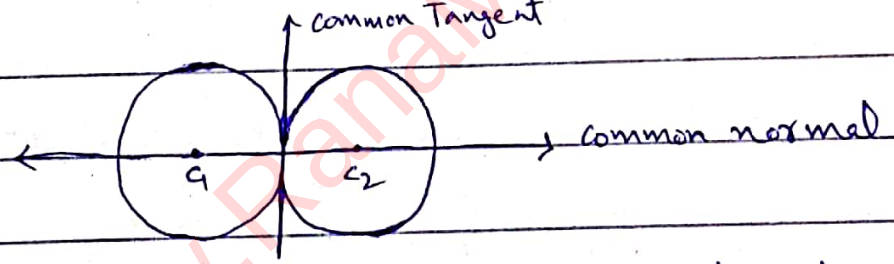
COLLISION ***

Collision:- When two particles strike against each other then collision is said to take place. There are two types of collision

(1) Head to head collision or direct collision

(2) Oblique Collision.

(1) Head to head Collision or Direct Collision:- when two bodies collide directly with each other such that the direction of motion of each body is along the common normal at the point of contact. This type of collision is called direct collision or head to head Collision.



(2) Oblique Collision:- When direction of motion either or both is not along the common normal then this type of collision is called oblique Collision.

Elastic Collision:- A collision in which Kinetic Energy before & after Collision remain same. Also momentum before & after

Momentum:- The product of mass & its velocity is called momentum.

collision remain same OR A collision in which Kinetic Energy & momentum conserved.

Inelastic Collision:- A collision in which momentum is conserved but Kinetic energy is not conserved OR A collision in which the K.E before collision is not equal to K.E after collision but momentum is equal before & after collision.

Newton's Law of Restitution OR Newton's Experimental Law:-

CASE I:- When two bodies collide directly then their relative velocities after collision is in a constant ratio with the relative velocity before collision & in opposite direction.

Let two bodies collide directly. let u & u' be their velocities before collision & v & v' be their velocities after collision then Newton's law of restitution

$$\frac{v - v'}{u - u'} = -e$$

$$v - v' = -e(u - u')$$

where e is constant quantity & called modules or coefficient of elasticity or restitution or resistance of bodies.

CASE II :- When two bodies collide obliquely then their relative velocity resolved along their common normal after impact is in a constant ratio to their relative velocity before impact resolved in the same direction & is in opposite direction. Thus if two bodies of masses m & m' moving with velocities u & u' before collision in the directions inclined angle α & β respect. to their common normal & v & v' be their velocities in direction θ & ϕ respectively after collision then.

$$\frac{v\cos\theta - v'\cos\phi}{u\cos\alpha - u'\cos\beta} = -e$$

★ If $e = 1$ the collision is called perfectly elastic

★ If $e = 0$ the collision is called perfectly inelastic.

★ If $0 < e < 1$ the collision is called elastic

Law of Conservation of Momentum:- Law of conservation of momentum for direct collision is $mv + m'v' = mu + m'u'$

Also for oblique collision law of conservation of momentum is

$mv\cos\theta + m'v'\cos\phi = mv\cos\alpha + m'v'\cos\beta$

Q :- A smooth sphere imping (collide) on a fix smooth plane with a velocity u. Show that it will rebound with velocity eu where e is coefficient of elasticity.

Solution:- Let u be the velocity of sphere before impact & v be the velocity of smooth sphere after impact. As plane is fixed so its velocity before & after impact is zero. Now.

Relative velocity before Impact $= u - 0 = u$

" " after Impact $= v - 0 = v$

So by Newton's law.

$$\frac{v-0}{u-0} = -e \Rightarrow \frac{v}{u} = -e$$

$$v = -eu$$

−ve sign show that sphere moves back on velocities with velocity eu

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Q :- A heavy elastic ball dropped on a horizontal floor from a height of 20 feet and after rebound twice it is observed to attain height of 10 feet find the coefficient of restitution.

Solution: Given $u = 0$

$a = g = 9.8\ m/s^2$

$= 32\ ft/s^2$

Height $= h = 20$ ft.

As we know that

$$2ax = v^2 - u^2$$

$$2(g)(h) = v^2 - u^2$$

$$2(32)(20) = v^2 - 0 \Rightarrow 1280 = v^2$$

$$\sqrt{v^2} = \sqrt{1280} \Rightarrow v = 16\sqrt{5}\ ft/sec$$

Let e be the coefficient of restitution

Velocity after 1st rebound $= e(16\sqrt{5})$

Also " " 2nd " $= e(e(16\sqrt{5}))$

$= 16\sqrt{5} \cdot e^2$

Now consider the motion after 2nd rebound

Initial velocity $= u = 16\sqrt{5}\, e^2$

height $= h = 10$ feet

$g = -32$ ft/s^2

$v = 0$

As $v^2 - u^2 = 2gh$

$0 - (16\sqrt{5}\, e^2)^2 = 2(-32)(10)$

$-1280\, e^4 = -640$

$e^4 = \frac{640}{1280}$

$e^4 = \frac{1}{2}$

$e = (\frac{1}{2})^{1/4}$

Q :- A rubber ball is dropped from a height 'h' after rebound twice from the ground it reaches a height $h/2$ find the co-efficient of restitution & what would be the coefficient of restitution have the ball reached the height $h/2$ after rebound 3rd time.

Solution:- Given $u = 0$

height $= S = h$

$g = 32$ ft/s^2

Now from 3rd equ- of motion

$v^2 - u^2 = 2aS$

$v^2 - 0 = 2gh$

$v^2 = 2gh \Rightarrow v = \sqrt{2gh}$

Let e be the coefficient of restitution. So

The velocity after first rebound $= e\sqrt{2gh}$

" " " 2nd " $= e^2\sqrt{2gh}$

Now concider motion after 2nd rebound.

$u = e^2\sqrt{2gh}$

$a = g = -32\,ft/s^2$, $v = 0$, $S = \frac{h}{2}$

So from third equation of motion

$v^2 - u^2 = 2aS$

$0 - (e^2\sqrt{2gh})^2 = 2aS$

$-e^4 2gh = 2aS$

$-e^4 = \frac{2aS}{2gh}$

$+e^4 = \frac{\cancel{2}(+g)\frac{h}{2}}{2gh\cdot\cancel{2}}$

$e^4 = \frac{1}{2}$

$e = (\frac{1}{2})^{1/4}$

Now velocity after 3rd rebound $= e(e^2\sqrt{2gh})$

$= e^3\sqrt{2gh}$

concider motion after 3rd rebound $= u = e^3\sqrt{2gh}$

$a = g = -32\,ft/s^2$, $v = 0$

$s = h/2$

From 3rd equation of motion

$v^2 - u^2 = 2aS$

$0 - [e^3(2gh)]^2 = 2(-g)(h/2)$

$$\cancel{r}e^6(2gh) = \cancel{r}gh$$

$$e^6 = \frac{gh}{2gh} \Rightarrow e^6 = \frac{1}{2}$$

$$e = \left(\frac{1}{2}\right)^{1/6}$$

Q :- A particle falls from height 'h' in time 't' upon a fixed horizontal plane rebound & reaches the maximum height 'h'' in tim 't''

(a) Show that $t' = et$ and $h' = e^2 h$

(b) Also prove that the whole distance up & down describe by the particle before it has finished rebounding is $\frac{1+e^2}{1-e^2}h$ and that the time ellapses is $\frac{1+e}{1-e}\sqrt{\frac{2h}{g}}$

Solution:- Let a particle acquires a velocity v after falling through height 'h' in time 't' as

$$v = v + at$$

Here $v = 0$, $a = g$

$$v = 0 + gt \Rightarrow v = gt$$

$$t = \frac{v}{g} \quad \text{--- (1)}$$

Also As $v^2 - u^2 = 2as$

$v^2 - 0 = 2gh \Rightarrow v^2 = 2gh \Rightarrow v = \sqrt{2gh}$ — ②

Now velocity after first rebound $\frac{ev}{e\sqrt{2gh}}$ $e\sqrt{2gh}$

In time t' the particle attains a height 'h''

after first rebound at height h'

$V = 0$

As $v = u + at$

Here $v = 0$, $u = e\sqrt{2gh}$, $a = -g$, $t = t'$

$0 = e\sqrt{2gh} + (-g)(t')$

$gt' = e\sqrt{2gh} \Rightarrow t' = \frac{e\sqrt{2gh}}{g}$

$t' = e\,v/g$ using equation ②

$t' = et$ " " ①

Also As $v^2 - u^2 = 2as$

Here $v = 0$, $u = e\sqrt{2gh}$, $a = -g$, $s = h'$

$0 - (e\sqrt{2gh})^2 = 2(-g)(h')$

$-e^2 2gh = -2gh'$

$e^2 h = h'$

$h' = e^2 h$

(b) The height to which the particle raise after first rebound is e^2h and after 2nd rebound e^4h & after 3rd rebound e^6h So the total distance covered before finishing rebound is equal to

$h + 2e^2h + 2e^4h + 2e^6h + \ldots\ldots$

$= h + 2e^2h(1 + e^2 + e^4 + \ldots\ldots$

$= h + 2e^2h\left(\frac{1}{1-e^2}\right) \longrightarrow S_\infty\left(\frac{a_1}{1-r}\right)$

$$\frac{(1-e^2)h + 2e^2h}{1-e^2}$$

$$\frac{h - e^2h + 2e^2h}{1-e^2}$$

$$\frac{h + e^2h}{1-e^2} \Rightarrow \frac{(1+e^2)h}{1-e^2}$$

Now the time taken by the particle to reach maximum height after first rebound is et after 2nd rebound is e^2t after 3rd rebound is e^3t. Thus the total time taken by the particle before finishing the rebound is

$= t + 2et + 2e^2t + 2e^3t + \ldots\ldots$

$= t + 2et(1 + e + e^2 + \ldots\ldots$

$t + 2et\left(\frac{1}{1-e}\right)$

$$\frac{t - et + 2et}{1-e} = \frac{t+et}{1-e}$$

$$\frac{(1+e)t}{1-e} \Rightarrow \frac{1+e}{1-e}\cdot\frac{v}{g} \quad \text{using equ. ①}$$

$\frac{1+e}{1-e}\frac{\sqrt{2gh}}{g}$ using equ ① $\Rightarrow \frac{1+e}{1-e}\sqrt{\frac{2h}{g}}$

Q :- An elastic ball of mass 'm' is projected vertically upward from a point on horizontal plane with velocity v. If 'e' is the coefficient of elasticity then find the total space describe by it and the time that ellapses upto instant of its nth rebound. What is the K.E after nth rebound.

Solution:- Let velocity of projection $= v$

And fixed velocity at heigest point $= v_f = 0$

$a = -g \quad , \quad S = h$

As $v_f^2 - v^2 = 2aS$

$0 - v^2 = 2(-g)h$

$-v^2 = -2gh$

$v^2 = 2gh \Rightarrow h = \frac{v^2}{2g}$

Now the distance covered upto first rebound

$= \frac{v^2}{2g} + \frac{v^2}{2g} \Rightarrow \frac{v^2}{g}$

Similarly the distance covered upto 2nd rebo-

$= e^2\left(\frac{v^2}{g}\right) = \frac{e^2v^2}{g}$

Therefore the total distance upto nth rebound is $= \frac{v^2}{g} + \frac{e^2v^2}{g} + \frac{e^4v^2}{g} + \ldots\ldots$ to nth rebound

$= \frac{v^2}{g}\left[\frac{1(1-(e^2)^n)}{1-e^2}\right] \Rightarrow \frac{v^2}{g}\left(\frac{1-e^{2n}}{1-e^2}\right)$

$S_n = \frac{a_1(1-r^n)}{1-r}$

Now for time As $v = v + at$

$0 = v - gt \Rightarrow gt = v$

$t = \frac{v}{g}$

So total time upto first rebound $= \frac{2v}{g}$

" " " 2nd " $= \frac{2ev}{g}$

" " " 3rd " $= \frac{2e^2v}{g}$

" " " nth " $=$

$\frac{2v}{g} + \frac{2ev}{g} + \frac{2e^2v}{g} + \dots\dots$ to nth rebound

$\frac{2v}{g}(1 + e + e^2 + \dots\dots$ to nth rebound

$= \frac{2v}{g}\left(\frac{1(1-e^n)}{1-e}\right) \Rightarrow \frac{2v}{g}\left[\frac{1-e^n}{1-e}\right]$

Now velocity upto nth rebound $= e^n v$

$\therefore$ K.E after nth rebound $= \frac{1}{2}m(e^n v)^2$

$= \frac{1}{2}m e^{2n} v^2$

Q : Two elastic spheres of masses m and m′ moving with velocities v & v' impact directly. If 'e' is the coefficient of elasticity then find their velocities after Impact.

Solution:- Let the two spheres of masses m and m' moving with velocity u & u' before impact. Let after impact their velocities as v & v'

Now by Newton's Experimental law

$$\frac{v-v'}{u-u'} = -e$$

$$v - v' = -e(u-u') \quad \text{——— (1)}$$

Also by law of conservation of momentum.

$$mv + m'v' = mu + m'u' \quad \text{——— (2)}$$

from equation 1

$$v - v' = eu' - eu$$

Multiply both sides by m'

$$m'v - m'v' = m'eu' - m'eu \quad \text{——— (3)}$$

Now Adding equation (2) and (3)

$$mv + \cancel{m'v'} = mu + m'u'$$

$$m'v - \cancel{m'v'} = m'eu' - m'eu$$

$$(m+m')v = (m - m'e)u + (1+e)m'u'$$

$$v = \frac{(m - m'e)u + (1+e)m'u'}{m+m'} \quad \text{——— (4)}$$

from equation 1

$$v - v' = eu' - eu$$

Multiply both sides by m

$$mv - mv' = meu' - meu \quad \text{——— (5)}$$

Subtracting equation (5) from (2)

$$m\cancel{v} + m'v' = mu + m'u'$$
$$\underline{-m'v \mp mv' = \mp meu \pm meu'}$$
$$(m+m')v' = (1+e)mu + (m'-me)u'$$

$$v' = \frac{(1+e)mu + (m'-me)u'}{m+m'} \quad \text{———} (6)$$

Equation No. 4 and equation No. 6 are required velocities after impact

Cor 1:- If two spheres are perfectly elastic and their masses are equal

i.e $e = 1$, $m = m'$

Then from equation No. 4

$$v = \frac{(m - m(1))u + (1+1)mu'}{m+m}$$

$$= \frac{0 + 2mu'}{2m} \Rightarrow v = \frac{2\cancel{m}u'}{2\cancel{m}}$$

$$\Rightarrow v = u'$$

Also from equation No. 6

$$v' = \frac{(1+1)mu + (m - m(1))u'}{m+m}$$

$$v' = \frac{2mu + 0}{2m} \Rightarrow v' = \frac{2\cancel{m}u}{2\cancel{m}}$$

$$v' = u$$

Thus after direct impact spheres interchange their velocities

Cor 2:- If spheres are elastic & the 2nd sphere is very much bigger than the first sphere is at rest.

i.e $e = 1$, $u' = 0$ & $m' >>>>>> m$

Thus from equation ④

$$v = \frac{m'(0)(1+1) + u(0 - m'.1)}{0 + m'}$$

$$= \frac{0 - m'u}{m'} = \frac{-\cancel{m'}u}{\cancel{m'}}$$

$$v = -u$$

Also from equation No. 6

$$v' = \frac{0+0}{0+m'} \Rightarrow v' = 0$$

Q :- An imperfect elastic ball is projected with velocity $\sqrt{gh}$ and in angle α with horizontal So that it strikes a vertical wall at a distance 'c' from the point of projection. Show that the coefficient of restitution b/w the ball & wall is $\frac{c}{h\sin 2\alpha - c}$.

Solution:- $v = \frac{S}{t}$

$vt = S$

Let the ball is Projected at angle α from a point 'A' at a distance c from the wall then

$$u(\cos\alpha)(t_1) = c$$

$$t_1 = \frac{c}{u\cos\alpha} \quad \text{——— } ①$$

Now after rebound its horizontal velocity becomes $eu\cos\alpha$ & t_2 by the time taken by the ball from P to A

Then $(eu\cos\alpha)t_2 = c$

$$t_2 = \frac{c}{eu\cos\alpha} \quad \text{——— } ②$$

Now in time t_1 & t_2 vertical distance becomes zero. Then by 2nd equation of motion

$$S = ut + \frac{1}{2}at^2$$

Here $S=0$, $u = u\sin\alpha$, $t = t_1+t_2$, $a = -g$

$$0 = (u\sin\alpha)(t_1+t_2) + \frac{1}{2}(-g)(t_1+t_2)^2$$

$$\frac{1}{2}g(t_1+t_2)^2 = (u\sin\alpha)(t_1+t_2)$$

$$\frac{1}{2}g(t_1+t_2) = u\sin\alpha \Rightarrow t_1+t_2 = \frac{2u\sin\alpha}{g}$$

using equation ① & ②

$$\frac{c}{u\cos\alpha} + \frac{c}{eu\cos\alpha} = \frac{2u\sin\alpha}{g}$$

$$\frac{c}{u\cos\alpha}\left(1+\frac{1}{e}\right) = \frac{2u\sin\alpha}{g}$$

$$1+\frac{1}{e} = \frac{2u^2\sin\alpha\cos\alpha}{g.c}$$

$$\frac{1}{e} = \frac{u^2\sin 2\alpha}{g.c} - 1$$

$$\frac{1}{e} = \frac{u^2\sin 2\alpha - gc}{gc}$$

$$e = \frac{gc}{u^2\sin^2\alpha - gc}$$

Given $u = \sqrt{gh}$

$$e = \frac{gc}{gh\sin^2\alpha - gc} \Rightarrow e = \frac{gc}{g(h\sin^2\alpha - c)}$$

$$e = \frac{c}{h\sin^2\alpha - c}$$

Q :- A particle is projected with velocity u from a point on ground so that it strikes the smooth vertical wall and impact the ground at midway b/w the point of projection & wall. Show that the angle of projection is $\frac{1}{2}\sin^{-1}\left(\frac{(1+2e)ag}{2eu^2}\right)$ where 'a' is the distance of the wall from the point of the projection & e be the coefficient of elasticity.

Solution:- Let the ball is projected at an angle α with velocity u from a point A at a distance 'a' from the wall & t_1 be the time taken from point A to point 'P'

Then $(u\cos\alpha)t_1 = a \Rightarrow t_1 = \frac{a}{u\cos\alpha}$ —— ①

Now after impact with the wall it

impact the ground in the midway between the point of projection and the wall then its horizontal velocity becomes $eu\cos\alpha$ & t_2 be the time taken by ball from point P to A.

Therefore $eu\cos\alpha\, t_2 = \frac{a}{2}$

$$t_2 = \frac{a}{2eu\cos\alpha} \quad \text{--- (2)}$$

So using 2nd equation of motion

$$S = ut + \frac{1}{2}at^2$$

Here $S=0$, $u = u\sin\alpha$, $t = t_1+t_2$, $a = -g$

$$0 = (u\sin\alpha)(t_1+t_2) - \frac{1}{2}g(t_1+t_2)^2$$

$$\frac{1}{2}g(t_1+t_2)^2 = (u\sin\alpha)(t_1+t_2)$$

$$\frac{1}{2}g(t_1+t_2) = u\sin\alpha$$

$$t_1+t_2 = \frac{2u\sin\alpha}{g}$$

using equation (1) & (2)

$$\frac{a}{u\cos\alpha} + \frac{a}{2eu\cos\alpha} = \frac{2u\sin\alpha}{g}$$

$$\frac{a}{u\cos\alpha}\left[1 + \frac{1}{2e}\right] = \frac{2u\sin\alpha}{g}$$

$$1 + \frac{1}{2e} = \frac{2u^2\sin\alpha\cos\alpha}{ag}$$

$$\frac{2e+1}{2e} = \frac{u^2\sin 2\alpha}{ag}$$

$$\frac{(2e+1)ag}{2eu^2} = \sin 2\alpha$$

$$2\alpha = \sin^{-1}\left[\frac{(2e+1)ag}{2eu^2}\right]$$

$$\alpha = \frac{1}{2}\sin^{-1}\left[\frac{(2e+1)ag}{2eu^2}\right]$$

Q :- A ball is projected from a point in horizontal plane maked one rebound. Show that if the 2nd range is equal to the greatest height at which the ball attains the angle of projection is $\tan^{-1}(4e)$.

Solution:- Let u be the velocity of projection & α be the angle of projection Then

Greatest height attain $= \dfrac{u^2\sin^2\alpha}{2g}$ ——— ①

As in projection motion, horizontal component of velocity remain same but vertical component of velocity changes. Therefore After 1st rebound.

The horizontal component of velocity $= u\cos\alpha$

& vertical " " " $= eu\sin\alpha$

$$\text{2nd range} = \frac{2(u\cos\alpha)(eu\sin\alpha)}{g}$$

$$\text{2nd range} = \frac{eu^2\,2\sin\alpha\cos\alpha}{g}$$

But given $\Rightarrow$ 2nd range = Greatest height

$$\frac{e u^2 2 \sin\alpha \cos\alpha}{g} = \frac{u^2 \sin^2\alpha}{2g}$$

$$4e \cos\alpha = \sin\alpha$$

$$4e = \frac{\sin\alpha}{\cos\alpha} \Rightarrow 4e = \tan\alpha$$

$$\alpha = \tan^{-1}(4e)$$

Q :- A projection of elacity e is projected in a distance which is to be vertical and bounds on a smooth horizontal plane makes one rebound & the range of 1st rebound is r. Show that the 2nd range is er.

Solution:- Let a projectile is projected from a point 'O' makes an angle α with x-axis.

Then the horizontal range $= \frac{(2u\cos\alpha)(u\sin\alpha)}{g}$

But given range $= r$

Therefor $r = \frac{2(u\cos\alpha)(u\sin\alpha)}{g}$ —— ①

where u is the velocity of projectile

Now after first rebound

Horizontal velocity $= u\cos\alpha$

& vertical $''$ $= eu\sin\alpha$

Then 2nd rang $= \dfrac{2(u\cos\alpha)(eu\sin\alpha)}{g}$

$= \dfrac{e(2u^2\sin\alpha\cos\alpha)}{g}$

$= er$

Q :- A ball vertically fall for two seconds and hits the plane inclined 30° to the horizontal. If the coefficient of restitution be 3/4. then show that the time ellapses before it hits the plane is 3 sec

Solution:- Let the ball strikes the inclined plane falling vertically for 2 seconds.

Therefore velocity of ball before impact $v = u + at$

$v = 0 + 32(2)$

$v = 64$ ft/s

Let $u = 64$ ft/s

After impact its component along & perpendicular to the plane are $u\sin 30$ & $eu\cos 30$

Now time of flight is

$t = \dfrac{2eu\cancel{\cos}30}{g\cancel{\cos}30} \Rightarrow \dfrac{\cancel{2}(3/4)\cancel{64}^{4}}{\cancel{32}_{16}}$

$t = 3$ sec

Q:- Loss of Kinetic Energy

vvs

Solution:- Let m & m' be the mass of two balls moving with velocity u & u' let after impact the velocities are v & v' respectively. let e be the coefficient of elasticity

Total K.E before impact $= \frac{1}{2}mu^2 + \frac{1}{2}m'u'^2$

& " " after " $= \frac{1}{2}mv^2 + \frac{1}{2}m'v'^2$

Now by law of conservation of momentum

$$mv + m'v' = mu + m'u' \quad \text{--- (1)}$$

$$m^2v^2 + m'^2v'^2 + 2mm'vv' = m^2u^2 + m'^2u'^2 + 2mm'uu' \quad \text{--- (2)}$$

Also by Newton's experimental law

$$v - v' = -e(u - u')$$

Squaring both sides

$$v^2 + v'^2 - 2vv' = e^2(u - u')^2$$

Multiply both sides by mm'

$$mm'v^2 + mm'v'^2 - 2mm'vv' = mm'e^2(u - u')^2 \quad \text{--- (3)}$$

Adding equation No. (2) & (3)

$$m^2v^2 + m'^2v'^2 + \cancel{2mm'vv'} = m^2u^2 + m'^2u'^2 + 2mm'uu'$$

$$mm'v^2 + mm'v'^2 - \cancel{2mm'vv'} = mm'e^2(u - u')^2$$

$$mv^2(m+m') + m'v'^2(m+m') = m^2u^2 + m'^2u'^2 + 2mm'uu' + mm'e^2(u-u')^2$$

$$\Rightarrow (m+m')(mv^2+m'v'^2) = m^2u^2 + m'^2u'^2 + 2mm'uu' + mm'(u-u')^2 - mm'(u-u')^2 + mm'e^2(u-u')^2$$

$$\Rightarrow (m+m')(mv^2+m'v'^2) = m^2u^2 + m'^2u'^2 + 2mm'uu' + mm'(u^2+u'^2-2uu') - mm'(u-u')^2(1-e^2)$$

$$\Rightarrow (m+m')(mv^2+m'v'^2) = m^2u^2 + m'^2u'^2 + \cancel{2mm'uu'} + mm'u^2 + mm'u'^2 - \cancel{2mm'uu'} - mm'(u-u')^2(1-e^2)$$

$$\Rightarrow (m+m')(mv^2+m'v'^2) = (m+m')mu^2 + (m+m')m'u'^2 - mm'(u-u')^2(1-e^2)$$

$$\Rightarrow (m+m')(mv^2+m'v'^2) = (m+m')(mu^2+m'u'^2) - mm'(u-u')^2(1-e^2)$$

Dividing both sides by $2(m+m')$

$$\frac{1}{2}mv^2 + \frac{1}{2}m'v'^2 = \frac{1}{2}mu^2 + \frac{1}{2}m'u'^2 - \frac{mm'(u-u')^2}{2(m+m')}(1-e^2)$$

$$\Rightarrow \frac{mm'(u-u')^2}{2(m+m')}(1-e^2) = \left(\frac{1}{2}mu^2 + \frac{1}{2}m'u'^2\right) - \left(\frac{1}{2}mv^2 + \frac{1}{2}m'v'^2\right)$$

$$\Rightarrow \left(\frac{1}{2}mu^2 + \frac{1}{2}m'u'^2\right) - \left(\frac{1}{2}mv^2 + \frac{1}{2}m'v'^2\right) = \frac{mm'(u-u')^2}{2(m+m')}(1-e^2)$$

$$(\text{K.E before impact}) - (\text{K.E after impact}) = \frac{mm'(u-u')^2}{2(m+m')}(1-e^2)$$

$$\text{Loss of K.E} = \frac{mm'(u-u')^2}{2(m+m')}(1-e^2)$$

Q :- Noreen Two elastic spheres of mass "m" collide directly. Show that energy loss during the impact is $\frac{1}{4}m(u^2-v^2)$ where u & v are relative velocities before & after impact.

