

$\Rightarrow A - S$ has first element

Let $t_0 \in A-S$ be the first element of $A-S$.

$\Rightarrow t_0 \in A-S$

$\Rightarrow t_0 \notin S$ $\quad \because$ by def of difference of set.

consider

$S(t_0) = \{x ; x \in A$ and $x \angle t_0\}$

let

$y \in S(t_0)$.

$\Rightarrow y \angle t_0$

$\Rightarrow y \notin A-S$. $\quad$ ($\because t_0$ is first element of $A-S$.)

$\Rightarrow y \in S$

$\Rightarrow S(t_0) \subseteq S$.

Then by condition (ii)

$t_0 \in S$

$\Rightarrow t_0 \notin A-S$, a contradiction, so our supposition $A \neq S$ is wrong

$\Rightarrow A = S$

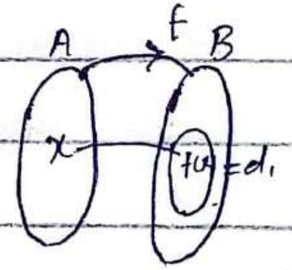
* Remark:-

$A = \mathbb{N}$ with natural order. Then principle of transfinite Induction changed into Mathematical Induction.

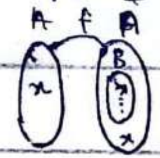
Imp (P) → Theorem :-

Let A be a well-ordered set and $B \subseteq A$ and $f: A \to B$ be a similarity mapping then for all $x \in A$.

$$x \underline{\alpha} f(x)$$



$x \underline{\alpha} f(x)$
$x = f(x)$
$x \alpha f(x)$.
$f(x) \not\alpha x$



$(\mathbb{N}, \leq)$
$2 \not< 2$
hold.

∵ D mai 1^{st} element jo hai us sy bra element mil jay ga or phr A well-ordered

Proof:-

Define a set

$$D = \{x : f(x) \alpha x\}$$

If $D = \phi$ then nothing is left to prove.

Suppose $D \neq \phi$

As $D \subseteq A$ and A is well-ordered set then

$\Rightarrow$ D has first element of D.

As $d_0 \in D$

$$\Rightarrow f(d_0) \alpha d_0 \quad \text{---}(1)$$

let $d_1 = f(d_0)$

$d_1 \propto d_0$ — (2

$\Rightarrow f(d_1) \propto f(d_0)$ $\because f$ is similarity mapping.

$\Rightarrow f(d_1) \propto d_1$

$\Rightarrow d_1 \in D$

$\Rightarrow$ which is contradiction to our supposition.

from (2) we have d_0 is not the first element of D.

$\Rightarrow D = \varphi$.

$x \propto f(x)$
$f(x) \not\propto x$

Theorem :-

(PU) imp

If A and B be two similar well-ordered sets there is one and only one similarity mapping from A to B.

Proof:-

Given $A \simeq B$ and A, B are well-ordered sets.

To prove $\exists$ only one similarity mapping b/w A and B

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Let $f: A \to B$ and $g: A \to B$ be Two Similarity mapping such That $f \neq g$.

As $f(x) \neq g(x)$ for Some $x \in A$

As $f(x), g(x) \in B$ and B is well-ordered set.

" Every well-ordered set is totally ordered set."

$\Rightarrow$ B is totally ordered set.

$\Rightarrow$ $f(x) \prec g(x)$ or $g(x) \prec f(x)$.

Say $f(x) \prec g(x)$

Since $g: A \to B$ is a similarity mapping.

$g^{-1}: B \to A$ is also Similarity mapping

$\Rightarrow$ $f(x) \prec g(x)$

$\Rightarrow$ $g^{-1}(f(x)) \prec g^{-1}(g(x))$ $\quad$ ∵ by apply g^{-1} on b.s.

$\Rightarrow$ $fog^{-1}(x) \prec x \quad \forall x \in A \longrightarrow *$ $\quad$ ∵ g is bijective function.

∵ $fog^{-1}(x) = g^{-1}(f(x))$

" Every images don't precedes elements.

∵ $f(x) \not\prec x$

$1 \to a$
$2 \to b$
$3 \to c$

$1 \to b$
$2 \to a$
$3 \to c$

we can find more than one bijective function but you aik similar function hog..

but fog^{-1} is similarity mapping from A to A.
we must have

$$x \not\leq fog^{-1}(x). \qquad \text{from } *.$$

This is a contradiction.
Thus the assumption $f \neq g$ is wrong.
$\Rightarrow f = g.$

→ Theorem:-

A is well ordered set cannot be similar to any of its initial segment. $\hookrightarrow x \alpha a$

proof:-

let A be a well ordered set and any $a \in A$.
consider

$$S(a) = \{x : x \in A \text{ and } x \alpha a\}$$

To prove $A \not\simeq S(a)$
on contrary; suppose $A \simeq S(a)$

$\Rightarrow$ $f : A \to S(a)$ be a similarity mapping

$f(a) \in S(a)$

$\Rightarrow$ $f(a) \prec a$

So image can't precede to any element.

$\because$ $f(x) \nprec x$.

This is a contradiction.

Hence our assumption is wrong. $A \sim S(a)$.

So $A \nsim S(a)$.

Therefore a well-ordered set cannot be similar to one of its initial segment.

Theorem:-

(PU) Two different initial segments of a well-ordered set cannot be similar.

proof:-

let A be a well-ordered set.

let $a, b \in A$ such that $a \neq b$. and A is well-ordered set.

" Every well-ordered set is totally ordered set."

$\Rightarrow$ $a \prec b$ or $b \prec a$ because $a \neq b$.

Say $a \prec b$ $\quad S(a) = \{x \in A : x \prec a\}$, $S(b) = \{x \in A, x \prec b\}$.

$\Rightarrow$ $a \in S(b)$

That is $S(a)$ is initial segment of well-ordered set $S(b)$.

$\Rightarrow$ $S(a) \subset S(b)$

$S(b) \subseteq A$.

by using

" If A is well-ordered then every subset of a well-ordered is well-ordered

$\Rightarrow$ As $S(b)$ is well-ordered set and $S(a)$ is initial segment of $S(b)$.

$S(a) = a$ a kor b ko strictly precede kry
$S(a) \subseteq S(b)$
but $a \notin S(a)$
$S(a) \subset S(b)$
by def of proper subset.

$a \prec b$ ju element a ko precede kry ga wo b ko bhi precede kry ga.

"A is well-ordered set cannot be similar to any of its segment".

Hence $S_{(a)} \not\simeq S(b)$.

(PU) V.imp

Theorem:-

let A be a well-ordered set and $S \subseteq A$ with the property that if $a, b \in A$ and $a \not< b$ and $b \in S$ then $a \in S$ then either $S = A$ or S is an initial segment of A.

Proof:-

Given that $S \subseteq A$, A is well ordered set. if $a, b \in A$ and $a \not< b$ and $b \in S$ then we have to find $S = A$.

If $S = A$ Then nothing is left to prove.

If $A \neq S \Rightarrow A - S$ is non-empty $(\because S \subseteq A)$.

$\Rightarrow A - S \subset A$ and A is well ordered set.

$\Rightarrow A - S$ has first element.

let $a_0 \in A - S$ be first element of $A - S$

$\Rightarrow a_0 \notin S$ by def of difference of set
$x \in A-B$ but $x \notin B$.

consider $S(a_0) = \{x : x \alpha a_0\}$.

we want to show that

$S = S(a_0)$

let $x \in S(a_0)$

$\Rightarrow x \alpha a_0$.

$\Rightarrow x \notin A-S$ ∵ a_0 first element of $A-S$. So we can't find an element that is less then a_0.

$\Rightarrow x \in S$

$\Rightarrow S(a_0) \subseteq S$ —— (A)

let $y \in S \Rightarrow y \neq a_0$ ($a_0 \in A-S$)

As $y, a_0 \in A$ and A is totally ordered set because A is well-ordered set.

$\Rightarrow$ either $y \alpha a_0$ or $a_0 \alpha y$.

if $y \alpha a_0 \Rightarrow y \in S(a_0)$ and

if $a_0 \alpha y$.

