Differentiation

Definition:

Let f be a function defined on an open interval containing point x. The derivative of f at x, is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \quad \text{provided the limit exists}$$

A function f is differentiable on an open interval $]a,b[$ if it is differentiable at each point of $]a,b[$.

A function f is differentiable on a closed interval $[a,b]$ if i) it is differentiable on the open interval $]a,b[$

ii) its right hand derivative at a and left hand derivative at b exist.

A function f is differentiable if and only if it is differentiable at each point of its domain.

Theorem:

If f is differentiable at a point $a \in \text{Dom} f$, then f is continuous at a.

Proof:

To prove that f is continuous at a, we need to show that

$$\lim_{x \to a} f(x) = f(a)$$

i.e

$$\lim_{x \to a} [f(x) - f(a)] = 0$$

Now.

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}(x-a)$$

$$= \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \lim_{x \to a}(x-a)$$

$$= f'(a) \cdot 0 = 0$$

Thus f is continuous at a.

NOTE:

The converse of this theorem is false. i.e a continuous function may not be differentiable.

Example:

Differentiate $f(x) = \frac{x}{x-1}$ (by definition).

Sol/ $f(x) = \frac{x}{x-1}$ and $f(x+h) = \frac{(x+h)}{(x+h)-1}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} = \lim_{h \to 0} \frac{\frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{(x^2 - x + xh - h) - x^2 - xh + x}{(x+h-1)(x-1)}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{-h}{(x+h-1)(x-1)}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)}$$

By applying limit we get

$$= -\frac{1}{(x-1)^2}$$

Notations:

Some common notations of derivatives are

$$f'(x),\quad y',\quad \frac{dy}{dx},\quad \frac{df}{dx},\quad \frac{d}{dx}f(x).$$

$$D(f(x)),\quad D_x f(x).$$

Derivative at specific number.

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a}.$$

Intermediate Value Theorem:

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

Q: If $f(x) = 4-x^2$; find $f'(-3)$, $f'(0)$, $f'(1)$

$$f(x) = 4-x^2$$

$$f(x+h) = 4-(x+h)^2$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{4-(x+h)^2-(4-x^2)}{h}$$

$$= \lim_{h\to 0} \frac{4-(x^2+2xh+h^2)-4+x^2}{h}$$

$$= \lim_{h\to 0} \frac{4-\cancel{x^2}-2xh-h^2-\cancel{4}+\cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{-2xh-h^2}{h^2} = \lim_{h\to 0} \frac{h(-2x-h)}{h}$$

$$= \lim_{h\to 0} (-2x-h) = -2x. \quad \text{(By applying limit)}$$

$f'(-3) = -2(-3) = 6$

$f'(0) = -2(0) = 0$

$f'(1) = -2(1) = -2.$

Q $f(s) = \sqrt{2s+1}$ find $f'(0), f'(1), f'(1/2)$

Sol

$f(s) = \sqrt{2s+1}$

$f(s+h) = \sqrt{2(s+h)+1}$

$$f'(s) = \lim_{h \to 0} \frac{f(s+h) - f(s)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2(s+h)+1} - \sqrt{2s+1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2(s+h)+1} - \sqrt{2s+1}}{h} \times \frac{\sqrt{2s+2h+1} + \sqrt{2s+1}}{\sqrt{2s+2h+1} + \sqrt{2s+1}}$$

$$= \lim_{h \to 0} \frac{(\sqrt{2s+2h+1})^2 - (\sqrt{2s+1})^2}{h(\sqrt{2s+2h+1} + \sqrt{2s+1}}$$

$$= \lim_{h \to 0} \frac{2s+2h+1 - 2s - 1}{h(\sqrt{2s+2h+1} + \sqrt{2s+1}}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{2s+2h+1}+\sqrt{2s+1}}$$

Applying limit

$$= \frac{2}{\sqrt{2s+1}+\sqrt{2s+1}} = \frac{2}{2(\sqrt{2s+1})}$$

$$f'(s) = \frac{1}{\sqrt{2s+1}}$$

$$f'(0) = \frac{1}{\sqrt{2(0)+1}} = \frac{1}{1} = 1$$

$$f'(1) = \frac{1}{\sqrt{2(1)+1}} = \frac{1}{\sqrt{3}}$$

$$f'(1/2) = \frac{1}{\sqrt{2(1/2)+1}} = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$$

Q Let $f(x) = \begin{cases} x & \text{if} \quad 0 \le x \le 1 \\ 2x-1 & \text{if} \quad 1 < x \le 2 \end{cases}$

Discuss the continuity and differentiability of f at $x=1$.

Sol: first we will discuss the continuity of f

Here $f(1) = 1$

$$f(1^-) = \lim_{x\to 1^-} f(x)$$

$$= \lim_{x\to 1^-} (x)$$

$$f(1^-) = 1$$

$$f(1^+) = \lim_{x\to 1^+} f(x)$$

$$= \lim_{x\to 1^+} (2x-1)$$

$$= 2(1)-1$$

$$= 1$$

Since $f(1^-) = f(1^+) = f(1)$

So f is continuous at $x=1$.

As

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

So $$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$Lf'(1) = \lim_{h\to 0^-} \frac{1+h-1}{h} = \frac{h}{h} = 1$$

$$Rf'(1) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0^+} \frac{[2(1+h)-1]-1}{h} = \lim_{h\to 0^+} \frac{[2+2h-1]-1}{h}$$

$$= \lim_{h\to 0^+} \frac{2h}{h} = 2.$$

Since

$Lf'(1) \neq Rf'(1)$.

So f is not differentiable at $x=1$.

Q If $f(x) = \begin{cases} x^2 \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$

Show that $f(x)$ is continuous and differentiable at $x=0$.

Sol

Here $f(0) = 0$

Now

$$f(0^-) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 \sin 1/x$$

$$= \lim_{h\to 0} (-h)^2 \sin(-1/h) = \lim_{h\to 0} -h^2 \sin 1/h$$

$$= -(0)^2 \underbrace{\sin(1/0)}_{\text{Some no. in } [-1,1]} = 0$$

$$f(0^-) = 0.$$

$$f(0^+) = \lim_{x\to 0^+}$$

$$= 0.$$

Since $f(0^-) = f(0^+) = f(0)$.

So f is continuous at $x = 0$.

As $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \Rightarrow f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$

Now.

$$Lf'(0) = \lim_{h\to 0^-} \frac{h^2 \sin 1/h - 0}{h}$$

$$= \lim_{h\to 0^-} (h \sin 1/h)$$

$$= 0 \text{ (some number)}$$

$$= 0.$$

$$Rf'(0) =$$

$$= 0$$

Since $Lf'(0) = Rf'(0)$.

So f is derivable at $x = 0$.

Q

$$f(x) = \begin{cases} (x-a)\sin\left(\frac{1}{x-a}\right) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$$

Is this fn continuous and differentiable at $x = a$?