Solution:- As

$$\text{loss of Kinetic Energy} = \frac{1}{2}\left(\frac{mm'}{m+m'}\right)(u-u')^2(1-e^2)$$

$$m = m'$$

$$\text{loss of K.E} = \frac{1}{2}\left(\frac{mm}{m+m}\right)(u-u')^2(1-e^2)$$

$$= \frac{1}{2}\frac{m^2}{2m}(u-u')^2(1-e^2)$$

$$= \frac{m}{4}(u-u')^2(1-e^2) \quad \text{——} (1)$$

Also given U = relative velocity before impact

$$U = u - u' \quad \text{——} (2)$$

Also V = relative velocity after impact

$$V = v - v' \quad \text{——} (3)$$

Now by experimental law

$$v - v' = -e(u-u')$$

$$V = -eU \quad \text{using equation (1) \& (2)}$$

$$V^2 = e^2U^2 \quad \text{——} (4)$$

So equation 1 becomes as

$$\text{Loss of K.E} = \frac{m}{4}U^2(1-e^2)$$

$$= \frac{m}{4}(U^2 - e^2U^2)$$

$$= \frac{m}{4}(U^2 - V^2) \quad \text{using eqn (4)}$$

Q :- If two inelastic spheres have direct impact then K.E loss by the impact is that of body whose mass is half of harmonic mean b/w those spheres & whose velocity is equal there relative velocity before impact.

Solution:- Since spheres are inelastic

Therefor $e = 0$

$$\text{Loss of K.E} = \frac{1}{2}\left(\frac{mm'}{m+m'}\right)(u-u')^2(1-e^2)$$

As $e = 0$

Loss of K.E $= \frac{1}{2}\left(\frac{mm'}{m+m'}\right)(u-u')^2(1-0)$

$$= \frac{1}{2}\left(\frac{mm'}{m+m'}\right)(u-u')^2 \quad \text{——— (1)}$$

Now harmonic mean b/w m & $m' = \frac{2mm'}{m+m'}$

Half of harmonic mean b/w m & $m' = \frac{1}{2}\left(\frac{2mm'}{m+m'}\right)$

$$= \frac{mm'}{m+m'}$$

Also as relative velocity before impact is $= u-u'$

Now:- K.E of the body whose mass is $\frac{mm'}{m+m'}$ and whose velocity is $u-u'$ is $= \frac{1}{2}\left(\frac{mm'}{m+m'}\right)(u-u')^2$ ——— (2)

which is same as equation (1)

From equation (1) & (2) we conclude that when two inelastic spheres have direct impact then the K.E loss by the impact is that of a body whose mass is half of harmonic mean between two spheres & whose velocities equals their relative velocity after impact.

Q :- A ball collide directly with another ball when is at rest and is itself reduce to rest by impact. If half of the initial K.E is destroyes in collision then find the coefficient of elasticity.

Solution:- Let u be the velocity of 1 st ball before impact and

v be the velocity of 1 st ball after impact.

Let m & m' be their masses then by Newton's experimental law

$$0 - v = -e(u - 0)$$
$$-v = -eu$$
$$v = eu \quad \text{--- (1)}$$

Also by law of conservation of momentum

$$m(0) + m'(v) = mu + m'(0)$$
$$m'v = mu \quad \text{--- (2)}$$

Now Initial K.E $= \frac{1}{2}mu^2 + \frac{1}{2}m'(0)^2$

$$= \frac{1}{2}mu^2$$

and final K.E $= \frac{1}{2}m(0)^2 + \frac{1}{2}m'(v)^2$

$$= \frac{1}{2}m'v^2 \quad \text{--- (3)}$$

$\therefore$ Loss of K.E $= \left(\frac{1}{2}mu^2 - \frac{1}{2}m'v^2\right)$

$$= \frac{1}{2}(mu^2 - m'v^2)$$

Given loss of K.E $= \frac{1}{2}$(Initial K.E)

$$\cancel{\tfrac{1}{2}}(mu^2 - m'v^2) = \cancel{\tfrac{1}{2}}\left(\frac{1}{2}mu^2\right)$$
$$mu^2 - m'v^2 = \frac{1}{2}mu^2$$
$$mu^2 - \frac{1}{2}mu^2 = m'v^2$$
$$\frac{1}{2}mu^2 = m'v^2$$

$$\frac{1}{2} m v^2 = (m'u) v$$

using equation ① & ②

$$\frac{1}{2} m v^2 = (mu)(eu)$$

$$\frac{1}{2} m v^2 = m e v^2$$

$$\boxed{e = 1/2}$$

Q :- Three perfectly elastic balls of mass m, $2m$ & $3m$ are placed in a straight line. The first ball collide directly on the 2nd with a velocity u & then the 2nd collide with 3rd. find the velocity of 3rd ball after impact.

Solution:- Let v and v' be the velocities of balls m & $2m$ after impact

Now by Newton's Experimental law. $v - v' = -e(u - u')$

$$v - v' = -1(u - 0)$$

$$v - v' = -u \quad \text{—— ①}$$

Also by law of conservation of momentum

$$mv + 2mv' = mu + 2m(0)$$

$$m(v + 2v') = mu$$

$$v + 2v' = u \quad \text{—— ②}$$

Subtracting equation ① from ②

$$\cancel{v} + 2v' = v$$
$$\underline{\cancel{v} \mp v' = \mp v}$$
$$3v' = 2v$$
$$\boxed{v' = \tfrac{2}{3}v}$$

Now v' be the velocity of 2nd ball before impact with the 3rd ball let V & V' be the velocities of 2nd & 3rd ball after impact resp. then by Newton's experimental law.

$$V - V' = -e(v' - 0)$$
$$V - V' = -ev' \Rightarrow V - V' = -v' \quad \because e = 1 \quad \text{— ③}$$

Also by law of conservation of momentum

$$2mV + 3mV' = 2mv' + 3m(0)$$
$$\cancel{m}(2V + 3V') = 2\cancel{m}v'$$
$$2V + 3V' = 2v' \quad \text{— ④}$$

Multiply ~~Subtracting~~ equation ③ by 2

$$2V - 2V' = -2v' \quad \text{— ⑤}$$

Sub equation ⑤ from ④

$$2\cancel{V} + 3V' = 2v'$$
$$\underline{2\cancel{V} \mp 2V' = \mp 2v'}$$
$$5V' = 4v'$$
$$V' = \frac{4}{5}\left(\frac{2v}{3}\right) \Rightarrow V' = \frac{8}{15}v$$

Q :- If the masses of two balls be 2:1 & the respective velocities 1:2 in opposite direction. Show that if the coefficient of elasticity is $5/6$ then each ball moves back after impact with $5/6$ th of its original velocity

Solution:- Since the masses are in ratio 2:1

Let the masses of two spheres be $2m$ & m with velocities u & $-2u$ respectively. If v & v' be their velocities after impact then by Newton's experimental law.

$$v - v' = -\frac{5}{6}(u-(-2u))$$

$$v - v' = -\frac{5}{6}(u+2u) \Rightarrow v - v' = -\frac{5}{\cancel{6}_2}\cancel{3}u$$

$$v - v' = -\frac{5}{2}u \quad \text{——} (1)$$

By law of conservation of momentum.

$$2mv + mv' = 2mu - 2mu$$

$$m(2v + v') = 0$$

$$2v + v' = 0 \quad \text{——} (2)$$

Adding (1) & (2)

$$3v = -\frac{5}{2}u$$

$$\boxed{v = -\frac{5}{6}(u)}$$

Putting in equation (1)

$$-\frac{5}{6}u - v' = -\frac{5}{2}u$$

$$-\frac{5}{6}u + \frac{5}{2}u = v'$$

$$v' = \frac{10}{6}u \Rightarrow \boxed{v' = \frac{5}{6}\, 2u}$$

VVImp

Q :- Two spheres of masses m and m' collide directly with velocities u & u' respectively. Show that momentum lossed by one and given by other is $\frac{mm'}{m+m'}(1+e)(u-u')$ where e is the co-efficient of elasticity.

Solution:- Let the two spheres of masses m and m' are moving with velocities u and u' before impact. Let after impact their velocities v & v' respectively.

Now by Newton's Experimental law.

$$\frac{v - v'}{u - u'} = -e$$

$$v - v' = -e(u - u')$$

$$v - v' = -eu + eu' \quad \text{——} \quad (1)$$

Also by law of conservation of momentum.

$$mv + m'v' = mu + m'u' \quad \text{——} \quad (2)$$

from equation No. 1

$$v - v' = eu' - eu$$

multiply both sides by m'

$$m'v - m'v' = m'eu' - m'eu \quad \text{—} \quad (3)$$

Adding equation (2) & (3)

$$\cancel{m v} + m'v' = mu + m'u'$$
$$m'v - \cancel{m'v'} = m'eu' - m'eu$$
$$(m+m')v = (m - m'e)u + (1+e)m'u'$$
$$v = \frac{(m - m'e)u + (1+e)m'u'}{m+m'} \quad \text{— (4)}$$

Also from equation (1)

$$v - v' = eu' - eu$$

multiply both sides by m

$$mv - mv' = meu' - meu \quad \text{— (5)}$$

Subtract equation (5) from (2)

$$\cancel{mv} + m'v' = mu + m'u'$$
$$\underset{-}{\cancel{mv}} \mathop{-}_{+} mv' = \mathop{-}_{+} meu \mathop{+}_{-} meu'$$
$$(m+m')v' = (1+e)mu + (m' - me)u'$$
$$v' = \frac{(1+e)mu + (m' - me)u'}{m+m'} \quad \text{— (6)}$$

equation (4) & (6) are required velocities after impact

Now momentum loss by sphere of mass

$$m = mu - mv$$
$$= m(u - v)$$
$$= m\left[u - \frac{(m - m'e)u + (1+e)m'u'}{m+m'}\right]$$
$$= m\left[\frac{\cancel{mu} + m'u - \cancel{mu} + m'eu - m'u' - m'u'e}{m+m'}\right]$$
$$= \frac{m}{m+m'}\left[(1+e)m'u - m'u'(1+e)\right]$$

$$= \frac{m(1+e)(m'u - m'u')}{m+m'} \quad \text{——— (7)}$$

Now momentum gain by 2nd sphere of mass m'

$$= m'v' - m'u'$$

$$= m'[v' - u']$$

$$= m'\left[\frac{(1+e)mu + (m'-me)u'}{m+m'} - u'\right]$$

$$= m'\left[\frac{mu + meu + \cancel{m'u'} - meu' - mu' - \cancel{m'u'}}{m+m'}\right]$$

$$=$$

$$= \frac{m(1+e)(m'u - m'u')}{m+m'} \quad \text{——— (8)}$$

from equation (7) & (8) it is called that

Momentum lost by 1st Sphere = Momentum gain by 2nd Sphere.

Q :- A bullet of mass m moving with velocity v strikes a block of mass M which is free to move in the direction of motion of bullet and is embadded in it. Show that a proportion $\frac{M}{m+M}$ of Kinetic Energy is lost.

(b) If block is afterward strikes by an equal bullet moving in the same direction with the same velocity show that there is a further loss of energy is equal to $\frac{M^2 m v^2}{2(M+2m)(M+m)}$

Solution:- Let v_1 be the speed of block after the bullet embadded (پیوست ہونا) in it.

Then by law of conservation of momentum

$$m v + M(0) = m(0) + (M+m)(v_1)$$

{Bullet کے لگنے سے (M+m) mass کا اس میں پیوست ہو گیا}

$$m v + 0 = 0 + (M+m) v_1$$

$$v_1 = \frac{m v}{M+m} \quad \text{——— (1)}$$

Also loss of K.E $= \frac{1}{2} m v^2 - \frac{1}{2}(M+m) v_1^2$

$$= \frac{1}{2} m v^2 - \frac{1}{2} \cancel{(M+m)} \left(\frac{m^2 v^2}{(M+m)^{\cancel{2}}} \right)$$

$$= \frac{1}{2} m v^2 - \frac{1}{2} \frac{m^2 v^2}{M+m}$$

$$= \frac{1}{2} m v^2 \left(1 - \frac{m}{M+m} \right)$$

$$= \frac{1}{2} m v^2 \left(\frac{M + \cancel{m} - \cancel{m}}{M+m} \right)$$

$$= \frac{1}{2} m v^2 \frac{M}{M+m}$$

Loss of kinetic energy $= \frac{1}{2} \frac{m M v^2}{m+M}$

$$= \frac{1}{2} \left(\frac{m M}{m+M} \right) v^2$$

Fraction of K.E loss $= \dfrac{\cancel{\frac{1}{2}}\left(\frac{\cancel{m} M}{m+M}\right)\cancel{v^2}}{\cancel{\frac{1}{2}}\,\cancel{m}\,\cancel{v^2}} = \dfrac{M}{m+M}$

(b) Let v_2 be the speed of block when the 2nd bullet is embadded in it.

Then by law of conservation of momentum.

$$mv + (M+m)v_1 = m(0) + (M+2m)v_2$$

$$mv + \cancel{M+m}\left[\frac{mv}{\cancel{M+m}}\right] = (M+2m)v_2$$

$$mv + mv = (M+2m)v_2$$

$$2mv = (M+2m)v_2$$

$$v_2 = \frac{2mv}{M+2m}$$

loss of K.E $= \frac{1}{2}mv^2 + \frac{1}{2}(M+m)v_1^2 - \frac{1}{2}(M+2m)v_2^2$

$$= \frac{1}{2}mv^2 + \frac{1}{2}(M+m)\left(\frac{mv}{M+m}\right)^2 - \frac{1}{2}(M+2m)\left(\frac{2mv}{M+2m}\right)^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}(M+m)\frac{m^2v^2}{(M+m)^2} - \frac{1}{2}(M+2m)\frac{4m^2v^2}{(M+2m)^2}$$

$$= \frac{1}{2}mv^2 + \frac{m^2v^2}{2(M+m)} - \frac{2m^2v^2}{(M+2m)}$$

$$= \frac{mv^2(M+m)(M+2m) + m^2v^2(M+2m) - 4m^2v^2(M+m)}{2(M+m)(M+2m)}$$

$$= \frac{mv^2(M^2+2mM+mM+2m^2) + m^2v^2(M+2m) - 4m^2v^2(M+m)}{2(M+m)(M+2m)}$$

$$= \frac{mv^2M^2 + 3m^2Mv^2 + 2m^3v^2 + m^2v^2M + 2m^3v^2 - 4m^2v^2M - 4m^3v^2}{2(M+m)(M+2m)}$$

$$= \frac{mv^2M^2}{2(M+m)(M+2m)} \quad \text{Ans.}$$

Q :- An imperfectly elastic sphere of mass 'm' moving with velocity 'v' impacts on other sphere of mass 'm'' at rest. the 2nd sphere afterwards strikes a vertical plane at right angle to its path. Show that there will be no further impact of sphere of mass $m(1+e'+ee')<em$ where e & e' are co-efficients of elasticity b/w spheres & plane.

Solution:- Let v_1 & v_2 be the velocities of the spheres after impact. Then by Newton's experimental law.

$$v_1 - v_2 = -ev \quad \text{——} ①$$

Also by law of conservation of momentum

$$mv_1 + m'v_2 = mv \quad \text{——} ②$$

Multiply equ ① by m

$$mv_1 - mv_2 = -mev \quad \text{——} ③$$

Subtract equ ③ from ②

$$\cancel{mv_1} + m'v_2 = mv$$

$$\underline{\cancel{mv_1} - mv_2 = -mev}$$

$$(m+m')v_2 = (1+e)mv$$

$$v_2 = \frac{(1+e)mv}{m+m'}$$

Putting value of v_2 in equ ①

$$v_1 - \frac{(1+e)mu}{m+m'} = -eu$$

$$v_1 = \frac{(1+e)mu}{m+m'} - eu$$

$$= \frac{mu + m\cancel{eu} - m\cancel{eu} - m'eu}{m+m'}$$

$$v_1 = \frac{(m-em')u}{m+m'}$$

The sphere of mass m' moves with velocity v_2 & strikes the vertical plane. It thus rebound with velocity $e'v_2 = \frac{e'(1+e)mu}{m+m'}$ away from the plane.

Now:- The velocity of sphere of mass m away from the plane =

$$-v_1 \Rightarrow \frac{-(m-em')u}{m+m'} \Rightarrow \frac{(em'-m)u}{m+m'}$$

∴ There will be no further impact if $e'v_2 < -v_1$

$$\frac{e'(1+e)m\cancel{u}}{(m+\cancel{m'})} < \frac{(em'-m)\cancel{u}}{(\cancel{m+m'})}$$

$$(1+e)me' < em'-m$$

$$m+(1+e)me' < em'$$

$$m\{1+(1+e)e'\} < em' \quad \text{Proved}$$

Q :- From a point on a smooth horizontal plane, a ball is projected with velocity u at angle α to the horizontal show that it will keep rebounding from the plane for a time $\frac{2u \sin}{g(1-e)}$ and will have a range $\frac{u^2 \sin 2\alpha}{g(1-e)}$, being the co-efficient of elasticity

Solution Let the ball is projected from a point 'O' The vertical and horizontal component of u are "$u \sin \alpha$" and "$u \cos \alpha$" respectively.

The ball at 1st describe a parabola and on strike the plane at its A, B, C, D.... striking

Thus the ball describes a series of parabola.

The vertical component of velocity becomes $eu \sin \alpha$ after 1st rebound at point 'A' but the horizontal component remain same $u \cos \alpha$ as there is no horizontal force is acting on ball.

So after 2nd, 3rd, 4th rebound vertical components changes i.e $u\sin\alpha$, $eu\sin\alpha$, $e^2u\sin\alpha$, $e^3u\sin\alpha$ ....
But horizontal components remain same.
Since time of flight $= \frac{2u\sin\alpha}{g}$

Therefore total time of flight taken by the ball comes to rest is

$$\frac{2u\sin\alpha}{g} + \frac{2eu\sin\alpha}{g} + \frac{2e^2u\sin\alpha}{g} + \dots$$

$$= \frac{2u\sin\alpha}{g}\left[1+e+e^2+\dots\right]$$

$$= \frac{2u\sin\alpha}{g}\left(\frac{1}{1-e}\right) \qquad \left\{S_\infty = \frac{a_1}{1-r}\right.$$

$$= \frac{2u\sin\alpha}{g(1-e)}$$

Also horizontal range = (Horizontal velocity)(time of flight)

$$= u\cos\alpha\left(\frac{2u\sin\alpha}{g(1-e)}\right)$$

$$= \frac{u^2\sin 2\alpha}{g(1-e)} \text{ Ans.}$$

Q :- A particle of elasticity drop from vertical height 'a' upon the highest

Point of a plane which is of length 'b' and is inclined at angle α to the horizontal and decend to the bottom in 3 jumps. Prove that $b = 4ae(1+e)(1+e+e^3)(1+e^2)\sin\alpha$

Solution:. Let a particle strikes the inclined plane at 'O' after falling vertically from a height 'a'

Therefore velocity of The particle before impact

$$\sqrt{2ga} = u \text{ (say)} \quad \text{——— ①}$$

acting vertically downward.

The component of velocity of impact are (i) $u\cos\alpha$ which is $\perp$ to plane

(ii) $u\sin\alpha$ which is along the plane

Then $u\sin\alpha$ remain same while the vertical component of velocity at point O, A, B are $eu\cos\alpha$, $e^2u\cos\alpha$, $e^3u\cos\alpha$ respectively. Therefore the time of flight of three jumps from O to A, A to B and B to C are $\frac{2eu\cos\alpha}{g\cos\alpha}$, $\frac{2e^2u\cos\alpha}{g\cos\alpha}$ & $\frac{2e^3u\cos\alpha}{g\cos\alpha}$ respectively.

This implies $\frac{2eu}{g}, \frac{2e^2u}{g}, \frac{2e^3u}{g}$ respectively.

Therefore time taken from O to C =

$$\frac{2eu}{g} + \frac{2e^2u}{g} + \frac{2e^3u}{g}$$

$$= \frac{2eu}{g}(1+e^2+e^3)$$

Now considering the motion from O to C parallel to the plane with acceleration $g\sin\alpha$ using 2nd equ of motion

$$s = ut + \frac{1}{2}gt^2$$

$$OC = (u\sin\alpha)t + \frac{1}{2}(g\sin\alpha)t^2$$

$$b = (u + \frac{1}{2}gt)t\sin\alpha$$

$$= \left[u + \frac{1}{2}g\left(\frac{2eu}{g}\right)(1+e+e^2)\right]\left[\left(\frac{2eu}{g}\right)(1+e+e^2)\sin\alpha\right]$$

$$= \left[u + eu(1+e+e^2)\right]\frac{2eu}{g}(1+e+e^2)\sin\alpha$$

$$= \frac{2eu^2}{g}(1+e+e^2)\sin\alpha + \frac{2e^2u^2}{g}(1+e+e^2)^2\sin\alpha$$

$$= \frac{2e(2ga)}{g}(1+e+e^2)\sin\alpha + \frac{2e^2(2ga)}{g}(1+e+e^2)^2\sin\alpha$$

$$= 4ae(1+e+e^2)\sin\alpha + 4ae^2(1+e+e^2)^2\sin\alpha$$

$$= 4ae(1+e+e^2)\sin\alpha\left[1 + e(1+e+e^2)\right]$$

$$= 4ae(1+e+e^2)\sin\alpha(1+e+e^2+e^3)$$

$$= 4ae(1+e+e^2)\sin\alpha(1(1+e) + e^2(1+e))$$

$$b = 4ae(1+e+e^2)\sin\alpha(1+e)(1+e^2)$$

Q. :- A particle after falling from rest through a distance small 'h' strikes a plane inclind an angle α to the horizontal. Show that if e is the co-efficient of elasticity then the distance between the first two points of impact is $4he(1+e)\sin\alpha$ and the whole range on the inclined plane ceases to rebound is $\frac{4he\sin\alpha}{(1-e)^2}$

Solution :- Let the particle 1st strikes at point on on inclined plane at angle α, After falling height 'h'. Then its vertical velocity at 'O' before impact is $\sqrt{2gh}$

let $v = \sqrt{2gh}$

Then the components of velocity along and perpendicular to the plane are $v\sin\alpha$ and $v\cos\alpha$ respectively. The particle strikes the plane at point O, A, B, C The component of velocity $\perp$ to plane at point O, A, B, C are $ev\cos\alpha, e^2v\cos\alpha, e^3v\cos\alpha, e^4v\cos\alpha$ respectively.