$A-S \neq S$
Jo element $A-S$ mai hai wo S mai nhi hogy.
$S \neq A-S$.
Jo elements S mai hai wo $A-S$ mai nhi hogy.

Then by given $a, b \in A$ and $a \alpha b$ and $b \in S \Rightarrow a \in S$

$\Rightarrow a_0 \in S$

but $a_0 \in A - S$.

which is not true $a_0 K y$.

$\Rightarrow y \alpha a_0 \quad \Rightarrow y \in S(a_0)$

$\Rightarrow S \subseteq S(a_0) \longrightarrow (B)$

From (A) and (B)

$S = S(a_0)$.

Comparison of Well Ordered sets:-

Let A and B be two well-ordered sets then either $A \simeq B$ or $A \not\simeq B$. If $A \not\simeq B$ and A is similar to some initial segment of B then A is said to be shorter than B and B is said to be longer than A.

E.g: $A = \{1, 2, 3, 4, 5\}$, $B = \{6, 7, 8, 9, 10, 11\}$ clearly $A \not\simeq B$

inital segment of B is $S(11) = \{6, 7, 8, 9, 10\}$.

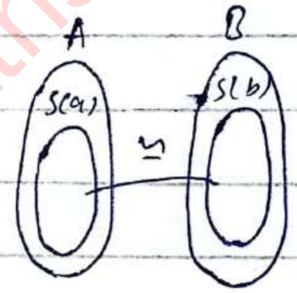
$A \simeq S(11)$. Then A is shorter than B and B is longer than A.

→ Theorem :-

Let A and B be two well-ordered sets and let an initial segment $S(a)$ of A is similar to an initial segment $S(b)$ of B. Then $S(a)$ is similar to a unique segment of B.

proof:-

let $b, b' \in B$ such that, $b \neq b'$



and $S(a) \simeq S(b)$ — (1)

and $S(a) \simeq S(b')$ — (2)

from (1) and (2)

$S(b) \simeq S(b')$ ($\because$ "$\simeq$" is equivalent relation).

a contradiction because two different initial segments of a well-ordered set cannot be similar. $\therefore b = b'$.

$\therefore S(a)$ is similar to a unique initial segment of B.

→ Theorem:-

Let A and B be two well-ordered sets and let an initial segment $S(a)$ of A is similar to an initial segment $S(b)$ of B. Then each initial segment of $S(a)$ is similar to an initial segment of $S(b)$.

proof:-

Given $S(a) \simeq S(b)$

$\Rightarrow$ $f: S(a) \to S(b)$ be a similarity mapping

let $a' \in S(a)$ such that

$f(a') = b'$, where $b' \in S(b)$

we claim $S(a') \simeq S(b')$

under restriction of f.

$f: S(a') \longrightarrow S(b')$

let $a^* \in S(a')$

$\Leftrightarrow \quad a^* < a'$.

$\Leftrightarrow \quad f(a^*) < f(a')$ $\quad$ ($\because f$ is similarity mapping) ($\because f(a') = b'$)

$\Leftrightarrow \quad f(a^*) < b'$

$\Leftrightarrow \quad f(a^*) \in S(b')$ $\quad$ ($\because S(b') = \{x : x < b'\}$)

$\Rightarrow \quad S(a') \simeq S(b')$

Theorem:-

Let A and B be two well-ordered sets and let $S = \{x : x \in A,\ S(x) \simeq S(y)$, where $y \in B\}$.
Then either $S = A$ or S is an initial segment of A.

proof:-

let $x \in S$ and $t < x$. $\quad$ $\because t \in S(x)$ then $t < x$.

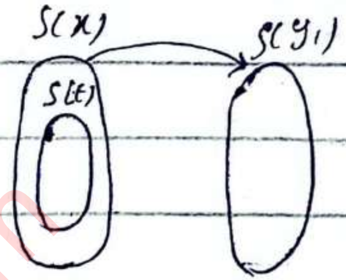
As $x \in A$ then by def. of S, $\exists$ some $y \in B$ such that $S(x) \simeq S(y)$.

As $t < x \Rightarrow S(t)$ is an initial segment of $S(x)$.

Then by Theorem ** (An initial segment $S(a)$ of A is similar to an ** initial segment $S(b)$ of B. Then each initial segment of $S(a)$ is

Similar to an initial segment of $S(b)$.

$\Rightarrow S(t) \simeq S(y_1)$, where $y_1 \in S(y)$ $\because y_1 \prec y$



Then by def. of S,

$\Rightarrow t \in S$

if $x \in S$ and $t \prec x$ then $t \in S$

So by

"let A be a well-ordered set and $S \subseteq A$ with the property that $a, b \in A$ and $a \prec b$ and $b \in S$ then $a \in S$ then either $S \simeq A$ or S is a initial segment of A".

$\Rightarrow S = A$ or S is an initial segment of A.

→ Theorem :-

Let A and B be two well-ordered sets and

$S = \{x : x \in A, S(x) \simeq S(y), \text{ for some } y \in B\}$

and $T = \{y : y \in B, S(y) \simeq S(x), \text{ for some } x \in A\}$

Then $S \simeq T$.

Proof:-

If $x \in S$ then $S(x) \simeq S(y)$ for some $y \in B$.

$\Rightarrow S(y) \simeq S(x)$ ("$\simeq$" is equivalent relation)

$\Rightarrow y \in T$ (By def of T)

Define $f : S \to T$

by $f(x) = y$, where $S(x) \simeq S(y)$

→ **f is one-one:-**

let $x_1, x_2 \in S$ with $x_1 \neq x_2$

$\Rightarrow S(x_1) \not\simeq S(x_2)$ — (1)

but $x_1, x_2 \in S$

Then $S(x_1) \simeq S(y_1)$, for some $y_1 \in B$

and $S(x_2) \simeq S(y_2)$, for some $y_2 \in B$

from (1)

$S(y_1) \not\simeq S(y_2)$

$\Rightarrow y_1 \not\simeq y_2$

$\Rightarrow$ $f(x_1) \cong f(x_2)$ $\because$ by def. of function.

$\Rightarrow$ f is one-one.

f is also onto because the def. of T.

→ **Ordered preserve:-**

let $x_1, x_2 \in S$ such that

$x_1 \propto x_2$

As $x_1, x_2 \in S$ $\Rightarrow$ $S(x_1) \cong S(y_1)$

and $S(x_2) \cong S(y_2)$, where $y_1, y_2 \in B$

Clearly $y_1, y_2 \in T$ because $S(y_1) \cong S(x_1)$ and $S(y_2) \cong S(x_2)$

As

$x_1 \propto x_2$ $\Rightarrow$ $x_1 \in S(x_2)$

$\Rightarrow$ $S(x_1) \subseteq S(x_2)$ $(\because S(x_1) \cong S(y_1))$

Also $S(y_1) \subseteq S(y_2)$ $(\because S(x_2) \cong S(y_2))$

$\Rightarrow$ $y_1 \in S(y_2)$

$\Rightarrow$ $y_1 \propto y_2$

by the def. of "f"

$\Rightarrow f(x_1) < f(x_2)$

$\Rightarrow$ f is Similar mapping.

$\Rightarrow S \simeq T$. D.Y.S

Theorem:-

Imp. Let $\mathcal{A}$ be the family of all initial Segments of a well ordered Set A. Then There is an initial Segment $S(a) \in \mathcal{A}$ which is Shorter than every other Segment in $\mathcal{A}$.

Proof:- As we know that

"let A be well-ordered set and $S \subseteq A$ with if $a < b$ and $b \in S$ then $a \in S$ then S is an initial Segment of A."

So $A \simeq S(A)$, the family of all initial Segments of element in A.

Since A is well-ordered set So "every Subset of well-ordered set is well-ordered". So S(A) being Similar To a well-ordered set A is well-ordered.

Since $\mathcal{A}$ is a collection of initial Segments of a well-ordered set A. So $\mathcal{A}$ is a subset of S(A) and hence $\mathcal{A}$ being Subset of a well-ordered set S(A). Contains a first element. let S(a) be the first element of $\mathcal{A}$ So $S(a) \subseteq S(x)$ for any other intial segment $S(x) \in \mathcal{A}$.

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Ordinal Number

Cardinal number of a well-ordered set is called its ordinal number

E.g: let $A = \{1, 2, a, b, \tau, J, \delta, S\}$ be an ordinary set.
Then cardinality of A is

$$\#(A) = 8.$$

but we can't find ordinal number of A. because set A is not well-ordered

Notation:-

If α is ordinal number of A then we denoted by

$$\text{ord}(A) = \alpha$$

Finite case of Ordinal Number :-

If a set is finite, then the ordinality of well-ordered set is equal to size of set.

$\#(\varphi) = 0$, $\#(\{1\}) = 1$, $\#(\{1,2\}) = 2$. (cardinals)

$\text{ord}(\varphi) = 0$, $\text{ord}(\{1\}) = 1$, $\text{ord}(\{1,2\}) = 2$ (ordinal number)

Example :-

let $A = \{1, 2, 3, 4\}$ with natural order "$(\leq)$"

A is well ordered set.

Then cardinality of $A = \#(A) = n$

$$\#(A) = 4.$$

ordinality of $A = \text{ord}(A) = 4.$

Infinite case of Ordinal Number :- (Transfinite set)

If a well-ordered set is Infinite then the ordinal number of well-ordered set is denoted by

$\text{Ord}(A) = \omega$ (omega) $\because$ If A is wellordered set.

Example :-

$N = \{1, 2, 3, \ldots\}$ with natural order. (set of Transfinite)

Then the ordinal number of $N =$ is

$$\text{Ord}(N) = \omega$$

(The ordinal no of all Transfinite sets in ω).

Note :-

Every cardinal number is not ordinal number but Every ordinal number is cardinal number.

Note

* Jo cheez cardinal no mai hold nhi krti wo ordinal no mai bhi hold nhi kry gi.
Jo cardinal mai hold kry gi wo zaruri nhi wo ordinal mai hold kry.

→ Inequalities in ordinal number:-

Let A and B be two well-ordered sets such that

$$\text{Ord}(A) = \lambda \quad \text{and} \quad \text{Ord}(B) = \mu.$$

Then $\lambda < \mu$ if A is shorter than B.

i.e A is similar to an initial segment of B.

⇒ Remark:-

If $A \simeq B \iff \text{Ord}(A) = \text{Ord}(B)$.

Imp → Theorem:-

If $\text{Ord}(A) = \lambda$ and $\mu < \lambda$ then there exist a unique initial segment say $S(a)$ of A such that

$$\mu = \text{Ord}(S(a))$$

Proof:-

let $S(a)$ and $S(a')$, where $a, a' \in A$ be two initial segments of A such that $\because a \neq a'$

$\mu = ord(S(a))$

$\mu = ord(S(a'))$

by using remark

"If $A \simeq B \iff ord(A) = ord(B)$"

$\Rightarrow ord(S(a)) = ord(S(a'))$

$\Rightarrow S(a) \simeq S(a')$. where $a, a' \in A$.

which is a contradiction because two different initial segments of a well-ordered set cannot be similar.

$\Rightarrow a = a'$

$\Rightarrow \exists$ a unique initial segments of A such that $ord(S(a)) = \mu$.

let $A = \{0,1,2,3,4,5,6,7\}$
$ord(A) = 8$
$S(6) = \{0,1,2,3,4,5\}$
$ord(S(6)) = 6$
$ord(A) = ord(S(a)) = …$
or koi bh element nhi js ka ordinal 6 ho.

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V. imp

→ Theorem :-

Let λ be an Ordinal number and $S(\lambda)$ be the set of all those ordinal numbers which are less than λ then $ord(S(\lambda)) = \lambda$.

$\lambda = 7$

$S(\lambda) = \{$

$ord(S(A)) = 7$

$\lambda = ord(S(\beta))$

Proof:-

Let $\lambda = ord(A)$ and $S(A)$ is the set of all initial segments of A.

First, we show that $A \simeq S(A)$.

$\because \phi$ (empty set) is well ordered set

Define $\varphi : A \to S(A)$,

by $\varphi(a) = S(a)$.

$A = \{1, 2, 3\}$.

$S(1) = —$

$S(2)$ —.

$S(3) =$ —

$S(A) = \{S(1), S(2), S(3)\}$.

To show φ is bijective.

→ φ is one-one:-

Let $a, b \in A$ such that

$a \neq b$

$\Rightarrow S(a) \neq S(b)$.

$\Rightarrow \varphi(a) \neq \varphi(b)$.

$\Rightarrow \phi$ is one-one.

→ ϕ is onto:-

Since for each $S(a) \in S(A)$ then there exist an element $a \in A$ such that :

$\phi(a) = S(a)$.

$\Rightarrow \phi$ is onto.

$\Rightarrow \phi$ is bijective.

→ ϕ preserves order:

let $a, a' \in A$ such that : $a \neq a'$

$a < a'$

$\Rightarrow a \in S(a')$

$\Rightarrow S(a)$ is an initial segment of $S(a')$.

i.e $S(a) \subset S(a')$ or

$\Rightarrow S(a) \leq S(a')$

$\Rightarrow \phi(a) < \phi(a')$

$\Rightarrow \phi$ is order preserving.

→ well-ordered set is totally ordered and totally ordered set hold transitive property

$\Rightarrow$ φ is a Similarity mapping.

$\Rightarrow$ $A \simeq S(A)$. $\quad \because$ If $A \simeq B \Leftrightarrow \text{ord}(A) = \text{ord}(B)$.

$\Rightarrow$ $\text{ord}(S(A)) = \text{ord}(A)$ $\quad \because \text{ord}(A) = \lambda$

$\Rightarrow$ $\text{ord}(S(A)) = \lambda$ — $(*)$

we claim $S(A) \simeq S(\lambda)$

Define $f : S(\lambda) \to S(A)$

by $f(\mu) = S(a)$ where $\mu = \text{ord}(S(a))$

The existence of Such $S(a)$ is guranted by Theorem That "If $\text{ord}(A) = \lambda$ and $\mu < \lambda$ Then $\exists$ a unique initial Segment Say $S(a)$ of A Such That $\mu = \text{ord}(S(a))$".

To show f is bijective.

→ **f is one-one:-**

let $\mu, \mu' \in S(\lambda)$ Such That

$\mu \neq \mu'$ $\quad \because \mu = \text{ord}(S(a))$

$\mu' = \text{ord}(S(a'))$, $\mu < \lambda$.

$\Rightarrow S(a) \neq S(a')$

$\Rightarrow f(\mu) \neq f(\mu')$

$A = \{1, 2, 3, 4\}$

$\text{ord}(A) = 4 = \lambda$.

$S(a_1) = S(4) = \{1, 2, 3\}$

$\mu_1 = \text{ord}S(a_1) = \text{ord}(S(4)) = 3$.

$S(a_2) = S(3) = \{1, 2\}$.

$\mu_2 = \text{ord}(S(a_2)) = \text{ord}(S(3))$
$= 2$.

$\mu_1 \neq \mu_2$

$\text{ord}(A) = \lambda = 4$

$S(\lambda) = S(4) = \{1, 2, 3\}$

$\mu_1, \mu_2 \in S(\lambda)$.

$\Rightarrow f$ is one-one. (For every different ordinal no.s we have different initial segments).

→ f is onto:-

Since for each $S(a) \in S(A)$ then there exist an $\mu \in S(\lambda)$ such that where $\mu = ord(S(a))$ then $\mu < \lambda$.

$$f(\mu) = S(a).$$

$\Rightarrow f$ is onto. (For every initial segments there exist a ordinal number)

$\Rightarrow f$ is bijective.

→ f preserves order:-

Let $\mu_1, \mu_2' \in S(\lambda)$ such that

$\mu_1 < \mu_2$ (if $u_1 < u_2$ ~~its means~~ $u_1 \measuredangle u_2$.)

where $\mu_1 = ord(S(a_1))$, $\mu_2 = ord(S(a_2))$

As $\mu_1 < \mu_2$

$\Rightarrow S(a_1)$ is an initial segment of $S(a_2)$

$\Rightarrow S(a_1) < S(a_2)$

$\Rightarrow f(\mu_1) < f(\mu_2)$

$\Rightarrow f$ preserves order.