Let t_1 be the time of flight from O to A. In this time the vertical distance is

$$0 = (ev\cos\alpha)t_1 - \frac{1}{2}(g\cos\alpha)t_1^2$$

$$\frac{1}{2}g\cos\alpha\, t_1^2 = ev\cos\alpha\, t_1$$

$$t_1 = \frac{2ev}{g}$$

So $OA = (v\sin\alpha)t_1 + \frac{1}{2}(g\sin\alpha)t_1^2$

$$= (v\sin\alpha)\left(\frac{2ev}{g}\right) + \frac{1}{2}g\sin\alpha\left(\frac{2ev}{g}\right)^2$$

$$= \frac{2ev^2\sin\alpha}{g} + \frac{1}{2}g\sin\alpha\frac{4e^2v^2}{g^2}$$

$$= \frac{2ev^2\sin\alpha}{g}(1+e)$$

$$= 2e\left(\frac{2gh}{g}\right)(1+e)$$

$$OA = 4eh\sin\alpha(1+e)$$

Now:- time of flight for all trajectories are $\frac{2ev}{g}, \frac{2e^2v}{g}, \frac{2e^3v}{g}, \ldots$

Total time of flight $= \frac{2ev}{g} + \frac{2e^2v}{g} + \frac{2e^3v}{g} + \cdots$

$$= \frac{2ev}{g}(1+e+e^2+\cdots)$$

$$= \frac{2ev}{g}\left(\frac{1}{1-e}\right) \qquad \left\{S_\infty = \frac{a}{1-r}\right.$$

Now the whole range on inclined plane till the particle cease to rebound

$$= (u\sin\alpha)t + \frac{1}{2}(g\sin\alpha)t^2$$

$$= u\sin\alpha\,\frac{2eu}{g(1-e)} + \frac{1}{2}\,g\sin\alpha\left[\frac{4e^2u^2}{g^2(1-e)^2}\right]$$

$$= \frac{2eu^2\sin\alpha}{g(1-e)} + \frac{2e^2u^2\sin\alpha}{g(1-e)^2}$$

$$= \frac{2eu^2\sin\alpha}{g(1-e)}\left[1+\frac{e}{1-e}\right]$$

$$= \frac{2e(2gh)\sin\alpha}{g(1-e)}\left[\frac{1-\cancel{e}+\cancel{e}}{1-e}\right]$$

$$= \frac{4eh\sin\alpha}{(1-e)^2}$$

Q :- Spheres of masses M & m impinges direct when moving in opposite direction with speed u & v respectively & the sphere of mass m is brought to rest by the collision. prove that $v(m-eM) = M(1+e)u$. After collision the sphere of mass M is acted on by a constant of which bring it to rest after travelling a distance a^3 prove that the magnitude of this force is $\frac{me^2(u+v)^2}{2a}$

Solution:- Let V be the velocity of sphere of mass M After impact then by law of conservation of momentum

$$MV + m(0) = Mu + m(-v)$$

$$MV = Mu - mv \quad \text{---} (1)$$

Also by Newton's Experimental law

$$V - 0 = -e[u - (-v)]$$

$$V = -e(u+v) \quad \text{---} (2)$$

Put in equ (1)

$$M(-e)(u+v) = Mu - mv$$

$$-Meu - Mev = Mu - mv$$

$$mv - Mev = Mu + Meu$$

$$(m - eM)v = M(1+e)u$$

Hence proved

Now after impact the sphere of mass M has initial velocity $V = -e(u+v)$ & final velocity is zero along with a distance a. Now by third equation of motion

$$v^2 - u^2 = 2fs \quad (\text{here } f \text{ is arc})$$

$$0 - [-e(u+v)]^2 = 2fa$$

$$f = \frac{-e^2(u+v)^2}{2a}$$

-ve sign show retradation

The force = mass x retradation

$$= M \times \left(\frac{-e^2 (u+v)^2}{2a} \right)$$

$$= \frac{-Me^2 (u+v)^2}{2a}$$

Magnitude of force $= \dfrac{Me^2 (u+v)^2}{2a}$

OBLIQUE CÕLLISION

Q :- A smooth sphere of mass m and whose co-efficient of restitution is e, impings obliquely on a fixed smoth plane. Find its sub sequent of motion

Solution:-

Let a sphere strike a fixed smooth plane with velocity u & make angle α with the normal ON & rebounds with velocity V makes angle θ with the normal ON. Since plane is fixed and smooth Therefore the resolved part of velocity ⊥ to the normal i.e along the tangent remain same i.e.

$$v\sin\theta = u\sin\alpha \quad \text{—— (1)}$$

Now by Newton's law of restitution

$$\frac{v\cos\theta - 0}{-u\cos\alpha - 0} = -e$$

$$\frac{v\cos\theta}{-u\cos\alpha} = -e$$

$$v\cos\theta = e(u\cos\alpha) \quad \text{—— (2)}$$

Squaring & Adding equ (1) & (2)

$$v^2\sin^2\theta + v^2\cos^2\theta = u^2\sin^2\alpha + e^2u^2\cos^2\alpha$$

$$v^2(\sin^2\theta + \cos^2\theta) = u^2(\sin^2\alpha + e^2\cos^2\alpha)$$

$$v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha) \quad \text{—— (3)}$$

Dividing equ (2) by (1)

$$\frac{\cancel{v}\cos\theta}{\cancel{v}\sin\theta} = \frac{e\cancel{u}\cos\alpha}{\cancel{u}\sin\alpha}$$

$$\cot\theta = e\cot\alpha \quad \text{—— (4)}$$

Equation (3) & (4) gives the velocities after impact in magnitude & direction

Corollary I:- when the impact is direct then we have $\alpha = 0$ so by equ (1)

$$v\sin\theta = u\sin 0 \Rightarrow v\sin\theta = 0$$

$$\sin\theta = 0 \Rightarrow \theta = \sin^{-1}(0)$$

$$\boxed{\theta = 0}$$

also from equ ②

$$v\cos\theta = eu\cos\theta$$

$$v = eu$$

Hence when a sphere impings directly on a smooth fixed plane the direction of motion of sphere is reversed & its velocity is reduced in the ratio $e:1$

$$\frac{v}{u} = \frac{e}{1} \Rightarrow v:u = e:1$$

Corollary II:- when the collision is perfectly elastic i.e $e = 1$ then from equ ③

$$v^2 = u^2(\sin^2\alpha + \cos^2\alpha)$$

$$v^2 = u^2 \Rightarrow v = u$$

velocity remains unchanged.

Corollary III:- when the velocity is in-elastic i.e $e = 0$

then from equ ②

$$v\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = 90^\circ$$

Also from equ ①

$$v\sin 90^\circ = u\cos\alpha \Rightarrow v(1) = u\sin\alpha$$

$$v = u\sin\alpha$$

sphere does not rebound. it simply slide along the plane.

Oblique impact of two smooth spheres.

Q :- Two smooth elastic sphere of mass m_1 & m_2 moving with velocities v_1 & v_2 collide so that their direction of motion before impact makes angle α & β with the line of centre. Find velocities & direction of motion after impact

Solution:-

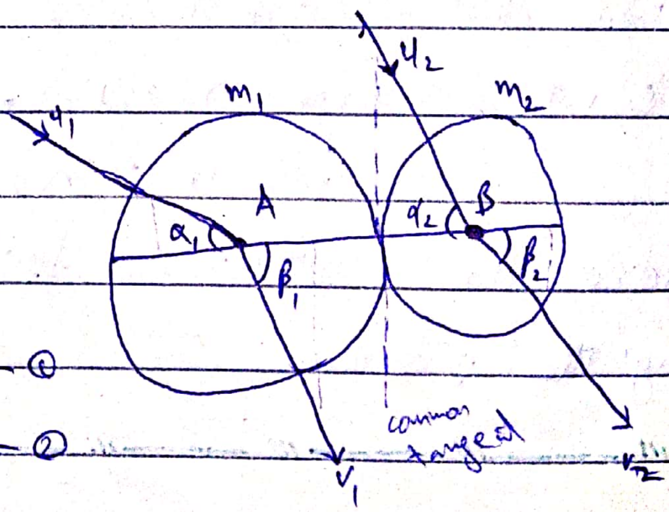
P.U 2001:- Discuss the impact of two smooth spheres which collide obliquely?

Solution Let m_1 & m_2 be masses of spheres with centre A & B at the time of impact. Let u_1 & u_2 be their velocities before impact. Let u_1 & u_2 makes angle α_1 & α_2 with line joining their centres (along the common normal). Let v_1 and v_2 be their velocities after the impact. which makes angle B_1 & B_2 with line joining the centers.

Equating velocities along the common tangent

$v_1 \sin\beta_1 = u_1 \sin\alpha_1$ — ①

$v_2 \sin\beta_2 = u_2 \sin\alpha_2$ — ②



By Law of Conservation of Linear Momentum along common normal.

$$m_1(v_1\cos\beta_1)+m_2(v_2\cos\beta_2)=m_1(u_1\cos\alpha_1)+m_2(u_2\cos\alpha_2) \quad \text{--- (3)}$$

By Newtons Law of Restitution along the common normal.

$$v_2\cos\beta_2-v_1\cos\beta_1=-e(u_2\cos\alpha_2-u_1\cos\alpha_1) \quad \text{--- (4)}$$

eleminate v_2 from equ (3) & (4)

multiply equ (4) by m_2 & subtract from (3)

$$m_1v_1\cos\beta_1+m_2v_2\cos\beta_2=m_1u_1\cos\alpha_1+m_2u_2\cos\alpha_2$$

$$\underset{+}{-}m_2v_1\cos\beta_1\underset{-}{+}m_2v_2\cos\beta_2=\underset{+}{-}em_2u_2\cos\alpha_2\underset{-}{+}em_2u_1\cos\alpha_1$$

$$(m_1+m_2)v_1\cos\beta_1=m_1u_1\cos\alpha_1+m_2u_2\cos\alpha_2+em_2u_2\cos\alpha_2-em_2u_1\cos\alpha_1$$

$$v_1\cos\beta_1=\frac{m_1u_1\cos\alpha_1+m_2u_2\cos\alpha_2+em_2u_2\cos\alpha_2-em_2u_1\cos\alpha_1}{m_1+m_2} \quad \text{--- (5)}$$

Now eleminate v_1 from (3) & (4)

Multiply (4) by m_1 & add to (3)

$$m_1v_1\cos\beta_1+m_2v_2\cos\beta_2=m_1u_1\cos\alpha_1+m_2u_2\cos\alpha_2$$

$$m_1v_1\cos\beta_1+m_1v_1\cos\beta_2=-em_1u_2\cos\alpha_2+em_1u_1\cos\alpha_1$$

$$(m_1+m_2)v_2\cos\beta_2=m_1u_1\cos\alpha_1+m_2u_2\cos\alpha_2-em_1u_2\cos\alpha_2+em_1u_1\cos\alpha_1$$

$$v_2\cos\beta_2=\frac{m_1u_1\cos\alpha_1+m_2u_2\cos\alpha_2+em_1u_1\cos\alpha_1-em_1u_2\cos\alpha_2}{m_1+m_2} \quad \text{--- (6)}$$

Squaring & adding (1) & (5)

$$v_1^2=u_1^2\sin^2\alpha_1+\left[\frac{m_1u_1\cos\alpha_1+m_2u_2\cos\alpha_2+em_2u_2\cos\alpha_2-em_2u_1\cos\alpha_1}{m_1+m_2}\right]^2 \quad \text{--- (7)}$$

which gives v_1 Ans

Squaring & Adding ② & ⑧

$$v_2^2 = u_2^2 \sin^2\alpha_2 + \left[\frac{m_1 u_1 \cos\alpha_1 + m_2 u_2 \cos\alpha_2 + e m_1 u_1 \cos\alpha_1 - e m_1 u_2 \cos\alpha_2}{m_1 + m_2}\right]^2 \quad (8)$$

which gives v_2.

Divide ① by ⑤ $\tan\beta_1 = \dfrac{(m_1+m_2)\, u_1 \sin\alpha_1}{m_1 u_1 \cos\alpha_1 + m_2 u_2 \cos\alpha_2 + e m_2 u_2 \cos\alpha_2 - e m_2 u_1 \cos\alpha_1}$

which gives β_1 — ⑨

Divide ② by ⑥ $\tan\beta_2 = \dfrac{(m_1+m_2)\, u_2 \sin\alpha_2}{m_1 u_1 \cos\alpha_1 + m_2 u_2 \cos\alpha_2 + e m_1 u_1 \cos\alpha_1 - e m_1 u_2 \cos\alpha_2}$

which gives β_2 — ⑩

Particular Cases:- ① Let sphere of mass m_2 be at rest

$\therefore u_2 = 0$

from ⑩ $\tan\beta_2 = 0 \Rightarrow \beta_2 = 0$

$\therefore$ Sphere of mass m_2 will move along the line joing their centres ater impact.

② If $u_2 = 0$ & $m_1 = e m_2$ then from equ ⑩

$\tan\beta_2 = 0$, $\beta_2 = 0$ & from equ ⑨

$\tan\beta_1 = \infty$, $\beta_1 = 90$

Sphere of mass m_1 moves at right angle to the line joining the centres.

③ If $m_1 = m_2$, $e = 1$ from equ ⑥

$v_1 \cos\beta_1 = u_2 \cos\alpha_2$ and from ⑦ $v_2 \cos\beta_2 = u_1 \cos\alpha_1$

i.e They exchange their velocities along the line joining centeres.

Kinetic Energy:- If m is mass of body which is moving with velocity v then Kinetic Energy of body is $K.E = \frac{1}{2}mv^2$

Potential Energy:- If a body of mass m gains a height h The potential energ $= mgh$

Direct Collision OR Collision in One Dimention

Case III :- For Elastic collision when masses not equal & both are moving.

$e = 1$ put in equ ④ & ⑤

$$v = \frac{m - m'}{m + m'}u + \frac{2m'u'}{m + m'} \quad \text{——— ⑥}$$

$$v' = \frac{2mu}{m + m'} + \frac{m' - m}{m + m'}u' \quad \text{——— ⑦}$$

Case IV :- For Elastic collision when masses not equal & one is at rest.

Suppose sphere of mass m' is at rest

$\therefore u' = 0$

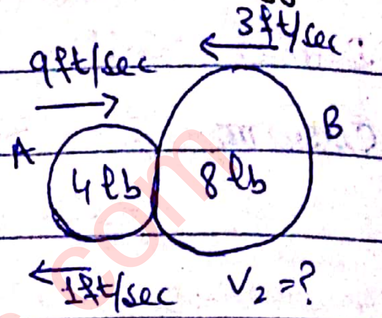
from equ ⑥ & ⑦

$$v = \frac{m - m'}{m + m'}u \quad \text{——— ⑧}$$

$$v' = \frac{2m}{m + m'}u \quad \text{——— ⑨}$$

Q :- Two spheres A & B of masses 4 and 8 lbs moving with velocities 9 ft/sec and 3 ft/sec in opposite direction. If A rebound with a velocity of 1 ft/sec. Find the velocity of B after impact, The co-efficient of elasticity & the loss of kinetic energy.

Solution



$m_1 = 4\,lb$, $m_2 = 8\,lb$

$u_1 = 9\,ft/sec$, $u_2 = -3\,ft/sec$

$v_1 = -1\,ft/sec$, $v_2 = ?$

By law of conservation of momentum

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$$

$$4(-1) + 8v_2 = 4(9) + 8(-3)$$

$$-4 + 8v_2 = 36 - 24$$

$$8v_2 = 12 + 4 \Rightarrow 8v_2 = 16$$

$$v_2 = 2\,ft/sec$$

By Newtons law of restitution.

$$v_2 - v_1 = -e(u_2 - u_1)$$

$$2 - (-1) = -e(-3 - 9)$$

$$3 = 12e \Rightarrow e = 3/12$$

$$\boxed{e = \frac{1}{4}}$$

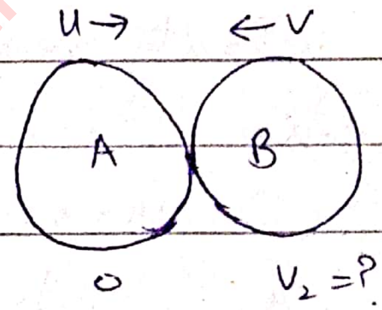
$$\text{Loss of K.E} = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(1 - e^2)(u_2 - u_1)^2$$

$$\text{Loss of K.E} = \frac{1}{2}\cdot\frac{(4)(8)}{4+8}\left(1-\frac{1}{16}\right)(-3-9)^2$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{32}^{16}}{\cancel{12}}\left(\frac{16-1}{16}\right)(\cancel{144}^{12})$$

$$= 180 \text{ foot-poundals.}$$

Q: A ball 'A' moving with velocity u, impinges directly on an equal ball B with velocity v in the opposite direction. If A be brought to rest by the impact show that $u : v' = 1+e : 1-e$ where e is the co-efficient of restitution.



Solution Let m be mass of each ball

$\therefore m_1 = m_2 = m$

$u_1 = u, \quad u_2 = -v, \quad v_1 = 0$

$v_2 = ?$

By law of conservation of momentum

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$$

$$m(0) + m v_2 = mu + m(-v)$$

$$0 + v_2 = u - v \Rightarrow v_2 = u - v \quad \text{——} \ (1)$$

By Newton's Law of restitution.

$$v_2 - v_1 = -e(u_2 - u_1)$$

$$v_2 - 0 = -e(-v - u) \Rightarrow v_2 = ev + eu \quad \text{——} \ (2)$$

from equ (1) & (2)

$$u - v = ev + eu$$

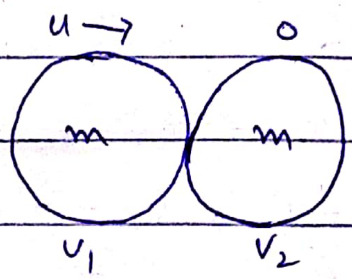
$$u - eu = v + ev \Rightarrow u(1-e) = v(1+e)$$

$$\frac{u}{v} = \frac{1+e}{1-e}$$

$$u : v = 1+e : 1-e \quad \text{Ans.}$$

Q :- A sphere impinges on an equal sphere at rest. If the co-efficient of restitution is e. Show that their velocities after impact are as $1-e : 1+e$. If the mass of 1st sphere be m & that of the 2nd be m'. Show that the 1st can not have its velocity reversed if $m > em'$

Solution Let m be mass of each sphere. Let u be velocity of 1st sphere before impact. the ~~seat~~ 2nd sphere is at rest.



$$m_1 = m_2 = m$$

$u_1 = u$, $u_2 = 0$. Let v_1 & v_2 be their velocities after impact.

By Law of conservation of Linear Momentum

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$$

$$mv_1 + mv_2 = mu + m(0)$$

$$v_1 + v_2 = u \quad \text{——— ①}$$

By Newton's Law of ~~conservation~~ Restitution

$$v_2 - v_1 = -e(u_2 - u_1)$$

$$v_2 - v_1 = -e(0 - u)$$

$$v_2 - v_1 = eu \quad \text{——} ①$$

By adding eqn ① & ②

$$2v_2 = u + eu$$

$$v_2 = \frac{u}{2}(1+e) \quad \text{——} ③$$

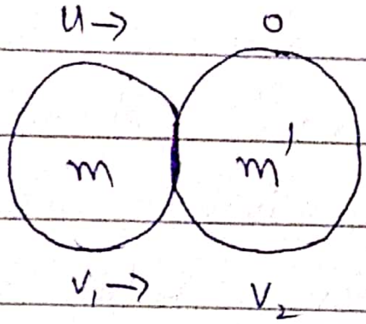
Subtract eqn ① from ②

$$2v_1 = u - eu \Rightarrow v_1 = \frac{u}{2}(1-e) \quad \text{——} ④$$

$$v_1 : v_2 = \frac{u}{2}(1-e) : \frac{u}{2}(1+e)$$

$$v_1 : v_2 = (1-e) : (1+e)$$

Part II



Given m & m' are masses of 1st & 2nd sphere.

formula

$$v_1 = \left(\frac{m_1 - em_2}{m_1 + m_2}\right)u_1 + \frac{(1+e)m_2u_2}{m_1 + m_2}$$

Put $m_1 = m$, $m_2 = m'$, $u_1 = u$, $u_2 = 0$

$$v_1 = \left[\frac{m - em'}{m + m'}\right]u + 0 \Rightarrow v_1 = \left[\frac{m - em'}{m + m'}\right]u$$

Which is velocity of 1st sphere after impact. After the collision the 1st sphere will not be reversed if $v_1 = +ve$

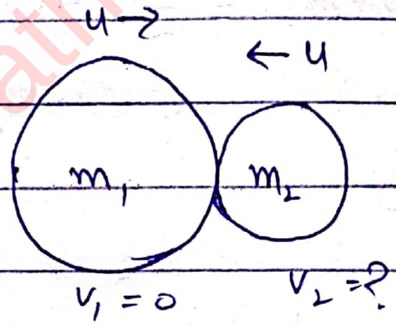
$$v_1 > 0$$

$$\left[\frac{m - em'}{m + m'}\right] u > 0 \Rightarrow m - em' > 0$$

$$m > em'$$

اگر +ve velocity
تو یہ +ve ہے اور
اگر -ve ہے تو -ve
اب یہ sphere Reversed نہیں ہوگا

Q :- Two elastic sphere impinge directly with equal & opposite velocities. Find the ratio of their masses so that one of them may be reduced to rest by the impact, the co-efficient of elasticity being e.



Solution Let m_1 & m_2 be masses of two spheres. Let u be velocity of spheres of mass m_1 before impact. Hence velocity of 2nd sphere is $-u$.

Given one of the sphere is brought to rest after impact. Let spher of m_1 is brought to rest after impact.

$$\therefore v_1 = 0$$

By Law of conservation of Linear Momentum

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$$

$$m_1(0) + m_2 v_2 = m_1 u + m_2(-u)$$

$m_2 v_2 = m_1 u - m_2 u$ ——— ①

By Newton's Law of restitution.

$v_2 - v_1 = -e(u_2 - u_1)$

$v_2 - 0 = -e(-u - u)$

$v_2 = 2eu$ ——— ② Put in eq ①

$m_2(2eu) = m_1 u - m_2 u \Rightarrow 2e m_2 = m_1 - m_2$

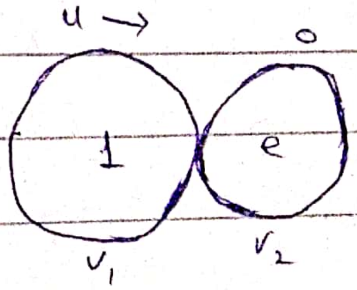
$m_1 = m_2 + 2e m_2 \Rightarrow m_1 = m_2(1+2e)$

$$\frac{m_1}{m_2} = \frac{1+2e}{1}$$

$$m_1 : m_2 = 1+2e : 1$$

Similarly if $v_2 = 0$ then $m_1 : m_2 = 1 : 1+2e$

Example :- A series of n elastic spheres whose masses $1, e, e^2$ etc. are at rest seprated by intervals with their centres on a straight line. The first is made to impinge directly on the second with velocity u. Show that finally the first $(n-1)$ spheres will be moving with the same velocity $(1-e)u$ & the last with velocity u. Prove also that the final K.E of the system is $\frac{1}{2}(1-e+e^n)u^2$



Solution :- Consider the impact of first & 2nd sphere. Let v_1 & v_2 be their velocities after impact respectively.

By Law of conservation of linear momentum

$$(1)v_1 + e(v_2) = 1u + e(0)$$

$$v_1 + ev_2 = u \quad \text{—— ①}$$

By Newton's Law of restitution

$$v_2 - v_1 = -e(0 - u) \Rightarrow v_2 - v_1 = eu \quad \text{—— ②}$$

add ① & ②

حل کرکے v_1 اور v_2 کی ویلیو معلوم کریں

$$ev_2 + v_2 = u + eu$$

$$v_2(1+e) = u(1+e)$$

$$\boxed{v_2 = u}$$ Put in eqn ①

$$v_1 + eu = u \Rightarrow v_1 = u - eu$$

$$v_1 = (1-e)u$$

The 1st sphere impinges upon the 2nd with velocity u & the velocity of 1st after impact is $(1-e)u$ & velocity of 2nd sphere after impact is u.

Similarly when the sphere impinges on the 3rd:

The velocity of 2nd sphere after impact $= (1-e)u$

and " " " 3rd " " " $= u$

Hence velocity of 1st $(n-1)$ spheres after impact are $(1-e)u$ & that of the nth

$\frac{a(1-r^n)}{1-r}$

sphere after impact is u.

mass of spheres are $1, e, e^2, e^3, \ldots, e^{n-1}$

Final K.E of system

$= \frac{1}{2}(1)[(1-e)u]^2 + \frac{1}{2}(e)[(1-e)u]^2 + \frac{1}{2}e^2[(1-e)u]^2 + \ldots$

$+ (n-1)\text{terms} + \frac{1}{2}(e^{n-1})u^2$

$= \frac{1}{2}(1-e)^2u^2\{1+e+e^2+\ldots(n-1)\text{terms}\} + \frac{1}{2}e^{n-1}u^2$

$= \frac{1}{2}(1-e)^2u^2 \cdot \left\{\frac{1-e^{n-1}}{(1-e)}\right\} + \frac{1}{2}e^{n-1}u^2$

$= \frac{1}{2}(1-e)u^2(1-e^{n-1}) + \frac{1}{2}e^{n-1}u^2$

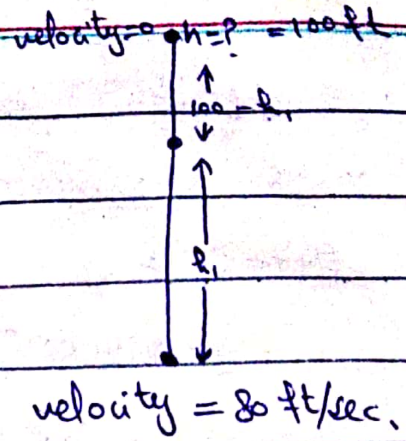
$= \frac{1}{2}u^2\{(1-e)(1-e^{n-1}) + e^{n-1}\}$

$= \frac{1}{2}u^2\{1-\cancel{e^{n-1}} - e + e^n + \cancel{e^{n-1}}\}$

$= \frac{1}{2}u^2(1-e+e^n)$ Ans.

Q:- A Ball is projected vertically with a velocity of 80 ft/sec. The moment it reaches the highest point, a second equal ball is thrown after it from the same point with the same velocity. How High will They collide. If the ~~Solution~~ co-efficient of restitution be $\frac{3}{5}$, find the time the second ball takes to reach the ground.

Solution Maximum height attaind by the 1st Ball



Let h be the maximum height attained by the 1st ball.

formula $\quad v^2 - u^2 = -2gh$

$$0 - (80)^2 = -2(32)\,h$$

$$h = \frac{80 \times 80}{2 \times 32}$$

$$h = 100 \text{ ft}$$

h_1 = Height of point where they meet each.

For <u>1st Ball</u>

formula $\quad x = ut + \frac{1}{2}gt^2$

$$100 - h_1 = (0)t + \frac{1}{2}(32)t^2$$

$$100 - h_1 = 16t^2 \quad \text{——— (1)}$$

For <u>2nd Ball</u>

formula $\quad x = ut - \frac{1}{2}gt^2$

$$h_1 = 80(t) - \frac{1}{2}(32)\,t^2$$

$$h_1 = 80t - 16t^2 \quad \text{——— (2)}$$

adding equ (1) & (2)

$$100 - \cancel{h_1} + \cancel{h_1} = \cancel{16t^2} + 80t - \cancel{16t^2}$$

$$t = \frac{100}{80} = \frac{5}{4} \text{ sec}$$

$\boxed{\text{5/4 second کے بعد دونوں ٹکرائیں گے}}$

put in equ (2)

$$h_1 = 80(5/4) - 16(5/4)^2$$

$$h_1 = 100 - \frac{16(25)}{16} \Rightarrow h_1 = 75 \text{ feet}$$

Velocities before Impact

Let u_1 & u_2 be velocities of 1st & 2nd ball before impact.