A $\to$ well-ordered
$\lambda = \text{ord}(A) \to$ well order
$S(A) \to$ well ordered set
$S(\lambda) \to$ well ordered

$\Rightarrow$ f is similar mapping.

$\Rightarrow$ $S(\lambda) \simeq S(A)$ $\quad \because$ If $A \simeq B \iff \text{ord}(A) = \text{ord}(B)$.

$\Rightarrow$ $\text{ord}(S(\lambda)) = \text{ord}(S(A))$ $\quad$ from $(*)$ $\text{ord}(S(A)) = \lambda$.

$\Rightarrow$ $\text{ord}(S(\lambda)) = \lambda$ $\quad$ (Proved).

→ Ordinal Addition :-

Let A and B be two well-ordered sets and are disjoint with $\text{ord}(A) = \lambda$ and $\text{ord}(B) = \mu$ then

$\text{ord}(\{A ; B\}) = \lambda + \mu$.

$\{A ! B\} \to$ order Union.

$\text{ord}(\{A ! B\})$
$\to$ ordinal of order union

→ Ordinal Multiplication :-

If $\text{ord}(A) = \lambda$ and $\text{ord}(B) = \mu$ then

$\text{Ord}(A \times B) = \lambda\mu$.

→ Remarks :-

(1) Ordinal addition is non-commutative.

(2) Under ordinal addition, element is zero.
↳ Identity

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Proof:-

(i) Ordinal addition is non-commutative.

let $w = ord(A)$, $A = N$ and $B = \{a_1, a_2, \ldots\ldots a_n\}$, $ord(B) = n$

$\{A;B\} = \{1, 2, \ldots\ldots a_1, a_2, \ldots\ldots a_n\}$ —— (1)

$\{B;A\} = \{a_1, a_2, \ldots\ldots a_n, 1, 2, \ldots\ldots\}$ —— (2)

Now $ord(\{A;B\}) = w+n$

$ord(\{B,A\}) = n+w$

Now from (1)

$S(a_1) = \{1, 2, 3, \ldots\ldots\} = N = A$

$\Rightarrow$ A is similar to an initial segments of $\{A,B\}$.

$\Rightarrow$ A is shorter than $(\{A,B\})$

$\Rightarrow ord(A) < ord(\{A;B\})$

$\Rightarrow w < w+n$ —— (*)

from (2)

Define a mapping

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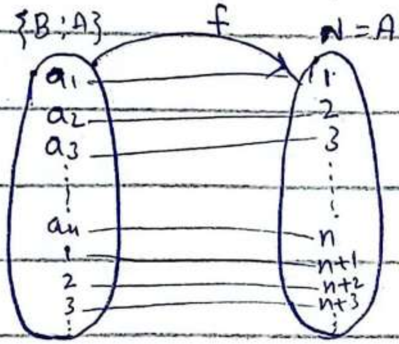
$f: (\{B;A\}) \to A$ by

$$f(x) = \begin{cases} i & \text{if } x = a_i \\ n+i & \text{if } x = i \end{cases}, \quad i = 1, 2, 3, \ldots$$



From above mapping function is bijective.

To show f is order preserve.

→ **f preserves order:-**

let $a_i, a_j \in \{B;A\}$ such that

$a_i < a_j$

$\Leftrightarrow f(a_i) < f(a_j)$

$\Leftrightarrow i < j$ by def of function if $x = a_i$ then i. $f(x) = f(a_i) = i$

$\Rightarrow f$ preserves order

$\Rightarrow$ f is similarity mapping.

$\Rightarrow \{B;A\} \simeq A$ $\quad\because$ If $A \simeq B \Leftrightarrow ord(A) = ord(B)$.

$\Rightarrow ord(\{B;A\}) = ord(A)$

$\Rightarrow n+w = w$ —— $(*)'$

From $(*)$ and $(*)'$,

$(*) \Rightarrow$ $w+n$ is greater than w. and From $(*)'$ $n+w$ is equal to w. Then $n+w$ ~~is~~ is not equal to $w+n$.

$\Rightarrow n+w \neq w+n.$

So ordinal addition is non-commutative.

(2) Under ordinal addition the identity element is zero.

let $Ord(A) = \lambda$, $ord(\varphi) = 0$

$\{A;\varphi\} = A$ and $\{\varphi : A\} = A$

$\Rightarrow ord(\{A;\varphi\}) = ord(\{\varphi;A\}) = ord(A).$

$\Rightarrow \lambda + 0 = 0 + \lambda = \lambda$

$\Rightarrow$ 0 is additional identity.

→ Remark :-

(1) Ordinal multiplication is non-commutative.
(2) Ordinal multiplication is Associative.

Proof:-

(1) Ordinal multiplication is non-commutative.

let $A = N$, $w = ord(A)$ and $B = \{a, b\}$, $ord(B) = 2$.

By ordinal multiplication :

$B \times A = \{(a,1), (b,1), (a,2), (b,2), \ldots\ldots\}$ — (1)

$A \times B = \{(1,a), (2,a), (3,a), \ldots (1,b), (2,b), (3,b) \ldots\}$ — (2)

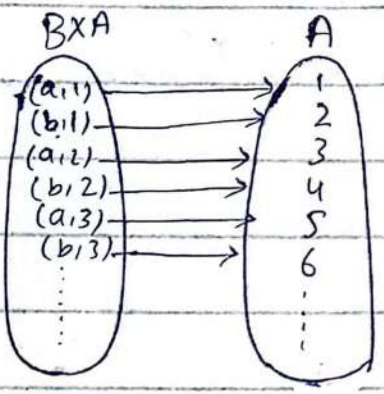
Now

$ord(B \times A) = 2w$

$ord(A \times B) = w2$.

From (1) Define a mapping $f: B \times A \to A$ by

$$f(x) = \begin{cases} 2i-1 & \text{if } x = (a,i) \\ 2i & \text{if } x = (b,i) \end{cases}, \quad i = 1,2,3,\ldots$$



$f(a,i) = 2i-1$
$i=1,$
$(a,1) = 2(1)-1$
$(a,1) = 1.$
$(b,1) = 2i$
$(b,1) = 2.$

From above mapping Function is bijective.

→ **f is Order preserve :-**

let $(a,n), (b,n) \in B \times A$ Such that

$(a,n) < (b,n)$

$\Leftrightarrow \quad f(a,n) < f(b,n)$ — by def. of function if $x=(a,i)$ then $2i-1$ if $x=(b,i)$ then $2i$

$\Leftrightarrow \quad 2n-1 < 2n$

$n = 9$
$f(a,9) < f(b,9)$
$2(9)-1 < 2(9)$
$17 < 18$

$\Rightarrow$ f preserve order

$\Rightarrow$ f is Similar mapping.

$(B \times A) \simeq A$ $\quad\because$ If $A \simeq B \Longleftrightarrow \text{ord}(A) = \text{ord}(B)$.

$\Rightarrow \text{ord}(B \times A) = \text{ord}(A)$

$\Rightarrow 2w = w$ —— $(*)$

From (2)

$S((1,b)) = \{(1,a),(2,a),(3,a)\ldots\} \Rightarrow N = A$

$\Rightarrow$ A is Similar to an initial segment of $A \times B$.

$\Rightarrow$ A is Shorter than $A \times B$.

$\Rightarrow \text{ord}(A) < \text{ord}(A \times B)$

$\Rightarrow w < w2$ —— $(*)'$

From $(*)$ and $(*')$

$(*) \rightarrow$ $2w$ is equal to w. and from $(*')$ $w2$ is greater than w

So $w2$ and $2w$ is not equal.

$\Rightarrow 2w \neq w2$.

So Ordinal multiplication is non-commutative.

(2) Ordinal multiplication is associative.

let $\lambda = ord(A)$, $\mu = ord(B)$ and $\eta = ord(C)$.

$ord(A \times (B \times C)) = \lambda(\mu\eta)$

$ord((A \times B) \times C) = (\lambda\mu)\eta$

To show $\lambda(\mu\eta) = (\lambda\mu)\eta$.

we have to show that $A \times (B \times C) \simeq (A \times B) \times C$.

Define a mapping $f : A \times (B \times C) \to (A \times B) \times C$. by

$f(a, (b, c)) = ((a, b), c)$

To show f is bijective.