For 1st Ball

formula $\quad v^2 - u^2 = 2gx$

$$u_1^2 - 0 = 2(32)(25)$$

$$u_1^2 = 1600$$

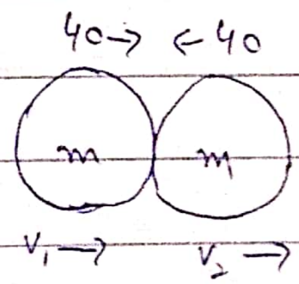
$$u_1 = 40 \text{ ft/sec}$$

For 2nd Ball

$$v^2 - u^2 = -2gx \Rightarrow u_2^2 - (80)^2 = -2(32)(75)$$

$$u_2^2 = (80)^2 - 2(32)(75) \Rightarrow u_2^2 = 1600$$

$$u_2 = 40 \text{ ft/sec}$$



Let m be mass of each ball. Let v_1 & v_2 be their velocities after impact.

By Law of Conservation of Linear Momentum

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$$

$$v_1 + v_2 = u_1 + u_2$$

$$v_1 + v_2 = 40 - 40 \Rightarrow v_1 + v_2 = 0 \quad \text{——— (3)}$$

By Newton's Law of Restitution

$$v_2 - v_1 = -e(u_2 - u_1)$$

$$v_2 - v_1 = -3/5(-40-40)$$

$$v_2 - v_1 = 48 \quad \text{—— (4)}$$

add equ (3) & (4)

$$2v_2 = 48 \Rightarrow v_2 = 24 \text{ ft/sec}$$

t = Time taken by 2nd ball to reach the ground.

formula. $x = ut + \frac{1}{2}gt^2$

$$75 = 24(t) + \tfrac{1}{2}(32)t^2$$

$$16t^2 + 24t - 75 = 0$$

$$t = \frac{-24 \pm \sqrt{(-24)^2 - 4(16)(-75)}}{2(16)}$$

$$t = 1.54 \,,\, -3.04$$

t is $+$ve

$\therefore t \neq -3.04$

$$t = 1.54$$

Total Time = Time taken before impact to reach at height 75 ft + time taken to reach the ground

$$= 5/4 + 1.54 = 1.25 + 1.54$$

$= 2.79$ second.

Impulse of Blow:

If a force $\vec{F}$ acts on a body for time t, then Impulse of force $\vec{I}$ is given by

$$\vec{I} = \vec{F}t$$

& its magnitude is $I = Ft$ ——— ①

We know that

$v = u + at \Rightarrow t = \frac{v-u}{a}$ ——— ②

also $F = ma$ ——— ③

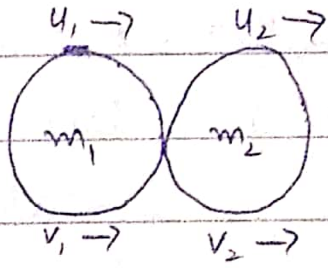
Put values in eqn ①

$$I = ma\left(\frac{v-u}{a}\right)$$

$$I = mv - mu$$

Impulse = Change in momentum.

Impulse of Blow on the sphere of mass m'.



Impulse of Blow on the sphere of mass m'

= Change in momentum of the sphere of mass m_1

$= m_1v_1 - m_1u_1$

$= m_1(v_1 - u_1)$

$$= m_1\left[\frac{(m_1 - em_2)u_1}{m_1+m_2} + \frac{(1+e)m_2u_2}{m_1+m_2} - u_1\right]$$

$$= m_1\left[\frac{m_1u_1 - em_2u_1 + m_2u_2 + em_2u_2 - m_1u_1 - m_2u_1}{m_1+m_2}\right]$$

$$= \frac{m_1 m_2}{m_1+m_2}\left[-eu_1 + u_2 + eu_2 - u_1\right]$$

$$= \frac{m_1 m_2}{m_1+m_2}\left[u_2(1+e) - u_1(1+e)\right]$$

$$= \frac{m_1 m_2}{m_1+m_2}(1+e)(u_2-u_1)$$

Distance b/w Spheres at time t after impact.

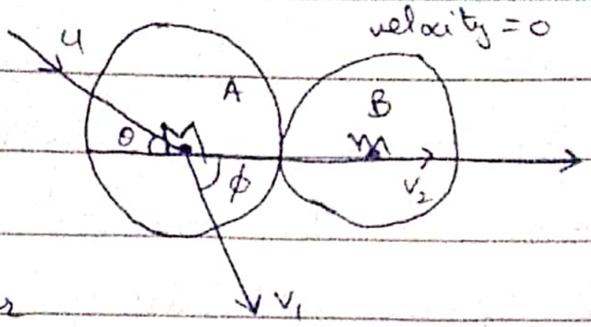
Distance b/w spheres at time t afte impact $\Big\}$ = (relative velocity)(time)

$$=(v_2-v_1)\,t$$

$$= -e(u_2-u_1)t \quad \Big\{ \text{By Newton's Law of Restitution.}$$

Example :- A body of mass M moving with a velocity u collide with another of mass m which rests on a table. Both are perfectly elastic and smooth and the body of mass m is driven in a direction making an angle θ with the previous line of motion of M. Show that its velocity is $\frac{2M}{m+M} u\cos\theta$

Solution The sphere of mass m is at rest. So after collision it will move along the line joinig the centres.



Let v_1 & V_2 be their velocities after impact.

Given V_2 makes angle θ with u. Let v_1 makes angle ϕ with line joining centres.

Given balls are perfectly elastic

$$\therefore \; e = 1$$

By Law of Conservation of Linear Momentum along common normal (Line joining the centres)

$$M(v_1 \cos\phi) + mV_2 = M(u\cos\theta) + 0 \quad \text{——— ①}$$

By Newton's Law of restitution along common normal

$$V_2 - v_1 \cos\phi = -1(0 - u\cos\theta)$$

$$V_2 - V_1 \cos\phi = u\cos\theta \quad \text{——— ②}$$

Multiply equ ② by M & add to ①

$$Mv_1 \cancel{\cos\phi} + mV_2 = Mu\cos\theta$$

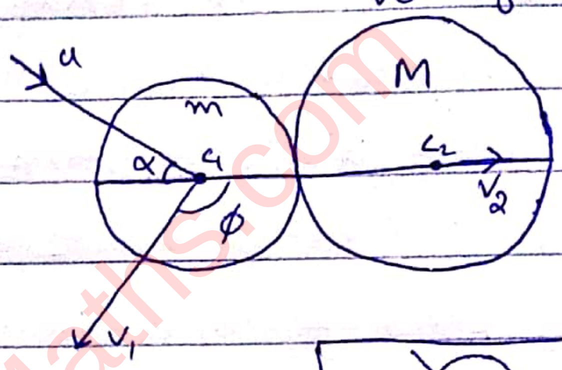
$$MV_2 - Mv_1 \cancel{\cos\phi} = Mu\cos\theta$$

$$(M+m)V_2 = 2Mu\cos\theta$$

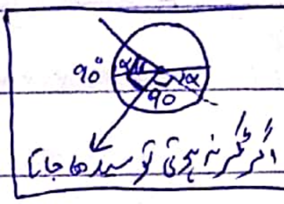
$$V_2 = \frac{2Mu\cos\theta}{M+m}$$

Q. A smooth sphere of mass m, travelling with velocity u, impinges obliquely on a smooth sphere of mass M at rest. Its original line of motion making an angle α with the line of centres. At the moment of impact show that the sphere of mass m will be deflected through a right angle if $\tan^2\alpha = \dfrac{eM - m}{M + m}$

Solution:- Let v_1 & v_2 be velocities of mass m & M after impact respectively.



Let v_1 makes angle ϕ with line joing centres.



The sphere of mass M is at rest. After collision it will moves along the line joing their centres.

The sphere of mass m is deflected at right angle $\therefore \phi = 90° + \alpha$

$$\cos\phi = \cos(90° + \alpha) = -\sin\alpha \quad \text{——} ①$$

$$\sin\phi = \sin(90° + \alpha) = \cos\alpha \quad \text{——} ②$$

By Law of conservation of Linear Momentum along common normal.

$$m(v_1 \cos\phi) + Mv_2 = m(u\cos\alpha) + M(0)$$

$$-m v_1 \sin\alpha + M v_2 = m u \cos\alpha \quad \text{——} ③ \quad \text{using } ①$$

By Law of restitution

$$V_2 - V_1 \cos\phi = -e(0 - u\cos\alpha)$$

$$V_2 + V_1 \sin\alpha = eu\cos\alpha \longrightarrow (4) \quad \text{using } (1)$$

Equating velocities of m along the common tangent

$$V_1 \sin\phi = u\sin\alpha$$

or $\quad V_1 \cos\alpha = u\sin\alpha \longrightarrow (5) \quad$ using (2)

from (4), $\quad V_2 = eu\cos\alpha - V_1\sin\alpha \quad$ Put in (3)

$$-mV_1\sin\alpha + M(eu\cos\alpha - V_1\sin\alpha) = mu\cos\alpha$$

$$-mV_1\sin\alpha + eMu\cos\alpha - MV_1\sin\alpha = mu\cos\alpha$$

$$(m+M)V_1\sin\alpha = (eM - m)u\cos\alpha$$

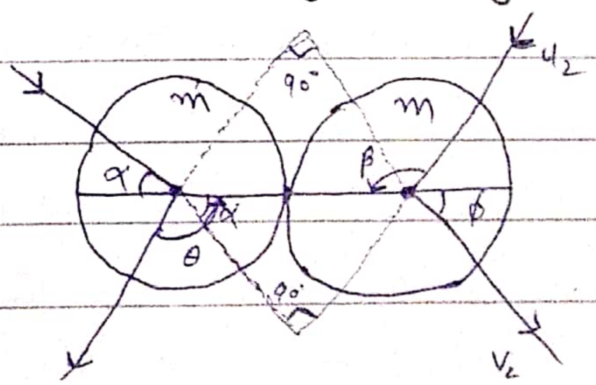
Put values of V_1 using (5)

$$(m+M)\left(\frac{u\sin\alpha}{\cos\alpha}\right)\sin\alpha = (eM-m)u\cos\alpha$$

$$\frac{\sin^2\alpha}{\cos^2\alpha} = \frac{eM-m}{m+M}$$

$$\tan^2\alpha = \frac{eM-m}{M+m}$$

Example :- Two equal, smooth & perfectly elastic spheres moving at right angle to one another impinge obliquely. Show that after impact, they will still move right angle to each angle.



Solution Let m be mass of each sphere. Spheres are perfectly elastic $\therefore e = 1$

Let u_1 & u_2 be their velocities before impact and v_1 & v_2 be their velocities after impact. Let u_1 & u_2 makes angle α & β with line joining their centres respectively. Let v_1 & v_2 make angle θ & ϕ with line joing centres

Given $u_1 \perp u_2 \Rightarrow \beta = 90° + \alpha$

$\cos\beta = \cos(90+\alpha) = -\sin\alpha$ —— ①

$\sin\beta = \sin(90+\alpha) = \cos\alpha$ —— ②

equating velocities along common tangent.

$v_1 \sin\theta = u_1 \sin\alpha$ —— ③

$v_2 \sin\phi = u_2 \sin\beta \Rightarrow v_2 \sin\phi = u_2 \cos\alpha$ —— ④ using ②

By Law of Conservation of Linear Momentum along common normal

$mv_1 \cos\theta + mv_2 \cos\phi = mu_1 \cos\alpha + mu_2 \cos\beta$

$\Rightarrow v_1 \cos\theta + v_2 \cos\phi = u_1 \cos\alpha - u_2 \sin\alpha$ —— ⑤ using ①

By Newton's Law of Restitution

$v_2 \cos\phi - v_1 \cos\theta = -1(u_2 \cos\beta - u_1 \cos\alpha)$

$v_2 \cos\phi - v_1 \cos\theta = u_2 \sin\alpha + u_1 \cos\alpha$ —— ⑥

add ⑤ & ⑥

$2v_2 \cos\phi = 2u_1 \cos\alpha$ —— ⑦

Subtract ⑥ from ⑤

$$2v_1 \cos\theta = -2u_2 \sin\alpha \quad \text{———} (8)$$

divide (3) by (8)

~~divide~~ $$\frac{v_1 \sin\theta}{v_1 \cos\theta} = \frac{u_1 \sin\alpha}{-u_2 \sin\alpha}$$

$$\tan\theta = -\frac{u_1}{u_2} \quad \text{———} (9)$$

Divide (4) by (7) $$\frac{v_2 \sin\phi}{v_2 \cos\phi} = \frac{u_2 \cos\alpha}{u_1 \cos\alpha}$$

$$\tan\phi = \frac{u_2}{u_1} \quad \text{———} (10)$$

from equ (9) & (10) $\tan\theta$ = Slope of v_1

$\tan\phi \tan\theta = -1$ $\tan\phi$ = Slope of v_2

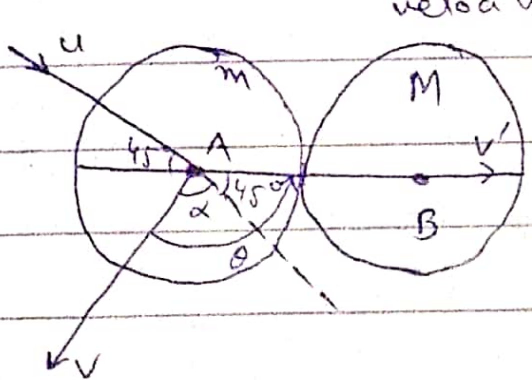
$\therefore v_2 \perp v_1$ A $\tan\theta \tan\phi = -1$

$\therefore v_1 \perp v_2$

Example : A smooth sphere of mass m impinges on another of mass M at rest, The direction of motion making an angle of 45° with the line of centres at the moment of impact. Show that if $e = \frac{1}{2}$ the direction of motion of sphere m is turned through an angle

$$\tan^{-1}\frac{3M}{4m+M}$$

Solution: Let u be velocity of sphere of mass m. before impact &



the sphere of mass M is at rest. Let v & v' be their velocities of m & M after impact respectively.

Given $e = \frac{1}{2}$

Let the sphere of mass m turns through an angle α. Let v makes angle θ with line joining centres. $\theta = \alpha + 45°$

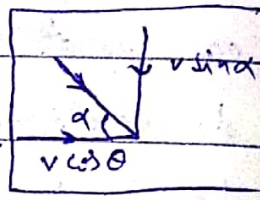
The sphere of mass M is at rest before impact. Hence after impact, it will moves along the line joining their centres.

Equating velocities along common tangent.

For the sphere of mass m

$v\sin\theta = u\sin 45 \Rightarrow v\sin\theta = \frac{1}{\sqrt{2}} u$ —— ①

By Law of conservation of Linear momentum along line joining centres



$m(v\cos\theta) + Mv' = m(u\cos 45) + M(0)$

$\Rightarrow mv\cos\theta + Mv' = \frac{1}{\sqrt{2}} mu$ —— ②

By Newton's Law of Restitution along line joining centres.

$v' - v\cos\theta = -\frac{1}{2}(0 - u\cos 45)$

$v' - v\cos\theta = \frac{1}{2\sqrt{2}} u$ —— ③

Multiply eqn ③ by M & subtract from ②

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Example, Two equal balls of radius "a" are in contact and are struck simultaneously by a ball of radius "c" moving in the direction of their common tangent if all the balls be of the same material, The coefficient of restitution being e. Prove that the imping ball will be reduced to rest if

$$2e = \frac{c^2(a+c)^2}{a^3(2a+c)}$$

Solution Let

A be centre of ball with radius c and M be its mass. Let m be mass of each ball at rest. Given a is radius of each of the ball at rest. Let AX be their common tangent at the point of contact D. All the balls are of same material.

Let ρ be the density.

$$M = \rho(\text{volume}) = \rho\left(\tfrac{4}{3}\pi c^3\right) = \tfrac{4}{3}\rho\pi c^3$$

$$m = \rho(\text{volume}) = \rho\left(\tfrac{4}{3}\pi a^3\right) = \tfrac{4}{3}\rho\pi a^3$$

Third ball is moving along the common tangent.

When two equal balls are in contact and are struck by a ball moving in the direction of their tangent as shown in figure. B & C are centres of balls at rest and A is centre of moving ball.

AX is common tangent to the balls at rest then.

① Along common tangent AX law of linear momentum holds.

② Along AB (or AC) law of restitution holds.

Let u be velocity of ball of mass M & its velocity after impact is zero. Each of ball of mass m are at rest before impact. Let v be their velocities after impact.

Let $\angle BAX = \angle CAX = \theta$

$AB = a + c$, $BD = a$

In $\triangle ABD$

$$\sin\theta = \frac{BD}{AB} = \frac{a}{a+c}$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$= 1 - \left(\frac{a}{a+c}\right)^2$$

$$= \frac{(a+c)^2 + a^2}{(a+c)^2}$$

$$\cos^2\theta = \frac{a^2+c^2+2ac-a^2}{(a+c)^2}$$

$$= \frac{c(c+2a)}{(a+c)^2}$$

By Law of conservation of Linear Momentum along AX

$$M(0) + m(v\cos\theta) + m(v\cos\theta) = Mu + m(0) + m(0)$$

$$2mv\cos\theta = Mu$$

$$2\left(\frac{4}{3}\rho\pi a^3\right) v\cos\theta = \left(\frac{4}{3}\rho\pi c^3\right)u$$

$$2a^3 v\cos\theta = c^3 u \quad \text{——} (1)$$

By Newton's Law of restitution along AB

$$v - 0 = -e(0 - u\cos\theta)$$

$$v = eu\cos\theta \quad \text{——} (2)$$

put value of v in equ (1)

$$2a^3(eu\cos\theta)\cos\theta = c^3 u$$

$$2ea^3\cos^2\theta = c^3$$

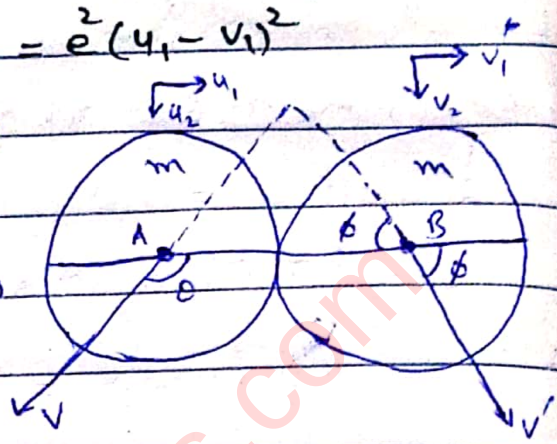
$$2ea^3 \frac{c(c+2a)}{(a+c)^2} = c^3$$

$$2e = \frac{c^3(a+c)^2}{a^3c(c+2a)}$$

$$2e = \frac{c^2(a+c)^2}{a^3(c+2a)} \quad \text{Ans.}$$

Example :- Two equal balls of elasticity e impinge have before impact resolved velocities

u_1, v_1 in the direction of the common normal and u_2, v_2 perpendicular to it. If their motion after impact are at right angle. Prove that $(u_1+v_1)^2 + 4u_2v_2 = e^2(u_1-v_1)^2$



Solution Let m be mass of each ball. Let v & v' be their velocities after impact which makes angle θ & φ with line joining centres of A & B respec-

Given v & v' are at right angle.

$$\theta = 90 + \phi$$

equating velocities along common tangent.

$$v\sin\theta = u_2 \quad \text{——} \ ①$$

$$v'\sin\phi = v_2 \quad \text{——} \ ②$$

By Law of Conservation of linear Momentum along common normal.

$$mv\cos\theta + mv'\cos\phi = mu_1 + mv_1$$

$$\Rightarrow v\cos\theta + v'\cos\phi = u_1 + v_1 \quad \text{——} \ ③$$

By Newton's law of restitution

$$v'\cos\phi - v\cos\theta = -e(v_1 - u_1)$$

$$\Rightarrow v'\cos\phi - v\cos\theta = -ev_1 + eu_1 \quad \text{——} \ ④$$

Subtract ④ from ③

$$2v\cos\theta = u_1 + v_1 + ev_1 - eu_1$$

$$v\cos\theta = \frac{1}{2}\{(u_1+v_1) - e(u_1 - v_1)\} \quad \text{—— (5)}$$

add equ (3) & (4)

$$v'\cos\phi = \frac{1}{2}\{(u_1+v_1) + e(u_1 - v_1)\} \quad \text{—— (6)}$$

divide equ (1) by (5)

$$\frac{v\sin\theta}{v\cos\theta} = \frac{u_2}{\frac{1}{2}\{(u_1+v_1) - e(u_1-v_1)\}}$$

$$\tan\theta = \frac{2u_2}{(u_1+v_1) - e(u_1-v_1)} \quad \text{—— (7)}$$

Divide equ (2) by (6)

$$\tan\phi = \frac{2v_2}{(u_1+v_1) + e(u_1-v_1)} \quad \text{—— (8)}$$

Given v & v' are at right angle

$$\theta = 90 + \phi$$

$$\tan\theta = \tan(90+\phi)$$

$$= -\cot\phi \Rightarrow \tan\theta = \frac{-1}{\tan\phi}$$

$$\tan\theta\tan\phi = -1$$

$$\left\{\frac{2u_2}{(u_1+v_1) - e(u_1-v_1)}\right\}\left\{\frac{2v_2}{(u_1+v_1) + e(u_1-v_1)}\right\} = -1$$

$$\frac{4u_2v_2}{(u_1+v_1)^2 - e^2(u_1-v_1)^2} = -1$$

$$4u_2v_2 = -(u_1+v_1)^2 + e^2(u_1-v_1)^2$$

$$(u_1+v_1)^2 + 4u_2v_2 = e^2(u_1-v_1)^2 \quad \text{Ans.}$$

بقية

$$mv\cos\theta + M\cancel{v'} = \frac{1}{\sqrt{2}} mu$$

$$M\cancel{v'} - Mv\cos\theta = -\frac{1}{2\sqrt{2}} Mu$$

$$(m+M)v\cos\theta = \frac{1}{\sqrt{2}} u\left(m - \frac{M}{2}\right)$$

$$(m+M)v\cos\theta = \frac{1}{\sqrt{2}} u\left(\frac{2m-M}{2}\right)$$

$$v\cos\theta = \frac{u(2m-M)}{2\sqrt{2}(m+M)} \quad \text{——— (4)}$$

divide equ ① by ④

$$\frac{v\sin\theta}{v\cos\theta} = \frac{\frac{1}{\sqrt{2}} u}{\frac{u(2m-M)}{2\sqrt{2}(m+M)}}$$

$$\tan\theta = \frac{2(m+M)}{2m-M}$$

AS $\theta = \alpha + 45^\circ \Rightarrow \therefore \alpha = \theta - 45^\circ$

$$\tan\alpha = \tan(\theta - 45^\circ)$$

$$= \frac{\tan\theta - \tan 45^\circ}{1 + \tan\theta \tan 45^\circ}$$

$$= \frac{\frac{2(m+M)}{2m-M} - 1}{1 + \frac{2(m+M)}{2m-M}(1)}$$

$$= \frac{\frac{2m+2M-2m+M}{\cancel{2m-M}}}{\frac{2m-M+2m+2M}{\cancel{2m-M}}}$$

$$\tan\alpha = \frac{3M}{4m+M}$$

$$\alpha = \tan^{-1}\left(\frac{3M}{4m+M}\right)$$

Q :- Two equal balls are lying in contact on a smooth table and a third equal bal moving along their common tangent strikes them simultaneously. Prove that $\frac{3}{5}(1-e^2)$ of its K.E is lost by impact, e being co-efficient of restitution.

Solution.- Let A be centre of impinging balls. B & C are centres of balls at rest. Let m be mass of each ball.

Let AX be common tangent to the balls at rest at point of contact D. Let u & u' be velocities of impinging ball before and after impact.

Let v be velocity of each ball at rest after impact.

$\therefore \Delta ABC$ is equilateral triangle.

$\angle BAC = 60°$

$\angle BAX = \angle CAX = 30°$

By Law of conservation of linear Momentum along AX (common tangent)

$$mu' + m(v\cos 30) + m(v\cos 30) = mu + m(0) + m(0)$$

$$u' + \frac{\sqrt{3}}{2}v + \frac{\sqrt{3}}{2}v = u$$

$$2\left(\frac{\sqrt{3}}{2}v\right) = u - u'$$

$$\sqrt{3}\,v = u - u' \quad \text{——— (1)}$$

By Newton's Law of restitution along AB

$$v - u'\cos 30 = -e(0 - u\cos 30°)$$

$$v - \frac{\sqrt{3}}{2}u' = \frac{\sqrt{3}}{2}eu$$

$$v = \frac{\sqrt{3}}{2}(eu + u') \quad \text{——— (2)}$$

Put value of v in equ (1)

$$\sqrt{3}\cdot\frac{\sqrt{3}}{2}(eu + u') = u - u'$$

$$\frac{3}{2}eu + \frac{3}{2}u' = u - u'$$

$$\frac{3}{2}u' + u' = u - \frac{3}{2}eu$$

$$\frac{5}{\cancel{2}}u' = \frac{2u - 3eu}{\cancel{2}}$$

$$u' = \frac{u(2-3e)}{5} \quad \text{——— (3)}$$

Put in equ (1)

$$\sqrt{3}\, v = u - \frac{u(2-3e)}{5}$$

$$= u\left[1 - \frac{2-3e}{5}\right] = u\left[\frac{5-2+3e}{5}\right]$$

$$v = \frac{3(1+e)u}{5\sqrt{3}}$$

$$v = \frac{\sqrt{3}}{5}(1+e)u \quad \text{——— (4)}$$

Loss of K.E $= \left\{\frac{1}{2}mu^2 + \frac{1}{2}m(0)^2 + \frac{1}{2}m(0)^2\right\} - \left\{\frac{1}{2}m(u')^2 + \frac{1}{2}mv^2 + \frac{1}{2}mv^2\right\}$

$$= \frac{1}{2}m\left\{u^2 - (u')^2 - v^2 - v^2\right\}$$

$$= \frac{1}{2}m\left\{u^2 - (u')^2 - 2v^2\right\}$$

$$= \frac{1}{2}m\left[u^2 - \frac{u^2(2-3e)^2}{25} - \frac{2(3)}{25}(1+e)^2u^2\right]$$

$$= \frac{1}{2}mu^2\left\{1 - \frac{(2-3e)^2}{25} - \frac{6}{25}(1+e)^2\right\}$$

$$= \frac{1}{2}mu^2\left[\frac{25-(2-3e)^2 - 6(1+e)^2}{25}\right]$$

$$= \frac{1}{2}mu^2\left[\frac{25-4-9e^2+12e-6-6e^2-12e}{25}\right]$$

$$= \frac{1}{2}mu^2\left[\frac{15-15e^2}{25}\right]$$

$$= \frac{1}{2}mu^2(15)\left(\frac{1-e^2}{25}\right) \Rightarrow \frac{3}{5}(1-e^2)\left(\frac{1}{2}mu^2\right)$$

$$= \frac{3}{5}(1-e^2)\{\text{K.E before impact}\}$$

2005, 2006

Example :- Two equal spheres of mass m are in contact on a smooth horizontal table. A third equal ball of mass m' impinges symmetrically on them and is reduced to rest. Prove that $e = \frac{2m'}{3m}$ and find the loss of K.E due to impact.