→ **f is one-one :-**

let $(a, (b, c)), (a', (b', c')) \in A \times (B \times C)$ and

$f(a, (b, c)) = f(a', (b', c'))$

$\Rightarrow ((a, b), c) = ((a', b'), c)$

$\Rightarrow (a, b) = (a', b')$ or $c = c'$. $\Rightarrow a = a'$, or $b = b'$ or $c = c'$

$\Rightarrow (a, b, c) = (a', b', c')$

$\Rightarrow (a, (b, c)) = (a', (b', c'))$

$\Rightarrow$ f is one-one.

$\rightarrow$ f is onto:-

Since for any $((a,b),c) \in (A\times B)\times C$ then there exist $(a,(b,c)) \in A\times(B\times C)$ such that $f(a,(b,c)) = ((a,b),c)$

$\Rightarrow$ f is onto.

$\Rightarrow$ f is bijective.

To show f is order preserve.

$\rightarrow$ f is order preserve:-

let $(a,(b,c)), (a',(b',c')) \in A\times(B\times C)$ and

$(a,(b,c)) \angle (a'(b',c'))$ — by associative multiplication property $(a,b),c = a,(b,c)$

$\Leftrightarrow$ $(a,(b,c)) \angle (a',(b',c'))$

$\Leftrightarrow$ $a \angle a'$ or $(b,c) \angle (b',c')$

$\Leftrightarrow$ $a \angle a'$ or $b \angle b'$, or $c \angle c'$

$\Leftrightarrow$ $a \angle a'$ or $b \angle b'$ or $c \angle c'$

$\Leftrightarrow$ $(a,b) \angle (a',b')$ or $c \angle c'$ — Now it form the pair (a,b)

$\Leftrightarrow$ $((a,b),c) \angle ((a',b'),c)$

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$\Rightarrow$ f preserve order.

$\Rightarrow$ f is Similarity mapping.

$\Rightarrow$ $A\times(B\times C) \simeq (A\times B)\times C$. $\quad\because$ If $A \simeq B \Leftrightarrow ord(A) = ord(B)$

$\Rightarrow$ $ord(A\times(B\times C)) = ord((A\times B)\times C)$.

$\Rightarrow$ $\lambda(\mu\eta) = (\lambda\mu)\eta$ (Proved).

So ordinal multiplication is associative.

→ **Note :-**

properties that does not hold in cardinal number not hold in ordinal number. and the properties that hold in cardinal number do not necessarily hold in ordinal number.

→ Remark :-

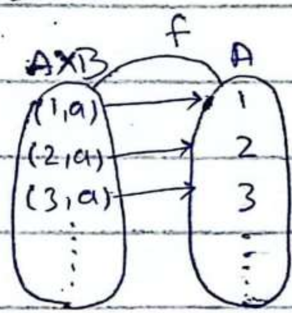
(1) Under ordinal multiplication, the identity element is 1.

proof:-

let $A = N$ and $B = \{a\}$ with $ord(A) = \omega$, $ord(B) = 1$.

$A \times B = \{(1,a), (2,a), (3,a) \ldots\}$ — (1)

$B \times A = \{(a,1), (a,2), (a,3) \ldots\}$ — (2)



(bijective + order preserving)

$\Rightarrow ord(A \times B) = \omega \cdot 1$

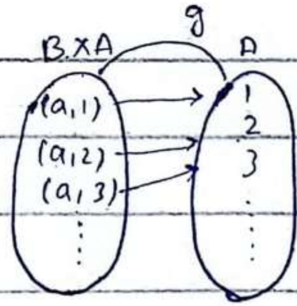
$\Rightarrow ord(B \times A) = 1 \cdot \omega$

From (1) we can see that

$A \times B \simeq A$

$\Rightarrow ord(A \times B) = ord(A)$ By using if $A \simeq B$ then $ord(A) = ord(B)$.

$\Rightarrow \omega \cdot 1 = \omega$ — (3)



(bijective + order preserving)

From (2) we can see that

$B \times A \simeq A$

$\Rightarrow ord(B \times A) = ord(A)$ by using if $A \simeq B$ then $ord(A) = ord(B)$.

$\Rightarrow 1 \cdot \omega = \omega$ — (4)

(2) Is Left distribution Law holds for ordinal numbers?

Proof:-

let $\lambda = ord(A)$, $\mu = ord(B)$ and $\eta = ord(C)$

we have to prove that

$$\lambda \cdot (\mu + \eta) = \lambda\mu + \lambda\eta$$

$\Rightarrow$ $ord(A \times \{B;C\}) = \lambda(\mu+\eta)$

$\Rightarrow$ $ord(A \times B) + ord(A \times C) = \lambda\mu + \lambda\eta$

As :

$\Rightarrow$ $ord(A \times \{B;C\}) = ord(\{A \times B ; A \times C\})$

$\Rightarrow$ $ord(A \times \{B;C\}) = ord(A \times B) + ord(A \times C)$

$\Rightarrow$ $\lambda(\mu+\eta) = \lambda\mu + \lambda\eta$.

Hence left distribution law holds for ordinal numbers.

→ **Note:-**

Right distribution property of multiplication over addition does not hold in general.

Example:-

$(1+1)\omega = 2\omega$

but $2\omega = \omega$ $\quad \because ord(B \times A) = ord(A).$

So $(1+1)\omega = \omega$ — (1)

Now

$1\omega + 1\omega = 1(\omega+\omega)$ $\quad \because 1\cdot\omega = \omega$ by identity element of multiplication.

$1\omega + 1\omega = \omega + \omega$

$= \omega\cdot 1 + \omega\cdot 1$ $\quad \because \omega = 1\cdot\omega$

$= \omega(1+1)$

$1\omega + 1\omega = \omega 2$ — (2)

From (1) $(1+1)\omega = \omega$ and From (2) $1\omega + 1\omega = \omega 2$

So $(1+1)\omega \neq 1\omega + 1\cdot\omega$

→ prove That if λ is any ordinal number then $\lambda+1$ will be the immediate Successor of λ.

Proof:-

let μ be the immediate successor of λ. $\quad \therefore \lambda < \mu$

Then by def of $S(\mu)$.

$$S(\mu) = S(\lambda) \cup \{\lambda\}$$

$$\Rightarrow \text{ord}(S(\mu)) = \text{ord}(S(\lambda) \cup \{\lambda\})$$

$$= \text{ord}(S(\lambda)) + \text{ord}(\{\lambda\})$$

Since $\text{ord}(S(\lambda)) = \lambda$ for any ordinal number λ.

$$\Rightarrow \mu = \lambda + 1.$$

As μ is immediate Successor of λ.

Hence $\lambda+1$ is an immediate successor of λ.

agr hm μ ka initial segment find kry us ma λ ay ga.

$\because \lambda \in S(\mu)$.

or agr λ ka initial segment find kry us mai λ khud nhi ayaga. $\lambda \notin S(\lambda)$

Iykn μ ky initial segment mai λ or $S(\lambda)$ ay gy.

Choice Function :-

Let $\{A_i\}_{i\in I}$ be a non-empty family of non-empty subsets of a set X Then a function

$$f : \{A_i\}_{i\in I} \to X$$

defined by

$$f(A_i) = a_i \in A \quad \text{for all } i \in I.$$

is type ky functions mai Jn ka image pre-image set sy belong krta hai wo choice function hoty hai.

Example:-

let $X = \{1,2,3,4,5\}$

$A_1 = \{1,2\}$, $A_2 = \{2,3,4\}$, $A_3 = \{5\}$, $A_4 = \{1,2,5\}$

$\{A_i\}_{i\in I}$, $I = \{1,2,3,4\}$

$$f : \{A_i\}_{i\in I} \to X$$

by $f(A_1) = 1$, $f(A_2) = 3$

$f(A_3) = 5$, $f(A_4) = 2$.

So we can see that in function in which every set choose one point as a image from all elements. So function is choice function.

→ Cartesian product :-

Let $\{A_i\}_{i \in I}$ be a non-empty family of non-empty subsets of a set then cartesian product of this family is denoted by $\prod_{i \in I} A_i$ and defined as the set of all choice functions.