Solution:- Let u be velocity of sphere of mass m' before impact. After impact it is brought to rest.

سائز ایک ہی ہے لیکن equal mass
نہیں ہے material is differ ہوگی

Let A be centre of sphere of mass m' & B and C are centres of spheres of mass m each.

$\therefore$ ABC is equilateral triangle. Let AX be common tangent to the sphere of mass m.

$$\angle BAC = 60^\circ$$

$$\angle BAX = \angle CAX = 30^\circ$$

Let v be velocity of sphere of mass m after impact.

By Law of Conservation of Linear Momentum along AX

$m(v\cos 30^\circ) + m(v\cos 30^\circ) + m'(0) = m'u + m(0) + m(0)$

$$2mv\cos 30^\circ = m'u$$

$$2mv\left(\frac{\sqrt{3}}{2}\right) = m'u \Rightarrow \sqrt{3}\,mv = m'u \quad \text{——} (1)$$

By Newton's law of restitution along AB

$$v - 0 = -e(0 - u\cos 30^\circ) \Rightarrow v = eu\cos 30^\circ$$

$$v = \frac{\sqrt{3}}{2}eu \quad \text{——} (2)$$

Put in equ (1)

$$\sqrt{3}\,m\left(\frac{\sqrt{3}}{2}eu\right) = m'u$$

$$\frac{3}{2}me = m'$$

$$e = \frac{2m'}{3m} \quad \text{——} (3)$$

Loss of K.E $= \frac{1}{2}m'u^2 - \left\{\frac{1}{2}mv^2 + \frac{1}{2}mv^2\right\}$

$$= \frac{1}{2}\left\{m'u^2 - mv^2 - mv^2\right\}$$

$$= \frac{1}{2}\left\{m'u^2 - 2mv^2\right\} = \frac{1}{2}\left\{m'u^2 - 2m\left(\frac{3}{4}e^2u^2\right)\right\}$$

$$= \frac{1}{2}\left\{m'u^2 - \frac{3}{2}me^2u^2\right\} = \frac{1}{2}\left\{m'u^2 - \frac{3}{2}\left(\frac{2m'}{3e}\right)e^2u^2\right\} \text{ using equ (3)}$$

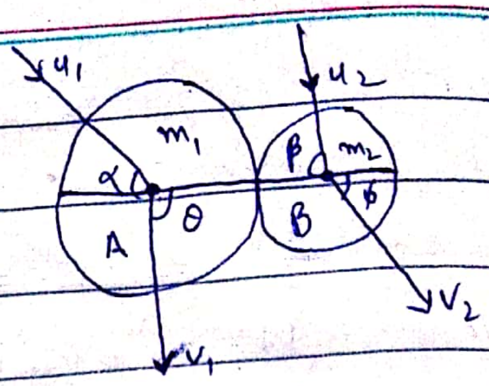
$$= \frac{1}{2}\left\{m'u^2 - m'eu^2\right\}$$

Loss of K.E $= \frac{1}{2}m'u^2(1-e)$

Q :- Prove that when two smooth spheres impinge oblique, the K.E is always lost by impact unless the elasticity is perfect.

Solution

Loss of K.E Oblique Impact



Let m_1 and m_2 be masses of spheres with centre A & B respectively. Let u_1 & u_2 be their velocities before impact which makes angle α & β with line joining their centres. Let v_1 & v_2 be their velocities after impact which makes angle θ & ϕ with line joining their centres.

Equating velocities along common tangent.

$$v_1 \sin\theta = u_1 \sin\alpha \quad \text{--- (1)}$$

$$v_2 \sin\phi = u_2 \sin\beta \quad \text{--- (2)}$$

By law of conservation of Linear Momentum along common normal.

$$m_1 v_1 \cos\theta + m_2 v_2 \cos\phi = m_1 u_1 \cos\alpha + m_2 u_2 \cos\beta \quad \text{--- (3)}$$

By Newton's Law of Restitution along common normal

$$v_2 \cos\phi - v_1 \cos\theta = -e(u_2 \cos\beta - u_1 \cos\alpha) \quad \text{--- (4)}$$

Squaring eqn (3) & (4)

$$(m_1 v_1 \cos\theta + m_2 v_2 \cos\phi)^2 = (m_1 u_1 \cos\alpha + m_2 u_2 \cos\beta)^2 \quad \text{--- (5)}$$

$$(v_2 \cos\phi - v_1 \cos\theta)^2 = e^2(u_2 \cos\beta - u_1 \cos\alpha)^2 \quad \text{--- (6)}$$

Multiply eqn (6) by $m_1 m_2$ & add to eqn (5)

$$(m_1 v_1 \cos\theta + m_2 v_2 \cos\phi)^2 + m_1 m_2 (v_2 \cos\phi - v_1 \cos\theta)^2 = (m_1 u_1 \cos\alpha + m_2 u_2 \cos\beta)^2 + e^2 m_1 m_2 (u \cos\beta - u_1 \cos\alpha)^2$$

$$(m_1 v_1 \cos\theta + m_2 v_2 \cos\phi)^2 + m_1 m_2 (v_2 \cos\phi - v_1 \cos\theta)^2 = (m_1 u_1 \cos\alpha + m_2 u_2 \cos\beta)^2 + m_1 m_2 (u_2 \cos\beta - u_1 \cos\alpha)^2 - m_1 m_2 (u_2 \cos\beta - u_1 \cos\alpha)^2 + e^2 m_1 m_2 (u_2 \cos\beta - u_1 \cos\alpha)^2 \quad \text{--- (7)}$$

Now simplify the L.H.S only

$$(m_1 v_1 \cos\theta + m_2 v_2 \cos\phi)^2 + m_1 m_2 (v_2 \cos\phi - v_1 \cos\theta)^2$$

$$= m_1^2 v_1^2 \cos^2\theta + m_2^2 v_2^2 \cos^2\phi + \cancel{2 m_1 m_2 v_1 v_2 \cos\theta \cos\phi} + m_1 m_2 v_2^2 \cos^2\phi + m_1 m_2 v_1^2 \cos^2\theta - \cancel{2 m_1 m_2 v_1 v_2 \cos\theta \cos\phi}$$

$$= m_1^2 v_1^2 \cos^2\theta + m_1 m_2 v_1^2 \cos^2\theta + m_2^2 v_2^2 \cos^2\phi + m_1 m_2 v_2^2 \cos^2\phi$$

$$= m_1 v_1^2 (m_1 + m_2) + m_2 v_2^2 \cos^2\phi (m_1 + m_2)$$

$$= (m_1 + m_2)(m_1 v_1^2 \cos^2\theta + m_2 v_2^2 \cos^2\phi) \quad \text{--- (8)}$$

~~Simplify~~ Similarly

$$(m_1 u_1 \cos\alpha + m_2 u_2 \cos\beta)^2 + m_1 m_2 (u_2 \cos\beta - u_1 \cos\alpha)^2$$

$$= (m_1 + m_2)(m_1 u_1^2 \cos^2\alpha + m_2 u_2^2 \cos^2\beta) \quad \text{--- (9)}$$

Put values in eqn (7) using (8) & (9)

$$(m_1 + m_2)(m_1 v_1^2 \cos^2\theta + m_2 v_2^2 \cos^2\phi) = (m_1 + m_2)(m_1 u_1^2 \cos^2\alpha + m_2 u_2^2 \cos^2\beta) - m_1 m_2 (u_2 \cos\beta - u_1 \cos\alpha)^2 + e^2 m_1 m_2 (u_2 \cos\beta - u_1 \cos\alpha)^2$$

Divide by $(m_1 + m_2)$

$$m_1 v_1^2 \cos^2\theta + m_2 v_2^2 \cos^2\phi = m_1 u_1^2 \cos^2\alpha + m_2 u_2^2 \cos^2\beta - \frac{m_1 m_2 (u_2 \cos\beta - u_1 \cos\alpha)^2 (1 - e^2)}{m_1 + m_2}$$

$$= m_1 u_1^2 \cos^2\alpha + m_2 u_2^2 \cos^2\beta - \left(\frac{m_1 m_2}{m_1 + m_2}\right)(1 - e^2)(u_2 \cos\beta - u_1 \cos\alpha)^2 \quad \text{--- (10)}$$

Squaring equ ① & ②

$$v_1^2 \sin^2\theta = u_1^2 \sin^2\alpha \quad \text{—— (11)}$$

$$v_2^2 \sin^2\phi = u_2^2 \sin^2\beta \quad \text{—— (12)}$$

Multiply (11) by m_1 & (12) by m_2 & add.

$$m_1 v_1^2 \sin^2\theta + m_2 v_2^2 \sin^2\phi = m_1 u_1^2 \sin^2\alpha + m_2 u_2^2 \sin^2\beta \quad \text{—— (13)}$$

add equ (10) & (13)

$$m_1 v_1^2(\cos^2\theta + \sin^2\theta) + m_2 v_2^2(\cos^2\phi + \sin^2\phi)$$
$$= m_1 u_1^2(\cos^2\alpha + \sin^2\alpha) + m_2 u_2^2(\cos^2\beta + \sin^2\beta)$$
$$- \left(\frac{m_1 m_2}{m_1 + m_2}\right)(1-e^2)(u_2\cos\beta - u_1\cos\alpha)^2$$

$$m_1 v_1^2 + m_2 v_2^2 = m_1 u_1^2 + m_2 u_2^2 - \left(\frac{m_1 m_2}{m_1+m_2}\right)(1-e^2)(u_2\cos\beta - u_1\cos\alpha)^2$$

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 - \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)(1-e^2)(u_2\cos\beta - u_1\cos\alpha)^2$$

$$\left(\frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2\right) - \left(\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2\right) = \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)(1-e^2)(u_2\cos\beta - u_1\cos\alpha)^2$$

$$\text{Loss of K.E} = \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)(1-e^2)(u_2\cos\beta - u_1\cos\alpha)^2$$

Corollary when $e < 1$

Loss of K.E $= +ve$

when $e = 1$ (or e is perfect)

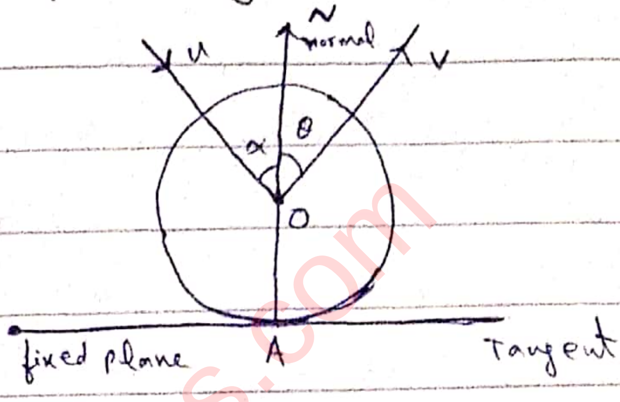
Loss of K.E $= 0$

Hence K.E always lost except when elasticity is perfect.

Impact with Smooth Plane

PU 2003

Q :- A smooth sphere of mass m and coefficient of friction e impinges obliquely against a smooth plane. Find the subsequence motion. Also find the loss of K.E energy & impulse of blow



Proof {we are considering motion of the centre of the sphere}

Suppose a smooth sphere of mass m moving with velocity u strikes against a smooth fixed plane in the direction making an angle α with the normal to plane at the point of contact A. Suppose the ball rebounds with velocity v making angle θ with normal AN.

angle of inclination is α & angle of reflection is θ

[illegible] mass b/ plane

Law of conservation [illegible] b/ [illegible] of moment

Let O be centre of sphere equating the velocity along the common tangent

$$v \sin\theta = u \sin\alpha \quad \text{——} (1)$$

By Newton's Law of restitution along common normal AN.

$$(v\cos\theta - 0) = -e(-u\cos\alpha - 0)$$

$$v\cos\theta = e\,u\cos\alpha \quad \text{——} (2)$$

$$v^2(\sin^2\theta + \cos^2\theta) = u^2(\sin^2\alpha + e^2\cos^2\alpha)$$

$v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha)$ —— ③

which gives v

divide ② by ①

$$\frac{v\cos\theta}{v\sin\theta} = \frac{eu\cos\alpha}{u\sin\alpha} \Rightarrow \cot\theta = e\cot\alpha \quad \text{—— ④}$$

~~which gives θ~~

Particular Cases

When the impact is direct $\alpha = 0$

from equ ① $\quad v\sin\theta = 0$

$\sin\theta = 0 \Rightarrow \boxed{\theta = 0}$

from equ ② when $\alpha = 0$ & $\theta = 0$

$$\boxed{v = eu}$$

When the impact is direct, the direction of motion is reversed & velocity after impact $= e$ (velocity of before impact)

(iii) when $e = 1$

from ④ $\quad \cot\theta = \cot\alpha$

$\theta = \alpha$

Put $\theta = \alpha$ and $e = 1$ in equ ②

$v = u$ {v & u velocity before & after Impact}

Loss of K.E During Impact with fixed smooth plane

$$\text{Loss of K.E} = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$

$$= \frac{1}{2}m(u^2 - v^2)$$

$$\text{Loss of K.E} = \frac{1}{2}m\{u^2 - u^2(\sin^2\alpha + e^2\cos^2\alpha)\} \quad \text{using } ③$$

$$= \frac{1}{2}mu^2(1-\sin^2\alpha - e^2\cos^2\alpha)$$

$$= \frac{1}{2}mu^2(\cos^2\alpha - e^2\cos^2\alpha)$$

$$= \frac{1}{2}mu^2(1-e^2)\cos^2\alpha$$

Impulse of the Blow on the sphere

$$\left.\begin{array}{l}\text{Impulse of the below} \\ \text{on the sphere}\end{array}\right\} = \begin{array}{l}\text{Change of momentum of sphere} \\ \text{in the direction of common normal}\end{array}$$

$$= mv\cos\theta - m(-u\cos\alpha)$$

$$= m(v\cos\theta + u\cos\alpha)$$

$$= m(eu\cos\alpha + u\cos\alpha) \quad \text{using } ②$$

$$= mu(1+e)\cos\alpha$$

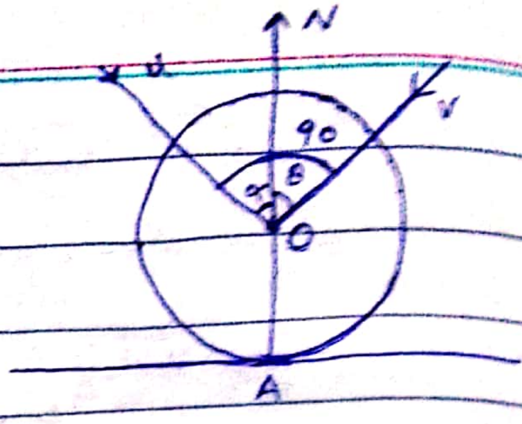
If the impact is direct

$\alpha = 0$, $\cos\alpha = 1$

Impulse of below $= mu(1+e)$

<u>Note</u>:- α is called angle of inclination and θ is called angle of reflection.

Q :- At what angle must a body whose elasticity is $\frac{1}{3}$ be inclined on a perfectly hard plane so that the angle b/w the direction before & after impact be a right angle?



Solution Let α be angle of incidence & θ be the angle of reflection.

We know that

$\cot\theta = e\cot\alpha$, given $\alpha + \theta = \pi/2$

$$\theta = \pi/2 - \alpha$$

$$\therefore \cot(\pi/2 - \alpha) = e\cot\alpha$$

$$\tan\alpha = e\cot\alpha$$

Given $e = 1/3 \Rightarrow \tan\alpha = \frac{1}{3}\cdot\frac{1}{\tan\alpha}$

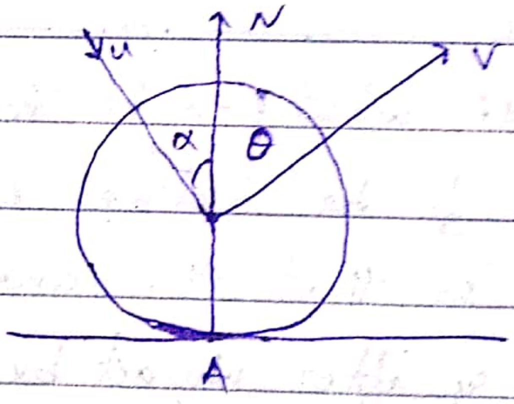
$$\tan^2\alpha = 1/3$$

$\tan\alpha = \frac{1}{\sqrt{3}}$ $\{0<\alpha<90^\circ$. In 1st quadrant $\tan\alpha = +ve\}$

$$\alpha = 30^\circ$$

P.U 2001

Q :- An imperfectly elastic sphere whose elasticity is equal to $\tan 30^\circ$ impinges upon a plane with a velocity such that the velocity after impact = The velocity before impact $\times \sin 45^\circ$. Find the angle of incidence and reflection.



Solution Let u & v be velocities before & after the impact. Let α

and θ be angle of incidence & angle of reflection respectively.

Given $e = \tan 30^\circ \Rightarrow e = \frac{1}{\sqrt{3}}$

Given:- Velocity after impact = (velocity before impact) $\sin 45^\circ$

$$v = u \sin 45^\circ$$

$$v = \frac{1}{\sqrt{2}} u \quad \text{——— (1)}$$

Equating velocities along the plane

$$v \sin\theta = u \sin\alpha$$

$$\frac{1}{\sqrt{2}} u \sin\theta = u \sin\alpha$$

$$\frac{1}{\sqrt{2}} \sin\theta = \sin\alpha \quad \text{——— (2)}$$

By Newton's Law of restitution along common normal

$$v\cos\theta - 0 = -e(-u\cos\alpha - 0)$$

$$\frac{1}{\sqrt{2}} u \cos\theta = \frac{1}{\sqrt{3}} u \cos\alpha$$

$$\frac{1}{\sqrt{2}} u \cos\theta = \frac{1}{\sqrt{3}} u \cos\alpha$$

$$\frac{1}{\sqrt{2}} \cos\theta = \frac{1}{\sqrt{3}} \cos\alpha \quad \text{——— (3)}$$

Squaring and adding equ (2) & (3)

$$\frac{1}{2}(\sin^2\theta + \cos^2\theta) = (\sin^2\alpha + \tfrac{1}{3}\cos^2\alpha)$$

$$\frac{1}{2} = \sin^2\alpha + \tfrac{1}{3}\cos^2\alpha$$

$$3 = 6\sin^2\alpha + 2\cos^2\alpha$$

$$3 = 6\sin^2\alpha + 2(1 - \sin^2\alpha)$$

$$3 = 6\sin^2\alpha + 2 - 2\sin^2\alpha$$

$$3-2 = 4\sin^2\alpha \Rightarrow \sin^2\alpha = 1/4$$

$$\sin\alpha = 1/2 \Rightarrow \boxed{\alpha = 30^\circ}$$

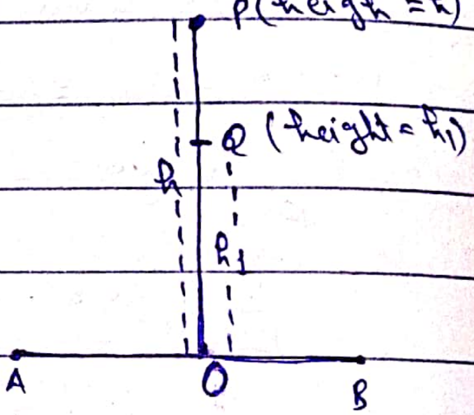
Put in eq ②

$$\frac{1}{\sqrt{2}}\sin\theta = \sin 30^\circ \Rightarrow \sin\theta = \sqrt{2}\left(\frac{1}{2}\right)$$

$$\sin\theta = 1/\sqrt{2}$$

$$\boxed{\theta = 45^\circ}$$

Q :- A ball is dropped on the floor from the height h. If the coefficient of restitution is e. Find the height of ball at the top of nth rebound.



<u>Solution</u>:- Let AB be floor. The ball is dropped from point P at height h and strikes the floor at point 'O'

formula $v^2 - u^2 = 2gx$

u = velocity at P $= 0$

v = velocity at O $= ?$

$$v^2 - 0 = 2gh$$

$$v = \sqrt{2gh}$$

{The ball strikes the ground with this velocity

velocity after impact = e(velocity before impact)

$= e\sqrt{2gh}$

Let the ball rises to point Q at height h_1 at the top of 1st rebound.

formula $\quad v^2 - u^2 = -2gx$

u = velocity at $O = e\sqrt{2gh}$

v = " " $Q = 0$

$$0 - (e\sqrt{2gh})^2 = -2gh_1$$

$$2e^2gh = 2gh_1 \Rightarrow h_1 = e^2h \quad \text{——} ①$$

Hence height attained after 1st rebound is e^2 time the previous fall (or previous height)

Let h_2 be height attained at the top of 2nd rebound.

$$h_2 = e^2 h_1 \Rightarrow h_2 = e^2(e^2h)$$

$$h_2 = (e^2)^2 h \quad \text{——} ②$$

similarly h_3 = height attained at the top of 3rd rebound.

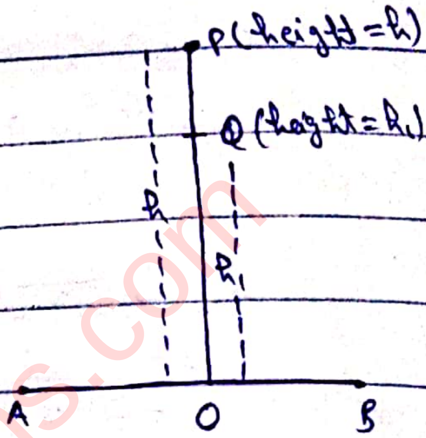
$h_3 = (e^2)^3 h$ & so on

h_n = height attained at the top of nth rebound. $\quad h_n = (e^2)^n h$

$= e^{2n} h$ Ans.

P.U 2001

Unseen :- A heavy ball drops from the ceiling of a room and after rebounding twice from the floor reaches a height equal to half of the height of ceiling. Show that $e = (1/2)^{1/4}$



Solution Let AB be floor. The ball is dropped from point P at height h on the ceiling strikes the floor at pt O

formula $v^2 - u^2 = 2gx$

$u =$ velocity at $P = 0$

$v =$ velocity at $O = ?$

$x = h$

$v^2 - 0 = 2gh \Rightarrow v = \sqrt{2gh}$

velocity after impact $= e$(velocity before impact)

$= e\sqrt{2gh}$

Let the ball rises to point Q at height h_1 at the top of 1st rebound.

formula $v^2 - u^2 = -2gx$

$u =$ velocity at $O = e\sqrt{2gh}$

$v =$ " " " $Q = 0$

$x = h_1$

$$0 - (e\sqrt{2gh})^2 = -2gh_1 \Rightarrow 2e^2gh = 2gh_1$$

$$h_1 = e^2 h \quad \text{——— (1)}$$

Hence height attained after 1st rebound is e^2 time the previous fall.

Let h_2 be height attained at the top of 2nd rebound.

$$h_2 = e^2 h_1 \Rightarrow h_2 = e^2(e^2 h)$$

$$h_2 = (e^2)^2 h \quad \text{——— (2)}$$

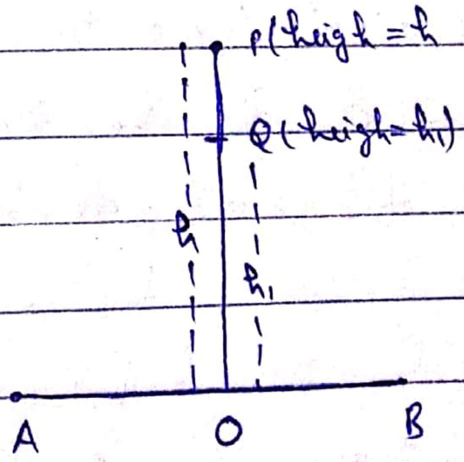
Given $h_2 = h/2$

$$(e^2)^2 h = h/2 \Rightarrow e^4 = 1/2$$

$$e = (1/2)^{1/4}$$

2003, 2004

Example :- A heavy elastic ball is dropped upon a horizontal floor from a height of 20ft and after rebounding twice, it is observed to attain a height of 10 ft. Find the coefficient of restitution.



Solution :- Let AB be floor. The ball is dropped from point P at height h and strikes the floor at point 'O'.

formula $v^2 - u^2 = 2gx$

u = velocity at $P = 0$

v = " " $O = ?$

$\therefore v^2 - 0 = 2gh \Rightarrow v = \sqrt{2gh}$

velocity after impact = e(velocity before impact)

$= e\sqrt{2gh}$

Let the ball rises to point Q at height h_1 at the top of 1st rebound.

formula $v^2 - u^2 = -2gx$

u = velocity at $O = e\sqrt{2gh}$

v = velocity at $Q = 0$

$x = h_1$

$(0) - (e\sqrt{2gh})^2 = -2gh_1$

$2e^2gh = 2gh_1$

$h_1 = e^2h$ —— ①

Hence height attained after 1st rebound is e^2 time the previous fall.