→ Axiom of choice :-

Cartesian product of non-empty family of non-empty sets is non-empty.

agr koi bhi hm non empty family of non empty sets ly uska disjoint hona zruri nhi is case mai cartesian product empty nhi .

OR

There exists a choice function for any non-empty family of non-empty sets.

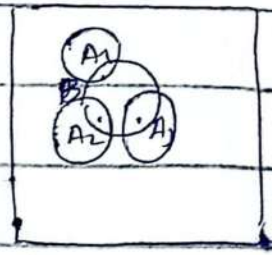
$\because$ If $A \times B = \varphi$

If $A = \varphi$ or $B = \varphi$.

→ Zermelo's postulate :-

Let $\{A_i\}_{i \in I}$ be a non empty family of non-empty disjoint sets, then there exists a subset B of $\bigcup_{i \in I} A_i$ such that the intersection of B and each A_i contains exactly one element.

Example:-



$A_1 = \{1,2,3\}$, $A_2 = \{a,b\}$, $A_3 = \{4,5\}$.

$A_1 \cup A_2 \cup A_3 = \{1,2,3,4,5,a,b\}$

$B = \{1,a,4\} \subseteq \bigcup_{i=3} A_i$.

Also, $B \cap A_1 = \{1\}$, $B \cap A_2 = \{a\}$

$B \cap A_3 = \{4\}$.

we can see that, we can find an B subset of Union set then by Taking intersection with A_i is equal to exactly one element.

hum non-empty family of non-empty sets lygy is case mai hwy inki Union ka aik subset mly ga ya subset hr set mai sy 1.1 element ly ga.

→ Theorem :-

Show That axiom of choice is equivalent to Zermelo's postulate.

Proof:-

Suppose axiom of choice is True.

To prove Zermelo's postulate is also true.

let $\{A_i\}_{i \in I}$ be a non-empty family of disjoint non-empty sets.

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As axiom of choice is True. So there exists a choice function $f: \{A_i\}_{i \in I} \to X$.

$f(A_1) = a_1 \in A$
$f(A_2) = a_2 \in A$
$\vdots$
To B set mai is trh ky elements iy ey fn images a rhy hai.

let $B = \{ f(A_i) : i \in I \}$

Now $B \cap A_i = f(A_i)$

$\Rightarrow$ B contains exactly one element from each of ... set.

Conversely:-

Suppose Zermelo's postulate is true.

To prove axiom of choice is True.

let $\{A_i\}_{i \in I}$ be a non-empty family of non-empty subsets of a set X.

put $A_i^* = A_i \times \{i\}$

then the family $\{A_i^*\}_{i \in I}$ is disjoint.

For this we have to prove $A_i^* \cap A_j^* = \phi$ for $i \neq j$.

Suppose $A_i^* \cap A_j^* \neq \phi$.

$\Rightarrow \exists \; x \in A_i^* \cap A_j^*$

$\Rightarrow x \in A_i^*$ and $x \in A_j^*$

$A_i^* = A_i \times \{i\}$

$\Rightarrow x = (a_i, j)$ and $x = (a_i, j)$

$A_i = \{a_{i1}, a_{i2}, a_{i3} \ldots$

~~$A_i \times \{i\} = \{(a_{i1}, i), (a_{i2},$~~ $\ldots \}$.

where $a_i \in A_i$ and $a_j \in A_j$

$\Rightarrow (a_i, j) = (a_i, j)$

$\Rightarrow i = j$ and $a_i = a_j$

which is a contradiction So $\{A_i^*\}_{i \in I}$ is disjoint family of non-empty sets.

$A_i^* = A_i \times \{i\}$ ↳ non-empty family of non-empty set so A_i^* is also nonempty. So A_i^* is disjoint family of non-empty sets.

So by Zermelo's postulates, $\exists$ a subset B of $\bigcup_{i \in I} A_i^*$ Such that $B \cap A_i^*$ contains exactly one element.

let $B \cap A_i^* = \{(a_i, i)\}$, where $a_i \in A_i$

$\Rightarrow (a_i, i) \in A_i \times \{i\}$

$\Rightarrow (a_i, i) \in A_i^*$.

So we can define

$f : \{A_i\}_{i \in I} \to X$

by $f(A_i) = a_i \in A_i$

$\Rightarrow f$ is a choice function.

According, Zermelo's postulates implies axiom of choice.

Chain:-

let $(A, \leq)$ be a ~~partially~~ ~~orderedly~~ ~~setered~~ set in which every let B be a subset of A. Then the set B is said to be chain of partial ordered set A. If every element of B are comparable.

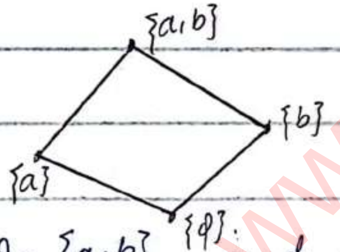
Antichain:-

let $(A, \leq)$ be a partially ordered set.
let B be a subset of A. Then the set B is said to be antichain of A. If every element of B are not comparable.

Examples:-

A poset
$B \subseteq A$
chain
if $a, b \in B$
$a \leq b$ or $b \leq a$.
B is chain.

(1)



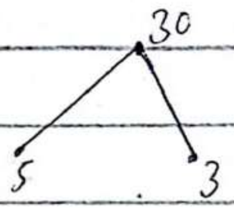
let $A = \{a, b\}$ and $(P(A), \subseteq)$ be a partial order set then

Chain	Anti chain
• $\{\phi, \{a\}, \{a, b\}\}$ is a chain	$\{\{a\}, \{b\}\}$ is an anti chain.
bcz $\phi \subseteq \{a\}$ and $\{a\} \subseteq \{a, b\}$.	bcz $\{a\}$ and $\{b\}$ are not comparable.

- $\{\{a\}, \{a,b\}\}$ is a chain.
- $\{\phi, \{a\}\}$ is also a chain.
- $\{\phi, \{b\}, \{a,b\}\}$ is also a chain.

(2)

let $B = \{3, 5, 30\}$ under "divides" relation.



Here $\{5, 30\}$ and $\{3, 30\}$ are chain.

$\{5, 3\}$ is anti chain.

→ Length of chain :-

The number of an elements in a chain is called the length of the chain.

$\{1, 3, 9, 18\}$ का length $= 4$.

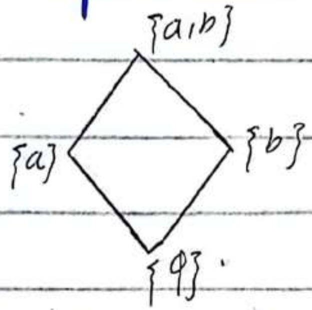
→ Maximal chain :-

A chain is maximal chain If no other element from partial ordered set is comparable with this chain.

→ Maximal Antichain :-

An Antichain is maximal antichain If no other element from the partial ordered set is not comparable with this antichain.

Example :-



let $A = \{a, b\}$ and $(P(A), \subseteq)$ be a partial order set. Then:

Maximal chain	Maximal antichain
$\{\varphi, \{a\}, \{a,b\}\}$ is maximal chain bcz no other element from A is comparable with this chain.	$\{\{a\}, \{b\}\}$ is maximal antichain. bcz no other element from A is not comparable with this antichain.

Example :-

let $P = \{1, 2, 3, 4, 5, 6, 9, 12, 18\}$. Under divides relation.

Chain	Anti chain.
(2,4), (3,9), (1,2,4), (1,3,9) (2,4,12), (3,9,18), (1,2,4,12) (1,3,9,18). are all chain.	(2,3), (3,5), (4,5), (6,9) (2,3,5), (5,6,9), (3,4,5). are all anti chain.

Maximal chain	Maximal antichain
(1,2,4,12) and (1,3,9,18) are maximal chain.	(2,3,5) and (5,6,9) are maximal antichain.

→ Zorn's Lemma:-

let X be a non-empty partially ordered set in which every chain has an upper bound in X. Then X contains at least maximal element.

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→ Well Ordering Theorem :- (Zermelo)

Show that every non-empty set can be well-ordered.

proof:-

let X be any set and $\mathcal{A}$ be the family of all well-ordered subsets of X.

If $w \in \mathcal{A}$ then $w=(B, \leq)$ where B is a subset of X and $\leq$ is a well-ordered defined in B.