Let h_2 be height attained at the top of 2nd rebound.

$h_2 = e^2h_1 \Rightarrow h_2 = e^2(e^2h)$

$h_2 = (e^2)^2 h$ —— ②

given $h = 20$ ft and $h_2 = 10$ ft

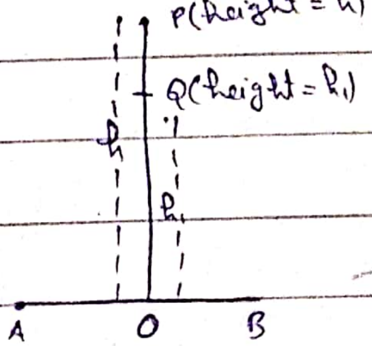
So eqn ② becomes

$10 = (e^2)^2 20 \Rightarrow e^4 = {}^{10}/_{20}$

$e^4 = \frac{1}{2} \Rightarrow e = (\frac{1}{2})^{1/4}$

Example :- A rubber ball drops from height h and after rebounding twice from the ground it reaches a height $\frac{h}{2}$. Find the co-efficient of restitution. What would be the co-efficient of restitution had the ball reached a height $\frac{h}{2}$ after rebounding three time.

Solution:- Let AB be floor. The ball is dropped from pt P at height h strikes the ground at point "O"



formula $v^2 - u^2 = 2gx$

$u =$ velocity at $P = 0$

$v =$ velocity at $0 = ?$

$x = h \quad \therefore v^2 - 0 = 2gh \Rightarrow v = \sqrt{2gh}$

velocity after impact $= e$(velocity before impact)

$= e\sqrt{2gh}$

Let the ball rises to point Q at height h_1 at the top of 1st rebound.

formula : $v^2 - u^2 = -2gx$

$u =$ velocity at $O = e\sqrt{2gh}$

$v =$ " " $Q = 0$

$x = h_1$

$$0 - (e\sqrt{2gh})^2 = -2gh_1$$

$$2e^2gh = 2gh_1$$

$$h_1 = e^2 h \quad \text{—— } ①$$

Hence height attained after 1st rebound is e^2 time the previous fall.

- Let h_2 be height attained at the top of 2nd rebound.

$$h_2 = e^2 h_1 \Rightarrow h_2 = e^2(e^2 h)$$

$$h_2 = (e^2)^2 h \quad \text{—— } ②$$

Given $h_2 = h/2$

$$(e^2)^2 h = h/2 \Rightarrow e^4 = 1/2$$

$$e = (1/2)^{1/4}$$

(ii) Let h_3 be height attained at the top of 3rd rebound.

$$h_3 = e^2 h_2 \Rightarrow h_3 = e^2(e^2)^2 h \quad \text{using ②}$$

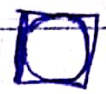
$$h_3 = e^6 h$$

put given $h_3 = \frac{h}{2}$

$$\therefore \frac{\cancel{h}}{2} = e^6 \cancel{h}$$

$e^6 = \frac{1}{2} \Rightarrow e = \left(\frac{1}{2}\right)^{1/6}$

P.U 2005:- A large smooth horizontal circular table has a vertical rim around its edge. Show that if a small body is projected. From a point P at the edge of the table, in a direction making an angle α with the radius to P, so that after two impacts the body returns to P then $\cot^2\alpha = \dfrac{1+e+e^2}{e^3}$



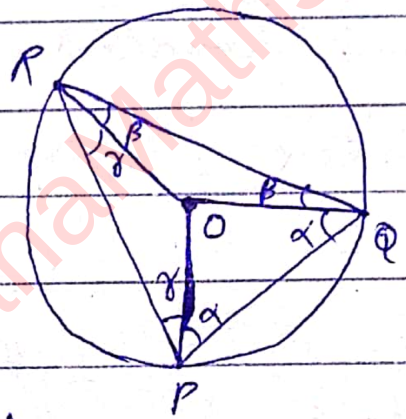
caram board

In Isocels triangle opposite angles are equal

: دو اضلاع برابر ہیں

<u>Solution</u>

Let P be point of projection. Let Q & R be points of impact. Let O be centre of circle.



Let $\angle OPQ = \alpha$, $\angle OQR = \beta$, $\angle ORP = \gamma$

Therefore $\angle OQP = \alpha$, $\angle ORQ = \beta$, $\angle OPR = \gamma$

For impact at Q

$\cot\beta = e\cot\alpha$ —— ① By Previous result

For impact at R

$\cot\gamma = e\cot\beta$

$= e(e\cot\alpha)$ using ①

$= e^2\cot\alpha$ —— ②

$$(\alpha+\gamma)+(\alpha+\beta)+(\beta+\gamma)=\pi$$

$$2\alpha+2\beta+2\gamma=\pi$$

$$\alpha+\beta+\gamma=\frac{\pi}{2} \Rightarrow \alpha+\beta=\frac{\pi}{2}-\gamma$$

$$\tan(\alpha+\beta)=\tan\left(\frac{\pi}{2}-\gamma\right)$$

$$\frac{\tan\alpha+\tan\beta}{1+\tan\alpha\tan\beta}=\cot\gamma$$

$$\frac{\dfrac{\tan\alpha}{\tan\alpha\tan\beta}+\dfrac{\tan\beta}{\tan\alpha\tan\beta}}{\dfrac{1}{\tan\alpha\tan\beta}-\dfrac{\tan\alpha\tan\beta}{\tan\alpha\tan\beta}}=\cot\gamma$$

$$\frac{\cot\beta+\cot\alpha}{\cot\alpha\cos\beta-1}=\cot\gamma$$

سارا ڈبا cot میں
جمع کرنا ہے ٹوٹل cot کی
value بھی دے

$$\frac{e\cot\alpha+\cot\alpha}{(\cot\alpha)(e\cot\alpha)-1}=e^2\cot\alpha \quad \text{using } ① \ \& \ ②$$

$$\frac{(e+1)\cot\alpha}{e\cot^2\alpha-1}=e^2\cot\alpha$$

$$e+1=e^3\cot^2\alpha-e^2$$

$$e^2+e+1=e^3\cot^2\alpha$$

$$\cot^2\alpha=\frac{e^2+e+1}{e^3}$$

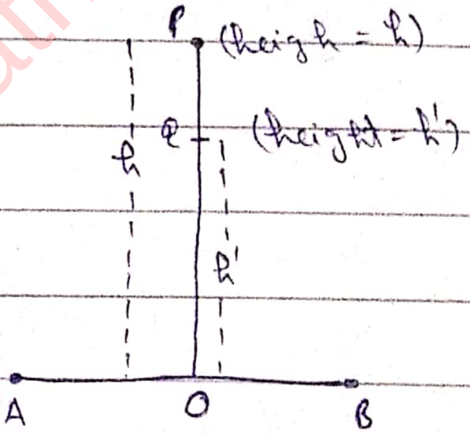
Example: A particle falls from a height 'h' in time "t" upon a fixed horizontal plane

It rebound and reaches the maximum height "h'" in time t'. Show that

(a) $t' = et$ & $h' = e^2 h$

(b) Also prove that the whole distance (up & down) describe by the particle before it has rebounding is $\frac{1+e^2}{1-e^2} h$

(c) and that the time that ellapses is
$$\frac{1+e}{1-e}\sqrt{\frac{2h}{g}}$$



Solution (a)

Let v be velocity of impact with plane.
Given time taken from point P at height h to the point of impact O on the plane it formula $v = u + gt$

$V = 0 + gt$ (time taken is t)

$$t = \frac{v}{g} \longrightarrow ①$$

which is time taken to fall from a height h.

formula: $v^2 - u^2 = 2gx$

put $u = 0$, $v = V$, $x = h$

$v^2 - 0 = 2gh \Rightarrow h = \frac{v^2}{2g}$ —— ②

velocity after impact = e (velocity before impact)

$= ev$

Let t' be time taken to reach point Q at height h'

formula $v = u - gt$

Put $u = ev$

v = velocity at highest point $Q = 0$

$t = t'$

$\therefore \quad 0 = ev - gt'$

$gt' = ev \Rightarrow t' = \frac{ev}{g}$ —— ③

formula $v^2 - u^2 = -2gx$

$0 - (ev)^2 = -2gh'$

$h' = \frac{e^2v^2}{2g}$ —— ④ Ans.

from eqn ① & ③

$t' = et$ —— ⑤

from ② & ④

$h' = e^2 h$ —— ⑥

(b) from ⑥ we see that

height attained at the top of each rebound $\Big\}$ = e^2(previous fall)

So height attained at the top of 2nd rebound $\Big\}$ = $e^2 h'$

$$= e^2(e^2 h)$$

$$= (e^2)^2 h$$

similarly:- Height attained at the top of 3rd rebound $= (e^2)^3 h$

so on.

Total distance covered before coming to rest (or stopped rebounding) $= h + 2(e^2 h) + 2\{(e^2)^2 h\} + 2\{(e^2)^3 h\} + \dots +$ up to infinity

$$= h + 2e^2 h(1 + e^2 + e^2 + \dots + \text{infinity times})$$

$$= h + 2e^2 h\left(\frac{1}{1-e^2}\right)$$

$$= h\left[1 + \frac{2e^2}{1-e^2}\right]$$

$$= h\left[\frac{1 - e^2 + 2e^2}{1-e^2}\right]$$

$$= \frac{h(1+e^2)}{1-e^2}$$

(c) from ⑤ $t' = et$ (1st rebound time)

Time taken to attain maximum height during each rebound $= e$ {time taken during previous fall}

" " " " " " " $= et'$

" " 2nd rebound $= e(et)$

$$= e^2 t$$

Similarly time taken to attain a maximum height during 3rd rebound $= e^3 t$

So on

Total time taken $= t + 2(et) + 2(e^2t) + 2(e^3t) + \dots +$ upto infinity

$= t + 2et(1 + e + e^2 + \dots$ upto infinity$)$

$= t + 2et\left[\frac{1}{1-e}\right]$

$= t\left[1 + \frac{2e}{1-e}\right]$

$= t\left[\frac{1-e+2e}{1-e}\right]$

$= t\left(\frac{1+e}{1-e}\right)$ —— (7)

from (2)

$v^2 = 2gh \Rightarrow v = \sqrt{2gh}$

from (1)

$t = \frac{v}{g}$ Put value of v

$t = \frac{\sqrt{2gh}}{g} \Rightarrow t = \sqrt{\frac{2h}{g}}$ Put in (7)

Total time taken $= \left(\frac{1+e}{1-e}\right)\sqrt{\frac{2h}{g}}$

Example :- An elastic ball of mass m is projected vertically upward from a point on a horizontal plane with velocity u. If e be the co-efficient of elasticity, Find the total space (Total distance) described by it and the time that elapses upto the instant of its nth rebound. What is its K.E after the n th rebound.

Solution :- velocity of projection $= u$

Let h be the height attained

Formula $\quad v^2 - u^2 = -2gx$

$$0 - u^2 = -2gh \Rightarrow h = u^2/2g \quad \text{—— (1)}$$

$$\text{Height attained} = \frac{(\text{velocity of projection})^2}{2g}$$

velocity after 1st rebound $= e(\text{velocity before impact})$

$$= eu$$

$$\text{Height attained at the top of 1st rebound} = \frac{(eu)^2}{2g} \quad \text{using (1)}$$

$$= \frac{e^2u^2}{2g} \quad \text{—— (2)}$$

velocity after second impact $= e(eu)$

$$= e^2u$$

Height attained at the top of the 2nd

$$\text{rebound} = \frac{(e^2u)^2}{2g} \quad \text{—— (3)}$$

So on

$$\left.\begin{array}{l}\text{Total distance at the}\\ \text{instant of nth rebound}\end{array}\right\} = 2\left(\frac{u^2}{2g}\right) + 2\left(\frac{e^2u^2}{2g}\right) + 2\left(\frac{e^4u^2}{2g}\right) + \dots + n \text{ terms}$$

at the instant of — Touch nth کا ہے nth rebound شامل نہیں ہے گزرا
nth rebound ہوتا ہے۔ یعنی جب nth شروع ہونے والا ہے

$$\left.\begin{array}{l}\text{Total distance at the} \\ \text{instant of nth rebound}\end{array}\right\} = \frac{2u^2}{2g}\left(1+e^2+e^4+\cdots n \text{ terms}\right)$$

$$= \frac{u^2}{g}\left(\frac{1\left(1-(e^2)^n\right)}{1-e^2}\right)$$

$$= \frac{u^2\left(1-e^{2n}\right)}{g\left(1-e^2\right)}$$

(ii) The ball is projected with velocity u and let t be time to attain maximum height.

formula $\quad v = u - gt$

$0 = u - gt \Rightarrow t = \frac{u}{g}$

$$\left.\begin{array}{l}\text{Time taken to attained} \\ \text{maximum height}\end{array}\right\} = \frac{\text{velocity of Projection}}{g}$$

Hence time taken to attain maximum height in 1st, 2nd, 3rd rebounds etc is

$$\frac{eu}{g}, \frac{e^2u}{g}, \frac{e^3u}{g}, \ldots$$

$$\text{Total time taken} = 2\left(\frac{u}{g}\right)+2\left(\frac{eu}{g}\right)+2\left(\frac{e^2u}{g}\right)+\cdots+n \text{ terms}$$

$$= \frac{2u}{g}\left(1+e+e^2+ n \text{ terms}\right)$$

$$= \frac{2u}{g}\left[\frac{1(1-e^n)}{1-e}\right]$$

(iii) velocity at the instant of each rebound are $eu, e^2u, e^3u, \dots$ n terms

So velocity at the instant of nth rebound

$$= e^n u$$

$$K.E = \frac{1}{2} m (e^n u)^2$$

$$= \frac{1}{2} m e^{2n} u^2$$

The 2nd part of previous question can also be written as.

<u>Solution</u> The ball is projected with velocity u. Let t be the time to attained max height

formula $\quad v = u - gt$

$0 = u - gt \quad \Rightarrow \quad t = u/g$ —— ①

$$\left.\begin{array}{l}\text{Time taken to attained} \\ \text{maximum height}\end{array}\right\} = \frac{\text{velocity of Projection}}{g}$$

velocity after impact $= e(\text{velocity before impact})$

$$= eu$$

$$\left.\begin{array}{l}\text{Time taken to reach at the} \\ \text{top of 1st rebound}\end{array}\right\} = \frac{eu}{g} \quad \text{—— ②}$$

velocity of 2nd impact $= e(eu)$

$$= e^2 u$$

$$\left.\begin{array}{l}\text{Time taken to reach at} \\ \text{the top of 2nd rebound}\end{array}\right\} = \frac{e^2u}{g} \quad \text{—— ③}$$

So on

Total time taken $= 2\left(\frac{u}{g}\right) + 2\left(\frac{eu}{g}\right) + 2\left(\frac{e^2u}{g}\right) + \dots + n$ terms

$= \frac{2u}{g}\left\{1 + e + e^2 + \dots + n \text{ terms}\right\}$

Total time taken $= \frac{2u}{g}\left\{\frac{1(1-e^n)}{1-e}\right\}$

Moment of Force "F" about "O":-

Let $\vec{r}$ be position vector of a particle of mass m at time t about a fixed origin O. Let F be force acting on the particle at time t. then

$I = \vec{r} \times \vec{F}$ is called Torque or moment of $\vec{F}$

Angular Momentum of a Particle:-

Let $\vec{r}$ be position vector of a particle of mass 'm' at time 't' about a fixed origin 'O' Then the angular momentum 'h' of the particle about "O" at time t is defined as moment of linear momentum

$\therefore \vec{h} = \vec{r} \times m\vec{v}$

Theorem:- The rate of change of angular momentum of a particle about a point "O" is equal to the torque about O of the force acting on the particle i.e

$$\frac{d\vec{h}}{dt} = \vec{\omega}$$

Solution The angular momentum $\vec{h}$ is given by

$$\vec{h} = \vec{r} \times m\vec{v}$$

$$\frac{d\vec{h}}{dt} = \frac{d}{dt}\{\vec{r} \times m\vec{v}\}$$

$$= \frac{d\vec{r}}{dt} \times (m\vec{v}) + \vec{r} \times \frac{d}{dt}(m\vec{v})$$

$$= \vec{v} \times m\vec{v} + \vec{r} \times \vec{F} \quad \{\text{Newton 2nd Law}$$

$$= m(\vec{v} \times \vec{v}) + \vec{\omega}$$

$$= 0 + \vec{\omega}$$

$$= \vec{\omega}$$

A particle of mass m moves along a curve defined by $\vec{r} = (a\cos\omega t)\hat{i} + (b\sin\omega t)\hat{j}$ Find the torque & angular momentum about origin.

Solution $\vec{r} = (a\cos\omega t)\hat{i} + (b\sin\omega t)\hat{j}$

$$\vec{v} = \frac{d\vec{r}}{dt} = -a\omega\sin\omega t\,\hat{i} + b\omega\cos\omega t\,\hat{j}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -a\omega^2\cos\omega t\,\hat{i} - b\omega^2\sin\omega t\,\hat{j}$$

$$= -\omega^2(a\cos\omega t\,\hat{i} + b\sin\omega t\,\hat{j})$$

$$= -\omega^2 r$$

$\vec{F} = m\vec{a} \Rightarrow \vec{F} = -m\omega^2\vec{r}$.

$\bar{L}$ = torque of $\bar{F}$ about O

$= \vec{r} \times \bar{F}$

$= \vec{r} \times (-m\omega^2 \bar{r})$

$= -m\omega^2 (\bar{r} \times \bar{r})$

$= 0$

Angular Momentum $\vec{h} = \vec{r} \times m\vec{v}$

$= m(\bar{r} \times \bar{v})$

$$= m \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\omega t & b\sin\omega t & 0 \\ -a\omega\sin\omega t & b\cos\omega t & 0 \end{vmatrix}$$

$= m\hat{k}(ab\omega\cos^2\omega t + ab\omega\sin^2\omega t)$

$= abm\omega\hat{k}(\cos^2\omega t + \sin^2\omega t)$

$= abm\omega\hat{k}$

Angular Momentum of a system of Particles.

The angular momentum about "O" of a system S of particles of masses m_1, $m_2, \ldots m_n$ placed at points whose position vectors are $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_n$

$$\vec{h} = \sum_{i=1}^{n} \bar{r}_i \times (m_i \vec{v}_i)$$

where $\bar{v}_i$ is the velocity of the ith particle

The sum of momentum of the external forces F_i about O is called external torque. The external torque $\overline{L}$ is

$$\overline{r}_1 \times m_1 \overline{V}_1 + \overline{r}_2 \times m_2 \overline{V}_2 + \overline{r}_3 \times m_3 \overline{V}_3 + \cdots$$

given by

$$\overline{L} = \sum_{i=1}^{n} \overline{r}_i \times \overline{F}_i$$

where F_i is external force on the particle.

External force are applied forces
Forces F_i زور لگاتے ہیں
وہ external ہوتی ہیں
اور وہ ہمیں پتہ ہوتی ہیں
تو ان کی Torque ہوتی ہے

But internal forces are unknown
weight اور friction جیسے
ہوں

Theorem:- The time rate of change of angular momentum of system S of particle about O is equal to the external torque of S about O i.e

$$\frac{d\vec{h}}{dt} = \overline{L}$$

Proof:-

$$\overline{h} = \sum_{i=1}^{n} \overline{r}_i \times (m_i \vec{V}_i)$$

$$\frac{d\vec{h}}{dt} = \frac{d}{dt} \sum_{i=1}^{n} \vec{r}_i \times (m_i \vec{V}_i)$$

$$= \sum_i \left\{ \frac{d\overline{r}_i}{dt} \times (m_i \overline{v}_i) + \overline{r}_i \times \frac{d}{dt}(m_i \vec{v}_i) \right\}$$

$$= \sum_i \left\{ \overline{v}_i \times m_i \overline{v}_i + \overline{r}_i \times \frac{d}{dt}(m_i \vec{v}_i) \right\}$$

$$= \sum_i \left\{ m_i (\overline{v}_i \times \overline{v}_i) + \vec{r}_i \times \frac{d}{dt}(m_i \vec{v}_i) \right\}$$

$$\frac{d\vec{h}}{dt} = \sum_i \{ 0 + \bar{r}_i \times \frac{d}{dt}(m_i \bar{v}_i) \}$$

$$= \sum_i \bar{r}_i \times \frac{d}{dt}(m_i \vec{v}_i)$$

$$= \sum_i \bar{r}_i \times (\bar{F}_i + \bar{F}_i')$$

wher $\bar{F}_i$ & $\bar{F}_i'$ are external & internal forces on the ith particle.

$$\frac{d\vec{h}}{dt} = \sum_i \bar{r}_i \times \bar{F}_i + \sum_i \bar{r}_i \times \bar{F}_i'$$

The internal forces of action & reaction occure in pair and are equal & opposite.

$$\therefore \sum_i \bar{r}_i \times \bar{F}_i' = 0$$

$$\therefore \frac{d\vec{h}}{dt} = \sum_i \bar{r}_i \times \bar{F}_i$$

$$= \bar{L}$$

wher $\bar{L} = \sum_i \bar{r}_i \times \bar{F}_i$

is sum of external torque.

Conservation Force:- Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be position vector of a particle at time t. Let $\vec{F}$ be force acting on the particle at this instant.

The field of force $\vec{F}$ is called conservation if their exist a scaler function V of x, y, z such that

$$\vec{F} = -\left[\frac{\partial v}{\partial x}\hat{i} + \frac{\partial v}{\partial y}\hat{j} + \frac{\partial v}{\partial z}\hat{k}\right]$$

$$= -\vec{\nabla} v \quad \text{or} \quad \vec{F} = -\text{grad } v$$

v is called potential of $\vec{F}$ or the potential energy of the particle.

we know that

$$\text{Curl}(\text{grad } v) = 0 \quad \text{——} \textcircled{1}$$

$$\vec{F} = -\vec{\nabla} V$$

$$\vec{F} = -\text{grad } v \Rightarrow \text{Curl } \vec{F} = -\text{Curl}(\text{grad } v)$$

$$= 0 \quad \text{using } \textcircled{1}$$

$\therefore \vec{F}$ is conservative if and only if $\text{curl } \vec{F} = 0$

P.U 2002, 2006 :- Prove that the force field $\vec{F} = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}$ is conservative & determine its potential

Solution :- $\vec{F} = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}$

$$\text{Curl } \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 - 2xyz^3 & 3 + 2xy - x^2z^3 & 6z^3 - 3x^2yz^2 \end{vmatrix}$$

$$\text{Curl } \vec{F} = \hat{i}\left\{\frac{\partial}{\partial y}(6z^3-3x^2yz^2) - \frac{\partial}{\partial z}(3+2xy-x^2z^3)\right\}$$

$$-\hat{j}\left\{\frac{\partial}{\partial x}(6z^3-3x^2yz^2) - \partial/\partial z\,(y^2-2xyz^3)\right\}$$

$$+\hat{k}\left\{\frac{\partial}{\partial x}(3+2xy-x^2z^3) - \partial/\partial y\,(y^2-2xyz^3)\right\}$$

$$= \hat{i}\{-3x^2z^2+3x^2z^2\} - \hat{j}\{-6xyz^2+6xyz^2\} + \hat{k}\{2y-2xz^3-2y+2xz^3\}$$

$$\text{Curl } \vec{F} = 0$$

$\therefore \vec{F}$ is conservative.

(ii) $\vec{F} = (y^2-2xyz^3)\hat{i} + (3+2xy-x^2z^3)\hat{j} + (6z^3-3x^2yz^2)\hat{k}$

$\because \vec{F}$ is conservative

$$\therefore \vec{F} = -\vec{\nabla}V$$

$$\therefore -\vec{\nabla}V = (y^2-2xyz^3)\hat{i} + (3+2xy-x^2z^3)\hat{j} + (6z^3-3x^2yz^2)\hat{k}$$

$$\vec{\nabla}V = (-y^2+2xyz^3)\hat{i} + (-3-2xyz^3)\hat{j} + (-6z^3+3x^2yz^2)\hat{k}$$

$$\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k} = (-y^2+2xyz^3)\hat{i} + (-3-2xy+x^2z^3)\hat{j} + (-6z^3+3x^2yz^2)\hat{k}$$

Equating components

$$\frac{\partial V}{\partial x} = -y^2+2xyz^3 \Rightarrow V = -xy^2 + \frac{2x^2yz^3}{2} + \phi_1(y,z)$$

$$\frac{\partial V}{\partial y} = -3-2xy+x^2z^3 \Rightarrow V = -3y - \frac{2xy^2}{2} + x^2yz^3 + \phi_2(x,z)$$

$$\frac{\partial V}{\partial z} = -6z^3+3x^2yz^2 \Rightarrow V = \frac{-6z^4}{4} + \frac{3x^2yz^3}{3} + \phi_3(x,y)$$

$\therefore V = -xy^2 + x^2yz^3 - 3y - \frac{3}{2}x^4 + c$

at origin $V = 0$

$V = mgh = P.E$
$\because h = 0$ so $V = 0$

When $x = 0$, $y = 0$, $z = 0$, then $V = 0$

$0 = 0 + 0 - 0 + 0 + c$

$\boxed{c = 0}$

$\therefore V = xy^2 + x^2yz^3 - 3y - \frac{3}{2}x^2$

Impulse:-

The Impulse I of a constant force $\vec{F}$ during the time interval t is defined as

$$\vec{I} = \vec{F}t \quad \text{(vector quantity)}$$

Its magnitudes is

$$I = Ft \quad \text{——— ①}$$

By Newton 2nd Law

$$F = ma$$

Also $v = u + at \Rightarrow \therefore t = \frac{v-u}{a}$

Put in eqn ①

$$I = ma\left[\frac{v-u}{a}\right] \Rightarrow I = mv - mu$$

$\therefore I =$ Change in momentum of particle

When force $\vec{F}$ is variable in magnitude but constant in direction, then Impulse I during the interval (t_1, t_2) is defined as

$$I = \int_{t_1}^{t_2} F\,dt \quad \text{(magnitude)}$$

$$I = \int_{t_1}^{t_2} ma\,dt$$

$$= m\int_{t_1}^{t_2} a\,dt = m\int_{t_1}^{t_2} \frac{dv}{dt}\cdot dt$$

$$I = m|v|_{t=t_1}^{t=t_2}$$ (v is function of t)

$$= m(v(t_2) - v(t_1))$$

$= m(v_2 - v_1)$ where v_1 & v_2 are velocities of particle at t_1 & t_2 respectively.

$$I = mv_2 - mv_1$$

= Change in momentum of particle.

We see that whether force is constant or variable the impulse is change of momentum of particle in both cases.

Impulsive Force:-

If the force $\vec{F} = X\hat{i} + Y\hat{j}$ increases without limit & the time interval $t = t_2 - t_1$ tends to zero such that the integral $\vec{I} = \int_{t_1}^{t_2} \vec{F}\,dt$ is finite then Impulse I is called impulsive Force or blow & it is denoted by $\hat{I}$

$\boxed{\hat{I} \text{ is not a unit vector}}$

If $\hat{I} = \hat{X}\hat{i} + \hat{Y}\hat{j}$ ($\hat{X}$ & $\hat{Y}$ are component of impulsive Force)

$$\hat{X} = \lim \frac{}{t_2 - t_1} \int_{t_1}^{t_2} X\, dt$$

$$\hat{Y} = \lim \frac{}{t_2 - t_1} \int_{t_1}^{t_2} Y\, dt$$

Example :- A body of mass $m_1 + m_2$ is split into two parts of masses m_1 & m_2 by an internal explosion which generates K.E E. Show that if after explosion the parts move in the same line as before, their relative speed is $\sqrt{\dfrac{2E(m_1+m_2)}{m_1 m_2}}$ $v_2 - v_1 = ?$

Solution :- Let u be velocity of body before explosion. Let v_1 & v_2 be velocities of m_1 & m_2 after explosion.

By Law of conservation of linear Momentum

$$(m_1 + m_2)u = m_1 v_1 + m_2 v_2 \quad \text{———} \quad ②$$

Energy generated by explosion is equal to increase in K.E

Given increase in K.E $= E$ $\boxed{\text{پہلے کم اور بعد میں زیادہ}}$

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{1}{2}(m_1 + m_2)u^2 = E$$

$$m_1v_1^2 + m_2v_2^2 - (m_1+m_2)u^2 = 2E \quad \text{———} ②$$

from ①

$$u = \frac{m_1v_1 + m_2v_2}{m_1+m_2} \quad \text{put in equ ②}$$

$$m_1v_1^2 + m_2v_2^2 - (m_1+m_2)\frac{(m_1v_1+m_2v_2)^2}{(m_1+m_2)^2} = 2E$$

$$m_1v_1^2 + m_2v_2^2 - \frac{(m_1^2v_1^2 + m_2^2v_2^2 + 2m_1m_2v_1v_2)}{m_1+m_2} = 2E$$

$$(m_1+m_2)(m_1v_1^2+m_2v_2^2) - (m_1^2v_1^2 + m_2^2v_2^2 + 2m_1m_2v_1v_2) = 2E(m_1+m_2)$$

$$m_1^2v_1^2 + m_1m_2v_2^2 + m_1m_2v_1^2 + m_2^2v_2^2 - m_1^2v_1^2 - m_2^2v_2^2 - 2m_1m_2v_1v_2 = 2(m_1+m_2)E$$

$$v_2^2 + v_1^2 - 2v_1v_2 = \frac{2(m_1+m_2)E}{m_1m_2}$$

$$(v_2 - v_1)^2 = \frac{2(m_1+m_2)E}{m_1m_2}$$

$$v_2 - v_1 = \sqrt{\frac{2(m_1+m_2)E}{m_1m_2}}$$

which is relative velocity.

The Principle Of Linear Momentum

The sudden Change in linear momentum of the system is equal to the total external impulsive force.

PROOF. Let m_i be mass of ith particle placed at $P_i(x_i, y_i)$. Let X_i & Y_i be the component of external force on it along x-axis & y-axis respectively. $\boxed{F = X\hat{i} + Y\hat{j}}$

Let r_i be position vector of $P_t(x_i, y_i)$

$$\vec{r_i} = x_i\hat{i} + y_i\hat{j}$$

$$\vec{v_i} = \dot{\vec{r_i}} = \dot{x}_i\hat{i} + \dot{y}_i\hat{j}$$

By Newton 2nd Law

$$\sum_i \frac{d}{dt}(m_i\dot{x}_i) = \sum_i X_i \quad \&$$

$$\sum_i \frac{d}{dt}(m_i\dot{y}_i) = \sum_i Y_i$$

By Newton 2nd law

$F = ma$

$= m\frac{dv}{dt}$

$F = \frac{d}{dt}(mv)$

Rate of change of momentum

$v = \dot{x}_i\hat{i} + \dot{y}_i\hat{j}$

x component of velocity of the ith particle

Rough

$\dot{x}_i$ = velocity of ith particle along x-axis

$m_i\dot{x}_i$ = momentum of ith particle along x-axis

$\frac{d}{dt}(m_i\dot{x}_i) = X_i$ [Rate of change of moment is equal to force i.e $F = \frac{d}{dt}(mv)$]

x component of rate of change of momentum is equal to the x component of force

Rate of change of momentum for a single particle

for whole System

$$\sum \frac{d}{dt}(m_i\dot{x}_i) = \sum X_i$$

let $\sum_i X_i = X$ and $\sum_i Y_i = Y$ $\boxed{X_1 + X_2 + \dots = X}$

$$\sum_i \frac{d}{dt}(m_i \dot{x}_i) = X \quad , \sum_i \frac{d}{dt}(m_i \dot{y}_i) = Y$$

Rate of change of momentum of all the particles along x-axis is equal to the total force along x-axis

Integrate w.r.t "t" over the interval (t_1, t_2)

$$\left|\sum_i m_i \dot{x}_i\right|_{t_1}^{t_2} = \int_{t_1}^{t_2} X\, dt$$

$$\left|\sum_i m_i \dot{y}_i\right|_{t_1}^{t_2} = \int_{t_1}^{t_2} Y\, dt$$

taking limit $t_2 \to t_1$

$$\lim_{t_2 \to t_1} \left|\sum_i m_i \dot{x}_i\right|_{t_1}^{t_2} = \lim_{t_2 \to t_1} \int_{t_1}^{t_2} X\, dt$$

&

$$\lim_{t_2 \to t_1} \left|\sum_i m_i \dot{y}_i\right|_{t_1}^{t_2} = \lim_{t_2 \to t_1} \int_{t_1}^{t_2} Y\, dt$$

So

$$\lim_{t_2 \to t_1} \left|\sum_i m_i \dot{x}_i\right|_{t_1}^{t_2} = \hat{X}$$

&

$$\lim_{t_2 \to t_1} \left|\sum_i m_i \dot{y}_i\right|_{t_1}^{t_2} = \hat{Y}$$

where $\hat{X}$ & $\hat{Y}$ are external impulsive

forces along x-axis & y-axis respectively.
L.H.S of these equations denot sudden change in linear momentum & R.H.S represents the total external impulsive forces in the direction of axes (?)

This prove the principle of linear momentum.

Rough work for Previos question

As $\lim_{t_2 \to t_1} \left| \sum_i m_i \dot{x}_i \right|_{t_1}^{t_2}$

upper limit − lower limit

$= \lim_{t_2 \to t_1} \left\{ \sum_i m_i \dot{x}_i(t_2) - \sum_i m_i \dot{x}_i(t_1) \right\}$

Total momentum of the system at t_2 along x-axis

Total momentum of the system at point t_1 along the x-axis

$= \lim_{t_2 \to t_1} \{\text{Change in momentum}\}$

= Sudden change (because time interval is almost zero)

Similarly we can prove for y-axis

The Principle of Angular Momentum.

The sudden change in angular momentum of a system about a fixed axis is equal to the moment of external impulsive forces about the axis.

PROOF We prove the principle for a single particle. The proof can be extended to a system of particles.

Let $\vec{r} = x\hat{i} + y\hat{j}$ be position of particle of mass m at time t.

Let $\vec{F} = X\hat{i} + Y\hat{j}$ be the external force on the particle.

$\bar{L}$ = moment of force $\vec{F}$ about origin

$$\bar{L} = \vec{r} \times \vec{F}$$

$$= \begin{vmatrix} i & j & k \\ x & y & 0 \\ X & Y & 0 \end{vmatrix}$$

$$\bar{L} = \hat{k}(xY - yX)$$

taking magnitude

$$L = xY - yX$$

$\boxed{\hat{k} \text{ is unit vector so } |\hat{k}| = 1}$

We know that

$$\frac{dh}{dt} = h \quad \Rightarrow \quad \frac{dh}{dt} = xY - yX$$

Integrate over the interval (t_1, t_2) and taking limit when $t_2 \to t_1$

$$\lim_{t_2 \to t_1} |h|_{t_1}^{t_2} = \lim_{t_2 \to t_1} \int_{t_1}^{t_2} (xY - yX)\, dt$$

$$= x \lim_{t_2 \to t_1} \int_{t_1}^{t_2} Y\, dt - y \lim_{t_2 \to t_1} \int_{t_1}^{t_2} X\, dt$$

All of the sudden the position of the particle is not changed it remains constant.
ایک چھوٹے سے لمحے میں particle اڑ کر دوسری جگہ نہیں
جا سکے گا۔ اس لیے x اور y constant ہے۔ کیونکہ
particle اپنی پوزیشن change نہیں کرے گا

x & y remains constant instantaneously

x & y are position of particle

$$\lim_{t_2 \to t_1} |h|_{t_1}^{t_2} = x\hat{Y} - y\hat{X} \quad \text{———} \ (1)$$

where $\hat{I} = \hat{X}\hat{i} + \hat{Y}\hat{j}$ is external Impulsive force on the particle

Now

$$\bar{r} \times \hat{I} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ \hat{X} & \hat{Y} & 0 \end{vmatrix}$$

$$\bar{r} \times \hat{I} = \hat{k}(x\hat{Y} - y\hat{X})$$

taking magnitude

$$|\bar{r} \times \hat{I}| = (x\hat{Y} - y\hat{X}) \quad \text{———} \ (2)$$

recoil $\Rightarrow$ Gun کا پیچھے کی طرف جھٹکا لگنا

from ① & ②

$$\lim_{t_2 \to t_1} \left| h \right|_{t_1}^{t_2} = \left| \bar{r} \times \hat{I} \right|$$

Sudden change in angular momentum = Moment of external Impulsive forces

h = angular momentum

As $\displaystyle\lim_{t_2 \to t_1} \left| h \right|_{t_1}^{t_2} = \lim_{t_2 \to t_1} \left| h(t_2) - h(t_1) \right|$

= Sudden change in angular momentum

Example :- A gun of mass M fires a shell of mass m horizontally and the energy of the explosion is such as would be sufficient to project the shell to a height h. Show that velocity of recoil is

$$\sqrt{\frac{2m^2gh}{M(M+m)}}$$

Solution :- Mass of gun = M

mass of shell = m

Let v_1 be the velocity of shell & v be the velocity of recoil of gun.

By principle of conservation of linear momentum

$$M(-v) + m(v_1) = M(0) + m(0)$$

Gun shell Gun shell

$$-Mv + mv_1 = 0 \Rightarrow mv_1 = Mv$$

$$v_1 = \frac{Mv}{m} \quad \text{---} \quad (1)$$

E = energy generated by explosion

= Change in K.E

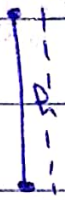
$$= \left[\frac{1}{2}M(-v)^2 + \frac{1}{2}mv_1^2\right] - \left[\frac{1}{2}M(0)^2 + \frac{1}{2}m(0)^2\right]$$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}mv_1^2 \quad \text{---} \quad (2)$$

Given energy generated by explosion = energy required to project the shell of mass m to height h

$$E = mgh \quad \text{---} \quad (3)$$

Total Energy = K.E + P.E



= 0 + mgh

at the highest point

انرجی اتنی زیادہ ہے کہ اگر کسی shell پر لگائی جائے تو وہ اسے height h تک پہنچا دے گی۔ اوپر جاتے وقت K.E کم ہوتے ہوتے highest point پر zero ہو جائے گی کیونکہ highest pt پر velocity زیرو ہوتی ہے اور P.E بڑھتی جائے گی اور mgh ہو جائے گی اس لیے highest pt پر ایک ہی انرجی ہوگی اور وہ P.E ہوگی

from (2) & (3)

$$\frac{1}{2}Mv^2 + \frac{1}{2}mv_1^2 = mgh$$

$$\frac{1}{2}Mv^2 + \frac{1}{2}m\left(\frac{Mv}{m}\right)^2 = mgh$$

$$Mv^2 + \frac{M^2v^2}{m} = 2mgh$$

$$Mv^2\left[1 + \frac{M}{m}\right] = 2mgh$$

$$Mv^2\left[\frac{m+M}{m}\right] = 2mgh$$

$$v^2 = \frac{2m^2gh}{M(M+m)}$$

$$V = \sqrt{\frac{2m^2gh}{M(M+m)}}$$

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Q :- A bullet of mass m, moving with velocity v, strikes a block of mass M, which is free to move in the direction of motion of the bullet & is embedded in it. Show that a proportion $\frac{M}{M+m}$ of the K.E is lost. If the block is afterwards struck by an equal bullet moving in the same direction with the same velocity, show that there is a further loss of energy equal to $\frac{M^2mv^2}{2(M+2m)(M+m)}$

Solution Let v_1 be the velocity of block after the first bullet get embedded in it.

By principle of conservation of linear momentum

$$(M+m)v_1 = M(0) + mv$$

after the fire bullet gets embedded in the block

Block is at rest before bullet hit it

bullet is moving with velocity v

$$v_1 = \frac{mv}{M+m} \quad \text{———} \ ①$$

Loss of K.E $= \frac{1}{2}mv^2 - \frac{1}{2}(M+m)v_1^2$

کرو گا

red block میں گئی آ کے آرام کا
K.E zero ہو گی

$$= \frac{1}{2}mv^2 - \frac{1}{2}(M+m)\frac{m^2v^2}{(M+m)^2}$$

$$= \frac{1}{2}mv^2\left[1 - \frac{m}{M+m}\right]$$

$$= \frac{1}{2}mv^2\left[\frac{M+m-m}{M+m}\right]$$

$$= \frac{1}{2}\frac{mMv^2}{M+m}$$

Proportion of K.E lost $= \dfrac{\text{Loss of K.E}}{\text{K.E before Impact}}$

$$= \frac{\frac{1}{2}\frac{mMv^2}{(M+m)}}{\frac{1}{2}mv^2}$$

$$= \frac{M}{M+m}$$

(ii) Let v_2 be velocity of block when 2nd bullet get embedded

By Principle of conservation of Linear Momentum

$$(M+2m)v_2 = (M+m)v_1 + mv$$

$$(M+2m)v_2 = (M+m)\frac{mv}{M+m} + mv$$

$$(M+2m)\,v_2 = 2mv$$

$$v_2 = \frac{2mv}{M+2m}$$

Further loss of K.E

$$= \frac{1}{2}mv^2 + \frac{1}{2}(M+m)v_1^2 - \frac{1}{2}(M+2m)v_2^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}(M+m)\frac{m^2v^2}{(M+m)^2} - \frac{1}{2}(M+2m)\frac{4m^2v^2}{(M+2m)^2}$$

$$= \frac{1}{2}mv^2\left[1 + \frac{m}{M+m} - \frac{4m}{M+2m}\right]$$

$$= \frac{1}{2}mv^2\left\{\frac{(M+m)(M+2m) + m(M+2m) - 4m(M+m)}{(M+m)(M+2m)}\right\}$$

$$= \frac{1}{2}mv^2\left\{\frac{M^2 + 2mM + mM + \cancel{2m^2} + mM + 2m^2 - 4mM - 4m^2}{(M+m)(M+2m)}\right\}$$

$$= \frac{1}{2}mv^2\left\{\frac{M^2}{(M+m)(M+2m)}\right\}$$

$$= \frac{1}{2}\,\frac{mM^2v^2}{(M+m)(M+2m)}$$

Q 2:- A 100 lb shell travelling at 1500 ft/sec bursts into two equal portions which continue to travel in the same line. If 200 ft tons of energy are generated by the explosion, find the subsequent (بعد کی) velocities.

Solution:- Let v_1 & v_2 be velocities of two equal portions.

By principle of conservation of linear momentum

$$50(v_1) + 50(v_2) = 100(1500)$$

$$v_1 + v_2 = 3000 \quad \text{--- (1)}$$

Given

E = Energy generated by explosion

= 200 ft - ton

= ~~200(2240) ft pound~~

= 200(2240) g ft poundal

Energy generated by explosion = Increase in K.E

$$200[2240g] = \frac{1}{2}(50)v_1^2 + \frac{1}{2}(50)v_2^2 - \frac{1}{2}(100)(1500)^2$$

$$(200)(2240)(32) = 25v_1^2 + 25v_2^2 - 50(1500)^2$$

$$v_1^2 + v_2^2 - 4500000 = 573440$$

$$v_1^2 + v_2^2 = 5073440 \quad \text{--- (2)}$$

we know that

$$(v_1 + v_2)^2 + (v_1 - v_2)^2 = 2(v_1^2 + v_2^2)$$

$$(3000)^2 + (v_1 - v_2)^2 = 2(5073440)$$

$$(v_1 - v_2)^2 = 1146880$$

$$v_1 - v_2 = 1070.92 \quad \text{--- (3)}$$

add (1) & (3)

$$2v_1 = 4070.92$$

$$v_1 = 2035.46 \text{ ft/sec}$$

subtract ② from ①

$2v_2 = 1929.08$

$v_2 = 964.54$ ft/sec

/// ——— /// ——— /// ——— /// ——— /// ——— ///

Q :- Two carriages each of weight ω tons, tightly coupled together, are running on level rails at v ft/sec. When the coupling is released, the system gains E ft tons of Energy from the released buffers. Show that the velocity of the front carriage is increased and that of the rear carriage is decreased by the same amount $\sqrt{\frac{Eg}{\omega}}$ ft/sec g being the acceleration due to gravity.

Solution :- Weight of each carriage $= \omega$ ton

$= (2240\,\omega)$ lb

Let v_1 & v_2 be their velocities after impact. By principle of conservation of Linear Momentum.

$$(2240\,w)v_1 + (2240\,w)v_2 = (2240)(2w)V$$

$$v_1 + v_2 = 2V \quad \text{—— ①}$$

Energy generated by impact $= E$ ft - ton

$= (2240)\, E$ ft pound

$= (2240\,g)\, E$ ft poundal

Increase in K.E = Energy generated by impact

$$\frac{1}{2}(2240w)v_1^2 + \frac{1}{2}(2240w)v_2^2 - \frac{1}{2}(2240)(2w)v^2 = 2240\,gE$$

$$\frac{1}{2}v_1^2 + \frac{1}{2}v_2^2 - v^2 = \frac{gE}{w}$$

$$v_1^2 + v_2^2 = \frac{2gE}{w} + 2v^2 \quad \text{--- (2)}$$

we know that

$$(v_1+v_2)^2 + (v_1-v_2)^2 = 2(v_1^2+v_2^2)$$

$$(2v)^2 + (v_1-v_2)^2 = 2\left\{\frac{2gE}{w} + 2v^2\right\}$$

$$\cancel{4v^2} + (v_1-v_2)^2 = \frac{4gE}{w} + \cancel{4v^2}$$

$$v_1 - v_2 = 2\sqrt{\frac{gE}{w}} \quad \text{--- (3)}$$

add (1) & (3)

$$2v_1 = 2v + 2\sqrt{\frac{gE}{w}}$$

$$v_1 = v + \sqrt{\frac{gE}{w}} \text{ ft/sec} \quad \text{--- (4)}$$

Subtract (3) from (1)

$$2v_2 = 2v - 2\sqrt{\frac{gE}{w}}$$

$$v_2 = v - \sqrt{\frac{gE}{w}} \text{ ft/sec} \quad \text{--- (5)}$$

from (4) & (5) we see that that velocity of front carriage is increased and that of the rear (v_1, v_2) Carriage is decreased by $\sqrt{\frac{gE}{w}}$.

Ghori P.U. 2001
CH # 7

Q :- A shell of mass M is moving with velocity v in a line. An internal explosion generates an moment of energy E & breaks the shell into two portion whose masses are in the ratio $m_1 : m_2$. The fregment continue to move in the origional line of motion of the shell show that their speeds are

$$v + \sqrt{\frac{2m_2E}{m_1M}} \quad \text{and} \quad v - \sqrt{\frac{2m_1E}{m_2M}}$$

Solution :- masse کی ratio دو ہوئی masses میں دیئے گئے
original mass کو constant سے multiply کر کے حاصل ہوتے ہیں۔

Given ratio of masses is $m_1 : m_2$

Let λm_1 & λm_2 be the masses of fragment

$$\lambda m_1 + \lambda m_2 = M \quad (\text{Total mass of shell})$$

$$\lambda = \frac{M}{m_1 + m_2}$$

$\therefore$ masses of the fragments are $\frac{Mm_1}{m_1+m_2}$ & $\frac{Mm_2}{m_1+m_2}$

Let v_1 & v_2 be their velocities after explosion. By Law of Conservation of Linear Momentum

$$\frac{Mm_1}{m_1+m_2} v_1 + \frac{Mm_2}{m_1+m_2} v_2 = Mv$$

$$v = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \quad \text{——— ①}$$

Energy generated by explosion = E

Increase in energy = Energy generated by explosion

$$\frac{1}{2}\left[\frac{M m_1}{m_1 + m_2}\right] v_1^2 + \frac{1}{2}\left[\frac{M m_2}{m_1 + m_2}\right] v_2^2 - \frac{1}{2} M v^2 = E \quad \text{——— ②}$$

Put value of v

$$\frac{1}{2}\left[\frac{M m_1}{m_1 + m_2}\right] v_1^2 + \frac{1}{2}\left[\frac{M m_2}{m_1 + m_2}\right] v_2^2 - \frac{1}{2} \frac{M(m_1 v_1 + m_2 v_2)^2}{(m_1 + m_2)^2} = E$$

multiply by $\frac{2(m_1 + m_2)^2}{M}$

$$(m_1 + m_2) m_1 v_1^2 + (m_1 + m_2) m_2 v_2^2 - (m_1 v_1 + m_2 v_2)^2 = \frac{2E(m_1 + m_2)^2}{M}$$

$$m_1^2 v_1^2 + m_1 m_2 v_1^2 + m_1 m_2 v_2^2 + m_2^2 v_2^2 - m_1^2 v_1^2 - m_2^2 v_2^2 - 2 m_1 m_2 v_1 v_2 = \frac{2E(m_1 + m_2)^2}{M}$$

$$v_1^2 + v_2^2 - 2 v_1 v_2 = \frac{2E(m_1 + m_2)^2}{m_1 m_2 M}$$

$$v_1 - v_2 = \sqrt{\frac{2E}{m_1 m_2 M}}\,(m_1 + m_2)$$

$$v_1 = v_2 + \sqrt{\frac{2E}{m_1 m_2 M}}\,(m_1 + m_2) \quad \text{——— ③}$$

equ ① becomes

$$(m_1 + m_2) v = m_1 v_1 + m_2 v_2$$

Put values of v_1

$$(m_1+m_2)v = m_1\left\{v_2 + \sqrt{\frac{2E}{m_1m_2M}}(m_1+m_2)\right\} + m_2v_2$$

$$(m_1+m_2)v = (m_1+m_2)v_2 + m_1\sqrt{\frac{2E}{m_1m_2M}}(m_1+m_2)$$

$$v = v_2 + \sqrt{\frac{2m_1E}{m_2M}}$$

$$v_2 = v - \sqrt{\frac{2m_1E}{m_2M}} \quad \text{——— (4)}$$

Put in eqn (3)

$$v_1 = v - \sqrt{\frac{2m_1E}{m_2M}} + m_1\sqrt{\frac{2E}{m_1m_2M}} + m_2\sqrt{\frac{2E}{m_1m_2M}}$$

$$= v - \cancel{\sqrt{\frac{2m_1E}{m_2M}}} + \cancel{\sqrt{\frac{2m_1E}{m_2M}}} + \sqrt{\frac{2m_2}{m_1M}}$$

$$v_1 = v + \sqrt{\frac{2m_2E}{m_1M}} \quad \text{——— (5)}$$

Q :- Assuming that in a cannon the force on the ball depends only on the volume of the gas generated by the gun powder. Show that the ratio of the final velocity of the ball when the gun is free to recoil to its velocity when the gun is fired is $\sqrt{\frac{M}{(m+M)}}$

Solution mass of cannon $= M$

mass of ball $= m$

Ist consider that cannon is free to recoil

Let v be velocity of recoil of cannon

& v_1 be velocity of ball in this case.

By Law of Conservation of Linear Momentum

$$M(-v) + mv_1 = M(0) + m(0)$$

$$-Mv + mv_1 = 0$$

$$v = \frac{mv_1}{M} \quad \text{——} \quad (1)$$

Let E be energy generated by the explosion

Change in K.E = energy generated by explosion

$$\left[\frac{1}{2}M(-v)^2 + \frac{1}{2}mv_1^2\right] - \left[\frac{1}{2}M(0)^2 + \frac{1}{2}m(0)^2\right] = E$$

$$\frac{1}{2}Mv^2 + \frac{1}{2}mv_1^2 = E$$

put value of v using (1)

$$\frac{1}{2}M\frac{m^2v_1^2}{M^2} + \frac{1}{2}mv_1^2 = E$$

$$\frac{1}{2}mv_1^2\left\{\frac{m}{M} + 1\right\} = E$$

$$\frac{1}{2}mv_1^2\left\{\frac{m+M}{M}\right\} = E \quad \text{——} \quad (2)$$

Now consider the cannon is fixed.