Given a partial order to $\mathcal{A}$ as follows

If $w_1, w_2 \in \mathcal{A}$ then $w_1 \preccurlyeq w_2$ if either $w_1 \leq w_2$ or w_1 is an initial segment of w_2.

If $w_1 \preccurlyeq w_2$ then $B_1 \subseteq B_2$

let $\{w_i\}_{i \in I}$, $w_i = (B_i, \leq)$ be chain of $\mathcal{A}$.

Then the family of sets $\{B_i\}_{i \in I}$ ordered by set inclusion is totally ordered.

Define $W = (B, \leq)$ as follows where $B = \bigcup_{i \in I} B_i$.

If $a, b \in B = \bigcup_{i \in I} B_i$ then $\exists$ $i, j \in I$ such that $a \in B_i$, $b \in B_j$

Since $\{B_i\}_{i \in I}$ is totally ordered so either $B_i \subseteq B_j$ or $B_j \subseteq B_i$.

let $B_j \subseteq B_i$ then $a, b \in B_i$

let $a \leq b$ as elements of W.

If $a < b$ as elements of B_i. $\because$ $W_i = (B_i, \leq)$.

So $W = (B, \leq)$ is well-ordered and belongs to the family $\mathcal{A}$.

Hence W is an upper bound of $\{W_i\}_{i \in I}$

So every chain has an upper bound in $\mathcal{A}$.

Then by Zorn's lemma

$\mathcal{A}$ has at least one maximal element say W^*.

It can be shown that

$W^* = X$

If $W^* \neq X$ then let $a \in X - W^*$

Consider the set $\{w^*, \{a\}\}$ which is well-ordered subset of X. and Hence belongs to the family $\mathcal{A}$.

Now w^* becomes an initial segment of $\{w^*, \{a\}\}$

i.e $w^* \prec \{w^*, \{a\}\}$.

which is a contradiction to the supposition that w^* is the maximal element of $\mathcal{A}$.

Hence $w^* = X$ Then X is well-ordered.

So every non-empty set can be well-ordered.

Theorem :-

Let X be a partially ordered set Then show that X contains a maximal chain.

proof:-

let $\mathcal{A}$ be the family of all chain of a partially ordered set X.

Given partial order to the family $\mathcal{A}$ by the set inclusion.

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~~let $\{A_i\}_{i\in I}$ be chain in family~~ of chain $\mathcal{A}$. $\because \{A_i\}_{i\in I} \subseteq \mathcal{A}$.

let $A = \bigcup_{i\in I} A_i$ then A will be chain. $\because$ any subset of a chain is also a chain

Define a order in A as if $a, b \in A$ then $\because$ the union of all chains is itself a chain.

$\exists$ A_i, A_j in $\mathcal{A}$.

Such that $a \in A_i$, $b \in A_j$.

Since $\{A_i\}$ is chain so one of A_i and A_j is contained in other.

let $A_i \subset A_j$, then $a, b \in A_j$.

Now $a \leq b$ as elements of A.

if $a \leq b$ as elements of A_j.

Thus A being a chain of X belongs to $\mathcal{A}$.

and A is the upper bound of $\{A_i\}_{i\in I}$.

Since $\{A_i\}_{i\in I}$ was taken to the arbitrary ordered subset of family of chain $\mathcal{A}$.

So every chain of $\mathcal{A}$ has an upper bound in $\mathcal{A}$.
Then by Zorn's lemma.
$\mathcal{A}$ has at least one maximal element. Say A_0.
Therefore A_0 is chain of X. which is not properly contained in any other chain.
So $\mathcal{A}$ contains a maximal element A_0 which is a maximal chain in X.
Hence A_0 is maximal chain in X.

Dual Order :-

Let $\underline{\alpha}$ be a partial ordering on a set A then $\succsim$ is also a partial ordering on A called dual order.

$\because$ $\succsim$ is inverse of $\underline{\alpha}$

Therefore $\underline{\alpha} = \succsim^{-1}$

→ Ordered Subset :-

let R be a partial order relation on A and B be any subset of A. Then the relation R on A induces a relation R_A on B defined by

$$R_A = R \cap (B \times B)$$

In other words,

every subset B of an ordered set is also ordered with the relation of B induced by the relation on A.

→ Quasi Order :-

Suppose that α is a relation on A satisfying the following two properties

(i) Irreflexive :-

for any $a \in A$, $a \not\alpha a$.

(ii) Transitive :

for any $a, b, c \in A$

if $a \alpha b$ and $b \alpha c$ then $a \alpha c$

Then α is called quasi-order on A.

Remark:

- There is a close relationship b/w partial orders and quasi order.
- If $\underline{\alpha}$ is a partial order on a set A and we define $a \alpha b$ by $a \underline{\alpha} b$ but $a \neq b$.
  Then α is a quasi order on A.
- Conversely if α is a quasi order on A and we define $a \underline{\alpha} b$ by either $a \alpha b$ or $a = b$, then $\underline{\alpha}$ is a partially ordered on A.

Usual Order:-

The relation "$\underline{\alpha}$" on a positive integer "Z^+" is a partial order on Z and infact a partial ordered on R (real no). and any subset of R. This order is said to be usual order.

→ Algebraic Number :-

A real number r is called an algebraic number if r is a root of a polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n = 0 \quad (a_n \neq 0)$$ with integral co-efficients.

→ Transcendental Number :-

A real number which is not Algebraic is called a transcendental number.

→ Properties of Algebraic and Transcendental Numbers:

- Algebraic numbers + Transcendental numbers = Real numbers.
- Every rational number is algebraic.
- Every Transcendental number is irrational.
- Irrational numbers may be algebraic or transcendental ($\sqrt{2}$ is algebraic and π is Transcendental).
- Set of all algebraic numbers is Denumerable.

- Set of all Transcendental numbers is uncountable.

Theorem:-

Prove that the set A of algebraic numbers is Denumerable.

proof:-

Firstly we show that the set P_n of polynomials of degree n with integer co-efficients.

we know that, Since there are infinite many primes say S.

$S \subseteq N$, Then S is denumerable.

because "S is infinite subset of N."

Define a mapping

$f: P_n \to Q^+$ given by

$f(a_0 + a_1 x + \cdots + a_n x^n) = 2^{a_0} 3^{a_1} \cdots P_{n+1}^{a_n}$

where P_{n+1} is the n^{th} prime and $a_n \neq 0$.

Now we have to show that P_n is bijective.

$\Rightarrow$ f is one-one :-

let $\quad f(a_0+a_1x+\cdots+a_nx^n) = f(b_0+b_1x+\cdots+b_nx^n)$

$\Rightarrow \quad 2^{a_0}3^{a_1}\cdots\cdots p_{n+1}^{a_n} = 2^{b_0}3^{b_1}\cdots p_{n+1}^{b_n}$

$\Rightarrow \quad 2^{a_0-b_0}\,3^{a_1-b_1}\cdots\, p_{n+1}^{a_n-b_n} = 1.$

$\Rightarrow \quad a_i - b_i = 0 \quad \forall\ i = 0,1,\cdots n$

$\Rightarrow \quad a_i = b_i \quad \forall\ i = 0,1\cdots n.$

we get

$a_0+a_1x+\cdots\cdots+a_nx^n = b_0+b_1x+\cdots+b_nx^n$

So f is one-one.

$\Rightarrow$ f is onto :-

Since for every $2^{a_0}3^{a_1}\cdots p_{n+1}^{a_n} \in Q^+$ then $\exists\ (a_0+a_1x+\cdots+a_nx^n) \in P_n$ such that

$f(a_0+a_1x+\cdots+a_nx^n) = 2^{a_0}3^{a_1}\cdots\cdots p_{n+1}^{a_n}$

$\Rightarrow$ f is onto. $\quad \because Q^+$ is denumerable.

$\Rightarrow$ f is bijective.

$\Rightarrow \quad P_n \sim Q^+.$

and $\quad P_n \sim f(P_n) \subseteq Q^+$

Hence $f(P_n)$ is infinite subset of Q^+.

"Every infinite set has denumerable subset".

Hence $f(P_n)$ is denumerable

So P_n is denumerable set.