Let v_2 be velocity of ball in This case (i.e. when is no recoil)

Change in K.E = Energy generated by explosion

$$\frac{1}{2}M(0)^2 + \frac{1}{2}mv_2^2 - 0 = E$$

$$\frac{1}{2}mv_2^2 = E \quad \text{--- (3)}$$

divide equ (2) by (3)

$$\frac{\frac{1}{2}mv_1^2\left[\frac{m+M}{M}\right]}{\frac{1}{2}mv_2^2} = \frac{E}{E}$$

$$\frac{v_1^2}{v_2^2}\left[\frac{m+M}{M}\right] = 1$$

$$\frac{v_1^2}{v_2^2} = \frac{M}{m+M}$$

$$\frac{v_1}{v_2} = \sqrt{\frac{M}{m+M}}$$

Example: Find the velocity acquired by a block of wood of mass M lb which is free to move (recoil) when it is stuck by a bullet of mass m lb, moving with velocity v, in a direction passing through the centre of gravity. If the bullet is embadded a ft. Show that the resistance of the wood to the bullet, suppos uniform is $\frac{Mmv^2}{2(M+m)ga}$ lb-wt and that the time of penetration is $\frac{2a}{v}$ sec, during which time the block will move $\frac{ma}{m+M}$ ft.

<u>Solution:-</u> Mass of block = M lb

mass of bullet = m lb

velocity of bullet = v

Let v_1 be common velocity of block & the bullet after impact.

By Law of conservation of Linear Momentum

$$(M+m)v_1 = M(0) + mv$$

$$v_1 = \frac{mv}{M+m} \quad \text{---}(1)$$

(ii) Let R lb-wt be common resistance of wood to the bullet

Force of resistance = R - lb - wt

$= Rg$ Poundals

Work done by the force of resistance = (force)(Displacement)

$= (Rg)(a)$

By Principle of energy & work

Work done = Chang in K.E

$$Rga = \frac{1}{2}mv^2 + \frac{1}{2}M(0)^2 - \frac{1}{2}(m+M)v_1^2$$

$$Rga = \frac{1}{2}mv^2 - \frac{1}{2}(m+M)\frac{m^2v^2}{(M+m)^2}$$

$$= \frac{1}{2}mv^2 - \frac{1}{2}\frac{m^2v^2}{(M+m)}$$

$$Rga = \frac{1}{2}mv^2\left\{1 - \frac{m}{M+m}\right\}$$

$$= \frac{1}{2}mv^2\left\{\frac{M+m-m}{M+m}\right\}$$ t.1.

$$= \frac{mMv^2}{2(M+m)}$$

$$R = \frac{mMv^2}{2ga(M+m)} \text{ lb-wt}$$

(iii) Let t be time of penetration (اندر گھسنے کا عمل)

$$\text{retardation} = \frac{\text{Force}}{\text{mass}} \qquad \left[\because F = ma, \quad a = \frac{F}{m}\right.$$

$$= \frac{Rg}{m}$$

$$= \frac{mMv^2}{2ga(M+m)} \cdot \frac{g}{m}$$

$$= \frac{Mv^2}{2a(M+m)}$$

formula $v = u - at$

$$v_1 = V - \frac{Mv^2}{2a(M+m)}t \Rightarrow \frac{mv}{m+M} = V - \frac{Mv^2}{2a(M+m)}t$$

$$\frac{Mv^2}{2a(M+m)}t = V - \frac{mv}{m+M}$$

$$= \frac{mv+Mv-mv}{m+M}$$

$$t = \frac{Mv}{m+M} \cdot \frac{2a(M+m)}{Mv^2}$$

$$t = \frac{2a}{v} \text{ sec}$$

Let s be distance covered by block during this time

By Principle of energy & work applied to the block.

Work done on the block = Change in the K.E of the block

(Force) (Distance) = Change in K.E of the block

$$(Rg)(s) = \frac{1}{2} M v_1^2 - 0$$

$$Rgs = \frac{1}{2} \frac{M m^2 v^2}{(M + m)^2}$$

$$\frac{m M v^2}{2ga(M+m)} g s = \frac{1}{2} \frac{M m^2 v^2}{(M+m)^2}$$

$$s = \frac{M m^2 v^2}{2(M+m)^2} \cdot \frac{2a(M+m)}{m M v^2}$$

$$s = \frac{ma}{M+m} \text{ ft}$$

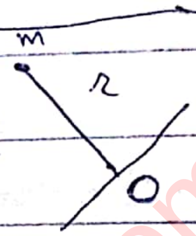
Moment of Inertia:-

If m is mass of particle of a body whose mass is M. Let r be distance of the particle from a line then mr^2 is called moment of inertia of the particle about the line is $\Sigma m r^2$ is called the moment of inertia of the body about the line.

If $\Sigma m r^2 = MK^2$ then K is called radius of gyration of the body about the line.

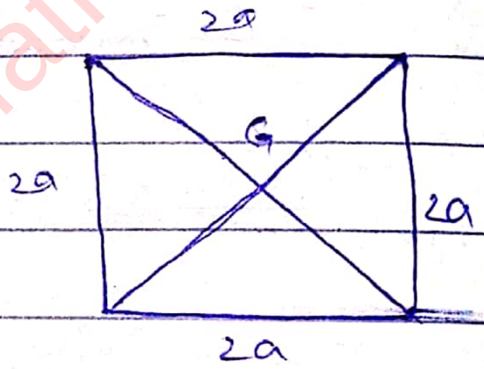
Note

K is some constant | Moment of inertia. mr^2



① Moment of Inertia of a Square Lamina

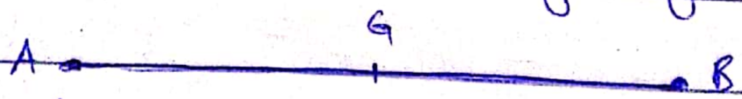
Let M be mass of a square lamina of side 2a. Let G be c.g of lamina



The moment of inertia of square lamina abount an axis throug G and ⊥ to lamina $= \frac{2}{3} Ma^2$

② Moment of Inertia (M.I) of rod.

Let 2a be length of rod AB of mass M. Let G be c.g of rod AB

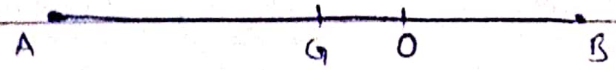


(i) M.I of rod about an axis throug

G and ⊥ to rod $= MK^2$ where K is radius of gyration rod

where $K^2 = \frac{1}{3}$ [half length of rod$]^2 = \frac{1}{3} a^2$

(ii) Let O be any other point of rod AB

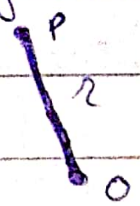


Then M.I of rod about line throug O & ⊥ to the rod $= M\{K^2 + (OG)^2\}$

$$= M\{\tfrac{1}{3} a^2 + (OG)^2\}$$

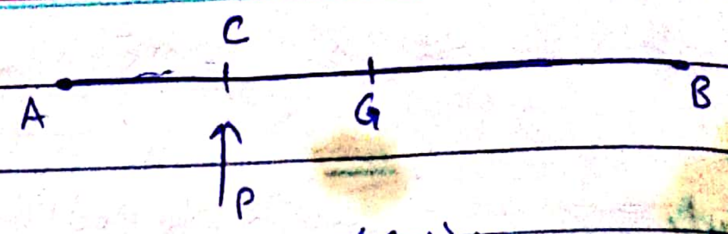
3) Angular momentum $= I\omega$ where I is momentum of inertia and ω is angular velocity.

4):- If u is linear velocity of particle placed at P & ω is its angular velocity about O.



Let OP = r, then $u = \omega r$

5) Let AB be rod & G is c.g of rod. Let c be any point b/w A & G. Let a blow P is stuck at c. Let u be linear velocity & ω be angular velocity of G.



velocity at end A $= u + (GA)\omega$

velocity at end B $= u - (GB)\omega$

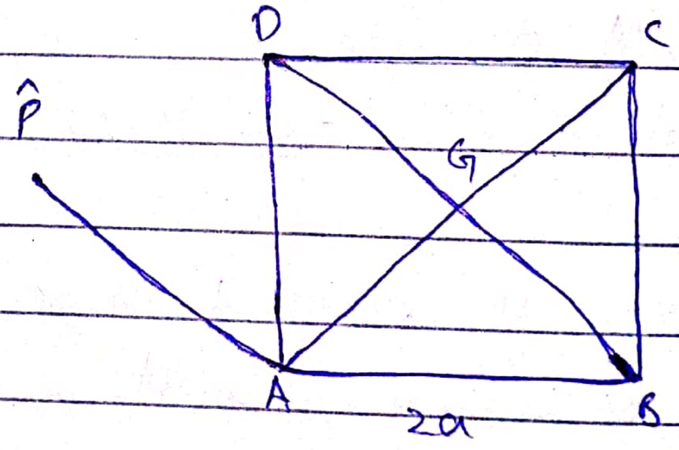
A end کی velocity بڑھے گی اور B کی گھٹے گی

2004

Q :- A uniform square plate of mass M of side $2a$, rests on a smooth horizontal table. A horizontal Impulsive force of magnitude $\hat{P}$ is applied at a corner in a direction $\perp$ to the diagonal at the corner. Show that the angular velocity generated by this impulsive force is $\dfrac{3\sqrt{2}\,\hat{P}}{2Ma}$

Solution

Let ABCD is a square plate of mass M.



Given length of each side $= 2a$

Let G be point of intersection of

AC & BD

By pathogoras theorem

$$(AC)^2 = (AB)^2 + (BC)^2$$

$$= (2a)^2 + (2a)^2$$

$$= 8a^2$$

$$\therefore AC = 2\sqrt{2}\, a$$

$$AG = \frac{1}{2} AC = \sqrt{2}\, a$$

I = moment of Inertia of square about an axis through G

$$I = \frac{2}{3} M a^2$$

Let ω be angular velocity

Angular momentum about $G = I\omega$

$$= \frac{2}{3} M a^2 \omega$$

Moment of $\hat{P}$ about $G = \hat{P}(AG)$

$$= \hat{P}(\sqrt{2}\, a)$$

$$= \sqrt{2}\, a \hat{P}$$

By principle of angular momtum

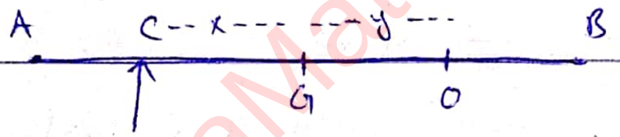
Change in angular momentum about G = momentum of external Impulsive force $\hat{P}$ about G.

$$\frac{2}{3} M a^2 \omega - 0 = \sqrt{2}\, a \hat{P}$$

$$\omega = \frac{3\sqrt{2}\, a \hat{p}}{2 M a}$$

Q :- A uniform rod at rest, is stuck by a blow at right angle to its length at a distance x from its centre. If the point about which it begins to turn is at a distance y from the centre G of the rod, Show that $y = \frac{a^2}{3x}$ where 2a is length of rod

Solution



Let AB be rod of length 2a & mass m with centre G

A below P is applied at C where CG = x. Let the rod rotates about pt O where OG = y

Let u be linear velocity of G & ω be angular velocity of G about O. after the below is stuck at C

$$u = \omega(OG) \quad \text{or} \quad u = \omega y$$

By principle of linear momentum

Change in linear Momentum = Total external Impulsive force.

$mu - 0 = P$

rest dy, a impact rod, 6 α, put velocity u

with

$m\omega y = P$ — ①

I = moment of Inertial about O

$I = m\{K^2 + (OG)^2\}$ where K is radius of gyration

$K^2 = \frac{1}{3}a^2$

$= m\{\frac{1}{3}a^2 + y^2\}$

Angular momentum about $O = I\omega$

$= m\omega\{\frac{1}{3}a^2 + y^2\}$

Moment of P about $O = P(OC) = P(x+y)$

By Principle of Angular momentum

Change in angular momentum about O = Momentum of external Impulsive force about O

$m\omega\{\frac{1}{3}a^2 + y^2\} - 0 = P(x+y)$

Put value of P

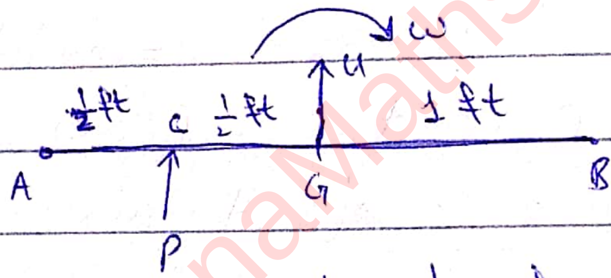
$m\omega(\frac{1}{3}a^2 + y^2) = m\omega y(x+y)$

$\frac{1}{3}a^2 + \cancel{y^2} = xy + \cancel{y^2}$

$y = \frac{a^2}{3x}$

Q :- A bar 2 ft long of mass 10 lb lies on a smooth horizontal table. It is stuck horizontaly at a distance of 6 inches from one end, The blow being ⊥ to the bar. The magnitude of the blow is such that it would impact a velocity of 3 ft/sec to a mass of 2 lb. Find the velocities of the ends of the bar just after it is stuck.

Solution



Let AB be rod of length 2 ft. & mass 10 lb. Let G be mid point of AB $\therefore AB = 2$ ft

$AG = BG = 1$ ft

A below P is stuck at C where

$AC = \frac{1}{2}$ ft

By principle of Linear momentum (Applied at mass of 2 lb moving with velocity 3 ft/sec)

Change in linear momentum = Total external impulsive force

$2(3) - 0 = P$

$P = 6$ unit

Below اتنی تیزی سے لگ کر وہ 2 lb mass کو لگائی جاتی ہے تو
وہ 3 ft کی velocity سے چلے۔ یہاں فرض کیا گیا ہے اور اس
فرض کیا گیا ہے principle of linear moment لگا کر

Suppose u is the linear velocity & ω be angular velocity of G just after the below is stuck.

By Principle of linear momentum applied to the rod.

Change in linear momentum = Total external impulsive force.

$10(u) - 0 = P \quad \Rightarrow 10u = 6$

$u = \frac{6}{10} = 0.6$ ft/sec

I = moment of initia of rod about G

$= Mk^2$

$= 10\left\{\frac{1}{3}(\text{half length of rod})^2\right\}$

$= \frac{10}{3}\left\{\frac{1}{2}(2)\right\}^2 = \frac{10}{3}$

Angular momentum $= I\omega = \frac{10}{3}\omega$

moment of P about G $= P(CG) = P\left(\frac{1}{2}\right) = \frac{P}{2}$

By Principle of angular momentum

Change in angular momentum about G = Moment of external Impulsive force about G.

$\frac{10}{3}\omega - 0 = \frac{P}{2}$

$\boxed{\omega = \frac{d\theta}{dt} \text{ rad/sec}}$

$\frac{10\omega}{3} = \frac{6}{2}$

$\omega = 0.9$ rad/se

velocity of end A $= u + (GA)\omega$

$\boxed{\text{velocity of centre}}$

جہاں دھکا لگے گا وہ velocity بڑھ جائے گی

$= 0.6 + 1(0.9)$

$= 0.6 + 0.9$

$= 1.5$ ft/sec

velocity of end B $= u - (GB)\omega$

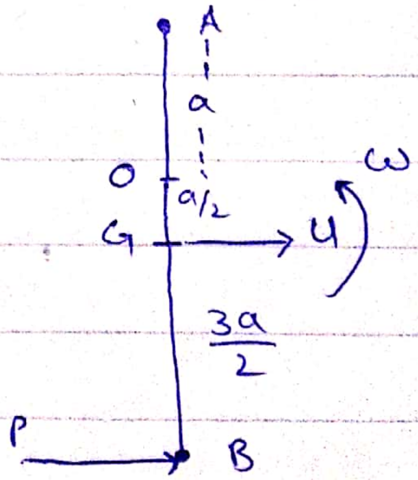
$= 0.6 - 1(0.9)$

$= -0.3$ ft/sec

Q :- A uniform rod of mass m & length 3a, hangs from a pin passing throug it at a distance a from the upper end. Find, in terms of m, a, g the magnitude of the smallest below, stuck at the lower end of the rod which will make the rod describe a complete revolution.

Solution :-

Below کتنی لگائی جائے کہ rod پورا چکر کاٹ جائے



Let AB be rod of mass m and length 3a with centre G. Let A be upper end and pin is passing through O where AO = a

$$\therefore AG = AG - AO$$

$$= \frac{3a}{2} - a = \frac{a}{2}$$

Let u be linear & ω be angular velocity of G just after the below is stuck at end B.

By principle of linear momentum

Change in linear momentum = Total external Impulsive force

$$mu - 0 = P$$

$$u = \frac{P}{m} \quad \text{—— (1)}$$

I = moment of inertia about O

$= m\{K^2 + (OG)^2\}$ where K is radius of gyration

$$= m\{\tfrac{1}{3}(\text{half length of rod})^2 + (OG)^2\}$$

$$= m\left\{\frac{1}{3}\left(\frac{3a}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right\}$$

$$= m\left\{\frac{3a^2}{4} + \frac{a^2}{4}\right\}$$

$$= ma^2$$

Angular momentum about O = Iω

$$= ma^2\omega$$

moment of P about O $= P(OB)$

$$= P\left(\frac{3a}{2} + \frac{a}{2}\right)$$

$$= 2aP$$

By Principle of Angular momentum

change in angular momentum about

O = moment of external force about O

$$ma^2\omega - 0 = 2aP$$

$$\omega = \frac{2aP}{ma^2} \quad , \quad \omega = \frac{2P}{ma}$$

velocity at end $B = u + (GB)\omega$

$$= u + \frac{3a}{2}\omega$$

$$= \frac{P}{m} + \frac{3a}{2}\left(\frac{2P}{ma}\right)$$

$$= \frac{P}{m} + \frac{3P}{m}$$

$$= \frac{4P}{m} \quad \text{—— (2)}$$

During the rotation about O

x = maximum height attained

by end B $= 2(OB)$

$$= 2(2a)$$

$$x = 4a$$

اب P کی value نکالی ہے
اس لیے دوسرے طریقے سے
End B پر velocity نکالی ہے

پورا چکر کاٹنے کے لیے B پر
velocity کم از کم اتنی ہو کہ rod کی
ٹانگیں اوپر ہو جائیں اور سر نیچے

velocity of end B is least if during rotation the velocity of B at the highest point is zero formula

$$v^2 - u^2 = -2gx$$

put $v = 0$, $x = 4a$

$$0 - u^2 = -2g(4a)$$

$$u^2 = 8ga$$

$$u = \sqrt{8ga} \longrightarrow (3)$$

from eqn (2) & (3)

$$\frac{4P}{m} = \sqrt{8ga}$$

$$P = \frac{m}{4} 2\sqrt{2}\sqrt{ga}$$

$$P = m\sqrt{\frac{ga}{2}}$$

Poisson's Hypothesis :-

According to poisson's Hypothesis when two bodie come into contact, there is a shot interval in which they undergo compression. It is called interval of compression. During this interval the centres of the bodies come closer & closer to each other as a result of diformation of two bodies.

At the instant of greatest compression the two bodies moves with the common velocity & their centres are nearest to each other.

This is followed by another short interval in which the orignal shape of the body is restored. It is called interval of restitution.

If I_1 & I_2 are Impulsive pressures(force) during compression & restitution respectively then according to poission hypothesis

$$I_2 : I_1 = e : 1$$

$$\frac{I_2}{I_1} = \frac{e}{1}$$

$$I_2 = e I_1$$

e is called co-efficient of restitution

Equivalence between Possion's Hypothesis and Newton Law of Restitution:-

Suppose u is the common velocities of masses m_1 and m_2 at the instant of greatest compression.

Let u_1 & u_2 be velocities of m_1 & m_2 before impact.

If I_1 is Impulsive pressure of m_1 on m_2 then impulsive pressure of m_2 on m_1 is I_1

For Compression,

Change of linear momentum of m_1 = Impulsive force on m_1

$$m_1 u - m_1 u_1 = -I_1 \quad \text{———} ①$$

Also change of linear momentum of m_2 = Impulsive force on m_2

$$m_2 u - m_2 u_2 = I_1 \quad \text{———} ②$$

Let V_1 & V_2 be the velocities of m_1 & m_2 after impact respectively.

Let I_2 be the impulsive pressure of m_1 on m_2.

∴ Impulsive pressure of m_2 on m_1 = $-I_2$

For restitution

$$m_1 V_1 - m_1 u = -I_2$$

& $$m_2 V_2 - m_2 u = I_2$$

By Poisson's hypothesis

$I_2 = eI_1$ put values of I_2

$$m_1 V_1 - m_1 u = -eI_1 \quad \text{———} ③$$

$$m_2 V_2 - m_2 u = eI_1 \quad \text{———} ④$$

eleminate u from ① & ②

Multiply ① by m_2 and ② by m_1 and subtract

$$\cancel{m_1 m_2 u} - m_1 m_2 u_1 = -m_2 I_1$$

$$\underset{-}{\cancel{m_1 m_2 u}} \underset{+}{-} m_1 m_2 u_2 \underset{-}{=} m_1 I_1$$

$$m_1 m_2 (u_2 - u_1) = -(m_1 + m_2) I_1 \quad \text{———} ⑤$$

similarly eleminate u from ③ & ④

multiply ③ by m_2 & ④ by m_1 & subtract

$$m_1 m_2 v_1 - m_1 \cancel{m_2} u = -m_2 e I_1$$

$$m_1 m_2 v_2 - m_1 \cancel{m_2} u = m_1 e I_1$$

$$- \quad + \quad -$$

$$-m_1 m_2 (v_2 - v_1) = -eI_1 (m_1 + m_2)$$

$$m_1 m_2 (v_2 - v_1) = eI_1 (m_1 + m_2) \quad \text{—— ⑥}$$

Divide ⑥ by ⑤

$$\frac{\cancel{m_1 m_2}(v_2 - v_1)}{\cancel{m_1 m_2}(u_2 - u_1)} = \frac{eI_1 \cancel{(m_1 + m_2)}}{-\cancel{(m_1 + m_2)} I_1}$$

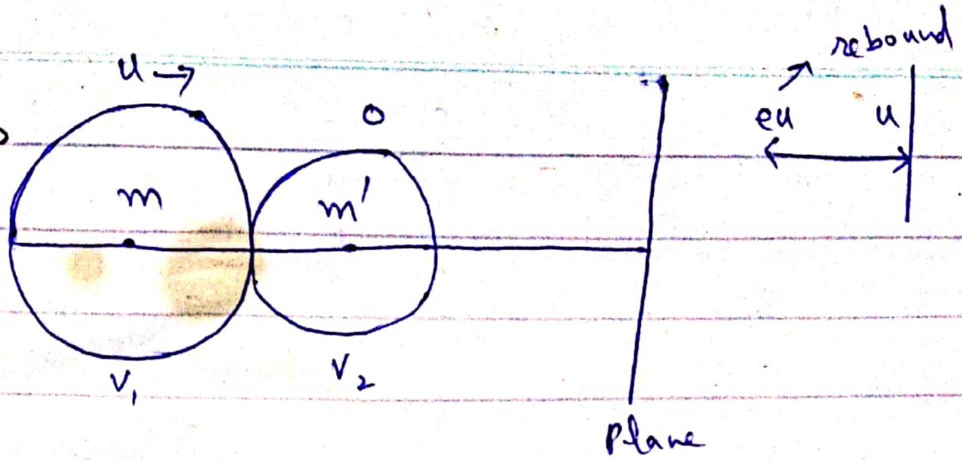
$$\frac{v_2 - v_1}{u_2 - u_1} = -e$$

$$v_2 - v_1 = -e(u_2 - u_1)$$

which is Newton law of restitution

Example :- An imperfectly elastic sphere of mass m moving with velocity u impinges on another sphere of mass m' at rest. The second sphere afterwards strikes a vertical plane at right angle to its path. Show that there will be no further impact of the sphere if $m(1 + e' + ee') < em'$ where e & e' are the coefficient of restitution b/w the spheres & plane respectively.

Solution



Let v_1 & v_2 be velocities of spheres of mass m & m' after impact respectively.

By Law of conservation of momentum

$$mv_1 + m'v_2 = mu + m'(0)$$

$$mv_1 + m'v_2 = mu \quad \text{——— (1)}$$

By Newton Law of restitution

$$v_2 - v_1 = -e(0 - u)$$

$$v_2 - v_1 = eu \quad \longrightarrow (2)$$

v_1 اور v_2 کی value نکالنی ہے

multiply equ (2) by m & add to (1)

$$mv_2 + m'v_2 = mu + meu$$

$$v_2 = \frac{mu(1+e)}{m+m'}$$

Put in equ (2)

$$\frac{mu(1+e)}{m+m'} - v_1 = eu$$

$$v_1 = \frac{mu(1+e)}{m+m'} - eu$$

$$v_1 = \frac{mu(1+e) - eu(m+m')}{m+m'}$$

$$v_1 = \frac{mu + emu - emu - em'u}{m + m'}$$

$$v_1 = \frac{(m - em')u}{m + m'}$$

Now,

The sphere of mass m' moves with velocity v_2 strikes the vertical plane & rebounds with velocity $e'v_2$ in the direction away from plane.

The velocity of sphere of mass m away from plane $= -v_1$

There will be no further impact if

$$e'v_2 < -v_1$$

$$\frac{e'(1+e)mu}{(\cancel{m+m'})} < \frac{-(m - em')u'}{(\cancel{m+m'})}$$

$$e'm + ee'm < -m + em'$$

$$m + e'm + ee'm < em'$$

$$m(1 + e' + ee') < em'$$

The end.

MUHAMMAD TAHIR WATTOO

COMSATS ISLAMABAD

03448563284