For each $p \in P_n$ let

$$A_n(P) = \{x \in R : P(x) = 0\}$$

by def, each element of $A_n(P)$ is an algebraic no since a polynomial of degree n has at most n real roots, $A_n(P)$ has at most n element and so is a finite set thus $A_n = \bigcup_{P \in P_n} A_n(P)$.

is a denumerable union of set each of which is finite.

$\because$ Denumerable union of denumerable set is denumerable.

Thus A_n is denumerable.

Thus $A_n \subseteq A_n(P)$.

let A denote the set of algebraic numbers.

Now $A = \bigcup_{n=1}^{\infty} A_n$. So $A_n \subseteq A$.

Examples :-

(a) Consider the set Q of rational numbers with the usual order and consider the subset $D = \{x \in Q : 8 < x^3 < 15\}$ Determine D is bounded above and bounded below. Do $\sup(D)$ and $\inf(D)$ exists?

Sol:-

Here Q is the set of rational number and $D = \{x \in Q : 8 < x^3 < 15\}$ is subset of Q.

let's find the cube roots of 8 and 15.

cube root of $8 = \sqrt[3]{8} = 2.$ $\quad\because 8 < x^3 < 15$

cube root of $15 = \sqrt[3]{15} \simeq 2.466.$ $\quad\therefore 2 < x < \sqrt[3]{15}$

So the rational number whose are b/w 8 and 15.

i.e b/w 2 and $\sqrt[3]{15}$.

2 and $\sqrt[3]{15}$ is not in the set.

Now $D = \{x \in Q : 2 < x < \sqrt[3]{15}\}.$

$x = (2, \sqrt[3]{15})$

$\Rightarrow D = \{$

- **Upper bound :-**

by def of upper bound $x \leq u, \forall u \in A, x \in B.$

As $D = \{x \in \mathbb{Q} : 2 < x < \sqrt[3]{15}\}$. is subset of $\mathbb{Q}$.

So upper bounds of D is $(\sqrt[3]{15}, \infty)$

- **Lower bound :-**

by def of lower bound $l \leq x, \forall x \in B, l \in A.$

As $D = \{x \in \mathbb{Q} : 2 < x < \sqrt[3]{15}\}$. is subset of $\mathbb{Q}$.

So lower bound of D is $(-\infty, 2)$.

Then D is bounded above and bounded below.

- **Supremum :-**

by def of supremum $x \leq u \Rightarrow u \leq u_1$

If a upper bound of D precedes every other upper bound of D.

Then it is supremum.

So $\sqrt[3]{15}$ is only upper bound that precedes other upper bound.

but $\sqrt[3]{15} \notin \mathbb{Q}$ because it is irrational.

So $\sup(D)$ does not exists.

- **Infimum:-**

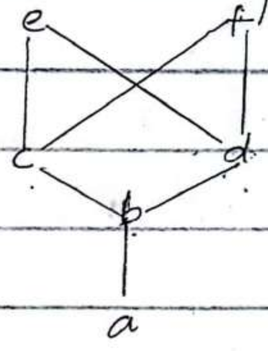
by def of Infimum $l \alpha x \Rightarrow l_1 \alpha l$.

(If a lower bound of D dominate every other lower bound of D. then it is Infimum.

So 2 is only lower bound of D. That dominates every other lower bound so 2 is infimum of D.

$\Rightarrow$ Inf (D) = 2.

(b) Let $X = \{a, b, c, d, e\}$ be ordered as show in the figure and $A = \{b, c, d\}$ be a subset of X. Find minimal and maximal element of X. Also find Sup(A) and Inf(A) in X.



Sol:-

let $X = \{a, b, c, d, e\}$ be ordered as shown in figure. and $A = \{b, c, d\}$ be subset of X.

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- **Minimal Element :-**

by def of minimal element if. $x \preceq m \Rightarrow x = m$ $x, m \in A$
That is if there is no element in A which precedes m.
So we can see that there is no element in X which precedes a.
So a is the minimal element of X.

- **Maximal Element :-**

by def of maximal Element if $l \preceq x \Rightarrow l = x.$, $x, m \in A$
That is if there is no element in A which dominate l.
So we can see that there is no element in X which dominates e and f.
So e and f are the maximal element of X.

- **upper bounds :-**

by def of upper bound $x \preceq u$, $\forall x \in B$.
As $A = \{b, c, d\}$. is subset of X.

So every element of A precedes e and f.

$\Rightarrow$ $b \preceq e$, $b \preceq f$, $c \preceq e$, $c \preceq f$, $d \preceq e$, $d \preceq f$.

So e and f are upper bound of A in X.

- **Supremum:**

by def of Supremum

If a upper bound of A precedes every other upper bound of A then it is Supremum.

As e and f are upper bound while e and f are not comparable.

(e and f are not related to each other).

So Sup (A) does not exists.

- **Lower bounds:**

by def of lower bound $l \preceq x \ \forall x \in B$

As $A = \{b, c, d\}$ is subset of X.

So every element of A which dominates a and b.

$\Rightarrow$ $b \preceq b$, $a \preceq b$

So a and b are lower bounds of A in X.

- **Infimum:-**

by def of Infimum

If a lower bound of A dominate every other lower bound of A that is infimum.

So b dominate every lower bound of A. $(a \leq b), b \leq b)$

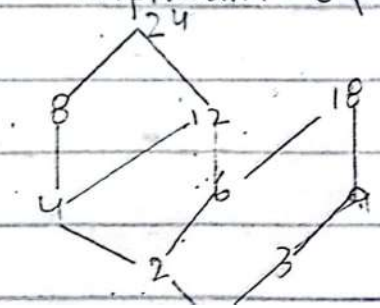
So b is infimum of A.

So $\text{Inf}(A)$ is b.

(c) Let $X = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be ordered as given in the picture and let $A = \{4, 6, 9\}$

(i) Find the maximal and minimal elements of X.

(ii) Find the Supremum and infimum of A.



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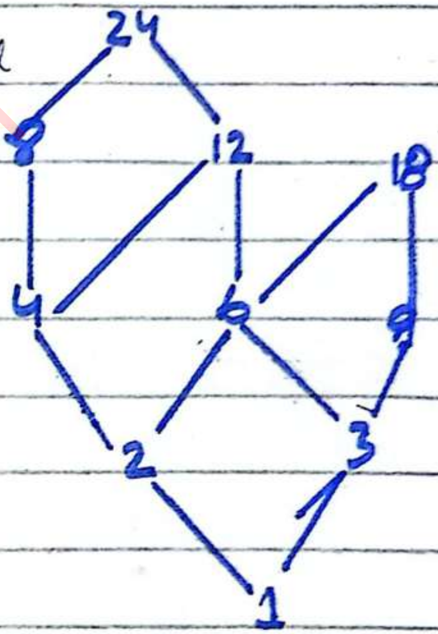
Sol:-

let $X = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be ordered as given in the picture and let $A = \{4, 6, 9\}$.



- **Minimal Element :-**

by def of minimal element

if $x \preceq m \Rightarrow x = m$, $x, m \in A$.

that is if there is no element in A which precedes m.

So we can see that there is no element in X which precedes 1.

So 1 is minimal element of X.

- **Maximal Element :-**

by def of maximal element.

if $l \preceq x \Rightarrow l = x$. , $x, l \in A$.

That is if there is no element in A which dominate l.

So we can see that in picture there is no element in X which dominates 18 and 24.

So 18, and 24 are maximal element.

- **Upper bound :-**

by def of upper bound. $x \preceq u$, $\forall\ x \in B$.

As $A = \{4, 6, 9\}$ be subset of X.

So we can see that there is no element which dominate the element of A.

So there is no upper bound of A.

- **Supremum :-**

there is no upper bound of A.

So supremum of A does not exist.

$\Rightarrow$ Sup(A) does not exist

• Lower Bound:-

by def of lower bound $l \leq x \ \forall x \in B$

As $A = \{4, 6, 9\}$ be subset of X.

So every element of A which dominate 1.

So 1 is lower bound of A.

• Infimum :-

by def of infimum

If a lower bound of A dominate every other lower bound of A that is Infimum.

So 1 is only lower bound of A.

So $\text{Inf}(A) = 1$.